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# SAHLQVIST'S THEOREM FOR BOOLEAN ALGEBRAS WITH OPERATORS

Maarten de Rijke
Yde Venema

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Faculteit der Wiskunde en Informatica
(Department of Mathematics and Computer Science)
Plantage Muidergracht 24
1018TV Amsterdam

Faculteit der Wijsbegeerte
(Department of Philosophy)
Nieuwe Doelenstraat 15
1012CP Amsterdam

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Maarten de Rijke
Yde Venema
Department of Mathematics and Computer Science
University of Amsterdam

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# Sahlqvist's Theorem for Boolean Algebras with Operators

Maarten de Rijke* & Yde Venema

Department of Mathematics and Computer Science
University of Amsterdam
Plantage Muidergracht 24, 1018 TV Amsterdam

## 1 *Introduction*

The aim of this note is to explain how a well-known result from Modal Logic, Sahlqvist's Theorem, can be applied in the theory of Boolean Algebras with Operators to obtain a large class of identities, called *Sahlqvist identities*, that are preserved under canonical embedding algebras. These identities can be specified as follows. Let $\sigma = \{ f_i : i \in I \}$ be a set of (normal) additive operations. Let an *untied term* over $\sigma$ be a term that is either (i) negative (i.e., in which every variable occurs in the scope of an odd number of complementation signs $-$ only), or (ii) of the form $g_1(g_2 \ldots (g_n(x)) \ldots)$, where the $g_i$s are duals of unary elements of $\sigma$ (i.e., $g_i$ is defined by $g_i(x) = -f_i(-x)$ for some unary operator in $\sigma$), or (iii) obtained from terms of type (i) or (ii) by applying $+, \cdot$ and elements of $\sigma$ only. Then, an equality is called a *Sahlqvist equality* if it is of the form $s = 1$, where $s$ is obtained from complemented untied terms $-u$ by applying duals of elements of $\sigma$ to terms that have no variables in common, and $\cdot$ only.

Examples of Sahlqvist identities are abundant in algebraic logic—in fact, *all* axioms for both relation and cylindric algebras can be brought in a Sahlqvist form. For instance, Johan van Benthem observed that the axiom $x^{\smile}; -(x;y) \leq -y$ in relation algebra has a Sahlqvist equivalent $-[(x^{\smile}; -(x;y)) \cdot y)] = 1$.

To prove that such Sahlqvist equalities are indeed preserved under canonical embedding algebras we will not have to prove any really new results, but we will merely have to order some known results in an appropriate way.

As this note is aimed primarily at algebraists, we assume that the reader is familiar with basic algebraic notions and facts; for algebraic details not explained in this note we refer the reader to [2]. We will be somewhat more explicit concerning the modal logical results and definitions we will need; most of them will be presented in §2. After that, in §3, we describe the modal counterparts of the above Sahlqvist equalities, and partially prove a Sahlqvist Theorem, which says that Sahlqvist formulas are both *canonical* and *first order*. From this the preservation of Sahlqvist equalities under canonical embedding algebras is easily derived. Finally, §4, which is essentially a part of the second author's dissertation [9], contains a detailed demonstration of the usefulness of the Sahlqvist Theorem. It shows that by the Sahlqvist Theorem the equivalence of two equations may be proved or disproved by reasoning on *modal frames* (or *atom structures*) rather then by manipulating these

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equations themselves; as an example illustrating this method Henkin's equation in cylindric algebras is proved to be equivalent to an equality of a simpler form.

The reader is advised to skip §2 upon a first reading, and only to return to it later on to look up a definition.

We would like to thank Johan van Benthem for stressing the importance of Sahlqvist's Theorem, and Andréka Hajnal, István Németi and Ildikó Sain for encouraging us to write this note.

## 2 Preliminaries

A *Boolean algebra with operators* (BAO) is an algebra $\mathfrak{B}$ of type $\{+,\cdot,-,0,1\}\cup\{f_i : i\in I\}$ such that $(B,+,\cdot,-,0,1)$ is a Boolean algebra, and the operators $\{f_i : i\in I\}$ are *(finitely) additive* (join preserving) in every argument; a BAO is called *normal* if for every $f_i$, $f_i(\vec{x}) = 0$ whenever one of the terms $x_j = 0$.

Let us quickly move on to the Stone Representations of BAO's, the so-called general frames. First, a *modal similarity type* is a pair $S = (O,\rho)$, where $O = \{\nabla_i : i \in I\}$ is a set of *modal operators*, and $\rho$ is a rank function for $O$. As variables ranging over modal operators we use $\nabla, \nabla_1, \ldots$; for monadic modal operators we use $\Diamond, \Diamond_1, \ldots$. For $\nabla_i \in S$ its *dual* operator $\lhd_i$ is defined as $\lhd_i(\phi_1,\ldots,\phi_{\rho(i)}) \equiv \neg\nabla_i(\neg\phi_1,\ldots,\neg\phi_{\rho(i)})$; the dual of a monadic operator $\Diamond_i$ is denoted $\Box_i$. A *modal language* is a pair $M = (S,Q)$, where $S$ is a modal similarity type, and $Q$ is a set whose elements are called proposition letters. From the modal and Boolean constants, and the proposition letters, the modal formulas are built up in the obvious way, using $\neg, \wedge$, and the operators in $S$. When no confusion arises we write $M(S)$ or even $M$ rather than $M(S,Q)$.

