Institute for Logic, Language and Computation

THE DECIDABILITY OF DEPENDENCY IN INTUITIONISTIC PROPOSITIONAL LOGIC

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ILLC Prepublication Series
for Mathematical Logic and Foundations ML-93-20

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For $(K_{n+3}^1)^+$: $g_{n+3}(q) \wedge ((p \rightarrow f_{n+1}(q)) \rightarrow p) \rightarrow p \ := \ h_n(p,q)$,
for $(K_n^1, K_{n+1}^1)^+$: $g_{n+3}(q) \wedge (p \leftrightarrow f_{n+1}(q)) \rightarrow p \ := \ k_n(p,q)$,
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$\neg\neg q \wedge ((p \rightarrow q) \rightarrow p) \rightarrow p$, $\neg\neg q \wedge (p \leftrightarrow q) \rightarrow p$, $\neg q \wedge \neg\neg p \rightarrow p$, $\neg(\neg q \wedge \neg p)$.
Here $g_n(q)$ and $f_n(q)$ are such that, for any 1-variable T-model L, $L \models g_n(q)$ iff $L \preccurlyeq K_n$, $L \models f_{n+1}(q)$ iff $L \preccurlyeq K_n$ or $L \preccurlyeq K_{n+1}$. The relevant properties of the $g_n(q)$ and $f_n(q)$ are:
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3.1 Theorem. If for no $n \in \mathbb{N}$, $\vdash_{IPC} g_{n+3}(B) \wedge ((A \rightarrow f_{n+1}(B)) \rightarrow A) \rightarrow A$ or $\vdash_{IPC} g_{n+3}(B) \wedge (A \leftrightarrow f_{n+1}(B)) \rightarrow A$, or $\vdash_{IPC} g_{n+3}(A) \wedge ((B \rightarrow f_{n+1}(A)) \rightarrow B) \rightarrow B$ or $\vdash_{IPC} g_{n+3}(A) \wedge (B \leftrightarrow f_{n+1}(A)) \rightarrow B$, or one of the above degenerate cases is provable in IPC for A, B, then A and B are independent over IPC.

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THE DECIDABILITY OF DEPENDENCY IN INTUITIONISTIC PROPOSITIONAL LOGIC

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ILLC Prepublications
for Mathematical Logic and Foundations
ISSN 0928-3315
Coordinating editor: Dick de Jongh

received November 1993

Abstract. A definition is given for formulae $A_1, \ldots, A_n$ in some theory T which is formalized in a propositional calculus S to be (in)dependent with respect to S. It is shown that, for intuitionistic propositional logic IPC, dependency (with respect to IPC itself) is decidable. This is an almost immediate consequence of Pitts' uniform interpolation theorem for IPC. A reasonably simple infinite sequence of IPC-formulae $F_n(p, q)$ is given such that IPC-formulae A and B are dependent if and only if at least one of the $F_n(A, B)$ is provable.

1. Introduction. We denote the intuitionistic propositional calculus by IPC. Let us call formulae $A_1, \ldots, A_n$ of some intuitionistic theory T IPC-*dependent over* T, or *dependent over* T for short, if, for some IPC-formula $F(p_1, \ldots, p_n)$, $\vdash_T F(A_1, \ldots, A_n)$, but $\nvdash_{IPC} F(p_1, \ldots, p_n)$. Otherwise $A_1, \ldots, A_n$ are called independent. In de Jongh (1982) the behavior of formulae of one propositional variable in intuitionistic arithmetic HA was discussed. The main result of that paper was that for arithmetic sentences A, if $\nvdash_{HA} \neg\neg A \rightarrow A$ and $\nvdash_{HA} \neg\neg A$, then A is independent over HA with respect to IPC. This result was generalized to formulae. We did not mention the fact that the result applies to the propositional calculus itself as well.

