Logic of Proofs with the strong provability operator

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Abstract

Logics with the strong provability operator "... *is true and provable*" together with the proof operators "*p is a proof of* ..." are axiomatized. Kripke-style completeness, decidability and arithmetical completeness of these logics are established.

1 Introduction

Logics with the provability operator "... *is provable*" and the proof operators "*p is a proof of* ..." corresponding to certain natural classes of proof predicates in Peano arithmetic **PA** were introduced in [1]. In some respect (see *e.g.* [2],[3]) the strong provability operator "... *is true and provable*" provides a better model for provability than the operator "... *is provable*". The logic of the strong provability operator is known ([4]) to coincide with Grzegorczyk logic **Grz**.

In this paper we present joint logics with both the strong provability operator and the proof operators. These logics are proved to be decidable; natural

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axiomatizations of them based on Grzegorczyk logic are offered, provided with Kripke-style semantics and proved to be arithmetically complete.

Fix a labeled modal language $\mathcal{L}$ containing a countable set of *proof variables*, a countable set of *propositional variables*, propositional connectives $\bot$, $\rightarrow$, the unary modal operator $\boxdot$ and for each proof variable p the unary labeled modal operator $\Box_p$. Formulas of $\mathcal{L}$ are defined as usual; in particular, if p is a proof variable and A is a formula of $\mathcal{L}$, then $\Box_p A$ is a formula of $\mathcal{L}$ (called quasi-atomic in [1]). Until further notice a modal formula will be understood to be a formula in the language $\mathcal{L}$. We shall let $\ulcorner\varphi\urcorner$ denote the numeral for the Gödel number of φ. We shall not distinguish between a natural number and its numeral.

We now need some definitions.

Let $Prf(x,y)$ be an arithmetical formula which is provably equivalent in **PA** to a srandard Δ_1 formula, and let $Prf(x,y)$ enumerate the theorems of **PA**, i.e. for each arithmetic formula φ

$$\mathbf{PA} \vdash \varphi \quad \Leftrightarrow \quad \exists x Prf(x, \ulcorner\varphi\urcorner) \text{ is true.}$$

Put $Pr(y) := \exists x Prf(x,y)$. We call $Prf(x,y)$ a *standard proof predicate* (cf.[1],[5]) if for any arithmetical variables φ and ψ

$$\mathbf{PA} \vdash (Pr(\ulcorner\varphi\urcorner) \& Pr(\ulcorner\varphi \rightarrow \psi\urcorner)) \rightarrow Pr(\ulcorner\psi\urcorner),$$

and for each Σ^0_1-arithmetical sentence σ

$$\mathbf{PA} \vdash \sigma \rightarrow Pr(\ulcorner\sigma\urcorner).$$

A standard proof predicate Prf is *functional* ([1]) if for all natural numbers l, m, n such that $Prf(l,m)$ and $Prf(l,n)$ we have $m = n$.

Let Prf be a standard proof predicate and let α be a function that assigns to each proof variable a natural number and to each propositional variable a sentence of the language of Peano arithmetic. We define a *realization* to be a pair (Prf, α); a realization is called *functional* if Prf is functional. The *arithmetical translation* A^* of a modal formula A under the realization $* = (Prf, \alpha)$ is defined as follows:

$\bot^* = (0 = 1)$,
$S^* := \alpha(S)$ for any propositional variable S,

$(A \to B)^* = A^* \to B^*$,
$p^* := \alpha(p)$ for any proof variable p,
$(\boxdot A)^* := A^* \& Pr(\ulcorner A^* \urcorner)$, and
$(\Box_p A)^* := Prf(p^*, \ulcorner A^* \urcorner)$.

There is no need to consider also the strong version of labeled modalities

$$(\Box_p A)^* := A^* \& Prf(p^*, \ulcorner A^* \urcorner)$$

since the last formula is obviously equivalent in **PA** to $Prf(p^*, \ulcorner A^* \urcorner)$.

We introduce the following logics with the strong provability operator and the proof operators:
the system **BGrz** which turns out to correlate with the class of all standard proof predicates,
the system **FGrz** corresponding to the class of all functional standard proof predicates,
the system **MGrz** describing the behaviour of the usual Gödel proof predicate.

Natural axiomatizations of these logics combine **Grz** axioms for the strong modality with the principles for the labeled modalities of the systems **B**, **F** and **M**, respectively, from [1].

Kripke models (see *e.g.* [6], [4]) for logics with the reflexive modality $\boxdot$ and labeled modalities are described here, including canonical models, the finite model property and the unification technique ([7], [1]). These yield the decidability of **BGrz**, **FGrz**, and **MGrz**.

