Relation Algebras can tile*

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Abstract

Undecidability of the equational theory of the class RA of relation algebras can easily be proved using the undecidability of the word-problem for semigroups. With some effort and ingenuity, one can push this proof through for the larger class SA. We provide another "cause" for undecidability which works for even larger classes than SA. The reason is that we can encode the tiling problem. In doing so we will meet very simple BAO-varieties with undecidable equational theories which might be useful in other undecidability proofs.

Our work is part of the research project which tries to establish the border between undecidability and decidability in relational type algebras, cf (Németi, 1987), (Maddux, 1994), (Andréka et al., 1995) and the references therein, (Marx, 1995). The ultimate goal of this research is to come up with versions of relational algebra which are still suitable for modern dynamic applications but whose equational theory is decidable or even tractable.

(Maddux, 1994): Corollary 16 shows that the equational theory of any variety K such that $\mathsf{RRA} \subseteq \mathsf{K} \subseteq \mathsf{SA}$ is undecidable. The ingenious proof goes by showing that for every quasi-equation q in the language of semigroups, there is an effectively obtainable equation q^* in the language of the class K such that

q is valid in all semigroups if and only if q^* is valid in K.

The theorem follows then immediately from the fact that the quasi-equational theory of semigroups is undecidable. (Maddux, 1980) uses the same problem

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to prove undecidability of Df_3 and CA_3. (Andréka et al., 1995) also use (versions of) the semigroup-wordproblem to show undecidability of weakenings of RA (but where composition is associative).

We prove a strengthening of Maddux' SA-theorem using a very different undecidable problem: $\mathbb{N} \times \mathbb{N}$ tiling. We also show that it will be very hard (maybe impossible) to obtain the same result using the quasi-equational theory of semigroups (Theorems 4.1 and 4.3).

This covers the first theme of the paper: find a "different cause" for undecidability in relational type algebras. This answers a question raised in (van Benthem, 1996) p. 174 about the connection between these two styles of undecidability-analysis regarding varieties of the relational algebraic type. We restrict the discussion here to the case in which all algebras are boolean algebras with operators (BAO's). So when we say that an axiom A causes undecidability, we really mean A plus the BAO-axioms causes undecidability.

The second theme is to show how easily we get undecidable varieties. If we look at the decidability/undecidability results in both relation and cylindric algebra theory, we see the following pattern: in relation algebras undecidability is connected to some form of associativity of composition and in cylindric algebras it is connected to some form of commutativity of the cylindrifications. If we do not require these axioms, decidability is possible: e.g., the class WA (cf. (Maddux, 1982)) defined by all RA equations, but associativity replaced by the weaker $(x;(y \wedge 1'));z = x;((y \wedge 1');z)$ has a decidable equational theory (Németi, 1987); the equational theory of the variety of cylindric relativized set algebras of any dimension is decidable (Németi, 1986), and the same holds for the variety of relativized relation set algebras (Marx, 1995). These last two classes can be axiomatized by sets of canonical equations which correspond on the atom-structure level to first-order universal Horn conditions. Looking at these and related decidability results in relativized cylindric and relation algebra theory, a tentative conclusion jumps to mind: the axioms which express existential conditions on the atom-structures are to blame for undecidability. An even stronger conclusion –which this author at least believed for some time– is that as long as such a variety does not validate any existential frame-conditions its equational theory is decidable.

We will give two examples showing that the last conclusion is blatantly false (Theorems 2.1 and 4.6).

The paper is organized as follows. In the first section we define the tiling problem and a simple BAO-variety Tile having an undecidable quasi-equational theory but whose equational theory is just as complex as that of boolean algebras. In the next section we show how to translate quasi-equations into equations using a deduction theorem. In the third section we interpret the variety Tile in RRA using the standard translation from modal logic. In the last section we put all our machinery to work. We end with some implications of our research, outlining a promising new research field, and stating some technical questions.

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Preliminaries. We assume familiarity with relation(al type) algebras. The class SA is defined as the class of all relational type algebras satisfying all RA -axioms, except instead of associativity it satisfies $(x;1);1 = x;1$. For $\mathfrak{F} = (W, R_i)_{i \in I}$ a first order structure, we define $\mathfrak{F}^+$ to be the *complex algebra* of $\mathfrak{F}$. It is the algebra $(Pow(W), \cup, \setminus, f_i)_{i \in I}$ where each f_i is a n-place operator if R_i is an n+1-ary relation with the following definition:
$f_i(X_1, \ldots, X_n) \;=\; \{w \in W \mid$ there exists $w_1, \ldots, w_n \in W$ s.t. $w_j \in X_j$ and $R_i w, w_1, \ldots, w_n$.
For K a class of structures, K^+ denotes the class of all complex algebras of structures in K. The variety generated by such a K^+ is called a *complex variety*, cf. (Goldblatt, 1989).

1 Undecidability by tiling

As shown in (Berger, 1966), the following problem is undecidable.

$\mathbb{N} \times \mathbb{N}$ **tiling**: Given a finite set $\mathcal{T}$ of tiles, can $\mathcal{T}$ tile $\mathbb{N} \times \mathbb{N}$?.

That is, does there exists a function t from $\mathbb{N} \times \mathbb{N}$ to $\mathcal{T}$ such that:

$$\begin{aligned} \text{right}(t(n,m)) &= \text{left}(t(n+1,m)), \text{ and} \\ \text{up}(t(n,m)) &= \text{down}(t(n,m+1))? \end{aligned}$$

Here "right, left, up and down" are functions determining the four "colors" of a tile. Tiles are assumed to be fixed in orientation. So the last condition says that colors of adjacent tiles match.

In the next definition we tailor-make a variety of boolean algebras with operators to the tiling problem. This variety is inspired by Theorem 4.2.1 and its proof in (Spaan, 1993), a highly recommended book for anyone interested in complexity-questions in BAO's.

Definition 1.1 *Let* Ti *denote the variety of boolean algebras with normal unary operators of the form* $(A, \vee, -, \langle\mathsf{right}\rangle, \langle\mathsf{up}\rangle)$ *satisfying the following axioms:*

$$\begin{array}{llcll} Ti1 & hx \wedge hy & \leq & h(x \wedge y) & \text{for } h \in \{\langle\mathsf{right}\rangle, \langle\mathsf{up}\rangle\} \\ Ti2 & \langle\mathsf{right}\rangle\langle\mathsf{up}\rangle x \wedge \langle\mathsf{up}\rangle\langle\mathsf{right}\rangle y & \leq & \langle\mathsf{right}\rangle\langle\mathsf{up}\rangle(x \wedge y). & \end{array}$$

Let $\mathfrak{Ti}$ *be the class of all first order structures* $\mathfrak{F} = (W, R^{\langle\mathsf{right}\rangle}, R^{\langle\mathsf{up}\rangle})$, *with the* R *binary relations on* W *satisfying*

$$\begin{array}{lll} Ti1' & \forall xyz((xR^h y \wedge xR^h z) \to y = z) & \text{for } h \in \{\langle\mathsf{right}\rangle, \langle\mathsf{up}\rangle\} \\ Ti2' & \forall xyzz'z''((zR^{\langle\mathsf{right}\rangle}z'R^{\langle\mathsf{up}\rangle}x \wedge zR^{\langle\mathsf{up}\rangle}z''R^{\langle\mathsf{right}\rangle}y) \to x = y). & \end{array}$$

