Algebraic characterizations of various Beth definability properties

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Abstract

In this paper it will be shown that the Beth definability property corresponds to surjectiveness of epimorphisms in abstract algebraic logic. This generalizes a result by I. Németi (cf. [HMT85, Theorem 5.6.10]). Moreover, an equally general characterization of the weak Beth property will be given. This gives a solution to Problem 14 in [Sai90]. Finally, the characterization of the projective Beth property for varieties of modal algebras by L. Maksimova (see [Mak97]) will be shown to hold for the larger class of semantically algebraizable logics.

1 Introduction

Abstract algebraic logic is typically concerned with equivalence theorems of the following kind,

$$\text{Logic } \mathcal{L} \text{ has property } P \iff \mathsf{Alg}(\mathcal{L}) \text{ has property } \mathsf{alg}(P),$$

where $\mathsf{Alg}(\mathcal{L})$ denotes the class of algebras associated to the logic $\mathcal{L}$ in some canonical way, on which we come to speak later. The main motivation for making these efforts is that it shows us that algebra and logic are, so to speak, nothing but two sides of the same coin; they study the same phenomena, some differences in form aside. Hence in attacking logical (resp. algebraic) problems, we now have *two* powerful machineries at our disposal, both logic and algebra, and we can live in the world best suited for the task at hand.
This paper contributes to these investigations. More specifically, we will study equivalences between various Beth definability properties and surjectiveness of (certain kinds of) epimorphisms. A link between these two properties was first laid in [Ném84]. A similar result by I. Németi in terms of surjectiveness of so-called *almost onto* epis can be found in [HMT85, Theorem 5.6.10]. In Section 4 we generalize these accomplishments in the sense that we characterize a more general definability property. Subsequently it will be shown that the proof of this characterization is generally applicable in proving equivalences, mutatis mutandis, between all kinds of definability properties and surjectiveness of various

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kinds of epimorphisms. This gives us for example equally general characterizations of the so-called weak Beth property, and projective Beth property. At this point we have to assure the reader that all the notions introduced so far will be explained in full detail in Section 2 and Section 3.
There is an extensive literature devoted to the study of the Beth property and surjectiveness of epimorphisms. Without having the illusion of being complete, let us in this connection just mention the following. For a comprehensive account on definability and interpolation in modal logics we refer to the work of L. Maksimova (cf. e.g. [Mak92a] and [Mak92b]). I. Sain has extensively studied definability properties via epimorphisms in polyadic, cylindric and relation algebras. See e.g. [Sai84] and [Sai90].

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2 Semantically algebraizable logic, general logic, algebraic counterpart of a logic

Notation 2.1 Throughout this paper we will use the notation of [AKNS94]. The one exception that affirms this rule is that for any homomorphism $h : \mathcal{A} \longrightarrow \mathcal{B}$, the homomorphic image of $\mathcal{A}$ along h will be denoted by $h[\mathcal{A}]$. Furthermore, for clearness' sake, let us mention the following. Algebras will be denoted by calligraphic letters like $\mathcal{A}, \mathcal{B}$, whereas the roman type A, B denote their respective universes. For any function f and subset X of the domain of f, $f\lceil X$ denotes the restriction of f to X, that is, $f\lceil X \stackrel{\text{def}}{=} \{\langle x, f(x)\rangle : x \in X\}$. For any function f, the *range* of f is defined by $Rg(f) \stackrel{\text{def}}{=} \{f(x) \ : \ x \text{ in the domain of } f\}$. For functions $f : \mathcal{A} \longrightarrow \mathcal{B}$, $g : \mathcal{B} \longrightarrow \mathcal{C}$, the *composition* $g \circ f : \mathcal{A} \longrightarrow \mathcal{C}$ is defined for all $x \in A$ by $(g \circ f)(x) \stackrel{\text{def}}{=} g(f(x))$. We would also like to point out that we will apply set-theoretical notation to classes. For instance, $\bigcup C$ will denote the union of all members of C, also when C is a *proper* class. ◀

The results of this paper apply to so-called *strongly nice* general logics, cf. [AKNS94], which we will call *semantically algebraizable* general logics. For readers more familiar with the Blok-Pigozzi approach we observe that this is basically the same as an *algebraizable* logic, cf. e.g. [BP89].[1] In order to keep the present paper self-contained, we will start by briefly recapitulating these notions.

Let us recall from e.g. [AKNS94] or [NA94] that by a **logic** $\mathcal{L}$ we understand an ordered quadruple, consisting of

[1]For a detailed comparison between the two approaches, we refer to [FJ94].

1. A set $F_{\mathcal{L}}$ of *formulas*. These formulas are recursively defined from some set P of *atomic formulas* by means of some fixed set $Cn(\mathcal{L})$ of *logical connectives*.

2. A class $M_{\mathcal{L}}$ of *models*. In general this will be a *proper* class.

3. A family $\{mng_{\mathcal{M}} : \mathcal{M} \in M_{\mathcal{L}}\}$ of functions with domain $F_{\mathcal{L}}$. For every $\mathcal{M} \in M_{\mathcal{L}}$, $mng_{\mathcal{M}}$ is called the *meaning function* of $\mathcal{M}$.

4. A relation $\models_{\mathcal{L}}$ (called the *validity relation*) between $M_{\mathcal{L}}$ and $F_{\mathcal{L}}$.

It should be emphasized that $\mathcal{F}_{\mathcal{L}} \stackrel{\text{def}}{=} \langle F_{\mathcal{L}}, Cn(\mathcal{L})\rangle$ is the absolutely free algebra generated by P, using the logical connectives as algebraic operations. $\mathcal{F}_{\mathcal{L}}$ is called the *formula algebra* of $\mathcal{L}$.

Henceforth we will omit the subscript $\mathcal{L}$ from $F_{\mathcal{L}}$, etc. when it causes no confusion.

In explicitly stating the parameter P which occurs in the above definition, we obtain the notion of a *general* logic.

DEFINITION 2.2 (general logic) A *general logic* is a class

$$\mathbf{L} \stackrel{\text{def}}{=} \langle \mathcal{L}^P : P \text{ is a set}\rangle,$$

where for every set P the quadruple $\mathcal{L}^P \stackrel{\text{def}}{=} \langle F^P, M^P, mng^P, \models^P\rangle$ is a logic in the above sense with P as its set of atomic formulas. Every $\mathcal{L}^P \in \mathbf{L}$ has the same set of logical connectives, which will be denoted by $Cn(\mathbf{L})$. Moreover the following conditions hold.

1. For any sets P and Q, if there is a bijection $f : P \longrightarrow Q$ then logic $\mathcal{L}^Q$ is an "isomorphic copy" of $\mathcal{L}^P$. That is, there exists a bijection $f^M : M^P \longrightarrow M^Q$ such that for all $\varphi \in F^P$ and all $\mathcal{M} \in M^P$,

$$mng^P_{\mathcal{M}}(\varphi) = mng^Q_{f^M(\mathcal{M})}(f^F(\varphi)), \text{ and}$$

$$\mathcal{M} \models^P \varphi \Leftrightarrow f^M(\mathcal{M}) \models^Q f^F(\varphi).$$

   Here f^F is the unique isomorphism from $\mathcal{F}^P$onto $\mathcal{F}^Q$ extending f.

2. For all sets $P \subseteq Q$, $\{mng^P_{\mathcal{M}} : \mathcal{M} \in M^P\} = \{(mng^Q_{\mathcal{M}}) \lceil F^P : \mathcal{M} \in M^Q\}$. ◀

Item 2.2.1 can be interpreted as saying that only the cardinality of the set of atomic formulas really matters, while item 2.2.2 says that $\mathcal{L}^P$ is the natural restriction of $\mathcal{L}^Q$, whenever P is a subset of Q. Again we will omit superscriptions like P, Q, etc. if the reading is unambiguous.

DEFINITION 2.3 (semantically algebraizable-, equivalential-, structural logic) A general logic $\mathbf{L}$ is called *semantically algebraizable* if in addition the following conditions are met:

1. For every set P and every $\mathcal{M} \in M^P$, the function $mng^P_{\mathcal{M}}$ is a homomorphism from $\mathcal{F}^P$ into some algebra.

2. For every set P, $(\forall \mathcal{M} \in M^P)(\forall h \in Hom(\mathcal{F}^P, mng^P_{\mathcal{M}}[\mathcal{F}^P]))(\exists \mathcal{N} \in M^P)\ mng^P_{\mathcal{N}} = h$.

