# A set-theoretic translation method for polymodal logics

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**Abstract**

The paper presents a *set-theoretic translation method* for polymodal logics that reduces the derivability problem of a large class of propositional polymodal logics to the derivability problem of a very weak first-order set theory $\Omega$. Unlike most existing translation methods, the one we proposed applies to any normal complete finitely-axiomatizable polymodal logic, regardless if it is first-order complete or if an explicit semantics is available for it. Moreover, the finite axiomatizability of $\Omega$ makes it possible to implement mechanical proof search procedures via the deduction theorem or more specialized and efficient techniques.

In the last part of the paper, we briefly discuss the application of *set $T$-resolution* to support automated derivability in (a suitable extension of) $\Omega$.

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# 1 Introduction

The paper deals with derivability in modal logic. Most inference systems for modal logic are defined in the style of sequent or tableaux calculi, e.g. [8, 20]. As an alternative, a number of translation methods for modal logic into classical first-order logic have been proposed in the literature (for an up-to-date survey see [16]). They allow one to use mechanical theorem provers for Predicate Calculus to implement modal theorem provers. These methods, compared with the direct approach of finding a proof algorithm for a particular modal logic, have the advantage of being *independent* from the modal logic under consideration: a single theorem prover may be used for any translatable modal logic.

In the simplest approach the first-order language $\mathcal{L}$ into which the translation is carried out contains a constant $\tau$ denoting the initial world in the frame, a binary relation $R(x, y)$ denoting the accessibility relation, and a denumerable amount of unary predicates $P_1(x), \ldots, P_n(x), \ldots$. The translation function $\pi$ is defined by induction on the structure of the modal formula as follows:

- $\pi(P_j, x) \equiv P_j(x)$;
- $\pi(-, x)$ commutes with the boolean connective;
- $\pi(\Box\, \psi, x) \equiv \forall y(xRy \rightarrow \pi(\psi, y))$;
- $\pi(\Diamond \psi, x) \equiv \exists y(xRy \wedge \pi(\psi, y))$.

Let $H$ be a normal modal logic. $H$ is *first-order complete* if there exists a first-order sentence $Axiom_H$ of $\mathcal{L}$, involving only the binary relational symbol $R(x, y)$ and the equality symbol, such that a modal formula $\phi$ is derivable from $H$ if and only if the formula $\phi$ is true in the initial world $\tau$ of all generated frames satisfying $Axiom_H$ [2, 10].

For these logics the following holds:

$$\vdash_H \phi \text{ iff } \quad \vdash Axiom_H \rightarrow \pi(\phi, \tau),$$

where $\vdash$ stands for derivability in classical Predicate Calculus. Therefore, as long as we have $Axiom_H$, at least in principle, a classical theorem prover can be used as a theorem prover for $H$.

Efficiency concerns have motivated further investigations on the above (relational) translation method. Such studies (e.g. [15]) suggested a "functional" semantics for modal logic and resulted in a family of more efficient and general translation methods. For a normal modal logic $H$ with a first-order semantics $Axiom_H$, this method guarantees a sound and complete correspondence between satisfiability with respect to $H$ and satisfiability with respect to first-order structures satisfying $Axiom_H$. From the computational point of view, the functional translation may still cause some problem when using a first-order theorem prover, due to the presence of equalities in $Axiom_H$. A way to limit the complexity induced by the introduction of equality using a mixed relational/functional translation proposed in [14].

A common feature of all the mentioned methods is that in order to be applied directly, the underlying modal logic must have a first-order semantic. Up to our knowledge any attempt to deal with logics not having a first-order semantic requires ad-hoc techniques. Moreover if the logic has a first-order semantic but it is implicitly specified with Hilbert axioms only, a preliminary step is necessary to find the corresponding first-order axioms. The question of automatically solve this last problem has been extensively studied and algorithms have been proposed, e.g. [2, 9].

The main motivation of the present work is to find a translation applicable to all complete modal logics, regardless of the first-order axiomatizability of their semantics. The set-theoretic translation we propose works for all normal complete finitely-axiomatizable modal logics. In particular, our method works also if the modal logic under consideration is specified only with Hilbert axioms.

The basic idea behind our translation is to represent any Kripke frame as a set, with the accessibility relation modeled using the membership relation $\in$. Moreover, in order to achieve a computationally valid result, we want to refer to a *finitely* (first-order) axiomatizable set theory. We succeeded in carrying out our translation in a very weak[1] set theory called $\Omega$. Given a modal formula $\phi(P_1, ..., P_n)$ we define as its translation the *set-theoretic term*, with variables $x, x_1, ..., x_n$, denoted by $\phi^*(x, x_1, ..., x_n)$ and built using $\cup, \setminus$, and $Pow$. Intuitively the term $\phi^*(x, x_1, ..., x_n)$ represents the set of those worlds (in the frame $x$) in which the formula $\phi$ holds. The inductive definition of $\phi^*(x, x_1, ..., x_n)$ is rather straightforward in all the cases but for $\Box\,\phi$, whose translation is defined as: $(\Box\,\phi)^* \equiv Pow(\phi^*)$ (see Section 2 for details).

We prove that for any normal complete finitely-axiomatizable modal logic $H = K + \psi(\alpha_{j_1}, .., \alpha_{j_m})$, where $\psi(\alpha_{j_1}, .., \alpha_{j_m})$ is an axiom schema, the following holds:

$$\vdash_H \phi(P_1, ..., P_n) \text{ iff } \Omega \vdash \forall x(Trans(x) \wedge Axiom_H(x) \rightarrow \forall x_1 ... \forall x_n(x \subseteq \phi^*(x, x_1, ...x_n))),$$

where $Trans(x)$ and $Axiom_H(x)$ stand for $\forall y\ (y \in x \rightarrow y \subseteq x)$ and $\forall x_{j_1}, .., \forall x_{j_m}(x \subseteq \psi^*(x, x_{j_1}, .., x_{j_m}))$, respectively. Instead of translating Hilbert axioms, one may use a set-theoretic semantics for $H$ when such a semantics is available. We will study the case of $G$ as an example of such an approach.

The proposed set-theoretic translation method is then generalized to polymodal logics. Such a generalization involves the revision of the definition of the translation function to cope with a set of different modal operators instead of a single one.

In the last part of the paper, we take into consideration the application of set $T$-resolution techniques to support derivability in $\Omega$. In order to apply such techniques, it is necessary to guarantee the decidability, with respect to $\Omega$, of the class of ground formulae written in any language which extends the one in which the axioms of $\Omega$ are written with Skolem functions. We succeeded in providing such a decidability result in a suitable extension of $\Omega$, and the main steps of the proof are outlined in Section 5 (the details can be found in [7]).

The paper is organized as follows. In Section 2, we introduce the proposed set-theoretic translation method by showing its application to the modal logic $G$. In this case the proofs are simpler, and a clear description of the main features of the translation method is possible; in the meanwhile, it gives an example of how the method applies to a logic provided with a non-first-order semantics. In Section 3, we consider the general case and we exploit the possibility of translating the Hilbert axioms of the logic. The proof of soundness of the translation is carried out using a particular universe of non-well-founded sets, and applies to a large class of extensions of $\Omega$. In Section 4, we generalize the proposed method to polymodal logics. Finally, in Section 5, we briefly discuss the application of set $T$-resolution techniques to support derivability in $\Omega$.

## 2 A set-theoretic translation of G

We first consider the case of the propositional modal logic $G$ obtained by adding the Löb's axiom schema $\Box(\Box\,\alpha \rightarrow \alpha) \rightarrow \Box\,\alpha$ to $K$. Our goal is to find a translation of $G$ formulae in the language of Set Theory and a finitely axiomatizable theory $\Omega$ such that, for any modal formula $\phi$, $\vdash_G \phi$ if and only if $\Omega$ proves the translation of $\phi$.

