Logic with Probabilistic Operators

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1 Introduction

A natural way of generalizing standard modal logic is to introduce quantitative modalities. For example, while the language of ordinary epistemic logic contains only operators B_i ("an agent i believes that ..."), one can add quantitative operators corresponding to *degrees* of belief.

In this paper we study languages which contain operators P_r: "the probability of ... is equal to r", where r is a rational number. Depending on the semantics we choose, the logic of such operators is a generalization of the modal logic KD (KD45). Namely, P_1 is a KD-modality (KD45-modality).

In Abadi & Halpern (1989), Bacchus (1991), Fagin, Halpern & Megiddo (1988), Halpern (1990), Voorbraak (1993) some richer languages are investigated. Corresponding logics are complete when probabilistic modalities are added to a propositional language and are proved to be non-axiomatizable in the first-order case. The latter result (cf. Abadi & Halpern (1989) or Halpern (1990)) suggests considering a less expressive language.

The present investigation is closely related to the work of Fattorosi-Barnaba & Amati (1987) and van der Hoek (1992). They studied a logic with operators $P_r^{>}$: "the probability of ... is greater than r", r is any real number from the interval [0,1]. A complete axiomatization is obtained only with respect to the class of models where probabilities which are assigned to formulas all come from a finite set $F \subset [0,1]$. For the general case a complete axiomatization is not found. One may expect that it cannot be obtained by standard means, i.e. within a language with finite formulas and standard derivation rules (cf. Keisler (1985): the generalized quantifiers corresponding to modalities in question ("the probability of the set of elements satisfying ... is greater than r") are axiomatized in an infinitary language).

A weaker logic containing only P_r-operators also resists attempts to axiomatize it by standard means. In this paper it is axiomatized by introducing an infinitary inference rule. The rest of the paper is organized as follows. In section 2 we give a complete axiomatization of a first-order logic with probabilistic operators and prove some other completeness results. In section 3 we study propositional probabilistic logics, and in section 4 discuss prospects of finding a standard axiomatization (without an infinitary rule).

2 Some Completeness Results

2.1 Parametrized models

Consider a first-order language extended by a set of unary modal operators $\{P_r$: r is a rational number $\}$. The restriction to rational-valued probabilities which is made in this paper, is motivated mainly by technical reasons. It allows to keep the language countable (which is useful in the Henkin construction below) and to establish an easy correspondence between qualitative probabilistic modalities and our operators in Section 3. But one can also argue that in "real-life" situations degrees of probability are always rational.

The definition of a well-formed formula is standard; if φ is a w.f.f., so is $P_r\varphi$. Φ denotes the set of all w.f.f.

The first logic which we consider is called PPL: Parametrized Probabilistic Logic. In a PPL-model every possible world has its own probability measure; therefore the truth of a modal formula depends on a possible world. We shall see that in PPL P_1 is a KD-modality.

Definition 1 *A model for PPL is $M = < W, D, I, \mu_w >$, where W is a non-empty set of possible worlds; D is a domain (the same for all worlds); I is an interpretation function (for every $w \in W$ and for every n-place predicate letter P, $I(w, P) \subseteq D^n$); μ_w - for every $w \in W$ - is a function defined on the set of all definable subsets of W, Θ (i.e. $\Theta = \{Q \subseteq W : \exists\varphi, \alpha(Q = \{w : M, w \models^\alpha \varphi\})\}$), such that:*

1. *For all $Q \in \Theta$, $\mu_w(Q) \in [0, 1] \cap \mathbf{Q}$*
2. *$\mu_w(W) = 1$;*
3. *(Finite additivity): $\mu_w(Q_1 \cup Q_2) = \mu_w(Q_1) + \mu_w(Q_2)$, where $Q_1 \cap Q_2 = \emptyset$.*

$$M, w \models^\alpha P_r\varphi \Leftrightarrow Q^\alpha_\varphi \in \Theta \wedge \mu_w(Q^\alpha_\varphi) = r,$$

where $Q^\alpha_\varphi = \{w' : M, w' \models^\alpha \varphi\}$. Since all definable subsets are in Θ, the first condition can be omitted.

The definitions of satisfiability etc. are standard.