A *general frame* $\mathfrak{F}$ of similarity type $S$ is a tuple $(W, \{R_i : i\in I\}, \mathcal{W})$ where $W \neq \emptyset$, $R_i \subseteq W^{\rho(i)+1}$, and $\mathcal{W} \subseteq \mathrm{Sb}(W)$ contains $\emptyset$, and is closed under $\cdot$, $-$, and the operators $\{f_{R_i} : i \in I\}$, where $f_{R_i} : \mathrm{Sb}(W)^{\rho(i)} \to \mathrm{Sb}(W)$ is defined by

$$(1)\quad f_{R_i}(Y_1,\ldots,Y_{\rho(i)}) = \{x_0 : \exists x_1 \ldots x_{\rho(i)}\, (R_i(x_0,x_1,\ldots,x_{\rho(i)}) \wedge \bigwedge_{1\leq j\leq\rho(i)} (x_i \in Y_i))\}.$$[1]

For future use we also define $g_{R_i} : \mathrm{Sb}(W)^{\rho(i)} \to \mathrm{Sb}(W)$, by putting $g_{R_i}(Y_1,\ldots,Y_{\rho(i)}) = -f_{R_i}(-Y_1,\ldots,-Y_{\rho(i)})$. A *Kripke frame* or *atom structure* of similarity type $S$ is a tuple $(W,\{R_i : i\in I\})$, with $W$ and $\{R_i : i \in I\}$ as before. A general frame $\mathfrak{F}$ defines a Kripke frame $\mathfrak{F}_\#$ via the forgetful functor $(\cdot)_\# : (W,\{R_i : i\in I\},\mathcal{W}) \mapsto (W,\{R_i : i \in I\})$. A Kripke frame $\mathfrak{F}$ defines the general frame $\mathfrak{F}^\#$ via $(\cdot)^\# : (W,\{R_i : i \in I\}) \mapsto (W,\{R_i : i \in I\},\mathrm{Sb}(W))$.

Given a general frame $\mathfrak{F} = (W,\{R_i : i \in I\},\mathcal{W})$ its *complex algebra* is the BAO $\mathfrak{F}^+ = (\mathcal{W},\cup,\cap,\emptyset,W,-,\{f_{R_i} : i \in I\})$, where $f_{R_i} : \mathrm{Sb}(W)^{\rho(i)} \to \mathrm{Sb}(W)$ is defined as in (1).

Given a BAO $\mathfrak{B}$ with operators $\{f_i : i \in I\}$, the general frame $\mathfrak{B}_+$ is the tuple $(X_\mathfrak{B},\{R_{f_i} : i\in I\},\mathcal{W})$, where $X_\mathfrak{B}$ is the set of ultrafilters on $\mathfrak{B}$, $R_{f_i} \subseteq X_\mathfrak{B}^{\rho(i)+1}$ is de-

[1] Algebraists may be accustomed to seeing the argument places reversed in the definition of the function $f_{R_i}(Y_1,\ldots,Y_{\rho(i)})$ as $\{x_0 : \exists x_1 \ldots x_{\rho(i)}\, (R_i(x_0,x_1,\ldots,x_{\rho(i)}) \wedge \bigwedge_{1\leq j\leq\rho(i)}(x_i \in Y_i))\}$ in (1). Being modal logicians we like to think that the modal notation is the more elegant one.

fined by

$$R_{f_i}(a_0, a_1, \ldots, a_{\rho(i)}) \text{ iff } \forall j\,(1 \le j \le \rho(i) \to x_j \in a_j) \text{ implies } f_i(x_1, \ldots, x_{\rho(i)}) \in a_0,$$

and $\mathcal{W} \subseteq \mathrm{Sb}(X_{\mathfrak{B}})$ is $\{\, \hat{x} : x \in B \,\}$ for $\hat{x} = \{\, a \in X_{\mathfrak{B}} : x \in a \,\}$. The *canonical structure* $\mathfrak{Cs}\,\mathfrak{B}$ of $\mathfrak{B}$ is the structure $(\mathfrak{B}_+)_\#$. By definition the complex algebra of the canonical structure of $\mathfrak{B}$ is called the *canonical embedding algebra* of $\mathfrak{B}$: $\mathfrak{Em}\,\mathfrak{B} = (\mathfrak{Cs}\,\mathfrak{B})^+$.[2] By a *canonical* variety we mean one that is closed under canonical embedding algebras.

A *valuation* on a general frame $\mathfrak{F}$ is a function $V$ taking proposition letters to elements of $\mathcal{W}$; a valuation on a Kripke frame $\mathfrak{F}$ is a valuation on $\mathfrak{F}^\#$. In algebraic terms: a valuation is an *assignment* to the variables of elements of the 'subcomplex' algebra $\mathcal{W}$. Truth of a modal formula in a *model* $(\mathfrak{F}, V)$ is then defined as follows: $(\mathfrak{F}, V), w_0 \models p$ iff $w_0 \in V(p)$; $(\mathfrak{F}, V), w_0 \models \neg\phi$ iff $(\mathfrak{F}, V), w_0 \not\models \phi$; $(\mathfrak{F}, V), w_0 \models \phi \wedge \psi$ iff both $(\mathfrak{F}, V), w_0 \models \phi$ and $(\mathfrak{F}, V), w_0 \models \psi$; and $(\mathfrak{F}, V), w_0 \models \nabla_i(\phi_1, \ldots, \phi_{\rho(i)})$ iff $\exists w_1, \ldots, w_{\rho(i)}\,(R_i(w_0, w_1, \ldots, w_{\rho(i)}) \wedge \bigwedge_{1 \le j \le \rho(i)} (\mathfrak{F}, V), w_j \models \phi_j))$. We write $(\mathfrak{F}, V) \models \phi$ for: for all $w \in W$, $(\mathfrak{F}, V), w \models \phi$; $\mathfrak{F}, w \models \phi$ is short for: for all valuations $V$ on $\mathfrak{F}$, $(\mathfrak{F}, V), w \models \phi$; and $\mathfrak{F} \models \phi$ is short for: for all $w \in W$, $\mathfrak{F}, w \models \phi$.

A modal formula $\phi$ in $n$ proposition letters induces an $n$-ary polynomial $h_\phi(x_1, \ldots, x_n)$ which may be defined as follows:

$$\begin{aligned}
h_{p_j}(x_1, \ldots, x_n) &\equiv x_j \\
h_{\neg\phi}(x_1, \ldots, x_n) &\equiv -h_\phi(x_1, \ldots, x_n) \\
h_{\phi\wedge\psi}(x_1, \ldots, x_n) &\equiv h_\phi(x_1, \ldots, x_n) \cdot h_\psi(x_1, \ldots, x_n) \\
h_{\nabla_i(\phi_1, \ldots, \phi_{\rho(i)})}(x_1, \ldots, x_n) &\equiv f_{R_i}(h_{\phi_1}(x_1, \ldots, x_n), \ldots, h_{\phi_{\rho(i)}}(x_1, \ldots, x_n)).
\end{aligned}$$

And conversely, each polynomial in a similarity type of BAO's is of the form $h_\varphi$ for some modal formula $\phi$ in a modal language of the appropriate type. This identification of formulas and terms is made explicit in the following proposition.

