Bisimulation and bounded bisimulation

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1 Introduction

In this paper the notion of bounded bisimulation is introduced to explore connections between work on Kripke models, non-well-founded sets and knowledge structures. The paper is organized as follows. The next section contains definitions of bisimulation and bounded bisimulation on Kripke models, which are the core notions in this paper. The next two sections contain definitions of non-well-founded models for modal logic, and of knowledge structures. Both of these have been proposed as alternatives to Kripke semantics. It is shown how these models can be seen as representing bisimulation classes and bounded bisimulation classes of Kripke models respectively. In the two sections after that we show how bounded bisimulation can be characterized by certain fragments of infinitary modal logic and by Ehrenfeucht games. The last section contains some examples of results proven by different authors in different contexts and shows how they can be seen as closely related using the notions introduced in this paper.

2 Bisimulation and bounded bisimulation

I start with some basic definitions to fix notation:

Definition 2.1 *Language*
Let a set of propositional variables $\mathcal{P}$ be given. The language of infinitary modal logic $\mathcal{L}_\infty$ is the smallest class such that: $\mathcal{P} \subseteq \mathcal{L}_\infty$, if ϕ in $\mathcal{L}_\infty$, then $\neg\phi$ and $\Diamond\phi$ are in $\mathcal{L}_\infty$, and if Φ is a subset of $\mathcal{L}_\infty$, then $\bigwedge \Phi \in \mathcal{L}_\infty$. □

Definition 2.2
A *Kripke model* K is a triple (W, R, V), where W is a set of indices, $R \subseteq W \times W$ is an accessibility relation, and $V : \mathcal{P} \mapsto \text{Pow}(W)$ is a valuation function that assigns to each atomic sentence in $\mathcal{P}$ a subset of W.

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A *pointed Kripke model* is a pair (K, w), where K is a Kripke model (W, R, V), and $w \in W$. $\Box$

From now on, K and K' will be silently understood to be the models (W, V, R) and (W', V', R') respectively.

We have defined here a Kripke model with a single accessibility relation. When Kripke models are used as models for multi-agent epistemic logic or transition systems, it is often useful to have more than just a single accessibility relation in the model. Most of the definitions and results in this paper can be easily generalized to apply to models with an arbitrary (finite) number of accessibility relations.

The definitions of satisfaction in a model are the standard ones:

$$\begin{array}{rcl} (K, w) \models p & \text{iff} & w \in V(p) \\ (K, w) \models \neg\phi & \text{iff} & (K, w) \not\models \phi \\ (K, w) \models \bigwedge\Phi & \text{iff} & (K, w) \models \phi \text{ for all } \phi \in \Phi \\ (K, w) \models \Diamond\phi & \text{iff} & \text{there is a } v \text{ such that } wRv \text{ and } (K, v) \models \phi \end{array}$$

Bisimulation has been proposed as a notion of equivalence between models in the context of modal logic by van Benthem (1976), and in the context of transition systems by Park (1981).

Definition 2.3 *Bisimulation*
A relation $B \subseteq W \times W'$ is a bisimulation between two models $K = (W, R, V)$ and $K' = (W', R', V')$ iff for all $w \in W$ and $w' \in W'$, if wBw' then:

1. $w \in V(p)$ iff $w' \in V'(p)$ for all $p \in \mathcal{P}$.
2. For all v such that wRv, there is a v' such that $w'R'v'$ and vBv'.
3. For all v' such that $w'R'v'$, there is a v such that wRv and vBv'.

Two pointed Kripke models (K, w) and (K', w') are *bisimilar*, $(K, w) \simeq (K', w')$, iff there is a bisimulation between K and K' connecting w with w'. $\Box$

Consider now the following construction:

Definition 2.4 *Bounded bisimulation.*[1]
Let α be an ordinal[2]. We define the notion of α-bisimulation by induction on α: two pointed Kripke models (K, w) and (K', w') are α-bisimilar, $(K, w) \simeq_\alpha (K', w')$, iff it holds that

1. $w \in V(p)$ iff $w' \in V'(p)$ for all $p \in \mathcal{P}$.

