Introducing Some Classical Elements of Modal Logic to the Propositional Logics of Qualitative Probabilities

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Abstract

This paper presents the construction of canonical models for a number of logics of qualitative probabilities by introducing a conservative infinitary rule into their axiomatizations. These logics include the minimal logic of qualitative probabilities [Seg71, Gar75] and a new multioperator propositional logic, that contains qualitative probabilistic analogons of operations essential to propositional dynamic logic. These operations capture some basic laws for stochastic processes into this new logic's semantics and are given a complete axiomatization.

The paper also presents qualitative probabilistic versions of the modal logic techniques of unravelling and filtration.

All the constructs and arguments in this paper are suitable to be combined with their counterparts for other propositional, e.g. modal logics, and can possibly be incorporated in proofs of completeness and decidability of logics containing both qualitative probabilistic and other propositional operators for application purposes.

Keywords: probabilistic logic, propositional logic, modal logic techniques, dynamic logic, stochastic processes.

1 Introduction

In this paper we develop versions of the canonical construction, of the filtration technique, of the unravelling technique and of some dynamic operations for the propositional logics of qualitative probabilities.

Under this name we collect the propositional logics that contain at least one binary operator $\leq$, $\varphi \leq \psi$ being read as 'ψ at least as probable as φ (from the standpoint of a certain possible world)'.

Two kinds of models have been studied for the minimal logic of qualitative probabilities with a single $\leq$-operator: probabilistic models [Seg71, Gar75], and Kripke models with finite yet nonempty sets of successors of every possible world (**FSKD**-models) [Hoe92].

A *probabilistic model* is introduced as a set of *possible worlds* W and a boolean algebra $B(W)$ of subsets of W with a family of probabilistic measures μ_w, one associated with every

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possible world $w \in W$ that are defined on the boolean algebra $B(W)$. $B(W)$ is also the range of the valuation v of propositional variables. The $\leq$-operator has the following interpretation:

$w \models \varphi \leq \psi$ iff $\mu_w(\{w' : w' \models \varphi\}) \leq \mu_w(\{w' : w' \models \psi\})$,

and for this reason $B(W)$ is additionally required to be closed under subsets of the form $\{w \in W : \mu_w(W') \leq \mu_w(W'')\}$, where $W', W'' \in B(W)$.

In the case of **FSKD**-models the interpretation of $\leq$ is based upon comparison of the *numbers* of successors of the current world that satisfy each operand:

$w \models \varphi \leq \psi$ iff $|\{w' \in R(w) : w' \models \varphi\}| \leq |\{w' \in R(w) : w' \models \psi\}|$.

Both these kinds of models satisfy the schema **D**: $\Box\varphi \Rightarrow \Diamond\varphi$ for $\Box$ being defined as follows: $\Box\varphi \rightleftharpoons \top \leq \varphi$[1].

Here is an axiomatic system for the minimal logic (unabbreviated from [Hoe92]):

$(A0)$ *all classical propositional tautologies*

$(A1)$ $\top \leq (\varphi \Leftrightarrow \varphi') \wedge \top \leq (\psi \Leftrightarrow \psi') \Rightarrow (\varphi \leq \psi \Rightarrow \varphi' \leq \psi')$

$(A2)$ $\varphi \leq \psi \vee \psi \leq \varphi$

$(A3)$ $\perp \leq \varphi$

$(A4)$ $\neg(\top \leq \perp)$

(B_m) $\top \leq (\varphi_1, \ldots, \varphi_m \mathsf{E} \psi_1, \ldots, \psi_m) \Rightarrow \left(\bigwedge_{i=1}^{m-1} \psi_i \leq \varphi_i \Rightarrow (\varphi_m \leq \psi_m)\right)$

$$(MP) \quad \frac{\varphi \Rightarrow \psi, \ \varphi}{\psi} \qquad (N) \quad \frac{\varphi}{\top \leq \varphi}$$

where E denotes the boolean expression of *there are exactly as many formulas satisfied from among* φ_i, $i = 1, \ldots, m$, *as from among* $\psi_i, i = 1, \ldots, m$.

In [Seg71] and [Gar75] we find variants of a completeness result for this logic stating that for an arbitraty nontheorem there exists a finite probabilistic model that refutes it.

In [Hoe92] we find a similar result with respect to **FSKD**-models, that is arrived at through a crucial step of 'adjusting' the measures μ_w that appear in the probabilistic refutation models of [Gar75] so that their values become rational, and modelling relations of the kind $\mu_w(\{w' : w' \models \varphi\}) = k/n$, where n is a suitable common denominator of all the values taken by the measures μ_w, by assigning to worlds w the corresponding numbers of successors that satisfy φ. Van der Hoek also did some forerunning of the study of correspondence issues for **FSKD**-frames[2].

We demonstrate the canonical construction and the filtration technique on the example of the minimal logic. Taken together these give us the strong finite model property and decidability of this logic, that we already know from [Gar75] and [Hoe92], and a little more: the canonical model satisfies also logically infinite consistent sets and offers the ease of incorporating other, e.g. modal operators at the expence of mostly technical effort.

We give semantical definitions of our qualitative-probabilistic versions of dynamic operations together with a complete axiomatic system, and a sketch of the canonical construction for the multioperator logic that thus appears. We demonstrate our version of the unravelling technique on a fragment of this logic that contains only the qualitative probabilistic version of composition. The definition of the latter matches the law of composition valid for certain classes of stochastic processes and gives urge to attempts at their description in terms of qualitative probabilities.

[1] We deliberately avoid this abbreviation below, because we expect non-probabilistic $\Diamond$-operators to mix up into some prospective sequel studies.

[2] A latest collection of results of van der Hoek and de Rijke on logics with operators, whose interpretation is based on counting the successors of possible worlds, the most popular one being the logic of graded modalities [FatCar85,87], can be found in [HoeRij93].

2 The Canonical Construction

Our canonical construction is of a probabilistic model and, as we point out below, it meets some natural obstacles, that are closely connected with the fact that the logics of qualitative probabilities are not compact. We propose a possible enhancement of the axiomatic system that helps to avoid the undesirable effects of lack of compactness.

Our reasons for calling what we construct a *canonical* model are that we take the steps met in the canonical constructions for other modal logics, and that our model refutes all the non-theorems of our logic. We carry out the construction to the step at which the obstacles occur and then give the enhancement that helps us overcome it.

For our construction we fix a language of the logic of qualitative probabilities and extend this initial language with a countably infinite set of fresh propositional variables $p_{i,j}$ where $0 \leq i < j$ and $j < \omega$, and consider the countably infinite set of formulas:

$$\Pi_j \rightleftharpoons \Big(\bigwedge_{i_1, i_2 < j} p_{i_1,j} \leq p_{i_2,j} \Big) \wedge \Big(\top \leq \bigvee_{i<j} p_{i,j} \Big) \wedge \Big(\bigwedge_{i_1 < i_2 < j} p_{i_1,j} \wedge p_{i_2,j} \leq \bot \Big)$$

As one can easily deduce, for any model with valuation v and containing a possible world w $w \models \Pi_j$ implies $\mu_w(v(p_{i,j})) = 1/j$. We also introduce the abbreviations:

$$R_{k,j} \rightleftharpoons \bigvee_{i=0}^{k-1} p_{i,j},$$

for which $w \models \Pi_j$ implies $\mu_w(v(R_{k,j})) = k/j$.[3].