Note that the last condition can be written equivalently using only 3 variables as

$$Ti2'^{*} \quad \forall xyz((zR^{\langle\mathsf{right}\rangle}yR^{\langle\mathsf{up}\rangle}x) \to \forall y(zR^{\langle\mathsf{up}\rangle}y \to \forall z(yR^{\langle\mathsf{right}\rangle}z \to x = z))).$$

Theorem 1.2 *(i)* $\mathsf{Ti} = \mathbf{S}\mathfrak{Ti}^{+}$ *is a canonical variety whose axioms correspond to universal Horn sentences.*
(ii) Ti *enjoys the strong amalgamation property.*
(iii) The quasi-equational theory of Ti *is undecidable.*
(iv) The equational theory of Ti *is decidable, in fact the satisfaction-problem is* NP*-complete.*

Ti is the variety we will use in the undecidability proofs to come. Since it is so simple, it might have applications for other undecidability results as well.

PROOF. (i) All the axioms are positive equations, whence the variety is canonical. The first order correspondents can be computed using e.g. a Sahlqvist algorithm (de Rijke and Venema, 1995). (ii) follows from Lemma 5.2.6 in (Marx, 1995), which says that every canonical BAO-variety SK^{+} with K universal Horn definable has strong amalgamation.

The proof of (iii) will be simple since Ti is deliberately designed to encode the tiling problem. It is sufficient to show that we can find a non-trivial algebra in Ti which satisfies a set of equations $\Sigma_{\mathcal{T}}$ if and only if $\mathcal{T}$ tiles $\mathbb{N} \times \mathbb{N}$. Because then we have shown that

$$\mathsf{Ti} \not\models \bigwedge \Sigma_{\mathcal{T}} \to 0 = 1 \text{ iff } \mathcal{T} \text{ tiles } \mathbb{N} \times \mathbb{N},$$

and we know that the tiling problem is undecidable.

Let $\mathcal{T} = \{t_1, \ldots, t_k\}$ be given. Define $\Sigma_{\mathcal{T}}$ as the following set of equations:

$$
\begin{array}{llcl}
A1 & \langle \mathsf{right}\rangle 1 & = & \langle \mathsf{up}\rangle 1 = 1 \\
A2_i & T_i & \leq & \bigwedge_{i \neq j} \neg T_j \\
A3 & 1 & \leq & \bigvee_{1 \leq i \leq k} T_i \\
A4_i & T_i & \leq & \langle \mathsf{right}\rangle \bigvee \{T_j \in \mathcal{T} \mid \mathrm{right}(t_i) = \mathrm{left}(t_j)\} \\
A5_i & T_i & \leq & \langle \mathsf{up}\rangle \bigvee \{T_j \in \mathcal{T} \mid \mathrm{up}(t_i) = \mathrm{down}(t_j)\}.
\end{array}
$$

Claim 1 If $\mathcal{T}$ tiles $\mathbb{N} \times \mathbb{N}$ then there exists a non-trivial Ti $\mathfrak{A}$, such that $\mathfrak{A} \models \Sigma_{\mathcal{T}}$.

PROOF OF CLAIM. Take a $\mathfrak{Ti}$-type structure $\mathfrak{F}$ with domain $\mathbb{N} \times \mathbb{N}$. Define the relations as follows:

$R^{\langle \mathsf{right}\rangle} = \{((k,m),(k,m+1)) \mid k,m \in \mathbb{N}\}$

$R^{\langle \mathsf{up}\rangle} = \{((k,m),(k+1,m)) \mid k,m \in \mathbb{N}\}$.

Clearly $\mathfrak{F} \in \mathfrak{Ti}$, whence $\mathfrak{F}^+ \in \mathsf{Ti}$. Define the following valuation of the variables T_i:

$(k,m) \in T_i$ iff t_i tiles (k,m).

It's simple to check that $\mathfrak{F}^+ \models \Sigma_{\mathcal{T}}$. ◀

Claim 2 Let $\mathfrak{A}$ be a non-trivial Ti validating $\Sigma_{\mathcal{T}}$. Then $\mathcal{T}$ tiles $\mathbb{N} \times \mathbb{N}$.

PROOF OF CLAIM. Let $\mathfrak{A}$ be a non-trivial Ti validating $\Sigma_{\mathcal{T}}$. By Theorem 1.2.(i), $\mathfrak{A}$ is a subalgebra of $\mathfrak{F}^+$, for some $\mathfrak{F} = (W, R^{\langle \mathsf{right}\rangle}, R^{\langle \mathsf{up}\rangle}) \in \mathfrak{Ti}$. By *Ti1′* and *A1*, the relations $R^{\langle \mathsf{right}\rangle}$ and $R^{\langle \mathsf{up}\rangle}$ are total functions. We will denote them by s_{right} and s_{up}, respectively. We claim that

$$\mathfrak{F} \models \forall x (s_{\mathsf{right}} s_{\mathsf{up}} x = s_{\mathsf{up}} s_{\mathsf{right}} x), \tag{1}$$

which follows immediately from *Ti2′*.

Let $w \in W$ be arbitrary (it exists, because $\mathfrak{A}$ is non-trivial). Define inductively a function $f : \mathbb{N} \times \mathbb{N} \longrightarrow W$ as

$f(0,0) = w$

if $f(k,m) = x$, then $f(k,m+1) = s_{\mathsf{up}}(x)$

if $f(k,m) = x$, then $f(k+1,m) = s_{\mathsf{right}}(x)$.

f is a well-defined function because of (1). Now define a tiling on $\mathbb{N} \times \mathbb{N}$ by

$$t_i \text{ tiles } (k,m) \text{ if } f((k,m)) \in T_i^{\mathfrak{A}}.$$

By *A2* and *A3*, every (k,m) is tiled by exactly one tile, and by *A4* and *A5* adjacent colors match. ◀

This proves part (iii) of the theorem. For part (iv) we only give a sketch. We will show that for any term τ, $\mathsf{Ti} \not\models \tau = 1$ iff there exists an $\mathfrak{F} \in \mathfrak{Ti}$ whose size is polynomial in the length of τ and $\mathfrak{F}^+ \not\models \tau = 1$. This gives us our result because the satisfaction-problem of boolean algebras is NP-complete.

So assume $\mathsf{Ti} \not\models \tau = 1$, for some τ. Then there is a $\mathfrak{F} = (W, R^{\langle\mathsf{right}\rangle}, R^{\langle\mathsf{up}\rangle}) \in \mathfrak{Ti}$ such that $\mathfrak{F}^+ \not\models \tau = 1$. That means there is a valuation of the variables v and a $w \in W$ such that $w \in -\tau_\mathsf{v}$.

We will generate a finite structure $\mathfrak{F}'$ from w and show that the restriction of the valuation v still ensures that $w \in -\tau_\mathsf{v}$, but now in the small structure.