We consider the theory $\Omega$ specified by the following axioms in the language with relational symbols $\in, \subseteq$, and functional symbols $\cup, \setminus, Pow$:

$x \in y \cup z \leftrightarrow x \in y \vee x \in z$;

$x \in y \setminus z \leftrightarrow x \in y \wedge x \notin z$;

[1] Compare this theory with more classical finite axiomatizations of Set Theory, such as NBG [13].

$x \subseteq y \leftrightarrow \forall z(z \in x \rightarrow z \in y)$;

$x \in Pow(y) \leftrightarrow x \subseteq y$.

Notice that $\Omega$ does not contain the extensionality axiom as well as the axiom of foundation. In the next section we will make an essential use of the latter fact, since we will model the accessibility relation with the membership relation, and in order to do this we will be forced to work in universes containing non-well-founded sets. As a matter of fact, it will be convenient to use universes satisfying AFA [1].

In the case of $G$ a standard (well-founded) model of Set Theory is sufficient to carry out the proof of soundness of the translation.

Given a modal formula $\phi(P_1, ..., P_n)$, its translation is the *set-theoretic term* $\phi^*(x, x_1, ..., x_n)$, with variables $x, x_1, ..., x_n$, inductively defined as follows:

- $P_i^* \equiv x_i$;
- $(\phi \vee \psi)^* \equiv \phi^* \cup \psi^*$;
- $(\phi \wedge \psi)^* \equiv \phi^* \cap \psi^*$;
- $(\neg\phi)^* \equiv x \setminus \phi^*$;
- $(\phi \rightarrow \psi)^* \equiv (x \setminus \phi^*) \cup \psi^*$;
- $(\Box\, \phi)^* \equiv Pow(\phi^*)$.

where $x$ is a different from $x_i$ for $i = 1, .., n$, $\phi^* \cap \psi^*$ stands for $\phi^* \setminus (\phi^* \setminus \psi^*)$, and $\Diamond$ is translated as $\neg\Box\neg$.

We will show that:

$$\vdash_G \phi \text{ iff } \Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \rightarrow \forall x_1 ... \forall x_n(x \subseteq \phi^*(x, x_1, ..., x_n))),$$

where $Trans(x)$ stands for $\forall y(y \in x \rightarrow y \subseteq x)$ ($x$ is transitive), and $Axiom_G(x)$ represents the conjunction of $\forall y(y \subseteq x \wedge \exists z(z \in y) \rightarrow \exists s(s \in y \wedge \forall v(v \notin s \cap y)))$ and $\forall z \forall w \forall y(z \in x \wedge w \in x \wedge y \in x \wedge z \in w \wedge w \in y \rightarrow z \in y)$ ($x$ is well-founded and $\in$ restricted to $x$ is transitive, respectively).

We prove that the proposed translation is complete and sound. The proof of completeness is straightforward; the proof of soundness relies on the characterization of $G$ by the class of all finite trees.

**THEOREM 2.1 (Completeness of the translation method)** *For each formula $\phi$ of $G$ involving $n$ propositional variables $P_1, .., P_n$*

$$\vdash_G \phi \textit{ implies } \Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \rightarrow \forall x_1 ... \forall x_n(x \subseteq \phi^*(x, x_1, ..., x_n))).$$

**Proof.** The proof is by induction on the derivation $\vdash_G \phi(P_1, ..., P_n)$.

The cases of tautologies and closure under modus ponens do not present any difficulty, and thus they are omitted (a proof can be found in [7]). We explicitly prove the thesis for K and Löb's axiom schemata, and for closure under necessitation. We first consider the K axiom schema:

$$\Box\,(\alpha \rightarrow \beta) \rightarrow (\Box\,\alpha \rightarrow \Box\,\beta),$$

where, without loss of generality, we suppose that $\alpha$ and $\beta$ involve the $n$ propositional variables $P_1, .., P_n$, and show that $\Omega$ derives its translation, namely:

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \rightarrow \forall x_1 ... \forall x_n(x \subseteq (\Box\,(\alpha \rightarrow \beta) \rightarrow (\Box\,\alpha \rightarrow \Box\,\beta))^*)).$$

By definition, $(\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta))^* \equiv (x \setminus t) \cup s$, where $t$ and $s$ are terms on variables $x, x_1, ..., x_n$ defined as $t = Pow((x \setminus \alpha^*) \cup \beta^*)$ and $s = (x \setminus Pow(\alpha^*)) \cup Pow(\beta^*)$, respectively. We want to prove that $\forall z(z \in x \to z \in (x \setminus t) \cup s)$, or, equivalently, that $\forall z(z \in x \wedge z \in t \to z \in s)$. Thus, it suffices us to show that $z \in x$ and $z \subseteq (x \setminus \alpha^*) \cup \beta^*$ implies that $z \in (x \setminus Pow(\alpha^*)) \cup Pow(\beta^*)$. This can be done by showing that $z \in x$, $z \subseteq (x \setminus \alpha^*) \cup \beta^*$ and $z \subseteq \alpha^*$ implies $z \subseteq \beta^*$. Since $z \subseteq \alpha^*$, for each $s$ if $s \in z$ then $s \in \alpha^*$; $s \in \beta^*$ then follows from $z \subseteq (x \setminus \alpha^*) \cup \beta^*$.

Notice that we never used the hypothesis that $x$ satisfies $Trans(x)$ and $Axiom_G(x)$.

Consider now the closure under necessitation: if $\vdash_G \phi$ then $\vdash_G \Box\phi$. In such a case, we suppose that:

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \to \forall x_1 ... \forall x_n(x \subseteq \phi^*)),$$

and prove that:

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \to \forall x_1 ... \forall x_n(x \subseteq Pow(\phi^*))).$$

For each $x$ satisfying $Trans(x)$ and $Axiom_G(x)$, we must prove that $\forall x_1, ..., \forall x_n(x \subseteq Pow(\phi^*))$, that is, for each $z$ if $z \in x$ then $z \in Pow(\phi^*)$ or, equivalently, $z \subseteq \phi^*$. Suppose $z \in x$ and $t \in z$. Since $x$ satisfies $Trans(x)$, $z \subseteq x$ and thus $t \in x$. From the hypotheses, it follows that $t \in \phi^*$.

Finally, let us show that $\Omega$ proves the translation of Löb's axiom, that is, we show that if $P_1, .., P_n$ are the $n$ variables occurring in $\phi$, then

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \to \forall x_1 ... \forall x_n(x \subseteq (\Box(\Box\phi \to \phi) \to \Box\phi)^*)).$$

The proof is nothing but the formalization in $\Omega$ of the proof of the validity of the Löb's axiom schema in any well-founded transitive frame (cfr, e.g., [19]).

By definition, $(\Box(\Box\phi \to \phi) \to \Box\phi)^* \equiv (x \setminus t) \cup Pow(\phi^*)$, where $t$ stands for the term $Pow((x \setminus Pow(\phi^*)) \cup \phi^*)$. We want to prove that if $x$ satisfies $Trans(x) \wedge Axiom_G(x)$, then $\forall s(s \in x \wedge s \in t \to s \in Pow(\phi^*))$. This is equivalent to show that there exists no set belonging to the subset $y$ of $x$ with $y = x \cap t \setminus Pow(\phi^*)$. We consider the formula:

$$\forall y(\forall s(s \in y \to \exists v(v \in s \cap y)) \to (y \subseteq x \to \forall z(z \notin y))),$$

which can be derived from the axiom stating the well-foundedness of $x$, and show that for $y = x \cap t \setminus Pow(\phi^*)$ the formula $\forall s(s \in y \to \exists v(v \in s \cap y))$ holds. Since $y \subseteq x$, this proves the thesis.

