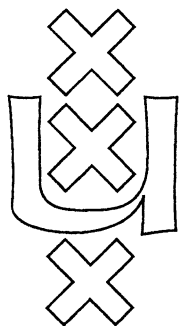


Institute for Logic, Language and Computation

**COMPARING MODELS OF THE
NON-EXTENSIONAL TYPED λ -CALCULUS**

Raymond Hoofman

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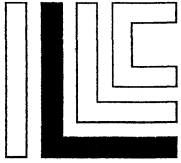


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COMPARING MODELS OF THE NON-EXTENSIONAL TYPED λ -CALCULUS

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Comparing Models of the Non-Extensional Typed λ -Calculus

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Abstract

In this paper we compare “powerset models” of the non-extensional typed lambda calculus. We show that the choice of a certain *minimal* interpretation (with respect to a certain class of interpretations) of the type-constructor \Rightarrow yields models with a *maximal* theory (in that class).

1 Introduction

As opposed to extensional lambda calculi, which require the interpretation of abstracted terms (within isomorphism) to be functions, non-extensional calculi allow a large degree of freedom in the choice of their models. This is already apparent for the untyped non-extensional lambda calculus. For example, the standard interpretation of a lambda abstracted term in a set-theoretical model like Engeler’s graph model is as follows

$$[\lambda x.t]_\rho = \{(X, b) \mid b \in [t]_{\rho[X/x]}, X \text{ finite}\}.$$

However, a (related) interpretation

$$[\lambda x.t]_\rho = \{(X, Y) \mid Y \subseteq [t]_{\rho[X/x]}, X, Y \text{ finite}\},$$

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would be equally justifiable (see e.g. [4]).

Similarly, in the typed non-extensional lambda calculus, there is in general no canonical choice for the interpretation of the type $\sigma \Rightarrow \tau$. This can be illustrated by considering the category \mathbf{Pow} of powersets and continuous functions, which supports various interpretations of the typed lambda calculus. For example, the above two untyped models are solutions of the recursive equation $D = (D \Rightarrow D)$ in \mathbf{Pow} , where we interpret \Rightarrow in the first case as

$$\mathcal{P}A \Rightarrow_m \mathcal{P}B = \mathcal{P}\{(X, b) \mid X \subseteq A \text{ finite}, b \in B\},$$

and in the second as

$$\mathcal{P}A \Rightarrow_n \mathcal{P}B = \mathcal{P}\{(X, Y) \mid X \subseteq A \text{ finite}, Y \subseteq B \text{ finite}\}.$$

Several questions arise concerning the canonicity of the various interpretations and their associated lambda-theories. In this paper we consider, as a particular case study, interpretations of the non-extensional typed lambda calculus in the category \mathbf{Pow} . We show that the interpretation \Rightarrow_m mentioned above is *minimal* for the class of *linear* interpretations (i.e., interpretations in which the application operator preserves arbitrary lubs in its first argument). As a consequence, the theory of \Rightarrow_m is *maximal* among the theories of linear interpretations (theorem 25).

2 Preliminaries

Let \mathbf{C} be a full subcategory of the category \mathbf{Dcpo} of directed complete partial orders (dcpo's) and continuous (i.e., directed lub preserving) functions. Given objects $D, E \in \mathbf{C}$, their function space $[D, E]$ (consisting of the continuous functions $D \rightarrow E$ ordered pointwise) is a dcpo but need not be an object in \mathbf{C} . Hence we are interested in “approximations” of the function space in \mathbf{C} , or, more formally, in objects $D \Rightarrow E \in \mathbf{C}$ having $[D, E]$ as a retract. In detail, such a retract is given by the following items:

- ◇ a continuous function $\bullet : (D \Rightarrow E) \times D \rightarrow E$,
- ◇ a continuous function $R : [D, E] \rightarrow (D \Rightarrow E)$,

satisfying the requirement

$$R(f) \bullet x = f(x).$$

In case $\mathcal{C} \subseteq \mathbf{Dcpo}$, the category-theoretically notion of a *semi-exponent* in \mathcal{C} exactly corresponds to the above notion of an object approximating a function space. Recall the following definition from [2, 3].

Definition 1 *Let \mathcal{C} be a category with finite products and $D, E \in \mathcal{C}$ objects. A semi-exponent of D, E is an object $D \Rightarrow E \in \mathcal{C}$ together with*

- ◇ an arrow $\varepsilon : (D \Rightarrow E) \times D \rightarrow E$ in \mathcal{C} ,
- ◇ an arrow $\Lambda(f) : D' \rightarrow (D \Rightarrow E)$ in \mathcal{C} ,
for each continuous $f : D' \times D \rightarrow E$,

satisfying the requirements

1. $\varepsilon \circ (\Lambda(f) \times id)$,
2. $\Lambda(\varepsilon \circ (f \times id))$.

Proposition 2 *For $\mathcal{C} \subseteq \mathbf{Dcpo}$, there is a bijective correspondence in \mathcal{C} between semi-exponents and objects having function spaces as retracts.*

Proof: Given a semi-exponent $D \Rightarrow E$, there is a retraction between $D \Rightarrow E$ and $[D, E]$ given by $\bullet = \varepsilon$ and

$$R(f) = \Lambda(R(1 \times D \xrightarrow{\pi_3} D \xrightarrow{f} E)).$$

The other way round, an object $D \Rightarrow E$ having the function space $[D, E]$ as a retract gives rise to a semi-exponent with $\varepsilon = \bullet$ and for $f : D' \times D \rightarrow E$,

$$\Lambda(f)(d') = R(f(d', -)).$$

It is easily checked that the above defines the required bijection. ■

Recall that a *weak cartesian closed* structure ([2, 3]) on a category \mathcal{C} assigns to each pair of objects $D, E \in \mathcal{C}$ a semi-exponent $D \Rightarrow E$. By the above proposition, a weak cartesian closed structure on a category $\mathcal{C} \subseteq \mathbf{Dcpo}$

chooses an “approximation” in \mathbf{C} of the function space $[D, E]$ for each pair $D, E \in \mathbf{C}$.