We define the set of possible worlds for our model to be the set W of all maximal consistent sets of formulas in our extended language. We define the boolean algebra $B(W)$ of measurable sets over W as the collection of all sets of the form:

$$m_\varphi \ {}_{def}{=} \{w \in W : \varphi \in w\},$$

where φ is a formula in our initial language and put $v(p) \ {}_{def}{=} m_p$ for all propositional variables p. What we have is obviously a boolean algebra, because $m_{\neg\varphi} = W - m_\varphi$ and $m_{\varphi\wedge\psi} = m_\varphi \cap m_\psi$. Now we define the probability measures μ_w that are to be associated with every possible world w as follows:

$$\mu_w(m_\varphi) \ {}_{def}{=} \inf\{k/j : \ \Pi_j \Rightarrow (\varphi \leq R_{k,j}) \in w\}.$$

In order to prove that μ_w are really probabilistic measures we need to establish the following properties:

(1) $(\varphi \wedge \psi) \leq \bot \in w$ implies $\mu_w(m_{\varphi\vee\psi}) = \mu_w(m_\varphi) + \mu_w(m_\psi)$

(2) $m_{\varphi_0} \supseteq m_{\varphi_1} \supseteq \ldots \supseteq m_{\varphi_n} \supseteq \ldots$ and $\bigcap_{n=0}^{\infty} m_{\varphi_n} = \emptyset$ imply $\lim_n \mu_w(m_{\varphi_n}) = 0$.

We leave (1) to the reader as an easy exercise and proceed to establish (2). The inclusion $m_{\varphi_i} \supseteq m_{\varphi_{i+1}}$ implies that $\varphi_{i+1} \Rightarrow \varphi_i \in x$ and that $\bigwedge_{i=0}^{n} \varphi_i \Leftrightarrow \varphi_n \in w$ for all maximal consistent sets w and hence $\vdash \bigwedge_{i=0}^{n} \varphi_i \Leftrightarrow \varphi_n$ for all n. Since the intersection of all the m_{φ_i} is empty, we have $\Phi = \{\varphi_i : i < \omega\} \not\subset w$ for all $w \in W$, i.e. Φ is inconsistent. This implies that $\vdash \neg(\bigwedge \Phi')$ for some finite $\Phi' \subset \Phi$, and hence we have $m_{\varphi_k} = \emptyset$ for all $k \geq \max\{i : \varphi_i \in \Phi'\}$. $\dashv$

It follows from the transitivity of $\leq$, which is an instance of B_3, that $\varphi \leq \psi \in w$ implies $\mu_w(v(\varphi)) \leq \mu_w(v(\psi))$. However the converse does not hold, and here we arrive at the principal obstacle before bringing our construction to an end. In a few more words we have the following picture:

[3]We use v to denote valuations of propositional variables and their standard extensions over compound formulas as well.

Due to $(A2)$ we have either $\varphi \leq \psi \in w$ or $\psi \leq \varphi \in w$ for every $w \in W$, and any formulas φ and ψ. This implies that within every w the $\leq$-operator defines a linear quasi-order on the set of all formulas. By fixing the values of μ_w on $v(R_{i,j})$ in $\mathbf{Q}$ we map this quasi-order onto an extension of the linear order of some subset of $\mathbf{Q}$. There are no premises in our logic to a conclusion that the isometric embedding of $\langle \mathbf{Q}, \leq_Q \rangle$ into $\langle \mathbf{R}, \leq_R \rangle$ can be appropriately extended to handle any of the possible extensions of $\mathbf{Q}$ as a linearly ordered set that this quasi-order can require. For a rather instructive example consider the set of formulas Ξ:

$$\xi_0 \rightleftharpoons \neg(\varphi \leq \bot), \qquad \xi_k \rightleftharpoons \Pi_k \wedge (\varphi \leq R_{1,k}) \text{ for all } k > 0.$$

Obviously Ξ is consistent (any finite part of it is satisfiable) and yet, following our definition, of μ_w we have $\mu_w(m_\varphi) = \mu_w(m_\bot) = 0$ for any maximal consistent $w \supset \Xi$.

This makes it rather difficult (and obviously even impossible in the case of **FSKD**-models) to think of the semantics of consistent sets that are not at least *logically finite*[Seg71][4].

There are two approaches towards resolving this situation, one of them being to extend the axiomatic system so that the *real-valued* probabilistic semantics is preserved, and the other - to provide semantics that would make sense of consistent sets like Ξ, the most extensively studied option being to resort to nonstandard numbers [Cha93]. We consider only the first approach. We add to the list of rules the following infinitary one:

$$(Ded) \qquad \frac{\forall\, j\ \left(\Pi_j \Rightarrow \left(\bigwedge_{i=0}^{j-1} (\psi \leq R_{i,j} \Rightarrow \varphi \leq R_{i,j})\right)\right)}{\varphi \leq \psi}$$

Now we repeat the same steps of construction, taking for our possible worlds the maximal consistent sets from among those that are also closed under the application of (Ded) and giving up the demonstration of (2) that was supplied for the 'naive' canonical construction. [5] The enhanced system now lets us immediately obtain the equivalence $\varphi \leq \psi \in w \leftrightarrow \mu_w(\varphi) \leq \mu_w(\psi)$ for all possible worlds w. Hence for our model holds what is usually the core of *truth lemma*: $\langle W, B(W), \langle \mu_w : w \in W \rangle, v \rangle, w \models \Gamma$ iff $w \supseteq \Gamma$.

In a few more words our model satisfies all the (Ded)-consistent sets of formulas. We shew that sets of formulas that are not (Ded)-consistent cannot be satisfied using real-valued measures only.

As a last remark to this section, let us say that the rule (Ded) is conservative. The simple way to demonstrate this is to pick an arbitrary nontheorem of the minimal logic without (Ded) and notice that the finite model that refutes it after [Gar75] validates (Ded) too.

3 Filtration and the Finite Model Property

Before going to our qualitative-probabilistic version of filtration itself, let us explain in a little more detail why the canonical construction and this filtration together give us more that a direct construction of a finite model, that we find in preceding works. It is well known that the minimal modal logic $\mathbf{K}$ admits a canonical model and finite filtration. Other (multi)modal

[4]In fact in both [Seg71] and [Gar75] this confusing circumstance is elegantly avoided by performing a filtration-like step before defining the probabilistic measures and thus arriving at logically finite sets of formulas and finite boolean algebras for the probabilistic measures to define on.

[5]Note that the abandoned argument establishes a case of *trivial* continuity, i. e. the sequences $\{\mu(m_{\varphi_k})\}_{k=0}^{\infty}$ do not have infinitely many nonzero members.

logics do so too. Their canonical models are constructed following the same basic steps as the ones followed here, and especially they share the carrier of their relational structures - the set of the maximal consistent sets of formulas in their languages. This enables us to use this kind of *compatibility* in order to design and prove the completeness of logics that combine modal operators and qualitative probabilities at the expense of quite little effort. Furthermore, a straightforward combination of the filtration techniques inherent to these logics and the one we introduce below can be employed to establish the strong finite model property for them.

Since the canonical constructions for the imaginable combined logics and the definition of filtration for them are achieved by just adding the steps and clauses that are inherent to their pure forms, we proceed with the filtration technique for $\leq$ alone.