3. There is a *derived binary connective*[2] Δ of $\mathbf{L}$ such that for every set P

$$(\forall \mathcal{M} \in M^P)(\forall \varphi, \psi \in F^P)[mng^P_{\mathcal{M}}(\varphi) = mng^P_{\mathcal{M}}(\psi) \Leftrightarrow \mathcal{M} \models^P \varphi \Delta \psi].$$

4. There are *derived unary connectives* ε, δ such that for every set P

$$(\forall \mathcal{M} \in M^P)(\forall \varphi \in F^P)[\mathcal{M} \models^P \varphi \Leftrightarrow \mathcal{M} \models^P \varepsilon(\varphi)\Delta\delta(\varphi)].$$

General logics satisfying condition 2.3.1 and 2.3.2 are called *structural*. Structural general logics satisfying condition 2.3.3 are called *equivalential*. ◀

Let us motivate the above definition. Condition 2.3.1 is one way of formulating the principle of compositionality of meaning. An equivalent way of putting the structurality condition 2.3.2 is to say that the class of meaning functions is closed under substitutions. Formally, for every set P, $(\forall \mathcal{M} \in M^P)(\forall \sigma \in Hom(\mathcal{F}^P, \mathcal{F}^P))(\exists \mathcal{N} \in M^P)\ mng^P_{\mathcal{N}} = mng^P_{\mathcal{M}} \circ \sigma$. Clause 2.3.3 and 2.3.4 require the existence of derived connectives that intuitively play the same role as the biconditional $\leftrightarrow$ and the constant $True$ do in classical propositional logic. For example, the $\{\wedge, \rightarrow\}$-fragment of this latter logic fulfills these clauses by setting $\varphi\Delta\psi \stackrel{\text{def}}{=} (\varphi \rightarrow \psi \wedge \psi \rightarrow \varphi)$, $\delta(\varphi) \stackrel{\text{def}}{=} \varphi$ and $\varepsilon(\varphi) \stackrel{\text{def}}{=} \varphi \rightarrow \varphi$.

In the literature one often finds a more general definition which takes into account logics in which the job here performed by Δ (or by ε, δ) is taken over by a (possibly infinite) set of connectives. For the sake of simplicity, we adhere to the above definition. However, the proofs of all our results will unproblematically carry over to this more general case.

We note that many of the well investigated sentential logics are semantically algebraizable. These include e.g. all fragments of classical (and intuitionistic) propositional logic containing either the connective $\leftrightarrow$, or the pair $\wedge$, $\rightarrow$, and all kinds of (multi-) modal and temporal logics, provided the semantic consequence relation is defined *globally* (cf. Remark 3.3). Also some "approximations" of ordinary first order logic are semantically algebraizable. One such logic is $\mathbf{L}_n$, i.e. first order logic restricted to n variables.

In the sequel the following fact will come in handy which immediately follows from the properties 2.3.3 and 2.3.4 of a semantically algebraizable logic. Here, for any set P and $\Sigma \subseteq F^P$, we denote by $Mod^P(\Sigma)$ the class of models of Σ. That is, the class of those models in M^P in which every $\sigma \in \Sigma$ is valid.

Fact 2.4 Let $\mathbf{L}$ be a semantically algebraizable general logic. For every set P, and every set of formulas $\Sigma \subseteq F^P$,

$$Mod^P(\Sigma) = Mod^P(\Sigma^*),$$

where $\Sigma^* \stackrel{\text{def}}{=} \{\varphi\Delta\psi : \varphi, \psi \in F^P \text{ and } (\forall \mathcal{M} \in Mod^P(\Sigma))\ mng^P_{\mathcal{M}}(\varphi) = mng^P_{\mathcal{M}}(\psi)\}$. ◀

[2]By "derived" we mean that Δ is not necessarily a member of $Cn(\mathbf{L})$. It might be built up from elements of $Cn(\mathbf{L})$.

In the Introduction we already hinted at some general method for associating to a logic a certain class of algebras. Now we are able to make this precise.

DEFINITION 2.5 (algebraic counterpart of a logic) Let $\mathbf{L}$ be a structural general logic. For any set P, and any class $K \subseteq M^P$ we define an equivalence relation $\sim_K$ on the formula algebra $\mathcal{F}^P$ as follows :

$$(\forall \varphi, \psi \in F^P)\ [\varphi \sim_K \psi \ \stackrel{\text{def}}{\Longleftrightarrow}\ (\forall \mathcal{M} \in K)\ mng^P_{\mathcal{M}}(\varphi) = mng^P_{\mathcal{M}}(\psi)].$$

Note that $\{\sim_K :\ K \subseteq M^P\}$ is a set (i.e. *not* a proper class), since F^P is a set. By condition 2.3.1, $\sim_K$ is a congruence relation. For $K \subseteq M^P$, let $\mathcal{F}^P/\sim_K$ denote the the factor algebra of $\mathcal{F}^P$ factorized by $\sim_K$. Now define

$$\begin{aligned} \mathsf{Alg}_{\models}(\mathcal{L}^P) &\stackrel{\text{def}}{=} \mathbf{I}\{\mathcal{F}^P/\sim_K :\ K \subseteq M^P\}, \\ \mathsf{Alg}_m(\mathcal{L}^P) &\stackrel{\text{def}}{=} \{mng^P_{\mathcal{M}}[\mathcal{F}^P] :\ \mathcal{M} \in M^P\}, \end{aligned}$$

where $\mathbf{I}$ denotes the operation of taking isomorphic copies. The *algebraic counterpart* of the general logic $\mathbf{L}$ (notation: $\mathsf{Alg}_{\models}(\mathbf{L})$), resp. *the class of meaning algebras* of $\mathbf{L}$ (notation: $\mathsf{Alg}_m(\mathbf{L})$), consists of the union of all $\mathsf{Alg}_{\models}(\mathcal{L}^P)$ (resp. $\mathsf{Alg}_m(\mathcal{L}^P)$) over all sets P. ◀

The class $\mathsf{Alg}_{\models}(\mathbf{L})$ can be seen as an abstraction of the well-known Lindenbaum-Tarski algebra. It is an *abstract* class of algebras, as for example the class of n-dimensional cylindric algebras ($=\mathsf{Alg}_{\models}(\mathbf{L}_n)$). On the other hand, the class $\mathsf{Alg}_m(\mathbf{L})$ is a *concrete* class; it is the algebraic counterpart of the *models* of the logic $\mathbf{L}$. As an example, note that the class of n-dimensional cylindric set algebras can be taken as $\mathsf{Alg}_m(\mathbf{L}_n)$.

Let us mention the following two results from [AKNS94] that we will gratefully make use of later.

LEMMA 2.6 ([AKNS94], Claim 3.2.17.1) *Let $\mathbf{L}$ be a structural general logic, $\mathcal{A} \in \mathsf{Alg}_m(\mathbf{L})$ and $h : \mathcal{F}^P \longrightarrow \mathcal{A}$ a homomorphism, for some set P. Then*

$$(\exists \mathcal{N} \in M^P)\ mng^P_{\mathcal{N}} = h.$$

Note that this lemma extends Definition 2.3.2, according to which there exists such a model $\mathcal{N}$ in case $\mathcal{A} \in \mathsf{Alg}_m(\mathcal{L}^P)$.

Now let us recall from e.g. [AKNS94] that for any class of similar algebras K,

$$\begin{aligned} \mathbf{S}\mathsf{K} &\stackrel{\text{def}}{=} \{\mathcal{M}\ :\ \mathcal{M} \text{ is a subalgebra of some } \mathcal{N} \in \mathsf{K}\}. \\ \mathbf{P}\mathsf{K} &\stackrel{\text{def}}{=} \mathbf{I}\{\textstyle\prod_{i \in I} \mathcal{A}_i\ :\ I \text{ is a set, and } (\forall i \in I)\ \mathcal{A}_i \in \mathsf{K}\}. \end{aligned}$$

Note that by the above definition the class $\mathbf{P}\mathsf{K}$ is closed under taking isomorphisms. Then the following equality holds.