The main example in this paper of a subcategory $\mathbf{C} \subseteq \mathbf{Dcpo}$ which is *not* closed under function spaces is the category \mathbf{Pow} of powersets (ordered by subset inclusion) and continuous functions. As the following example shows however, we can define various kinds of semi-exponents in \mathbf{Pow} .

Example 3 Define semi-exponents \Rightarrow_m , \Rightarrow_n , and \Rightarrow_s for a set S on \mathbf{Pow} as follows:

- $\diamond \mathcal{P}A \Rightarrow_m \mathcal{P}B = \mathcal{P}\{(X, b) \mid X \subseteq A \text{ finite}, b \in B\},$
 $\phi \bullet x = \{b \mid \exists (X, b) \in \phi(X \subseteq x)\},$
 $R(f) = \{(X, b) \mid b \in f(X), X \text{ finite}\}.$
- $\diamond \mathcal{P}A \Rightarrow_n \mathcal{P}B = \mathcal{P}\{(X, Y) \mid X \subseteq A \text{ finite}, Y \subseteq B \text{ finite}\},$
 $\phi \bullet x = \bigcup \{Y \mid \exists (X, Y) \in \phi(X \subseteq x)\},$
 $R(f) = \{(X, Y) \mid Y \subseteq f(X), X, Y \text{ finite}\}.$
- $\diamond \mathcal{P}A \Rightarrow_s \mathcal{P}B = \mathcal{P}(\{(X, b) \mid X \subseteq A \text{ finite}, b \in B\} \cup S),$
 $\phi \bullet x = \{b \mid \exists (X, b) \in \phi(X \subseteq x)\},$
 $R(f) = \{(X, b) \mid b \in f(X), X \text{ finite}\} \cup S.$

Many more semi-exponents exist in \mathbf{Pow} . As we will see later on, the semi-exponent $D \Rightarrow_m E$ is *minimal* for a certain class of semi-exponents in the sense that it is a retract of each member $D \Rightarrow E$ of that class. Intuitively, the semi-exponent \Rightarrow_m gives a *best* approximation (with respect to the class) of the function space in \mathbf{Pow} .

The full subcategory $\mathbf{Alg} \subseteq \mathbf{Dcpo}$ of *algebraic* dcpo’s provides a further example of a category lacking function spaces. Recall that an element $x \in D$ is *compact* iff for each directed subset $S \subseteq D$, $x \leq \bigvee S$ implies $\exists y \in S (x \leq y)$. A dcpo D is *algebraic* iff for each $x \in D$ the set of compact elements below x is directed and has x as least upperbound. It is well-known that for algebraic dcpo’s D, E the function space $[D, E]$ need not be algebraic. Semi-exponents, however, can easily be found in \mathbf{Alg} . For example, for algebraic dcpo’s $D, E \in \mathbf{Alg}$ take

- $\diamond D \Rightarrow_{fun} E = \mathcal{P}\{f : D \rightarrow E \mid f \text{ continuous}\},$
- $\diamond \phi \bullet x = \bigvee \{f(x) \mid f \in \phi\},$

$$\diamond R(f) = \{c \mid c \leq f \text{ \& } c \text{ compact}\},$$

where a function $D \rightarrow E$ is compact iff it is compact as an element of the dcpo $[D, E]$. Note that the subcategory $\text{Pow} \subseteq \text{Alg}$ is closed under the semi-exponent \Rightarrow_{fun} .

3 Models of the Typed Lambda Calculus

For each weak cartesian closed structure $(\Rightarrow, \bullet, R)$ on a full subcategory $\mathbb{C} \subseteq \text{Dcpo}$, we define an interpretation of the typed lambda calculus (with a base type o). First, fix an object $D \in \mathbb{C}$. Then, assign to each type σ an object $D^\sigma \in \mathbb{C}$ as follows:

$$\begin{aligned} \diamond D^o &= D, \\ \diamond D^{\sigma \Rightarrow \tau} &= D^\sigma \Rightarrow D^\tau. \end{aligned}$$

An environment ρ is a function $Var \rightarrow \bigcup_\sigma D^\sigma$ (where Var is the set of (typed) variables) satisfying the requirement $\rho(x^\sigma) \in D^\sigma$. By $\rho[d/x]$ we denote the environment equal to ρ except that it yields d for x . For each lambda term t^σ and each environment ρ , we define an element $[t]_\rho \in D^\sigma$ by the following inductive clauses:

$$\begin{aligned} \diamond [x]_\rho &= \rho(x), \\ \diamond [st]_\rho &= [s]_\rho \bullet [t]_\rho, \\ \diamond [\lambda x.t]_\rho &= R([t]_{\rho[-/x]}), \end{aligned}$$

where $[t]_{\rho[-/x]}$ is the (continuous) function given by $[t]_{\rho[-/x]}(d) = [t]_{\rho[d/x]}$. It is left to the reader to check that the above interpretation is well-defined, but note that it corresponds to the (general) notion of an interpretation of the typed lambda calculus in a weak cartesian closed category [2]. We call $\mathcal{D} = (\{D^\sigma\}, [\cdot])$ the interpretation *based* on D and the semi-exponent \Rightarrow .