Definition 2 *Weak PPL is the smallest set of formulas closed with respect to the following schemata of axioms and derivation rules:*

A0. *First-order logic;*

A1. $P_1(\varphi \vee \neg\varphi)$;

A2. $P_r(\varphi \wedge \psi) \wedge P_s(\varphi \wedge \neg\psi) \rightarrow P_{r+s}\varphi$;

A3. $\neg P_r\varphi$, $r \notin [0, 1]$;

A4. $P_r\varphi \rightarrow \neg P_s\varphi$, $r \neq s$;

MP. $\vdash \varphi, \vdash \varphi \rightarrow \psi \Rightarrow \vdash \psi$;

R1. $\vdash \varphi \leftrightarrow \psi \Rightarrow \vdash P_r\varphi \leftrightarrow P_r\psi$

Claim 1 *Weak PPL is not complete with respect to the class of PPL-models.*

Proof can be found in section 4. □

Claim 2 *Weak PPL is complete with respect to the class of models where μ_w is defined on an arbitrary set $\Theta_w \subseteq \mathcal{P}(W)$ such that $W \in \Theta_w$ and Θ_w is closed with respect to disjoint union.*

Proofsketch. Consistency is obvious. For completeness, construct by Henkin argument all maximally consistent in the sense of weak PPL sets of formulas. Those will be worlds. The domain (the set of Henkin witnesses) is the same in all worlds. The interpretation function is defined as usual: $< d_1, \ldots, d_n > \in I(w, P) \Leftrightarrow P(d_1, \ldots, d_n >$. For Θ_w, put $\{w' : \varphi \in w'\} \in \Theta_w$ iff there exists an operator P_r such that $P_r\varphi \in w$. It is easy to see that Θ_w contains W (since $P_1(\varphi \vee \neg\varphi) \in w$) and is closed with respect to disjoint unions of definable subsets (due to A2). Finally,

$$\mu_w(\{w' : \varphi \in w'\}) = r \Leftrightarrow P_r\varphi \in w.$$

A4 guarantees that μ_w is a function. The truth lemma does not now present any difficulty:

$$\varphi \in w \Leftrightarrow w \models \varphi.$$

□

However, this result is of little interest: Θ_w is not a Boolean algebra, and we do not have any reason to call μ_w probability function.

The last intermediate claim gives a pleasant exception and demonstartes that PPL is closely related to KD:

Claim 3 *If the language contains only one operator P_1, the following logic is complete with respect to the class of models where Θ_w is an arbitrary Boolean algebra of subsets of W:*

0 *classical logic*

K $P_1(\varphi \to \psi) \to (P_1\varphi \to P_1\psi)$;

D $P_1\varphi \to \neg P_1\neg\varphi$;

R'1 $\vdash \varphi \Rightarrow \vdash P_1\varphi$.

Proof. Consistency is again easy to check. The proof of completeness is analogous to the proof of the claim above. Denote $P_1\neg\varphi$ as $P_0\varphi$. The following theorems are derivable:

$P_1\varphi \wedge P_1\psi \to P_1(\varphi \wedge \psi)$;

$P_0\varphi \to P_0(\varphi \wedge \psi)$;

$P_1\varphi \to P_1(\varphi \vee \psi)$;

$P_0\varphi \wedge P_0\psi \to P_0(\varphi \vee \psi)$.

Construct the model as in the previous proof. The derivabilities above guarantee that Θ_w is a Boolean algebra. □

To obtain the general completeness result we strengthen weak PPL as follows.

Definition 3 *The logic PPL is the smallest set of formulas closed with respect to weak PPL and*

R2. $\{\Sigma \vdash \neg P_r\varphi : r \in [0,1] \cap \mathbf{Q}\} \Rightarrow \Sigma \vdash \bot$.

The standard notion of inference should be now extended to allow infinite premises.

The structure of inference can be represented as follows. An inference with one application of R2 contains countably many branches of finite length:

$$\begin{array}{c} \underbrace{\vdots\vdots\cdots\vdots}_{\mathbf{R2}} \quad \text{for every } r \in [0,1] \cap \mathbf{Q} \\ \vdots \end{array}$$

Such inference can be a part of another inference where R2 is used more than once (or even infinitely many times):

$$\begin{array}{c} \underbrace{\begin{array}{cccc} \underbrace{\vdots\vdots\cdots\vdots}_{\mathbf{R2}} & \underbrace{\vdots\vdots\cdots\vdots}_{\mathbf{R2}} & \cdots & \underbrace{\vdots\vdots\cdots\vdots}_{\mathbf{R2}} \\ \vdots & \vdots & & \vdots \end{array}}_{\mathbf{R2}} \\ \vdots \end{array}$$

We shall write $\vdash \varphi$ and $\Gamma \vdash \varphi$ for φ is provable in PPL and φ is derivable from the set of premises Γ in PPL.

The consistency of this system is obvious. The rule $R2$ preserves validity since for every definable subset the probability is defined.

We shall construct a canonical model for PPL, but to do it we must change the standard procedure of constructing a maximally consistent set (basically because in our logic $\Sigma \vdash \bot$ does not imply that there is a finite subset of Σ which is contradictory; a contradiction can be derived by $R2$ if Σ contains, for some φ, $\neg P_r\varphi$ for every rational r in [0,1]).

This shows that we should modify the procedure a little bit.