If $s \in y$ then $s \in x$, $s \in t = Pow((x \setminus Pow(\phi^*)) \cup \phi^*)$, and $s \notin Pow(\phi^*)$. From the last conjunct, we derive that $\exists v(v \in s \wedge v \notin \phi^*)$. Since $x$ satisfies $Trans(x)$ and $Axiom_G(x)$ (in particolar, the transitivity of $\in$ with respect to $x$ holds), from $v \in s$ we can prove that $v \subseteq s$, i.e. $\forall e(e \in v \to e \in s)$. First of all, since $s \in x$, from $Trans(x)$ we can conclude that $s \subseteq x$. From $s \subseteq x$ and $v \in s$, we derive that $v \in x$, and then from $Trans(x)$ it follows that $v \subseteq x$. For each $e$ such that $e \in v$, $v \subseteq x$ implies that $e \in x$. Hence $e \in s$ follows from $e \in x$, $v \in x$, $s \in x$, $e \in v$, $v \in s$, and the transitivity of $\in$ on $x$.

Now, from $s \in Pow((x \setminus Pow(\phi^*)) \cup \phi^*)$ and $v \subseteq s$, it follows that $v \in Pow((x \setminus Pow(\phi^*)) \cup \phi^*)$, that is, $v \in t$. Finally, from $v \in s$, $s \subseteq (x \setminus Pow(\phi^*)) \cup \phi^*$, and $v \notin \phi^*$ it follows that $v \in x \setminus Pow(\phi^*)$, and then $v \notin Pow(\phi^*)$. From $v \in x$, $v \in t$, and $v \notin Pow(\phi^*)$ we can conclude that $v \in x \cap t \setminus Pow(\phi^*) = y$ that proves the thesis.

(T2.1) ∎

**THEOREM 2.2 (Soundness of the translation method)** *For each formula $\phi$ of $G$ involving $n$ propositional variables $P_1, ..., P_n$*

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \to \forall x_1 ... \forall x_n(x \subseteq \phi^*(x, x_1, ..., x_n))) \text{ implies } \vdash_G \phi.$$

**Proof.** The proof is based on the finite tree completeness of $G$. By a finite tree is meant a frame $(W, R, r)$ in which $W$ is a finite set containing the element $r$ (the root), $R$ is transitive and asymmetric and the set

of $R-$predecessors of any element contains $r$ and is linearly ordered by $R$. The Completeness Theorem says that $\vdash_G \phi$ iff $\phi$ is valid in every finite tree (see [19] for details).

Let $HF^A$ be the structure for the language of $\Omega$ consisting of all hereditarily finite sets built from atoms in $A = \{a_0, a_1, .....\}$, with the natural set-theoretic interpretation of the relational and functional symbols $\in, \subseteq, \cap, \cup, \setminus, Pow$. $HF^A$ is a model for $\Omega$[2]. Then, for every term $t(x_0, ..., x_n)$ and for every $h_0, ..., h_n$ in $HF^A$, we may consider the element $t(h_0, ..., h_n)$ in $HF^A$; in particular, if $\phi(P_1, ..., P_n)$ is a modal formula, the term $\phi^*(x, x_1, ..., x_n)$ may be evaluated in the elements $h_0, ..., h_n$ giving an element of $HF^A$.

We will proceed as follows. Given a finite tree $(W, R, r)$ we determine an element $W^*$ of $HF^A$ such that:

1. $Trans(W^*) \wedge Axiom_G(W^*)$ holds in the model $HF^A$, and
2. given a modal formula $\phi(P_1, ..., P_n)$, if $\forall x_1 ... \forall x_n (W^* \subseteq \phi^*(W^*, x_1, ..., x_n))$ holds in $HF^A$, then $\phi(P_1, ..., P_n)$ is valid in $(W, R, r)$.

Fix an injection $\pi$ from the leaves of $W$ (i.e. nodes without any successor) to $A$. We define $W^*$ in $HF^A$ as follows. For every node $w \in W$ let

$$w^* = \begin{cases} \pi(w) & \text{if } w \text{ is a leaf of } W, \\ \{v^* : wRv\} & \text{otherwise.} \end{cases}$$

Define $W^*$ as $r^*$. First, notice that $w^* \in HF^A$ for every $w \in W$. Moreover, it is not difficult to see that $HF^A \models Trans(W^*) \wedge Axiom_G(W^*)$. It is possible to prove by induction on the height $h(w)$ of the node $w$ in the tree $(W, R, r)$ that for any $v \in W$, if the elements $w^*$ and $v^*$ are equal in $HF^A$, then $w = v$ (the details of the proof are given in [7]).

Let $\models$ a valuation of the propositional variables $P_1, ..., P_n$ on $W$. Consider the elements $P_1^*, ..., P_n^*$ given by $P_i^* = \{w^* \in W^* : w \models P_i\}$. Since $W$ is finite, we have that $P_1^*, ..., P_n^*$ belong to $HF^A$.

To verify second condition we use the following lemma:

**LEMMA 2.1** *For all $w \in W$ and for all formula $\phi(P_1, ..., P_n)$,*

$$w \models \phi(P_1, ..., P_n) \text{ iff } w^* \in \phi^*(W^*, P_1^*, ..., P_n^*) \text{ holds in } HF^A.$$

**Proof.** By induction on the structure of the formula $\phi(P_1, ..., P_n)$.

If $\phi(P_1, ..., P_n) \equiv P_i$ and $w \models P_i$ then by definition of $P_i^*$ one has $w^* \in P_i^*$. Viceversa, if $w^* \in P_i^*$ then $w^* = z^*$ for some $z \in W$ with $z \models P_i$. But in this case we proved that $w = z$ and so $w \models P_i$. The proof for boolean connectives is straightforward [7]. Consider now the formula $\Box \phi(P_1, ..., P_n)$:

$w \models \Box \phi(P_1, ..., P_n)$ iff, for all $z \in W$, $wRz$ implies $z \models \phi(P_1, ..., P_n)$ iff, for all $z \in W$, $wRz$ implies $z^* \in \phi^*(W^*, P_1^*, ..., P_n^*)$ iff $\{z^* : wRz\} \subseteq \phi^*(W^*, P_1^*, ..., P_n^*)$ iff (by definition of $w^*$), $w^* \subseteq \phi^*(W^*, P_1^*, ..., P_n^*)$ iff $w^* \in Pow^*(\phi^*(W^*, P_1^*, ..., P_n^*))$ iff $w^* \in (\Box \phi)^*(W^*, P_1^*, ..., P_n^*)$.

(L2.1) ■

From Lemma (2.1), it follows that $\phi(P_1, ..., P_n)$ is valid in the model $(W, R, \models)$ if and only if the corresponding set $W^*$ in $HF^A$ is a subset of $\phi^*(W^*, P_1^*, ..., P_n^*)$.

To conclude the proof of Theorem (2.2), suppose that

$$\Omega \vdash \forall x (Trans(x) \wedge Axiom_G(x) \rightarrow \forall x_1 ... \forall x_n (x \subseteq \phi^*(x, x_1, .., x_n))).$$

If $(W, R, r)$ is a finite tree, the corresponding set $W^*$ in $HF^A$ satisfies $Trans(W^*) \wedge Axiom_G(W^*)$; then, from $\Omega \vdash \forall x (Trans(x) \wedge Axiom_G(x) \rightarrow \forall x_1 ... \forall x_n (x \subseteq \phi^*))$, it follows that for all elements $h_1, ..., h_n$ in $HF^A$

[2] Actually, $HF^A$ is also a model for the extensionality axiom for all non-atomic elements, even though we do not require such an axiom in $\Omega$.

one has $W^* \subseteq \phi^*(W^*, h_1, ..., h_n)$. In particular, for all valuations $\models$ of the propositional variables $P_1, ..., P_n$ on $W$, this is true for the sets $P_1^*, ..., P_n^*$ defined as in Lemma (2.1). From the same lemma one deduces that $\phi(P_1, ..., P_n)$ is valid in the model $(W, R, \models)$, and from the finite tree Completeness Theorem it follows $\vdash_G \phi$. (T2.2) ∎

# 3 The set-theoretic translation method

In this section we generalize the proposed set-theoretic translation method to any normal complete finitely-axiomatizable modal logic, possibly specified with Hilbert axioms only.