Let T be the smallest set of terms which contains all subterms of τ and is closed under the rule $\langle\mathsf{right}\rangle\langle\mathsf{up}\rangle\sigma \in T \iff \langle\mathsf{up}\rangle\langle\mathsf{right}\rangle\sigma \in T$. Clearly T is finite and its size is polynomial in $|\tau|$ (we use this to denote the length of τ). We will build step by step a labeled substructure $\mathfrak{F}'$.

In step 0, we put $w \in W'$ and set $l(w) = \{\sigma \in T \mid w \in \sigma_\mathsf{v}\}$ (l assigns this label to w).

In step $n+1$, we consider all elements of W thrown in in the previous step. Let v be such an element. If there exists a term $\langle\mathsf{right}\rangle\sigma \in l(v)$, then we include the point $v' \in W$ such that $R^{\langle\mathsf{right}\rangle}vv'$ and set (if $v' \notin W_n$) $l(v') = \{\sigma \in T \mid \langle\mathsf{right}\rangle\sigma \in l(v)\}$. Similarly when we find $\langle\mathsf{up}\rangle\sigma \in l(v)$. If we find no such formula we do nothing.

This process terminates after m steps where m is the level of $\langle\mathsf{right}\rangle$ and $\langle\mathsf{up}\rangle$ nesting in τ. Now let $\mathfrak{F}'$ be the substructure of $\mathfrak{F}$ with the domain created in the above construction. $\mathfrak{F}' \in \mathfrak{Ti}$, since $\mathfrak{Ti}$ is closed under taking substructures, and the size of $\mathfrak{F}'$ is polynomial in the length of τ. It is almost trivial to see that the valuation v restricted to $\mathfrak{F}'$ will enforce that $w \notin \tau_\mathsf{v}$. Whence $\mathfrak{F}'^+ \not\models \tau = 1$. QED

Instead of working with the variety Ti it is more convenient to work with a slightly smaller variety Tile, which is the class of all algebras in Ti satisfying $\langle\mathsf{right}\rangle 1 = \langle\mathsf{up}\rangle 1 = 1$. From the above proof we can distill the following facts.

Corollary 1.3 *(i)* Tile *is a canonical variety.*
(ii) Tile *can be defined by Ti1, Ti3 and Ti4*
Ti3 $\quad \langle\mathsf{right}\rangle 1 = \langle\mathsf{up}\rangle 1 = 1$
Ti4 $\quad \langle\mathsf{right}\rangle\langle\mathsf{up}\rangle x = \langle\mathsf{up}\rangle\langle\mathsf{right}\rangle x.$
(iii) $\mathsf{Tile} = \mathbf{S}\mathfrak{Tile}^+$, *where* $\mathfrak{Tile}$ *is the class of all structures* $(W, s_{\mathsf{right}}, s_{\mathsf{up}})$ *in which* s_{right} *and* s_{up} *are commuting total functions.*
(iv) Tile*-validity of quasi-equations of the form* $\bigwedge e_i \to 0 = 1$ *is undecidable.*

Note that the class $\mathfrak{Tile}$ can be axiomatized with a first order sentence in the language $(R^{\langle\mathsf{right}\rangle}, R^{\langle\mathsf{up}\rangle})$ containing only 3 variables.

2 Deduction theorem

We will expand the variety Ti with a seemingly harmless unary operator, and show that this boosts the complexity of the equational decision problem from NP-complete to undecidable. On the other hand, all the other properties of Ti (canonicity, strong amalgamation and universal Horn definable) are inherited by the expansion. We will see that the crucial difference between the two varieties is that the expansion has a deduction theorem.

Let $\mathsf{Ti}+\Diamond$ be the variety of all Ti algebras expanded with a new normal unary operator $\Diamond$, satisfying

Ti4 $\Diamond\Diamond x \leq \Diamond x$

Ti5 $\langle\mathsf{right}\rangle x \leq \Diamond x$ and $\langle\mathsf{up}\rangle x \leq \Diamond x$.

Let $\mathfrak{Tile}+R^{\Diamond}$ be the class of all $\mathfrak{Ti}$-structures expanded with a binary relation $R^{\Diamond}$ satisfying

Ti4′ $\forall xyz((R^{\Diamond}xy \wedge R^{\Diamond}yz) \to R^{\Diamond}xz)$

Ti5′ $\forall xy(R^{\langle\mathsf{right}\rangle}xy \to R^{\Diamond}xy)$ and $\forall xy(R^{\langle\mathsf{up}\rangle}xy \to R^{\Diamond}xy)$.

Theorem 2.1 *(i)* $\mathsf{Ti}+\Diamond = \mathbf{S}(\mathfrak{Tile}+R^{\Diamond})^{+}$ *is a canonical variety whose axioms correspond to universal Horn sentences.*
(ii) $\mathsf{Ti}+\Diamond$ *enjoys the strong amalgamation property.*
(iii) $\mathsf{Ti}+\Diamond$ *has a deduction theorem.*
(iv) The equational theory of $\mathsf{Ti}+\Diamond$ *is undecidable.*

Before we prove the theorem we give a general result about deduction terms.

Definition 2.2 *Let* K *be a class of algebras. We say that* K *has a deduction theorem, if for any quasi-equation* q*, there exists an effectively obtainable equation* q^* *such that* $\mathsf{K} \models q$ *iff* $\mathsf{K} \models q^*$.

(Blok and Pigozzi, 1991) and the references therein contains a host of results about deduction theorems. A deduction theorem is useful for several reasons. For example, by Birkhoff's theorem, any **SPUp**-closed class with a deduction theorem is a variety. Every discriminator variety of BAO's has a deduction theorem (one can define a complemented closure operator which extends all other operators, a special case of Theorem 2.3 below).

The following theorem is a generalization of a folklore result in modal logic (cf. e.g., (van Benthem, 1983): Lemma 2.33). For $\Diamond$ a unary operation we abbreviate $\Diamond\ldots\Diamond x$ (n-times $\Diamond$) by $\Diamond^n x$.

Theorem 2.3 *Let* K *be a variety of normal boolean algebras with operators. If* K *has a term-definable normal operator* $\Diamond$ *and there is some natural number* l *such that (2) and (3) below hold, then* K *has a deduction theorem.*

$$\mathsf{K} \models \Diamond^k x \leq \Diamond^l x, \text{ for some } k > l. \tag{2}$$

For all $1 \leq i \leq n$, and for every n-place operator f in the similarity type of K,

$$\mathsf{K} \models f(x_1, \ldots, x_n) \leq \Diamond^m x_i, \textit{ for some } m \leq l. \tag{3}$$

We find lots of examples in algebraic logic: the simplest are closure algebras (BAO varieties with one unary operator f satisfying $ffx \leq fx$). In the class RA, $\Diamond x$ can be defined as $1;x;1$, in CA_n, as $\mathsf{c}_0 \ldots \mathsf{c}_{n-1}x$. In a BAO variety with two conjugated closure operators F and P, $\Diamond x$ can be defined as $Fx \vee Px$.