Lemma 1 *Every consistent (with respect to PPL) set of sentences Δ can be extended to a maximally consistent set Σ with the Henkin property and the property*
(*) *for every sentence φ there exists an $r \in [0,1] \cap \mathbf{Q}$ such that $P_r\varphi \in \Sigma$.*

Proof Take two orderings: an ordinary ordering of Φ and an ordering of all sentences of the form $P_r\varphi$ in the new alphabet (expanded with a countable set of new constants - Henkin witnesses). Let us call those orderings I and II, respectively. The construction goes as follows:

$\Sigma_1 = \Delta$,

$\Sigma_{2n} = \Sigma_{2n-1} \cup \{\varphi_n\}$, where φ_n is the nth formula from the ordering I, if $\Sigma_{2n-1} \cup \{\varphi_n\}$ is consistent, and $\Sigma_{2n} = \Sigma_{2n-1}$ otherwise; if Σ_{2n-1} is consistent with φ_n, and φ_n is of the form $\exists x\psi(x)$, $\Sigma_{2n} = \Sigma_{2n-1} \cup \{\varphi_n\} \cup \{\psi(d)\}$, where d is a new constant.

$\Sigma_{2n+1} = \Sigma_{2n} \cup \{P_r\varphi_n\}$ where $P_r\varphi_n$ is the first formula in the ordering II consistent with Σ_{2n}. There is always such a formula (otherwise Σ_{2n} is inconsistent due to R2). If on the previous step $\psi(d)$ was added, analogous argument shows that for some s $\Sigma_{2n} \cup \{P_r\varphi_n\} \cup \{P_s\psi(d)\}$ can be consistently added.

That is, at every even step we perform the usual procedure of constructing a Henkin model, and at every odd step we add a probabilistic statement, ensuring that every sentence in Σ_{2n} has a probability.

Finally, we take a countable union Σ of Σ_k. A contradiction can not be derivable from Σ by R2, since for every φ in $\cup_k\Sigma_k$, for some r $P_r\varphi \in \cup_k\Sigma_k$. Therefore a contradiction can be derivable only from a finite set of premises, and this implies that one of Σ_k was already inconsistent. Thus Σ is consistent. Due to our procedure, (*) holds. $\square$

Theorem 1 *PPL is complete:* $\Gamma \models \varphi \Rightarrow \Gamma \vdash \varphi$

Proof. As usual, we show that every consistent formula is satisfiable (has a model). It is enough to prove this for sentences only, since a formula is satisfiable if and only if its existential closure is.

The canonical model is defined as usual. The set of possible worlds consists of all maximally consistent sets with the Henkin property and (*), the universe is the set of Henkin witnesses. μ_Σ is defined for all sets $\{\Sigma' : \varphi \in \Sigma'\}$ as follows:

$$\mu_\Sigma(\{\Sigma' : \varphi \in \Sigma'\}) = r \Leftrightarrow P_r\varphi \in \Sigma$$

Since every Σ has the property (*), for every sentence φ in the extended language $P_r\varphi \in \Sigma$ for some r. Every object in the domain has a name; therefore, all definable subsets are sententially definable. This means that the probability is defined for every definable subset.

Comment. Note that this is the only point where we need (*) and therefore the only point where we need R2.

A3 guarantees that μ_Σ takes values in $[0,1] \cap \mathbf{Q}$, and A4 and R1 - that it is a function. The property $\mu_\Sigma(W) = 1$ follows from A1, additivity - from A2.

The interpretation function is constructed as usual, to provide (together with the definition of μ) that for every φ,

$$\varphi \in \Sigma \Leftrightarrow \Sigma \models \varphi.$$

Therefore we have: if $\Gamma \cup \{\neg\varphi\}$ is consistent, then $\Gamma \cup \{\neg\varphi\}$ has a model.
$\square$

The same method can be used to obtain more completeness results. For example, can enrich the language with a binary modality $\perp$, which means "is independent from". The motivation for introducing this modality was given in Pearl (1988). Pearl wrote that independence of two events is a basic property which can be known without knowing precise numerical values of probabilities of two events and of their conjunction. However, when the language does contain numerical modalities, $\perp$ does not add much (and encreases complexity, as it will be easy to see in the case of non-parametrized probabilistic logic). Consider the following two axioms:

A5 $P_r\varphi \wedge P_r\psi \wedge P_{rs}(\varphi \wedge \psi) \rightarrow \varphi \perp \psi$

A6 $P_r\varphi \wedge P_r\psi \wedge \varphi \perp \psi \rightarrow P_{rs}(\varphi \wedge \psi)$

The truth definition for $\perp$ is

$$M, w \models^{\alpha} \varphi \perp \psi \Leftrightarrow \mu_w(Q^{\alpha}_{\varphi})\mu_w(Q^{\alpha}_{\psi}) = \mu_w(Q^{\alpha}_{\varphi\wedge\psi})$$