**Proposition 2.1** *Let $S$ be a modal similarity type. Let $\mathfrak{F}$ be a general frame of type $S$. Let $\phi$ be a formula in $M(S)$. Then $\mathfrak{F} \models \phi$ iff $(\mathfrak{F})^+ \models h_\phi = 1$.*

A *(normal) modal logic* in a language $M(S)$ is a subset $\Lambda$ of the set of formulas in $M(S)$ that contains as axioms all propositional tautologies (PL), as well as

$$\text{(DB)} \quad \begin{aligned} &\nabla_i(p_1, \ldots, p_{j-1}, p, p_{j+1}, \ldots, p_{\rho(i)}) \vee \nabla_i(p_1, \ldots, p_{j-1}, p', p_{j+1}, \ldots, p_{\rho(i)}) \leftrightarrow \\ &\qquad \nabla_i(p_1, \ldots, p_{j-1}, p \vee p', p_{j+1}, \ldots, p_{\rho(i)}), \end{aligned}$$

and that is closed under the following derivation rules:

(MP) if $\phi, \phi \to \psi \in \Lambda$ then $\psi \in \Lambda$
(UG) if $\phi \in \Lambda$ then $\neg\nabla_i(\phi_1, \ldots, \phi_{j-1}, \neg\phi, \phi_{j+1}, \ldots, \phi_{\rho(i)}) \in \Lambda$
(SUB) if $\phi \in \Lambda$ then all substitution instances of $\phi$ are in $\Lambda$.

For a logic $\Lambda$ a *canonical general frame* for $\Lambda$ is defined by $\mathfrak{F}_\Lambda(\alpha) = (\mathfrak{A}_\Lambda(\alpha))_+$, where $\mathfrak{A}_\Lambda(\alpha)$ is the free algebra (on $\alpha$ generators) of the variety $\mathsf{V}_\Lambda$, where $\mathfrak{A} \in \mathsf{V}_\Lambda$ iff $\mathfrak{A} \models h_\varphi = 1$, for all $\varphi \in \Lambda$. For a class of general or Kripke frames $\mathsf{K}$, let $\mathrm{Th}(\mathsf{K}) = \{\, \phi : \text{for all } \mathfrak{F} \in \mathsf{K}, \mathfrak{F} \models \phi \,\}$. We call a logic $\Lambda$ *sound* with respect to a class of general or Kripke frames $\mathsf{K}$ if $\Lambda \subseteq \mathrm{Th}(\mathsf{K})$, and *complete* with respect to $\mathsf{K}$ if $\mathrm{Th}(\mathsf{K}) \subseteq \Lambda$. A logic $\Lambda$ is called *canonical* if $(\mathfrak{F}_\Lambda(\alpha))_\# \models \Lambda$, for every canonical general frame $\mathfrak{F}_\Lambda(\alpha)$.

[2] In [3] the canonical embedding algebra of $\mathfrak{B}$ is called the *Stone extension* of $\mathfrak{B}$; these, in turn, form a special case of the arbitrary extensions dealt with in [5], of a kind called *perfect extension*.

$L_0(S)$ is the first order language of type $S$; it has relation symbols $R_i$ ($i \in I$) of arity $\rho(i)+1$. $L_1(S)$ is $L_0(S)$ extended with unary predicate symbols $P_j$ corresponding to the proposition letters of our modal language. $L_2(S)$ is the language of monadic second order logic with relation symbols $R_i$ ($i \in I$) of arity $\rho(i)+1$, and variables $P_j$s ranging over sets. A modal formula $\phi$ *locally corresponds* to an formula $\alpha(x)$ if for all Kripke frames $\mathfrak{F}$ of the appropriate type, $\mathfrak{F}, w \models \phi$ iff $\mathfrak{F} \models \alpha[w]$. A modal formula $\phi$ *corresponds* to a sentence $\alpha$ if for all Kripke frames $\mathfrak{F}$ of the appropriate type, $\mathfrak{F} \models \varphi$ iff $\mathfrak{F} \models \alpha$. When interpreted on frames modal formulas correspond to $L_2(S)$-formulas (cf. [1]).

## 3 A Sahlqvist theorem

To describe the modal counterparts of the earlier Sahlqvist equalities we need the following definition.

**Definition 3.1** Let $S$ be a modal similarity type. *Positive* and *negative* occurrences of a proposition letter $p$ are defined as usual by: (i) $p$ occurs positively in $p$, (ii) a positive (negative) occurrence of $p$ in $\phi$ is a negative (positive) occurrence of $p$ in $\neg\phi$ and in $\phi \to \psi$, and a positive (negative) one in $\phi \vee \psi$, $\phi \wedge \psi$, $\nabla_i(\phi_1, \ldots, \phi, \ldots, \phi_{\rho(i)})$, $\lhd_i(\phi_1, \ldots, \phi, \ldots, \phi_{\rho(i)})$ ($\nabla_i \in S$). A formula $\phi$ in $M(S)$ is *positive* (*negative*) if every proposition letter occurs only positively (negatively) in $\phi$. $\phi$ is *monotone* in the proposition letter $p$ if for every model $(\mathfrak{F}, V)$ and every valuation $V'$ on $\mathfrak{F}$ with $V(p) \subseteq V'(p)$ and otherwise the same as $V$, $(\mathfrak{F}, V), w \models \varphi$ implies $(\mathfrak{F}, V'), w \models \varphi$.

Note that in a positive formula *negations* of modal or Boolean constants are allowed. Also, if $\phi$ is positive then $\phi$ is monotone in all proposition letters.

**Definition 3.2** Fix a modal similarity type $S$. A formula $\phi$ in $M(S)$ is a *Sahlqvist antecedent* if it is built up from formulas that are either negative or of the form $\Box_{i_1} \ldots \Box_{i_n} p$, using only $\vee, \wedge$ and $\nabla_i$, where $\Diamond_{i_1}, \ldots, \Diamond_{i_n}, \nabla_i \in S$.