THEOREM 2.7 ([AKNS94], Thm 3.2.17) *For a structural general logic $\mathbf{L}$,*

$$\mathsf{Alg}_{\models}(\mathbf{L}) = \mathbf{SP}\mathsf{Alg}_m(\mathbf{L}).$$

3 Beth definability property and epimorphisms

Let $\mathcal{L}^{P\cup\{r\}}$ be a logic, and $\Sigma \subseteq F^{P\cup\{r\}}$. We say that Σ explicitly defines r in terms of P iff there exists a formula $\varphi_r \in F^P$ which is equivalent to r in every model for Σ. As A. Padoa already noticed in 1900 (cf. [Pad00]), in order to show that such a formula does *not* exist it suffices to find a model for Σ in which it is possible to change only the interpretation of r while Σ remains true. In modern terminology we say that in this case Σ does not implicitly define r in terms of P. Half a century later E.W. Beth proved that in first order logic this condition is also necessary (cf. [Bet53]). In the framework of the present paper, these considerations give rise to the following definitions.

Below, by $\Sigma' \subseteq F^{P\cup R'}$ we understand the set of formulas obtained from Σ by replacing each $r \in R$ by the corresponding $r' \in R'$.

DEFINITION 3.1 (implicit definition, explicit definition) Let $\mathbf{L}$ be a structural general logic. Let $P, R, R' \stackrel{\text{def}}{=} \{r' : r \in R\}$ be disjoint sets. We say that a set $\Sigma \subseteq F^{P\cup R}$ *implicitly defines R in terms of P* iff

$$(\forall \mathcal{M} \in Mod^{P\cup R\cup R'}(\Sigma,\ \Sigma'))(\forall r \in R)\ mng_{\mathcal{M}}(r) = mng_{\mathcal{M}}(r').$$

We say that Σ *explicitly defines R in terms of P* iff

$$(\forall r \in R)(\exists \varphi_r \in F^P)(\forall \mathcal{M} \in Mod^{P\cup R}(\Sigma))\ mng_{\mathcal{M}}(r) = mng_{\mathcal{M}}(\varphi_r).$$

In this case we call the formula φ_r an *explicit definition* of r. ◀

Obviously, any $\Sigma \subseteq F^{P\cup R}$ that explicitly defines R in terms of P also implicitly defines R in terms of P. The converse is called the Beth (definability) property. That is, a general logic is said to have the **Beth (definability) property** iff for all Σ, P and R as above,

$$\Sigma \text{ implicitly defines } R \text{ in terms of } P \Longrightarrow \Sigma \text{ explicitly defines } R \text{ in terms of } P.$$

Logics which have the Beth property include classical (and intuitionistic) propositional logic, first order logic, but also numerous modal logics, like the minimal normal modal logic K or the well-known logics $K4$ and $S5$. Examples of logics which do not have this property are the implicational fragment of classical (and intuitionistic) propositional logic, or $\mathbf{L_3}$, i.e. first order logic restricted to 3 variables.

Remark 3.2 Definition 3.1 is a little more general than the one that can be found in standard textbooks on this subject, like [CK90] or [BF85]. The difference is that we do not restrict ourselves to the case where the set R is a singleton; R can be an arbitrary finite (or even an infinite) set. These three possibilities for the cardinality of R induce three Beth properties which we will, with some 'abus de language', baptize as the *singleton Beth property*, the *finite Beth property* and the *infinite Beth property*. For the purposes of the present paper it makes sense to concentrate on infinite Beth; as we will shortly see, it allows for the most natural algebraic characterization from the proof of which characterizations for the other two Beth properties can easily be obtained. We will come back to this in Remark 4.5. For completeness' sake let us anticipate Corollary 4.6 saying that in the present context we actually only need to distinguish two of the forementioned properties; a semantically algebraizable general logic has singleton Beth iff it has finite Beth. ◀

We note that in [AKNS94, Appendix B] a set $\Sigma \subseteq F^{P \cup R}$ is called an implicit definition of R in terms of P if any two $P \cup R$-models for Σ have the same meaning function as soon as their interpretation of formulas in F^P coincides. This alternative formulation is strictly stronger than Definition 3.1 and hence induces a strictly weaker Beth property. A characterization in the line of Theorem 1 for this alternative Beth property can only be given under the additional assumption that the logic has the so-called patchwork-property (cf. [AKNS94, Appendix B]). Indeed, for logics with the patchwork-property the two quoted formulations of an implicit definition are equivalent.

Remark 3.3 It is important to note that the definition of the Beth property essentially depends upon the notion of logical consequence. This gives us some elbow-room in the formulation of this property. For example, in modal logics a finite set $\Sigma \subseteq F^{P \cup R}$ can equally well be considered an implicit definition of R in terms of P if for every $r \in R$ either one of the two conditions below hold:

1. $\Sigma, \Sigma' \models r \leftrightarrow r'$,
2. $\models (\bigwedge_{\sigma \in \Sigma} \sigma \wedge \bigwedge_{\sigma' \in \Sigma'} \sigma') \rightarrow (r \leftrightarrow r')$.

Similar formulations are possible for explicit definitions. As shown in e.g. [Mak92a], the resulting definability properties are very different. In Definition 3.1 we formulated the global version, in the same spirit as the first formulation above. The reason for this is twofold. First this places us in one line with the algebraic tradition where the global consequence relation is the usual paradigm. Second, a consideration more to the content, is that concerning the Beth property the global perspective seems to be more natural: defining a primitive notion is not something that happens locally, as is the case in the second formulation above. Note the striking contrast with the related Craig interpolation property, which is almost always studied in its local form. ◀

As we will shortly see, the algebraic equivalent of the Beth property is surjectiveness of epimorphisms. Here an **epimorphism** of a class of similar algebras K is a homomorphism $f : \mathcal{A} \longrightarrow \mathcal{B}$, where $\mathcal{A}, \mathcal{B} \in \mathsf{K}$, such that for every $\mathcal{C} \in \mathsf{K}$ and every pair of homomorphisms $h, h' : \mathcal{B} \longrightarrow \mathcal{C}$ we have that $(h \circ f = h' \circ f) \Rightarrow h = h'$.

An important example of epimorphisms are surjections, and for some classes K these are the only K-epimorphisms. Henceforth these classes are said to have the property ES. For example the class of Boolean Algebras has ES. So has the class of Heyting Algebras. On the other hand, for other classes K there exist non-surjective K-epimorphisms. Examples include the class of semi-groups, or the class of distributive lattices.

4 An algebraic characterization of the Beth property

This section is devoted to the proof of the main theorem of this paper.

THEOREM 1 (Algebraic characterization of the Beth property) *Let* $\mathbf{L}$ *be a semantically algebraizable general logic. Then*

$$\begin{gathered}\mathbf{L} \textit{ has the Beth definability property} \\ \Longleftrightarrow \\ \textit{all the epimorphisms of } \mathsf{Alg}_{\models}(\mathbf{L}) \textit{ are surjective.}\end{gathered}$$

Some intuition The key idea underlying this theorem is that in some intuitive sense:

epimorphisms correspond to implicit definitions,
surjections correspond to explicit definitions.

Let us ponder one minute about this. The fact that the homomorphism $f : \mathcal{A} \longrightarrow \mathcal{B}$ is an epi(morphism) means that any homomorphism on $\mathcal{B}$ is determined by its restriction to the range of f. On the other hand, Σ being an implicit definition of R in terms of P means that every $P \cup R$-model for Σ is determined by its interpretation of P. Both surjections and explicit definitions somehow make the above determinations explicit. The key steps in the proofs below will therefore be (dependent on whether we consider the direction ES $\Rightarrow$ BP, or BP $\Rightarrow$ ES) to transform an implicit definition (resp. epi) into an epi (resp. implicit definition) in such a way that the explicit description on the algebraic (resp. logical) side (which exists by ES, resp. BP) allows us to extract an explicit description on the logical (resp. algebraic) side, that is, gives us an explicit definition (resp. surjection). This will be made precise in the following proof.

Proof of Theorem 1: Let $\mathbf{L}$ be a semantically algebraizable general logic.