Our canonical construction guarantees that the following theorem holds:

Theorem 2 *PPL + A5, A6 is complete for the intended interpretation of $\perp$.*

2.2 Non-parametrized models

If possible worlds are understood as particular outcomes of an experiment, and probability is understood as a relative frequency, it does not make sense to associate a measure with every world: we need only one measure in the model. This implies that every probabilistic statement is either true in all possible worlds, or false in all possible worlds. That is why PL (non-parametrized probabilistic logic) which differs from PPL only by the fact that μ does not depend on a possible world, is axiomatized by adding to PPL the following axioms:

A7. $P_r(\varphi \wedge P_s\psi) \equiv (P_r\varphi \wedge P_s\psi) \vee (\neg P_s\psi \wedge P_r\perp)$

A8. $P_r(\varphi \wedge \neg P_s\psi) \equiv (P_r\varphi \wedge \neg P_s\psi) \vee (P_s\psi \wedge P_r\perp)$

Further on, we shall study the propositional fragment of PL (so, I shall use "PL" to refer to the propositional fragment of PL).

Theorem 3 *(Normal form theorem)* [1] *Every formula of PL has an equivalent in a* normal form, *that is, a disjunction of conjunctions of propositional formulas and modal formulas without nestings of probability operators.*

Proof Consider any formula of the form $(\neg)P_r\theta$, where θ contains nested operators. Let us show that the number of nestings of the operators in θ can be decreased by 1. Let $P_s\psi$ be a subformula of θ with the greatest number of nestings. If θ is of the form $\varphi \wedge (\neg)P_s\psi$ (where φ can be a tautology), then A7 (A8) can be applied; for example,

$$P_r(\varphi \wedge P_s\psi) \equiv (P_r\varphi \wedge P_s\psi) \vee (\neg P_s\psi \wedge P_r\perp)$$

If θ is of the form $\varphi \vee (\neg)P_s\psi$, then $\neg\theta$ is of the right form and all we need in order to be able to apply A7 (A8) is to derive

$$P_r\theta \equiv P_{1-r}\neg\theta,$$

which follows from A2 and R2. (If we add

A2(b). $P_r(\varphi \wedge \psi) \wedge P_s\varphi \rightarrow P_{s-r}(\varphi \wedge \neg\psi)$

to the system, R2 is not needed for the proof of the normal form theorem).

A simple induction on the number of nestings of modal operators proves that in any formula of PL the nestings can be eliminated. By propositional reasoning, every formula can be rewritten in a normal form. $\square$

Next, we shall see that the satisfiability problem for PL can be reduced to the problem of solving a system of linear equations.

[1] The formulation of the normal form theorem is due to F.Voorbraak.

Consider an arbitrary formula φ. Due to the normal form theorem we can assume that φ is in a normal form. A disjunction is satisfiable iff at least one of the disjuncts is. That is why without loss of generality we can restrict attention to a formula φ of the form

$$\psi_0 \wedge P_{r_1}\psi_1 \wedge \ldots \wedge P_{r_n}\psi_n \wedge \neg P_{r_{n+1}}\psi_{n+1} \wedge \ldots \wedge \neg P_{r_m}\psi_m,$$

$0 \le n \le m$, where $\psi_0, \ldots, \psi_m$ do not contain probability operators. Consider all state descriptions (all possible conjunctions of propositional variables and their negations) built from the propositional variables in $\psi_0, \ldots, \psi_m$; let us denote them $\alpha_1, \ldots, \alpha_k$.

Each ψ_i is provably equivalent to some disjunction of α_j; hence by $R1$

$$\vdash P_{r_1}\psi_1 \equiv P_{r_1}(\alpha_{11} \vee \ldots \vee \alpha_{1u})$$

$$\vdots$$

$$\vdash P_{r_n}\psi_n \equiv P_{r_n}(\alpha_{n1} \vee \ldots \vee \alpha_{nv})$$

$$\vdash \neg P_{r_{n+1}}\psi_{n+1} \equiv \neg P_{r_{n+1}}(\alpha_{n+1,1} \vee \ldots \vee \alpha_{n+1,w})$$

$$\vdots$$

$$\vdash \neg P_{r_m}\psi_m \equiv \neg P_{r_m}(\alpha_{m1} \vee \ldots \vee \alpha_{mz})$$

Intuitively, α_i will correspond to the possible world w_i. Since the α_i (and w_i) are mutually disjoint, φ is satisfiable iff there exists a model where