PROOF. Define $cx = \bigvee_{0 \leq n \leq l} \Diamond^n x$. c^∂ denotes the dual of c, that is $c^\partial = -c-$. Then c is a normal operator and, by (2) and (3), we have that

$$\begin{aligned} x &\leq cx \quad (4)\\ ccx &\leq cx \quad (5)\\ \text{for all } 1 \leq i \leq n, \text{ for every } n\text{-place operator } f,\ f(x_1, \ldots, x_n) &\leq cx_i. (6) \end{aligned}$$

Because we have the booleans it is sufficient to look at quasi-equations of the form $\tau_1 = 1 \rightarrow \tau_2 = 1$. We claim that

$$\mathsf{K} \models \tau_1 = 1 \rightarrow \tau_2 = 1 \text{ iff } \mathsf{K} \models c^\partial \tau_1 \leq \tau_2. \tag{7}$$

From right to left is immediate by normality of c. For the other side, suppose there exists $\mathfrak{A} \in \mathsf{K}$ such that $\mathfrak{A} \not\models c^\partial \tau_1 \leq \tau_2$. Let $\mathfrak{B}$ be the relativization of $\mathfrak{A}$ to $c^\partial \tau_1$, so it has domain

$$B = \{x \wedge c^\partial \tau_1 \mid x \in A\},$$

and the operations are defined as follows: $x \vee^\mathfrak{B} y = x \vee^\mathfrak{A} y$, $-^\mathfrak{B} x = c^\partial \tau_1 \wedge^\mathfrak{A} -^\mathfrak{A} x$, $f^\mathfrak{B}(x_1, \ldots, x_n) = f^\mathfrak{A}(x_1, \ldots, x_n) \wedge^\mathfrak{A} c^\partial \tau_1$ (cf. (Henkin et al., 1971 1985): Def.2.2.1).

We claim that (i) $\mathfrak{B} \models \tau_1 = 1$, (ii) $\mathfrak{B} \not\models \tau_2 = 1$, and (iii) $\mathfrak{B} \in \mathsf{K}$.

(i) follows from the fact that $\mathfrak{B} \models c^\partial \tau_1 = 1$ and (4). (ii) is immediate because in $\mathfrak{A}$ we have $c^\partial \tau_1 \wedge -\tau_2 \neq 0$. For (iii) we show that the function $h : A \longrightarrow B$ defined as $h(x) = x \wedge c^\partial \tau_1$ is a surjective homomorphism. This is sufficient because, $\mathfrak{A} \in \mathsf{K}$ and we assumed that K was a variety. Surjectivity is immediate by the definition of B. For the boolean operations, h is always a homomorphism. For convenience we do the proof for a unary operator f.

$$\begin{array}{lll} hf^\mathfrak{A} x & = & \\ f^\mathfrak{A} x \wedge c^\partial \tau_1 & \leq & \text{by } (5) \\ f^\mathfrak{A} x \wedge c^\partial c^\partial \tau_1 & \leq & \text{by } (6) \\ f^\mathfrak{A} x \wedge -f - c^\partial \tau_1 & \leq & \text{by BAO} \\ f^\mathfrak{A}(x \wedge c^\partial \tau_1) \wedge c^\partial \tau_1 & = & f^\mathfrak{B} hx. \end{array}$$

The other side follows by monotonicity of f. QED

Using this machinery the proof of Theorem 2.1 becomes very simple.

PROOF OF THEOREM 2.1. (i) is immediate because the extra equations are positive as well. The correspondents can easily be computed. (ii) follows as before from the fact that $\mathfrak{Tile}+R^{\Diamond}$ is universal Horn definable. (iii) is immediate by Theorem 2.3. Now (iv) follows because if we could decide the equational theory of $\mathsf{Ti}+\Diamond$, we could –by the deduction theorem– decide the quasi-equational theory of $\mathsf{Ti}+\Diamond$, whence also that of Ti. But that is undecidable by Theorem 1.2. QED

3 Interpreting BAO-equations in RRA

We show that RRA is strong enough to interpret many other varieties of BAO's, in particular the variety Tile. This immediately leads to undecidability of RRA-equations. That result itself is not particularly shocking, but in the next section we will see that only a tiny bit of the power of RRA is used for this interpretation. Interpretations of modal logics in (R)RA are studied in (Orlowska, 1991).

Tile is a modal variety (a BAO-variety with unary operators), so we can use many techniques from modal logic. One useful technique is the notion of a *standard translation* ST, translating modal formulas to first order formulas (cf. e.g. (van Benthem, 1983) p.39). We can equally well use this translation for terms of modal varieties.

For modal varieties $\mathbf{S}\mathsf{K}^+$, where K is a class of first order structures defined by Σ, we have

$$\Sigma \models_{FO} ST(\tau) \text{ iff } \mathbf{S}\mathsf{K}^+ \models \tau = 1, \tag{8}$$

and similarly for quasi-equations,

$$\Sigma \wedge \bigwedge_{i\in I} ST(\tau_i) \models_{FO} ST(\tau) \text{ iff } \mathbf{S}\mathsf{K}^+ \models \bigwedge_{i\in I} \tau_i = 1 \rightarrow \tau = 1. \tag{9}$$

Since $\mathbf{S}\mathsf{K}^+$ is a modal variety, the standard translation of any term goes to a first order formula with 2 variables and unary and binary predicates. As we saw above, in the case of the class $\mathfrak{Tile}$, Σ is finite and can be written using three variables and binary predicates.

Since RRA is equally expressive as first oder logic with 3 variables and binary predicate-symbols, (a modified) (8) and (9) also hold for RRA and the variety $\mathsf{Tile} = \mathbf{S}\mathfrak{Tile}^+$. So we obtain undecidability of RRA-equations by the fact that RRA has a deduction theorem (since it is a discriminator variety) and the undecidability of Tile-quasi-equations.

In the next section, we will show that we can push this argument through for much weaker super-varieties of RRA. We will now give the translation from Tile-terms to relational-type terms, show how to state the Tile axioms in the relational type language and prove that Tile-quasi-equations can be interpreted in RRA. The following abbreviation will be very useful

$$\mathsf{d}_1 x = x;0'.$$

(Here and elsewhere, $0'$ is an abbreviation for $-1'$.) We call an element x in an algebra $\mathfrak{A}$ *functional* if $\mathfrak{A} \models x \leq -\mathsf{d}_1 x$. The reason for this terminology is that if $\mathfrak{A}$ is a full relational set algebra, then $\{(a,b),(a,c)\} \subseteq x$ implies that $b = c$.

We will first translate the $\mathfrak{Tile}$ axioms. Instead of $R^{\langle \mathsf{right} \rangle}$ and $R^{\langle \mathsf{up} \rangle}$, we use p_x and p_y respectively. Then we can write them as

$$\begin{array}{ll} A1 & p_x \leq \neg \mathsf{d}_1 p_x, \;\; p_y \leq \neg \mathsf{d}_1 p_y \\ A2 & 1 = p_x;1 = p_y;1 \\ A3 & p_x;p_y = p_y;p_x. \end{array}$$

Let $\mathfrak{M}$ be a full relation set algebra satisfying these equations. Then *A1* and *A2* say that p_x and p_y are total functions, which commute by *A3*.