[1] Bounded bisimulation for ordinals smaller or equal to ω have been defined in Hennessy and Milner (1985) and several other places places. The generalization of the notion to arbitrary ordinals is new, as far as I know.

[2] We use von Neuman ordinals, i.e. if α is an ordinal, then $\alpha = \{\beta \mid \beta < \alpha\}$

2. For all $\beta < \alpha$, and for all v such that wRv, there is a v' such that $w'R'v'$ and $(K, v) \simeq_\beta (K, v')$,

3. For all $\beta < \alpha$, and for all v' such that $w'R'v'$, there is a v such that wRv and $(K, v) \simeq_\beta (K', w')$. □

Where it is not likely to lead to confusion, I will write $w \simeq_\alpha w'$ instead of $(K, w) \simeq_\alpha (K', w')$, and even write w instead of (K, w). Note that for each limit ordinal α it holds that $(K, w) \simeq_\alpha (K', w')$ iff $(K, w) \simeq_\beta (K', w')$ for all $\beta < \alpha$.

Bisimulation and bounded bisimulation are closely related. The following proposition shows that α-bisimulations are proper approximations of bisimulation.

Proposition 2.5

- $w \simeq w'$ iff $w \simeq_\alpha w'$ for all ordinals α.
- If $\alpha \leq \beta$, and B is a β-bisimulation between w and w', then B is an α-simulation between w and w'.
- For each α, there are w and w' such that $w \simeq_\alpha w'$ but $w \not\simeq w'$.

proof: For the first statement, it is easily seen that any relation that is an α-bisimulation for each α is a bisimulation. For the other direction, show by induction on α that each bisimulation is an α-bisimulation.

The proof of the second statement is completely straightforward, and we will proof the third statement together with proposition 7.4. □

3 Sets

Interestingly, there have been proposals to 'simplify' Kripke semantics by identifying bisimilar Kripke models. Examples are Aczel's (1988) models for process algebra, and Gerbrandy's (to appear) models for (multi-modal) epistemic logic. This idea finds its most elegant expression in the use of sets *simpliciter* as a model for modal logic in Barwise and Moss (1996).

Consider $V_{afa}[\mathcal{P}]$, Aczel's non-well-founded set-theoretic universe with a set $\mathcal{P}$ of urelements. We can interpret a modal language on such sets as follows: if an atomic sentence p is an element of a set, it is true in that set, otherwise, it is false, and the modal operator is interpreted by the 'element-of' relation between sets:

Definition 3.1 Let a be a set in $V_{afa}[\mathcal{P}]$.

$$\begin{array}{lcl} a \models p & \text{iff} & p \in a \\ a \models \neg\phi & \text{iff} & a \not\models \phi \end{array}$$

$$a \models \bigwedge\Phi \quad \text{iff} \quad a \models \phi \text{ for each } \phi \in \Phi$$
$$a \models \Diamond\phi \quad \text{iff} \quad b \models \phi \text{ for each set } b \in a$$

The following definition specifies a relation between sets and Kripke models:

Definition 3.2 Let (K, w) be a pointed Kripke model.
A *labeling* l of K is a function l that assigns to each index w in K a set $\{p \mid w \in V(p)\} \cup \{l(v) \mid wRv\}$. If l is a labeling of K and w is in K, then $l(w)$ is a *solution* of (K, w), and that (K, w) is a *picture* of $l(w)$. □

Now, assuming Aczel's axiom of anti-foundation (Aczel, 1988), we get the following results:

Proposition 3.3

- Each pointed Kripke model has exactly one set as its solution.
- Each set has a model as its picture.
- Two models are bisimilar iff they are pictures of the same set.
- If (K, w) is a picture of a, then for each $\phi \in \mathcal{L}_\infty$: $(K, w) \models \phi$ iff $a \models \phi$.

So we can see non-well-founded models as representing bisimulation equivalence classes of Kripke models.