Let Ψ be a set of formulas and let us denote a filtration of a model $\langle W, B(W), \langle \mu_w : w \in W\rangle, v\rangle$ with respect to Ψ by $\langle |W|, B(|W|), \langle \mu_{|w|}, |w| \in |W|\rangle, |v|\rangle$, where $|w|$ denote equivalence classes and $|W| = \{|w| : w \in W\}$. Let us also denote the mapping $w \to |w|$ by F. Here are the constraints on F in our definition and the corresponding filtration property:

$(P1)$ for all $m \in B(|W|)$ $F^{-1}(m) \in B(W)$

$(P2)$ for all $m_1, m_2 \in B(|W|)$, $w \in W$ $\mu_w(F^{-1}(m_1)) \leq \mu_w(F^{-1}(m_2))$ iff $\mu_{|w|}(m_1) \leq \mu_{|w|}(m_2)$.

Lemma: *(filtration lemma)* Assume that $F^{-1}(|v|(\psi)) = v(\psi)$ for all $\psi \in \Psi$ and Ψ is closed under subformulas. Then for all $\psi \in \Psi$ $w \models \psi$ iff $|w| \models \psi$.

Proof: Induction on the construction of ψ. For propositional variables p we have $w \models \psi$ iff $w \in v(\psi)$ iff $|w| \in |v|(\psi)$ iff $|w| \models \psi$. The induction is trivial over boolean connectives. For the case of $\psi \doteq \chi \leq \theta$ we have $w \models \psi$ iff $\mu_w(v(\chi)) \leq \mu_w(v(\theta))$ iff $\mu_{|w|}(|v|(\chi)) \leq \mu_{|w|}(|v|(\chi))$ by $(P2)$ and hence iff $|w| \models \psi$.$\dashv$

We perform the filtration that gives us the desired finite model property with respect to the following set of formulas:

$\Psi_\varphi \ {}_{def}= \ \Psi^0_\varphi \cup \{\psi \leq \chi : \ \psi, \chi \in \Psi^0_\varphi \cup \{\bot, \top\}\}$,

where

$\Psi^0_\varphi \ {}_{def}= \ Subf(\varphi) \cup \{\theta : \ \theta$ is a boolean combination of elements of $Subf(\varphi)$ in disjunctive normal form$\}$.

We put

$|w| \ {}_{def}= \{w' \in W : \ \forall \psi \in \Psi_\varphi \ (w' \models \psi \leftrightarrow w \models \psi)\}$,

and

$|v|(p) \ {}_{def}= \{|w| \in |W| : \ w \models p\}$ for all propositional variables p.

In order to define the measures $\mu_{|w|}$ let us note that for any class $|w|$ and any two formulas $\psi, \chi \in \Psi^0_\varphi$ we have either $\psi \leq \chi \in w'$ for all $w' \in |w|$ or $\chi \leq \psi \in w'$ for all $w' \in |w|$ or both. Now consider the following system of linear equations and inequations:

$\sum\limits_{i=0}^{k} x_{\psi_i} = 1$ for all subsets $\{\top \leq \bigvee\limits_{i=0}^{k} \psi_i, \top \leq \neg\psi_{i_1} \vee \neg\psi_{i_2}, 0 \leq i_1 < i_2 \leq k\}$ of $\Psi^{|w|}_\varphi$;

$x_{\psi_1} = x_{\psi_2}$ for all subsets $\{\psi_1 \leq \psi_2, \psi_2 \leq \psi_1\}$ of $\Psi^{|w|}_\varphi$;

$x_{\psi_1} < x_{\psi_2}$ for all $\psi_1, \psi_2 \in \Psi^0_\varphi$ such that $\psi_1 \leq \psi_2 \in \Psi^{|w|}_\varphi$ and $\psi_2 \leq \psi_1 \notin \Psi^{|w|}_\varphi$,

where $\Psi^{|w|}_\varphi \ {}_{def}= \{\psi \in \Psi_\varphi : \ w \models \psi\}$.

It is easy to find that $x_\psi = \mu_{w'}(v(\psi))$, $\psi \in \Psi^0_\varphi$, is a solution of this system for any $w' \in |w|$. We define the measure $\mu_{|w|}$ by picking an arbitrary solution $\langle x_\psi : \ \psi \in \Psi_\varphi\rangle$ of this system and putting $\mu_{|w|}(\{|w''|\}) = x_{\theta_{|w''|}}$, where $\theta_{|w''|} \in \Psi^0_\varphi$ is suitably chosen so that $|v|(\theta_{|w''|}) = \{|w''|\}$. This gives us a probabilistic measure to associate with $|w|$ that satisfies exactly those formulas of the kind $\psi \leq \chi$, where $\psi, \chi \in \Psi_\varphi$, that are satisfied at any of the worlds in $|w|$. The construction of our mapping $F : w \to |w|$ implies that it satisfies $(P1)$ and the explicit assumption from the filtration lemma. For the proof $(P2)$ we just need to notice that, since we have either $\mu_{w'}(v(\psi)) \leq \mu_{w'}(v(\chi))$ for all $w' \in |w|$, or the reverse for all $w' \in |w|$ again, or both, and therefore this relation is expressed in our system for the measure $\mu_{|w|}$. Hence we have the same relation for $\mu_{|w|}$ and $|v|$ and vice versa.

Now due to the finiteness of Ψ_φ we obtain that the filtrated model is finite. Moreover, the cardinality of $|W|$ does not exceed a number that depends only on the syntactic structure of φ. Hence we conclude that this filtration, being performed on the canonical model, entails the strong finite model property and hence the decidability of our logic.

Remark. The use of linear systems can be kept in mind for the case of infinite sets of formulas in the place of Ψ_φ as well, because the original model provides the solvability of the systems. Let us also notice that in the finite case we can consider only rational-valued solutions: something that gives us the 'adjustment' step towards the relational interpretation of the $\leq$-operator by means of **FSKD**-models [Hoe92] and gives an idea of the possible definition of filtration of **FSKD**-models for $\leq$.

4 Multiple $\leq$-operators: some dynamic operations

As we claimed above, the components of our construction of canonical model and those of our filtration technique can make part of corresponding constructions for logics that possibly contain other propositional operators. These components, if reproduced in the appropriate amount, do for the minimal logic of an arbitrary number of $\leq$-operators as well.

In this supersection we study a distinguished case of a logic with an infinite set of $\leq$-operators, but which moreover satisfy the meaningful relations among each other that we introduce as qualitative probabilistic counterparts to the dynamic operations as known in modal logic.

The dynamic nature of our definitions is best manifested on the case of composition, where we just bring the law of composition of Markov processes into our semantics. Consider two $\leq$-operators, $\leq_\sigma$ and $\leq_\tau$, and the following dynamic reading of $w \models \varphi \leq_\pi \psi$ for π being σ and π being τ: 'the *transition* π takes the *state* w to a state that satisfies ψ at least as probably as to one that satisfies φ'. Then our definition of the composition of the two operators, the operator $\leq_{\sigma\star\tau}$, justifies the same reading for π being $\sigma \star \tau$, provided that we denote so the sequence of transitions σ and τ taken as a compound transition.

We carry out the canonical construction for this logic too. We also use a fragment of it to demostrate a qualitative-probabilistic version of the *unravelling technique* that was introduced in [Sah75] on.