Now we define the standard translation. Our proposed definitions for the operators $\langle \mathsf{right} \rangle z$ and $\langle \mathsf{up} \rangle z$ are

$$\begin{array}{lcl} \langle \mathsf{r} \rangle z & = & (p_x;(1' \wedge z));1 \\ \langle \mathsf{u} \rangle z & = & (p_y;(1' \wedge z));1. \end{array}$$

In terms of first order logic, this translation amounts to $\exists v_1(p_x(v_0,v_1) \wedge z(v_1,v_1))$, which is in essence the standard translation ST.

Let $(\cdot)^t$ be the translation from Tile terms to relational type terms defined as follows:

$$\begin{array}{lcll} (x)^t & = & (x \wedge 1') & \text{for all variables } x \\ (-\sigma)^t & = & -(\sigma)^t \wedge 1' & \\ (\sigma \wedge \tau)^t & = & \sigma^t \wedge \tau^t \wedge 1' & \\ (\langle \mathsf{right} \rangle \sigma)^t & = & \langle \mathsf{r} \rangle \sigma^t \wedge 1' & \\ (\langle \mathsf{up} \rangle \sigma)^t & = & \langle \mathsf{u} \rangle \sigma^t \wedge 1'. & \end{array}$$

For $\sigma = \tau$ a Ti-equation, $(\sigma = \tau)^t$ denotes $\sigma^t = \tau^t$.

Proposition 3.1 *Let e_i be arbitrary* Ti*-equations. Then*

$$\mathsf{Tile} \models \bigwedge_{i \in I} e_i \rightarrow 0 = 1 \textit{ iff } \mathsf{RRA} \models \bigwedge_{i \in I} e_i^t \wedge A1 \wedge A2 \wedge A3 \rightarrow 0 = 1.$$

PROOF. In an obvious way we can turn $\mathfrak{Tile}$-structures into full relational set algebras and vice versa, preserving truth of the equations involved (here we use the fact that the translation always goes below the identity). We leave the easy details to the reader.

QED

4 Undecidability in weakened relation algebras

Now we will prove the analogue of Proposition 3.1 for a much larger class than RRA. We use the following abbreviations:

$$\begin{aligned} \mathsf{c}_1 x &= x;1 \\ \mathsf{c}_0 x &= 1;x \\ \Diamond x &= \mathsf{c}_0\mathsf{c}_1 x \vee \mathsf{c}_0 x \vee \mathsf{c}_1 x. \end{aligned}$$

Define URA as the class of all relational type algebras $\mathfrak{A} = (A, \vee, -, ;, \smile, 1')$ satisfying the following axioms:

URA0 $\mathfrak{A}$ is a boolean algebra where ; is a normal operator
URA1 $x;1' \leq x, 1';x \leq x$
URA2 $(1;(y \wedge 1'));z \leq 1;((y \wedge 1');z)$
URA3 $(x;1);1 \leq x;1,\ \ 1;(1;x) \leq 1;x,\ \ (1;x);1 = 1;(x;1)$.

A few words about the axioms. *URA1* and *URA2* are weakenings of RA-axioms. *URA2* is an instance of Maddux' *weak associativity*. *URA3* gives us all instances of Maddux' SA-axiom; it enforces that c_1 and c_0 are commuting closure operators.

Theorem 4.1 *Let* K *be a variety such that* $\mathsf{RRA} \subseteq \mathsf{K} \subseteq \mathsf{URA}$. *Then the equational theory of* K *is undecidable.*

Before we prove the theorem, we compare it –and the way we prove it– to the similar result of (Maddux, 1991), with SA instead of URA.

Theorem 4.2 $\mathsf{RRA} \subsetneq \mathsf{RA} \subsetneq \mathsf{SA} \subsetneq \mathsf{URA}$.

PROOF. The first two inequalities follow from (Maddux, 1991). For the last, we first show that all URA-axioms are SA-valid. The Theorems used in the proof are from (Maddux, 1991). *URA0* follows from Theorem 8.(i) and (ii). *URA1* is one of the SA-axioms. *URA2* and *URA3* are instances of Theorem 13.(iv).

URA is strictly larger than SA because none of the SA-axioms mentioning converse are URA-valid.

QED

As mentioned before we will prove Theorem 4.1 by interpreting the variety Tile. This theorem with SA instead of URA is proved by interpreting the quasi-equational theory of semigroups. The proof for SA uses the equality $(x;y);1 = x;(y;1)$ in a crucial way. We will show that in URA this equation fails. This indicates that an undecidability proof in the spirit of Maddux' proof for SA using the semigroup-wordproblem is impossible, and indeed that it might be very hard to interpret the wordproblem directly in URA. Since URA strictly contains SA one might tentatively conclude that the tiling problem is a "deeper cause" of undecidability in relational type algebras than the semigroup-wordproblem. This partly solves a problem raised in (van Benthem, 1996) p.174 about the connection between these two styles of undecidability-analysis in relational type algebras.

Theorem 4.3 *There exists an* URA *in which* $(x;y);1 = x;(y;1)$ *fails.*

In fact our counterexample will be a bit stronger. The algebra we provide also validates the following RA-(quasi-)equations

$$x;1' = 1';x = x \quad (x \vee y)^{\smile} = x^{\smile} \vee y^{\smile} \quad (x^{\smile})^{\smile} = x \quad x \leq 1' \rightarrow x^{\smile} = x.$$

The fact that we can create the counterexample crucially depends on the failure of the Peircean or cycle laws in URA.

PROOF. We will define an URA $\mathfrak{A}$ with seven non-identity atoms and one identity atom. The identity atom is denoted by $1'$. The seven non-identity atoms are $\{a, b, c, x, y, w_1, w_2\}$. In order to define the algebra it is sufficient to say how ; and $^{\smile}$ work on the atoms. For every atom $x \in A$ we set $x^{\smile} = x$ and $x;1' = 1';x = x$. In $\mathfrak{A}$ we only set the following equations between non-identity atoms.

$$\begin{aligned} a &\leq b;c && (10)\\ b &\leq x;y && (11)\\ a &\leq x;w_1 && (12)\\ a &\leq b;w_2 && (13)\\ w_2 &\leq y;c. && (14) \end{aligned}$$

We claim that

$$\mathfrak{A} \not\models (x;y);1 = x;(y;1). \tag{15}$$

By (10) and (11), $a \leq (x;y);1$. But the only way that $a \leq x;(y;1)$ would be by (12) and $w_1 \leq y;1$. But this last equation does not hold. Next we show that $\mathfrak{A} \in \mathsf{URA}$. $\mathfrak{A}$ trivially validates *URA1*, and *URA2* by our definition and the fact that $1'$ is an atom. *URA3* is a bit harder. From $a \leq (x;y);1 \leq (x;1);1$ we must have $a \leq x;1$, which we have by (12). From $a \leq (1;y);1$ we want $a \leq 1;(y;1)$, which we have by (13) and (14). From (13) and (14) we have $a \leq 1;(1;c)$, so we want $a \leq 1;c$, but we have that already by (10). These are all the cases to check for *URA3*. QED

We will now prove Theorem 4.1.