A

$$\mu(w_1) + \cdots + \mu(w_k) = 1$$

$$\mu(w_{11}) + \cdots + \mu(w_{1u}) = r_1$$

$$\vdots$$

$$\mu(w_{n1}) + \cdots + \mu(w_{nv}) = r_n$$

B

$$\mu(w_{n+1,1}) + \cdots + \mu(w_{n+1,w}) \neq r_{n+1}$$

$$\vdots$$

$$\mu(w_{m1}) + \cdots + \mu(w_{mz}) \neq r_m$$

$$\mu(w_i) \ge 0$$

If this system of equations has a solution, the model is constructed: μ is defined by the system of equations, ρ_{w_i} corresponds to α_i, ψ_0 is satisfiable at least in one world (otherwise $\vdash \neg\psi_0$ and therefore $\vdash \neg\varphi$). And conversely, if φ has a model, then this system must have a solution. Thus,

Theorem 4 *Propositional fragment of PL is decidable.*

Coming back to the independence operator, one can easily see that its adding to the logic increases complexity: to see whether a formula has a model one has to solve a system of equations which involves products. (For example,

$$(\alpha_1 \vee \alpha_2) \perp (\alpha_1 \vee \alpha_3)$$

is satisfiable if

$$(x_1 + x_2)(x_1 + x_3) = x_1)$$

Before proving the completeness theorem for PL, we need the following lemma:

Lemma 2 *The following rule, for every n, is derivable in PL (and also in PPL):*

$$\frac{\{\Sigma \vdash \neg P_{r_1}\varphi_1 \vee \ldots \vee \neg P_{r_n}\varphi_n : \forall r_1, \ldots, r_n \in [0,1] \cap \mathbf{Q}\}}{\Sigma \vdash \perp}$$

Proof

1. $\Sigma \vdash \neg P_{r_1}\varphi_1 \vee \ldots \vee \neg P_{r_n}\varphi_n$ for every $r_1, \ldots, r_n \in [0,1] \cap \mathbf{Q}$
2. $\Sigma, P_{r_1}\varphi_1, \ldots, P_{r_{n-1}}\varphi_{n-1} \vdash \neg P_{r_n}\varphi_n$ for every $r_1, \ldots, r_n \in [0,1] \cap \mathbf{Q}$
3. $\Sigma, P_{r_1}\varphi_1, \ldots, P_{r_{n-1}}\varphi_{n-1} \vdash \perp$ for every $r_1, \ldots, r_{n-1} \in [0,1] \cap \mathbf{Q}$ (R2)
4. $\Sigma, P_{r_1}\varphi_1, \ldots, P_{r_{n-2}}\varphi_{n-2} \vdash \neg P_{r_{n-1}}\varphi_{n-1}$ for every $r_1, \ldots, r_{n-1} \in [0,1] \cap \mathbf{Q}$

...

$2n$. $\Sigma \vdash \neg P_{r_1}\varphi_1$ for every $r_1 \in [0,1] \cap \mathbf{Q}$

$2n+1$. $\Sigma \vdash \perp$ R2

□

Finally, we can prove

Theorem 5 *Propositional fragment of PL is complete.*

Proof It suffices to prove that if the system of equations corresponding to φ has no solution, $\vdash \neg\varphi$.

Our system has k unknowns (probabilities of $\alpha_1, \ldots, \alpha_k$).

For all possible $s_1, \ldots, s_k$

$$P_{s_1}\alpha_1 \wedge \ldots \wedge P_{s_k}\alpha_k \wedge \varphi \to \perp \tag{1}$$

(just apply A2 to obtain the values of $\alpha_{11} \vee \ldots \vee \alpha_{1u}$, ..., $\alpha_{n1} \vee \ldots \vee \alpha_{nv}$, $\alpha_{n+1,1} \vee \ldots \vee \alpha_{n+1,w}$, ..., $\alpha_{m1} \vee \ldots \vee \alpha_{mz}$. Since the system has no solution, at least one of those values (sums of the corresponding s_i's) shall be different from one of the $r_1, \ldots, r_n$, respectively, or equal to one of the $r_{n+1}, \ldots, r_m$. In the first case we obtain a contradiction by A4, in the second case by propositional reasoning.

(1) implies

$$\varphi \to \neg P_{s_1}\alpha_1 \vee \ldots \vee \neg P_{s_k}\alpha_k \tag{2}$$

which, by Lemma 2 , implies

$$\vdash \neg\varphi$$

□

As one can see from the proof above, allowing derivations with *only one* application of R2 is enough to have completeness of PL.

3 Decidability

We have already seen that the propositional fragment of PL is decidable. The same holds for the propositional fragment of PPL:

Theorem 6 *Propositional fragment of PPL is decidable.*

Proof Satisfiability problem for the propositional fragment of PPL is reducible to satisfiability problem for a decidable system QM (qualitative modalities) as stated, for example, in van der Hoek (1992).

The models for QM are the same as the models of PPL, but instead of our operators P_r, for every rational r, the language of QM contains a binary modality $\succeq$ with $M, w \models \varphi \succeq \psi \Leftrightarrow \mu_w(Q_\varphi) \geq \mu_w(Q_\psi)$.