Define the set of *Sahlqvist formulas* in $M(S)$ as being the smallest set $X$ such that if $\phi$ is a Sahlqvist antecedent, and $\psi$ is a positive formula, then $\phi \to \psi \in X$; if $\sigma_1, \sigma_2 \in X$ then $\sigma_1 \wedge \sigma_2 \in X$; and if $\sigma_1, \ldots, \sigma_{\rho(i)} \in X$ have no proposition letters in common, then $\lhd_i(\sigma_1, \ldots, \sigma_{\rho(i)}) \in X$.

For a modal similarity type $S$ that contains only unary operators several definitions exist of what it is for a formula in $M(S)$ to be a Sahlqvist formula; however, all are equivalent to (or are covered by) the restriction of 3.2 to such similarity types.

We believe that the generalization to arbitrary similarity types is in fact ours. One may wonder whether this is the obvious generalization from the 'unary case', e.g., why are boxes (i.e., duals of unary normal, additive operations) allowed in Sahlqvist antecedents, while for $n \geq 2$ duals of $n$-ary operations in $S$ are not? The reason why we are interested in Sahlqvist formulas is that they may be shown to be complete and to define certain first order properties of the underlying relations in (generalized) frames. A look at the kind of formulas forbidden in Sahlqvist antecedents in the unary case in order to guarantee these properties, shows that they typically include combinations of the form $\Box(\ldots \vee \ldots)$, or, in first order terms, $\forall(\ldots \vee \ldots)$. But these are precisely the combinations that pop up when we have $n$-ary boxes ($n \geq 2$) around! (In fact, if $\nabla$ is a binary modal operator, and $\lhd$ is its dual, then $(p \lhd p) \lhd p \to (p \nabla p) \nabla p$ may already be shown to be non-elementary.)

Before proving an important property of Sahlqvist formulas we recall that for a binary relation $R$, $\breve{R} = \{ (y, x) : Rxy \}$. To each modal formula $\phi$ we associate a set operator $F^\phi$ as follows. Let $P_1, \ldots, P_k$ be sets and let $\vec{P}$ abbreviate $P_1, \ldots, P_k$. $F^{p_j} = P_j$ $(1 \leq j \leq k)$, while $F^{\neg\phi}(\vec{P}) = (F^\phi(\vec{P}))^c$, and $F^{\phi\wedge\psi}(\vec{P}) = F^\phi(\vec{P}) \cap F^\psi(\vec{P})$. $F^{\nabla_i(\phi_1,\ldots,\phi_{\rho(i)})}(\vec{P}) = f_{R_i}(F^{\phi_1}(\vec{P}), \ldots, F^{\phi_{\rho(i)}}(\vec{P}))$, while $F^{\triangleleft_i(\phi_1,\ldots,\phi_{\rho(i)})}(\vec{P}) = g_{R_i}(F^{\phi_1}(\vec{P}), \ldots, F^{\phi_{\rho(i)}}(\vec{P}))$. We assume that the set operator corresponding to Boolean or modal constants is provided by the context in which these constants occur.

**Theorem 3.3** *Let $S$ be a modal similarity type. Let $\chi$ be a Sahlqvist formula in $M(S)$. Then $\chi$ corresponds to an $L_0(S)$-sentence $\alpha_\chi$ effectively obtainable from $\chi$.*

*Proof.* This is more or less similar to the proof of [7, Theorem 8] (cf. also [1, Theorem 9.10]). Assume that $\chi$ has the form $\phi \rightarrow \psi$.

Let $p_1, \ldots, p_k$ be the proposition letters occurring in $\chi$. Having $\mathfrak{F} = (W, \{ R_i : i \in I \}) \models \chi$ means having $\mathfrak{F} \models \forall\vec{P}\forall x\, (x \in F^\chi(\vec{P}))$. By assumption the latter formula has the form

$$\forall\vec{P}\forall x \left(x \in F^\phi(\vec{P}) \rightarrow x \in F^\psi(\vec{P})\right), \tag{2}$$

where $\phi$ is a Sahlqvist antecedent, and $\psi$ is a positive formula. Next, using such equivalences as

$$\forall\cdots \left((\Phi \wedge x \in F^{\phi_1\vee\phi_2}(\vec{P})) \rightarrow \Psi\right) \leftrightarrow \bigwedge_{j=1,2} \forall\cdots \left((\Phi \wedge x \in F^{\phi_j}(\vec{P})) \rightarrow \Psi\right), \tag{3}$$

$$\begin{aligned} &\forall\cdots \left((\Phi \wedge x \in F^{\nabla_i(\phi_1,\ldots,\phi_{\rho(i)})}(\vec{P})) \rightarrow \Psi\right) \leftrightarrow \\ &\qquad \forall\cdots\forall y_1\ldots y_{\rho(i)} \left((\Phi \wedge R_i x y_1 \ldots y_{\rho(i)} \wedge \bigwedge_j (y_j \in F^{\phi_j}(\vec{P}))) \rightarrow \Psi\right), \end{aligned} \tag{4}$$

and

$$\forall\cdots \left((\Phi \wedge x \in F^\nu(\vec{P})) \rightarrow \Psi\right) \leftrightarrow \forall\cdots \left(\Phi \rightarrow (\Psi \vee x \in F^{\neg\nu}(\vec{P}))\right), \tag{5}$$

(2) can be rewritten as a conjunction of formulas of the form

$$\forall\vec{P}\forall x\forall\vec{y}\vec{z} \left((\Phi \wedge \bigwedge_{j=1}^{k} \bigwedge_{l=1}^{m_j} (y_{lj} \in g_{R_{n_{lj}}} \cdots g_{R_{1_{lj}}}(P_j))) \rightarrow \bigvee_{j=1}^{h} (z_j \in F^{\psi_j}(\vec{P}))\right), \tag{6}$$

where $\Phi$ is a quantifier free $L_0$-formula ordering its variables in a certain way, and where all the $\psi_j$s are monotone. If a predicate variable $P$ occurs only in the consequent $\bigvee_{j=1}^{h}(z_j \in F^{\psi_j}(\vec{P}))$ in (6), then, by the monotonicity of the $\psi_j$s, it can be replaced by $\bot$, and the quantifier binding $P$ may be deleted. Thus we may assume that every predicate letter occurs in the consequent of (6) only if it occurs in the antecedent of (6).