ES $\Rightarrow$ BP:

Assume that all the epimorphisms of $\mathsf{Alg}_{\models}(\mathbf{L})$ are surjective. Let $P, R, R' \stackrel{\text{def}}{=} \{r' : r \in R\}$ be disjoint sets, and let $\Sigma \subseteq F^{P \cup R}$ be an implicit definition of R in terms of P. As described above, the crucial point is to find an $\mathsf{Alg}_{\models}(\mathbf{L})$-epimorphism f such that from the surjectiveness of f one can derive that Σ explicitly defines R. Hereto, let us consider the natural mapping $g : \mathcal{F}^{P \cup R} \longrightarrow \mathcal{F}^{P \cup R} / \sim_{Mod(\Sigma)}$. Here, $\sim_{Mod(\Sigma)}$ is the congruence relation as defined in Definition 2.5. That is, for all $\varphi, \psi \in F^{P \cup R}$,

$$g(\varphi) = g(\psi) \Leftrightarrow (\forall \mathcal{M} \in Mod^{P \cup R}(\Sigma))\ mng_{\mathcal{M}}(\varphi) = mng_{\mathcal{M}}(\psi). \tag{1}$$

For convenience, let us fix some notation. Let $\mathcal{G} \stackrel{\text{def}}{=} g[\mathcal{F}^{P \cup R}]$, $k \stackrel{\text{def}}{=} g\lceil F^P$ and $\mathcal{K} \stackrel{\text{def}}{=} k[\mathcal{F}^P]$. Let id be the identity from $\mathcal{F}^P$ into $\mathcal{F}^{P \cup R}$, and let id_K be the identity from $\mathcal{K}$ into $\mathcal{G}$. This situation is depicted in Figure 1.[3] Claim 4.2 and Claim 4.1 together show that id_K is an epimorphism of the kind we are looking for.

Claim 4.1 *If* id_K *is surjective, then* Σ *defines* R *explicitly in terms of* P.

[3]In the figures, the symbol $*$ indicates that the diagram commutes. The fact *that* these diagrams commute follows immediately from the definitions of the homomorphisms involved.

$$
\begin{tikzcd}
\mathcal{F}^P \arrow[r, "id"] \arrow[d, "k"'] \arrow[dr, phantom, "*"] & \mathcal{F}^{P\cup R} \arrow[d, "g"] \arrow[rrd, "mng_{\mathcal{N}}\lceil F^{P\cup R}"] \arrow[dr, phantom, "*"] & & \\
\mathcal{K} \arrow[r, "id_K"'] & \mathcal{G} \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & \mathcal{A} \arrow[r, "\pi_i"] & \mathcal{A}_i \\
& \mathcal{F}^{P\cup R'} \arrow[rru, "mng_{\mathcal{N}}\lceil F^{P\cup R'}"'] & &
\end{tikzcd}
$$

Figure 1: Ad ES $\Rightarrow$ BP.

Proof of Claim 4.1:
Let $r \in R$. By surjectiveness of id_K, there exists some $\varphi_r \in F^P$ such that $g(r) = g(\varphi_r)$. By (1) this implies that φ_r is an explicit definition of r. $\blacksquare$

By assumption, it now suffices to prove the following claim.

Claim 4.2 *id_K is an $\mathsf{Alg}_{\models}(\mathbf{L})$-epimorphism.*

Proof of Claim 4.2:
Let $\mathcal{A} \in \mathsf{Alg}_{\models}(\mathbf{L})$ and let $h, h' : \mathcal{G} \longrightarrow \mathcal{A}$ be two homomorphisms such that $h \circ id_K = h' \circ id_K$. Assume that $h \neq h'$. Note that for every $p \in P$

$$h(g(p)) = h'(g(p)), \tag{2}$$

hence there exists some $r_0 \in R$ such that $h(g(r_0)) \neq h'(g(r_0))$. We will try to derive a contradiction out of this. By Theorem 2.7, $\mathcal{A} \in \mathbf{SP}\mathsf{Alg}_m(\mathbf{L})$. Hence there is some $\mathcal{A}_i \in \mathsf{Alg}_m(\mathbf{L})$ and a homomorphism $\pi_i : \mathcal{A} \longrightarrow \mathcal{A}_i$ such that $\pi_i(h(g(r_0))) \neq \pi_i(h'(g(r_0)))$. Let us denote by χ the result of replacing in a formula $\chi' \in F^{P\cup R'}$ each occurrence of an atomic formula $r' \in R'$ by the corresponding $r \in R$. In particular, for any $\chi' \in F^P$ we have $\chi = \chi'$. Now let us consider the unique homomorphism $\alpha : \mathcal{F}^{P \,\cup\, R \,\cup\, R'} \longrightarrow \mathcal{A}_i$ defined such that $\alpha(q) = \pi_i(h(g(q)))$, $\forall q \in (P \cup R)$, and $\alpha(q') = \pi_i(h'(g(q)))$, $\forall q' \in P \cup R'$. By (2), α is well-defined and by Lemma 2.6 there exists some model $\mathcal{N} \in M^{P \,\cup\, R \,\cup\, R'}$ such that $\alpha = mng_{\mathcal{N}}^{P \,\cup\, R \,\cup\, R'}$. We will show that

$$\mathcal{N} \in Mod(\Sigma, \Sigma'). \tag{3}$$

Since Σ implicitly defines R in terms of P, it follows from (3) that for all $r \in R$, $mng_{\mathcal{N}}(r) = mng_{\mathcal{N}}(r')$. Hence $\pi_i(h(g(r_0))) = mng_{\mathcal{N}}(r_0) = mng_{\mathcal{N}}(r'_0) = \pi_i(h'(g(r_0)))$. A contradiction.

Now let us prove (3). Let $\varphi, \psi \in F^{P\cup R}$ be such that $(\forall \mathcal{M} \in Mod^{P\cup R}(\Sigma))\ mng_{\mathcal{M}}(\varphi) =$

$$
\begin{tikzcd}
\mathcal{F}^P \arrow[r, "id"] \arrow[d, "\widetilde{id}_B\lceil \mathcal{F}^P"'] \arrow[dr, phantom, "*"] & \mathcal{F}^{P\cup R} \arrow[d, "\widetilde{id}_B"'] \arrow[dr, "mng_{\mathcal{M}}\lceil F^{P\cup R}"] \arrow[dr, phantom, "*", near start, shift right=3] & \\
\mathcal{P} \arrow[r, "id_P"'] & \mathcal{B} \arrow[r, shift left, "g"] \arrow[r, shift right, "g'"'] & mng_{\mathcal{M}}[\mathcal{F}^{P\cup R\cup R'}] \\
& \mathcal{F}^{P\cup R'} \arrow[ur, "mng_{\mathcal{M}}\lceil F^{P\cup R'}"'] &
\end{tikzcd}
$$

Figure 2: Ad BP $\Rightarrow$ ES.

$mng_{\mathcal{M}}(\psi)$. Then $g(\varphi) = g(\psi)$ and hence $mng_{\mathcal{N}}(\varphi) = mng_{\mathcal{N}}(\psi)$. By condition 2.3.3 and Fact 2.4 this shows that $\mathcal{N} \in Mod(\Sigma)$. It remains to show that $\mathcal{N} \in Mod(\Sigma')$. Hereto consider $\varphi', \psi' \in F^{P\cup R'}$ such that $(\forall \mathcal{M} \in Mod^{P\cup R'}(\Sigma'))\ mng_{\mathcal{M}}(\varphi') = mng_{\mathcal{M}}(\psi')$. We will show that $g(\varphi) = g(\psi)$. As before, this implies that $mng_{\mathcal{N}}(\varphi') = mng_{\mathcal{N}}(\psi')$, and we are done. Therefore, let us pick an arbitrary $\mathcal{M} \in Mod^{P\cup R}(\Sigma)$. Since $\mathbf{L}$ is a general logic, there exists an "isomorphic copy" $\mathcal{M}_{\text{iso}} \in M^{P\cup R'}$ in the sense of Definition 2.2. Then $\mathcal{M}_{\text{iso}} \in Mod(\Sigma')$. We picked φ' and ψ' in such a way that this implies that $mng_{\mathcal{M}_{\text{iso}}}(\varphi') = mng_{\mathcal{M}_{\text{iso}}}(\psi')$. By 2.2.1, then $mng_{\mathcal{M}}(\varphi) = mng_{\mathcal{M}}(\psi)$. We conclude that $g(\varphi) = g(\psi)$.
Q.E.D. (Claim 4.2)
Q.E.D. (ES $\Rightarrow$ BP)

BP $\Rightarrow$ ES:

Assume that $\mathbf{L}$ has the Beth definability property. Let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be an $\mathsf{Alg}_{\models}(\mathbf{L})$-epimorphism. We have to show that $Rg(f) = B$.

Let us write $\mathcal{P} \stackrel{\text{def}}{=} f[\mathcal{A}]$, and let $id_P : \mathcal{P} \longrightarrow \mathcal{B}$ be the identity. Since $\mathrm{Rg}(f) = \mathrm{Rg}(id_P)$ it follows that id_P is an $\mathsf{Alg}_{\models}(\mathbf{L})$-epimorphism.