Let $\psi(\alpha_{j_1}, ..., \alpha_{j_m})$ be an axiom schema and $H$ be the modal logic obtained by adding $\psi(\alpha_{j_1}, ..., \alpha_{j_m})$ to $K$. Furthermore, let us suppose that $H$ is a complete modal logic, that is, for each modal formula $\phi$, $\vdash_H \phi$ if and only if $\phi$ is valid in each frame in which the axiom schema $\psi$ is valid. We prove the soundness and completeness of the proposed translation method for any normal complete finitely-axiomatizable modal logic. Formally, we prove that for any formula $\phi$ involving $n$ propositional variables $P_1, .., P_n$ and any normal complete finitely-axiomatizable modal logic $H = K + \psi(\alpha_{j_1}, .., \alpha_{j_m})$ the following holds:

$$\vdash_H \phi \text{ iff } \Omega \vdash \forall x(Trans(x) \wedge Axiom_H(x) \rightarrow \forall x_1 ... \forall x_n(x \subseteq \phi^*(x, x_1, ... x_n))),$$

where $Trans(x)$ is the formula $\forall y\ (y \in x \rightarrow y \subseteq x)$ and $Axiom_H(x)$ is the formula $\forall x_{j_1}, .., \forall x_{j_m}(x \subseteq \psi^*(x, x_{j_1}, .., x_{j_m}))$. As in the case of $G$, the proof of completeness is much easier than the proof of soundness and it can be found in [7].

**THEOREM 3.1 (Soundness of the translation method)** *For each formula $\phi$ of $H$ involving n propositional variables $P_1, .., P_n$*

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_H(x) \rightarrow \forall x_1 ... \forall x_n(x \subseteq \phi^*(x, x_1, .., x_n))) \text{ implies } \vdash_H \phi$$

**Proof.** Hereinafter, let $\mathcal{U}$ denote a universe of hypersets satisfying the axioms $ZF^- + AFA$ [1].
We begin proving the following lemma:

**LEMMA 3.1** *Let $\alpha$ be an ordinal, $V_\alpha$ be the set of all well-founded sets of rank less than $\alpha$, and $\mathcal{U} \setminus V_\alpha$ be the universe of all hypersets not belonging to $V_\alpha$. The structure for the language of $\Omega$ with support (domain) $\mathcal{U} \setminus V_\alpha$ and interpretation function $_'$ defined as follows[3]:*

$x \in' y$ *iff* $x \in y$;

$x \cup' y = x \cup y$;

$x \subseteq' y$ *iff* $x \setminus V_\alpha \subseteq y$;

$$x \setminus' y = \begin{cases} x \setminus y & \text{if } x \setminus y \notin V_\alpha \\ V_\alpha & \text{otherwise.} \end{cases};$$

$Pow'(y) = \{x : x \setminus V_\alpha \subseteq y\}$.

*is a model of $\Omega$.*

[3] We denote the defined interpretation of symbols $\in, \cup, \setminus, \subseteq, Pow$ in $\mathcal{U} \setminus V_\alpha$ with $\in', \cup', \setminus', \subseteq', Pow'$, and the *standard* interpretation in $\mathcal{U}$ simply by $\in, \cup, \setminus, \subseteq, Pow$.

**Proof.** We first show that $\setminus'$, $Pow'$, and $\cup'$ are well-defined over $\mathcal{U} \setminus V_\alpha$.

The proof for $\setminus'$ directly follows from its definition since either $x \setminus y$ does not belong to $V_\alpha$ and $\setminus'$ is equal to $x \setminus y$, or it is equal to $V_\alpha$ which does not belong to $V_\alpha$.

The case of $Pow'$ is also straightforward: by contradiction, suppose that $y \notin V_\alpha$ and $Pow'(y) \in V_\alpha$. By definition it follows that $y \in Pow'(y)$, and from the hypothesis $Pow'(y) \subseteq V_\alpha$ since $V_\alpha$ is transitive. Hence $y \in V_\alpha$ while we assumed $y \notin V_\alpha$.

Finally, for $x \cup' y$ notice that $x \cup' y$ is $x \cup y$ by definition, and $x \cup y \in V_\alpha$ if and only if $x \in V_\alpha$ and $y \in V_\alpha$.

To complete the proof we must show that the proposed interpretation verifies the axioms of $\Omega$.

Since $x \cup' y$ and $\in'$ are defined as $x \cup y$ and $\in$, respectively, the verification of the first axiom is trivial.

Consider now the second axiom and let $x, y, z$ belong to $\mathcal{U} \setminus V_\alpha$. If $y \setminus z \in V_\alpha$ then $y \setminus' z = V_\alpha$, and thus there are no $x \in \mathcal{U} \setminus V_\alpha$ such that $x \in' y \setminus' z$. This is correct since from $y \setminus z \in V_\alpha$ it follows that $y \setminus z \subseteq V_\alpha$ and then there are no $x \in \mathcal{U} \setminus V_\alpha$ such that $x \in' y$ and $x \notin' z$. In case $y \setminus z \notin V_\alpha$ we have that $y \setminus' z$ is equal to $y \setminus z$ and therefore the axiom is verified.

For the third axiom, suppose that $x, y, z$ belong to $\mathcal{U} \setminus V_\alpha$. By definition, $x \subseteq' y$ if and only if $x \setminus V_\alpha \subseteq y$, which is equivalent to say that for all $z$ in $\mathcal{U} \setminus V_\alpha$, if $z \in x$ then $z \in y$, namely $\forall z(z \in' x \rightarrow z \in' y)$ holds in $\mathcal{U} \setminus V_\alpha$.

Finally consider the fourth axiom, and let $x$ belong to $\mathcal{U} \setminus V_\alpha$. From the definition of $Pow'(y)$ it follows that $x \in' Pow'(y)$ if and only if $x \setminus V_\alpha \subseteq y$, but $x \setminus V_\alpha \subseteq y$ if and only if $x \subseteq' y$ and therefore the axiom is verified.

(L3.1) ∎

Given a frame $(W, R)$, we want to interpret it in the universe $\mathcal{U} \setminus V_\alpha$, for a suitable ordinal $\alpha$. Let us associate a set $a\!\downarrow$ in $\mathcal{U}$ with each world $a \in W$. From the axiom *AFA* it follows that, for each $a \in W$, there exists a unique labeled decoration $^*$ such that $a^* = \{b^* : aRb\} \cup a\!\downarrow$ (cf. [1]). Moreover, it is possible to define $a\!\downarrow$ in such a way that, for every $a, b$ in $W$, $a^* \notin b\!\downarrow$ and $a \neq b$ in $W$ implies $a^* \neq b^*$. For this purpose, let us consider a set $\tilde{W}$ in $\mathcal{U}$, whose elements are well-founded sets of the same rank $\alpha$, and such that there exists a bijection between $\tilde{W}$ and $W$. For each $a \in W$, we denote the image of $a$ in $\tilde{W}$ with $\tilde{a}$, and define $a\!\downarrow = \{\tilde{a}\}$. The following lemma can be easily proved.

