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MODAL DERIVATION RULES

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Modal Derivation Rules

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Abstract

We define the non-ξ rule as a derivation rule for multi-modal logics. This rule is a generalized version of Gabbbay's irreflexivity rule. We prove a meta-theorem on completeness, of the following kind:
If Λ is a derivation system having a set of axioms that are special Sahlqvist formulas, and Λ^+ is the extension of Λ with a set of non-ξ rules, then Λ^+ is strongly sound and complete with respect to a class of frames determined by the axioms and the rules.

Contents

1. Introduction

Let us for the moment consider the simplest tense similarity type[1] with two operators F and P. It is well-known that the logic K^tL, being the extension of the basic tense logic K^t with the axioms

(4) $FFp \rightarrow Fp$

(H) $Fp \rightarrow G(Fp \vee p \vee Pp)$

$(H^\smile)$ $Pp \rightarrow H(Pp \vee p \vee Fp)$

completely axiomatizes the class LO of linear orders. Adding the axiom (T) $Gp \rightarrow p$ then gives a complete axiomatization of the class LR of *reflexive* linear orderings.

Now suppose we want to axiomatize the class LI of *irreflexive* linear frames. There is no modal or tense formula corresponding to irreflexivity in the same manner as (4), (H) and (T) correspond to resp. transitivity, linearity and reflexivity. So (in principle) LI is much harder to axiomatize than LR or LO. The usual procedure, establishing the completeness of K^tL itself[2] for LI, consists of starting with some model M for a consistent set of formulas Σ and then transforming M into an *irreflexive* model M' for Σ.

A different road was taken by Gabbay in [5], where he suggested to add to K^tL a special derivation rule, which he baptized the *irreflexivity rule*. This rule can be formulated as follows:

(IR) If $\vdash \neg(Gp \rightarrow p) \rightarrow \phi$ and p does not occur in ϕ, then $\vdash \phi$.

Gabbay's completeness proof then consists of constructing a linear irreflexive model right away, without passing models that may be bad in the sense that they have reflexive points.

This idea was followed by many authors who wanted to give axiomatizations for classes of frames defined by conditions which are not directly expressible in the intensional language. Examples include [4, 31] for branching-time temporal logics, [24, 27] for modal logics of intervals, [14, 25, 26] for many-dimensional modal logics, and [7, 17, 18]. There is an independent Bulgarian line of papers [16, 8, 9, 8] where similar rules are used in a context of enriched modal formalisms. Finally, in the temporal logic of program verifications there is a related concept called 'clock rule' (cf. [1, 21] and the references therein).

So the question naturally arises whether anything general can be said about

[1] We follow the conventions in non-classical logics and its semantics as laid down in e.g. [10, 6]. However, we want to be quite general in the sense that we consider languages with arbitrarily many operators, sometimes of arbitrary adity. We use the term *(modal) similarity type* to range over these situations. The reader unfamiliar with non-monadic modal operators may just read 'set of diamonds' instead of 'similarity type'.

[2] In this sense the example is not representative: For K^tL, the irreflexivity rule is *conservative* (cf. section 7).

logics having rules like the irreflexivity rule. Let us first have a closer look at (IR); we suggest to concentrate on the 'converse' statement, i.e.

> If ϕ is consistent and does not use p,
> then $\phi \wedge \neg(Gp \to p)$ is consistent.

In other words, to a consistent formula ϕ we may always add a conjunct of the form $\neg(Gp \to p)$ *witnessing* irreflexivity.

More general, we set

Definition 1.1.
Let ξ be a modal formula in the proposition letters $p_0, \ldots, p_{n-1}$. For a class K of frames, set $\mathsf{K}_{-\xi}$ as the class of *non-ξ frames* in K, i.e. the frames $F = (W, R)$ in K such that for no world w in W, $F, w \models \xi$. For Ξ a set of formulas, $\mathsf{K}_{-\Xi}$ is the intersection of the $\mathsf{K}_{-\xi}$, $\xi \in \Xi$. ⊞

Note that in general, the following three classes of frames, all defined using the negation of ξ, are *distinct*:
(i) $\mathsf{K}_{\neg\xi}$ (i.e. the class of frames with $F \models \neg\xi$.)
(ii) $\overline{\mathsf{K}_{\xi}}$ (i.e. the complement of K_{ξ}.)
(iii) $\mathsf{K}_{-\xi}$
For, F is in $\mathsf{K}_{\neg\xi}$ iff *for all* valuations V and *all* worlds w, $F, V, w \models \neg\xi$, F is in the second class iff *there are* a valuation V and a world w with $F, V, w \models \neg\xi$, and $F \in \mathsf{K}_{-\xi}$ means that *for every* world w *there is* a valuation V with $F, V, w \models \neg\xi$. This means, so to speak, that $-\xi$ 'corresponds' to the second order formula

$$\forall x_0 \exists P_0 \ldots P_{n-1} \neg\xi^1(x_0)$$

where $\xi^1(x_0)$ is the local first order model correspondent[3] of ξ, every monadic predicate P_i being the first order counterpart of the propositional variable p_i in ξ. Thus we are studying classes of frames that are definable in a version of second order logic where we have a restricted possibility to use existential quantification over monadic predicates.

As an example, consider the formula $\xi = Gp \to Pp$ which is locally equivalent on the frame level to $\exists y(Rxy \wedge \breve{R}xy)$. So $\mathsf{K}_{-\xi}$ is the class of frames F with $F \models \forall x \forall y(Rxy \to \neg\breve{R}xy)$ i.e. the class of *asymmetric* frames, while $\overline{\mathsf{K}_{\xi}}$ is the class of frames with $F \models \exists x \forall y(Rxy \to \neg\breve{R}xy)$. The negation $Gp \wedge H\neg p$ of ξ can be shown to be globally equivalent to the formula $\neg\exists x \exists y Rxy$, so $\mathsf{K}_{\neg\xi}$ finally is the class of frames with empty R.
As another example, one can show $\mathsf{K}_{-(Gp \to FFp)}$ to be the class of *intransitive* frames. In these two examples the second order definition of $\mathsf{K}_{-\xi}$ can be replaced by a first order one, but this need not always be the case.

[3] cf. [2, 3]. If $\xi(p_1) \equiv \Diamond\Box p$, then $\xi^1(x_0) \equiv \exists x_1(R_\Diamond x_0 x_1 \wedge \forall x_2(R_\Diamond x_1 x_2 \to P_1 x_2))$.

Now suppose we want to axiomatize the logic $\Theta(\mathsf{K}_{-\xi})$ consisting of all formulas valid in $\mathsf{K}_{-\xi}$. Let ϕ be a $\Theta(\mathsf{K}_{-\xi})$-consistent formula, then there is a model $M = (F, V)$ such that F is in $\mathsf{K}_{-\xi}$ and with a world w in M where $M, w \models \phi$. Let $p_0, \ldots, p_{n-1}$ be *new* propositional variables, in the sense that they are not elements of $Dom(V)$. As $F, w \not\models \xi$, there is a valuation V' such that $F, V', w \models \neg\xi(p_0, \ldots, p_{n-1})$. Now let V'' be defined by

$$\begin{array}{rcll} V''(q) & = & V(q) & \text{if } q \in Dom(V) \\ V''(p_i) & = & V'(p_i) & \text{for } i = 0, \ldots, n-1. \end{array}$$

then clearly we have $(F, V''), w \models \phi \wedge \neg\xi$.

This means that

> $\phi \wedge \neg\xi(p_0, \ldots, p_{n-1})$ is $\Theta(\mathsf{K}_{-\xi})$-consistent if ϕ is $\Theta(\mathsf{K}_{-\xi})$-consistent and none of the p_i occurs in ϕ.

Taking the converse again of the above proposition, we have a formulation of the $\neg\xi$-consistency rule:

Definition 1.2.
Let $\xi(p_0, \ldots, p_{n-1})$ be a modal formula. The $\neg\xi$- *consistency rule*, or shorter: the *non-ξ rule* is the following derivation rule:

$(N\xi R)$ $\quad\vdash \neg\xi(p_0, \ldots, p_{n-1}) \rightarrow \phi \;\Rightarrow \vdash \phi$
provided none of the p_i occurs in ϕ.

⊞

The paragraph above definition 1.2 can be seen as a proof of the *soundness* of NξR with respect to $\mathsf{K}_{-\xi}$: if $\mathsf{K}_{-\xi} \models \neg\xi(p_0, \ldots, p_{n-1}) \rightarrow \phi$ and no p_i occurs in ϕ, then $\mathsf{K}_{-\xi} \models \phi$.
The aim however is of course to try and show *completeness* for non-ξ rules; this will be the main subject of this paper. We should note at this moment that in general we do not have an isolated NξR added to a minimal (tense) logic, but a situation in which we add possibly more than one NξR to a logic having other axioms besides the basics.
So the general context is the following: we have a similarity type S, an S-logic Λ which is (strongly) sound and complete with respect to a class of frames K, and a set of formulas Ξ. Let Λ^+ be the logic obtained by adding the non-ξ rule to Λ for all $\xi \in \Xi$. The question now is the following

> Is Λ^+ strongly complete with respect to $\mathsf{K}_{-\Xi}$?

In [5], Gabbay proves a generalized irreflexivity lemma stating that a Λ^+-consistent set Σ of formulas has a model M with $M \models \Theta(K_{\Lambda,-\Xi})$. Unfortunately, this is not enough to prove completeness, for we have to find a model

M such that the underlying *frame* is in $\mathsf{K}_{-\Xi}$.
In general this seems to be difficult and maybe even impossible to establish. Therefor we concentrate on logics with a special, nice kind of axioms, viz. so-called Sahlqvist formulas. For some of these logics we can get a positive answer to the above question. The answer we obtain is partial because our proof method will turn out to be highly sensitive to the similarity type of the logic. In particular, and maybe surprisingly, there is a striking difference in our approach between *tense* similarity types (i.e. where the language has a 'converse' operator for each of its diamonds) and uni-directional ones (where no operator has a converse).
Furthermore, we feel our proofs become more perspicuous if we add a special operator, the so-called *difference operator*, to the language. In many applications this will turn out to be only an apparent extension of the language because the operator is *definable* in the old language, at least over the class of frames that we want to axiomatize.

The organization of this paper is as follows:

In the next section we define and discuss the set of formulas that are admissible as axioms in our completeness result. As a corollary of the proof method, we can give a perspicuous formulation of the algorithm producing the first order equivalent of a Sahlqvist formula. After that, we give some definitions and facts concerning the difference operator. Section 4 contains the basic idea of our approach, for tense similarity types. In section 5 we show what goes wrong in a context where not every diamond has its converse in the language. In section 6 we state and prove the general theorem, and in the last section we draw our conclusions, give a motivating example and pose some questions.