From now on we will view the universe P (resp. B) of the *algebra* $\mathcal{P}$ (resp. $\mathcal{B}$) as a set of atomic formulas, thus obtaining the *logic* $\mathcal{L}^P$ (resp. $\mathcal{L}^B$). In order to be in accordance with previous notation, let $R \stackrel{\text{def}}{=} B \setminus P$, and let $R' \stackrel{\text{def}}{=} \{r' \ : \ r \in R\}$ be a set of fresh atomic formulas. As before, for any $\chi \in F^{P\cup R}$ we denote by χ' the result of substituting in χ every occurrence of an atomic formula $r \in R$ by the corresponding $r' \in R'$. Finally, let $\widetilde{id}_B$ be the usual extension of id_B to $\mathcal{F}^B$ ($= \mathcal{F}^{P\cup R}$). See also Figure 2, where one can safely ignore everything that has not been defined yet.

This time the crux is to find an implicit definition Σ such that from the fact that Σ is an explicit definition one can infer the surjectiveness of f. In Claim 4.3 and Claim 4.4 it will be proved that for this purpose we can take the set $\Sigma \subseteq F^{P\cup R}$ defined by:

$$\Sigma \stackrel{\text{def}}{=} \{\varphi\Delta\psi : \varphi, \psi \in F^{P\cup R} \text{ and } \widetilde{id}_B(\varphi) = \widetilde{id}_B(\psi)\}. \tag{4}$$

Claim 4.3 *Σ defines R implicitly in terms of P.*

Proof of Claim 4.3:
Let $\mathcal{M} \in Mod^{P\,\cup\,R\,\cup\,R'}(\Sigma, \Sigma')$. We need to show that $mng_{\mathcal{M}}(r) = mng_{\mathcal{M}}(r')$, for every $r \in R$. Hereto let us define the homomorphism $g : \mathcal{B} \longrightarrow mng_{\mathcal{M}}[\mathcal{F}^{P\,\cup\,R\,\cup\,R'}]$ as follows:

$$(\forall b \in B)\ g(b) = mng_{\mathcal{M}}(\varphi_b),$$

where $\varphi_b \in F^{P\cup R}$ is such that $\widetilde{id}_B(\varphi_b) = b$. Since $\widetilde{id}_B$ is surjective, we can always find such φ_b. Moreover, for every $\varphi_b, \psi_b \in F^{P\cup R}$ such that $\widetilde{id}_B(\varphi_b) = \widetilde{id}_B(\psi_b)$ it holds that $mng_{\mathcal{M}}(\varphi_b) = mng_{\mathcal{M}}(\psi_b)$, since $\mathcal{M} \in Mod(\Sigma)$. We conclude that g is well-defined. Analogously we define the homomorphism $g' : \mathcal{B} \longrightarrow mng_{\mathcal{M}}[\mathcal{F}^{P\,\cup\,R\,\cup\,R'}]$ by setting $(\forall b \in B)\ g'(b) = mng_{\mathcal{M}}(\varphi'_b)$, where $\varphi'_b \in F^{P\cup R'}$ is such that $\widetilde{id}_B(\varphi_b) = b$. Since $\mathcal{M} \in Mod(\Sigma')$ also g' is well-defined. The situation described so far has been sketched in Figure 2. Note that for any $b \in B$, the formula b itself is one such φ_b that appears in the definition of g. Similarly, b' can be taken as the φ'_b in the definition of g'. This means that for every $b \in B$, $g(b) = mng_{\mathcal{M}}(b)$ and $g'(b) = mng_{\mathcal{M}}(b')$. Recall that $(\forall p \in P)\ p = p'$. Hence $(\forall p \in P)\ g(p) = mng_{\mathcal{M}}(p) = mng_{\mathcal{M}}(p') = g'(p)$. Since id_P is an $\mathsf{Alg}_{\models}(\mathbf{L})$-epimorphism, it follows that $g = g'$. Hence for every $r \in R$, $mng_{\mathcal{M}}(r) = g(r) = g'(r) = mng_{\mathcal{M}}(r')$.
Q.E.D. (Claim 4.3)

Since $\mathbf{L}$ has the Beth property, it follows that Σ explicitly defines R in terms of P.

Claim 4.4 *id_P is surjective.*

Proof of Claim 4.4 :
Let $r \in R$. Since Σ defines R explicitly in terms of P,

$$(\exists\varphi_r \in F^P)(\forall\mathcal{M} \in Mod^{P\cup R}(\Sigma))\ mng_{\mathcal{M}}(r) = mng_{\mathcal{M}}(\varphi_r). \tag{5}$$

We will show that $r = \widetilde{id}_B(\varphi_r)$. This implies that id_P is surjective.

Assume $r \neq \widetilde{id}_B(\varphi_r)$. By Theorem 2.7, $\mathcal{B} \in \mathbf{SP}\mathsf{Alg}_m(\mathbf{L})$. Hence there exists a $\mathcal{B}_i \in \mathsf{Alg}_m(\mathbf{L})$ and a homomorphism $\pi_i : \mathcal{B} \longrightarrow \mathcal{B}_i$ such that $\pi_i(\widetilde{id}_B(r)) \neq \pi_i(\widetilde{id}_B(\varphi_r))$. By Lemma 2.6, there exists some model $\mathcal{N} \in M^{P\cup R}$ such that $mng_{\mathcal{N}} = \pi_i \circ \widetilde{id}_B$. Hence $mng_{\mathcal{N}}(\varphi_r) \neq mng_{\mathcal{N}}(r)$. By (5) it follows that $\mathcal{N} \notin Mod^{P\cup R}(\Sigma)$. On the other hand, consider some $\varphi\Delta\psi \in \Sigma$. Then $mng_{\mathcal{N}}(\varphi) = \pi_i(\widetilde{id}_B(\varphi)) = \pi_i(\widetilde{id}_B(\psi)) = mng_{\mathcal{N}}(\psi)$. Hence $\mathcal{N} \in Mod^{P\cup R}(\Sigma)$. Contradiction. ■

Since $\mathrm{Rg}(f) = \mathrm{Rg}(id_P)$, we can conclude that f is surjective.

Q.E.D. (BP $\Rightarrow$ ES)
Q.E.D. (Theorem 1)

Remark 4.5 After having seen the proof of Theorem 1, the result by I. Németi (cf. [HMT85, Theorem 5.6.10]) mentioned in the Introduction can be better understood. Besides the different framework, the main difference is that Németi characterizes a different definability property, namely the singleton Beth property (cf. Remark 3.2) which assures explicit definability of only those implicit definitions which define one single primitive notion r at a time. If we want to carry the proof of Theorem 1 over to this case, we will get stuck after Claim 4.3 when we apply the Beth property to infer the explicit definability of R, which need not be a singleton at all.

This is why I. Németi considered on the algebraic side so-called *almost-onto* epimorphisms. These are epis $f : \mathcal{A} \longrightarrow \mathcal{B}$ such that $\mathcal{B}$ is generated by the range of f together with one extra element $b_0 \in B$. Theorem 5.6.10 in [HMT85] states the equivalence of singleton Beth to surjectiveness of these almost-onto epimorphisms. Let us briefly sketch how this follows as a special case from the proof of Theorem 1. For the direction BP $\Rightarrow$ ES, note that if the epimorphism $id_P : \mathcal{P} \longrightarrow \mathcal{B}$ is in addition almost-onto, then the formula algebra generated by the union of P and the one extra element b_0 can be mapped onto $\mathcal{B}$. Similar to the proof of Claim 4.3 it can then be shown that the kernel of this homomorphism is an implicit definition of $\{b_0\}$ in terms of P. Applying singleton Beth, and following the proof of Claim 4.4, we obtain that $b_0 \in Rg(id_P)$. As $\mathcal{B}$ is generated by $\{b_0\} \cup Rg(id_P)$, surjectivity of id_P follows. Note that our proof for the other direction goes through in its original form, as the epimorphism id_K considered in Claim 4.2 will be almost-onto in case $|R|= 1$. ◀

Following the above heuristics, it turns out that the finite Beth property (cf. Remark 3.2) corresponds to surjectiveness of those epimorphisms $f : \mathcal{A} \longrightarrow \mathcal{B}$ such that $\mathcal{B}$ is generated by the range of f together with some finite subset. It can easily be seen that this last property is equivalent to surjectiveness of almost-onto epimorphisms. Hence from the proof of Theorem 1 we obtain the following corollary.