As usual we say that $\mathcal{D}, \rho \models s = t$ iff $[s]_\rho = [t]_\rho$ in the interpretation based on D and \Rightarrow . Furthermore, $\mathcal{D} \models s = t$ iff $\mathcal{D}, \rho \models s = t$ holds for all environments ρ . By general results of [2], all the equalities of the typed $\lambda\beta$ -calculus hold in \mathcal{D} . Moreover since the η -rule need not be satisfied, \mathcal{D} is a model of the the *non-extensional* typed lambda calculus.

Let the *theory* $Th_{\mathcal{D}}$ denote the set of equalities $\{s = t \mid \mathcal{D} \models s = t\}$. In this paper we are interested in comparing the theories based on distinguished weak cartesian closed structures \Rightarrow and \Rightarrow' on \mathbf{Pow} . The following examples shows that in general these theories need not be the same.

Example 4 Consider $\mathcal{P}\emptyset^{\circ\Rightarrow\circ}$ in the model based on \Rightarrow_m . It is easy to see that this is equal to $\mathcal{P}\emptyset$ and hence that all terms $t^{\circ\Rightarrow\circ}$ have identical interpretations in this model. In particular, $(x^{\circ\Rightarrow\circ} = \lambda y^{\circ}.xy)$ holds in the model.

Next consider $\mathcal{P}\emptyset^{\circ\Rightarrow\circ}$ in the model based on \Rightarrow_n . A simple calculation shows that this is equal to $\mathcal{P}\{\langle\emptyset, \emptyset\rangle\}$. Fix an environment ρ satisfying $\rho(x^{\circ\Rightarrow\circ}) = \emptyset$, then the interpretation of $\lambda y.xy$ in this environment is $\{\langle\emptyset, \emptyset\rangle\}$, whereas the interpretation of x is \emptyset . Hence $(x^{\circ\Rightarrow\circ} = \lambda y.xy)$ does not hold in the model.

4 The Semi-Exponent \Rightarrow_m is Minimal

In this section we show that each *linear* semi-exponent in \mathbf{Pow} “contains” the semi-exponent \Rightarrow_m . First, a semi-exponent $\mathcal{P}A \Rightarrow \mathcal{P}B$ in \mathbf{Pow} is called *linear* iff the associated function $\bullet : (\mathcal{P}A \Rightarrow \mathcal{P}B) \times \mathcal{P}A \rightarrow \mathcal{P}B$ preserves arbitrary lubs in its first argument, i.e., $(\bigcup S)\bullet x = \bigcup_{\phi \in S}(\phi\bullet x)$. All semi-exponents mentioned till now are linear. Here are two examples of non-linear semi-exponents.

Example 5 Think of $\mathcal{P}A \Rightarrow \mathcal{P}B$ as a set of automaton which take input from $\mathcal{P}A$ and yield output in $\mathcal{P}B$. Each automaton ϕ is determined by a set of instructions of the form (X, b) (“on input X yield output b ”) and can furthermore be switched on or off. Accordingly, we define

- ◇ $\mathcal{P}A \Rightarrow_{aut} \mathcal{P}B = \mathcal{P}(\{(X, b) \mid X \subseteq A \text{ finite}, b \in B\} \cup \{\mathbf{on}\})$,
- ◇ $\phi\bullet x = \{b \mid \exists(X, b) \in \phi(X \subseteq x) \ \& \ \mathbf{on} \in \phi\}$,
- ◇ $R(f) = \{(X, b) \mid b \in f(X), X \text{ finite}\} \cup \{\mathbf{on}\}$.

Note that each function f is represented by an enabled automaton.

Example 6 Fix an element $a \in A$. Define

- ◇ $\mathcal{P}A \Rightarrow_a \mathcal{P}B = \mathcal{P}A \Rightarrow \mathcal{P}B$,

$$\diamond \phi \bullet x = \{b \mid \exists (X, b) \in \phi (X \subseteq x \ \& \ (X \cup \{a\}, b) \in \phi)\},$$

$$\diamond R(f) = R_m(f).$$

Second, we introduce the notion of *elementary* members of a semi-exponent (on Pow).

Definition 7 For a continuous function $f : \mathcal{P}A \rightarrow \mathcal{P}B$ define $[f] \in \mathcal{P}A \Rightarrow_m \mathcal{P}B$ by

$$[f] = \{(X, b) \mid b \in f(X) \ \& \ \mu(X, f, b)\},$$

where

$$\mu(X, f, b) \Leftrightarrow (Y \subseteq X \ \& \ b \in f(Y) \Rightarrow Y = X).$$

For an arbitrary $\phi \in \mathcal{P}A \Rightarrow \mathcal{P}B$, we write $[\phi]$ for $[f_\phi]$ (where $f_\phi(x) = \phi \bullet x$).

Proposition 8 The operator $[\cdot]$ satisfies the following items:

1. $[f] \bullet x = f(x)$ (i.e., “[f] represents f ”),
2. $f \leq g \ \& \ [g]$ finite $\Rightarrow [f]$ finite.

We leave the (simple) proof of this proposition to the reader.

Recall that a continuous function $f : \mathcal{P}A \rightarrow \mathcal{P}B$ is *compact* iff f is compact as an element of the dcpo $[\mathcal{P}A, \mathcal{P}B]$.