For our needs we extend our set of useful formulas Π_j and infinitary rule (Ded) by denoting by Π_j^σ and (Ded_σ) respectively the result of the substitution of all occurrences of $\leq$ with $\leq_\sigma$ in them. As for models, the measure that is associated with a world w for an operator $\leq_\sigma$ we denote by μ_w^σ. We also use the abbreviation:

$\varphi \equiv_\sigma \psi \rightleftharpoons \varphi \leq_\sigma \psi \wedge \psi \leq_\sigma \varphi.$

4.1 The dynamic operations

The operations from propositional dynamic logic that we develop probabilistic analogons of are composition, union, the identity operator, and iteration. For an arbitrary pair of operators

$\leq_\sigma$ and $\leq_\tau$ we denote their composition by $\leq_{\sigma\star\tau}$ and their sum by $\leq_{\sigma+\tau}$ [6]. The identity operator we denote by $\leq_{Id}$, and the iteration of an arbitrary operator $\leq_\sigma$ - by $\leq_{\sigma^*}$. Here are the equations by which we define the modelling measures of these operators:

$$(\star) \qquad \mu_{w_0}^{\sigma\star\tau}(W') = \int\limits_{w\in W} \mu_w^\tau(W')d\mu_{w_0}^\sigma$$

$$(+) \qquad \mu_w^{\sigma+\tau}(W') = \frac{1}{2}(\mu_w^\sigma(W') + \mu_w^\tau(W'))$$

$$(Id) \qquad \mu_w^{Id}(W') = \chi_{W'}(w)$$

$$(*) \qquad \mu_{w_0}^{\sigma^*}(W') = \frac{1}{2}\mu_{w_0}^{Id}(W') + \frac{1}{2}\int\limits_{w\in W} \mu_w^{\sigma^*}(W')d\mu_{w_0}^\sigma$$

Here W' denotes an arbitrary measurable set and $\chi_{W'}$ denotes its characteristic function. For a definition of the integral that occurs in $(\star)$ and in $(*)$ to work with we take:

$$(\star\star) \qquad \int\limits_{w\in W} \mu_w^\tau(W')d\mu_{w_0}^\sigma = \inf_{A_1,\ldots,A_n} I_{A_1,\ldots,A_n}^{sup} = \sup_{A_1,\ldots,A_n} I_{A_1,\ldots,A_n}^{inf},$$

where

$$I_{A_1,\ldots,A_n}^{sup} {}_{def} = \sum_{i=1}^{n} \sup_{w\in A_i} \mu_w^\tau(W')\mu_{w_0}^\sigma(A_i),$$

$$I_{A_1,\ldots,A_n}^{inf} {}_{def} = \sum_{i=1}^{n} \inf_{w\in A_i} \mu_w^\tau(W')\mu_{w_0}^\sigma(A_i),$$

and $\{A_i : \ i = 1, \ldots, n\}$ range over the set of all the *partititions* of W over $B(W)$, provided that

$$\inf_{A_1,\ldots,A_n} I_{A_1,\ldots,A_n}^{sup} = \sup_{A_1,\ldots,A_n} I_{A_1,\ldots,A_n}^{inf}.$$

For $\leq_{\sigma\star\tau}$, $\leq_{\sigma+\tau}$ and $\leq_{Id}$ being this way defined we have the well-known modal schemata:
$\top \leq_\sigma (\top \leq_\tau \varphi) \Leftrightarrow \top \leq_{\sigma\star\tau} \varphi$
$\top \leq_{\sigma+\tau} \varphi \Leftrightarrow \top \leq_\sigma \varphi \wedge \top \leq_\tau \varphi$
$\varphi \Leftrightarrow \top \leq_{Id} \varphi$
and, if possible, converting probabilistic models into Kripke-ones after [Seg71, Gar75] would give that the accessibility relations for $\leq_{\sigma\star\tau}$, $\leq_{\sigma+\tau}$ and $\leq_{Id}$, are the composition and the union of those for $\leq_\sigma$ and $\leq_\tau$ and the identity relation respectively. As for iteration, the equation $(*)$ needs a little more explanation. It is easy to find out that if there exists a measure $\mu_w^{\sigma^*}$ that satisfies $(*)$ then it is unique. This unique solution of $(*)$ can be obtained as the limit of the sequence $\{\mu_{w,k}^{\sigma^*}\}_{k=0}^\infty$, where:

[6] Observe that here $+$ *is not associative.*

$$\mu^{\sigma^*}_{w_0,k+1}(W')\ {}_{def}= \frac{1}{2}\mu^{Id}_{w_0}(W') + \frac{1}{2}\int\limits_{w\in W} \mu^{\sigma^*}_{w,k}(W')d\mu^{\sigma}_{w_0}$$

and $\mu^{\sigma^*}_{w_0,0}$ can be chosen arbitrarily, e. g.:

$$\mu^{\sigma^*}_{w_0,0}(W')\ {}_{def}= \mu^{Id}_{w_0}(W')$$

An explicit form of the solution can be given by the sum:

$$\mu^{\sigma^*}_{w_0}(W') = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}}\mu^{\sigma^k}_{w_0}(W'),$$

where $\sigma^k \rightleftharpoons Id\underbrace{\star\sigma\star\ldots\star\sigma}_{k\ times}$. This can be checked by direct substitution into $(*)$, using that $\mu^{\sigma\star Id}_{w} = \mu^{\sigma}_{w}$ for all σ as follows from the definition of $\leq_{Id}$ and $(\star)$. The sum above also shows that $\leq_{\sigma^*}$ has exactly the same modal properties as the iteration as known for Kripke semantics. For example the axioms of Segerberg [Seg77] hold.

4.2 An Axiomatization and the Canonical Model

In order to achieve a plain form of axiomatization for $\star$, and in particular to avoid connectives of arbitrary arity like E, we introduce an auxiliary set of ternary operators $\leq^{\varphi}_{\sigma,\tau}$. The semantic definition for them, that, as far as formal argument is concerned, is used only for the establishment of soundness of our axiom schemata, is as follows:

$$w_0 \models \varphi \leq^{\psi}_{\sigma,\tau} \chi\ \ {}_{def}\leftrightarrow \int\limits_{w\in v(\psi)} \mu^{\tau}_{w}(v(\varphi))d\mu^{\sigma}_{w_0} \leq \mu^{\sigma\star\tau}_{w_0}(v(\chi))$$

Here are our axiom schemata:

$(M_{i,j,k,l})$ $((\Pi^{\sigma}_{j} \Rightarrow \psi \leq_{\sigma} R_{i,j}) \wedge \top \leq_{\sigma} ((\psi \wedge \Pi^{\tau}_{l}) \Rightarrow (\varphi \leq_{\tau} R_{k,l}))) \Rightarrow (\Pi^{\sigma\star\tau}_{jl} \Rightarrow \varphi \leq^{\psi}_{\sigma,\tau} R_{ik,jl})$

(S) $(\varphi \leq^{\psi_1}_{\sigma,\tau} \chi_1 \wedge \varphi \leq^{\psi_2}_{\sigma,\tau} \chi_2 \wedge \top \leq_{\sigma} ((\chi_1 \wedge \chi_2) \leq_{\tau} \bot)) \Rightarrow \varphi \leq^{\psi_1 \vee \psi_2}_{\sigma,\tau} (\chi_1 \vee \chi_2)$

(E) $\varphi \leq^{\psi}_{\sigma,\tau} \chi \wedge \top \leq_{\sigma} \psi \Leftrightarrow \varphi \leq_{\sigma\star\tau} \chi$

Note that $(M_{i,j,k,l})$ is meant to give approximation of the values of individual expressions of the kind $\mu^{\tau}_{w}(W')\mu^{\sigma}_{w_0}(A_i)$, where $v(\psi) = A_i$, (S) serves to sum these expressions, and (E) serves to eliminate auxiliary operators.