Before defining the translation from PPL to QM, add a countable set of new propositional variables to the language of QM:

$$\{q_{11}\} \cup \{q_{21}, q_{22}\} \cup \{q_{31}, q_{32}, q_{33}\} \cup \ldots \cup \{q_{n1}, q_{n2}, \ldots, q_{nn}\} \cup \ldots$$

and define a formula Q_n, for every natural number $n \geq 2$, as

$$Q_n =_{df} \bigwedge_{1 \geq i,j \geq n, i \neq j} \neg(q_{ni} \wedge q_{nj}) \succeq \top \wedge \bigwedge_{1 \geq i,j \geq n} (q_{ni} \simeq q_{nj}) \wedge (q_{n1} \vee \ldots \vee q_{nn}) \succeq \top$$

(where $\top$ is any propositional tautology, and $\varphi \simeq \psi$ is defined as $\varphi \succeq \psi \wedge \psi \succeq \varphi$).

One can easily check that Q_n is true in a world of some model if and only if the measure associated to this world assigns zero probability to the intersection of propositions corresponding to q_{ni} and q_{nj}, for every such pair with $i \neq j$; that all such propositions are equiprobable; and that their union is equal to 1. It follows immediately that this measure assigns $\frac{1}{n}$ to every such proposition, and $\frac{m}{n}$ to a union of m such propositions. We shall use it to translate formulas with numeric operators.

One more technical detail: so far I assumed that every rational number corresponds to just one operator: that some sort of standard way to represent rational numbers is used, and there is no operator, say, $P_{\frac{2}{4}}$. To be very precise in what follows, and at the same time to make the proof of the theorem below easier, an intermediate translation can be used: from the original language to the language which contains operators for every fraction $\frac{m}{n}$. It is obvious that this translation preserves satisfiability.

Now, the translation $*$ is defined as follows. Given a formula φ of PPL, translate it in the language with operators for every rational fraction and find an equivalent formula where all operators have common denominator, say, n. Then $(\varphi)^* = Q_n \wedge \varphi'$, $'$ commutes with classical connectives and atomic formulas, and

$$(P_{\frac{m}{n}}\varphi)' = \varphi' \simeq (q_{n1} \vee \ldots \vee q_{nm})$$

Theorem 7 $\models_{PPL} \varphi$ *if and only if* $\models_{QM} \varphi^*$.

Proof

(*if*): Assume there is a model M and a world w which satisfy φ^*. Induction on complexity of φ proves that the same world satisfies φ. (Note that $M, w \models \varphi^*$ means that $M, w \models Q_n$, where n is the smallest common denominator for the operators in φ, and $M, w \models \varphi'$). The only non-trivial case: $\varphi' = \psi' \simeq (q_{n1} \vee \ldots \vee q_{nm})$, that is, $\varphi = P_{\frac{m}{n}}\psi$. Since M, w satisfy Q_n,

$$M, w \models \psi' \simeq (q_{n1} \vee \ldots \vee q_{nm}) \Leftrightarrow \mu_w(\{w' : M, w' \models \psi'\}) = \frac{m}{n}$$

By the inductive hypothesis, $\{w' : M, w' \models \psi'\} = \{w' : M, w' \models \psi\}$. By the truth definition, $M, w \models P_{\frac{m}{n}}\psi$.

Now we have $M, w \models \varphi^* \Rightarrow M, w \models \varphi' \Rightarrow M, w \models \varphi$.

(*only if*): Assume there is a model M and a world w which satisfy φ, and the common denominator in all operators occurring in φ is n. Let M' be as follows: W' consists of n copies of W; for every $Q \subseteq W$, $\mu'_w(Q) = \mu_w(Q)/n$. The valuation is as before: $v \in V'(p) \Leftrightarrow v \in V(p)$, only we have now n copies of each world. Each of q_{ni}'s is assigned its own copy of W (which has probability $\frac{1}{n}$). So, Q_n is satisfied, and $q_{n1} \vee \ldots \vee q_{nm}$ has probability $\frac{m}{n}$. Again, an induction on the complexity of φ shows that φ^* is satisfied in M' and w. Let $\varphi = P_{\frac{m}{n}}\psi$. $M, w \models \varphi$ iff $\mu_w(Q_\psi) = \frac{m}{n}$; that is, by the construction of μ'_w, it assigns to the same set $\frac{m}{nn}$. There are n disjoint copies of Q_ψ is M', so the probability of the whole set of worlds satisfying ψ in M' is $\frac{m}{n}$. By the inductive hypothesis, $Q_\psi = \cup_n Q_{\psi^*}$, so $\mu'_w(Q_{\psi^*}) = \frac{m}{n}$. But this means that $M', w \models Q_n \wedge \psi^* \simeq (q_{n1} \vee \ldots \vee q_{nm})$. $\square$

Corollary 1 *The satisfiability problem for the propositional fragment of PPL is decidable.*

4 Incompleteness

In this section we will see that without $R2$ neither PL nor PPL are complete. First of all, even

A2(b). $P_r(\varphi \wedge \psi) \wedge P_s\varphi \rightarrow P_{s-r}(\varphi \wedge \neg\psi)$

is not derivable; but we shall add it as an axiom. There are still infinitely many valid formulas which are not derivable.