The main results of this paper were obtained in the winter of 1990/1991, while the author was visiting the Department of Computing at Imperial College in London, on a grant from the Erasmus schedule nr. ICP–90–NL–0211. We would like to thank Dov Gabbay, Ian Hodkinson and the other members of the temporal logic group for stimulating discussions on modal derivation rules and providing a very encouraging research climate. We are also indebted to Maarten de Rijke for a thorough and inspiring discussion on the ins and outs of Sahlqvist's theorem.

Putting the finishing touch to this paper, we received a letter from V. Goranko announcing a similar and even more general result than ours. In the future we hope to be able to compare our proof with his.

2. Sahlqvist tense formulas.

It is well-known[4] that on the level of frames every formula ϕ locally and globally has a second order equivalent ϕ^2. In many important cases however, it turns out that this formula ϕ^2 has a much simpler first order equivalent (in L_S). Well-known examples include reflexivity for $p \to \Diamond p$ and the Church-Rosser propeerty for $\Diamond\Box p \to \Box\Diamond p$. A general theorem in this direction was found by Sahlqvist (cf. [19]). The *correspondence* part of Sahlqvist's theorem gives a decidable set of modal S-formulas having a local equivalent in L_S. In [3], van Benthem provided a quite perspicuous algorithm to find this first order correspondent ϕ^s of a Sahlqvist formula ϕ. (At the end of this section, we will give our version of this *substitution method.*) The second, *completeness* part of the Sahlqvist theorem states that adding a set Σ of Sahlqvist axioms to the minimal S-logic K_S, we obtain a complete axiomatization for the class of frames K_Σ. An accessible version of the proof of this part can be found in [22], from which we have borrowed much of the terminology in this section. The essential observation in the proof is that Sahlqvist logics are canonical, or equivalently, that Sahlqvist formulas have the following property:

> If $G = (F, A)$ is a descriptive general frame such that $G \models \sigma$, then $F \models \sigma$.

In this section we will prove a related result, for a subset of the Sahlqvist formulas. In fact we will show that van Benthem's substitution method (which deals with Kripke frames) also works for a certain class of *general* frames to be defined later on.

In the remainder of this section we fix a similarity type S. A crucial distinction will be made among the *diamonds* of S, between the uni-directional ones and those of which the converse diamond also belongs to S.

Definition 2.1.
Assume that a subset T of the diamonds of S is given as $T = \{F_j, P_j \mid j \in J\}$. Diamonds in this set are called *tense diamonds*, their duals *tense boxes*. We call F_j the *converse* of P_j and the other way round. If $\Diamond$ is a tense diamond, its converse is denoted by $\Diamond\check{}$. A diamond that is not in T is called *uni-directional.* If a similarity type has only tense operators, we call it a *tense similarity type.* A frame $(F, R_\nabla)_{\nabla\in S}$ for S is called a *tense frame* if for every $\Diamond \in T$, the accessibility relations of $\Diamond$ and $\Diamond\check{}$ are each other's converse, i.e. $R_{\Diamond\check{}} = (R_\Diamond)\check{}$. ⊞

With emphasis, we want to note that the above definition should be understood

[4] cf. [2, 3]. If ξ^1 is the model correspondent of $\xi(p_1)$, then ξ^2 is (locally) $\forall P_1\xi^1(x_0)$, (globally) $\forall x_0\forall P_1\xi^1(x_0)$.

as to include the case where a modal operator is its *own* converse.

Definition 2.2.
A *strongly t(ense)-positive formula* is a conjunction of formulas $\Box_1 \ldots \Box_m p_i$ $(m \geq 0)$ where every $\Box_j$ is a tense box. A formula is *positive* (*negative*) if it is obtained from propositional variables (resp. negations of propositional variables), constants and negations of constants, by applying $\vee$, $\wedge$, existential modal operators and their duals. A modal formula is *untied tense* if it is obtained from strongly positive formulas and negative ones by applying only $\wedge$ and arbitrary existential modal operators. Formulas of the form $\phi \rightarrow \psi$ with ϕ an untied tense formula and ψ a positive one, are called *basic Sahlqvist tense* formulas. *Sahlqvist tense* formulas, or shortly: *St*-formulas, finally, are of the form $\Box^m\sigma$ with σ a basic Sahlqvist tense formula.
Sahlqvist formulas are St-formulas with a weakened condition so that *all* boxes may appear in strongly positive formulas. ⊞

A typical example of a Sahlqvist formula which is not an St-formula, is given by $\Diamond\Box p \rightarrow \Box\Diamond p$ (at least, if $\Diamond\check{}$ is not an S-operator.) Note that the 'tense axiom' $(p \rightarrow \Box\check{}\Diamond p)$ itself can be replaced by the St-form $(\Diamond\Box\check{}\neg q \wedge q) \rightarrow \bot$.

Sahlqvist formulas are defined in a syntactic mannner, but in fact the important constraint on the consequent is a semantic one, viz. monotonicity:

Definiton 2.3.
Let V and V' be two valuations on a frame F. V' is *wider than* V, notation: $V \leq V'$, if for all atoms p, $V(p) \subseteq V'(p)$. A modal formula is *monotone* if for all frames F, V, V' and w:

$$F, V, w \models \phi \text{ and } V \leq V' \text{ imply } F, V', w \models \phi$$

⊞

We also need related concepts for the first order model-language.

Definition 2.4.
Let Q be the set of propositional variables of the language. $L_{S,Q}$ denotes the first order language with S-accessibility predicates and a monadic predicate P_i for every propositional variable $p_i \in Q$. The *sign* of an occurrence of a predicate T in a formula ϕ is defined by induction to ϕ: T occurs positively in the atomic formula $Tx_0 \ldots x_{n-1}$. If T occurs positively (negatively) in ϕ, then it occurs negatively (positively) in $\neg\phi$, and positively (negatively) in $\phi \vee \psi$ and $\exists x\phi$. An $L_{S,Q}$-formula is *positive (negative)* if all occurrences of Q-predicates are positive (negative).
An $L_{S,Q}$-formula $\phi(x_1, \ldots, x_n)$ is *monotone* if for all valuations V, V' and all n-tupels $w_1, \ldots, w_n$:

$$F, V \models \phi[w_1, \ldots, w_n] \text{ and } V \leq V' \text{ imply } F, V', w \models \phi[w_0, \ldots, w_n]$$ ⊞

Note that in the above definition it does not matter how the *accessibility* predicates occur in a formula. There is a lot to be said about the above concepts, but we confine ourselves to the following:

Lemma 2.5.

(i) If ϕ is a positive (negative) formula, then so is ϕ^1.

(ii) Negations of positive (negative) formulas are equivalent to negative (positive) ones.

(iii) A formula is monotone if it has a positive equivalent.

Proof.
Standard. ⊞

Before stating the main theorem of this section we need one more definition:

Definition 2.6.
A general frame $G = (F, A)$ is *discrete* if for all worlds w in F, $\{w\} \in A$. ⊞

So a general frame is discrete iff it contains all singletons. Note that ordinary Kripke frames, seen as full general frames, are discrete. The theorem below states that St-formulas are *persistent* with respect to the class of discrete frames. For a motivation of its importance we refer the reader to the next section.

Theorem 2.7.
Let $G = (F, A)$ be a discrete general tense frame and σ a Sahlqvist tense formula such that $G \models \sigma$. Then $F \models \sigma$.

The remainder of this section will be devoted to the proof of theorem 2.7. From now on we fix the St-formula σ and the general frame $G = (F, A)$, $F = (W, R_\nabla)_{\nabla \in S}$. To establish the validity of σ in F, we must prove that for every valuation V, the model $F, V \models \sigma$. So let us start with defining a set of valuations for which we already know that $F, V \models \sigma$:

Definition 2.8.
A valuation V is *admissible* if $V(p) \in A$ for all atoms p.

Lemma 2.9.
For all admissible valuations V, $F, V \models \sigma$.

Proof.
Standard. ⊞Lemma 2.9.

We now proceed to define a second kind of valuations, intuitively those forming the *minimal* valuations needed to make the strongly positive formulas, (these being the 'real' antecedent of the Sahlqvist formula σ,) true in a world of W.

Definition 2.10.
First we define *basic rudimentary formulas*, or short, br-formulas: a basic rudimentary formula of *length* 0 is of the form $\beta(x,y) \equiv x = y$. If $\beta(x, x_n)$ is a basic rudimentary formula of length n and $R_\diamond$ is the accessibility relation of a tense diamond, then $\exists x_n(\beta(x, x_n) \wedge R_\diamond x_n y)$ is a basic rudimentary formula of length $n+1$.
A *rudimentary* formula, or short, an r-formula, is of the form

$$\rho(x_1, \ldots, x_n, y) \equiv \bigvee_{1 \leq i \leq n} \beta_i(x_i, y)$$

where every β_i is a disjunction of basic rudimentary formulas in x_i and y.
A subset X of W is *rudimentary in* $w_1, \ldots, w_n \in W$ if for some rudimentary formula $\rho(x_1, \ldots, x_n, y)$, $X = \{v \in W \mid F \models \rho(w_1, \ldots, w_n, v)\}$.
A valuation V is *rudimentary* if for all atoms p, $V(p)$ is rudimentary. ⊞

Note that, intuitively, a basic rudimentary formula $\beta(x, y)$ of length n describes the existence and form of an path from x to y following tense accessibility relations. A rudimentary formula $\rho(x_1, \ldots, x_n, y)$ describes the position of y with respect to $x_1, \ldots, x_n$ in the frame, in terms of 'tense paths' leading from x_i to y, for every x_i.

Lemma 2.11.
Rudimentary valuations on discrete general tense frames are admissible.

Proof.
It is sufficient to prove that for every r-formula $\rho(x_1, \ldots, x_n, y)$, the sets $X_{\rho,\vec{w}} = \{v \in W \mid F \models \rho(w_1, \ldots, w_n, v)\}$ are in A for all n-tupels $\vec{w} = (w_1, \ldots, w_n)$ of worlds in W. Because A is closed under finite unions, we can do with showing the above for *basic* rudimentary formulas. By induction to the length k of a basic formula $\beta(x, y)$ we prove the following claim:

For every $w \in W$, $X_{\beta,w} \in A$.