By an easy argument we have that $\bigwedge_{l=1}^{m_j}(y_{lj} \in g_{R_{n_{lj}}} \cdots g_{R_{1_{lj}}}(P_j))$ if and only if we have $\bigcup_{l=1}^{m_j} f_{\breve{R}_{1_{lj}}} \cdots f_{\breve{R}_{n_{lj}}}(\{ y_{lj} \}) \subseteq P_j$. Thus by universal instantiation (6) implies the first order formula

$$\forall x\forall\vec{y}\vec{z} \left(\Phi \rightarrow \bigvee_{j=1}^{h} z_j \in F^{\psi_j}(\bigcup_{l=1}^{m_1} f_{\breve{R}_{1_{l1}}} \cdots f_{\breve{R}_{n_{l1}}}(\{ y_{l1} \}), \ldots, \bigcup_{l=1}^{m_k} f_{\breve{R}_{1_{lk}}} \cdots f_{\breve{R}_{n_{lk}}}(\{ y_{lk} \}))\right). \tag{7}$$

But, conversely, by the monotonicity of the functions $F^{\psi_j}$ (7) implies (6), and we are done.

To prove the general case one may argue inductively. If the Sahlqvist formulas $\chi_1, \chi_2$ have been shown to correspond to $\alpha_1, \alpha_2$, respectively, then $\chi_1 \wedge \chi_2$ corresponds to $\alpha_1 \wedge \alpha_2$; and if $\chi_1, \ldots, \chi_{\rho(i)}$ are Sahlqvist formulas that have no proposition letters in common, and that have been shown to correspond to $\forall x\, \alpha_1, \ldots, \forall x\, \alpha_{\rho(i)}$, then $\triangleleft_i(\chi_1, \ldots, \chi_{\rho(i)})$ corresponds to $\forall x \vec{y}\, (R_i x y_1 \ldots y_{\rho(i)} \rightarrow \alpha_1(y_1) \vee \ldots \vee \alpha_{\rho(i)}(y_{\rho(i)}))$. ⊣

Two remarks are in order. First, in the above result we may in fact replace 'corresponds' by 'locally corresponds'. But given the algebraic application we have in mind the *global* version is more natural. Second, although the algorithm in the above general proof may seem somewhat intractable or even obscure, in particular examples it is quite manageable, as is witnessed in §4.

**Theorem 3.4** *Let $S$ be a modal similarity type. For $j \in J$, let $\chi_j$ be Sahlqvist formulas in $M(S)$. Let $\Lambda$ be the modal logic axiomatized by $\{\, \chi_j : j \in J \,\}$. Then $\Lambda$ is canonical. Hence $\chi$ is complete with respect to the class of Kripke frames defined by $\{\, \alpha_{\chi_j} : j \in J \,\}$.*

*Proof.* The case where $S$ contains only unary modal operators is [7, Theorem 19]. To prove the general case one may use the same arguments together with the canonical frame construction for modal logics of arbitrary similarity type as found in [9, Chapter 2]. (An alternative proof of the unary case may be found in [8].) ⊣

We leave it to the reader to check that every Sahlqvist formula induces a Sahlqvist identity, and conversely.

**Theorem 3.5** *Let $\Sigma$ be a set of Sahlqvist equalities. Let $\mathsf{V}_\Sigma$ be the variety defined by $\Sigma$. Then $\mathsf{V}_\Sigma$ is canonical.*

*Proof.* Let $\widehat{\Sigma}$ be the set of modal translations of the elements of $\Sigma$. So $\widehat{\Sigma}$ is a set of Sahlqvist formulas. Now, to prove the theorem, let $\mathfrak{B} \in \mathsf{V}_\Sigma$. Let $\mathfrak{A}_\Sigma(|B|)$ be the free $\Sigma$-algebra on $|B|$ generators. Then $\mathfrak{A}_\Sigma(|B|) \twoheadrightarrow \mathfrak{B}$, and hence $\mathfrak{Em}\,\mathfrak{A}_\Sigma(|B|) \twoheadrightarrow \mathfrak{Em}\,\mathfrak{B}$, by [2, Corollary 3.2.5(6)]. So we are done once we have shown that $\mathfrak{Em}\,\mathfrak{A}_\Sigma(|B|) \in \mathsf{V}_\Sigma$.

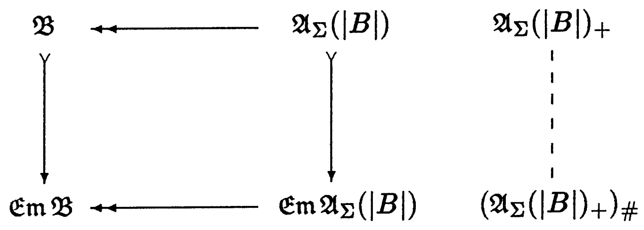


Figure 1.

Since $\mathfrak{A}_\Sigma(|B|) \models \Sigma$, $\mathfrak{A}_\Sigma(|B|)_+ \models \widehat{\Sigma}$. So by 3.4 $\mathfrak{Cs}\,\mathfrak{A}_\Sigma(|B|) = (\mathfrak{A}_\Sigma(|B|)_+)_\# \models \widehat{\Sigma}$. But then $\mathfrak{Em}\,\mathfrak{A}_\Sigma(|B|) = ((\mathfrak{A}_\Sigma(|B|)_+)_\#)^+ \models \Sigma$, i.e. $\mathfrak{Em}\,\mathfrak{A}_\Sigma(|B|) \in \mathsf{V}_\Sigma$. ⊣

**Remark 3.6** In [5] Jónsson and Tarski also describe a class of equalities that are preserved under canonical embedding algebras. The class they define contains all equalities $h_1 = h_2$, where both $h_1$ and $h_2$ are symbols for functions that are either additive, or obtained from additive ones by using composition only. Obviously, all Jónsson and Tarski equalities may be seen as (a conjunction of two) Sahlqvist equalities; but conversely, not every Sahlqvist equality is a Jónsson and Tarski equality. (As an example, $\Diamond\Box p \to \Box\Diamond p$ is a Sahlqvist formula, and hence its algebraic counterpart is a Sahlqvist equality; it is not a Jónsson and Tarski equality, however.) Hence, the class of Sahlqvist equalities forms a strict superset of the class of Jónsson and Tarski equalities.