Consider the following example (let $\varphi_1, \varphi_2, \varphi_3$ be pairwise disjoint formulas):

$$P_{r_1}(\varphi_1 \vee \varphi_2) \wedge P_{r_2}(\varphi_1 \vee \varphi_3) \wedge P_{r_3}(\varphi_2 \vee \varphi_3)$$

This corresponds to the following system of equations:

$$x_1 + x_2 = r_1$$

$$x_1 + x_3 = r_2$$

$$x_2 + x_3 = r_3$$

which has a unique solution:

$$x_1 = \frac{r_1 + r_2 - r_3}{2}$$

The following formula (I shall call it S_3) is therefore valid for all r_1, r_2 and r_3 (given that φ_i are disjoint[2]):

$$P_{r_1}(\varphi_1 \vee \varphi_2) \wedge P_{r_2}(\varphi_1 \vee \varphi_3) \wedge P_{r_3}(\varphi_2 \vee \varphi_3) \rightarrow P_r\varphi_1,$$

where $r = \frac{r_1+r_2-r_3}{2}$. S_3 is not derivable without $R2$. To show this, it suffices to slightly change the semantics as follows: instead of having all definable subsets of W measurable, take the following set $\Delta \subseteq \mathcal{P}(W)$ to serve as a domain of μ_w:

i $W \in \Delta$

[2]I could have written instead of φ_1, φ_2 and φ_3 - $\varphi_1 \wedge \neg\varphi_2 \wedge \neg\varphi_3$, $\varphi_2 \wedge \neg\varphi_1 \wedge \neg\varphi_3$, $\varphi_3 \wedge \neg\varphi_2 \wedge \neg\varphi_1$

ii $Q_1, Q_2 \in \Delta, Q_1 \cap Q_2 = \emptyset \Rightarrow Q_1 \cup Q_2 \in \Delta$;

iii $Q_1 \in \Delta, Q_1 \cap Q_2 \in \Delta \Rightarrow Q_1 - Q_2 \in \Delta$.

Now $M, w \models P_r\varphi \Leftrightarrow Q_\varphi \in \Delta$ & $\mu_w(Q_\varphi) = r$.

It is easy to check that our axioms are sound. S_3 is not. Let for example $Q \in \Delta$ if and only if the cardinality of Q is even, and μ assigns each set its cardinality. If $|W|$ is even, then the conditions (i)-(iii) hold and Ax is true. But S_3 is not: for example, let $|Q_1| = 1$, $|Q_2| = 3$ and $|Q_3| = 5$, and these sets are pairwise disjoint. Then any union of the two of them has an even cardinality and thus is measurable, but neither of the Q_i's is; the antecedent of S_3 is true and the consequent is false.

This example can be generalized: for arbitrary n disjoint formulas φ_i, if the probability of every union of $n-1$ formulas is defined, then the probability of each φ_i is defined. Let φ_i not occur in the kth disjunction (every unkown variable must occur in precisely $n-1$ disjunctions). Then S_n states

$$\bigwedge_{j=1}^{n} P_{r_j}(\varphi_{j1} \vee \ldots \vee \varphi_{j(n-1)}) \rightarrow P_r\varphi_i,$$

where

$$r = \frac{\Sigma_{j \neq k}^{n} r_j - (n-2)r_k}{n-1}$$

S_n is valid in our initial semantics. One can show that for every n S_n is not derivable from $Ax + S_3, \ldots, S_{n-1}$ (Ax is PL (PPL) without $R2$).

S_n becomes valid if we add to the definition of Δ the following condition:

S(n) Let $Q_1, \ldots, Q_n \subseteq W$, $Q_i \cap Q_j = \emptyset$ for $i \neq j$, $1 \leq i, j \leq n$. Let also for every i, $1 \leq i \leq n$, the set $\cup_{j \neq i} Q_j$ is in Δ. Then every Q_i is in Δ.