The arithmetical completeness theorem for **BGrz** is obtained by an easy reduction to that for **B**: a straightforward correspondence between reflexive **Grz** models and irreflexive **GL** models is extended to the labeled modal language. This method, however, does not go through in the functional cases, i.e. for **FGrz** and **MGrz**. The arithmetical completeness of **FGrz** is proved by a Solovay-style construction ([4]) modified for reflexive models and the Artemov fixed point for the proof predicate from [1]. The arithmetical completeness theorem for **MGrz** is obtained by the interaction of the modified Solovay constuction with the realization of proof variables for the monotone system from [1].

2 Description of systems

Definition. Axioms of **BGrz**:

(A1)	tautologies in the language $\mathcal{L}$	
(A2)	$\boxdot(A \to B) \to (\boxdot A \to \boxdot B)$	distributivity
(A3)	$\boxdot A \to \boxdot \boxdot A$	transitivity
(A4)	$\boxdot A \to A$	reflexivity
(A5)	$\boxdot(\boxdot(A \to \boxdot A) \to A) \to A$	Grzegorczyk scheme
(A6)	$\Box_p A \to A$	quasi-reflexivity
(A7)	$\Box_p A \to \boxdot \Box_p A$	stability
(A8)	$\neg \Box_p A \to \boxdot(\neg \Box_p A)$	stability,

where p is a proof variable, A and B are formulas.

Rules of **BGrz**:

(R1) $$\frac{A \qquad A \to B}{B}$$

(R2) $$\frac{A}{\boxdot A}$$

The system **BGrz** is clearly a combination of the logic **Grz** in the language with labeled modalities (axiom schemes **A1-A5**) and principles of quasi-reflexivity and stability.

We shall use the unification technique (see *e.g.* [7]) adapted in [1] for the labeled modal language. By "$\equiv$" we mean syntactical coincidence. Let A, B, C, D be modal formulas. Then

$$\text{“}C = D \pmod{A = B}\text{”}$$

will stand for

$$\text{“for every substitution } \theta \ (A\theta \equiv B\theta \ \Rightarrow \ C\theta \equiv D\theta)\text{”.}$$

Note that the relation "$C = D \pmod{A = B}$" is decidable.

Definition. The system **FGrz** is **BGrz** together with the *functionality* axiom scheme

(A9) $\Box_p A \& \Box_p B \to (C \to D)$ if $C = D \pmod{A = B}$,

for any a proof variable p and modal formulas A, B, C, D.

Definition. The system **MGrz** is **FGrz** together with the *acyclicity* axiom scheme

(A10) $\neg[\Box_{p_1} A_2(p_2) \& \Box_{p_2} A_3(p_3) \& \ldots \& \Box_{p_n} A_1(p_1)]$

for each n, any proof variables $q_1, \ldots, q_n$, and any modal formulas $A_1(q_1), \ldots, A_n(q_n)$, where $A_i(q_i)$ is a modal formula in which the proof variable q_i occurs $(i = 1, \ldots n)$[1].

3 Kripke-style semantics and decidability

We assume the reader to be familiar with the basic notions of Kripke models (see *e.g.* [6], [4]). Henceforth by *frame* we shall understand a finite reflexive partial ordering $(K, \preceq)$ (any **Grz** frame ([4]) would be here). A *model* is a triple $(K, \preceq, \{W_x\}_{x \in K})$, where $(K, \preceq)$ is a frame and $\{W_x\}_{x \in K}$ is a family of finite sets of atomic and quasi-atomic formulas. We define

$$x \Vdash Q \qquad \text{as} \qquad Q \in W_x$$

for any $x \in K$ and any atomic and quasi-atomic Q, and then extend $\Vdash$ to all modal formulas by the following *forcing conditions*:

1. $\Vdash$ respects propositional connectives at each node,
2. for every modal formula A and each node x
   $x \Vdash \boxdot A$ iff $\forall y \succeq x \; y \Vdash A$

Definition. A model is **BGrz** *model* if it also satisfies

3. quasi-reflexivity: for every quasi-atomic formula $\Box_p A$ and each node x
   $x \Vdash \Box_p A \;\Rightarrow\; x \Vdash A$,
4. quasi-stability: for every quasi-atomic formula $\Box_p A$ and and all $x \preceq y$
   $x \Vdash \Box_p A$ iff $y \Vdash \Box_p A$,

As usual a modal formula A is *valid in a model* $\mathcal{K} = (K, \preceq, \{W_x\}_{x \in K})$ if A holds at every node.