For $k = 0$, we have $X_{\beta,w} = \{w\}$ in A by the discreteness of G.
For $k = m+1$, let $\beta(x, y)$ be of the form $\exists x_n(\beta'(x, x_n) \wedge R_\diamond x_n y)$ where $\diamond$ is a tense diamond.
Now $X_{\beta,w} = \{v \in W \mid F \models \beta(w, v)\}$ is the set of worlds v such that there is a $u \in W$ with $F \models \beta'(w, u)$ and $F \models R_\diamond uv$.
So $X_{\beta,w}$ contains precisely the worlds having an $R_\diamond$-*predecessor* in $X_{\beta',w}$, or

$$X_{\beta,w} = \{v \in W \mid v \text{ has an } (R_\diamond)\check{}\text{-successor in } X_{\beta',w}\}$$

By the induction hypothesis, $X_{\beta',w}$ is in A, and by the fact that we are in a *tense* frame, $(R_\diamond)\check{}$ is the accessibility relation of $\diamond\check{}$. So $X_{\beta,w}$ is in A.
⊞Lemma 2.11

Note that in the above proof it is essential to have *tense* operators in *tense* frames.

Lemma 2.12.
Let ψ be an untied formula. Then its first order model-equivalent $\psi^1(x_0)$ is equivalent to

$$\exists x_1 \dots x_n \big(\pi \wedge \bigwedge_{i<k} \forall y(\rho_i(\vec{x}, y) \rightarrow P_i y) \wedge \bigwedge_{j<m} N_j(u_j)\big)$$

where the x_i's are distinct variables different from x_0, all the variables u_i are among $x_0, \dots, x_n$, π is a conjunction of atomic $L_S(x_0, \dots, x_n)$-formulas (i.e. atomic accessibility formulas of the form $R_\nabla(x_{i_0}, \dots, x_{i_{\ell(\nabla)}})$ with ∇ an arbitrary S-operator and every variable in $\{x_0, \dots, x_n\}$), the ρ_i's are suitable rudimentary formulas, and the N_j's are negative.

Proof.
By a straightforward induction to the complexity of untied formulas, cf. [22].
⊞Lemma 12

Lemma 2.13.
Let $\sigma = \psi_1 \rightarrow \psi_2$ be a basic Sahlqvist formula. Then $\sigma^1(x_0)$ is equivalent to

$$\forall x_1 \dots x_n \Big(\big(\pi \wedge \bigwedge_{i<k} \forall y(\rho_i(\vec{x}, y) \rightarrow P_i y)\big) \rightarrow \gamma_2(x_0, \dots, x_n)\Big)$$

where the antecedent is as in the previous lemma and the consequent γ_2 is some monotone formula.

Proof.
Let $N(x_0, \dots, x_n)$ be the formula $\bigwedge_{j<m} N_j(u_j)$, then N is negative. By lemma 2.12, the local model correspondent $\sigma^1(x_0)$ of σ is equivalent to

$$\forall x_1 \dots x_n \Big(\big(\pi \wedge \bigwedge_{i<k} \forall y(\rho_i(\vec{x}, y) \rightarrow P_i y) \wedge N\big) \rightarrow \psi_2^1(x_0)\Big)$$

So, by moving the negative N from the antecedent to the consequent, we obtain

$$\forall x_1 \dots x_n \Big(\big(\pi \wedge \bigwedge_{i<k} \forall y(\rho_i(\vec{x}, y) \rightarrow P_i y)\big) \rightarrow \big(\neg N \vee \psi_2^1(x_0)\big)\Big)$$

where the antecedent is already as desired, and the consequent is monotone as it is a disjunction of two monotone formulas.

⊞Lemma 13

Proof of theorem 2.8.
Let σ be of the form $\Box_1 \dots \Box_m(\psi_1 \rightarrow \psi_2)$, where every $\Box_i$ is a tense box, ψ_1 is

an untied formula and ψ_2 is a positive formula.
We assume $m = 0$. (The wider case is a straightforward generalization.)
Using the notation of the previous lemmas, set

$$\gamma_1(x_0, \ldots, x_n) \equiv \pi \wedge \bigwedge_{i<k} \forall y(\rho_i(\vec{x}, y) \rightarrow P_i y)$$

Obviously, $\sigma^1(x_0)$ is equivalent to $\forall x_1 \ldots x_n(\gamma_1 \rightarrow \gamma_2)$, where γ_2 is monotone.
So by the fact that $G = (F, A) \models \sigma$ we get

$$\text{for all } \textit{admissible} \text{ valuations } V, \quad F, V \models \forall x_0 \ldots x_n(\gamma_1 \rightarrow \gamma_2) \tag{1}$$

Our aim is to show that this implies $F \models \sigma$, or equivalently

$$\text{for } \textit{all} \text{ valuations } V, F, V \models \forall x_0 \ldots x_n(\gamma_1 \rightarrow \gamma_2) \tag{†}$$

So let a valuation V be given, together with worlds $w_0, w_1, \ldots, w_n \in W$ for which we have

$$F, V \models \gamma_1(w_0, w_1, \ldots, w_n) \tag{2}$$

Now let V^- be the rudimentary valuation that precisely 'fits' in γ_1, i.e. $V^-(p_i) = \{v \in W \mid F \models \rho_i(\vec{w}, v)\}$, then

$$F, V^- \models \gamma_1(w_0, w_1, \ldots, w_n) \tag{3}$$

V^- is admissible by lemma 2.11, so (1) and (3) give

$$F, V^- \models \gamma_2(w_0, w_1, \ldots, w_n) \tag{4}$$

But by (2) and definition of V^-, we have $V^- \leq V$. Together with the fact that γ_2 is monotone, this yields

$$F, V \models \gamma_2(w_0, w_1, \ldots, w_n) \tag{5}$$

which ensures (†).

⊞Theorem 2.7

As a matter of fact, from this proof it is only a minor step to give the algorithm producing the correspondent $\sigma^s(x_0)$ of an arbitrary (i.e. not necessarily tense) Sahlqvist formula:

Definition 2.14.
For a Sahlqvist formula σ, let $\sigma^s(x_0)$ be the L_S-formula

$$\forall x_1 \ldots x_n(\pi \rightarrow (\gamma_2(x_0, \ldots, x_n)[\rho_i(\vec{x}, u)/P_i u]))$$

(i.e. we substitute, everywhere in γ_2, $\rho_i(\vec{x}, u)$ for the atomic formula $P_i u$.) ⊞

Theorem 2.15 (Sahlqvist correspondence).
Let σ be an arbitrary Sahlqvist formula, w a world in a frame F. Then

$$F, w \models \sigma \iff F \models \sigma^s(x_0)$$

Proof.

From left to right:
Let $w_1, \ldots, w_n$ be such that $F \models \pi[w_0, \ldots, w_n]$. This implies that, with V^- the valuation such that

$$V^-(p_i) = \{v \in W \mid F \models \rho_i(\vec{w}, v)\},$$

we have
$$F, V^- \models \pi \wedge \forall y(\rho_i(\vec{x}, y) \rightarrow P_i y))[w_0, \ldots, w_n]$$
So by lemma 2.13, $F, V^- \models \gamma_2(x_0, \ldots, x_n)$. By definition of V^- this immediately gives
$$F \models (\gamma_2(x_0, \ldots, x_n)[\rho_i(\vec{x}, u)/P_i u]))[w_0, \ldots, w_n)]$$
which is what we desired.

From right to left:
Here we can copy the proof of theorem 2.8, after making the observation that now
$$F, V^-, w_0 \models \sigma$$
by definition of σ^s and the assumption $F \models \sigma^s[w_o]$.

⊞Theorem 2.15

3. The D-operator.

An important rôle in this paper is played by the so-called *difference* operator D. This operator is special in having the *inequality* relation as its intended accessibility relation:

Definition 3.1.
Let S be a similarity type containing the monadic operator D. An S-frame $F = (W, R_\nabla)_{\nabla \in S}$ is called *(D-)standard* if

$$R_D = \{(s,t) \in {}^2W \mid s \neq t\}$$

As abbreviations we use $\underline{D}\phi \equiv \neg D \neg \phi$, $O\phi \equiv \phi \wedge \underline{D}\neg\phi$, $E\phi \equiv \phi \vee D\phi$.
For K a class of S-frames, we denote the class of standard frames in K by $\mathsf{K}^{\neq}$.

In the sequel we will occasionally drop the adjective 'standard' when referring to the intended semantics, explicitly using the term 'non-standard' for the frames with $R_D \neq \{(s,t) \in^2 W \mid s \neq t\}$. Note that in a standard model we have

$M, w \models D\phi$ iff there is a $v \neq w$ with $M, v \models \phi$.
$M, w \models O\phi$ iff w is the *only* world with $M, w \models \phi$.
$M, w \models E\phi$ iff there is a world v with $M, v \models \phi$

In many examples the D-operator is *definable* in the poorer language; for example, over the class LI of irreflxive linear orderings we have

$$\mathsf{LI} \models D\phi \leftrightarrow (F\phi \vee P\phi)$$

In the last section of this paper we give another example and there we show how we can use such a definition to get rid of the D-operator altogether.

The D-operator was introduced independently by various authors, cf. [13, 20, 9]. A nice feature of this new operator, and the main reason for its introduction, is the fact that it greatly increases the expressive power of the language. For example, *irreflexivity* is easily seen to be characterized by the formula $\Diamond p \rightarrow Dp$. Maarten de Rijke proved many results on the expressiveness and completeness of modal and tense logics having a D-operator, cf. [17]. We only need the following:

Definition 3.2.
Let S be a similarity type containing D. For Λ an S-logic, ΛD denotes the logic Λ extended with the following axioms:
(D1) $p \rightarrow \underline{D}Dp$
(D2) $DDp \rightarrow (p \vee Dp)$
(D3) $\nabla(p_0, \ldots, p_{n-1}) \rightarrow \bigwedge Ep_i$
ΛD^+ is the logic ΛD extended with the irreflexivity rule for D:

(IR_D) If $\vdash Op \rightarrow \phi$ and p does not occur in ϕ, then $\vdash \phi$.

Instead of $K_{\{D\}}$ (the minimal D-logic), we write K_D, instead of $K_D D$: KD. ⊞

Theorem 3.3.
For any similarity type S, $K_S D$ and $K_S D^+$ are both strongly sound and complete with respect to the class of standard S-frames.