For $+$, Id and * we take the following axiom schemata:

(U) $(\varphi \leq_{\sigma} \psi \wedge \varphi \leq_{\tau} (\psi \vee \chi_1 \vee \chi_2) \Rightarrow (\varphi \leq_{\sigma+\tau} (\psi \vee \chi_1) \vee \varphi \leq_{\sigma+\tau} (\psi \vee \chi_2)))$
$\wedge(\varphi \leq_{\tau} \psi \wedge \varphi \leq_{\sigma} (\psi \vee \chi_1 \vee \chi_2) \Rightarrow (\varphi \leq_{\sigma+\tau} (\psi \vee \chi_1) \vee \varphi \leq_{\sigma+\tau} (\psi \vee \chi_2)))$

(I) $\varphi \Leftrightarrow \top \leq_{Id} \varphi$

(F) $\varphi \leq_{\sigma^*} \psi \Leftrightarrow \varphi \leq_{Id+(\sigma\star\sigma^*)} \psi$

The construction of the canonical model for this logic follows closely that for the minimal logic of a single $\leq$-operator: we take the maximal consistent sets that contain all instances of our axiom schemata and are closed under the infinitary rules (Ded_{σ}) as our possible worlds.

What we have to add to our argument is that the schemata $(M_{i,j,k,l})$, (S) and (E) together entail $(\star)$ and (U), (I) and (F) entail $(+)$, (Id) and $(*)$ respectively, that we require to hold between the measures μ^σ_w. We proceed to do this in the order of our introducing them above.

$(M_{i,j,k,l})$, (S), and (E) entail $(\star)$.

We first establish the equality

$$\inf_{A_1,\ldots,A_n} I^{sup}_{A_1,\ldots,A_n} = \sup_{A_1,\ldots,A_n} I^{inf}_{A_1,\ldots,A_n},$$

that is required for the meaningfulness of $(\star\star)$. It is easy to deduce that for all partititions $\{A_i:\ i=1,\ldots,n\}$ $I^{inf}_{A_1,\ldots,A_n} \leq I^{sup}_{A_1,\ldots,A_n}$, and hence we have $\sup I^{inf} \leq \inf I^{sup}$. So it is enough to prove, that for an arbitrary $\varepsilon > 0$ there exists a partitition $\{B_0,\ldots,B_k\}$ such that:

$$I^{sup}_{B_0,\ldots,B_k} - I^{inf}_{B_0,\ldots,B_k} < \varepsilon$$

For this purpose let $k > 2/\varepsilon$ and $W' = m_\varphi = v(\varphi)$ according to our definition of $B(W)$ and v. Now consider the formulas:

$\alpha_{i,k} \rightleftharpoons \Pi^\tau_k \Rightarrow \varphi \leq_\tau R_{i,k}$ for $i=0,\ldots,k$,

$\beta_{i,k} \rightleftharpoons \Pi^\tau_k \Rightarrow \neg(\varphi \leq_\tau R_{i-1,k})$ for $i=1,\ldots,k$,

and

$\beta_{0,k} \rightleftharpoons \Pi^\tau_k \Rightarrow \bot \leq_\tau \varphi$.

The sets $m_{\alpha_{i,k}}$ are measurable, because substituting $p_{0,k},\ldots,p_{k-1,k}$ by any k propositional variables $q_0,\ldots,q_{k-1}$ from the initial language that do not occur in φ in $\alpha_{i,k}$ gives formulas that are derivably equivalent to $\alpha_{i,k}$ and belong to the initial language. The same holds for $m_{\beta_{i,k}}$. Obviously $B_i \ _{def} = m_{\alpha_{i,k}\wedge\beta_{i,k}}$, $i=0,\ldots,k$ form a partition of W. Moreover, according to the definition of μ^τ_w we have $\mu^\tau_w(W') \in (\frac{i-1}{k}, \frac{i}{k}]$ for all $w \in B_i$, $i>0$, $\mu^\tau_w(W') = 0$ for all $w \in B_0$, and, since $\sum\limits_i \mu^\sigma_{w_0}(B_i) = 1$, we have

$$\sum_{i=0}^{k} \sup_{w\in B_i} \mu^\tau_w(W')\mu^\sigma_{w_0}(B_i) - \sum_{i=0}^{k} \inf_{w\in B_i} \mu^\tau_w(W')\mu^\sigma_{w_0}(B_i) \leq \frac{1}{k} < \varepsilon.$$

For the establishment of the equality $(\star)$ let us again assume that $W' = v(\varphi)$. Now, by substituting $\gamma_{k,l} \rightleftharpoons \alpha_{k,l} \wedge \beta_{k,l}$ for ψ in $(M_{i,j,k,l})$, for an arbitrary w_0 we obtain:

(M_k) $\quad \vdash (\Pi^\sigma_j \Rightarrow \gamma_{k,l} \leq_\sigma R_{i_k,j}) \Rightarrow (\Pi^{\sigma\star\tau}_{jl} \Rightarrow \varphi \leq^{\gamma_{k,l}}_{\sigma,\tau} R_{i_k k,jl})$

(γ) $\quad \vdash \bigvee\limits_{k=0}^{l} \gamma_{k,l}$

In order to enable the application of the (S) schema on the formulas $(\varphi \leq^{\gamma_{k,l}}_{\sigma,\tau} R_{i_k k,jl})$ in the consequents of (M_k) we have to supply disjoint propositions to substitute for χ_i, $i=1,2$ on the right sides of $\leq^{\gamma_{k,l}}_{\sigma,\tau}$. For this purpose we extend our set of abbreviations

$R_{k_1,k_2,j} \rightleftharpoons \bigvee\limits_{i=k_1}^{k_2-1} p_{i,j}$ for $k_1 \leq k_2$,

and notice that:

$\vdash \Pi^\sigma_j \Rightarrow R_{k_1,k_2,j} \equiv_\sigma R_{k_2-k_1,j}$,

$\vdash \Pi^\sigma_j \Rightarrow R_{k_1,k_2,j} \vee R_{k_2,k_3,j} \equiv_\sigma R_{k_1,k_3,j}$ for $k_1 \leq k_2 \leq k_3$,

$\vdash \top \leq_\sigma (\Pi^\tau_j \Rightarrow (R_{k_1,k_2,j} \wedge R_{k_2,k_3,j}) \leq_\tau \bot))$.