**LEMMA 3.2** *For each $a, b$ in $W$, $a^* \notin b\!\downarrow$. Furthermore, from $a \neq b$ in $W$ it follows that $a^* \neq b^*$. Finally, for each $a \in W$, $a^* \notin V_{\alpha+1}$ and $a^* \setminus V_{\alpha+1} = \{b^* : aRb\}$.*

**Proof.** If $a^* \in b\!\downarrow$, then $b\!\downarrow = \{\tilde{b}\}$ implies that $a^* = \tilde{b}$. Since $\tilde{a} \in a\!\downarrow$ and $a\!\downarrow \subseteq a^*$, it follows that $\tilde{a} \in \tilde{b}$, but this is impossible because $\tilde{a}$ and $\tilde{b}$ have the same rank $\alpha$. Let us show now that if $a \neq b$ in $W$ then $a^* \neq b^*$. Suppose that $a^* = b^*$. From such an assumption it follows that $\tilde{a} \in b^*$, with $b^* = \{c^* : bRc\} \cup b\!\downarrow$. If $\tilde{a} \in b\!\downarrow$ then $\tilde{a} = \tilde{b}$, contradicting $a \neq b$. If $\tilde{a} = c^*$, for a given $c$ such that $bRc$, then $\tilde{c} \in \tilde{a}$ (contradiction). The remaining part of the proof easily follows.

(L3.2) ∎

Consider now a valuation $\models$ of the propositional variables $P_1, ..., P_n$ over the frame $(W, R)$, and let $W^*$ be equal to $\{a^* : a \in W\}$, where $^*$ is the previously introduced labeled decoration. $W^*$ does not belong to $V_{\alpha+1}$ because $V_{\alpha+1}$ is transitive and, for each $a \in W$, $a^* \notin V_{\alpha+1}$. Furthermore, let $P_i^*$ be equal to $\{a^* \in W^* : a \models P_i\}$ if such a set is not empty, and to $V_{\alpha+1}$ otherwise. The following lemma holds.

**LEMMA 3.3** *For each $a \in W$ and each formula $\phi(P_1, ..., P_n)$*

$$a \models \phi(P_1, ..., P_n) \text{ iff } a^* \in' \phi^*(W^*, P_1^*, ..., P_n^*) \text{ in the universe } \mathcal{U} \setminus V_{\alpha+1}.$$

**Proof.** By induction on the structure of the formula $\phi(P_1, ..., P_n)$. The cases of propositional variables and boolean combinations of formulae are left to the reader (see [7] for details). In the case of a formula of the form $\Box\, \phi(P_1, ..., P_n)$, we have that:

$a \models \Box\, \phi(P_1, ..., P_n)$ iff for all $b \in W$, $aRb$ implies $b \models \phi(P_1, ..., P_n)$ iff for all $b \in W$, $aRb$ implies $b^* \in' \phi^*(W^*, P_1^*, ..., P_n^*)$ iff $a^* \setminus V_{\alpha+1} \subseteq \phi^*(W^*, P_1^*, ..., P_n^*)$ iff $a^* \in' Pow'(\phi^*(W^*, P_1^*, ..., P_n^*))$ iff $a^* \in' (\Box\, \phi)^*(W^*, P_1^*, ..., P_n^*)$.

(L3.3) ∎

From Lemma (3.3), it follows that a formula $\phi(P_1, ..., P_n)$ is valid in a model $(W, R, \models)$ if and only if the corresponding hyperset $W^*$ is a subset of $\phi^*(W^*, P_1^*, ..., P_n^*)$ in the model $\mathcal{U} \setminus V_{\alpha+1}$. This result can be generalized to frames.

**LEMMA 3.4** *A formula $\phi(P_1, ..., P_n)$ is valid in the frame $(W, R)$ if and only if for the corresponding hyperset $W^*$, $\forall x_1, ..., x_n(W^* \subseteq' \phi^*(W^*, x_1, ..., x_n))$ holds in $\mathcal{U} \setminus V_{\alpha+1}$.*

**Proof.** For each $a \in W$ and $x_1, .., x_n \in \mathcal{U} \setminus V_{\alpha+1}$, it is straightforward to prove that

$$a^* \in' \phi^*(W^*, x_1, ..., x_n) \text{ iff } a^* \in' \phi^*(W^*, x_1 \cap' W^*, ..., x_n \cap' W^*).$$

The proof is by induction on the structure of the formula $\phi$ and we only present the proof of the inductive step for $\phi \equiv \Box\, \beta$ (see [7] for details):

$a^* \in' (\Box\, \beta)^*(W^*, x_1, ..., x_n)$ iff $a^* \in' Pow'(\beta^*(W^*, x_1, ..., x_n))$ iff $a^* \setminus V_{\alpha+1} \subseteq \beta^*(W^*, x_1, ..., x_n)$ iff for all $b \in W$, if $aRb$ then $b^* \in' \beta^*(W^*, x_1, ..., x_n)$ iff for all $b \in W$, if $aRb$ then $b^* \in' \beta^*(W^*, x_1 \cap' W^*, ..., x_n \cap' W^*)$ iff $a^* \setminus V_{\alpha+1} \subseteq \beta^*(W^*, x_1 \cap' W^*, ..., x_n \cap' W^*)$ iff $a^* \in' (\Box\, \beta)^*(W^*, x_1 \cap' W^*, ..., x_n \cap' W^*)$.

Given $n$ hypersets $x_1, ..., x_n$ belonging to $\mathcal{U} \setminus V_{\alpha+1}$, we define a valuation $\models$ of the propositional variables $P_1, ..., P_n$ such that, for each $a \in W$, $a \models P_i$ if and only if $a^* \in' x_i \cap' W^*$. It is straightforward to see that, if $P_1^*$, .., $P_n^*$ are the hypersets defined in Lemma (3.3) on the basis of the valuation $\models$, then $P_i^*$ and $x_i \cap' W^*$ have the same elements in the model $\mathcal{U} \setminus V_{\alpha+1}$. From this, it is easy to verify by induction on $\phi$ that $\phi^*(W^*, P_1^*, .., P_n^*)$ and $\phi^*(W^*, x_1 \cap' W^*, ..., x_n \cap' W^*)$ have the same elements. Suppose now that a formula $\phi$ is valid in the frame $(W, R)$; then it is also valid in the model $(W, R, \models)$, and from Lemma (3.3) it follows that, for all $a \in W$, $a^* \in' \phi^*(W^*, x_1 \cap' W^*, ..., x_n \cap' W^*)$. This allows us to conclude that $a^* \in' \phi^*(W^*, x_1, ..., x_n)$, and, therefore, for all hypersets $x_1, ..., x_n$ in $\mathcal{U} \setminus V_{\alpha+1}$, $W^* \subseteq' \phi^*(W^*, x_1, ..., x_n)$.

The vice versa can be easily proved by associating the hypersets $P_1^*$, .., $P_n^*$ (where $P_i^*$ is equal to $\{a^* \in' W^* : a \models P_i\}$ if such a set is not empty, and to $V_{\alpha+1}$ otherwise) with each valuation $\models$ of the propositional variables $P_1, ..., P_n$.

(L3.4) ∎

To conclude the proof of Theorem (3.1), let us suppose that

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_H(x) \rightarrow \forall x_1 ... \forall x_n(x \subseteq \phi^*(x, x_1, .., x_n))).$$

Let $(W, R)$ be a frame in which the axiom schema $\psi(\alpha_{j_1}, ..., \alpha_{j_m})$ characterizing the normal modal logic $H$ is valid. In particular, the formula $\psi(P_{j_1}, ..., P_{j_m})$ is valid in $(W, R)$, and from Lemma (3.4), it follows that the formula $\forall x_{j_1}, ..., \forall x_{j_m} W^* \subseteq' \psi^*(W^*, x_{j_1}, ..., x_{j_m})$ is true in the universe $\mathcal{U} \setminus V_{\alpha+1}$. Furthermore, it is easy to prove that $Trans(W^*)$ holds as well. Since $\mathcal{U} \setminus V_{\alpha+1}$ is an $\Omega$ model, from the hypotheses it follows that the formula $\forall x_1 ... \forall x_n$ $(W^* \subseteq' \phi^*(W^*, x_1, .., x_n))$ is true in $\mathcal{U} \setminus V_{\alpha+1}$, and then, again from Lemma (3.4), that the formula $\phi$ is valid in $(W, R)$. From the completeness of the normal modal logic $H$, it follows that $\vdash_H \phi$.