4 Knowledge structures

Knowledge structures are models for epistemic logic that were proposed in Fagin, Halpern and Vardi (1991). It turns out that knowledge structures can be identified with α-bisimulation classes of pointed models in the same way as sets can be seen as representing bisimulation classes.

Definition 4.1 *Knowledge structures*
For each ordinal $\alpha \geq 0$, an α-knowledge structure is a function f on $\mathcal{P} \cup \alpha$ such that $f(p) \in \{0, 1\}$ for each $p \in \mathcal{P}$ (i.e. f restricted to $\mathcal{P}$ is a classical possible world), and for each $\beta < \alpha$, $f(\beta)$ is a set of β-knowledge structures. Moreover, each α-knowledge structure f has to satisfy *extension*, which is the following requirement:

- For all ordinals β and γ such that $\beta < \gamma < \alpha$ it holds that $g \in f(\beta)$ iff there is an $h \in f(\gamma)$ such that $g = h_{<\beta}$. (Where $h_{<\beta}$ is the restriction of h to $\mathcal{P} \cup \beta$.) □

In fact, Fagin et al. use a slightly more complex notion of a knowledge structure: they consider models for more more than one agent, and apart from extension, they postulate two other restrictions. On these structures, a satisfaction relation is defined (for a multi-modal language with common knowledge operators). The satisfaction relation has a rather cumbersome definition, which I will omit. However, the satisfaction relation suggests the following relation of correspondence between knowledge structures and Kripke models.

Definition 4.2 Let (K, w) be a model. By induction, we specify for each ordinal $\alpha > 0$ which α-knowledge structure f corresponds with (K, w).[3]

- The 0-knowledge structure corresponding with (K, w) is that function f on $\mathcal{P}$ such that $f(p) = 1$ iff $w \in V(p)$, and $f(p) = 0$ otherwise.
- Let g be the α-knowledge structure corresponding with (K, w). The $\alpha+1$-knowledge structure f that corresponds with (K, w) is such that $f_{<\alpha} = g$ and $f(\alpha)$ is the set of α-knowledge structures that correspond with (K, v) for each v such that wRv.
- If λ is a limit ordinal, then the λ-knowledge structure f that corresponds with (K, w) is such that for each $\beta < \lambda$: $f_{<\beta}$ is the β-knowledge structure corresponding with (K, w). □

It is not hard to check that all this is well-defined, and extension is satisfied where it needs to be.

We now have the following relation between knowledge structures and Kripke models, and we have a new characterization for α-bisimulation.

Proposition 4.3 For each ordinal α:

- Each Kripke model corresponds with a unique α-knowledge structure.
- For each α-knowledge structure there is a corresponding Kripke model.
- $(K, w) \simeq_\alpha (K', w')$ iff (K, w) and (K', w') correspond with the same α-knowledge structure.
- For each α and each $\phi \in \mathcal{L}_\alpha$ (cf. definition 5.1, $(K, w) \models \phi$ iff ϕ is satisfied in the corresponding α-knowledge structure.

proof: All these results are proven by Fagin (1994), except for the third item, which has a straightforward proof. □

This shows that knowledge structures are related in the same way to bounded bisimulation equivalence classes as non-well-founded models are related to bisimulation equivalence classes.

[3] Basically the same construction can be found in Fagin (1994).

5 Languages

Bounded bisimulation is naturally characterized by certain fragments of $\mathcal{L}_\infty$: two Kripke models are α-bisimilar just in case they satisfy the same sentences of $\mathcal{L}_\infty$ up to depth α:

Definition 5.1

- Define the depth of a formula of $\mathcal{L}_\infty$ inductively as follows: depth(p) = 1; depth$(\neg\phi)$ = depth(ϕ), depth$(\Diamond\phi)$= depth(ϕ) + 1 and depth$(\bigwedge\Phi) = \bigcup\{$depth$(\phi) \mid \phi \in \Phi\}$.
- For each ordinal α,the language $\mathcal{L}_\alpha$ is the class of all sentences of $\mathcal{L}_\infty$ of depth $\leq \alpha$.