Corollary 4.6 *Let* **L** *be a semantically algebraizable general logic. Then*

$$\mathbf{L} \textit{ has the finite Beth property} \Leftrightarrow \mathbf{L} \textit{ has the singleton Beth property.}$$

5 The weak Beth property

Historically the first definability property to be investigated was the Beth property that we have treated in this paper so far. However, with a growing interest in logics other than first order logic, other properties too float up the surface. Just as in the context of abstract model theory properties like countable compactness or Δ-interpolation are considered to be more natural than compactness or interpolation, it turned out that the so-called *weak Beth property* actually better captures our intuition concerning definability than the Beth property (cf. [Fri73]). The point is that the usual formal notion of an implicit definition is for some logics too weak. To make this precise, let us introduce some terminology. For sets P, R, and $\Sigma \subseteq F^{P \cup R}$ we understand by $Cons^P(\Sigma)$ the set of P-consequences of Σ. That is,

$$Cons^P(\Sigma) \stackrel{\text{def}}{=} \{\varphi \in F^P \ : \ (\forall \mathcal{M} \in M^{P \cup R})[\mathcal{M} \in Mod(\Sigma) \Rightarrow \mathcal{M} \in Mod(\varphi)]\}.$$

Definition 3.1 can be rephrased as saying that a set $\Sigma \subseteq F^{P\cup R}$ implicitly defines R in terms of P if every P-model for $Cons^P(\Sigma)$ can be extended in *at most* one way to a $P \cup R$-structure in which Σ holds. But what if some P-model for $Cons^P(\Sigma)$ can not be extended at all in such a way? It can be argued that in this case Σ does not meet our intuitive expectations with regard to "being a definition". This leads to the additional requirement that every P-structure in which $Cons^P(\Sigma)$ holds, can be extended in *at least* one way to a $P \cup R$-model for Σ. This results in the notion of a *strong implicit definition.* Now the *weak Beth definability property* assures the existence of an explicit definition for just these strong implicit definitions. For examples showing that the weak Beth property applies to a strictly larger class of logics than the Beth property, the reader is referred to [BF85, page 74].

DEFINITION 5.1 (strong implicit definition, weak Beth property) Let $\mathbf{L}$ be a structural general logic. Let $P, R, R' \stackrel{\text{def}}{=} \{r' : r \in R\}$ be disjoint sets. We call $\Sigma \subseteq F^{P\cup R}$ a *strong implicit definition of R in terms of P* iff

$$\Sigma \text{ implicitly defines } R \text{ in terms of } P \text{, and}$$

$$(\forall \mathcal{M} \in Mod^P(Cons^P(\Sigma)))(\exists \mathcal{N} \in Mod^{P\cup R}(\Sigma))\; mng_{\mathcal{N}}\lceil F^P = mng_{\mathcal{M}}. \tag{6}$$

We say that $\mathbf{L}$ has the *weak Beth definability property* iff for all P, R, Σ as above,

$$\begin{array}{ccc} & \Sigma \text{ is a strong implicit definition of } R \text{ in terms of } P & \Rightarrow \\ \Rightarrow & \Sigma \text{ explicitly defines } R \text{ in terms of } P. & \blacktriangleleft \end{array}$$

Since the weak Beth property requires only explicit definability of some special implicit definitions, a counterstep on the algebraic side would consist in demanding surjectivity of a suitable subclass of epimorphisms. A possible subclass is the following.

DEFINITION 5.2 (K_0-extensible) Let $\mathsf{K}_0 \subseteq \mathsf{K}$ be two classes of similar algebras. Let $\mathcal{A}, \mathcal{B} \in \mathsf{K}$ and $h : \mathcal{A} \longrightarrow \mathcal{B}$ be a homomorphism. h is said to be K_0*-extensible* iff for every algebra $\mathcal{C} \in \mathsf{K}_0$ and every *surjection* $f : \mathcal{A} \longrightarrow \mathcal{C}$ there exists some $\mathcal{D} \in \mathsf{K}_0$ and $g : \mathcal{B} \longrightarrow \mathcal{D}$ such that $\mathcal{C} \subseteq \mathcal{D}$ and $g \circ h = f$. $\blacktriangleleft$

For readers familiar with [AKNS94] we point out that in comparison with that paper, we made in Definition 5.2 the additional requirement for f to be *onto.*

Now we have the following equivalence.

THEOREM 2 (Algebraic characterization of the weak Beth property) *Let $\mathbf{L}$ be a semantically algebraizable general logic. Then*

$$\begin{array}{c} \mathbf{L} \textit{ has the weak Beth definability property} \\ \Longleftrightarrow \\ \textit{all the } \mathsf{Alg}_m(\mathbf{L})\textit{-extensible epimorphisms of } \mathsf{Alg}_{\models}(\mathbf{L}) \textit{ are surjective.} \end{array}$$

Proof of Theorem 2: Let $\mathbf{L}$ be a semantically algebraizable general logic. We will closely follow the proof of Theorem 1, making alterations where necessary.

$$
\begin{tikzcd}
\mathcal{F}^P \arrow[r, "id"] \arrow[d, "k"] \arrow[dr, phantom, "*"] \arrow[dd, shift right=3ex, "mng_{\mathcal{N}}"'] & \mathcal{F}^{P\cup R} \arrow[d, "g"'] \arrow[dd, shift left=3ex, "mng_{\mathcal{N}'}"] \\
\mathcal{K} \arrow[r, "id_K"] \arrow[d, "f"] & \mathcal{G} \arrow[d, "h"'] \\
\mathcal{C} \arrow[r, phantom, "\subseteq"] & mng_{\mathcal{N}'}[\mathcal{F}^{P\cup R}]
\end{tikzcd}
$$

Figure 3: cf. Claim 5.3.

ExtES $\Rightarrow$ WeakBP:

Assume that every $\mathsf{Alg}_m(\mathbf{L})$-extensible epimorphism of $\mathsf{Alg}_{\models}(\mathbf{L})$ is surjective. Consider disjoint sets $P, R, R' \stackrel{\text{def}}{=} \{r' : r \in R\}$, and let $\Sigma \subseteq F^{P\cup R}$ be a strong implicit definition of R in terms of P. Our aim is to prove the following strengthening of Claim 4.2, where $g, \mathcal{G}, k, \mathcal{K}$ and id_K are defined as before.

Claim 5.3 *id_K is $\mathsf{Alg}_m(\mathbf{L})$-extensible.*

Reasoning as in the proof of Theorem 1, this enables us to conclude that Σ explicitly defines R in terms of P.

Proof of Claim 5.3:
Consider an arbitrary algebra $\mathcal{C} \in \mathsf{Alg}_m(\mathbf{L})$, and a surjection $f : \mathcal{K} \longrightarrow \mathcal{C}$. By Lemma 2.6, there exists some $\mathcal{N} \in M^P$ such that $f \circ k = mng_{\mathcal{N}}$. See also Figure 3. Note that $\mathcal{C} = mng_{\mathcal{N}}[\mathcal{F}^P]$, by surjectiveness of f. We claim that $\mathcal{N} \in Mod^P(Cons^P(\Sigma))$. Hereto, consider $\varphi, \psi \in F^P$ such that

$$(\forall \mathcal{M} \in Mod^P(Cons^P(\Sigma)))\ mng_{\mathcal{M}}(\varphi) = mng_{\mathcal{M}}(\psi). \tag{7}$$

If $k(\varphi) = k(\psi)$, then $mng_{\mathcal{N}}(\varphi) = f(k(\varphi)) = f(k(\psi)) = mng_{\mathcal{N}}(\psi)$, and from Fact 2.4 it follows that $\mathcal{N} \in Mod^P(Cons^P(\Sigma))$. In order to show that $k(\varphi) = k(\psi)$, consider some $\mathcal{M} \in Mod^{P\cup R}(\Sigma)$. What needs to be established is that $mng_{\mathcal{M}}(\varphi) = mng_{\mathcal{M}}(\psi)$. By 2.2.2 there exists some $\mathcal{M}' \in M^P$ such that $mng_{\mathcal{M}}\lceil F^P = mng_{\mathcal{M}'}$. By 2.3.3 and 2.3.4, this implies that $\mathcal{M}' \in Mod^P(Cons^P(\Sigma))$, and hence from (7) it follows that $mng_{\mathcal{M}'}(\varphi) = mng_{\mathcal{M}'}(\psi)$. We conclude that $mng_{\mathcal{M}}(\varphi) = mng_{\mathcal{M}'}(\varphi) = mng_{\mathcal{M}'}(\psi) = mng_{\mathcal{M}}(\psi)$, and we are done.