Proof.
Cf. [17]. ⊞

As a corollary of this completeness theorem some nice semantic properties of the operators are also *provable*:

Lemma 3.4.
(i)

$$KD^{(+)} \vdash E(Op \wedge \phi) \wedge E(Op \wedge \neg\phi) \rightarrow \bot$$

(ii) If ∇ is an S-operator, then

$$K_S D^{(+)} \vdash (\nabla(\ldots, Op \wedge \phi, \ldots) \wedge \nabla(\ldots, Op \wedge \neg\phi, \ldots)) \rightarrow \bot$$

(iii) If ∇ is an S-operator, then

$$K_S D^{(+)} \vdash \bigwedge_i \nabla(\ldots, Op \wedge \phi_i, \ldots) \rightarrow \nabla(\ldots, Op \wedge \bigwedge_i \phi_i, \ldots)$$

Proof.
By showing that the above schemes of formulas are semantically *valid* in standard S-frames, and then using the completeness theorem for $KD^{(+)}$. For (ii) and (iii) one also needs axiom (D3). ⊞

Combining the notions of Sahlqvist (tense) formulas and the D-operator, we seem to have two options. Because of the general result on Sahlqvist correspondence, we know that every S(t)-formula σ has a local correspondent $\sigma^{s'}(x_0)$ in the language L_S where R_D is the symbol for the accessibility relation of D. However, we are almost exclusively interested in the way this equivalence works out for the *standard* S-frames; this means that we will only consider interpretations where R_D is the inequality relation. It is then very natural to let this preference be reflected in the syntax, by a slight abuse of notation:

Definition 3.5.
Let S be a similarity type and σ a Sahlqvist formula. If S does not contain the D-operator, $\sigma^s(x_0)$ denotes the ordinary first order Sahlqvist equivalent of σ given in Definition 2.13. If S does contain D, $\sigma^{s'}(x_0)$ denotes this ordinary first order equivalent, $\sigma^s(x_0)$ is $\sigma^{s'}(x_0)$ with every occurrence of R_D replaced by $\neq$. ⊞

As an example, the Sahlqvist correspondent of $\Diamond p \rightarrow Dp$ is not $\forall x_1(Rx_0x_1 \rightarrow R_D x_0 x_1)$, but $\forall x_1(Rx_0x_1 \rightarrow x_0 \neq x_1)$, (or even better: $\neg Rx_0x_0$.) Note that with this notation we have equivalence of σ and σ^s for the standard frames:

Theorem 3.6.
Let σ be a Sahlqvist formula, w a world in a standard frame F. Then

$$F, w \models \sigma \iff F \models \sigma^s(w_0).$$

Proof.
Straightforward by Theorem 2.14 and the definitions of σ^s and standard frames. ⊞

However, by restricting our attention to standard frames we loose the automatic completeness of Sahlqvist's theorem: where we do have, for a set of Sahlqvist axioms Σ,

$$K_S D(\Sigma) \text{ is strongly sound and complete w.r.t. } \mathsf{K}_\Sigma,$$

we are not (yet) sure whether

$$K_S D^+(\Sigma) \text{ is strongly sound and complete w.r.t. } \mathsf{K}_\Sigma^{\neq}.$$

In the next section we will prove the above statement, for Sahlqvist *tense* axioms.

4. The main proof.

This subsection contains the main idea of the proof on the Sahlqvist theorem in a context with modal derivation rules. To keep notation as simple as possible, we consider a tense similarity type S having besides the difference operator D only one pair $\{F, P\}$ of tense operators. We let $\diamond$ range over the monadic modal operators, $\Box$ is the dual of $\diamond$, and $\diamond^\smile$ is the converse of $\diamond$, i.e. $F^\smile = P$, $P^\smile = F$ and $D^\smile = D$. Note that for this similarity type there is no distinction between ordinary Sahlqvist formulas and Sahlqvist tense formulas. We intend to prove the following theorem, keeping some generalizations and corollaries for later subsections.

Theorem 1: SD-theorem.
Let σ be a Sahlqvist form in S. Then $K^t D^+(\sigma)$ is strongly sound and complete with respect to K_σ^+.

Recall that $K^t D^+(\sigma)$ has the following axioms:

(CT)	all classical tautologies
(DB)	$\diamond(p \vee q) \rightarrow (\diamond p \vee \diamond q)$
(IV)	$\phi \rightarrow HFp$
(D1)	$p \rightarrow \underline{D}Dp$
(D2)	$DDp \rightarrow (p \vee Dp)$
(D3)	$\diamond p \rightarrow p \vee Dp$
(σ)	σ

Its derivation rules are

(MP)	Modus Ponens
(UG)	Universal Generalization
(SUB)	Substitution

and the irreflexivity rule for D:

(IR_D) $\qquad \vdash Op \rightarrow \phi \;\Rightarrow \vdash \phi$, provided p does not occur in ϕ.

Note that the above theorem is not an automatic corollary of the ordinary Sahlqvist theorem, because of the special interpretation for the accessibility relation of D that we have in mind, namely the inequality relation, and the fact that the axiom system has the unorthodox derivation rule (IR_D). The difference with the ordinary Sahlqvist case shows itself in the fact that the logic $K^tD^+(\sigma))$ is *not* canonical:
Consider the set

$$\{\phi \rightarrow D\phi \mid \phi \text{ a formula }\}$$

This set is consistent, so it must be contained in a maximal consistent set Δ which is a world in the canonical frame. Clearly however, Δ is R_D-reflexive, so inequality is *not* the canonical D-accessibility relation. In other words: the canonical frame is not standard.

So it turns out that the canonical model is bad because it contains R_D-reflexive worlds. A naive approach to this problem is to simply throw them out of the canonical universe. This is not sufficient however: consider the set

$$\{p_0 \wedge \underline{D}\neg p_0\} \cup \{F\top\} \cup \{G(\phi \rightarrow \underline{D}\phi) \mid \phi \text{ a formula }\}$$

It is consistent, so it has a MC extension $\Delta \in W^c$. Δ itself is not R_D-reflexive, but all of its R_F-successors are. So Δ, having at least one R_F-successor, is an unwelcome inhabitant of the canonical frame too.
Now instead of successively throwing bad MCSs out of the canonical frame, we feel it is better to follow a more constructive path, defining a canonical-like model consisting only of good MCSs. To give this notion of a 'good' MCS, we need some auxiliary definitions. The first one is meant to provide us with a unique representation

$$\phi_0 \wedge \Diamond_1(\phi_1 \wedge \ldots \Diamond_{n-1}(\phi_{n-1} \wedge \Diamond_n\phi_n)),$$

for every formula ϕ.

Definition 4.2: Diamond Forms.
For notational elegance we add the *dummy diamond* $\odot$ to the set of monadic operators. This operator has the following interpretation:

$$M, w \models \odot\phi \iff M, w \models \phi$$

Formula paths and their *lengths* are defined by induction:
(0) If ϕ s a formula, $\langle\phi\rangle$ is a formula path of length 0.
(1) For a formula ϕ, $\Diamond \in \{F, P, D, \odot\}$ and t a formula path of length n, $\langle(\phi, \Diamond), t\rangle$ is a formula path of length $n+1$.
For t a formula path, the formula $\Phi\mu(t)$ is defined as
(0) $\Phi\mu(\langle\phi\rangle) = \phi$
(1) $\Phi\mu(\langle(\psi, \Diamond), t\rangle) = \psi \wedge \Diamond\Phi\mu(t)$

Notions like 'consistency' apply to formula paths as if they were formulas.
For ϕ a formula, its *path representation* $Pr(\phi)$ is the following formula path:

$$
\begin{array}{lll}
(\mathrm{at}) & Pr(p) & = \langle p \rangle \\
(\neg) & Pr(\neg\psi) & = \langle \neg\psi \rangle \\
(\wedge) & Pr(\psi \wedge \chi) & = \begin{cases} \langle(\psi, \Diamond), Pr(\chi')\rangle & \text{if } \chi \equiv \Diamond\chi', \Diamond \in \{F, P, D\} \\ \langle(\psi, \odot), Pr(\chi)\rangle & \text{otherwise} \end{cases} \\
(\Diamond) & Pr(\Diamond\psi) & = \langle(\top, \Diamond), Pr(\psi)\rangle
\end{array}
$$

The *diamond form* $N(\phi)$ of a formula ϕ is a representation of ϕ as $\Phi\mu(Pr(\phi))$, viz.

$$\phi_0 \wedge \Diamond_1(\phi_1 \wedge \ldots \Diamond_{n-1}(\phi_{n-1} \wedge \Diamond_n\phi_n))$$

Let t be a formula path, ζ a formula and m a natural number. By a nested induction to m and t we define $W^p(\zeta, m, t)$ as the following formula path:

$$
\begin{array}{lll}
W^p(\zeta, 0, \langle\phi\rangle) & = & \langle\zeta \wedge \phi\rangle \\
W^p(\zeta, 0, \langle(\psi, \Diamond), t\rangle & = & \langle((\zeta \wedge \psi, \Diamond), t\rangle \\
W^p(\zeta, m+1, \langle\phi\rangle) & = & \langle\phi\rangle \\
W^p(\zeta, m+1, \langle(\psi, \Diamond), t\rangle) & = & \langle(\psi, \Diamond), W^p(\zeta, m, t)\rangle
\end{array}
$$

For ζ and ϕ formulas and m a natural number, we set

$$W(\zeta, m, \phi) = \Phi\mu(W^p(\zeta, m, Pr(\phi)))$$

⊞

The intuitive meaning of $W(\zeta, m, \phi)$ is the following: let ϕ have a diamond form

$$\phi_0 \wedge \Diamond_1(\phi_1 \wedge \ldots \Diamond_{n-1}(\phi_{n-1} \wedge \Diamond_n\phi_n))$$

then $W(\zeta, m, \phi)$ is ϕ with ζ added as a witness at level m, viz.

$$\phi_0 \wedge \Diamond_1(\phi_1 \wedge \ldots \Diamond_m(\zeta \wedge \phi_m \wedge \Diamond_{m+1}(\phi_{m+1} \wedge \ldots \Diamond_{n-1}(\phi_{n-1} \wedge \Diamond_n\phi_n)\ldots))\ldots),$$

if $m \leq n$. Otherwise $W(\zeta, m, \phi) = \phi$.

Definition 4.3: Distinguishing theories.
A set of formulas Σ is *distinguishing*, or a *d-theory* if
(i) it is maximal consistent and
(ii) for every ϕ in Σ and natural number m, there is a propositional p with $W(Op, m, \phi)$ in Σ. ⊞

Note that as d-theories are MCSs, the canonical accessibility relations R^c_F, R^c_P and R^c_D for F, P and D have the ordinary meaning:

$$R^c_\Diamond\Sigma\Delta \text{ iff for all } \phi \in \Delta,\ \Diamond\phi \in \Sigma$$

We want to take the d-theories as the possible worlds in our version of the canonical model. A minimal constraint which a canonical-ish model must meet

is that every consistent set of formulas is somehow to be found as (part of) a possible world. In our setting this means that every consistent set must have a distinguishing extension.
First we need a lemma of a rather technical nature:

Lemma 4.4.

$$\vdash W(Op, m, \phi) \rightarrow \eta \Rightarrow \vdash \phi \rightarrow \eta$$

provided p does not occur in ϕ or η.