Now l applications of (S) on (M_k), $k=0,\ldots,l$ give

$$\vdash \left(\Pi_j^\sigma \Rightarrow \bigwedge_{k=0}^{l}(\gamma_{k,l} \leq_\sigma R_{i_k,j})\right) \Rightarrow \left(\Pi_{jl}^{\sigma\star\tau} \Rightarrow \varphi \leq_{\sigma,\tau}^{\left(\bigvee\limits_{k=0}^{l}\gamma_{k,l}\right)} R_{s,jl}\right),$$

where $s = \sum\limits_{k=0}^{l} i_k k$, and hence by (E) and (γ)

$$\vdash \left(\Pi_j^\sigma \Rightarrow \bigwedge_{k=0}^{l}(\gamma_{k,l} \leq_\sigma R_{i_k,j})\right) \Rightarrow \left(\Pi_{jl}^{\sigma\star\tau} \Rightarrow \varphi \leq_{\sigma\star\tau} R_{s,jl}\right)$$

By substituting

$$i_{k,w_0\ def} = \min\{i_k :\ w_0 \models \Pi_j^\sigma \Rightarrow \gamma_{k,l} \leq_\sigma R_{i_k,j}\}$$

in the above we obtain

$$w_0 \models \Pi_{jl}^{\sigma\star\tau} \Rightarrow \varphi \leq_{\sigma\star\tau} R_{s_{w_0},jl},$$

where $s_{w_0} = \sum\limits_{k=0}^{l} i_{k,w_0} k$. The latter gives the approximation

$$\mu_{w_0}^{\sigma\star\tau}(W') \leq I_{v(\gamma_0),\ldots,v(\gamma_l)}^{sup} + \varepsilon,$$

where ε can be taken arbitrarily small by letting j take arbitrary big values. The corresponding approximation from below is obtained similarly by substituting $\neg\varphi$ for φ in the instances of $(M_{i,j,k,l})$, (S) and (E) and in the definition of $\gamma_{k,l}$ as well[7]. This is sufficient for our needs, because

$$I_{v(\gamma_0),\ldots,v(\gamma_l)}^{inf} + \frac{1}{l} \geq \int\limits_{w\in W} \mu_w^\tau(W')d\mu_{w_0}^\sigma \geq I_{v(\gamma_0),\ldots,v(\gamma_l)}^{sup} - \frac{1}{l}$$

(U) **entails** $(+)$.

Here we give a sketch of a more deduction-like argument making extensive use of the completeness of the minimal logic. What we hop-trip-and-jump at the derivation of are the schemata

$(\Pi_k^\sigma \Rightarrow \varphi \leq_\sigma R_{i,k}) \wedge (\Pi_k^\tau \Rightarrow \varphi \leq_\tau R_{i+i',k}) \Rightarrow (\Pi_{4k}^{\sigma+\tau} \Rightarrow \varphi \leq_{\sigma+\tau} R_{4i+2i',4k})$ for all k, $i+i' < k$,

where the application of (U) negotiates that $R_{2i+2i',2k}$ is but the disjunction of $R_{2i,2k}$, $R_{2i,2i+i',2k}$ and $R_{2i+i',2i+2i',2k}$ in the places of ψ, χ_1 and χ_2 respectively, and $4k$ is motivated by the convenience of eliminating unwanted antecedents as shown below. Recalling the definitional equality from the canonical construction helps see that to derive these schemata is sufficient for the establishment of $(+)$.

Here follow two sets of auxiliary theorems that we involve in our derivation. The first one is

$\vdash \Pi_{2s}^{\sigma+\tau} \wedge \Pi_s^{\sigma+\tau} \Rightarrow R_{i,i+2,2s} \equiv_{\sigma+\tau} p_{j,s}$ for all s, $i < 2s-1$, and $j < s$.

By substituting $\perp$ for χ_i, $i = 1, 2$, in (U) we get the second one

[7] The possibility of such a substitution is what makes the occurences of arbitrary propositions in our axiom schemata only on one side of $\leq$ in other cases too.

$\vdash \Pi^\sigma_{2k} \wedge \Pi^\tau_{2k} \Rightarrow \Pi^{\sigma+\tau}_{2k}$ for all k.

Now we assume the antecedents of our schemata

$[\Pi^\sigma_k \Rightarrow \varphi \leq_\sigma R_{i,k}] \qquad [\Pi^\tau_k \Rightarrow \varphi \leq_\tau R_{i+i',k}]$,

and using our first theorems we obtain

$\Pi^\sigma_{2k} \wedge \Pi^\tau_{2k} \Rightarrow \varphi \leq_\sigma R_{2i,2k} \wedge \varphi \leq_\tau R_{2i+2i',2k}$

and hence by (U) and by the second set of auxiliary theorems

$\Pi^\sigma_{2k} \wedge \Pi^\tau_{2k} \Rightarrow \Pi^{\sigma+\tau}_{2k} \wedge (\varphi \leq_{\sigma+\tau} R_{2i+i',2k} \vee \varphi \leq_{\sigma+\tau} (R_{2i,2k} \vee R_{2i+i',2i+2i',2k}))$.

Having in mind that

$\vdash \Pi^{\sigma+\tau}_{2k} \Rightarrow R_{2i+i',2k} \equiv_{\sigma+\tau} (R_{2i,2k} \vee R_{2i+i',2i+2i',2k})$,

we obtain

$\Pi^\sigma_{2k} \wedge \Pi^\tau_{2k} \Rightarrow \Pi^{\sigma+\tau}_{2k} \wedge \varphi \leq_{\sigma+\tau} R_{2i+i',2k}$.

Now we harmlesly add into the conjunctions on both sides of the implication

$\Pi^{\sigma+\tau}_{4k} \wedge \Pi^\sigma_{2k} \wedge \Pi^\tau_{2k} \Rightarrow \Pi^{\sigma+\tau}_{4k} \wedge \Pi^{\sigma+\tau}_{2k} \wedge \varphi \leq_{\sigma+\tau} R_{2i+i',2k}$,

and apply our first auxiliary theorems having in mind the definition of $R_{2i+i',2k}$ and arrive at

$\Pi^{\sigma+\tau}_{4k} \wedge \Pi^\sigma_{2k} \wedge \Pi^\tau_{2k} \Rightarrow \varphi \leq_{\sigma+\tau} R_{4i+2i',4k}$.

Discharging the assumptions and a few boolean transformations give us

$\vdash \Pi^\sigma_{2k} \wedge \Pi^\tau_{2k} \Rightarrow ((\Pi^\sigma_k \Rightarrow \varphi \leq_\sigma R_{i,k}) \wedge (\Pi^\tau_k \Rightarrow \varphi \leq_\tau R_{i+i',k}) \Rightarrow (\Pi^{\sigma+\tau}_{4k} \Rightarrow \varphi \leq_{\sigma+\tau} R_{4i+2i',4k}))$.

At this point the trip from $2k$ to $4k$ has already given that the variables that occur in the outmost antecedents $\Pi^\sigma_{2k} \wedge \Pi^\tau_{2k}$ of what we just derived occur only in these subformulas. Using this we prove below that assuming the satisfiability of the antecedents of the formulas above and the negations of their consequents (i. e. te schemata that we need) separately implies their simultaneous satisfiability. This being a contradiction with the derivability of the formulas above gives that the negation of the consequents are not satisfiable, i. e. these consequents are valid, and hence, by the completeness of the minimal logic - theorems. Our proof employs a semantical consideration.

Proposition: Let α and β be satisfiable formulas that share no propositional variables. Then $\alpha \wedge \beta$ is satisfiable too.

Proof: Assume that for $\mathbf{A}\ _{def} = \langle A, \langle \mu^\sigma_a : \ a \in A,\ \sigma \in \Sigma\rangle, v_A\rangle$ and $\mathbf{B}\ _{def} = \langle B, \langle \mu^\sigma_b : \ b \in B,\ \sigma \in \Sigma\rangle, v_B\rangle$ we have that

$\mathbf{A}, a_0 \models \alpha$ and $\mathbf{B}, b_0 \models \beta$.