Proof of Theorem 4.1. We will show that the theorem already holds if we consider the (varieties generated by) the converse-free reducts of RRA, K and URA. The theorem then follows since every equation without converse is K-valid iff it is valid in the converse free reduct of K. V^* denotes the variety generated by the converse-free reduct of the relational type variety V.

Our strategy is to prove that (i) K^* has a deduction theorem and (ii) Proposition 3.1 holds with K^* substituted for RRA. Undecidability of the K^*-equations then follows from the fact that the set of Tile quasi-equations of the form $\bigwedge e_i \to 0 = 1$ is undecidable.

We first need a lemma, telling us a bit more about URA.

Lemma 4.4 *The following (quasi)-equations are all* URA*-valid and* URA^**-valid.*

Mon	$x \leq y$	$\Rightarrow$	$x;z \leq y;z$ *and* $z;x \leq z;y$
URAt0	ffx	$\leq$	fx *for* f *any of* $\mathsf{c}_1, \mathsf{c}_0, \Diamond$
URAt1	$x;(1' \wedge y)$	$=$	$x \wedge 1;(1' \wedge y)$ *and* $(1' \wedge y);x = x \wedge (1' \wedge y);1$
URAt2	$\mathsf{c}_1 x$	$\leq$	$x \vee \mathsf{d}_1 x$
URAt3	$x \leq -\mathsf{d}_1 x$	$\Rightarrow$	$(x \wedge y) \wedge \mathsf{c}_1(x \wedge z) \leq (x \wedge y \wedge z)$
URAt4	$x \leq -\mathsf{d}_1 x$	$\Rightarrow$	$\mathsf{c}_1(x;y) \wedge x \wedge z \leq \mathsf{c}_1((x \wedge z);y)$
URAt5	$1 = 1';1$	$\Rightarrow$	$1' \wedge \mathsf{c}_1 x \wedge \mathsf{c}_1 y \leq \mathsf{c}_1(x \wedge \mathsf{c}_1 y)$.

Proof. *Mon* follows from the distribution of ; over $\vee$. *URAt0* follows immediately from *URA3*. *URAt1* can be derived using *URA1*, *Mon*, and distribution of ; over $\vee$. For *URAt2*, $\mathsf{c}_1 x = x;1 = x;(1' \vee 0') = x;1' \vee x;0' \overset{URA1}{\leq} x \vee \mathsf{d}_1 x$.

The next two quasi-equations deal with functional elements. *URAt3* is immediate by *URAt2*. For *URAt4* we reason as follows, $\mathsf{c}_1(x;y) = \mathsf{c}_1((x \wedge z);y \vee (x \wedge -z);y) = \mathsf{c}_1((x \wedge z);y) \vee \mathsf{c}_1((x \wedge -z);y)$. But $\mathsf{c}_1((x \wedge -z);y) \overset{Mon}{\leq} ((x \wedge -z);1);1 \overset{URA3}{\leq} (x \wedge -z);1 = \mathsf{c}_1(x \wedge -z)$. So by *URAt3*, $\mathsf{c}_1((x \wedge -z);y) \wedge x \wedge z \leq 0$. So both disjuncts are smaller than $\mathsf{c}_1((x \wedge z);y)$, as desired.

For *URAt5*, if $1 = 1';1$, then by *URAt1*, $1' \wedge \mathsf{c}_1 x \wedge \mathsf{c}_1 y = 1' \wedge \mathsf{c}_1(1';x) \wedge \mathsf{c}_1 y$. This equals by additivity of c_1,

$$[1' \wedge \mathsf{c}_1 y \wedge \mathsf{c}_1((1' \wedge \mathsf{c}_1 y);x)] \vee [1' \wedge \mathsf{c}_1 y \wedge \mathsf{c}_1((1' \wedge -\mathsf{c}_1 y);x)]$$

The second disjunct equals 0, because $1' \wedge \mathsf{c}_1 y \wedge \mathsf{c}_1((1' \wedge -\mathsf{c}_1 y);x \overset{URA3,Mon}{\leq} 1' \wedge \mathsf{c}_1 y \wedge (1' \wedge -\mathsf{c}_1 y);1 \overset{URAt1}{=} (1' \wedge \mathsf{c}_1 y);(1' \wedge -\mathsf{c}_1 y) \overset{URA1}{=} 0$.

So continuing with the first disjunct, $\mathsf{c}_1((1' \wedge \mathsf{c}_1 y);x) \overset{URAt1}{=} \mathsf{c}_1(x \wedge (1' \wedge \mathsf{c}_1 y);1) \overset{Mon}{\leq} \mathsf{c}_1(x \wedge (\mathsf{c}_1 y);1) = \mathsf{c}_1(x \wedge (y;1);1) \overset{URA3,Mon}{\leq} \mathsf{c}_1(x \wedge y;1) = \mathsf{c}_1(x \wedge \mathsf{c}_1 y)$. QED

It will be useful to define one more abbreviation: $\mathsf{s}^0_1 x = 1;(1' \wedge x) = \mathsf{c}_0(1' \wedge x)$. We claim that

$$\text{In } \mathsf{URA}^*,\ \mathsf{s}^0_1 \text{ distributes over } \wedge, \tag{16}$$

because $\mathsf{s}^0_1 x \wedge \mathsf{s}^0_1 y = 1;(1' \wedge x) \wedge 1;(1' \wedge y) \overset{URAt1}{\leq} [1;(1' \wedge x)];(1' \wedge y) \overset{URA2}{\leq} 1;[(1' \wedge x);(1' \wedge y)] \overset{URA1}{\leq} 1;[1' \wedge x \wedge y] = \mathsf{s}^0_1(x \wedge y)$.

Now by *URAt1*,

$$\langle \mathsf{r} \rangle x = (p_x;(1' \wedge x));1 = \mathsf{c}_1(p_x \wedge \mathsf{s}^0_1 x). \tag{17}$$

Claim 1 Any $\mathsf{K}^* \subseteq \mathsf{URA}^*$ has a deduction theorem.

Proof of Claim. Recall that we defined $\Diamond x$ as $x;1 \vee 1;x \vee (1;x);1$. Because ; is a normal operator, $\Diamond$ is a normal operator as well. By *URAt0*, $\Diamond$ satisfies (2). Since ; is the only operator of K^*, $\Diamond$ also satisfies (3), by *Mon*. But then, by Theorem 2.3, K^* has a deduction theorem. ◀

How can we ensure that also in a class as weak as URA, the above term-defined operators $\langle \mathsf{r} \rangle$ and $\langle \mathsf{u} \rangle$ are functional and commuting? Let φ denote the conjunction of the following equations.

$$\begin{array}{lcl}
A1 & = & p_x \leq \neg \mathsf{d}_1 p_x,\ \ p_y \leq \neg \mathsf{d}_1 p_y \\
A2 & = & 1 = \mathsf{c}_1 p_x = \mathsf{c}_1 p_y \\
A3 & = & p_x;p_y = p_y;p_x \\
A4 & = & 1 = \mathsf{c}_0 1' = \mathsf{c}_1 1' \\
A5 & = & 1 = \mathsf{c}_1 p_x;p_y \\
A6 & = & p_y;p_x \leq -\mathsf{d}_1(p_y;p_x).
\end{array}$$

We saw before that in a full relational set algebra the first three axioms imply that the defined operations $\langle \mathsf{r} \rangle$ and $\langle \mathsf{u} \rangle$ behave as commuting functional operators. In RRA, *A4* is trivially true, and *A5* and *A6* follow from *A1*$-$*A2*. The last three are only there because URA is so much weaker than RRA.