Proof.
By induction to m.
If $m = 0$, $W(Op, m, \phi)$ is equivalent to $Op \wedge \phi$, so $\vdash W(Op, m, \phi) \rightarrow \eta$ implies $\vdash Op \rightarrow (\phi \rightarrow \eta)$, whence by an application of (IR_D) we obtain $\vdash \phi \rightarrow \eta$.
If $m = k + 1$, distinguish two cases:
If ϕ is an atom or a negation, then $W(Op, m, \phi) = \phi$, so the claim is immediate.
In the other case we have $Pr(\phi) = \langle(\psi, \Diamond), Pr(\chi)\rangle$ for some ψ, χ (where $\Diamond \in \{F, P, D, \odot\}$), so $W(Op, k+1, \phi) = \psi \wedge \Diamond W(Op, k, \chi)$. The claim is now proved as follows:

$$\begin{array}{llr} & \vdash (\psi \wedge \Diamond W(Op, k, \chi)) \rightarrow \eta & \text{(assumption)} \\ \Rightarrow & \vdash (\neg(\psi \wedge \neg\eta) \wedge \Diamond W(Op, k, \chi)) & \text{(propositional logic)} \\ \Rightarrow & \vdash \neg(\Diamond\check{}(\psi \wedge \neg\eta) \wedge W(Op, k, \chi)) & \text{(tense logic)} \\ \Rightarrow & \vdash W(Op, k, \chi) \rightarrow \Box\check{}(\psi \rightarrow \eta) & \text{(modal logic)} \\ \Rightarrow & \vdash \chi \rightarrow \Box\check{}(\psi \rightarrow \eta) & \text{(induction hypothesis)} \\ \Rightarrow & \vdash \neg(\Diamond\check{}(\psi \wedge \neg\eta) \wedge \chi) & \text{(modal logic)} \\ \Rightarrow & \vdash \neg(\psi \wedge \neg\eta) \wedge \Diamond\chi) & \text{(tense logic)} \\ \Rightarrow & \vdash (\psi \wedge \Diamond\chi) \rightarrow \eta & \text{(propositional logic)} \end{array}$$

⊞Lemma 4.4

The following proposition is our version of Gabbay's generalized irreflexivity lemma (cf. [5]):

Lemma 4.5.
Let Σ be a consistent set in which the variable p does not occur, and $\phi \in \Sigma$. Then $\Sigma \cup \{W(Op, m, \phi)\}$ is consistent for all m.

Proof.
Suppose otherwise, then

$$\vdash W(Op, m, \phi) \rightarrow \neg\psi$$

for some $m \in \omega$ and $\psi \in \Sigma$. By Lemma 4.4 this would imply

$$\vdash \phi \rightarrow \neg\psi$$

contradicting the consistency of Σ.

⊞Lemma 4.5

Lemma 4.6: Extension Lemma.
If Σ is a consistent set, then there is a distinguishing Σ' containing Σ.

Proof.
Let Q be the set of propositional variables in Σ, assume that $p_0, p_1, \ldots$ are mutually distinct propositional variables not in Q, and set, for $0 \leq \xi \leq \omega$, $Q_\xi = Q \cup \{p_i \mid i < \xi\}$.
For a set Δ of formulas in Q_ω, let $PV(\Delta)$ be the set of propositional variables appearing in (formulas of) Δ. A theory Δ is called an *approximation* if Δ is consistent, $\Sigma \subseteq \Delta$ and $PV(\Delta) = Q_n$ for some $n < \omega$. In this case p_{n+1} is called the *new* variable for Δ and denoted by p_Δ.
Now let Δ be an approximation, ϕ a formula and $m \in \omega$. The pair (ϕ, m) is called a *shortcoming* for Δ if $\phi \in \Delta$ while no witness $W(Op, m, \phi)$ is in Δ. Assume that we have a wellordering $\mathcal{W}$ of $\Phi(M) \times \omega$. If Δ has shortcomings, let (ϕ_Δ, m_Δ) be the *first* (in $\mathcal{W}$) of Δ's shortcomings. Now set

$$\Delta^+ = \begin{cases} \Delta & \text{if } \Delta \text{ has no shortcomings} \\ \Delta \cup \{W(Op_\Delta, m_\Delta, \phi_\Delta)\} & \text{otherwise} \end{cases}$$

We claim that if Δ is an approximation, then so is Δ^+:
Δ^+ is consistent by lemma 4.5; the other conditions are straightforward.
We now define the following sequence of theories $\Sigma_0, \Sigma_1, \ldots$:

$$\begin{aligned} \Sigma_0 &= \Sigma \\ \Sigma_{2n+1} &= \begin{cases} \Sigma_{2n} \cup \{\phi_n\} & \text{if } \Sigma_{2n+1} \cup \{\phi_n\} \text{ is consistent} \\ \Sigma_{2n} \cup \{\neg\phi_n\} & \text{otherwise} \end{cases} \\ \Sigma_{2n+2} &= \begin{cases} (\Sigma_{2n+1})^+ & \text{if } \Sigma_{2n+1} \text{ has shortcomings} \\ \Sigma_{2n+1} & \text{otherwise} \end{cases} \end{aligned}$$

and set $\Sigma' = \bigcup_{n<\omega} \Sigma_n$.
It is then straightforward to prove the following:
(0) $(\Sigma_n)_{n<\omega}$ is an increasing sequence.
(1) Every Σ_n is an approximation.
(2) For every Q_ω-formula ϕ, either ϕ or $\neg\phi$ is in Σ'.
(3) For every Q_ω-formula ϕ and $m \in \omega$, there is a witness $W(Op, m, \phi)$ in Σ'.

This gives all the desired properties of Σ'. ⊞Lemma 4.6

The fact that any consistent set is contained in a d-theory, means that in a certain sense there are *enough* distinguishing sets. Note however, that we needed to extend the language to prove lemma 4.6. This could mean that problems might arise if we want to show that every d-theory Γ containing a formula $\Diamond\phi$ has a distinguishing $\Diamond$-successor Δ with $\phi \in \Delta$. For, in context of ordinary maximal consistent sets, this proposition is proved by showing that the set

$$\{\phi\} \cup \{\psi \mid \Box\psi \in \Gamma\}$$

has a maximal consistent extension. We might do the same here, but then we have to show that this set has a distinguishing extension *in the same proposition letters.* We choose a different proof, using the fact that because the language has the O-operator, the distinguishing Γ contains a complete description of Δ:

Lemma 4.7.
If Γ is a d-theory and $\Diamond\phi \in \Gamma$, then there is a d-theory Δ with $\phi \in \Delta$ and $R^c_\Diamond \Gamma\Delta$.

Proof.
As $\Diamond\phi$ is in Γ, so is $\Diamond(\phi \wedge Op)$ for some atom p. Let Δ be the set $\{\psi \mid \Diamond(Op \wedge \psi) \in \Gamma\}$. Δ is consistent, for assume otherwise, then there are $\psi_1, \ldots, \psi_n$ in Δ with every $\Diamond(Op \wedge \psi_i)$ in Γ and

$$\vdash (\bigwedge_i \psi_i) \to \bot$$

By lemma 3.4 we have

$$\vdash \bigwedge_i (\Diamond(Op \wedge \psi_i)) \to \Diamond(Op \wedge \bigwedge_i \psi_i)$$

So $\Diamond(Op \wedge \bigwedge_i \psi_i)$ and hence $\Diamond\bot$ is in Γ, contradicting its consistency.
As $\Diamond Op \in \Gamma$, for every ψ either $\Diamond(Op \wedge \psi)$ or $\Diamond(Op \wedge \neg\psi)$ is in Γ, so clearly Δ is maximal. The fact that $R^c_\Diamond \Gamma\Delta$ is immediate by definition of Δ.
To prove that Δ is distinguishing, let $\phi \in \Delta$, and $m \in \omega$. We have to show that for some q, $W(Oq, m, \psi)$ is in Δ:
By definition of Δ, $\top \wedge \Diamond(Op \wedge \psi) \in \Gamma$. As Γ is distinguishing, there is a q with

$$W(Oq, m+2, \top \wedge \Diamond(Op \wedge \psi))$$

in Γ. But a simple evaluation shows this formula to be equal to

$$\top \wedge \Diamond(Op \wedge W(Oq, m, \psi))$$

whence $W(Oq, m, \psi) \in \Delta$.

⊞Lemma 4.7

It will turn out that these two lemmas are sufficient to establish that there are *enough* d-theories. There is still one difference with the ordinary case which we need to discuss: suppose we would take the set of *all* distinguishing sets to form the universe of our canonical model. Then there would be *too many* worlds, for consider two D-theories Δ, Δ' with $p \wedge \underline{D}\neg p \in \Delta$, $p \wedge \underline{D}p \in \Delta'$. If both were to be in our 'canonical' model, the underlying frame would be non-standard, for Δ' is not an R_D-successor of Δ, while clearly $\Delta \neq \Delta'$. This inspires the following definition:

Definition 4.8.
Two distinguishing theories Γ and Δ are *connected*, notation: $\Gamma \sim_D \Delta$, if either $\Gamma = \Delta$ or $R^c_D \Gamma \Delta$. A *set* of d-theories is called *connected* if all pairs of its members are.

Lemma 4.9.
$\sim_D$ is an equivalence relation.

Proof.
Reflexivity of $\sim_D$ is immediate.
For symmetry, let $\Gamma \sim_D \Delta$. If $\Gamma = \Delta$, we are finished. If not, we have $R^c_D \Gamma \Delta$. Now R^c_D is a symmetric relation (this is an immediate consequence of having the Sahlqvist axiom (D1) in the logic). So we have $R^c_D \Delta \Gamma$, implying $\Delta \sim_D \Gamma$.
For transitivity of $\sim_D$, it suffices to show that R^c_D is *pseudo-transitive*:

$$\forall x \forall y \forall z ((xRy \wedge yRz) \rightarrow (x = z \vee xRz))$$

But this is immediate by the fact that pseudo-transitivity is the Sahlqvist correspondent of axiom (D3), and the completeness part of Sahlqvist's theorem. ⊞Lemma 4.9.

Definition 4.10: d-canonical structures.
A *d(istinguishing)*-canonical frame is of the form $F^d = (W^d, R^d_F, R^d_P, R^d_D)$ where W^d is a connected set of distinguishing theories, and the R^d's are the R^c's restricted to W^d.
Define also *d-canonical models* $M^d = (F^d, V^d)$ and *d-canonical general frames* $G^d = (F^d, A^d)$, where V^d is V^c restricted to W^d and A is given by $X \in A^d$ iff $X = V^d(\phi)$ for some ϕ. ⊞

In the sequel we will have a particular d-canonical model, frame, etc. in mind, viz. the one consisting of all worlds connected to a fixed d-theory Σ. Therefor, we will frequently speak about *the* d-canonical model, frame, etc.
We need several nice properties of the d-canonical model. The easiest to establish are the truth lemma, via the fact that the d-canonical frame is a tense frame and standard:

Lemma 4.11.
Let F^d be a d-canonical frame, then
(i) R^d_F and R^d_P are each others converse.
(ii) R^d_D is the inequality relation.