1.1 Theorem. If $\nvdash_{IPC} \neg\neg A \rightarrow A$ and $\nvdash_{IPC} \neg\neg A$, then A is independent over IPC.

In fact, the proof in §2 of the article mentioned above applies immediately to this case. Naturally, for IPC there is no immediate reason to look for a more constructive proof, as we did for HA in the major part of that paper. A fortiori of course, the result implies that dependency is decidable for the one variable case: it can be checked whether an arbitrary formula A is dependent by checking whether $\neg\neg A \rightarrow A$ or $\neg\neg A$ is provable. We call theorem 1.1 a *minimal provability result:* if anything non-trivial propositional is provable about A, $\neg\neg A \rightarrow A$ or $\neg\neg A$ is. The result leads to a characterization of the monadic propositional functions F for which there exist A such that exactly $\vdash F(A)$. This result holds for HA as well as for IPC. To remind the reader of the definition for n propositional variables:

1.2 Definition.
Exactly $\vdash F(A_1, \ldots, A_n)$ iff $\vdash F(A_1, \ldots, A_n)$ and, for all propositional G, $\vdash G(A_1, \ldots, A_n)$ $\Rightarrow \vdash F(p_1, \ldots, p_n) \rightarrow G(p_1, \ldots, p_n)$.

This leads to the following classification of formulas of one propositional variable in HA as well as in IPC.

1.3 Theorem. To each formula exactly one of the following cases applies (non-constructively, of course, in the case of HA),

(I) exactly $\vdash$ A

(II) exactly $\vdash \neg$A

(III) exactly $\vdash \neg\neg$A

(IV) exactly $\vdash \neg\neg A \rightarrow A$

(V) exactly $\vdash A \rightarrow A$ (A is independent)

Examples exhibiting the five cases in IPC are respectively, (I) $p \rightarrow p$, (II) $p \wedge \neg p$, (III) $\neg\neg p \rightarrow p$, (IV) $\neg p$, (V) p.

In this note we will show that for n variables the decidability of dependency is a consequence of Pitts' uniform interpolation theorem (Pitts, 1992). Moreover, we will give an analogue for two propositional variables of theorem 1.1. A general analogue for theorem 1.3 seems much harder (see de Jongh-Visser, 1993, however, for some results). We will not go into that here except for remarking that in the case of arithmetic there are easy analogues of theorem 1.3 for restricted cases, e.g. if one restricts oneself to Π^0_1-sentences. In the monadic case we have for Π^0_1-sentences A, exactly $\vdash$ A or exactly $\vdash \neg$A, or exactly $\vdash \neg\neg A \rightarrow A$ (i.e., in particular, an Π^0_1-sentence is never an independent one). In the binary case, if not $\vdash$ A, $\vdash \neg$A, $\vdash$ B or $\vdash \neg$B, then exactly

$\vdash (\neg\neg A \rightarrow A) \wedge (\neg\neg B \rightarrow B)$ or

$\vdash (A \rightarrow B) \wedge (\neg\neg A \rightarrow A) \wedge (\neg\neg B \rightarrow B)$ or

$\vdash (B \rightarrow A) \wedge (\neg\neg A \rightarrow A) \wedge (\neg\neg B \rightarrow B)$ or

$\vdash (A \leftrightarrow B) \wedge (\neg\neg A \rightarrow A) \wedge (\neg\neg B \rightarrow B)$. The only non-trivial relationship between Π^0_1-sentences is apparently the one of implication, as it is in classical arithmetic. We thank Albert Visser for many discussions on the subject.

2. Decidability of dependency over IPC. Pitts (1992) proved, among other things, that, for any IPC-formula $A(\vec{p},\vec{r})$, there is a formula $\exists\vec{r}\, A(\vec{p},\vec{r})$ such that, for any formula $B(\vec{p})$, $A(\vec{p},\vec{r}) \vdash_{IPC} B(\vec{p}) \Leftrightarrow \exists\vec{r}\, A(\vec{p},\vec{r}) \vdash_{IPC} B(\vec{p})$.

Consider the formulae $A_1, \ldots, A_n$ in the variables $\vec{r}$. From Pitts' Theorem it follows that

$\vdash (A_1 \leftrightarrow p_1) \wedge \ldots \wedge (A_n \leftrightarrow p_n) \rightarrow B(\vec{p}) \Leftrightarrow \vdash \exists\vec{r}\, ((A_1 \leftrightarrow p_1) \wedge \ldots \wedge (A_n \leftrightarrow p_n)) \rightarrow B(\vec{p})$.