Now consider the model

$\mathbf{A}\times\mathbf{B}\ _{def} = \langle A \times B, \langle \mu^\sigma_{\langle a,b\rangle} : \ a \in A,\ b \in B,\ \sigma \in \Sigma\rangle, v_{A\times B}\rangle$,

where $\mu^\sigma_{\langle a,b\rangle}$ are defined on the smallest boolean algebra that contains all the cartesian products of measurable subsets A' and B' of A and B respectively so that

$\mu^\sigma_{\langle a,b\rangle}(A' \times B') = \mu^\sigma_a(A')\mu^\sigma_b(B')$,

and the valuation $v_{A\times B}$ is defined as follows:

$v_{A\times B}(p)\ _{def} = v_A(p) \times B$ for the propositional variables p that occur in α;

$v_{A\times B}(p)\ _{def} = A \times v_B(p)$ otherwise.

Now two simple inductions on the construction of α and β respectively give that for an arbitrary subformula γ of its we have $\mathbf{A}\times\mathbf{B}$, $\langle a,b\rangle \models \gamma$ iff γ is a subformula of α and $\mathbf{A}, a \models \gamma$ or γ is a subformula of β and $\mathbf{B}, b \models \gamma$. Finally we get that $\mathbf{A}\times\mathbf{B}$, $\langle a_0, b_0\rangle \models \alpha \wedge \beta$. $\dashv$

Remark. Let us note that the choice of equal weights of μ^σ and μ^τ in $(+)$ is not the only possible one. In fact for an arbitrary natural p and q, $0 < p < q$, we can axiomatize the semantical definitions of $\leq_{\frac{p}{q}\sigma+\frac{q-p}{q}\tau}$:

$$(+_{p,q}) \qquad \mu_w^{\frac{p}{q}\sigma+\frac{q-p}{q}\tau}(W') = \frac{p}{q}\mu^\sigma_w(W') + \frac{q-p}{q}\mu^\tau_w(W'))$$

with the schemata

$$(U_{p,q}) \qquad (\varphi \leq_\sigma \psi \wedge \varphi \leq_\tau (\psi \vee \bigvee_{i=1}^{q} \chi_i) \Rightarrow \bigvee_{1 \leq n_1 < \ldots < n_{q-p} \leq q} \varphi \leq_{\frac{p}{q}\sigma + \frac{q-p}{q}\tau} (\psi \vee \bigvee_{i=1}^{q-p} \chi_{n_i}))$$
$$\wedge (\varphi \leq_\tau \psi \wedge \varphi \leq_\sigma (\psi \vee \bigvee_{i=1}^{q} \chi_i) \Rightarrow \bigvee_{1 \leq n_1 < \ldots < n_p \leq q} \varphi \leq_{\frac{p}{q}\sigma + \frac{q-p}{q}\tau} (\psi \vee \bigvee_{i=1}^{p} \chi_{n_i}))$$

The operators so defined and axiomatized give healthy fixpoint-defined versions of iteration too.

(I) **entails** (Id) **and** (F) **entails** $(*)$**.**

These two cases are trivially coped with using just the special form that the canonical model definitions of μ_w^σ have.$\dashv$

At the end of this section we should note that our arguments do not answer the question whether the above are examples of frame correspondence, because for the establishment of the frame properties we rely on our frame's being the canonical one.

5 The Unravelling Technique

The results of this section are what we propose as a version of Sahlqvist's technique[Sah75] for the probabilistic models of our logic. A minor difference from the prototype is that we *unravel* our model only as much as a following step of filtration[8] with respect to a given formula could possibly profit from. Performing filtration on the special kind of models that unravelled models are hints of more restrictions on the finite models to be tested by an imaginable decision procedure. An obvious advantage is that the multitude of arbitrary intersections of unrelated to each other equivalence classes, that increases the upper bound of the number of possible worlds in filtrates, shrinks. As with respect to our dealing with a logic containing composition, unravelling enables the reduction of the number of $(\star)$ equations to be satisfied in filtrates relative to the amount of measures among which these equations hold (by increasing the amount of measures, actually) to a reasonable minimum. We proceed with the precise construction.

Consider a formula φ. For an arbitrary occurence of a subformula ψ in φ consider the finite and possibly empty sequence of subformulas of φ $\chi_1 \leq_{\sigma_1} \theta_1, \ldots, \chi_k \leq_{\sigma_k} \theta_k$ such that:

i. $\chi_{i+1} \leq_{\sigma_{i+1}} \theta_{i+1}$ occurs either in χ_i or in θ_i or in both, but not under a $\leq$-operator for $1 \leq i < k$;

ii. $\chi_1 \leq_{\sigma_1} \theta_1$ occurs in φ, but not under a $\leq$-operator, and so does ψ either in χ_k or in θ_k or in both.

This sequence is unique with respect to a given occurence of ψ in φ, and defines unambiguously the expression $\Lambda \star \sigma_1 \star \ldots \star \sigma_k$ (here Λ stands in order to make the empty sequence also visible). We include this expression into our notation for occurence of a subformula in a formula like this: $\psi \nearrow_{\Lambda \star \sigma_1 \star \ldots \star \sigma_k} \varphi$. We also put $\psi \leq_{\tau_2} \chi \nearrow_{\Lambda \star \sigma_1 \star \ldots \star \sigma_k \star \tau_1} \varphi$ whenever $\psi \leq_{\tau_1 \star \tau_2} \chi \nearrow_{\Lambda \star \sigma_1 \star \ldots \star \sigma_k} \varphi$. We define the set:

$\Sigma_{\varphi\ def} = \{\Lambda \star \sigma_1 \star \ldots \star \sigma_k : \psi \nearrow_{\Lambda \star \sigma_1 \star \ldots \star \sigma_k} \varphi\}$

[8] The author has so far failed to complete this step with the *exact* satisfaction of $(\star)$ by the filtrate and thus accomplish a finite and comparatively efficient filtation for a logic with composition. Doing this could easily give the decidability of a logic containing not only $\star$, but also $+$ and Id.

Now consider a model $\mathbf{W} = \langle W, B(W), \langle \mu^\sigma_w : w \in W, \sigma \in \Sigma\rangle, v\rangle$ and a formula φ. We define the *unravelled* model $\mathbf{W}^{\mathbf{\Sigma}_\varphi} = \langle W \times \Sigma_\varphi, B(W \times \Sigma_\varphi), \langle \mu^\sigma_{\langle w,\Lambda\star\tau\rangle} : w \in W, \sigma, \tau \in \Sigma\rangle, v^{\Sigma_\varphi}\rangle$ by putting:

$v^{\Sigma_\varphi}(p) \ _{def} = v(p) \times \Sigma_\varphi$,

and

$\mu^\sigma_{\langle w_0,\Lambda\star\tau\rangle}(V) \ _{def} = \mu^\sigma_{w_0}(\{w \in W : \langle w, \Lambda \star \tau \star \sigma\rangle \in V\})$

for $\Lambda \star \tau \star \sigma \in \Sigma_\varphi$,

$\mu^\sigma_{\langle w_0,\Lambda\star\tau\rangle}(V) \ _{def} = \mu^\sigma_{w_0}(\{w \in W : \langle w, \Lambda \star \tau\rangle \in V\})$

for $\Lambda \star \tau \star \sigma \notin \Sigma_\varphi$, yet $\Lambda \star \tau \in \Sigma_\varphi$ and σ prime, and take the identities $(\star)$ for the definition of $\mu^\sigma_{\langle w_0,\Lambda\star\tau\rangle}$ otherwise. The latter definition is unambiguous and the defined above is really a model for our logic due to the equalities:

$$\mu^{\sigma_1\star\sigma_2}_{\langle w_0,\Lambda\star\tau\rangle}(V) = \mu^{\sigma_1\star\sigma_2}_{w_0}(\{w \in W : \langle w, \Lambda \star \tau \star \sigma_1 \star \sigma_2\rangle \in V\}) =$$

$$= \int\limits_{w\in W} \mu^{\sigma_2}_w(\{w \in W : \langle w, \Lambda \star \tau \star \sigma_1 \star \sigma_2\rangle \in V\}) d\mu^{\sigma_1}_{w_0} =$$

$$= \int\limits_{\langle w,\Lambda\star\tau\star\sigma_1\rangle \in W \times \{\Lambda\star\tau\star\sigma_1\}} \mu^{\sigma_2}_{\langle w,\Lambda\star\tau\star\sigma_1\rangle}(V) d\mu^{\sigma_1}_{\langle w_0,\Lambda\star\tau\rangle} =$$

$$= \int\limits_{\langle w,\Lambda\star\sigma\rangle \in W \times \Sigma_\varphi} \mu^{\sigma_2}_{\langle w,\Lambda\star\sigma\rangle}(V) d\mu^{\sigma_1}_{\langle w_0,\Lambda\star\tau\rangle}$$

the last of which holds, because $\mu^{\sigma_1}_{\langle w_0,\Lambda\star\tau\rangle}(W \times (\Sigma_\varphi \backslash \{\Lambda \star \tau \star \sigma_1\})) = 0$.[9]

Proposition: Assume that $\psi \nearrow_{\Lambda\star\sigma} \varphi$. Then $\mathbf{W}, w \models \psi$ iff $\mathbf{W}^{\mathbf{\Sigma}_\varphi}, \langle w, \Lambda \star \sigma\rangle \models \psi$.

Proof: Induction on the construction of ψ. $\dashv$

Corrolary: $\mathbf{W}, w \models \varphi$ iff $\mathbf{W}^{\mathbf{\Sigma}_\varphi}, \langle w, \Lambda\rangle \models \varphi$. $\dashv$

6 Concluding Remarks

Summarizing what we have obtained, we should note that there are only few points in our proofs that are neither versions of standard techniques of dealing with similar problems in modal logic, nor just straightforward. Some elements of our constructions, e.g. linear equations, have been extensively used in other probabilistic formalisms, for examples see [FagHalMeg90]. What we contribute is the ease of using canonical models at the expense of an infinitary rule added to the axiomatic system. This enables combining qualitative probabilities with other propositional formalisms and also studying logics that have multiple qualitative probabilistic operators to pass the stage of establishing completeness and, sometimes, decidability, as a

[9] Let us note that we do not construct (a root-containing fragment of) a *free* unravelled model. This is because $\star$ is associative, and we assume no parentheses in $\star$-containing expressions. A consequence of this is that measures like $\mu^{\sigma_2}_{\langle w,\Lambda\star\tau\star\sigma_1\rangle}$ and $\mu^{\sigma_1\star\sigma_2}_{\langle w_0,\Lambda\star\tau\rangle}$ are concentrated over the same subsets of $W \times \Sigma_\varphi$.

routine step. The filtration technique that we have extended to capture the $\leq$-operator and the finite models that can be obtained for combined modal-and-probabilistic logics by using it also show that the infinitary rule is hardly a real burden.

The problem of fixing the **R** as the domain of probabilistic measures was also been encountered by Alëshina, who studied a logic with explicit probability operators $P_x,\ x \in [0,1]$, $P_x\varphi$ having the meaning of 'φ has probability x' [Ale93]. She chose to introduce the following rule:

$$\frac{\forall\ s \in [0,1] \setminus \{r\}\ \vdash P_{r_1}\varphi_1 \wedge \cdots \wedge P_{r_n}\varphi_n \to \neg P_s\psi}{\vdash P_{r_1}\varphi_1 \wedge \cdots \wedge P_{r_n}\varphi_n \to P_r\psi}$$

which just drives the details of the structure of **R** out of the scope of the axiomatic system. The rule we introduce, roughly speaking, enables a reproduction of Dedekind's construction of **R** by one-sided cuts in our canonical model.

We would also like to mention that the construction of measures in filtrations by solving linear systems suggests a simple proof of the rationalizing lemma, as it was stated and first proved by van der Hoek [Hoe92], for the study of **FSKD**-models, and enables a rather straightforward transfer of our filtration axioms to Kripke-semantics.

As for the logic containing multiple $\leq$-operators and probabilistic analogons of some of the dynamic operations, giving up the **D** axiom would also allow the introduction of an analogon of $\Box_{\varphi?}$. The latter is implicitly present in the definition of the supplementary ternary operators $\leq^{\varphi}_{\sigma,\tau}$, which can be viewed as analogical to the superpositions $\Box_\sigma\Box_{\varphi?}\Box_\tau$. Further extending this logic would let us treat as a decidable and application-oriented fragment of it the probabilistic PDL studied by Kozen [Koz85], whose work inspired ours.

What can also prove useful with respect to extensions is that (logically) infinite sets of propositions significantly extend the potential expressive power of our language. For instance any real probability value is definable in the following way:

$v, w \models \{\Pi_j \Rightarrow (R_{i,j} \leq \varphi) : \frac{i}{j} < \sqrt{2}\} \cup \{\Pi_j \Rightarrow (\varphi \leq R_{i,j}) : \frac{i}{j} > \sqrt{2}\}$ defines $\mu_w(v(\varphi)) = \sqrt{2}$.

This opens the way to translating probabilistic formalisms with explicit probabilies into the logic of qualitative probabilities now taken by Alëshina [Ale94].

Further, it would be interesting to study a formalism that treats infinite sets like $\{\Pi_j \Rightarrow (R_{i,j} \leq \varphi) : \frac{i}{j} < \sqrt{2}\}$ as single propositions that can be *approximated* by finite conjunctions of elements of theirs in a fashion similar to the one known for exponentials in linear logic [Tro92]. It would be interesting to study a deduction system for such a formalism and clarify the connection between derivable formulas containing this special kind of propositions and formulas, where their occurences are replaced by finite approximations. It would be a completion of the alanogy between this sketch and the achievements in linear logic to arrive at a result informally readable as: "For a given (desirable) approximation of the probabilities of some given propositions it is sufficient to have a certain (computable) approximation of the probabilities of some other given propositions, provided that the exact values of the probabilities of the second ones determine the exact values of the probabilities of the first ones.", or as: "A derivable formula that contains occurences of the special kind of subformulas remains derivable after replacing (some of) these occurences with approximations. The approximations can be arbitrary for specified occurences. For the other occurences the good enough approximations can be computed depending on the ones arbitrarily chosen."

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8 References

We give no references to basic texts in modal logic for facts we believe familiar to every specialist and reserve this distinguished entries in our list for the unpublished *Special Courses of Nonclassical Logics* of prof. Dimiter Vakarelov, whom we owe our first knowledge of modal logic.

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