Proposition 9 If f is compact, then $[f]$ is a finite set.

Proof: Suppose that $[f]$ is an infinite set. The set $S = \{f_\phi \mid \phi \subseteq [f] \ \& \ \phi \text{ finite}\}$ is directed and has f as lub. However, by proposition 8.2 the function f is not below any element in S , and hence f is not compact. ■

The other way round,

Proposition 10 If $\phi \subseteq \mathcal{P}A \Rightarrow_m \mathcal{P}B$ is finite, then f_ϕ is compact.

Proof: Suppose that $f_\phi \leq \bigcup S$ with S directed, then $\phi \bullet x = f_\phi(x) \subseteq \bigcup S(x) = \bigcup_{g \in S} g(x)$ for all x . Hence if $(X, b) \in \phi$, then $b \in \phi \bullet X$ and there exists $g^{(X, b)} \in S$ such that $b \in g^{(X, b)}(X)$. By assumption the set $\{g^{(X, b)} \mid (X, b) \in \phi\}$ is finite and hence has upperbound (say) $g \in S$. For arbitrary x and $b \in f_\phi(x)$ we have $b \in \phi \bullet x$, hence $\exists (X, b) \in \phi (X \subseteq x)$ and

$b \in g^{(X,b)}(X) \subseteq g(X) \subseteq g(x)$. We conclude that $f_\phi \leq g$. ■

From the above two proposition follows:

Corollary 11 *The continuous function f is compact iff $[f]$ is a finite set.*

In general, the (analogue of) proposition 10 need not hold for an arbitrary semi-exponent. We call a semi-exponent $\mathcal{P}A \Rightarrow \mathcal{P}B$ *elementary* iff each finite subset $\phi \subseteq \mathcal{P}A \Rightarrow \mathcal{P}B$ represents a compact function (i.e., f_ϕ is compact). In other words, a semi-exponent is elementary iff for each finite subset $\phi \subseteq \mathcal{P}A \Rightarrow \mathcal{P}B$ we have that $[\phi]$ is a finite set. Except for \Rightarrow_{fun} , all the examples of semi-exponents on Pow we have seen are elementary.

For an arbitrary linear semi-exponent \Rightarrow , we call $\phi \in \mathcal{P}A \Rightarrow \mathcal{P}B$ *elementary* iff for all $n \in \phi$ we have that $[\{n\}]$ is finite. Note that the fact that $[\phi]$ is finite, implies that ϕ is elementary. For finite sets ϕ , the reverse of this implication also holds. In elementary semi-exponents, all elements ϕ are elementary, and vice-versa.

We now show that the semi-exponent \Rightarrow_m can be embedded in each linear semi-exponent \Rightarrow . First we define a function $r : (\mathcal{P}A \Rightarrow \mathcal{P}B) \rightarrow (\mathcal{P}A \Rightarrow_m \mathcal{P}B)$ by

$$r(\phi) = \bigcup \{[n] \mid n \in \phi\},$$

where $[n]$ denotes $[\{n\}]$.

Proposition 12 *The function r has the following properties:*

1. For all sets V , $r(\bigcup V) = \bigcup_{\phi \in V} r(\phi)$.
2. If ϕ is an elementary finite set, then $r(\phi)$ is finite.
3. For arbitrary x , $\phi \bullet x = r(\phi) \bullet_m x$.
4. For all continuous functions f , $rR(f) \subseteq R_m(f)$.

Proof: We leave the proofs of 1 and 2 as exercises to the reader and consider 3. Suppose $b \in \phi \bullet x$, then by linearity of \bullet in its first argument and continuity in its second, there exists a minimal finite $X \subseteq x$ and $n \in \phi$ such that $b \in \{n\} \bullet X$. Hence $(X, b) \in r\{n\} \subseteq r(\phi)$ and $b \in r(\phi) \bullet_m x$. The other way

round, suppose that $b \in r(\phi) \bullet x$, then there exists $(X, b) \in r(\phi)$ such that $X \subseteq x$. Hence, by definition of r , $b \in \phi \bullet X \subseteq \phi \bullet x$.

For the proof of 4, suppose that $(X, b) \in rR(f)$, then there exists $n \in R(f)$ such that $b \in \{n\} \bullet X$. Hence $b \in R(f) \bullet X$, from which it follows that $b \in f(X)$. By definition of R_m , we then have $(X, b) \in R_m(f)$. ■

Next we show that there exists a right-inverse $s : (\mathcal{P}A \Rightarrow_m \mathcal{P}B) \rightarrow (\mathcal{P}A \Rightarrow \mathcal{P}B)$ for r .

Proposition 13 *Suppose $\mathcal{P}A \Rightarrow \mathcal{P}B$ is a linear semi-exponent, $X \subseteq \mathcal{P}A$ finite, and $b \in B$. Then there exists $n \in \bigcup(\mathcal{P}A \Rightarrow \mathcal{P}B)$ such that $\{n\} \bullet x = \{b\}$ if $X \subseteq x$ and \emptyset otherwise. Moreover, if $b \in f(X)$, then $n \in R(f)$.*

Proof: Let g denote the continuous function defined by $g(x) = \{b\}$ if $X \subseteq x$ and \emptyset otherwise. Then there exists $n \in R(g)$ such that $\{n\} \bullet X = \{b\}$. It is easy to see that in fact $f_{\{n\}} = g$. Furthermore, for arbitrary f , suppose that $b \in f(X)$, then $g \leq f$, hence $n \in R(g) \subseteq R(f)$. ■

Fix for each (X, b) an element $n_{(X,b)}$ such as given by the above proposition (there may be many of them), and define

$$s(\phi) = \{n_{(X,b)} \mid (X, b) \in \phi\}.$$

Note that in the definition of s the axiom of choice is actually needed.