(T3.1) ∎

**Important Remark.** From the preceding proof, it should be clear that the proposed translation method works for any theory $\Omega^*$ extending $\Omega$, provided that the model $\mathcal{U} \setminus V_\alpha$ of Lemma (3.1) is a model of $\Omega^*$. This fact will play an essential role in Section 5, where we will prove the decidability results needed to apply the machinery of T-theorem proving to a theory $\Omega'$, which is stronger than $\Omega$, but still has $\mathcal{U} \setminus V_\alpha$ as a model.

One could observe that this remark does not apply to theories with extensionality and/or foundation. As far as extensionality is concerned, it is possible to show that we can deal with theories with extensionality by making a little change in the translation. The status of the axiom of foundation is entirely different; at least as long as one wants to represent the accessibility relation using the membership relation, the *anti-foundation* axiom does seem to be the best possible choice.

# 4 The generalization to polymodal logics

In this section we generalize the proposed set-theoretic translation method to polymodal logics. The main problem involved in such a generalization is the problem of mapping a polymodal frame, consisting of a set $U$ endowed with $k$ accessibility relations $R_1, .., R_k$, with $k > 1$, into a set provided with the membership relation only.

We solved this problem by preliminarly providing polymodal logics with an alternative semantics that turns the plurality of accessibility relations $R_1, .., R_k$ into a single accessibility relation $R$ and $k$ subsets $U_1, .., U_k$ of $U$.

## 4.1 An alternative semantics for polymodal logics

Let us first introduce the alternative semantics for polymodal logics, called *p-semantics*, and the relevant notions of frame, valuation, validity, and so on. To distinguish such a notions from the standard ones, we add the prefix $p$ to the usual terms (e.g. p-valuation, p-model, p-frame).

**DEFINITION 4.1** *A p-frame $\mathcal{F}$ is a $(k+2)$-tuple $(U, U_1, ..., U_k, R)$, where $U, U_1, ..., U_k$ are sets and $R$ is a binary relation on $U \cup U_1 \cup ... \cup U_k$, such that, for all $u, v, t$ in $U \cup U_1 \cup ... \cup U_k$, if $u \in U$, $uRv$ and $vRt$ then $t \in U$ (let us denote such a property as $Trans^2(U)$).*

It is worth noting that, according to the given definition of p-frame, from $u \in U$ and $uRv$ we cannot conclude that $v \notin U$.

**DEFINITION 4.2** *A p-valuation $\models_p$ is a subset of $U \times \Phi$, where $\Phi$ is the set of propositional variables. A p-valuation thus assigns a truth value to propositional variables at worlds belonging to $U$ only.*

The p-valuation $\models_p$ may be lifted to the set of all polymodal formulae in a canonical fashion in the case of boolean combinations, while, for all $u \in U$ and $i = 1, ...k$, we state that

$$u \models_p \Box_i \phi \text{ iff } \forall v(uRv \wedge v \in U_i \rightarrow \forall t(vRt \rightarrow t \models_p \phi)).$$

**DEFINITION 4.3** *A polymodal formula $\phi$ is p-valid in a p-frame $(U, U_1, ..., U_k, R)$ iff, for all p-valuation $\models_p$ and worlds $u \in U$, $u \models_p \phi$ holds.*

On the basis of the given definitions, it is possible to prove the following lemma:

**LEMMA 4.1** *Given a p-frame $(U, U_1, ..., U_k, R)$, there exists a classical polymodal frame $(U, \lhd_1, ..., \lhd_k)$ based on the set $U$ that validates all and only the formulae $\phi$ which are p-valid in $(U, U_1, ..., U_k, R)$.*

**Proof.** First of all, we define $\lhd_1, ..., \lhd_k$ as follows:

$$u \lhd_i v \text{ iff } \exists t(t \in U_i \wedge uRt \wedge tRv).$$

Then, note that any p-valuation $\models_p$ on the p-frame $(U, U_1, ..., U_k, R)$ may be interpreted as a valuation on $(U, \lhd_1, ..., \lhd_k)$, and vice versa. For any $u \in U$ and any polymodal formula $\phi$, it can be shown that:

$$u \models_p \phi \text{ iff } u \models \phi.$$

The proof is by induction on $\phi$. We confine ourselves to the case of $\Box_i$ operators, that is, $\phi$ of the form $\Box_i \psi$ (the proof of the other cases is straightforward).

Suppose that $u \models_p \Box_i \psi$. We want to prove that $u \models \Box_i \psi$, that is, $\forall w(u \lhd_i w \rightarrow w \models \psi)$. Consider a world $w$ such that $u \lhd_i w$. By definition of $\lhd_i$, it holds that $\exists t(t \in U_i \wedge uRt \wedge tRw)$. But $u \models_p \Box_i \psi$ means that $\forall v(uRv \wedge v \in U_i \rightarrow \forall t(vRt \rightarrow t \models_p \psi))$. Hence, $w \models_p \psi$ and $w \models \psi$ by induction.

Suppose now that $u \models \Box_i \psi$. If $v \in U_i$ is such that $uRv$, then, for all $t$ such that $vRt$, it follows that $u \lhd_i t$ (by definition of $\lhd_i$). From the hypothesis, it follows $t \models \psi$ and, by induction, $t \models_p \psi$.

Thus, if the formula $\phi$ is p-valid in $(U, U_1, ..., U_k, R)$, then, given any classical valuation $\models$ on $(U, \lhd_1, ..., \lhd_k)$, it follows that, for all $u \in U$, $u \models_p \phi$ holds in the corresponding p-model $(U, U_1, ..., U_k, R, \models_p)$, and, thus, $u \models \phi$ in $(U, \lhd_1, ..., \lhd_k)$. Since this is true for all $u \in U$ and all valuation $\models$, it follows that $\phi$ is classically valid in the frame $(U, \lhd_1, ..., \lhd_k)$. Symmetrically, it is possible to prove that, if $\phi$ is valid in $(U, \lhd_1, ..., \lhd_k)$, then it is p-valid in the p-frame $(U, U_1, ..., U_k, R)$.

(L4.1) ∎

**LEMMA 4.2** *For every classical polymodal frame $(U, \lhd_1, ..., \lhd_k)$ there exists a p-frame $(U, U_1, ..., U_k, R)$ that p-validates all and only the formulae $\phi$ which are valid in $(U, \lhd_1, ..., \lhd_k)$.*

**Proof.** Let us define $U_1, ..., U_k, R$ as follows:

- let $U_1, ..., U_k$ be pairwise disjoint sets isomorphic to $U$, each one disjoint from $U$ (let us denote by $u \mapsto u_i$ the correspondence between $U$ and $U_i$);
- for $i = 1, ..., k$ and $u, v \in U$, $u_i Rv$ iff $u \lhd_i v$;
- for all $u \in U$ and $i = 1, ..., k$, $uRu_i$.

First of all, it is easy to show that $Trans^2(U)$ holds in $(U, U_1, ..., U_k, R)$. Moreover, as before, any valuation $\models$ on $(U, \lhd_1, ..., \lhd_k)$ can be seen as a p-valuation $\models_p$ on $(U, U_1, ..., U_k, R)$. It is possible to prove that, for any $u \in U$ and any polymodal formula $\phi$, it holds that:

$$u \models \phi \text{ iff } u \models_p \phi$$

Again, the verification of boolean combinations is left to the reader. We consider only the case in which $\phi$ is of the form $\Box_i \psi$.