On the other hand, $\vdash B(\vec{A}) \Leftrightarrow \vdash (A_1 \leftrightarrow p_1) \wedge \ldots \wedge (A_n \leftrightarrow p_n) \rightarrow B(\vec{p})$. Hence $\exists\vec{r}\, ((A_1 \leftrightarrow p_1) \wedge \ldots \wedge (A_n \leftrightarrow p_n))$ axiomatizes the propositional theory of $A_1, \ldots, A_n$. In consequence, $A_1, \ldots, A_n$ is independent $\Leftrightarrow \vdash \exists\vec{r}\, ((A_1 \leftrightarrow p_1) \wedge \ldots \wedge (A_n \leftrightarrow p_n))$, and the latter is decidable.

One may be of the opininion that A and B would more properly be defined to be dependent if, for some F, $\vdash F(A,B)$ and, for no G, H such that $\vdash G(A)$ and $\vdash H(B)$, $G(p), H(q) \vdash F(p,q)$. With this alternative definition e.g. $\neg p$ and q would be independent, while under definition 1.2 they are dependent, since $\neg\neg\neg p$ is provable. (V. Shavrukov suggested this alternative to us.) It will be clear that decidability follows from the above proof for the alternative definition just as well. It seems to us that both definitions describe relevant concepts.

It is clear, of course, that, for any propositional logic S for which a uniform interpolation theorem holds, dependency is decidable for the logic itself. In fact the uniform interpolation theorem has been proved for the provability logic L by Shavrukov (1993), completely independently of the result by Pitts and by a completely different proof. Hence, dependency is decidable for L. Unfortunately, for most modal logics not even a standard interpolation theorem holds (see e.g. Maksimova, 1982), so, for many logics a completely different method will have to be found if one wants to study the problem.

3. A minimal provability result for two variables. We first recall some facts about Kripke models for intuitionistic propositional logic.

(i) For each A of **IPC**, if $\nvdash_{IPC} A$, there is a finite tree-ordered Kripke model $K=\langle W, \leq, \Vdash \rangle$ such that $K \nvDash A$. (There is no essential reason to restrict oneself to tree-ordered Kripke models, but these are more easily described.)

(ii) We write $w\uparrow$ for $\{w' \in W \mid w \leq w'\}$. The model K restricted to $w\uparrow$ is called a *generated submodel* of K.

(iii) A *p-morphism* from a Kripke model $K=\langle W, \leq, \Vdash \rangle$ to a Kripke model $K'=\langle W', \leq', \Vdash' \rangle$ is a surjection $\phi: W \to W'$ such that:

(a) $w \leq w' \Rightarrow \phi(w) \leq' \phi(w')$

(b) $\phi(w) \leq' \phi(w') \Rightarrow \exists w'' \geq w' (\phi(w'') = \phi(w))$

(c) for all $w \in W$ and all propositional variables p, $\phi(w) \Vdash p \Leftrightarrow w \Vdash p$

It is easily shown then that (c) applies to all formulas.

(iv) A finite tree-ordered Kripke model is called *irreducible* if all its p-morhic images to tree-ordered Kripke models are isomorphic. We will call such a model here *T-model* for short.

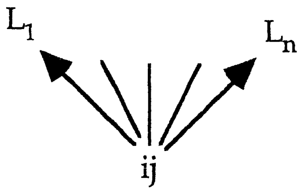
(v) (Jankov 1968, de Jongh 1970) For each T-model K there are formulas C_K and D_K such that

(a) $K \vDash C_K$, $K \nvDash D_K$

(b) For each T-model L such that $L \vDash C_K$, L is isomorphic to a generated submodel of K ($L \preccurlyeq K$).

(c) For each T-model L such that $L \nvDash D_K$, $K \preccurlyeq L$.