Proposition 14 *The function s has the following properties:*

1. For all sets V , $s(\bigcup V) = \bigcup_{\phi \in V} s(\phi)$.
2. If ϕ a finite set, then $s(\phi)$ is a finite set.
3. For arbitrary x , $\phi \bullet_m x = s(\phi) \bullet x$.
4. For all continuous functions f , $sR_m(f) \subseteq R(f)$.
5. For all ϕ , $s(\phi)$ is elementary.

Furthermore, we have

Proposition 15 $r \circ s = id$

Hence, there exists an *application preserving* embedding of \Rightarrow_m in an arbitrary linear semi-exponent on Pow . Informally, we can say that \Rightarrow_m is a *best* (linear) approximation of the function space in Pow .

5 The Theory of Linear Semi-Exponents

Fix an arbitrary linear semi-exponent \Rightarrow on Pow and an object $\mathcal{P}A \in \text{Pow}$. Let $\mathcal{L} = (\{L^\sigma\}, [\cdot])$ denote the lambda model based on $\mathcal{P}A$ and \Rightarrow , while $\mathcal{M} = (\{M^\sigma\}, \langle \cdot \rangle)$ denotes the corresponding model based on \Rightarrow_m (hence, $L^\sigma = M^\sigma = \mathcal{P}A$). In this section we will show that the theory of \mathcal{L} is included in the theory of \mathcal{M} .

To begin with, say that an element $\phi \in L^\sigma$ is *hereditary elementary* (or *h-elementary*) iff

1. $\sigma = o$, or
2. $\sigma = \sigma_1 \Rightarrow \sigma_2$, ϕ is elementary, and for all $x \in L^{\sigma_1}$ we have that x h-elementary implies $\phi \bullet x$ is h-elementary.

Proposition 16 *If $\phi' \subseteq \phi \in L^\sigma$ and ϕ h-elementary, then ϕ' is h-elementary. Moreover, if $W \subseteq L^\sigma$, and each $\phi \in W$ is h-elementary, then $\bigcup W$ is h-elementary.*

The proof of this proposition is left to the reader (but observe that the linearity of \bullet is crucial).

We show that the interpretation of the lambda calculus is closed under the property of h-elementariness. First we need the following lemma.

Lemma 17 *For all continuous functions f , $R(f)$ is elementary.*

Proof: We have to show that $\forall n \in \phi([n]$ is finite). By general domain-theory, the continuous function f is the lub of the directed set of compact functions below f . Hence $R(f) = R \vee \{c \mid c \leq f \ \& \ c \text{ compact}\} = \bigcup \{R(c) \mid c \leq f \ \& \ c \text{ compact}\}$ by continuity of R . It follows that for $n \in R(f)$ there exists a compact function $c \leq f$ such that $\{n\} \subseteq R(c)$. As a consequence we have that $[c]$ is a finite set and $f_{\{n\}} \leq c$. By proposition 8(2), $[n]$ is a finite set. ■

We say that a \mathcal{L} -environment ρ is h-elementary iff $\rho(x)$ is h-elementary for each x .

Proposition 18 *If ρ is h-elementary, then $\langle t \rangle_\rho$ is h-elementary.*

Proof: By induction on t . We consider the case that $t = \lambda x.s$, then $\langle t \rangle_\rho = R(\langle s \rangle_{\rho[-/x]})$. By the previous lemma, this set is elementary. Furthermore, for a h-elementary S , $\langle t \rangle_\rho \bullet S = R(\langle s \rangle_{\rho[-/x]} \bullet S) = \langle s \rangle_{\rho[S/x]}$, which is h-elementary by induction hypothesis. It follows that $\langle \lambda.s \rangle_\rho$ is h-elementary. ■

Observe that, as a consequence of this proposition, interpretations of *closed* lambda terms are h-elementary. Intuitively, non h-elementary elements do not play any role in the semantics.

Next we define a function $r^\sigma : L^\sigma \rightarrow M^\sigma$ by induction on σ as follows:

$$\diamond r^o(\phi) = \phi,$$

$$\diamond r^{\sigma \Rightarrow \tau}(\phi) = \{(r^\sigma(X), c) \mid \exists(X, b) \in r(\phi)(c \in r^\tau\{b\} \ \& \ X \text{ h-elementary})\}.$$

It is easy to see that r^σ is well-defined (use proposition 12(2)). Furthermore, for each type σ the function r^σ preserves arbitrary lubs (and hence is monotone).

Proposition 19 *Suppose that $x \in L^\sigma$ is h-elementary and $\phi \in L^{\sigma \Rightarrow \tau}$, then $r^\tau(\phi \bullet x) \subseteq r^{\sigma \Rightarrow \tau}(\phi) \bullet_m r^\sigma(x)$.*

Proof: The proof is by induction on the type τ . For the basis of the induction, assume that $\tau = o$. If $n \in r^o(\phi \bullet x) = \phi \bullet x$, then $n \in r(\phi) \bullet x$ by proposition 12(3). Hence there exists $(X, n) \in r(\phi)$ such that $X \subseteq x$ and hence X is h-elementary. By the monotonicity of r^σ , it follows that $r^\sigma(X) \subseteq r^\sigma(x)$. We have

$$\begin{aligned} r^{\sigma \Rightarrow o}(\phi) \bullet_m r^\sigma(x) &= \{(r^\sigma(X), b) \mid (X, b) \in r(\phi) \ \& \ X \text{ h-elementary}\} \bullet r^\sigma(x) \\ &= \{b \mid \exists(X, b) \in r(\phi)(r^\sigma(X) \subseteq r^\sigma(x) \ \& \ X \text{ h-elementary})\} \end{aligned}$$

hence $n \in r^{\sigma \Rightarrow o}(\phi) \bullet_m r^\sigma(x)$.