We shall construct a model, for every n, where S(n) is false and S(k), for every $k < n$, is true. Let $W = \{0, \ldots, n\}$ and Δ include W, $\emptyset$, every set with two elements which contains n, and the complements of these sets. (For $n = 4$ $W = \{0, 1, 2, 3, 4\}$ and

$$\Delta = \{W, \emptyset, \{0,1,2\}, \{0,1,3\}, \{0,2,3\}, \{1,2,3\}, \{0,4\}, \{1,4\}, \{2,4\}, \{3,4\}\})$$

Δ satisfies the conditions (i)-(iii). Take $Q_1 = \{0\}, \ldots, Q_n = \{n-1\}$. These sets are pairwise disjoint, and every union of $n-1$ of them is in Δ. But they themselves are not in Δ. So, S(n) is violated. It can be shown that all S_k, $k < n$, are trivialy true in this model.

Let for some $k < n$ S(k) is false. This would mean that there are disjoint $R_1, \ldots, R_k$ such that for every i, $1 \leq i \leq k$ $\cup_{j \neq i} R_j \in \Delta$, but neither of R_i is in Δ. Each such $\cup_{j \neq i} R_j$ can be either W, or $\emptyset$, or $\{l, n\}$, or its complement.

1. $\cup_{j \neq i} R_j = \emptyset$: every R_j (except possibly for R_i) is empty. It is clear that S(k) cannot be refuted this way.

2. $\cup_{j \neq i} R_j = W$, that is, R_i is empty. We shall not consider the case when one of the R_i is empty, because then it becomes reducible to S(k-1).

3. Assume that for some i $\cup_{j \neq i} R_j = \{l, n\}$. $k = 2$ is a trivial case: each union of $k-1$ sets is just one of those sets, so if the unions belong to Δ, so do the sets themselves. Let $k > 2$. Then $\{l, n\}$ is a union of more then one set. We assume that neither of R_i

is empty. Therefore, $\{l, m\}$ can be only the union of $\{l\}$ and $\{n\}$, and k can only be equal to 3. Is there any R_3 such that both $R_3 \cup \{l\}$ and $R_3 \cup \{n\}$ are in Δ, and R_3 is disjoint with both $\{l\}$ and $\{n\}$? An easy check shows that there is none. So, none of the unions of $k-1$ sets are of the form $\{l, n\}$.

4. Assume that for some i $\cup_{j \neq i} R_j$ is one of the sets with $n-1$ elements. Since $k < n$, and we must represent this set as a union of $k-1$ sets, it is clear that one of the R_i must have more than one element. From (2) and (3) we know that n belongs to neither of R_i. Therefore the union of R_j must be equal to $\{0, \ldots, n-1\}$. Now, if R_i has more than one element, $\cup_{j \neq i} R_j$ has less than $n-2$ elements from the set $\{0, \ldots, n-1\}$. No such set, except for the empty one, belongs to Δ. So, again this case is impossible.

□

We just have proved that S_n is independent form Ax and all S_m with $m < n$. A natural question to ask is whether $Ax + \{S_n : n \in \mathbf{N}\}$ is complete. The answer is no: there are more valid sentences (saying that some system of equations has a single solution) which are not derivable even in the presence of S_n. Consider for example the following matrix:[3]

$$\begin{matrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

The corresponding system of equations has a single solution; for disjoint $\varphi_1, ..., \varphi_4$

$$P_{r_1}(\varphi_2 \vee \varphi_4) \wedge P_{r_2}(\varphi_3 \vee \varphi_4) \wedge P_{r_3}(\varphi_1 \vee \varphi_4) \wedge P_{r_4}(\varphi_1 \vee \varphi_2 \vee \varphi_3) \rightarrow P_r \varphi_1$$

where r is a function of $r_1, \ldots, r_4$, must be derivable. However, in a structure, where **i, ii, iii** and $\mathbf{S(n)}$ are valid, and for some disjoint sets $Q_1, \ldots, Q_4$ sets $Q_2 \cup Q_4$, $Q_3 \cup Q_4$, $Q_1 \cup Q_4$ and $Q_1 \cup Q_2 \cup Q_3$ are measurable, Q_1 does not have to be measurable. This shows that the implication above is not derivable from Ax together with all schemata S_n.

Completeness problem for PL (PPL) without R2 is reducible therefore to the problem whether there are very simple (expressible in the language of $(P)PL$) conditions on 0-1 matrices guaranteeing that a corresponding system of equations has a single solution. (Which is not very likely to be the fact).

5 Conclusions

We have seen that first-order PPL (and first-order PL also: the proof is analogous) can be given complete axiomatization, and the same holds for PPL with the independence operator $\perp$ added. Another positive (although not very surprising) result is that propositional fragments of both PL and PPL are decidable.

However, completeness was achieved by adding the infinitary rule R2. A natural question to ask is whether one can find a better axiomatization.

The last incompleteness result does not say much more than that dropping R2 we obtain an incomplete system, altough it suggests that $(P)PL$ without R2 cannot be axiomatized by a finite number of schemata. The question whether $(P)PL$ can be axiomatized without an infinitary rule is still open.

[3] This example is due to C. de Vreugd.

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