(vi) If we consider a Kripke model for the language consisting of the two propositional variables p and q, the values of p and q at the root of a model K are respectively i and j and the generated submodels corresponding to the immediate successors of the root are $L_1, \ldots, L_n$, then we denote K by



Each T-model with a domain of more than one element has such a form with $L_1, \ldots, L_n$ irreducible and none of the L_j isomorphic to a generated submodel of any of the others. In case n=1, the root of L_1 has a forcing relation distinct from ij. All finite T-models can be obtained from the four irreducible p-q-models with one-element domains: 00, 11, 10, 11 by repeatedly adjoining roots with proper valuations to finite sets of $\preccurlyeq$-incomparable T-models already obtained.

Let us now suppose that we have that $\nvdash D_K(A_1, A_2)$. Then in a sense, the model K is "available" for A_1, A_2, because any counter-model to $D_K(A_1, A_2)$ (and such a model has to exist) has to contain K in its valuations for A_1, A_2. Any counter-examples to formulas which can be given on K or its generated submodels then give rise to underivable formulas as well.

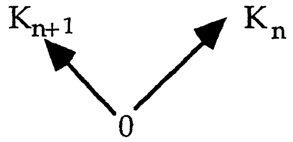
If we have a finite set of D_L's which are not derivable for A_1, A_2, then we may also construct models by taking the set of the L's and adjoining a root below them. If it happens to be the case that the forcing on the root is automatically 00, then the model thus obtained is a model that gives rise to underivable formulas in its turn. This is so, if among the old roots at least one value 00 occurs or if both 10 and 01 occur. More exactly, if such a case applies and the model K arises in the construction from the models $L_1, \ldots, L_n$ and $D_{L_1}(A_1, A_2), \ldots, D_{L_n}(A_1, A_2)$, are not derivable in IPC, then neither is $D_K(A_1, A_2)$. In this case we will denote the newly obtained model by $(L_1, \ldots, L_n)^+$ and say that $(L_1, \ldots, L_n)^+$ has been obtained by *rooting* from $L_1, \ldots, L_n$.

Let us recall the one-variable case.

$K_0 = 1$ $\qquad\qquad$ $K_1 = 0$

$$K_2 = \begin{matrix} 1 \\ \uparrow \\ 0 \end{matrix}$$

And, in general for any $n \geq 0$, $K_{n+3} =$



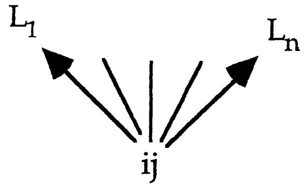
This means that all K_n can be constructed from $X=\{K_0, K_1, K_2\}$ by the second method of rooting models. Also, K_1 is a generated submodel of K_2. The proof of theorem 1.1 is then actually contained in the above sketch, but then applied to the 1-variable case.

In the 2-variable case a set X of Kripke-models which is sufficient for the construction of all models is the set of all T-models with 00 *only* occurring at the root. All T-models with 00 occurring at the root can be obtained from X by repeatedly rooting models, and all other T-models are generated submodels of models in this set.

A simpler such set, however, is the following set X^*:
Let us denote by K^1_n, K_n preceded by 1 everywhere, i.e. K_n with 0 replaced by 10 and 1 by 11. Similarly K^2_n will denote K_n followed by 1 everywhere, i.e. K_n with 0 replaced by 01 and 1 by 11.

Now take $X^* = \left\{ \overset{K^i_n}{\underset{00}{\uparrow}} \mid n \in \mathbb{N}, i=1,2 \right\} \cup \left\{ \overset{K^i_n \qquad K^i_{n+1}}{\nwarrow_{00} \nearrow} \mid n \in \mathbb{N}, i=1,2 \right\}$.

To show that this set suffices it is sufficient to generate the original set X from X^* by taking generated submodels and rooting them. Take an arbitary member



of X. If a root of one of the L_i is 11, then that $L_i = K^1_0 = K^1_1$.
In general, any T-model with a root 11, 10 or 01 is a K^i_n. If among the L_i no root 01 occurs, then all the L_i's are K^1_n's and we actually have one of the cases $(K^1_n)^+$ or $(K^1_n, K^1_{n+1})^+$, similarly, if no root 10 occurs. If both 01 and 10 occur on the roots of the L_i, then ij=00 is forced and the model is obtained by rooting the L_i, and the L_i themselves are generated submodels of models $(K^1_n)^+$ and $(K^1_n, K^1_{n+1})^+$.