For the induction step, assume that $\tau = \tau_1 \Rightarrow \tau_2$. We now have $r^\tau(\phi \bullet x) = \{(r^{\tau_1}(Y), c) \mid \exists(Y, b) \in r(\phi \bullet x)(c \in r^{\tau_2}\{b\} \ \& \ Y \text{ h-elementary})\}$. Suppose that $(r^{\tau_1}(Y), c) \in r^\tau(\phi \bullet x)$. From $(Y, b) \in r(\phi \bullet x)$, it follows by monotonicity of r and proposition 12(3) that $(Y, b) \in r(r(\phi) \bullet x)$. Hence by linearity of r there exists $d \in r(\phi) \bullet x$ such that $(Y, b) \in r\{d\}$. Furthermore, by definition of \bullet_m , we find $(X, d) \in r(\phi)$ satisfying $X \subseteq x \ \& \ (Y, b) \in r\{d\}$. From $(Y, b) \in r\{d\}$, Y h-elementary and $c \in r^{\tau_2}\{b\}$ it follows that $(r^{\tau_1}(Y), c) \in r^{\tau_1 \Rightarrow \tau_2}\{d\}$, while

from $X \subseteq x$ it follows that $r^\sigma(X) \subseteq r^\sigma(x)$ and X h-elementary. If we now write out

$$\begin{aligned}
r^{\sigma \Rightarrow \tau}(\phi) \bullet_m r^\sigma(x) &= \{n \mid \exists(Z, n) \in r^{\sigma \Rightarrow \tau}(\phi)(Z \subseteq r^\sigma(x))\} \\
&= \{n \mid \exists(Z, n) \in \{(r^\sigma(X), e) \mid \exists(X, d) \in r(\phi)(e \in r^\tau\{d\} \\
&\quad \& X \text{ h-elementary})\}(Z \subseteq r^\sigma(x))\} \\
&= \{e \mid \exists(X, d) \in r(\phi)(e \in r^\tau\{d\} \& r^\sigma(X) \subseteq r^\sigma(x) \\
&\quad \& X \text{ h-elementary})\}
\end{aligned}$$

then we see that $(r^{\tau_1}(Y), c) \in r^{\sigma \Rightarrow \tau}(\phi) \bullet_m r^\sigma(x)$. ■

Proposition 20 *For each term t^σ and h-elementary environment ρ we have*

$$r^\sigma\langle t \rangle_\rho \subseteq [t]_{r\rho},$$

where $r\rho(x^\tau) = r^\tau(\rho(x))$.

Proof: By induction on t . The case that t is a variable is trivial. Now suppose that $t = t_1 t_2$. We have

$$\begin{aligned}
r^\sigma\langle t_1 t_2 \rangle_\rho &= r^\sigma(\langle t_1 \rangle_\rho \bullet \langle t_2 \rangle_\rho) \\
&\subseteq r^{\tau \Rightarrow \sigma}\langle t_1 \rangle_\rho \bullet r^\tau\langle t_2 \rangle_\rho \\
&\subseteq [t_1]_{r\rho} \bullet [t_2]_{r\rho} \\
&= [t_1 t_2]_{r\rho}
\end{aligned}$$

where the second step is by proposition 19, and the third by the induction hypothesis.

Next consider the case that $t = \lambda x.s$. We have

$$\begin{aligned}
r^{\sigma_1 \Rightarrow \sigma_2}\langle \lambda x.s \rangle_\rho &= \{(r^{\sigma_1}(X), c) \mid \exists(X, b) \in r\langle \lambda x.s \rangle_\rho(c \in r^{\sigma_2}\{b\} \\
&\quad \& X \text{ h-elementary})\} \\
&= \{(r^{\sigma_1}(X), c) \mid \exists(X, b) \in rR\langle s \rangle_{\rho[-/x]}(c \in r^{\sigma_2}\{b\} \\
&\quad \& X \text{ h-elementary})\} \\
&\subseteq \{(r^{\sigma_1}(X), c) \mid \exists(X, b) \in R_m\langle s \rangle_{\rho[-/x]}(c \in r^{\sigma_2}\{b\} \\
&\quad \& X \text{ h-elementary})\}
\end{aligned}$$

where the last inclusion is by proposition 12(4). Suppose that $(r^{\sigma_1}(X), c) \in r^{\sigma_1 \Rightarrow \sigma_2} \langle \lambda x. s \rangle_\rho$, then from $(X, b) \in R_m \langle s \rangle_{\rho[-/x]}$ it follows that $b \in R_m \langle s \rangle_{\rho[-/x]} \bullet_m X = \langle s \rangle_{\rho[X/x]}$. Hence we have

$$\begin{aligned} c &\in r^{\sigma_2} \{b\} \\ &\subseteq r^{\sigma_2} \langle s \rangle_{\rho[X/x]} \\ &\subseteq [s]_{r(\rho[X/x])} \\ &= [s]_{r\rho[r^{\sigma_1}(X)/x]} \end{aligned}$$

where the third step holds by the induction hypothesis and the fact that X is h-elementary. It follows that $(r^{\sigma_1}(X), c) \in R_m[s]_{r\rho[-/x]} = [\lambda x. s]_{r\rho}$. ■