It should be noted that unlike our result the Jónsson and Tarski result applies also to *non*-normal (but additive) BAO's. In a paper by Henkin [3], one can also find a description of a class of equalities whose validity is preserved under canonical embedding algebras; however, the BAO's considered there need not even be additive.

## *4 An example: simplifying Henkin's equation*

As an application of theorems 3.3 and 3.5 we show that, in order to prove that two Sahlqvist equations are equivalent over a canonical variety $\mathsf{V}$, it suffices to show the equivalence (in $\mathbf{At}\,\mathsf{V}$) of their first order translations. This means that reasoning can be done in the Kripke frames, which is usually much easier than manipulating algebraic equations.

**Proposition 4.1** *Let* $\mathsf{V}$ *be a canonical variety, and* $\eta_1$ *and* $\eta_2$ *two Sahlqvist equations with first order correspondents* $\alpha_1$ *and* $\alpha_2$*. Then*

$$\mathbf{At}\,\mathsf{V} \models \alpha_1 \leftrightarrow \alpha_2 \iff \mathsf{V} \models \eta_1 \leftrightarrow \eta_2.$$

*Proof.* From left to right: let $\mathfrak{A}$ be an algebra in $\mathsf{V}$ with $\mathfrak{A} \models \eta_i$. By the fact that $\eta_i$ is a Sahlqvist equation, $\eta_i$ holds in $\mathfrak{Em}\,\mathfrak{A} = (\mathfrak{Cs}\,\mathfrak{A})^+$. This gives $\mathfrak{Cs}\,\mathfrak{A} \models \alpha_i$, so by assumption $\mathfrak{Cs}\,\mathfrak{A} \models \alpha_j$. But then again $\mathfrak{Em}\,\mathfrak{A} \models \eta_j$, so $\eta_j$ holds in $\mathfrak{A} \leq \mathfrak{Em}\,\mathfrak{A}$.

From right to left: let $\mathfrak{F}$ be a frame in $\mathbf{At}\,\mathsf{V}$ with $\mathfrak{F} \models \alpha_i$. Then $\mathfrak{F}^+ \models \eta_i \Rightarrow \mathfrak{F}^+ \models \eta_j \Rightarrow \mathfrak{F} \models \alpha_j$. $\dashv$

We assume familiarity with the notion of a cylindric algebra (cf. [6], [4]), but we modify some notation and definitions. Without loss of generality we may confine ourselves to the two-dimensional case. The algebraic language $\mathcal{L}_2$ has a constant $d_{01}$ and two unary operators $c_0$ and $c_1$, which we write as $\Diamond_0$ and $\Diamond_1$, respectively, if we want to stress the modal aspects of the subject. A cylindric-type *frame* is a quadruple $\mathfrak{F} = (W, \sim_0, \sim_1, D)$ with $\sim_i$ a binary accessibility relation (for $\Diamond_i$) on $W$, and $D$ the subset of $W$ where $d_{01}$ holds. In the following table we list the modal versions of the axioms governing the variety of cylindric algebras, together with their first order equivalents ($i \in \{0,1\}$):

| | | | |
|---|---|---|---|
| $(C1_i)$ | $p \to \Diamond_i p$ | $(N1_i)$ | $\forall x\, x \sim_i x$ |
| $(C2_i)$ | $p \to \Box_i \Diamond_i p$ | $(N2_i)$ | $\forall xy\, (x \sim_i y \to y \sim_i x)$ |
| $(C3_i)$ | $\Diamond_i p \to \Diamond_i \Diamond_i p$ | $(N3_i)$ | $\forall xyz\, ((x \sim_i y \wedge y \sim_i z) \to x \sim_i z)$ |
| $(C4_i)$ | $\Diamond_i \Diamond_j p \to \Diamond_j \Diamond_i p$ | $(N4_i)$ | $\forall xyz\, ((x \sim_i y \wedge y \sim_j z) \to \exists u\, (x \sim_j u \wedge u \sim_i z))$ |
| $(C5_i)$ | $\Diamond_i d_{01}$ | $(N5_i)$ | $\forall x \exists y\, (x \sim_i y \wedge Dy)$ |
| $(C6_i)$ | $\Diamond_i(d_{01} \wedge p) \to \Box_i(d_{01} \to p)$ | $(N6_i)$ | $\forall xyz\, ((x \sim_i y \wedge x \sim_i z \wedge Dy \wedge Dz) \to y = z)$. |

We define $C1 = C1_0 \wedge C1_1$, etc. A *cylindric algebra* is an algebra $\mathfrak{A} = (A, +, -, c_0, c_1, d_{01})$ such that $(A, +, -)$ is a Boolean Algebra, $c_0$ and $c_1$ are normal and additive, and $C1, \ldots, C6$ are valid in $\mathfrak{A}$. The variety of cylindric algebras is denoted by $\mathsf{CA}$.

A *cylindric frame* is a cylindric type frame $\mathfrak{F}$ such that $N_1, \ldots, N6$ are valid in $\mathfrak{F}$. So a frame $\mathfrak{F} = (W, \sim_0, \sim_1, D)$ is cylindric iff $\sim_0$ and $\sim_1$ are equivalence relations ($N1$, $N2$ and $N3$ for respectively reflexivity, symmetry and transitivity), every $\sim_i$-equivalence class contains precisely one 'diagonal' element in $D$ ($N5$ for existence, $N6$ for unicity), and $\sim_0$ and $\sim_1$ permute ($N4$). Below these facts may be used without notice.

The following proposition is immediate by the Sahlqvist form of $C1, \ldots, C_6$, and theorems 3.3 and 3.4.

**Proposition 4.2** (i) *$\mathfrak{F}$ is a cylindric frame iff $\mathfrak{F}^+$ is a cylindric algebra.*
(ii) CA *is a canonical variety.*

Besides the axioms $C1, \ldots, C6$ governing the variety of cylindric algebras, additional equations play an important rôle, especially *Henkin's equation*

$$(\eta) \qquad c_0(x \cdot -y \cdot c_1(x \cdot y)) \leq c_1(-d_{01} \cdot c_0 x).$$

For example, it can be shown that adding $\eta$ to $C1, \ldots, C6$, one obtains a complete equational axiom system for the set of equations valid in the variety of *representable* cylindric algebras. (This is only true in the two-dimensional case; in the higher dimensional case the rôle of $\eta$, though important, is not decisive.) One might wonder why the authors of [4] decided against giving $\eta$ the status of a CA-axiom. One of the reasons may have been that $\eta$ is less transparent than the other seven. In the remainder of this section we will show that $\eta$ has a simpler equivalent (over the variety CA), and that the equivalence is very easy to prove using the Sahlqvist form of the equations.