Proposition 5.2 Let $(K, w) \equiv_\alpha (K', w')$ iff for each sentence ϕ of $\mathcal{L}_\alpha$, $(K, w) \models \phi$ iff $(K', w') \models \phi$. Then $(K, w) \equiv_\alpha (K', w')$ iff $(K, w) \simeq_\alpha (K', w')$.

proof:

[$\Rightarrow$] We prove by induction on the structure of sentences ϕ of $\mathcal{L}_\infty$ that if ϕ is of depth α, $w \models \phi$ and $w \simeq_\alpha w'$, then $w' \models \phi$.

The only nontrivial case is $\Diamond\phi$. Let depth$(\phi) = \alpha$. Then depth$(\Diamond\phi)$ is $\alpha+1$. Assume $w \simeq_{\alpha+1} w'$. If $w \models \Diamond\phi$ then there is a v such that wRv and $v \models \phi$. Since w and w' are α-bisimilar, there is a v' such that $w' \simeq_\alpha v'$ and $w'R'v'$, and hence, by induction hypothesis, $v' \models \phi$. So $w' \models \Diamond\phi$.

[$\Leftarrow$] We show by induction on α that $\equiv_\alpha$ is an α-bisimulation.

$\alpha = 0$: immediate.

$\alpha+1$: Assume $w \equiv_{\alpha+1} w'$. Take any v such that wRv. Assume that for no v' such that $w'Rv'$ it holds that $v \simeq_\alpha v'$. Then, by induction hypothesis, for each v' such that $w'Rv'$ there is a $\phi_{v'} \in \mathcal{L}_\alpha$ such that $v \models \phi_{v'}$ but $v' \not\models \phi_{v'}$. Then $w \models \Diamond \bigwedge\{\phi_{v'} \mid w'Rv'\}$, which is an $\mathcal{L}_{\alpha+1}$-sentence. But $w' \not\models \Diamond \bigwedge\{\phi_{v'} \mid w'Rv'\}$, contradicting our assumption.

Assume $w \equiv_\lambda w'$ for some limit ordinal λ. Then, a forteriori, $w \equiv_\beta w'$ for all $\beta < \lambda$. So, by induction hypothesis, $w \simeq_\beta w'$ for all $\beta < \lambda$, from which it follows by definition of λ-bisimulation that $w \simeq_\lambda w'$. □

The language $\mathcal{L}_\omega$ is of course the classical fragment of $\mathcal{L}_\infty$. It is well-known that the classical modal language $\mathcal{L}_\omega$ can distinguish between any two non-ω-bisimilar models, and $\mathcal{L}_\infty$ corresponds to bisimulation. We get these results as corollaries of the above:

Corrolary 5.3

- $(K, w) \simeq_\omega (K', w')$ iff for all $\phi \in \mathcal{L}_\omega$: $(K, w) \models \phi$ iff $(K', w') \models \phi$.
- $(K, w) \simeq (K', w')$ iff for all $\phi \in \mathcal{L}_\infty$: $(K, w) \models \phi$ iff $(K', w') \models \phi$. (cf. e.g. van Benthem (1991))

6 Ehrenfeucht Games

Bounded bisimulations can also be characterized by Ehrenfeucht games.

We define an *Ehrenfeucht game bounded by* α (or the α-game for short) for each ordinal α by induction on α.[4] An α-game is played with two pointed models (K, w) and (K', w'), by two players (they are called the *spoiler* and the *duplicator*).

A 0-game simply consists of the spoiler choosing w and the duplicator answering w'. The duplicator wins iff $V(w) = V'(w')$.

An $\alpha + 1$-game is played as follows: the spoiler must choose a u from one of the models such that either wRu or $w'Ru$, and the duplicator must reply with an index from u' the other model. The players then proceed to play the α-game on (K, u) and (K', u') (or on (K, u') and (K', u), depending on the spoiler's choice). The duplicator wins just in case $V(w) = V'(v)$ and he chooses u' such that vRu' and he wins the α-game on (K, u) and (K', u') and

For limit ordinals α, the spoiler starts by choosing any ordinal $\beta < \alpha$, and the players play the β-game. The duplicator wins iff he wins this β-game.