Fix an arbitrary left-inverse s of r as in the previous section. We define a (collection of) function(s) s^σ running into the opposite direction of r^σ . For each type σ , define $s^\sigma : M^\sigma \rightarrow L^\sigma$ by the following inductive clauses:

- ◇ $s^\sigma(\phi) = \phi$,
- ◇ $s^{\sigma \Rightarrow \tau}(\phi) = s\{(s^\sigma(X), c) \mid \exists(X, b) \in \phi(c \in s^\tau\{b\})\}$.

It is easy to see that s^σ is well-defined and preserves arbitrary lubs. Furthermore, s^σ always yields h-elementary results.

Proposition 21 *For all $\phi \in M^\sigma$, $s^\sigma(\phi)$ is h-elementary.*

Proof: By induction to σ . The base of the induction is trivial. For the induction step, assume that $\sigma = \sigma_1 \Rightarrow \sigma_2$. As $s^\sigma(\phi)$ is of the form $s(\psi)$ (for some ψ), it clearly is elementary (proposition 14(5)). Furthermore, for an h-elementary $x \in L^{\sigma_1}$, we have

$$\begin{aligned} s^\sigma(\phi) \bullet x &= s\{(s^{\sigma_1}(X), c) \mid \exists(X, b) \in \phi(c \in s^{\sigma_2}\{b\})\} \bullet x \\ &= \{(s^{\sigma_1}(X), c) \mid \exists(X, b) \in \phi(c \in s^{\sigma_2}\{b\})\} \bullet x \\ &= \{c \mid \exists(X, b) \in \phi(c \in s^{\sigma_2}\{b\}) \ \& \ s^{\sigma_1}(X) \subseteq x\} \\ &= \bigcup_{(X, b) \in \phi \ \& \ s^{\sigma_1}(X) \subseteq x} s^{\sigma_2}\{b\} \end{aligned}$$

By the induction hypothesis and proposition 16, it follows that $s^\sigma(\phi) \bullet x$ is h-elementary. ■

Proposition 22 For all σ , $r^\sigma s^\sigma(\phi) = \phi$.

Proof: The proof is by induction to σ . If $\sigma = o$, then the proposition trivially holds. If $\sigma = \sigma_1 \Rightarrow \sigma_2$ we reason as follows:

$$\begin{aligned}
r^\sigma s^\sigma(\phi) &= \{(r^{\sigma_1}(X), c) \mid \exists(X, b) \in r s^\sigma(\phi)(c \in r^{\sigma_2}\{b\} \ \& \ X \text{ h-elementary})\} \\
&= \{(r^{\sigma_1}(X), c) \mid \exists(X, b) \in \{(s^{\sigma_1}(X), e) \mid \exists(X, d) \in \phi(e \in s^{\sigma_2}\{d\})\} \\
&\quad (c \in r^{\sigma_2}\{b\} \ \& \ X \text{ h-elementary})\} \\
&= \{(r^{\sigma_1} s^{\sigma_1}(X), c) \mid \exists(X, d) \in \phi(b \in s^{\sigma_2}\{d\} \ \& \ c \in r^{\sigma_2}\{b\})\} \\
&= \{(X, c) \mid \exists(X, d) \in \phi(c \in r^{\sigma_2} s^{\sigma_2}\{d\})\} \\
&= \{(X, c) \mid \exists(X, d) \in \phi(c \in \{d\})\} \\
&= \phi
\end{aligned}$$

■

Lemma 23 Suppose $\phi \in M^{\sigma \Rightarrow \tau}$ and $x \in M^\sigma$, then $s^\tau(\phi \bullet_m x) \subseteq s^{\sigma \Rightarrow \tau}(\phi) \bullet s^\sigma(x)$.

Proof: For the sake of convenience, we write ψ for the set $\{(s^\sigma(X), c) \mid \exists(X, b) \in \phi(c \in s^\tau\{b\})\}$. Hence $s^{\sigma \Rightarrow \tau}(\phi) = s(\psi)$.

The proof is by induction on τ . First assume that $\sigma = o$. Suppose that $n \in s^o(\phi \bullet_m x) = \phi \bullet_m x$, then $n \in \{b \mid \exists(X, b) \in \phi(X \subseteq x)\}$, hence $\exists(X, n) \in \phi(X \subseteq x)$. It follows that $(s^\sigma(X), n) \in \psi$ and $s^\sigma(X) \subseteq s^\sigma(x)$. Hence $n \in \psi \bullet s^\sigma(x) \subseteq s(\psi) \bullet s^\sigma(x) = s^{\sigma \Rightarrow \tau}(\phi) \bullet s^\sigma(x)$.