Definition. A **BGrz** model $\mathcal{K} = (K, \preceq, \{W_x\}_{x \in K})$ is **FGrz** *model* if it satisfies an extra forcing condition:

5. functionality: $x \Vdash \neg(\Box_p B \& \Box_p C)$ if B, C are not unifiable, else
   $x \Vdash \Box_p B \& \Box_p C \Rightarrow x \Vdash D \to E$ for every D, E s.t. $D = E \pmod{B = C}$.

[1] In [1] **A10** is called a *monotonicity* axiom.

Definition. A **FGrz** model $(K, \preceq, \{W_x\}_{x \in K})$ is **MGrz** *model* if the following forcing condition holds at each node $x \in K$:

6. acyclicity:
$x \not\Vdash \Box_{q_1} A_2(q_2) \& \Box_{q_2} A_3(q_3) \& \ldots \& \Box_{q_n} A_1(q_1)$

for each n, any proof variables $q_1, \ldots, q_n$, and any modal formulas $A_1(q_1), \ldots, A_n(q_n)$, where $A_i(q_i)$ is a modal formula in which the proof variable q_i occurs $(i = 1, \ldots n)$.

3.1 Theorem. *For* **T**$\in \{\mathbf{B}, \mathbf{F}, \mathbf{M}\}$

$$\mathbf{TGrz} \vdash A \quad \textit{iff} \quad A \textit{ is valid in all } \mathbf{TGrz} \textit{ models.}$$

Proof. Correctness goes by a straightforward induction on the derivations in **TGrz**.

Completeness will be proved via canonical models and filtrations. The proof goes simultaneously for all three logics **BGrz**, **FGrz**, and **MGrz**.

A set α of modal formulas is **TGrz**-*consistent* if for no $A_1, \ldots, A_n \in \alpha$ $TG \vdash \neg(A_1 \& \ldots \& A_n)$; α is a maximal set if either $B \in \alpha$ or $\neg B \in \alpha$ for every modal formula B. It is clear that each consistent set is included into some maximal consistent set, and that every maximal consistent set contains all theorems of **TGrz** and is closed under *modus ponens*. Let Ω be the class of all maximal **TGrz**-consistent sets, and let the binary relation $\alpha < \beta$ on Ω be defined as

- for every modal formula F if $\boxdot F \in \alpha$, then $\boxdot F \in \beta$,
- there exists $F \in \beta$ such that $\boxdot F \in \beta$, but $\boxdot F \notin \alpha$,

and let $\leq$ on Ω be a reflexive ordering which corresponds to $<$. Finally, we define

$$\alpha \Vdash Q \quad \text{iff} \quad Q \in \alpha$$

for every atomic and quasi-atomic modal formula Q and extend the relation $\Vdash$ to all modal formulas according to the forcing conditions 1 and 2. The following standard semantical lemma holds:

3.2 Lemma. *For every $\alpha \in \Omega$ and every modal formula F*

$$\alpha \Vdash F \quad \textit{iff} \quad F \in \alpha.$$

Proof. Induction on formulas. The basis, i.e. the cases of atomic and quasi-atomic formulas, holds by the definition, propositional connectives steps are standard. Consider the case $F \equiv \boxdot B$.

Let $\boxdot B \in \alpha$ and $\alpha \leq \beta$, then $\boxdot B \in \beta$, thus $B \in \beta$, and, by the induction hypothesis, $\beta \Vdash B$. Therefore $\alpha \Vdash \boxdot B$.

Let $\boxdot B \notin \alpha$; we distinguish two subcases. If $B \notin \alpha$, then, by the induction hypothesis, $\alpha \nVdash B$, and we are done, because $\leq$ is reflexive. Suppose now that $B \in \alpha$. Take X to be the set of all formulas from α of the form $\boxdot C$, together with $\{\boxdot(B \to \boxdot B),\ \neg B\}$. Note that X is **TGrz** consistent; otherwise we would have

$$\mathbf{TGrz} \vdash \boxdot(B \to \boxdot B) \& \boxdot C_1 \& \ldots \& \boxdot C_n \to B$$

for some $\boxdot C_1, \ldots, \boxdot C_n \in \alpha$. Thus

$$\mathbf{TGrz} \vdash \boxdot C_1 \& \ldots \& \boxdot C_n \to (\boxdot(B \to \boxdot B) \to B),$$

by the straightforward use of **A2** and **R2**

$$\mathbf{TGrz} \vdash \boxdot(\boxdot C_1 \& \ldots \& \boxdot C_n) \to \boxdot(\boxdot(B \to \boxdot B) \to B).$$