Note that, interestingly, each of these bounded games is finished after a finite number of moves. The following proposition shows that α-games can be used to characterize α-bisimulation:

Proposition 6.1 For each ordinal α:
The duplicator has a winning strategy for the α-game on (K, w) and (K', w') iff $(K, w) \simeq_\alpha (K', v)$

proof:

[$\Rightarrow$]. We prove this by induction on α. If $\alpha = 0$, then we get immediately that $(K, w) \simeq_0 (K', w')$. Assume that duplicator has a winning strategy for the $\alpha + 1$-game. Then, for each successor v of w there is a v' that is a successor of w' such that the duplicator has a winning strategy for the α-game on (K, v) and (K', v'), and vice versa. So, by induction hypothesis and the definition of $\alpha+1$-bisimulation, (K, w) and (K', w') are $\alpha+1$-bisimilar. If α is a limit ordinal, then the duplicator has a winning strategy for the α game just in case for each $\beta < \alpha$, he has a winning strategy for each β-game. But then, by induction hypothesis, $(K, w) \simeq_\beta (K', w')$ for each $\beta < \alpha$, and hence, $(K, w) \simeq_\alpha (K', w')$.

[$\Leftarrow$]. We show by induction on α that if $(K, w) \simeq_\alpha (K', w')$, then the duplicator has a winning strategy for the α-game on (K, w) and (K', w'). If $\alpha = 0$, the result is immediate. For an $\alpha + 1$-game, assume that $(K, w) \simeq_{\alpha+1} (K', w')$, and the spoiler chooses v from K in his second move. Then wRv by the rules of the game, so, since $(K, w) \simeq_{\alpha+1} (K', w')$, the duplicator can find a v' such that $w'Rv'$ and (K, v) and (K', v') are α-bisimilar. By induction hypothesis, then, the duplicator has a winning strategy for the α-game played on (K, v) and (K', v'), and we are finished.

[4] In his dissertation, Kees Doets defines similar games for first-order predicate logic.

If α is a limit ordinal and $(K, w) \simeq_\alpha (K', w')$ then, $(K, w) \simeq_\beta (K', w')$ for each $\beta < \alpha$. By induction hypothesis, the duplicator has a winning strategy for each β-game, where $\beta < \alpha$. So no matter which β is chosen by the spoiler, the duplicator wins the β-game, and hence, he wins the α-game. □

An unbounded game is defined simply as follows: the spoiler chooses any ordinal α, and the players play the α-game.

Proposition 6.2 The duplicator has a winning strategy for the unbounded game on (K, w) and (K', w') iff $(K, w) \simeq (K', w')$.

proof: Immediate from propositions 6.1 and 2.5. □

7 Some results

This section contains two examples of results that can be seen to be related by the interconnections noted above.

Many of the following results regarding the notion of rank are similar to those proven in Barwise and Moss (1996) about the 'degree' of a set (which is defined as the minimal depth of a characterizing sentence), and in Fagin (1994) about 'uniqueness ordinals' of Kripke structures (which is defined in terms of the length of its shortest characterizing knowledge structure).

Definition 7.1 The *rank* of a model (K, w), $\text{rank}(K, w)$, is the smallest ordinal α such that for all (K', w'): $(K, w) \simeq_\alpha (K', w')$ iff $(K, w) \simeq (K', w')$. □

A notion similar to that of rank occurs under the name 'uniqueness ordinal' in Fagin (1994), which is defined as the least ordinal α such that if f is the α-structure corresponding to (K, w), then for each $\beta \geq \alpha$, there is a unique β-knowledge structure g such that $g_{<\alpha} = f$. With the results of proposition 4.3, it is easily seen that the uniqueness ordinal of a model is exactly the rank of that model. Barwise and Moss define the *degree* of a set a as the least ordinal α such that $\mathcal{L}_\alpha$ characterizes a up to uniqueness. From propositions 3.3 and 5.2 it follows that the rank of a model is exactly the degree of the corresponding set. In both of the just quoted papers a result like the following is proven.