So let us define the intended simplification of Henkin's equation:

$$(\eta') \qquad d_{01} \cdot c_0(-x \cdot c_1 x) \leq c_1(d_{01} \cdot c_0 x).$$

Clearly both $\eta$ and $\eta'$ are Sahlqvist equations. Let us compute their first order equivalents.

**Definition 4.3** Let $\alpha$, $\alpha'$ be the formulas

$$(\alpha) \quad \forall u \forall v \forall w \Big( (u \sim_0 v \sim_1 w \wedge v \neq w) \rightarrow \exists x (\neg Dx \wedge u \sim_1 x \wedge (x \sim_0 v \vee x \sim_0 w)) \Big)$$

$$(\alpha') \quad \forall u \forall v \forall w \Big( (Du \wedge u \sim_0 v \sim_1 w \wedge v \neq w) \rightarrow \exists x (\neg Dx \wedge u \sim_1 x \sim_0 w) \Big).$$

The following pictures explain the meaning of $\alpha$ and $\alpha'$ for cylindric frames:

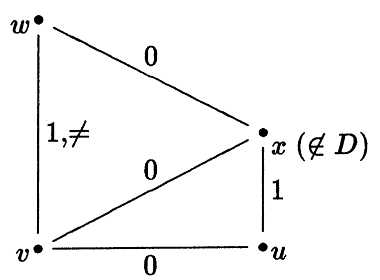


Figure 2: $\alpha$

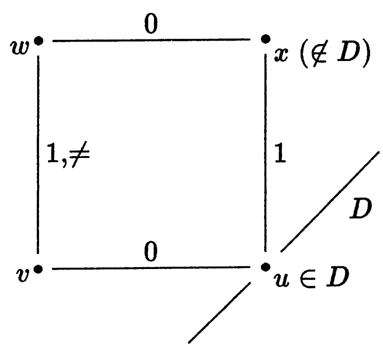


Figure 3: $\alpha'$

**Proposition 4.4** *Let $\mathfrak{F}$ be a frame of the appropriate type. Then $\mathfrak{F} \models \alpha \iff \mathfrak{F}^+ \models \eta$ and $\mathfrak{F} \models \alpha' \iff \mathfrak{F}^+ \models \eta'$.*

*Proof.* For $\eta$, we will spell out the algorithm of theorem 3.3 to find its first order correspondent. First consider its modal variant

$$(\chi) \qquad \Diamond_0(p \wedge \neg q \wedge \Diamond_1(p \wedge q)) \rightarrow \Diamond_1(\neg d_{01} \wedge \Diamond_0 p).$$

Let $\phi$ and $\psi$ be respectively the antecedent $\Diamond_0(p \wedge \neg q \wedge \Diamond_1(p \wedge q))$ and the consequent $\Diamond_1(\neg d_{01} \wedge \Diamond_0 p)$ of this formula. Clearly $\chi$ is a Sahlqvist formula, as $\phi$ is a Sahlqvist antecedent and $\psi$ is positive.
Now let $\mathfrak{F} = (W, \sim_0, \sim_1, D)$ be a Kripke frame for the language, then $\mathfrak{F} \models \chi$ iff

$$(8) \qquad \mathfrak{F} \models \forall x \forall P \forall Q (x \in \mathfrak{F}^\chi(P, Q)).$$

Now the formula $x \in F^\chi(P, Q)$ is by definition equivalent to

$$(9) \qquad x \in F^\phi(P, Q) \rightarrow x \in F^\psi(P, Q).$$

Step by step we will rewrite (9), abbreviating $u \in P$ by $Pu$. Starting with the antecedent of (9), we obtain

$$\exists y_1(x \sim_0 y_1 \wedge y_1 \in F^{p \wedge \neg q \wedge \Diamond_1(p \wedge q)}(P, Q)) \rightarrow x \in F^\psi(P, Q),$$

or better

$$\forall y_1\Big((x \sim_0 y_1 \wedge y_1 \in F^{p \wedge \neg q \wedge \Diamond_1(p \wedge q)}(P, Q)) \rightarrow x \in F^\psi(P, Q)\Big),$$

yielding the effect of (4). Then we get

$$\forall y_1\Big((x \sim_0 y_1 \wedge Py_1 \wedge \neg Qy_1 \wedge y_1 \in F^{\Diamond_1(p \wedge q)}(P, Q)) \rightarrow x \in F^\psi(P, Q)\Big),$$

and (5) gives

$$\forall y_1\Big((x \sim_0 y_1 \wedge Py_1 \wedge y_1 \in F^{\Diamond_1(p \wedge q)}(P, Q)) \rightarrow (x \in F^\psi(P, Q) \vee Qy_1)\Big).$$

Using (4), we obtain

$$(10) \quad \forall y_1 \forall y_2\Big((x \sim_0 y_1 \wedge Py_1 \wedge y_1 \sim_1 y_2 \wedge Py_2 \wedge Qy_2) \rightarrow (x \in F^\psi(P, Q) \vee Qy_1)\Big).$$

So we have $\mathfrak{F} \models \chi$ iff the following formula holds in $\mathfrak{F}$:

$$\forall x \forall P \forall Q \forall y_1 \forall y_2\Big((x \sim_0 y_1 \wedge y_1 \sim_1 y_2 \wedge Py_1 \wedge Py_2 \wedge Qy_2) \rightarrow (x \in F^\psi(P, Q) \vee Qy_1)\Big).$$

Comparing this formula with (6), we observe that for both $y_1$ and $y_2$ the sequence $g_{R_{n_{1j}}} \cdots g_{R_{1_{1j}}}$ of (6) is empty, so the universal instantiation mentioned just above (7) simply means replacing $Pu$ by $u \in \{y_1, y_2\}$ (or better, by $(u = y_1 \vee u = y_2)$), and $Qu$ by $(u = y_2)$.