Now we commute $\boxdot$ and &, apply axioms **A3** and **A5** together with some tautologies to derive

$$\mathbf{TGrz} \vdash \boxdot C_1 \& \ldots \& \boxdot C_n \to B,$$

similarly

$$\mathbf{TGrz} \vdash \boxdot C_1 \& \ldots \& \boxdot C_n \to \boxdot B,$$

which implies $\boxdot B \in \alpha$, i.e. a contradiction with our assumption that $\boxdot B \notin \alpha$. Let β be a maximal consistent extension of X. It is clear now that $\alpha < \beta$, as all the formulas of the form $\boxdot C$ from α are in X, thus in β, also $\boxdot(B \to \boxdot B) \in \beta$ and $\boxdot(B \to \boxdot B) \notin \alpha$. If $\boxdot(B \to \boxdot B) \in \alpha$, then also $B \to \boxdot B \in \alpha$, thus $\boxdot B \in \alpha$, contradiction. So, $\beta \in \Omega$, $B \notin \beta$, and, by the induction hypothesis, $\beta \nVdash B$, thus $\alpha \nVdash \boxdot B$.
■

Let $Sb\ A$ denote the set of all subformulas of A.

3.3 Corollary. *The forcing conditions quasi-reflexivity and quasi-stability hold at each $\alpha \in \Omega$.*

Proof. Indeed, each $\alpha \in \Omega$ contains axioms **A6, A7, A8** and, by lemma 3.2, they are all true at α.
■

Now for a given A such that **TGrz** $\nvdash A$, by a standard method (cf. [1]), we construct a finite tree **TGrz** countermodel $\mathcal{CM} = (K, \preceq, \Vdash)$. K will consist of finite sequences of the elements of Ω, $\prec$ will be the usual strict ordering by extension of finite sequences, and $\preceq$ will be a irreflexive ordering which corresponds to $\prec$.

- Stage 0. Pick (α_0) such that $\alpha_0 \in \Omega$ and $\alpha_0 \nVdash A$ as the root node of $\mathcal{CM}$. Such (α_0) is a maximal **TGrz** consistent set, extending $\{\neg A\}$, which is consistent, by the assumption **TGrz** $\nvdash A$.
- Stage $n+1$. For each sequence $(\alpha_0, \ldots, \alpha_n)$ already in K, look at $Y = \{\boxdot B \in Sb\ A \mid \alpha_n \nVdash \boxdot B \text{ and } \alpha_n \Vdash B\}$. If this set is empty, $(\alpha_0, \ldots, \alpha_n)$ is an end node. Otherwise, for each such $\boxdot B \in Y$, choose a node $\beta \in \Omega$ such that $\alpha_n < \beta$ and $\beta \nVdash B$. Such β exists: indeed, note that $\boxdot B \notin \alpha_n$ and $B \in \alpha_n$, take X and β as in the proof of lemma 3.2. Add $(\alpha_0, \ldots, \alpha_n, \beta)$ to K and define

$$(\alpha_0, \ldots, \alpha_n, \beta) \Vdash D \quad \text{iff} \quad \beta \Vdash D$$

  for any modal formula D.

It is obvious that this is a tree with the root (α_0). Finiteness follows from König's Lemma: the tree is finitely branching because branches are correlated with elements of the finite set $Sb\ A$ and there are no infinite paths because the succession from $(\alpha_0, \ldots, \alpha_n)$ to $(\alpha_0, \ldots, \alpha_n, \alpha_{n+1})$ results in at least one additional formula $\boxdot(B \to \boxdot B)$ for some $\boxdot B \in Sb\ A$ being forced by α_{n+1}. Forcing conditions 1 and 3, hold because they regulate a pointwise behaviour of the forcing relation, which, by definition, is similar at $(\alpha_0, \ldots, \alpha_n)$ and α_n. Condition 2 is fulfilled by the special choice of nodes of Ω extending sequences in K. The quasi-stability condition 4 holds because it takes place in the canonical model.

Now, $\alpha_0 \nVdash A$, and thus $(\alpha_0) \nVdash A$.

As the final stage of the proof we have to define the finite evaluation of atomic and quasi-atomic formulas at the nodes of the frame $(K, \preceq)$. This is trivial in the case of **BGrz**. For each $x \in K$ and every atomic or quasi-atomic formula Q put

$$Q \in W_x \quad \text{iff} \quad (Q \in Sb\ A \text{ and } x \Vdash Q).$$

Then forcing conditions 3 and 4 obviously hold for the model $(K, \preceq, \{W_x\}_{x\in K})$ since they are both closed under subformulas; it is also clear that the truth value of each subformula of A is preserved at each node. Thus
$\mathcal{K} = (K, \preceq, \{W_x\}_{x\in K})$ is a **BGrz** countermodel for A.