Proof.
(i) is immediate by the fact that F^d is a substructure of the canonical frame.
For (ii), the connectedness of F^d implies that $\Gamma \neq \Delta \Rightarrow R^d_D \Gamma \Delta$. The fact that every d-theory contains a witness $p \wedge \underline{D} \neg p$ ensures that no element of W^d is R^d_D-reflexive, so R^d_D is contained in the inequality relation. ⊞Lemma 4.11.

Lemma 4.12: Truth lemma.

$$M^d \models \phi\ [w] \text{ iff } \phi \in w$$

Proof.
By a formula induction, of which we only give the induction step for the modal operators:
Let ϕ be of the form $\Diamond\psi$.
First, suppose $M^d, w \models \phi$. We show that this implies the existence of a v with $R^d_\Diamond wv$ and $M^d, v \models \psi$: for $\Diamond \in \{F, P\}$ this is immediate by lemma 4.7, for $\Diamond = D$ we also need lemma 4.11(ii), namely the fact that v is an R^d_D-successor of w if $v \neq w$. By the induction hypothesis then, we get: there is a v with $R^d_\Diamond wv$ and $\psi \in v$. So by definition of $R_\Diamond$ we get $\Diamond\psi \in w$.
For the other direction, suppose $\Diamond\psi \in w$. By Lemma 4.7 there is a v with $R^d_\Diamond wv$ and $\psi \in v$. By the induction hypothesis $M^d, v \models \psi$. Again, for $\Diamond \in \{F, P\}$ this immediately implies $M^d, w \models \Diamond\psi$, for $\Diamond = D$ we need lemma 4.11(ii) once more (now we use $R_D \subseteq\ \neq$.) In both cases we find the desired $M^d, w \models \phi$.
⊞Lemma 4.12.

So it is left to prove that the underlying d-canonical frame is in K_σ, or, equivalently, to show that $F^d, V \models \sigma$ for all valuations V. This is immediate by the following lemma and theorem 2.8.

Lemma 4.13.
d-canonical general frames are discrete.

Proof.
Let w be a d-theory or world in a d-canonical general frame $G^d = (F^d, A^d)$. Let p be the propositional variable such that $Op \in w$, then by the truth lemma w is the *only* d-theory of G^d with $Op \in w$. So $\{w\} = V^d(Op) \in A^d$.
⊞Lemma 4.13

Proof of theorem 4.1.
Soundness is immediate.
For completeness, suppose $\Sigma \not\vdash \phi$, then $\Sigma \cup \{\neg\phi\}$ is consistent, so by lemma 4.6 there is a d-theory Σ' with $\Sigma \cup \{\neg\phi\} \subseteq \Sigma'$.
Let $M^d = (F^d, V^d)$ be the d-canonical model with $\Sigma' \in W^d$. By lemma 4.13 and theorem 2.8, $F^d \models \sigma$ and by the truth lemma, $M^d \models \psi$ for all $\psi \in \Sigma \cup \{\neg\phi\}$. So we obtained $\Sigma \not\models_{K^{\neq}_\sigma} \phi$.
⊞Theorem 4.1

5. Why it is not trivial.

Let us now consider a monadic similarity type S which is not omnidirectional. It suffices to take the case where we have only one diamond F besides D. We would like to extend the results of the previous section to this case, but there seem to be two problems:

The first of these was already noted by Gabbay [5] and is also discussed in Gargov and Goranko [8].
The point is the following. In the previous section we showed that it is not sufficient to prove completeness by purging the canonical frame of R_D-reflexive points: their predecessors also needed to be kicked out, and the predecessors of those, ad infinitum. In our 'constructive' approach this problem arises in the following way: it is not sufficient to show that $Op \wedge \phi$ is consistent if ϕ is so, we must also prove that $\phi_0 \wedge \Diamond_1(\phi_1 \wedge Op)$ is all right if $\phi_0 \wedge \Diamond_1\phi_1$ is, etc. In the tense-logical situation, we can do this by changing our 'perspective' on the formula, namely by moving the ϕ_1-position to the top level: we look at $\phi_1 \wedge \breve{\Diamond}_1\phi_0$ (which is consistent iff $\phi_0 \wedge \Diamond_1\phi_1$ is so), then we insert Op, obtaining $(Op \wedge \phi_1) \wedge \breve{\Diamond}_1\phi_0$. Returning to the old 'perspective' we see that indeed $\phi_0 \wedge \Diamond_1(\phi_1 \wedge Op)$ is consistent if $\phi_0 \wedge \Diamond_1\phi_1$ is consistent. It will be clear that *tense operators* are indespensable instuments for this surgery.

We will now prove that it really goes wrong in the uni-directional case:

Definition 5.1.
Let ρ be the formula $G(p \rightarrow Dp)$, ρ' the formula $\rho \wedge F\top$. ⊞

Note that ρ is a Sahlqvist formula, its equivalent $\rho^{s'}$ is $\forall x\forall y(Rxy \rightarrow R_Dyy)$. So ρ says: all R-successors are R_D-reflexive.

Now, recall that $K_FD^+(\rho')$ is the axiom system with the following axioms:

(CT) all classical tautologies
(DB) $\Diamond(p \vee q) \rightarrow (\Diamond p \vee \Diamond q)$
(D1) $p \rightarrow \underline{D}Dp$
(D2) $DDp \rightarrow (p \vee Dp)$
(D3) $\Diamond p \rightarrow p \vee Dp$
(ρ') ρ'

Its derivation rules are MP, UG, SUB and IR_D. If we had an analogon of theorem 4.1 for this logic, $K_FD^+(\rho')$ should be inconsistent, for we have

Proposition 5.2.
$\mathsf{K}^{\neq}_{\rho'} = \emptyset$.

Proof.
It suffices to show that ρ' has only non-standard frames.
Assume $F = (W, R, R_D) \models \rho'$, w a world of F. By $F, w \models \Diamond\top$ w has a successor v, by $F \models \rho^s(w)$, v is R_D-reflexive. But then F is not standard. ⊞

But, $K_F D^+(\rho')$ is *not* inconsistent, as we can easily show by considering non-standard frames again:

Proposition 5.3.

$$K_F D^+(\rho') \not\vdash \bot$$

Proof.
We will define a consistent set Δ with $K_F D^+(\rho') \subseteq \Delta$. Consider the following non-standard frame $F = (W, R, R_D)$:

$$\begin{array}{lcl} W & = & \{w, v\} \\ R & = & \{(w, v)\} \\ R_D & = & \{(w, v), (v, w), (v, v)\} \end{array}$$

and set $\Delta = \{\phi \mid F, w \models \phi\}$. Clearly then $\bot \notin \Delta$. To prove our claim that $K_F D^+(\rho') \subseteq \Delta$, we first show that Δ contains the axioms of $K_F D^+(\rho')$. This is fairly trivial: for instance, ρ' is in Δ as

$$F \models \forall y(Rxy \rightarrow R_D yy)[w]$$

Concerning the rules, the only thing worth treating is that Δ is closed under (IR_D): but this is immediate by the fact that w itself is R_D-irreflexive.
⊞Proposition 5.3

This problem is not difficult to mend: a close inspection of the completeness proof in the previous section reveals that the essential property that we need and which omnidirectional similarity types automatically give us, is the *deep insertion property*

$$(DIP) \qquad \begin{array}{l} \vdash W(Op, m, \phi) \rightarrow \eta \;\Rightarrow\; \vdash \phi \rightarrow \eta \\ \text{for all } m \in \omega \text{ and } p \text{ not occurring in } \phi \text{ or } \eta \end{array}$$

The idea is now to extend the definition of the irreflexivity rule so as to obtain a logic in which the extension lemma holds again:

Definition 20.
Define the following set of derivation rules:

$$(IR_D^*) \qquad \begin{array}{l} \vdash \neg W(Op, m, \psi) \;\Rightarrow\; \vdash \neg\psi \\ \text{for all } m \in \omega \text{ and } p \text{ not occurring in } \psi \end{array}$$

Lemma 5.5. (Deep Insertion Lemma)
Let Λ be a logic having (IR_D^*). Then Λ has (DIP).

Proof.
By the following chain of consequences (where we assume that p does not occur in ϕ or in η):

$$\begin{array}{llr} & \vdash W(Op, m, \phi) \to \eta & \text{(assumption)} \\ \Rightarrow & \vdash \neg(\neg\eta \wedge W(Op, m, \phi)) & \text{(PL)} \\ \Rightarrow & \vdash \neg W(Op, m+1, \neg\eta \wedge \phi) & \text{(evaluation of } W) \\ \Rightarrow & \vdash \neg(\neg\eta \wedge \phi) & (\mathrm{IR}_D^*) \\ \Rightarrow & \vdash \phi \to \eta & \text{(PL)} \end{array}$$

⊞Lemma 5.5

So for a similarity type where not all diamonds have converses, it is necessary to have the rule (IR_D^*) instead of (IR_D). This was already noted by Gabbay [5] and by Gargov and Goranko [8], from which we derived the above example. It is not yet clear whether this extension is also *sufficient* to prove the analogon of the SD-theorem, at least if we want to consider axiom systems with *arbitrary* Sahlqvist axioms. For, there is another difference between the tense logical case and the unimodal one.

This second problem seems to be more serious; assume that, analogous again to the previous section, we have constructed a d-canonical model M^d for a MCS Σ. We want to prove $F^d \models \sigma$, where σ is the Sahlqvist axiom added to the logic $K_S D^+$. In the tense logical case, we could do this, by using a special kind of valuations which we called *rudimentary*. We showed that for such a valuation

$$F^d, V \models \sigma.$$

This path however can only be taken if we have the converse diamond of F in the language (cf. the proof of Lemma 2.1); in the unidirectional case rudimentary valuations need not be *admissible*. It even turns out that not every d-canonical frame validates σ. We consider an example:

Definition 5.6.
Let γ be the formula $\sigma = FGp \to GFp$. ⊞

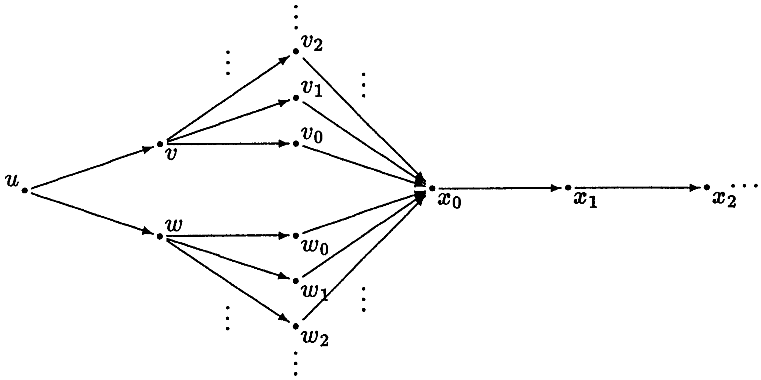
Clearly then γ is a Sahlqvist formula; its first order equivalent is the *Church-Rosser* formula

$$\gamma^s(x) = \forall y \forall z (Rxy \wedge Rxz \to \exists t (Ryt \wedge Rzt))$$

We will give a distinguishing theory Δ with $F^d \not\models \gamma(\Delta)$ for the d-canonical frame of Δ.