We are ready for the final claim.

Claim 2 Let $\mathsf{RRA}^* \subseteq \mathsf{K}^* \subseteq \mathsf{URA}^*$. Let e_i be arbitrary Tile-equations. Then

$$\mathsf{Tile} \models \bigwedge_{i\in I} e_i \to 0 = 1 \text{ iff } \mathsf{K}^* \models \bigwedge_{i\in I} e_i^t \wedge \mathit{A1} \wedge \ldots \wedge \mathit{A6} \to 0 = 1.$$

Proof of Claim. The direction from right to left follows from Proposition 3.1 and the fact that $\mathsf{RRA} \subseteq \mathsf{K}$.

For the other direction we have to turn a non-trivial K^* algebra satisfying $\bigwedge_{i\in I} e_i^t \wedge \mathit{A1} \wedge \ldots \wedge \mathit{A6}$ into a non trivial Tile algebra satisfying $\bigwedge_{i\in I} e_i$. We can do this in the same way as before with the class RRA. To be explicit, let $\mathfrak{A} = (A, \vee, -, ;, 1') \in \mathsf{URA}^*$. Define a Tile-type algebra $\mathfrak{B}$ as the relativization of the $(A, \vee, -, \langle\mathsf{r}\rangle, \langle\mathsf{u}\rangle)$ reduct of $\mathfrak{A}$ with the identity. The function $f : A \longrightarrow B$ defined as $f(x) = x \wedge^{\mathfrak{A}} 1'$ is clearly a surjective homomorphism, whence $\mathfrak{B} \models \bigwedge_{i\in I} e_i$, because the translation $(\cdot)^t$ puts all terms below the identity. Now $\mathfrak{B} \in \mathsf{Tile}$, if relativized to $1'$, $\langle\mathsf{r}\rangle$ and $\langle\mathsf{u}\rangle$ are normal operators satisfying *Ti1*, *Ti3* and *Ti4* in $\mathfrak{A}$.

We first show that $\langle\mathsf{r}\rangle$ is a functional operator. The same argument works for $\langle\mathsf{u}\rangle$. $\langle\mathsf{r}\rangle$ is a normal operator because ; is one. *Ti3* follows immediately from *A2* and *A4*. For *Ti1* we reason as follows

$$\begin{array}{lll}
1' \wedge \langle\mathsf{r}\rangle y \wedge \langle\mathsf{r}\rangle z & = & \\
1' \wedge \mathsf{c}_1(p_x \wedge \mathsf{s}_1^0 y) \wedge \mathsf{c}_1(p_x \wedge \mathsf{s}_1^0 z) & \leq & (\mathit{A4}, \mathit{URAt5}) \\
\mathsf{c}_1(p_x \wedge \mathsf{s}_1^0 y \wedge \mathsf{c}_1(p_x \wedge \mathsf{s}_1^0 z)) & \leq & (\mathit{A1}, \mathit{URAt3}) \\
\mathsf{c}_1(p_x \wedge \mathsf{s}_1^0 y \wedge \mathsf{s}_1^0 z) & \leq & (16) \\
\mathsf{c}_1(p_x \wedge \mathsf{s}_1^0 (y \wedge z)) & = & \\
\langle\mathsf{r}\rangle(y \wedge z). & &
\end{array}$$

The proof for commutativity is a bit more involved. We have to show that $1' \wedge \langle\mathsf{r}\rangle\langle\mathsf{u}\rangle z = 1' \wedge \langle\mathsf{u}\rangle\langle\mathsf{r}\rangle z$. Since $\langle\mathsf{r}\rangle$ and $\langle\mathsf{u}\rangle$ are normal and satisfy *Ti1* below $1'$, it is sufficient to show that $1' \wedge \langle\mathsf{r}\rangle\langle\mathsf{u}\rangle z \wedge \langle\mathsf{u}\rangle\langle\mathsf{r}\rangle - z = 0$, i.e, writing out definitions, that

$$1' \wedge \mathsf{c}_1(p_x \wedge \mathsf{s}_1^0 \mathsf{c}_1(p_y \wedge \mathsf{s}_1^0 z)) \wedge \mathsf{c}_1(p_y \wedge \mathsf{s}_1^0 \mathsf{c}_1(p_x \wedge \mathsf{s}_1^0 - z)) = 0. \tag{18}$$

Now

$$
\begin{array}{lcr}
\mathsf{c}_1(p_x \wedge \mathsf{s}^0_1\mathsf{c}_1(p_y \wedge \mathsf{s}^0_1 z)) & = & (A5) \\
\mathsf{c}_1(\mathsf{c}_1(p_x;p_y) \wedge p_x \wedge \mathsf{s}^0_1\mathsf{c}_1(p_y \wedge \mathsf{s}^0_1 z)) & \leq & (A1,\ URAt4) \\
\mathsf{c}_1\mathsf{c}_1([p_x \wedge \mathsf{s}^0_1\mathsf{c}_1(p_y \wedge \mathsf{s}^0_1 z)];p_y) & \leq & (URAt0) \\
\mathsf{c}_1([p_x \wedge \mathsf{s}^0_1\mathsf{c}_1(p_y \wedge \mathsf{s}^0_1 z)];p_y) & \leq & (Mon) \\
\mathsf{c}_1(p_x;p_y \wedge [\mathsf{s}^0_1\mathsf{c}_1(p_y \wedge \mathsf{s}^0_1 z)];p_y) & \leq & (URA2 \text{ is } (\mathsf{s}^0_1 x);y \leq \mathsf{c}_0[(1' \wedge x);y]\ ) \\
\mathsf{c}_1(p_x;p_y \wedge \mathsf{c}_0[[1' \wedge \mathsf{c}_1(p_y \wedge \mathsf{s}^0_1 z)];p_y]) & \leq & (Mon,\ URA1,\ \text{def of } \mathsf{c}_1) \\
\mathsf{c}_1(p_x;p_y \wedge \mathsf{c}_0[p_y \wedge \mathsf{c}_1\mathsf{c}_1(p_y \wedge \mathsf{s}^0_1 z)]) & \leq & (A1,\ URAt0,\ URAt3) \\
\mathsf{c}_1(p_x;p_y \wedge \mathsf{c}_0(p_y \wedge \mathsf{s}^0_1 z) & \leq & (URAt0 \text{ and } Mon) \\
\mathsf{c}_1(p_x;p_y \wedge \mathsf{s}^0_1 z) & = & (A3) \\
\mathsf{c}_1(p_y;p_x \wedge \mathsf{s}^0_1 z). & &
\end{array}
$$

Similarly we find $\mathsf{c}_1(p_y \wedge \mathsf{s}^0_1\mathsf{c}_1(p_x \wedge \mathsf{s}^0_1 - z)) \leq \mathsf{c}_1(p_y;p_x \wedge \mathsf{s}^0_1 - z))$. But, by *A6*, $p_y;p_x$ is functional, whence (using the same proof as above for *Ti1*) the left hand side of (18) is smaller than $\mathsf{c}_1(p_y;p_x \wedge \mathsf{s}^0_1 z \wedge \mathsf{s}^0_1 - z)$ which equals 0 by (16) and normality. ◀

This finishes the proof of the theorem. QED

The main ingredients in the proof were the defined closure operator $\Diamond$ and the "vertical difference operator" d_1, which enabled us to express that a relation was a function. We will now make these operators explicit and show that if we add them to the similarity type, we get undecidability of the equational theory using only axioms which correspond to universal sentences.