The cases **FGrz** and **MGrz** can now be treated exactly like the cases of **F** and **M** in [1].
■

In the cases **FGrz** and **MGrz** by the use of the technique of Kripke models with unification similarly to [1] we can get the following lemma, which will be used later.

3.4 Lemma. *For* **T**$=$ **F**, **M** *if* **TGrz** $\nvdash A$*, then there exist a substitution* τ *and a* **FGrz** *countermodel* $\mathcal{K}$ *for* $A\tau$ *such that*

1. $(Sb\ A)\tau = Sb\ A\tau$,
2. $Sb\ A\tau$ *is functional in* $\mathcal{K}$*, i.e. if* $\Box_p B \& \Box_p C$ *holds in* $\mathcal{K}$*, then* $B \equiv C$.

3.5 Corollary. **BGrz**, **FGrz**, *and* **MGrz** *are decidable.*

Proof. Now together with the finite model property for **TGrz** (**T=B,F,M**) we have delivered an algorithm, which gives a recursive enumeration of all finite models and thus all non-theorems of **TGrz**; by the standard Post argument, **TGrz** is decidable. Our proof of the Kripke style completeness for **TGrz** provides effectively a finite set of "possible countermodels" to A. This gives an upper bound on the computational time for a decision procedure of order 2^{cn}, with n the length of the formula, and c fixed.
■

4 Arithmetical completeness

In this section we will prove the arithmetical completeness theorems for **BGrz**, **FGrz**, and **MGrz**. In all three cases we deliver essentially different proofs. For **BGrz** the proof will be an easy reduction to the arithmetical completeness theorem for the logic of proofs **B** with the modality $\Box$ from [1]. In case of **FGrz** we will give a direct arithmetical completeness proof, which will use the Artemov fixed point construction from [1] together with a special modification of the Solovay argument for **Grz**. The arithmetical completeness of **MGrz** is proved by the same version of the Solovay argument combined with a construction of the realization for proof variables similar to that from the proof of the arithmetical completeness of the logic **M** in [1].

4.1 Theorem. $\mathbf{BGrz} \vdash A$ *iff* $\mathbf{PA} \vdash A^*$ *for every realization* $*$

Proof. The part "*only if*" is a standard correctness statement, which is proved by a straightforward induction on A; every case is in fact proved either in [4] (for **Grz** axioms) or in [1] (for axioms **A6, A7, A8**).

The part "*if*". There is a clear correspondence between Kripke models for provability logics with the godelian provability operator in the language with the modality $\Box$ and Kripke models for logics with the strong provability operator in the language with the modality $\boxdot$ (cf.[4]). We extend this correspondence to the language with labeled modalities.

Let $\widehat{A}$ denote the result of a substitution of each subformula $\boxdot B$ in A by $B\&\Box B$, $\widehat{X} = \{\widehat{x} \mid x \in X\}$. As usual let $\prec$ be the irreflexive ordering corresponding to $\preceq$.

Definition. If $\mathcal{K} = (K, \preceq, \{W_x\}_{x\in K})$ is a **BGrz** model, then by $\widehat{\mathcal{K}}$ we mean the triple $(K, \prec, \{V_x\}_{x\in K})$, where

$$Q \in V_x \quad \text{iff} \quad \text{there exists } R \text{ such that } Q \equiv \widehat{R} \text{ and } R \in W_x;$$

we also define the relation $\Vdash'$ as

$$x \Vdash' \boxdot A \quad \text{iff} \quad Q \in V_x,$$

and then extend $\Vdash'$ to all modal formulas in the language with the modality $\Box$ respecting propositional connectives at each node and according to

$$x \Vdash' \Box A \quad \text{iff} \quad \forall y \succ x \; y \Vdash' A.$$

4.2 Lemma. *For any* $x \in K$ *and modal formula* A

$$x \Vdash A \quad \textit{iff} \quad x \Vdash' \widehat{A}.$$

Proof. Induction on A. The basis, i.e. the case of atomic and quasi-atomic A, is covered by the definition of $\Vdash'$. The induction step in the case of "$\to$" is trivial. Let $A \equiv \boxdot B$, then $\widehat{A} \equiv \widehat{B} \& \Box \widehat{B}$. If $x \Vdash A$, then $\forall y \succeq x \; y \Vdash B$, pick an arbitrary y such that $y \succ x$ or $y = x$. In both cases $y \Vdash' \widehat{B}$, by the induction hypothesis, and thus $x \Vdash' \widehat{B} \& \Box \widehat{B}$. If $x \nVdash A$, then $y \nVdash B$ at some $y \succeq x$. By the induction hypothesis, $y \nVdash' \widehat{B}$. Hence $x \nVdash' \widehat{B} \& \Box \widehat{B}$.
■

4.3 Lemma. *The following irreflexive variant of quasi-stability: for any $x \prec y$*

$$x \Vdash' Q \quad \textit{iff} \quad y \Vdash' Q,$$

and quasi-reflexivity hold in $\widehat{\mathcal{K}}$.