Next assume that $\tau = \tau_1 \Rightarrow \tau_2$. Suppose $(K, n) \in s^\tau(\phi \bullet x) = s\{(s^{\tau_1}(X), c) \mid \exists(X, b) \in \phi \bullet x(c \in s^{\tau_2}\{b\})\}$. Then, by linearity of s , there exists $(X, b) \in \phi \bullet x$ such that $(K, n) \in s\{(s^{\tau_1}(X), c)\}$ and $c \in s^{\tau_2}\{b\}$. By definition of application \bullet_m , there exists $(Y, (X, b)) \in \phi$ such that $Y \subseteq x$ and satisfying the above two statements. Because

$$\begin{aligned}
s^{\tau_1 \Rightarrow \tau_2}\{(X, b)\} &= s\{(s^{\tau_1}(Z), e) \mid \exists(Z, d) \in \{(X, b)\}(e \in s^{\tau_2}\{d\})\} \\
&= s\{(s^{\tau_1}(X), e) \mid e \in s^{\tau_2}\{b\}\}
\end{aligned}$$

it follows that $\exists(Y, (X, b)) \in \phi(Y \subseteq x \ \& \ (K, n) \in s^{\tau_1 \Rightarrow \tau_2}\{(X, b)\})$. Hence $(s^\sigma(Y), (K, n)) \in \psi$ and $s^\sigma(Y) \subseteq s^\sigma(x)$. Finally $(K, n) \in \psi \bullet_m s^\sigma(Y) \subseteq \psi \bullet_m s^\sigma(x) \subseteq s(\psi) \bullet s^\sigma(x) = s^{\sigma \Rightarrow \tau}(\phi) \bullet s^\sigma(x)$. ■

Proposition 24 For each term t^σ and environment ρ we have

$$s^\sigma[t]_\rho \subseteq \langle t \rangle_{s\rho},$$

where $s\rho(x^\tau) = s^\tau(\rho(x))$.

Proof: The proof is by induction on t . We consider the case that $t = \lambda x.s$ of type $\sigma_1 \Rightarrow \sigma_2$. Then

$$\begin{aligned} s^\sigma[\lambda x.t]_\rho &= s\{(s^{\sigma_1}(X), c) \mid \exists(X, b) \in [\lambda x.s]_\rho(c \in s^{\sigma_2}\{b\})\} \\ &= s\{(s^{\sigma_1}(X), c) \mid \exists(X, b) \in R_m[s]_{\rho[-/x]}(c \in s^{\sigma_2}\{b\})\} \\ &= s\{(s^{\sigma_1}(X), c) \mid \exists b \in [s]_{\rho[X/x]}(c \in s^{\sigma_2}\{b\})\} \\ &\subseteq s\{(s^{\sigma_1}(X), c) \mid \exists b(s^{\sigma_2}\{b\} \subseteq s^{\sigma_2}[s]_{\rho[X/x]} \ \& \ c \in s^{\sigma_2}\{b\})\} \\ &\subseteq s\{(s^{\sigma_1}(X), c) \mid c \in s^{\sigma_2}[s]_{\rho[X/x]}\} \\ &\subseteq s\{(s^{\sigma_1}(X), c) \mid c \in \langle s \rangle_{s\rho[s^{\sigma_1}(X)/x]}\} \\ &= sR_m\langle s \rangle_{r\rho[-/x]} \\ &\subseteq R\langle s \rangle_{r\rho[-/x]} \\ &= \langle \lambda x.s \rangle_{r\rho} \end{aligned}$$

■

Finally we state as our main theorem that the theory of \mathcal{L} is included in the theory of \mathcal{M} .

Theorem 25 For arbitrary lambda terms s, t , we have that $\mathcal{L} \models s = t$ implies $\mathcal{M} \models s = t$.

Proof: Suppose that $\mathcal{L} \models s = t$, then $\langle s \rangle_\rho = \langle t \rangle_\rho$ for all \mathcal{L} -environments ρ . For an arbitrary \mathcal{M} -environment π , we have

$$\begin{aligned} [s]_\pi &= r^\sigma s^\sigma[s]_\pi \\ &\subseteq r^\sigma(\langle s \rangle_{s\pi}) \\ &= r^\sigma(\langle t \rangle_{s\pi}) \\ &\subseteq [t]_{rs\pi} \\ &= [t]_\pi, \end{aligned}$$

where the fourth step holds by proposition 20 and 24. Analogously we can show that for each \mathcal{M} -environment $[t]_\pi \subseteq [s]_\pi$. It follows that $[t]_\pi = [s]_\pi$ and

$\mathcal{M} \models s = t.$ ■

Hence, the theory of \Rightarrow_m is *maximal* among the theories of the linear semi-exponents on Pow.

6 Conclusion

In this paper we showed that the theory associated to a linear semi-exponent (on Pow) always is included in the theory of a particular “minimal” semi-exponent \Rightarrow_m . Many interesting questions concerning extensions and generalizations of this result remain open.

For example, what can we say about *non-linear* semi-exponents? The semi-exponent \Rightarrow_m can be embedded in the non-linear semi-exponent \Rightarrow_{aut} from example 5, hence our proof can probably be generalized to these kind of “pseudo-linear” semi-exponents. However, the semi-exponent \Rightarrow_a from example 6 can in general *not* be shown to contain \Rightarrow_m .

Further questions concern the exact nature of the theory of the minimal semi-exponent. Although it clearly does not hold for finite base sets, it does not seem impossible that the theory based on \Rightarrow_m and an *infinite* base set $\mathcal{P}A$ precisely is the set of provable lambda equations. By the results of our paper, this would imply that the theory of *each* elementary linear semi-exponent (with infinite base set) is complete for $\lambda\beta$.

More generally, we can study approximations of function spaces in a subcategory $\mathbf{C} \subseteq \mathbf{Dcpo}$ (e.g., \mathbf{Alg}). Does there always exist a (class of) best approximation(s) of the function space in \mathbf{C} ? And what is the relation between distinguished best approximations of a function space: do they have to be isomorphic?

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