If $u \models \Box_i \psi$, then, to prove that $u \models_p \Box_i \psi$, take $v, t$ in $U \cup U_1 \cup ... \cup U_k$ such that $v \in U_i$, $uRv$ and $vRt$. By definition of $R$, there exists $u_i \in U_i$ such that $v = u_i$ ($uRv$ and $u$ is different from any $u_i$), and, thus, $vRt$ can be rewritten as $u_i Rt$. By definition of $R$, $u_i Rt$ (if and) only if $u \lhd_i t$. Then, from the hypothesis, it follows $t \models \psi$, and, by induction, $t \models_p \psi$.

Suppose now that $u \models_p \Box \psi$, and let $v$ be $\lhd_i$-related to $u$, that is, $u \lhd_i v$. By definition of $R$, it follows that $uRu_i$ and $u_i Rv$; then, by definition of $\models_p$, it follows that $v \models_p \psi$, and, by induction, $v \models \psi$.

The thesis easily follows (as in Lemma (4.1)). (L4.2) ∎

It is worth noting that Lemma 4.1 and Lemma 4.2 together show that any p-frame $\mathcal{F} = (U, U_1, .., U_k, R)$ can be reduced to a p-frame $\mathcal{F}' = (U, U'_1, .., U'_k, R')$ whose set components $U, U'_1, .., U'_k$ are pairwise disjoint.

The previous two lemmas allow us to prove the following theorem.

**THEOREM 4.1** *If $H$ is a complete k-dimensional polymodal logic extending $K \otimes ... \otimes K$, then $\vdash_H \phi$ iff $\phi$ is p-valid in all p-frames in which all axioms of $H$ are p-valid.*

**Proof.** Since $H$ is complete, $\vdash_H \phi$ iff $\phi$ is valid in all classical frame in which all axioms of $H$ are valid. Then apply Lemma 4.2 and Lemma 4.1. (T4.1) ∎

## 4.2 A set-theoretic translation method for polymodal logics

As in the soundness proof for the monomodal case, we interpret any p-frame $(U, U_1, ..., U_k, R)$ as a $(k+1)-tuple$ $U^*, U_1^*, ..., U_k^*$ of "sets" in a particular $\Omega$ model such that, for all elements $t^*$ of the model that are $\in$-related to $U^* \cup U_1^* \cup ... \cup U_k^*$, it holds that:

$$t^* = \{s^* : tRs\}.$$

As in the monomodal case, every p-valuation of the propositional variables $P_1, ..., P_n$ on the p-frame is interpreted in terms of $n$ subsets $P_1^*, ..., P_n^*$ of $U^*$. Moreover, for each polymodal formula $\phi$, we want to define its translation as a term $\phi^*(x, y_1, ..., y_k, x_1, ..., x_n)$ such that:

$$u \models_p \phi \text{ iff } u^* \in \phi^*(U^*, U_1^*, ..., U_k^*, P_1^*, ...P_n^*).$$

Under such a constraint, the definition of the translation of $\Box_i\phi$ directly follows from the definition of $\models_p$ (and induction):

$u \models_p \Box_i\phi$ iff $\forall v(uRv \wedge v \in U_i \rightarrow \forall t(vRt \rightarrow t \models_p \phi))$ iff $\forall v(v^* \in u^* \wedge v^* \in U_i^* \rightarrow \forall t(t^* \in v^* \rightarrow t^* \in \phi^*))$ iff $u^* \cap U_i^* \subseteq P(\phi^*)$ iff $u^* \subseteq (U^* \setminus U_i^*) \cup P(\phi^*)$ iff $u^* \in P((U^* \setminus U_i^*) \cup P(\phi^*))$.

Thus, the translation of the term $\Box_i\phi(P_1, ..., P_n)$ must be defined as follows:

$$(\Box_i\phi)^* \equiv P((x \setminus y_i) \cup P(\phi^*))$$

Let us now prove the soundness and completeness of the translation method for polymodal logics.

**THEOREM 4.2** *(**Soundness of the translation method**) Let $H$ be a k-dimensional polymodal logic extending $K \otimes ... \otimes K$ by means of the axiom schema $\psi(\alpha_{j_1}, .., \alpha_{j_m})$. If $H$ is complete, then for each polymodal formula $\phi(P_1, .., P_n)$,*

$$\Omega \vdash \forall x \forall y_1 ... \forall y_k (Trans^2(x) \wedge Axiom_H(x, y_1, ..., y_k) \rightarrow \forall x_1 ... \forall x_n (x \subseteq \phi^*(x, y_1, ..., y_k, x_1, .., x_n)))$$

*implies*

$$\vdash_H \phi$$

*where $Axiom_H(x, y_1, ..., y_k)$ is, as usual, $\forall x_1 ... \forall x_m (x \subseteq \psi^*(x, y_1, ..., y_k, x_1, .., x_m)))$, and $Trans^2(x)$ stands for $\forall y \forall z\ (y \in z \wedge z \in x \rightarrow y \subseteq x)$, that is, $x \subseteq Pow(Pow(x))$.*

**Proof.** Since $H$ is complete, to show that $\vdash_H \phi$ it suffices to prove that $\phi$ is p-valid in all p-frame in which $\psi$ is p-valid (Theorem 4.1).

Let $(U, U_1, ..., U_k, R)$ be a p-frame in which $\psi$ is p-valid. Then, we can reason as in the monomodal case to prove that in a model of $\Omega$ there are $k$ sets $U^*, U_1^*, ..., U_k^*$ such that $Trans^2(U^*)$ holds and that, for any polymodal formula $\alpha(P_1, ..., P_n)$, $\forall x_1, .., x_n(U^* \subseteq \alpha^*(U^*, U_1^*, ..., U_k^*, x_1, ..., x_n))$ holds in the model if and only if $\alpha(P_1, ..., P_n)$ is p-valid in $(U, U_1, ..., U_k, R)$. Hence $Axiom_H(U^*, U_1^*, ..., U_k^*)$ holds in the model, and, by hypotesis, it follows that $\forall x_1, .., x_n(U^* \subseteq \phi^*(U^*, U_1^*, ..., U_k^*, x_1, ..., x_n))$. This allows us to conclude that $\phi$ is p-valid in $(U, U_1, ..., U_k, R)$. (T4.2) ∎

**THEOREM 4.3** *(**Completeness of the translation method**) Let $H$ be a k-dimensional polymodal logic extending $K \otimes ... \otimes K$ by means of the axiom schema $\psi(\alpha_{j_1}, .., \alpha_{j_m})$. If $H$ is complete, then, for each polymodal formula $\phi(P_1, .., P_n)$,*

$$\vdash_H \phi$$

*implies*

$$\Omega \vdash \forall x \forall y_1 ... \forall y_k (Trans^2(x) \wedge Axiom_H(x, y_1, ..., y_k) \rightarrow \forall x_1 ... \forall x_n (x \subseteq \phi^*(x, y_1, ..., y_k, x_1, .., x_n)))$$

**Proof.** The proof is by induction on the derivation $\vdash_H \phi$. The cases of tautologies, closure under substitution rule and modus ponens, and closure under the $H$-axiom are the same as in the monomodal case, and thus omitted.

We prove the closure under the $K$-axiom and necessitation rule of the modalities $\Box_i$. The closure under the $K$-axiom $\Box_i(\alpha \rightarrow \beta) \rightarrow (\Box_i \alpha \rightarrow \Box_i \beta)$ can be proved by showing that

$$\Omega \vdash \forall x \forall y_1 ... \forall y_k (Trans^2(x) \wedge Axiom_H(x, y_1, ..., y_k) \rightarrow \forall x_1 ... \forall x_n (x \subseteq (x \setminus u) \cup (x \setminus v) \cup P((x \setminus y_i) \cup P(\beta^*)))$$

where $u$ and $v$ stand for $P((x \setminus y_i) \cup P((x \setminus \alpha^*) \cup \beta^*)))$ and $P((x \setminus y_i) \cup P(\alpha^*))$, respectively.