Proof. Indeed, if $x \Vdash' Q$, Q quasi-atomic, and $x \prec y$, then $Q \equiv \widehat{R}$ and $x \Vdash R$ for some R. It is clear that R is quasi-atomic, and thus, by the quasi-stability of $\mathcal{K}$, $y \Vdash R$, and thus $y \Vdash' Q$. The same argument demonstrates that $y \Vdash' Q$ implies $x \Vdash' Q$.

Now let $x \Vdash' \Box_p A$. By the definition of $\Vdash'$, there exists R such that $\widehat{R} \equiv \Box_p A$ and $x \Vdash R$. This R is necessarily quasi-atomic, i.e. $R \equiv \Box_r B$ for some B such that $\widehat{B} \equiv A$. By the quasi-reflexivity property of $\mathcal{K}$, $x \Vdash B$, thus, by lemma 3.3, $x \Vdash' A$.
■

4.4 Corollary. *$\widehat{\mathcal{K}}$ is a model for the logic $\mathbf{B}^-$ with the modality $\Box$ from [1].*

Now we use a multiplication-of-a-reflexive-node trick, and then apply the Kripke and arithmetical completeness theorems for the logic $\mathbf{B}$ from [1].

Let $\mathbf{BGrz} \nvdash A$, then, by theorem 3.1, there exists a $\mathbf{BGrz}$ model $\mathcal{K}$ such that A is false at the root node r of $\mathcal{K}$. We define a new model $\mathcal{K}_N$ for a sufficiently large natural number N by adding new nodes $0, 1, \ldots, N$ to K, putting

$$0 \prec 1 \prec \ldots \prec N \prec r,$$

and defining

$$i \Vdash Q \quad \text{iff} \quad r \Vdash Q$$

for all $i = 0, 1, \ldots, N$ and for every atomic and quasi-atomic Q. It is evident that for every modal formula H

$$i \Vdash H \quad \text{iff} \quad r \Vdash H$$

(an easy induction on H). By lemma 4.2, $\widehat{A}$ is false at the root node 0 of $\widehat{\mathcal{K}_N}$, moreover, by corollary 4.4, $\widehat{\mathcal{K}_N}$ is a $\mathbf{B}^-$ model. In order to be able to apply the arithmetical completeness theorem for $\mathbf{B}$, we have to construct an $\widehat{A}$-sound countermodel for $\widehat{A}$, i.e. a countermodel for $\widehat{A}$, where $\Box B \to B$ holds at the root node for all subforlulas $\Box B$ of $\widehat{A}$. Each formula $\Box B \to B$ can be false at one node of the chain $0 \prec 1 \prec \ldots \prec N$ at the most. So, if N is sufficiently large, then, by the pigeonhole principle, there is $n \in \{0, 1, \ldots, N\}$ such that $\Box B \to B$ holds at n for every subformula $\Box B$ the formula $\widehat{A}$. The restriction of the model $\widehat{\mathcal{K}_N}$ to the set of its nodes

$$\{x \mid n = x \quad \text{and} \quad n \prec x\}$$

gives $\widehat{A}$-sound countermodel for $\widehat{A}$.

By the arithmetical completeness theorem for $\mathbf{B}$ ([1]), there exists a realization $*$ such that $\mathbf{PA} \nvdash \widehat{A}^*$. Since $(\widehat{A})^* \equiv A^*$, we have got $\mathbf{PA} \nvdash A^*$.
■

4.5 Theorem. $\mathbf{FGrz} \vdash A$ *iff* $\mathbf{PA} \vdash A^*$ *for every functional realization* $*$

Proof. We proceed mainly along the lines of the proof of the arithmetical completeness of $\mathbf{F}$ from [1]. Both fixed points from that proof – the Solovay one for the realization of the propositional variables, and the Artemov fixed point for proof predicate – are taken directly as they are. The only difference appears in the induction steps corresponding to the modality $\boxdot$ in the proof of lemma 4.8 below.