Definition 5.7.
Consider the following non-standard frame $F = (U, R, R_D)$:
The set of possible worlds is given as $W = \{u, v, w\} \cup \{v_n, w_n, x_n \mid n \in \omega\}$.

The accessibility relation R holds as follows: Ruv, Ruw,Rvv_n and Rww_n, all n, Rv_nx_0 and Rw_nx_0, all n, and Rx_nx_{n+1}, all n, viz.



Finally, we define a model M on F. Let the propositional variables of the language be named $p, q, r, p_0, p_1, p_2, \dots$
The valuation V is defined by

$$\begin{array}{lll} V(p) = \{u\} & V(q) = \{v\} & V(r) = \{w\} \\ V(p_{3n}) = \{v_n\} & V(p_{3n+1}) = \{w_n\} & V(p_{3n+2}) = \{x_n\} \end{array}$$

⊞

Lemma 5.8.
$M \models \sigma\gamma$ for all substitutions σ.

Proof.
It is our aim to show that for all formulas ϕ and $t \in U$:

$$M, t \models FG\phi \to GF\phi$$

For $t \neq u$ this is immediate by $F \models \gamma^s(t)$.
For $t = u$, let $V_\phi = \{n \in \omega \mid M, v_n \models \phi\}$ and $W_\phi = \{n \in \omega \mid M, w_n \models \phi\}$.
By a straightforward induction to ϕ we can show:

V_ϕ and W_ϕ are either both finite or both cofinite.

Now assume $M, u \models FG\phi$; without loss of generality we suppose that $M, v \models G\phi$. So V_ϕ contains *all* v_n, but then V_ϕ and W_ϕ are both infinite. This implies $M, w \models F\phi$. As we have $M, v \models F\phi$ too, we obtain $M, u \models GF\phi$.

⊞Lemma 5.8

Definition 5.9.
Let for $t \in W$, Δ_t be the set $\{\phi \mid M, t \models \phi\}$. ⊞

Lemma 5.10.
For every t in W, Δ_t is distinguishing.

Proof.
By induction to m we will prove:

For all $t \in W$, $\phi \in \Delta_t$, there is a p such that $W(Op, m, \phi) \in \Delta_t$.

For $m = 0$, let $t \in U$. By definition of the valuation V, there is a propositional variable p_t such that $V(p_t) = \{t\}$. So $M, t \models Op_t$, giving $W(Op_t, 0, \phi) \in \Delta_t$.
For $m = k+1$, let $t \in W$ and $\phi \in \Delta_t$. The only interesting case is where ϕ has the form $\psi \wedge \Diamond\chi$.
If $M, t \models \psi \wedge \Diamond\chi$, there is a t' with $R_\Diamond tt'$ and $M, t' \models \chi$. By the induction hypothesis, there is a p with $M, t' \models W(Op, k, \chi)$. But this means

$$\psi \wedge \Diamond W(Op, k+1, \chi) = W(Op, k, \phi) \in \Delta_t.$$

⊞Lemma 5.10

Lemma 5.11.
Let F^d be the d-canonical frame of Δ_u. Then $F^d \not\models \gamma^s(\Delta_u)$.

Proof.
It is straightforward to verify that in M^d, Δ_v and Δ_w are R_F-successors of Δ_u. Let Σ be a maximal consistent R_F-successor of both Δ_v and Δ_w. We can prove that such a Σ cannot be distinguishing by showing that for each propositional variable s

$$Gs \to s \in \Sigma$$

For, if $s \in \{p, q, r\} \cup \{p_{3n+1}, p_{3n+2} \mid n \in \omega\}$, we have $G(Gs \to s)$ in Δ_v, so by the truth lemma $M^d, \Delta_v \models G(Gs \to s)$, immediately giving the above claim. For $s \in \{p_{3n} \mid n \in \omega\}$ we can prove something similar, now using Δ_w.

⊞Lemma 5.11

Note that in the situation above, we have an example of a Sahlqvist formula which is not persistent with respect to the class of discrete frames: let $G = (F, A)$ be the general frame with F as defined in 5.7 and $X \in A$ if either X or its complement finite. Then G is discrete, $G \models \gamma$, while $F \not\models \gamma$.

Sahlqvist *tense* formulas however are still persistent for discrete general frames. Note that for a uni-directional similarity type, atoms are the only strongly positive formulas, so the set of St-formulas is rather small. Still, for this restricted set we do have a completeness theorem:

Definition 5.12.
Let S be an arbitrary similarity type of constants and diamonds. $K_S D^*$ is the basic S-logic extended with the set of rules (IR^*_D). ⊞

Theorem 5.13
Let S be an arbitrary similarity type of constants and diamonds, and Σ a set of Sahlqvist tense formulas. Then

$$K_S D^*(\Sigma) \text{ is strongly sound and complete for } K_\Sigma^{\neq}.$$

Proof.
An copy of the proof in section 4, using lemma 5.5 instead of 4.6. ⊞

Other soothing results will appear in the extended version of this paper and in [28]. We conjecture that for any *individual* set of Sahlqvist axioms, the completeness like in Theorem 5.13 can be shown to hold, but we are doubtful whether there is a uniform proof (analogous to that of Theorem 4.2) taking care of all Sahlqvist axiomatizations at once.

6. The SNξ-theorem.

We are now ready to prove a completeness theorem for a tense logic having other non-ξ rules besides (IR_D).

Definition 6.1.
Let S be a tense similarity type containing the D-operator, Σ a set of St-formulas and Ξ a set of arbitrary formulas. $K_S^t D^+(\Sigma, -\Xi)$ is the logic $K_S^t D^+$ extended with the axioms Σ and the non-ξ rules for all $\xi \in \Xi$. ⊞

Recall that the above definition implies that $K_S^t D^+(\Sigma, -\Xi)$ has the following axioms:

(CT)	all classical tautologies
(DB)	$\Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$
(IV)	$\phi \to \Box\breve{}\Diamond p$
(D1)	$p \to \underline{D} D p$
(D2)	$DDp \to (p \vee Dp)$
(D3)	$\Diamond p \to p \vee Dp$
(Σ)	Σ

Its rules are:

(MP), (UG), (SUB), (IR_D) and $\{\mathrm{N}\xi\mathrm{R} \mid \xi \in \Xi\}$.

Recall too that the class $\mathsf{K}_{(\Sigma,-\Xi)}^{\neq}$ was defined as the class of standard S-frames with

$$
\begin{array}{ll}
F \models \sigma & \text{for all } \sigma \text{ in } \Sigma \\
F, w \not\models \xi & \text{for all } w \text{ in } F,\ \xi \text{ in } \Xi
\end{array}
$$

Note that if all ξ's have local first order equivalents ξ^fon the frame level (for example, if they are Sahlqvist formulas), then $\mathsf{K}^{\neq}_{(\Sigma,-\Xi)}$ is elementary, defined by

$$
F \text{ in } \mathsf{K}_{(\Sigma,-\Xi)} \iff \begin{cases} F \models \forall x \sigma^s(x) & \text{for all } \sigma \in \Sigma \\ F \models \forall x \neg \xi^f(x) & \text{for all } \xi \in \Xi \end{cases}
$$

So, the theory below takes care of many classes of frames, for example the asymmetric or intransitive frames (cf. the characterizations given in the introduction).

Theorem 6.2 (SNξ-Theorem).
Let S, Σ and Ξ be as in definition 6.1. Then

$$
K^t_S D^+(\Sigma, -\Xi) \text{ is strongly sound and complete for } \mathsf{K}^{\neq}_{(\Sigma,-\Xi)}
$$

The proof of theorem 6.2 is in fact a straightforward adaptation of the proof in section 4. There we started with a MCS Δ and inserted in Δ, for every $m \in \omega$ and formula $\phi \in \Delta$, formulas $W(Op, m, \phi)$, in order to witness the R_D-irreflexivity of all worlds connected to Δ. Here we will also add formulas namely of the form $W(\neg\xi(p_1, \dots, p_n), m, \phi)$, this time in order to ensure that the canonical-like general frame we end with is not only standard (with respect to R_D), but also in $\mathsf{K}_{-\Xi}$. So we set

Definition 6.3.
A set Δ of S-formulas is *witnessing* if it satisfies
(i) Δ is maximal consistent.
(ii) Δ is distinguishing.
(iii) for all natural numbers m, formulas $\phi \in \Delta$ and $\xi \in \Xi$, there are propositional variables $p_1, \dots, p_n$ with $W(\neg\xi(p_1, \dots, p_n), m, \phi) \in \Delta$.

Lemma 6.4.
Every maximal consistent set Δ has a witnessing extension Δ'.

Proof.
An straightforward analogon of Lemma 4.6. ⊞

Definition 6.5.
A *w(itnessing)-canonical frame* is of the form $F^w = (W^w, R^w_\nabla)_{\nabla \in S}$ where W^w is a $\sim_D$-connected set of witnessing theories and R^w_∇ is the canonical accessibility relation of ∇, restricted to W^w. *Witnessing models* and *witnessing general frames* are also defined in the obvious way. For a w-theory Δ, *the* w-canonical frame (model, etc.) of Δ is the w-canonical frame with $\Delta \in W^w$. If we want to make the set Ξ explicit, we use the term *w-canonical structure witnessing against* Ξ. ⊞

Lemma 6.6: Truth Lemma.
Let M^w be a w-canonical model, Δ a world of M^w. Then

$$M^w, \Delta \models \phi \iff \phi \in \Delta.$$

Proof.
In the same manner as in section 4, we prove that for every w-theory Δ and for every diamond $\Diamond$ we have

$$\Diamond\phi \in \Delta \iff \text{there is a w-theory } \Delta' \text{ with } (\Delta, \Delta') \in R^w_\Diamond \text{ and } \phi \in \Delta'$$

As we can also show that F^w is standard in the sense that for every $\Diamond$ the accessibility relations of $\Diamond$ and $\Diamond^\smile$ are each other's converse, and R^w_D is the inequality relation, the truth lemma follows by a straightforward formula induction. ⊞Lemma 6.6

Lemma 6.7.
Let $G^w = (F^w, A^w)$ be a w-canonical general frame witnessing against Ξ. Then F^w is in $\mathsf{K}^{\neq}_{-\Xi}$.