Proposition 7.2 Each model has a rank.

proof: Take any model (K, w). Since W is a set, we can use proposition 2.5 to conclude that there must be some ordinal γ such that for each w and v in K: $w \simeq_\gamma v$ iff $w \simeq v$. Let α be the smallest limit ordinal $> \gamma$. We show that for each (K', w') it holds that $(K, w) \simeq_\alpha (K', w')$ iff $(K, w) \simeq (K', w')$.

Assume $(K, w) \simeq_\alpha (K', w')$. We show that $\simeq_\alpha$ is a bisimulation between (K, w) and (K', w').

Take any v such that wRv. Since $(K, w) \simeq_\alpha (K', w')$ there must be a v' such that $v \simeq_\gamma v'$ and $w'Rv'$. We show that $v \simeq_\alpha v'$. Since α is a limit ordinal,it is enough to show that $v \simeq_\beta v'$ for each $\beta < \alpha$.

Clearly, $v \simeq_\beta v'$ for all $\beta \leq \gamma$. So take any β such that $\gamma < \beta < \alpha$. Then, since $w \simeq_\alpha w'$, there is a u such that $u \simeq_\beta v'$. But then, $u \simeq_\gamma v$ (since $\gamma < \beta$), so $u \simeq v$ (that's how we chose γ), so $v \simeq_\beta v'$.

For the other bisimulation clause, take any v' such that $w'R'v'$. Then, since $\gamma < \alpha$, there is a v such that wRv and $v \simeq_\gamma v'$. We show that $v \simeq_\alpha v'$. For take any β such that $\gamma < \beta < \alpha$. Then there is a $u \simeq_\beta v'$ such that wRu. But then $u \simeq_\gamma v$, so $u \simeq v$, and hence $v \simeq_\beta v'$. □

Corrolary 7.3 Each (K, w) can be characterized up to bisimulation by a single $\mathcal{L}_\infty$-sentence (and each set can be characterized up to uniqueness by a single sentence (Barwise and Moss), and each Kripke model can be completely characterized by a single knowledge structure (Fagin))

proof: Because the class of all α-bisimulation classes is a set, we can use proposition 5.2 and conclude that $\mathcal{L}_\alpha$ is a set, up to semantical equivalence. Let $\text{rank}(K, w) = \alpha$. Then, using proposition 5.2, (K, w) is characterized up to bisimulation by its $\mathcal{L}_\alpha$-theory $\{\phi \in \mathcal{L}_\alpha \mid (K, w) \models \phi\}$. Since $\mathcal{L}_\alpha$ is, up to equivalence, a set, $\bigwedge\{\phi \in \mathcal{L}_\alpha \mid (K, w) \models \phi\}$ is a sentence characterizing (K, w) up to bisimulation. □

Proposition 7.4
For each successor ordinal α, there is a model with rank α.

proof: We will proof this proposition, and proposition 2.5 in one go. The question whether there is a model of rank α for each limit ordinal α is still open.

We will use sets instead of Kripke models to proof the proposition: in fact, we will show that each ordinal α has degree $\alpha + 1$. By proposition 3.3, we can find a Kripke model that corresponds to α, and using the same proposition, we know that also this Kripke model has rank $\alpha + 1$.

We will show that (1) if $\alpha \leq \beta$ then $\alpha \simeq_\alpha \beta$, and that (2) if $\alpha > \beta$ then $\alpha \not\simeq_\alpha \beta$. This proves the third part of proposition 2.5. Moreover, we will show that (3) $\text{rank}(\alpha) = \alpha + 1$.

(1) If $\alpha \leq \beta$, then $\alpha \simeq_\alpha \beta$.
Take any α and β such that $\alpha \leq \beta$. Then $\alpha \subseteq \beta$, so one direction of the bisimulation clause is immediate. For the other direction, we need to show that for each $\gamma < \alpha$ and each element b of β, we can find a element a of α such that $a \simeq_\gamma b$. So take any $\gamma < \alpha$ and any $b \in \beta$. Then $b = \delta$ for some $\delta < \beta$. Distinguish two cases: (i) $\delta < \gamma$, but then δ is an element of α, and we are finished, and (ii) $\delta \geq \gamma$, but then we can simply choose γ as our witness, which is an element of α and which by induction hypothesis is γ-bisimilar to δ.