A URD-type algebra is a relational type algebra expanded with two unary operations d_1 and $\Diamond$.

We define $\mathsf{RRA}^{\{\Diamond,\mathsf{d}_1\}}$ as the class of all URD-type algebras $\mathfrak{A} = (A, \vee, -, ;, \smile, 1', \mathsf{d}_1, \Diamond)$ whose $\mathsf{d}_1, \Diamond$-free reduct is in RRA and which validate $\mathsf{d}_1 x = x;0'$ and $\Diamond x = 1;x;1$ for all $x \in A$. So RRA and $\mathsf{RRA}^{\{\Diamond,\mathsf{d}_1\}}$ are term-definable equivalent.

The variety URD contains all algebras $\mathfrak{A} = (A, \vee, -, ;, \smile, 1', \mathsf{d}_1, \Diamond)$ satisfying the following $\mathsf{RRA}^{\{\Diamond,\mathsf{d}_1\}}$-valid equations:

URA0'	$\mathfrak{A}$ is a boolean algebra with normal operators
URA1	$x;1' \leq x,\ \ 1';x \leq x$
URA2'	$(1;(y \wedge 1'));z \leq 1;((y \wedge 1');z),\ \ ((x \wedge 1');1);1 \leq (x \wedge 1');1$
U1	$\Diamond\Diamond x \leq \Diamond x$
U2	$\mathsf{d}_1 x \leq \Diamond x$
U3	$x;y \leq (x \vee \mathsf{d}_1 x) \wedge \Diamond y$
D1	$\mathsf{d}_1\mathsf{d}_1 x \leq x \vee \mathsf{d}_1 x$
D2	$x \wedge \mathsf{d}_1 y \leq \mathsf{d}_1(y \wedge \mathsf{d}_1 x)$.

Theorem 4.5 *(i) The class* URD *is a canonical variety whose axioms correspond to universal first-order frame-conditions.*
(ii) The converse free reduct of URD *has a deduction theorem.*

PROOF. (i) follows by standard Sahlqvist reasoning, cf. (de Rijke and Venema, 1995).
(ii) Define $\Diamond' x$ as $\Diamond x \vee x$. Then $\Diamond'\Diamond' x \leq \Diamond' x$ follows from *U1* and $\mathsf{d}_1 x \leq \Diamond' x$ and $x;y \leq \Diamond' x \wedge \Diamond' y$ from *U2* and *U3* respectively. Whence by Theorem 2.3, URD has a deduction theorem. QED

Theorem 4.6 *The equational theory of any class* K *of* URD*-type algebras such that* $\mathsf{RRA}^{\{\Diamond,\mathsf{d}_1\}} \subseteq \mathsf{K} \subseteq \mathsf{URD}$ *is undecidable.*

PROOF. Define $\mathsf{c}_1 x$ as the abbreviation of $x \vee \mathsf{d}_1 x$. By axioms *D1* and *D2*, c_1 is a complemented closure operator (in particular, *URAt5* holds). We can prove the theorem by a precise copy of the previous undecidability proof. We won't go into details of the derivations again; they all follow the same argument. The only place where we need the new axiom *URA2′* is in the derivation below (18) at the place where we used *URAt0* and *Mon*. QED

Corollary 4.7 *The equational theory of the class* $\mathbf{Rl}\mathsf{RRA}^{\{\Diamond,\mathsf{d}_1\}}$ *(all relativizations of algebras in* $\mathsf{RRA}^{\{\Diamond,\mathsf{d}_1\}}$*) is undecidable.*

PROOF. If we relativize the variety $\mathsf{RRA}^{\{\Diamond,\mathsf{d}_1\}}$ –which is term-definable equivalent to RRA– and generate it's variety, we obtain a class which is between $\mathsf{RRA}^{\{\Diamond,\mathsf{d}_1\}}$ and URD (because URD is the complex variety of a universally defined class of atom-structures). QED

Final Remarks

Can we now answer the question which aspects of relation algebras are to blame for its undecidability? The results of Maddux and Andréka et al mentioned in the beginning seem to point at associativity of composition in combination with a full boolean algebra in which composition is an operator. This is enforced by the decidability for relativized relation algebras (cf. (Németi, 1987; Marx, 1995); their most prominent missing axiom is associativity and instead there is the much weaker *URA2*. On the other hand, in $\mathbf{Rl}\mathsf{RRA}^{\{\Diamond,\mathsf{d}_1\}}$ associativity is only valid in the weak sense of *URA2*. So our evidence seems to point more in the direction of 1) being able to express functionality and commutativity of two relations, 2) a deduction theorem, and 3) enough expressive power to encode modal logic.

Another research direction which can give answers to this question and has potentially powerful applications is to consider (generalized) reducts of RRA. As we mentioned before, all algebras we considered are boolean algebras with operators. If we weaken the ("static") boolean structure, undecidability might disappear. Strong results are known here already, witness decidability of the Lambek Calculus, Propositional Dynamic Logic, and the positive RRA-fragment. In fact the classical RRA-similarity type seems much to "static truth" inspired for modern dynamic applications of relation algebras (in for instance computer science and formal semantics of natural language). A basis for designing such *"dynamic reducts"* might be the bisimulation invariance of the operators. This would lead us to abolish e.g. conjunction, negation and converse, and instead have e.g. the residuals (as in sequential algebras) and Hollenberg's dynamic negation (Hollenberg, 1996). It seems not even necessary to stick with operators term-definable in RRA: both propositional dynamic logic and the μ-calculus have second order operators and are decidable.

The grand old lady RRA need not be sent to the history books of mathematics, but for modern applications where decidability/tractability is a must, we better have a close look at her potential offspring in the form of these dynamic reducts.

We end with two technical questions related to the present paper.

• (R. Maddux) Can we strengthen Theorem 4.1 as follows: the set of K-valid equations in n variables is undecidable? If so, what is the lowest bound on n? (The set of SA-valid equations in just one variable is already undecidable (Maddux, 1994).)

• (I. Németi) Can we replace RRA by "all finite RRA" in Theorem 4.1?

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