Let us take $x, y_1 ..., y_k$ such that both $Trans^2(x)$ and $Axiom_H(x, y_1, ..., y_k)$ hold, and fix $x_1, ..., x_n$. Suppose that $t \in x$, $t \in u$, and $t \in v$; we must prove that $t \subseteq (x \setminus y_i) \cup P(\beta^*)$, that is, if $s \in t$ and $s \in y_i$ then $s \in P(\beta^*)$. It immediately follows from $t \in u$ and $t \in v$, because from the former we obtain $s \subseteq (x \setminus \alpha^*) \cup \beta^*$, and from the latter $s \subseteq \alpha^*$.

In order to prove the closure under necessitation rule, we must show that from

$$\Omega \vdash \forall x \forall y_1 ... \forall y_k (Trans^2(x) \wedge Axiom_H(x, y_1, ..., y_k) \rightarrow \forall x_1 ... \forall x_n (x \subseteq \phi^*(x, y_1, ..., y_k, x_1, .., x_n)));$$

it follows that

$$\Omega \vdash \forall x \forall y_1 ... \forall y_k (Trans^2(x) \wedge Axiom_H(x, y_1, ..., y_k) \rightarrow \forall x_1 ... \forall x_n (x \subseteq (\Box_i \phi)^*(x, y_1, ..., y_k, x_1, .., x_n))).$$

Again, let us take $x, y_1 ..., y_k$ such that both $Trans^2(x)$ and $Axiom_H(x, y_1, ..., y_k)$ hold, and fix $x_1, ..., x_n$. By hypothesis, it follows that $x \subseteq \phi^*(x, y_1, ..., y_k, x_1, .., x_n)$. Thus, $P(x) \subseteq P(\phi^*) \subseteq (x \setminus y_i) \cup P(\phi^*)$ and then $P(P(x)) \subseteq P((x \setminus y_i) \cup P(\phi^*))$. From $Trans^2(x)$, it follows that $x \subseteq P(P(x))$, and, therefore, $x \subseteq P((x \setminus y_i) \cup P(\phi^*)) = (\Box_i \phi)^*$. (T4.3) ∎

# 5 On the application of set $T$-resolution

As we said in the introduction, on the ground of the results presented in the preceding sections it is possible to automatically test modal derivability–from modal theories in the specified class–using a classical first-order theorem prover.

Recently a more specialized technique (called $T$-theorem proving) for automated theorem proving in first-order theories has been proposed (see [18]). Based on the translation method introduced above, a special case $T$-theorem proving in which the underlying theory $T$ is $\Omega$ (or similar) can now be considered as an alternative for automatically testing modal derivability. In this section we briefly consider the problem of applying set $T$-resolution together with translation method.

A prerequisite to employ $T$-resolution in the context of a given theory $T$, is the decidability, with respect to $T$, of the class of ground formulae written in any language which extends the one in which the axioms of $T$ are written, with Skolem (uninterpreted) function symbols. In [18], it has been shown that the satisfiability problem with respect to any theory $T$ of ground formulae on a given language $\mathcal{L}^*$ obtained from $\mathcal{L}(T)$ adding

an arbitrary amount of functional and constant symbols, is equivalent to the $T$-satisfiability of the class of purely existential formulae written in $\mathcal{L}(T)$. Therefore we are interested in this last problem in the case of $\Omega$, whose language ($\mathcal{L}(\Omega)$ from now on) consists of the symbols $\emptyset, \cup, \setminus, \subseteq, \in$, and $Pow$.

Before commenting on the above mentioned problem, notice that the decidability of classes very similar to the one we want to deal with has already been proved by Cantone, Schwartz, and Ferro [3, 4, 5]. Unfortunately, the mentioned results–among the most complex in the field of Computable Set Theory–cannot be applied to our context, the problem being the background set theory upon which they rest. Our theory $\Omega$ is very weak, while it can be easily checked that the proofs in [3, 5] make an essential use of assumptions such as regularity, existence of the transitive closure of sets, extensionality, etc., which are certainly not derivable from $\Omega$.

We succeeded in providing a proof of the decidability result we need for a theory $\Omega'$ slightly stronger (but having essentially the same language) than $\Omega$. $\Omega'$ differs from $\Omega$ since it assumes as axiom some simple consequences of Cantor theorem on the number of subsets of a given set, which are not derivable from $\Omega$ [7].

The proof is based on a technique firstly introduced in [6, 17]. The main idea is the following: in order to establish whether there exists a model of $\Omega'$ satisfying a formula $\varphi(x_1, \ldots, x_n)$ (unquantified formula written in $\mathcal{L}(\Omega')$), we assume that there exists $M \models \Omega'$ such that $M \models \varphi(x_1, \ldots, x_n)$, and we concentrate our attention on $n$ elements $a_1, \ldots, a_n$ in the support of $M$ satisfying $\varphi$. Our goal is to show that under this hypothesis we can build another $n$-tuple $a_1^*, \ldots, a_n^*$ of elements in the support of a model $M'$ of $\Omega'$ that continue to satisfy $\varphi$. Moreover $a_1^*, \ldots, a_n^*$ will be completely described by a graph $G$ whose size is bounded by a function of $n$, in the sense that in order to test the existence of ($M'$ and) $a_1^*, \ldots, a_n^*$ it will be sufficient to test the existence of $G$, and this result will guarantee the decidability.

The problem of determining $a_1^*, \ldots, a_n^*$ is combinatorially non-trivial. First of all, notice that if in the formula $\varphi(x_1, \ldots, x_n)$ we had no conjuncts of the form $Pow(x_i) = x_j$, then we could define $a_1^*, \ldots, a_n^*$ simply as a $n$-tuple satisfying $a_i^* = \{a_j^* \mid a_j \in^M a_i\}$, and it would be easy to check that all our requirements would be satisfied (recall that we do not have to deal with the extensionality axiom). As a matter of fact we can think of the map $^*$ as a way of *marking* some of the elements in each of the $a_i$ (the marked elements being those of the form $a_j$) and then take $a_i^*$ as the set of (images with respect to $*$ of) marked elements in $a_i$. In order to deal with a literal of the form $Pow(x_i) = x_j$, we need to mark more elements: at least all those elements that are subsets of the set of marked elements in $a_i$. Notice that if one simply does so and marks all such elements (subsets of $a_i$) without "care", new elements can turn out marked in $a_i$, and the marking process may not terminate. We solved the above problem processing the $a_i$'s in an order compatible with their size and applying the simple consequences of Cantor's theorem that were forced to hold in $\Omega'$ just for this purpose (the details of the proof are given in [7]).

It may be interesting to note that it is still an open problem whether the class of purely existential formulae of $\mathcal{L}(\Omega)$ is decidable with respect to $\Omega$. In other words it is not known whether $T$-theorem proving can be applied directly to $\Omega$; hence, up to this point, despite its simplicity, $\Omega$ seems to be a less suitable theory for computational purposes, than a more complex one (i.e. $\Omega'$).

# Conclusions

In the paper we proposed a new translation method mapping polymodal formulae into set-theoretic terms of the very week set theory $\Omega$. The method can be used for any normal complete finitely-axiomatizable polymodal logic, possibly specified with Hilbert axioms only, and applies to a large class of theories extending $\Omega$. A very important and interesting line of investigations at this point, is the generalization of the proposed method to first-order (poly)modal logics.

We are also investigating the possibility of exploiting our translation method to reduce undecidable decision problems for particular propositional polymodal logics, e.g. [11], to the derivability problem with

respect to $\Omega$ of formulae of type $\forall^*\exists$, thereby showing the undecidability of the latter problem.

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