Now to get a generalization of Theorem 1.1 it suffices to ascribe D-formulae to the T-models $(K^1_n)^+$ and $(K^1_n, K^1_{n+1})^+$. These could be found by applying a general method (Jankov 1968, de Jongh 1970), but the following formulae are nicer:

for $(K^1_n)^+$: $g_{n+2}(q)\wedge((p\to f_{n+2}(q))\to p)\to p$:= $h_n(p,q)$

for $(K^1_n, K^1_{n+1})^+$: $g_{n+3}(q)\wedge((p\to g_{n+1}(q))\to p)\wedge((p\to g_{n+2}(q))\to p)\to p$:= $k_n(p,q)$.

3.1 Theorem. If for no $n\in\mathbb{N}$, $\vdash_{IPC} g_{n+2}(B)\wedge((A\to f_{n+2}(B))\to A)\to A$ or $\vdash_{IPC} g_{n+3}(B)\wedge((A\to g_{n+1}(B))\to A)\wedge((A\to g_{n+2}(B))\to A)\to A$, then A and B are independent over IPC.

It is to be noted that just as Theorem 1.1 this theorem immediately applies to **HA**, since rooting the models is applicable in the case of **HA** as well by adjoining the standard model $\mathbb{N}$ to the new root (see Smoryński, 1973). That this theorem is in a sense best possible can be demonstrated by showing that $h_n(p,q)$ and $k_n(p,q)$ are exactly provable. (Again this then applies to **HA** as well, now by the uniform version of the arithmetic completeness of **IPC** over **HA**, see Smoryński, 1973.)

3.2 Theorem. (a) The formula $h_n(p,q)$ is exactly provable for $(g_{n+2}(q)\wedge((p\to f_{n+2}(q))\to p))\vee p$ and q.

(b) The formula $k_n(p,q)$ is exactly provable for $(g_{n+3}(q)\wedge((p\to g_{n+1}(q))\to p)\wedge((p\to g_{n+2}(q))\to p))\vee p$ and q.

Proof. Actually, we will show in general that

(i) $C\wedge((p\to D)\to p)\to p$ where C and D do not contain p is exactly provable with the substitution $(C\wedge((p\to D)\to p))\vee p$ for p and the identity for the other variables,

(ii) $C\wedge((p\to D)\to p)\wedge((p\to E)\to p)\to p$ where C, D and E do not contain p is exactly provable with the substitution $(C\wedge((p\to D)\to p)\wedge((p\to E)\to p))\vee p$ for p and the identity for the other variables.

Of course, it suffices to prove (ii). We first show that the required formulae are actually provable. We apply the easily verified **IPC**-equivalence of $A\to B$ to $((A\to B)\to A)\to B$. Let us write p^* for $(C\wedge((p\to D)\to p)\wedge((p\to E)\to p))\vee p$.

$p^*\to D$ is equivalent to $(C\wedge((p\to E)\to p)\to(p\to D))\wedge(p\to D)$ and hence to $p\to D$.

$(p^*\to D)\to p^*$ is equivalent to

$(p\to D)\to((C\wedge((p\to D)\to p)\wedge((p\to E)\to p))\vee p)$ and hence implies $(p\to D)\to p$.

Similarly $(p^*\to E)\to p^*$ implies $(p\to E)\to p$. That

$C\wedge((p^*\to D)\to p^*)\wedge((p^*\to E)\to p^*)$ implies p^* is now trivial.

Next we have to show that no stronger formulae are provable. For that it is sufficient to note that in any Kripke model validating

$C\wedge((p\to D)\to p)\wedge((p\to E)\to p)\to p$ changing the valuation of p to that of $(C\wedge((p\to D)\to p)\wedge((p\to E)\to p))\vee p$ will leave all forcing relations as they are. This is obvious, because in any such Kripke model $(C\wedge((p\to D)\to p)\wedge((p\to E)\to p))\vee p$ is actually equivalent to p.

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