Correctness goes as usual by induction on a proof of A in $\mathbf{FGrz}$. After Theorem 4.1, it only remains to verify the correctness of the functionality axiom **A9**, which is in fact done in [1].

Completeness. By 3.4, take a substitution τ and a $\mathbf{FGrz}$ countermodel $\mathcal{K}$ for $A\tau$ such that $(Sb\ A)\tau$ is functional in $\mathcal{K}$. We assume that $K = \{1, \ldots, n\}$ and 1 is the root node. We define a new model $\mathcal{K}'$ by adding a node 0 to K, putting $0 \prec i$ $(1 \le i \le n)$, and $W_0 = W_1$. It is clear that $\mathcal{K}'$ is again a $\mathbf{FGrz}$ model, for all B

$$0 \Vdash B \quad \text{iff} \quad 1 \Vdash B,$$

and thus $(Sb\ A)\tau$ is functional in $\mathcal{K}'$ too.

We shall denote the usual Gödel proof predicate as $\mathit{Proof}(x, y)$ and the usual provability predicate as $\mathit{Provable}(y)$, thus $\mathit{Provable}(y) \equiv \exists x \mathit{Proof}(x, y)$.

Now for the model $\mathcal{K}'$ and for the usual Gödel proof predicate $\mathit{Proof}(x, y)$ we define a Solovay function $h(t)$:

- $h(0) = 0$;
- if $h(m) = i$ and $\mathit{Proof}(m, \ulcorner l \neq j \urcorner)$ for some $j \succ i$, set $h(m+1) = j$; otherwise put $h(m+1) = h(m)$,

where "$l = j$" is a natural arithmetical formula for "j is a limit of $h(t)$".

We shall need the following Solovay lemma ([4]):

4.6 Lemma.

1. $\mathbf{PA} \vdash$ "$0 \leq l \leq n$"*;*
2. "$l = 0$" *is true, but each of the theories* $\mathbf{PA}$ + "$l = i$" *is consistent for* $i = 0, 1 \ldots, n$*;*
3. $\mathbf{PA}$ + "$l = i$" $\vdash \mathit{Provable}(\ulcorner$"$l \neq i$"$\urcorner)$, $\ i = 1, 2, \ldots, n$*;*
4. $\mathbf{PA}$ + "$l = i$" $\vdash \neg\mathit{Provable}(\ulcorner$"$l \neq j$"$\urcorner)$, $\ i = 0, 1, \ldots, n,\ i \prec j$*;*
5. $\mathbf{PA}$ + "$l = i$" $\vdash \mathit{Provable}(\ulcorner$"$l \neq j$"$\urcorner)$, $\ i = 1, 2, \ldots, n,\ i \not\prec j$*.*

4.7 Corollary.

$$\mathbf{PA} \vdash \text{“}l = i\text{”} \to \mathit{Provable}(\ulcorner \bigvee_{i \prec j} \text{“}l = j\text{”} \urcorner), \quad i = 1, 2, \ldots, n.$$

Let $T = \{\Box_{p_{m_0}} A_0, \Box_{p_{m_1}} A_1, \ldots, \Box_{p_{m_q}} A_q\}$ be the list of all quasi-atomic formulas from $Sb\ A\tau$ which are valid in $\mathcal{K}'$.

In order to define an arithmetical realization $\sharp = (\mathit{Prf}, \alpha)$ we first introduce α for proof and propositional variables occurring in $Sb\ A\tau$:

$$\alpha(S_i) := [\bigvee_{j \text{ forces } S_i} \text{“}l = j\text{”}] \wedge i = i,$$

$$\alpha(p_j) := 2j.$$

The following fixed point equation (FPE) is provable in $\mathbf{PA}$:

$$\begin{array}{llllll}
\mathit{Prf}(u,v) & \longleftrightarrow & \forall r \leq u & & & \\
& \Big[\ u = 2r+1 & \to & \mathit{Proof}(r,v) & \& & \\
& u = 2r & \to & [\quad r = m_0 & \to\quad v = \ulcorner A_0{}^{\sharp} \urcorner & \& \\
& & & & \vdots & \\
& & & r = m_q & \to\quad v = \ulcorner A_q{}^{\sharp} \urcorner & \& \\
& & & \bigwedge_{i=0}^{i=q} r \neq m_i & \to\quad v = \ulcorner \forall x_0 (x_0 = x_0) \urcorner &]\Big],
\end{array}$$

where ${A_i}^\sharp$ is the arithmetical translation of A_i under the realization $\sharp = (Prf, \alpha)$.