Proof.
Let Δ be a world of F^w. As Δ is a w-theory of the logic, there are propositional variables $\vec{p}$ with $\neg\xi(\vec{p}) \in \Delta$. By the truth lemma then, for all ξ

$$M^w, u \models \neg\xi(\vec{p})$$

So $F^w, \Delta \not\models \xi$, for all $\xi \in \Xi$. The proof that F^w is standard runs just like in section 4.

⊞

Proof of theorem 6.2.
Soundness is already proved in the introduction to this paper. For completeness, let Δ be a $K^t_S D^+(\Sigma, -\Xi)$- consistent set of formulas. By the extension lemma, Δ is contained in a w-theory Δ'. Let M^w be the w-canonical model of Δ'. By the truth lemma,

$$M^d, \Delta' \models \phi \text{ for all } \phi \in \Delta'.$$

By lemma 6.7, F^w is in $\mathsf{K}^{\neq}_{-\Xi}$. It is in K_Σ by the facts that G^w is discrete (every w-theory is distinguishing!) and that $G^w \models \Sigma$. So we have found Δ a model based on a frame in the intended class $\mathsf{K}^{\neq}_{(\Sigma,-\Xi)}$.

⊞Theorem 6.2

Just as in the previous section, we can prove a poorer version of Theorem 6.2 for arbitrary (not tense) similarity types, but we leave this for the extended version of this paper, and for [28].

7. Conclusions and Questions.

7.1 General Conclusions.

For a wide multi-modal setting, we have generalized versions of Gabbay's Irreflexivity Rule and in the same way as the irreflexivity rule characterizes the irreflexive frames, the non-ξ rule is sound precisely on the class $\mathsf{K}_{-\xi}$ of frames F with for all worlds $F, w \not\models \xi$. In general, this class $\mathsf{K}_{-\xi}$ will not be definable in the modal language itself.

The main result of this paper, the SNξ-theorem 6.2, is a meta-theorem on completeness stating that for a certain, well-defined set St of formulas,

> *any* extension of the minimal modal logic
> with a set Σ of axioms from this set St
> and a set '$-\Xi$' of non-ξ rules,
> is strongly sound and complete
> forthe class of frames $\mathsf{K}_{(\Sigma,-\Xi)}$ 'characterized' by Σ and Ξ.

We would like to stress the fact that classes of the form $\mathsf{K}_{(\Sigma,-\Xi)}$ naturally occur, and that the main advantage of the SRξ-theorem is that the work of axiomatizing such classes becomes easier, as one can *split up* the proof, in the following parts:

(1) finding the proper characterization.
(2) applying the SRξ-theorem, immediately obtaining a strongly sound and complete derivation system.
(3) trying to simplify this system.

An example of this procedure is given below.

We would like to mention the fact that, although we confined ourselves in the sections 4–6 to similarity types having only constants and monadic modal operators, the statement above also holds for languages with operators of higher adity. To prove this claim we have to develop the tense logic of polyadic modal operators. These matters will be dealt with in the extended(!) version of this paper and in [28], of which it is a chapter. In the above-mentioned versions of this paper we will also discuss the implications of the SNξ-theorem for Boolean Algebras with (Additive) Operators.

7.2. Conservativity.

An interesting point which has not been discussed yet concerns the question whether non-ξ rules add new theorems to a logic.

Some scattered results are known:
In the introduction we saw an example where a rule is *conservative*: the logic K^tL already axiomatizes the class of irreflexive linear orderings, so adding (IR) does not produce any new theorem.
On the other hand, adding (IR) to $K^tL(Gp \to p)$ makes this logic inconsistent, so here (IR) is not conservative. In [32], Zanardo showed that the irreflexivity rule used in [4] to axiomatize a branching-time temporal logic, can be replaced by infinitely many axioms. An similar case is found in cylindric modal logic and the modal logic of relation algebras (cf. [25, 26]), where adding a non-ξ rule to a finite set of axioms creates a finite derivation system for a logic which is known not to be finitely axiomatizable when only the orthodox derivation rules (MP), (UG) and (SUB) are allowed. A striking difference between a uni-directional similarity type and its tense counterpart concerns the modal logic of the two-dimensional 'domino relation', where an axiomatization of the uni-directional modal logic needs *both* infinitely many axioms *and* a non-ξ rule (cf. [14]), while the tense logic allows a finite and orthodox axiomatization (cf. [29]).

The general question

> Are there natural (syntactic/recursively enumerable) criteria deciding when a non-ξ rule is conservative over a derivation system?

lies (almost) completely open. We have one minor result: recall that a formula is *closed* if it does not contain propositional variables (only constants), and that a logic Λ has the *interpolation property* if $\Lambda \vdash \phi \to \psi$ implies the existence of an *interpolant* χ in the common language of ϕ and ψ, such that $\Lambda \vdash \phi \to \chi$ and $\Lambda \vdash \chi \to \psi$.

Proposition 7.2.1
Let Λ be a logic and ξ a formula meeting the following constraints:
(i) Λ has the interpolation property.
(ii) for every closed formula γ, either $\Lambda \vdash \gamma$ or $\Lambda \vdash \neg\gamma$.
(iii) $\Lambda(-\xi)$ is consistent.
Then NξR is conservative over Λ.

Proof.
If Λ is inconsistent, then so is $\Lambda(-\xi)$, so assume Λ is consistent, and that is has (i) and (ii). Denote derivability in Λ by $\vdash$. To show that NξR is conservative over Λ, we must prove

$$\vdash \neg\xi(\vec{p}) \to \phi \;\;\Rightarrow\;\; \vdash \phi, \;\; \text{if no } p_i \text{ occurs in } \phi$$

So suppose $\vdash \neg\xi(\vec{p}) \to \phi$. Let γ be the interpolant of $\neg\xi(\vec{p})$ and ϕ. Clearly γ is a closed formula, so either $\vdash \gamma$ or $\vdash \neg\gamma$.

In the first case we have $\vdash \phi$ by $\vdash \gamma \to \phi$.
Otherwise, by $\vdash \neg\xi(\vec{p}) \to \gamma$ we obtain $\vdash \xi(\vec{p})$. So in this case, $\Lambda(-\xi)$ is inconsistent.

⊞Proposition 7.2.1

7.3. An example.

Suppose we want to axiomatize a simple *two-dimensional tense* logic.
The set of diamond we have in mind is $S = \{\Diamond_{\rightarrow}, \Diamond_{\leftarrow}, \Diamond_{\uparrow}, \Diamond_{\downarrow}\}$. An intended *two-dimensional frame* F is a cartesian product of two irreflexive linear temporal orders $(T_0, <'_0)$ and $(T_1, <'_1)$, i.e. $F = (T_0 \times T_1, <_0, <_1)$ where

$$
\begin{array}{lll}
(x_0, x_1) <_0 (y_0, y_1) & \text{if} & x_1 = y_1 \text{ and } x_0 <'_0 y_0 \\
(x_0, x_1) <_1 (y_0, y_1) & \text{if} & x_0 = y_0 \text{ and } x_1 <'_1 y_1
\end{array}
$$

The semantics of the diamonds is the obvious one, e.g.

$$F, V, (x_0, x_1) \models \Diamond_{\uparrow} \phi \text{ iff there is a } x'_1 \in T_1 \text{ with } x_1 <'_1 x'_1 \text{ and } F, V, (x_0, x'_1) \models \phi$$

To axiomatize the set of formulas valid on these two-dimensional frames, we first consider the class K^t of tense S-frames here given as $F = (W, R_0, R_1)$. Let LI^2 be the class of such frames that are isomorphic to disjoint unions of two-dimensional frames.
It is straightforward to show that F is in LI^2 iff

> Each R_i is transitive, linear and irreflexive
> $R_i \,|\, R_j = R_j \,|\, R_i$ and $\breve{R}_i \,|\, R_j = R_j \,|\, \breve{R}_i$, for $\{i, j\} = \{0, 1\}$.

where $R_i \,|\, R_j$ denotes the composition of R_i and R_j.
Let S^+ be the extension of S with the difference operator D. An *immediate consequence* of the SD-theorem 4.1 is now that the following logic A^+ is strongly sound and complete with respect to LI^2:

A^+ has the following axioms:

(CT)	all classical tautologies	
(DB)	$\Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$	$(\Diamond \in S^+)$
(CV)	$\Diamond \breve{\Box} p \to p$	$(\Diamond \in S)$
(D)	D1 $\wedge$ D2 $\wedge$ D3	
(TR)	$\Diamond\Diamond p \to \Diamond p$	$(\Diamond \in S)$
(LI)	$\Diamond p \to \Box(\Diamond p \vee p \vee \breve{\Diamond} p)$	$(\Diamond \in S)$
(IR)	$\Diamond p \to D p$	$(\Diamond \in S)$
(CP)	$\Diamond_1 \Diamond_2 p \to \Diamond_2 \Diamond_1 p$	$(\Diamond_1, \Diamond_2 \in S)$

and its rules are:

(MP), (UG), (SUB) and (IR_D).

Now we show how to get rid of the D-operator again; the easiest way is to show first that the D-operator is *definable* on the class of two-dimensional frames: set

$$D'\phi \equiv \Diamond_{\leftarrow}\phi \vee \Diamond_{\rightarrow}\phi \vee \Diamond_{\uparrow}\phi \vee \Diamond_{\downarrow}\phi \vee \Diamond_{\leftarrow}\Diamond_{\uparrow}\phi \vee \Diamond_{\leftarrow}\Diamond_{\downarrow}\phi \vee \Diamond_{\rightarrow}\Diamond_{\uparrow}\phi \vee \Diamond_{\rightarrow}\Diamond_{\downarrow}\phi$$

For a two-dimensional frame F we then have

$$F \models D\phi \leftrightarrow D'\phi$$

We proceed by defining a translation $^{\circ}$ from S^{+}-formulas to S-formulas by setting

$$\begin{array}{rcl} (p_i)^{\circ} & = & p_i \\ (D\phi)^{\circ} & = & D'\phi \end{array}$$

and letting $^{\circ}$ respect $\vee$ and $\neg$.
Finally we define the axiom system A by giving

axioms:	(CT), (DB), (CV), (TR), (LI) and (CP).
rules:	(MP), (UG), (SUB) and ($\text{IR}_{D'}$).

and the only thing left to prove is, for all S-formulas ϕ:

$$A^{+} \vdash \phi \iff A \vdash \phi^{\circ}$$

But this is rather trivial, using induction to the length of derivations.

We want to stress the point that the advantage of applying the SRD-theorem here lies in the fact that as soon as we have found a nice characterization of the intended class of frames, completeness is immediate, and that to prove that this characterization is correct, we can use first order logic, which makes life considerably easier.

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