Now that we know that $\mathcal{N} \in Mod^P(Cons^P(\Sigma))$, it follows from the fact that Σ is a strong implicit definition of R that there exists some $\mathcal{N}' \in Mod^{P\cup R}(\Sigma)$ such that

$$
\begin{tikzcd}
 & \mathcal{F}^B \arrow[d, "\widetilde{id}_B"'] \arrow[dd, bend left, "mng_{\mathcal{M}'}"] \\
\mathcal{P} \arrow[r, "id_P"] \arrow[d, "h"'] & \mathcal{B} \arrow[d, "g"'] \\
mng_{\mathcal{M}}[\mathcal{F}^P] \arrow[r, phantom, "\subseteq"] & \mathcal{D}
\end{tikzcd}
$$

Figure 4: cf. Claim 5.4.

$mng_{\mathcal{N}'}\lceil F^P = mng_{\mathcal{N}}$. We define the homomorphism $h : \mathcal{G} \longrightarrow mng_{\mathcal{N}'}[\mathcal{F}^{P\cup R}]$ as follows:

$$(\forall x \in G)\ h(x) = mng_{\mathcal{N}'}(\varphi_x),$$

where $\varphi_x \in F^{P\cup R}$ is such that $g(\varphi_x) = x$. Since g is surjective, there is always such φ_x. Furthermore, if $g(\varphi_x) = g(\psi_x)$, for some $\varphi_x, \psi_x \in F^{P\cup R}$, then $mng_{\mathcal{N}'}(\varphi_x) = mng_{\mathcal{N}'}(\psi_x)$, since $\mathcal{N}' \in Mod^{P\cup R}(\Sigma)$. Hence h is well-defined. Note that $\mathcal{C} = mng_{\mathcal{N}}[\mathcal{F}^P] \subseteq mng_{\mathcal{N}'}[\mathcal{F}^{P\cup R}]$. We will show that $h \circ id_K = f$, hereby completing the proof of Claim 5.3. Let $y \in K$. Then $y = k(\varphi_y)$, for some $\varphi_y \in F^P$, and $f(y) = f(k(\varphi_y)) = mng_{\mathcal{N}}(\varphi_y) = mng_{\mathcal{N}'}(\varphi_y) = h(g(\varphi_y)) = h(y) = h(id_K(y))$. ■

Q.E.D. (ExtES $\Rightarrow$ WeakBP)

WeakBP $\Rightarrow$ ExtES:

Assume that $\mathbf{L}$ has the weak Beth property. Let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be an $\mathsf{Alg}_m(\mathbf{L})$-extensible epimorphism of $\mathsf{Alg}_{\models}(\mathbf{L})$. As before, we write $\mathcal{P} \stackrel{\text{def}}{=} f[\mathcal{A}]$, and $R \stackrel{\text{def}}{=} B \setminus P$. The reader can easily verify that $id_P : \mathcal{P} \longrightarrow \mathcal{B}$ is $\mathsf{Alg}_m(\mathbf{L})$-extensible. We have to show that f is surjective. As an inspection of the proof of Theorem 1 shows, it suffices to prove the following strengthening of Claim 4.3, where P, R and Σ are defined as before (cf. (4)).

Claim 5.4 *Σ defines R implicitly in terms of P in a strong sense.*

Proof of Claim 5.4:
Let $\mathcal{M} \in Mod^P(Cons^P(\Sigma))$. By Claim 4.3 it suffices to find an $\mathcal{M}' \in Mod^{P\cup R}(\Sigma)$ such that $mng_{\mathcal{M}'}\lceil F^P = mng_{\mathcal{M}}$. Define the homomorphism $h : \mathcal{P} \longrightarrow mng_{\mathcal{M}}[\mathcal{F}^P]$ by

$$h(p) = mng_{\mathcal{M}}(\varphi_p),$$

where $\varphi_p \in F^P$ is such that $\widetilde{id}_B(\varphi_p) = p$. See Figure 4. Note that for $p \in P$, one such φ_p is the formula p itself. Hence $(\forall p \in P)\ h(p) = mng_{\mathcal{M}}(p)$. Since $\mathcal{M} \in Mod^P(Cons^P(\Sigma))$ it follows that h is well-defined and onto. Now we use the fact that id_P is $\mathsf{Alg}_m(\mathbf{L})$-extensible to infer the existence of some $\mathcal{D} \in \mathsf{Alg}_m(\mathbf{L})$ and a homomorphism $g : \mathcal{B} \longrightarrow \mathcal{D}$ such that $mng_{\mathcal{M}}[\mathcal{F}^P] \subseteq \mathcal{D}$ and $g \circ id_P = h$. By Lemma 2.6, there exists a model $\mathcal{M}' \in M^{P\cup R}$ such that $g \circ \widetilde{id}_B = mng_{\mathcal{M}'}$. We claim that $\mathcal{M}'$ is the desired model. For, $(\forall p \in P)\ mng_{\mathcal{M}}(p) =$

$h(p) = g(id_P(p)) = g(p) = g(\widetilde{id}_B(p)) = mng_{\mathcal{M}'}(p)$. Since $mng_{\mathcal{M}'}$ and $mng_{\mathcal{M}}$ are homomorphisms we conclude that $mng_{\mathcal{M}} = mng_{\mathcal{M}'} \lceil F^P$. Furthermore, let $\varphi, \psi \in F^{P \cup R}$ such that $\widetilde{id}_B(\varphi) = \widetilde{id}_B(\psi)$. Then $mng_{\mathcal{M}'}(\varphi) = g(\widetilde{id}_B(\varphi)) = g(\widetilde{id}_B(\psi)) = mng_{\mathcal{M}'}(\psi)$. That is, $\mathcal{M}' \in Mod^{P \cup R}(\Sigma)$. ■

Q.E.D. (WeakBP $\Rightarrow$ ExtES)

Q.E.D. (Theorem 2)

Remark 5.5 Other characterizations of the weak Beth property due to J. Madarász, I. Németi and I. Sain can be found in [Sai97]. ◀

The results of this paper apply to semantically algebraizable general logics, i.e. structural general logics with derived connectives $\delta, \varepsilon, \Delta$ that, in a sloppy way of speaking, play the role of the biconditional and the constant *True* in classical propositional logic. As the reader can verify, in the direction (weak)BP$\Rightarrow$(ext)ES, the connectives δ, ε have not been used. The purport of Theorem 4 below is that in the proofs of the other direction, both of Theorem 1 and Theorem 2, these connectives are really needed. Theorem 3 justifies our use of the special connective Δ in proving Theorem 1 and Theorem 2. These results were obtained in cooperation with J. Madarász. Proofs can be found in [Hoo96].

THEOREM 3 *There exists a non-equivalential structural general logic* $\mathbf{L}$ *such that* $\mathbf{L}$ *has the Beth property, but* not *all the* $\mathsf{Alg}_m(\mathbf{L})$*- extensible epimorphisms of* $\mathsf{Alg}_{\models}(\mathbf{L})$ *are surjective.*

THEOREM 4 *There exists an equivalential general logic* $\mathbf{L}$ *such that* $\mathbf{L}$ *does* not *have the weak Beth definability property, but all the epimorphisms of* $\mathsf{Alg}_{\models}(\mathbf{L})$ *are surjective.*

6 The projective Beth property

Let us shift our attention to yet another definability property, and let us hereto focus on the role of the set of atomic formulas P in an implicit definition Σ of r in terms of P. Assuming the logic under consideration to have the Beth property, we can divide the set P into the set P_0 of atoms that occur in the explicit definition φ_r and the set P_1 of those atoms that do not. Note that this does not imply that the restriction of Σ to $F^{P_0 \cup \{r\}}$ is already an implicit definition of r in terms of P_0. The formulas in Σ in which atoms from P_1 occur may very well be necessary to cause Σ to be an implicit definition, but these formulas are somehow in the 'background': they do not occur in the explicit definition. In the following definition this distinction has been made explicit. For more information about this definability property the reader is referred to [BF85, page 75].