(2) If $\beta < \alpha$, then $\beta \not\simeq_\alpha \alpha$.
Take any β and α such that $\beta < \alpha$, and assume to the contrary that $\beta \simeq_\alpha \alpha$. Because $\beta < \alpha$, $\beta \in \alpha$, and so there must be an $\gamma \in \beta$ such that $\gamma \simeq_\beta \beta$. But $\gamma \in \beta$ implies that $\gamma < \beta$, while the induction hypothesis says that $\gamma \simeq_\beta$ implies that $\beta \leq \gamma$.

(3) $\text{rank}(\alpha) = \alpha + 1$.
Let α be any ordinal, and take any set a and assume that $a \simeq_{\alpha+1} \alpha$. Then for each element of a there is an α-bisimilar element of α and vice versa. But by induction hypothesis each $\beta \in \alpha$ is of rank smaller or equal to α, so it follows that each element of a must be bisimilar to some successor of α and vice versa. Hence, α and a are bisimilar. So, $\text{rank}(\alpha) \leq \alpha + 1$.

From (1), it follows that $\text{rank}(\alpha) > \alpha$, so we have shown that the rank of α is $\alpha + 1$. □

From proposition 2.5, we have the corresponding negative result that no $\mathcal{L}_\alpha$-language characterizes bisimulation:

Corrolary 7.5 For each ordinal α, there are (K, w) and (K', w') such that $(K, w) \equiv_\alpha (K', w')$, but $(K, w) \not\simeq (K, v)$ (and similarly for sets) □

We now proceed to look at another result:

Proposition 7.6
$\text{rank}(K, w) \leq \omega$ iff (K, w) is bisimilar to a finitely branching model.

Similar results have been proven, independently as far as I know, by Alexandru Baltag in the context of non-well-founded set theory (reported in Barwise and Moss, 1996), by Fagin (1994) where the proposition is about knowledge structures, and in Hollenberg (1995) , where the proposition appears as a statement about Hennessy-Milner classes.

I will first give a direct proof of proposition 7.6 (loosely based on that of Fagin, 1994), then mention the different formulations of this theorem and explain how they are related to the present one using the techniques defined in this paper.

proof:

[$\Rightarrow$] Assume $\text{rank}(K, w) \leq \omega$, but (K, w) is not bisimilar to a finitely branching model, and assume without loss of generality that w has infinitely many successors. Let f be the $\omega + 1$-knowledge structure corresponding to (K, w). Since w has infinitely many non-bisimilar successors and $\text{rank}(K, w) \leq \omega$, w has infinitely many non-ω-bisimilar successors. Therefore, $f(\omega)$ contains infinitely ω-knowledge structures g (cf. proposition 4.3). Consider the function h on ω given by: $h(n) = \{g_{<n} \mid \text{there are infinitely many } g' \in f(\omega) \text{ such that } g'_{<n} = g\}$. Because there are only finitely many different n-knowledge structures for each $n < \omega$, and $f(\omega)$ is infinite, h satisfies extension, and hence it is an ω-knowledge structure.

Consider now the $\omega + 1$-knowledge structures f^1 and f^2 given by $f^1_{<\omega} = f^2_{<\omega} = f_{<\omega}$, and $f^1(\omega) = f(\omega) \cup \{h\}$, $f^2(\omega) = f(\omega)/\{h\}$. Then either f^1 or f^2 is our original f, so one of them is the $\omega + 1$-structure corresponding to (K, w); say it is f^1. By proposition 4.3, f^2 corresponds to a model (K', w'). Since $f^2_{<\omega} = f^1_{<\omega}$, (K, w) and (K', w') correspond to the same ω-knowledge structure, so they are ω-bisimilar. But they are not $\omega+1$-bisimilar, contradicting our assumption that $\mathrm{rank}(K, w) \leq \omega$.