So (10) is equivalent to the following instance of (7), viz.

$$\forall x \forall y_1 \forall y_2\Big((x \sim_0 y_1 \wedge y_1 \sim_1 y_2) \rightarrow (x \in F^\psi(\{y_1, y_2\}, \{y_2\}) \vee (y_1 = y_2))\Big),$$

which really means

$$\forall x \forall y_1 \forall y_2 \Big( (x \sim_0 y_1 \wedge y_1 \sim_1 y_2) \rightarrow \\ \Big( y_1 = y_2 \vee \exists z_1 (x \sim_1 z_1 \wedge \neg D z_1 \wedge \exists z_2 (z_1 \sim_0 z_2 \wedge (z_2 = y_1 \vee z_2 = y_2))) \Big) \Big).$$

Transporting $(y_1 = y_2)$ back to the antecedent, and after some straightforward formula manipulation, we finally obtain

$$\forall x \forall y_1 \forall y_2 \Big( (x \sim_0 y_1 \wedge y_1 \sim_1 y_2 \wedge y_1 \neq y_2) \rightarrow \exists z_1 (x \sim_1 z_1 \wedge \neg D z_1 \wedge (z_1 \sim_0 y_1 \vee z_1 \sim_0 y_2)) \Big),$$

which is what we were after. $\dashv$

**Proposition 4.5** *Let $\mathfrak{A}$ be a cylindric algebra. Then $\mathfrak{A} \models \eta \iff \mathfrak{A} \models \eta'$.*

*Proof.* By the previous two propositions it is sufficient to show that for a cylindric frame $\mathfrak{F}$, $\mathfrak{F} \models \alpha \iff \mathfrak{F} \models \alpha'$.
($\Leftarrow$) Assume that $\mathfrak{F} \models \alpha'$. To prove that $\mathfrak{F} \models \alpha$, let $u, v$ and $w$ be worlds in $\mathfrak{F}$ with $u \sim_0 v \sim_1 w$ and $v \neq w$. We have to find an $x$ with $x \notin D$, $u \sim_1 x$ such that $x$ is in the 0-equivalence class with $v$ or with $w$. Distinguish the following cases:

Case 1: $u \in D$.
Then $\mathfrak{F} \models \alpha'$ immediately gives us the desired $x$, with $x \sim_0 w$.

Case 2: $u \notin D$.
Then $u$ itself is the desired $x$, as $u \sim_0 v$ and $u \sim_1 w$.

($\Rightarrow$) For the other direction, we assume that $\mathfrak{F} \models \alpha$, we consider arbitrary $u, v$ and $w$ in $\mathfrak{F}$ with $u \notin D$, $u \sim_0 v \sim_1 w$ and $v \neq w$, and set ourselves the task to find an $x$ with $x \notin D$ and $u \sim_1 x \sim_0 w$, viz. Figure 3.

Since $\mathfrak{F} \models \alpha'$, there is a $y \notin D$ with $u \sim_1 y$ and $y \sim_0 v$ or $y \sim_0 w$. Distinguish

Case 1: $y \sim_0 w$.
This means we are finished immediately: take $x = y$.

Case 2: $y \sim_0 v$.
Since $\mathfrak{F} \models N4$, there is a $z$ in $\mathfrak{F}$ with $u \sim_1 z \sim_0 w$, as in Figure 4:

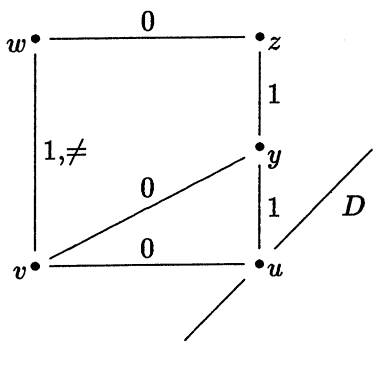


Figure 4.

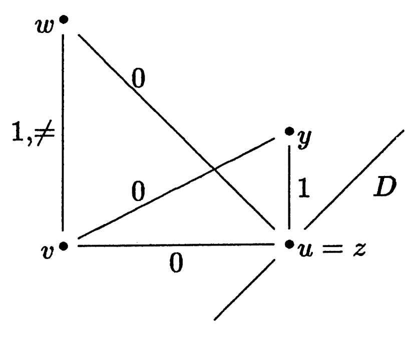


Figure 5.

Distinguish

Case 2.1: $z \notin D$.
Again we are finished: take $x = z$.

Case 2.2: $z \in D$.
This implies $z = u$ because $\mathfrak{F} \models N6$, so we have the situation depicted in Figure 5. We now have $w \sim_0 z = u \sim_0 v \sim_0 y$, so $y \sim_0 w$ after all, and we are back in case 1: take $x = y$. $\dashv$

## *References*


[1] J. van Benthem. *Modal Logic and Classical Logic.* Bibliopolis, Naples, 1983.

[2] R.I. Goldblatt. Varieties of complex algebras. *Annals of Pure and Applied Logic*, 38:173–241, 1989.

[3] L. Henkin. Extending Boolean operations. *Pacific Journal of Mathematics*, 32:723–752, 1970.

[4] L. Henkin, J.D. Monk, and A. Tarski. *Cylindric Algebras. Part 1. Part 2.* North-Holland, Amsterdam, 1971, 1985.

[5] J. Jónsson and A. Tarski. Boolean algebras with operators, Part I. *American Journal of Mathematics*, 73:891–939, 1952.

[6] I. Németi. Algebraizations of quantifier logics: an introductory overview. *Studia Logica*, 1991. To appear.

[7] H. Sahlqvist. Completeness and correspondence in the first and second order semantics for modal logic. In S. Kanger, editor, *Proceedings of the Third Scandinavian Logic Symposium. Uppsala 1973*, pages 110–143, Amsterdam, 1975. North-Holland.

[8] G. Sambin and V. Vaccaro. A topological proof of Sahlqvist's theorem. *The Journal of Symbolic Logic*, 54:992–999, 1989.

[9] Y. Venema. *Many-Dimensional Modal Logic.* PhD thesis, Department of Mathematics and Computer Science, University of Amsterdam, 1991. To appear.