DEFINITION 6.1 (projective Beth property) Let $\mathbf{L}$ be a structural general logic. Let $P, Q, R, Q' \stackrel{\text{def}}{=} \{q' : q \in Q\}, R' \stackrel{\text{def}}{=} \{r' : r \in R\}$ be disjoint sets. We call $\Sigma \subseteq F^{P \cup Q \cup R}$ a *projective implicit definition of* R *in terms of* P *via* Q iff

$$(\forall \mathcal{M} \in Mod^{P \cup Q \cup Q' \cup R \cup R'}(\Sigma,\ \Sigma'))(\forall r \in R)\ mng_{\mathcal{M}}(r) = mng_{\mathcal{M}}(r').$$

We say that Σ *explicitly defines R in terms of P via Q* iff

$$(\forall r \in R)(\exists \varphi_r \in F^P)(\forall \mathcal{M} \in Mod^{P\cup Q\cup R}(\Sigma))\ mng_{\mathcal{M}}(r) = mng_{\mathcal{M}}(\varphi_r).$$

L is said to have the *projective Beth property* (ProjBeth) iff for all P, Q, R and Σ as above

$$\begin{array}{ll} \Sigma \text{ is a projective implicit definition of } R \text{ in terms of } P \text{ via } Q, & \Longrightarrow \\ \Longrightarrow \quad \Sigma \text{ explicitly defines } R \text{ in terms of } P \text{ via } Q. & \blacktriangleleft \end{array}$$

Obviously, the Beth property is an instance of projBP by taking $Q = \emptyset$. As L. Maksimova showed (see [Mak97, Theorem 4]), in varieties of modal algebras the projective Beth property corresponds to surjectiveness of what I propose to call *weak epimorphisms.*

DEFINITION 6.2 Let K be a class of similar algebras. By a *weak epimorphism* of K we understand a homomorphism $f : \mathcal{A} \longrightarrow \mathcal{B}$, where $\mathcal{A}, \mathcal{B} \in \mathsf{K}$, such that either f is surjective, or there exists some $x \in B \setminus f[A]$ such that

$$(\forall \mathcal{C} \in \mathsf{K})(\forall h, h' \in \mathrm{Hom}(\mathcal{B}, \mathcal{C}))[(h \circ f = h' \circ f) \Rightarrow h(x) = h'(x)]. \tag{8}$$

In this case we call x a *witness* (of the fact that f is a weak epimorphism). We say that K has *StrongES* iff all weak epimorphisms of K are surjective. $\blacktriangleleft$

Note that for any epimorphism f, (8) holds *for all* $x \in B \setminus f[A]$. Hence StrongES is indeed stronger than ES. Now we have the following generalization of the forementioned result by L. Maksimova.

THEOREM 5 (Algebraic characterization of projective Beth property) *Let **L** be a semantically algebraizable general logic. Then*

$$\begin{array}{c} \mathbf{L} \textit{ has the projective Beth property} \\ \Longleftrightarrow \\ \mathsf{Alg}_{\models}(\mathbf{L}) \textit{ has StrongES.} \end{array}$$

Proof of Theorem 5: Let **L** be a semantically algebraizable general logic. We will follow the outline of the proof of Theorem 1, mutatis mutandis.

StrongES $\Rightarrow$ ProjBP:
Assume that every weak epimorphism of $\mathsf{Alg}_{\models}(\mathbf{L})$ is surjective. Let P, Q, R be disjoint sets, and let $\Sigma \subseteq F^{P\cup Q\cup R}$ be a projective implicit definition of R in terms of P via Q. Let g be the natural mapping $g : \mathcal{F}^{P\cup Q\cup R} \longrightarrow \mathcal{F}^{P\cup Q\cup R}/ \sim_{Mod(\Sigma)}$. As before, set $\mathcal{G} \stackrel{\text{def}}{=} g[\mathcal{F}^{P\cup Q\cup R}]$, $k \stackrel{\text{def}}{=} g\lceil F^P$, $\mathcal{K} \stackrel{\text{def}}{=} k[\mathcal{F}^P]$, and let id_K be the identity from $\mathcal{K}$ into $\mathcal{G}$. As in the proof of Theorem 1 we see that Σ explicitly defines R in terms of P if id_K is surjective. Hence by assumption it suffices to show that id_K is a weak $\mathsf{Alg}_{\models}(\mathbf{L})$-epimorphism.

Claim 6.3 *For $r \in R$, $g(r)$ witnesses the fact that id_K is a weak $\mathsf{Alg}_{\models}(\mathbf{L})$-epimorphism.*

Proof of Claim 6.3: Let $r_0 \in R$. Assume there exists some $\mathcal{C} \in \mathsf{Alg}_{\models}(\mathbf{L})$ and homomorphisms $h, h' : \mathcal{G} \longrightarrow \mathcal{C}$ such that $h \circ id_K = h' \circ id_K$, but $h(g(r_0)) \neq h'(g(r_0))$. Analogous to the proof of Claim 4.2 this leads to a contradiction. ■
Q.E.D. (StrongES ⇒ ProjBP)

ProjBP ⇒ StrongES:
Assume that **L** has the projective Beth property. Let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a weak epimorphism of $\mathsf{Alg}_{\models}(\mathbf{L})$. Then $id_P : \mathcal{P} \longrightarrow \mathcal{B}$ is a weak epimorphism of $\mathsf{Alg}_{\models}(\mathbf{L})$, where $\mathcal{P} \stackrel{\text{def}}{=} f[\mathcal{A}]$. Reasoning to contraposition, let us assume that id_P is not surjective. By Definition 6.2 this implies the existence of some $r \in B \backslash P$ for which (8) holds. Let $Q \stackrel{\text{def}}{=} B \backslash (P \cup \{r\})$, and let Σ be defined as in Claim 4.3. That is, $\Sigma \stackrel{\text{def}}{=} \{\varphi \Delta \psi \ : \ \varphi, \psi \in F^B \text{ and } \widetilde{id}_B(\varphi) = \widetilde{id}_B(\psi)\}$. We have the following analogue of Claim 4.3.

Claim 6.4 *Σ is a projective implicit definition of $\{r\}$ in terms of P via Q.*

Proof of Claim 6.4: Let $\mathcal{M} \in Mod^{P \cup Q \cup Q' \cup \{r, r'\}}(\Sigma, \Sigma')$. We have to show that $mng_{\mathcal{M}}(r) = mng_{\mathcal{M}}(r')$. Define the homomorphisms $g, g' : \mathcal{B} \longrightarrow mng_{\mathcal{M}}[\mathcal{F}^{P \cup Q \cup Q' \cup \{r, r'\}}]$ as in the proof of Claim 4.3. Following that proof, we see that $g \circ id_P = g' \circ id_P$. By (8), then $g(r) = g'(r)$. The definitions of g, g' imply that then $mng_{\mathcal{M}}(r) = mng_{\mathcal{M}}(r')$. ■

Since **L** has ProjBP, it follows that $\{r\}$ has some explicit definition $\varphi_r \in F^P$. As the proof of Claim 4.4 shows, this implies that $r = \widetilde{id}_B(\varphi_r)$, hence $r \in Rg(\widetilde{id}_B \lceil F^P) = Rg(id_P) = P$. Quod non. This contradicts our assumption that id_P was not surjective, and leads to the conclusion that on the contrary id_P (and hence f) is surjective.
Q.E.D. (ProjBP ⇒ StrongES)
Q.E.D. (Theorem 5)

Let *singleton projective Beth* and *infinite projective Beth* denote the obvious variants w.r.t. the cardinality of R. The above proof shows that strongES implies the infinite projective Beth property defined in Definition 6.1, but is already implied by the singleton projective Beth (cf. Claim 6.4). This observation yields the following corollary.

Corollary 6.5 *Let* **L** *be a semantically algebraizable general logic. Then*

L *has infinite projective Beth* ⇔ **L** *has singleton projective Beth.*

Further work The results in this paper apply to semantically algebraizable general logics. Theorem 3 and Theorem 4 show that in various extensions of this class of logics the link between the Beth property and surjectiveness of epis of the corresponding class of algebras no longer holds. However, this does not yet mean that outside the boundaries of algebraizable logics the Beth property can not properly be studied via surjectiveness of epimorphisms. The keyword here is matrix-semantics. In e.g. [BP92] this notion is introduced as an algebra-like semantics that proves to be adequate for a much larger class of logics than the algebraizable ones. [CP96] is an extensive study of interpolation properties in this framework. Future work, on which some progress has been made by W. Blok and the present author, is to investigate the Beth property in this setting.

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