[$\Leftarrow$]. Assume that (K, w) is bisimilar to a finitely branching model. Take this finitely branching model, and proceed as in the proof of proposition 7.2. $\Box$

Barwise and Moss formulate the following result, which they call Baltag's second theorem:

- A set a is uniquely characterized by a theory of $\mathcal{L}_\omega$ iff $a \in HF^1[\mathcal{P}]$.

The class $HF^1[\mathcal{P}]$ is the largest class of hereditary finite sets with $\mathcal{P}$ as urelements, i.e. it is the largest class of (non-well-founded) sets built from $\mathcal{P}$ such that $a \in HF^1[\mathcal{P}]$ iff a is finite, and for all $b \in a$: $b \in HF^1[\mathcal{P}]$.

To see what this result has to do with our proposition, note that by proposition 5.3 that a model is characterized up to bisimulation by a theory of $\mathcal{L}_\omega$ iff it is of rank $\leq \omega$. By proposition 3.3 it follows that the solution of a model is uniquely characterized by an $\mathcal{L}_\omega$ theory iff the model is of rank $\leq \omega$. Since the class $HF^1[\mathcal{P}]$ is exactly the class of solutions of models that are bisimilar to a finitely branching model, the result above follows.

A Hennessy-Milner class is any class of models $\mathcal{C}$ such that for all (K, w) and (K', w') in $\mathcal{C}$, $(K, w) \simeq (K', w')$ iff $(K, w) \equiv_\omega (K', w')$. A class $\mathcal{C}$ of models is a maximal Hennessy-Milner class iff for each model (K, w) that is not in $\mathcal{C}$, $\mathcal{C} \cup \{(K, w)\}$ is not a Hennessy-Milner class.

One of the results in Hollenberg (1995) is the following:

- The intersection of all maximal Hennessy-Milner classes is exactly the class of models that are bisimilar to an image-finite model.

This proposition is related to ours in the following way. First note that being 'image-finite' is another term for being 'finitely branching'. We show that the class of models of rank $\leq \omega$ is the intersection of all maximal Hennessy-Milner classes.

Assume $\mathrm{rank}(K, w) \leq \omega$. Let $\mathcal{C}$ be any maximal Hennessy-Milner class. Since $\mathrm{rank}(K, w) \leq \omega$, it holds that for each $(K', w') \in \mathcal{C}$ that $(K, w) \equiv_\omega (K', w')$ iff $(K, w) \simeq (K', w')$. So $\mathcal{C} \cup \{(K, w)\}$ is a Hennessy-Milner class, and hence $(K, w) \in \mathcal{C}$ by maximality.

Assume $\mathrm{rank}(K, w) > \omega$. Then there is a model (K', w') such that $(K, w) \equiv_\omega (K', w')$, but $(K, w) \not\simeq (K', w')$. Let $\mathcal{C}$ be a Hennessy-Milner class that contains (K', w'). Then clearly, $(K, w) \notin \mathcal{C}$, and hence (K, w) is not in the intersection of all maximal Hennessy-Milner classes.

Fagin (1995, page 232) formulates and proofs the following result:

- The uniqueness ordinal of a Kripke model with at most finite fanout is at most ω. Conversely, if K is a nonflabby Kripke model with uniqueness ordinal at most ω, then K has finite fanout.

We have seen earlier how the 'uniqueness ordinal' is defined: it corresponds to our notion of rank. A Kripke-model is in the extension of the rather unfortunate term 'nonflabby' iff for all indices w, w' in K, there is an $\mathcal{L}_\infty$-sentence ϕ such that $w \models \phi$, but $w' \not\models \phi$. The notion of 'having finite fanout' is of course the same as 'being finitely branching'.

That Fagin's proposition follows from ours can be seen by showing for the left-to-right part that each model is bisimilar to a 'nonflabby' model by simply 'dividing out' by $\simeq$, which results in a nonflabby model by proposition 5.3. For the other direction, note that if two Kripke models are bisimilar, then they have the same 'uniqueness ordinal'.