It is easy to see now that $Prf(u,v)$ is primitive recursive and functional, and that $\sharp$ is injective. We will demonstrate that $Prf(u,v)$ is a standard proof predicate for **PA** after lemma 4.8 below, but we can state now that

$$\mathbf{PA} \vdash \forall y[Pr(y) \;\leftrightarrow\; Provable(y) \vee \bigvee_{i=0}^{i=q} y = \ulcorner {A_i}^\sharp \urcorner].$$

4.8 Lemma. *For any modal formula $B \in Sb\ A\tau$ and all $k = 0, 1, \ldots, n$*

$$k \Vdash B \quad \Rightarrow \quad \mathbf{PA} \vdash \text{“}l = k\text{”} \to B^\sharp,$$

$$k \nVdash B \quad \Rightarrow \quad \mathbf{PA} \vdash \text{“}l = k\text{”} \to \neg B^\sharp.$$

Proof. By induction on formulas. The basis in the case when B is an atomic formila is trivial. Let B be a quasi-atomic formula. If $k \Vdash B$, then for some $B \equiv \Box_{p_{m_i}} A_i \in T$ we have $B^\sharp \equiv Prf(2m_i, \ulcorner {A_i}^\sharp \urcorner)$, which is a true primitive recursive formula according to FPE. Then $\mathbf{PA} \vdash B^\sharp$ and $\mathbf{PA} \vdash$ “$l = k$” $\to B^\sharp$. Let $k \nVdash B$ and $B \equiv \Box_{p_i} H$ for some H. Consider two cases:

$i \in \{m_0, \ldots, m_q\}$. Then $H \not\equiv A_{m_i}$, $i = 0, \ldots, q$, and, by the injectivity of $\sharp$, $H^\sharp \not\equiv {A_{m_i}}^\sharp$, thus $\ulcorner H^\sharp \urcorner \neq \ulcorner {A_{m_i}}^\sharp \urcorner$ and $Prf(2i, \ulcorner H^\sharp \urcorner)$ is false. But $B^\sharp \equiv Prf(2i, \ulcorner H^\sharp \urcorner)$, thus $\mathbf{PA} \vdash \neg B^\sharp$ and $\mathbf{PA} \vdash$ “$l = k$” $\to \neg B^\sharp$.

$i \notin \{m_0, \ldots, m_q\}$. Then $B^\sharp \equiv Prf(2i, \ulcorner H^\sharp \urcorner)$ and thus $B^\sharp$ is false according to FPE. Indeed, $Prf(u,v)$ is functional, $Prf(2i, \ulcorner \forall x_0(x_0 = x_0) \urcorner)$ is true and $\forall x_0(x_0 = x_0) \not\equiv H^\sharp$ for any modal formula H.

Now we proceed with inductions on formulas, first for $k > 0$, and then for $k = 0$.

Let $1 \le k \le n$. The induction step in case of “$\to$” is straightforward ([4]).
Let us make the induction step in case $G \equiv \Box H$
If $k \Vdash \Box H$, then for all $j \succeq k$ $\mathbf{PA} \vdash$ “$l = j$” $\to H^\sharp$, thus

$$\mathbf{PA} \vdash \bigvee_{k \prec j} \text{“}l = k\text{”} \to H^\sharp$$

and

$$\mathbf{PA} \vdash Provable(\bigvee_{k \prec j} \text{“}l = k\text{”}) \to Provable(H^\sharp).$$

By 4.7,
$$\mathbf{PA} \vdash \text{“}l = k\text{”} \to Provable(\ulcorner \bigvee_{k \prec j} \text{“}l = j\text{”} \urcorner),$$
thus
$$\mathbf{PA} \vdash \text{“}l = k\text{”} \to Provable(\ulcorner H^{\sharp} \urcorner),$$
and since
$$\mathbf{PA} \vdash \forall y (Provable(y) \to Pr(y)),$$
we have
$$\mathbf{PA} \vdash \text{“}l = k\text{”} \to Pr(\ulcorner H^{\sharp} \urcorner),$$
and finally $\mathbf{PA} \vdash \text{“}l = k\text{”} \to G^{\sharp}$.

If $k \not\Vdash \Box H$, then for some $j \succeq k$ $j \not\Vdash H$, by the induction hypothesis,
$$\mathbf{PA} \vdash \text{“}l = j\text{”} \to \neg H^{\sharp}.$$
We distinguish two cases. If $j = k$, then
$$\mathbf{PA} \vdash \text{“}l = j\text{”} \to \neg (H^{\sharp} \& Pr(\ulcorner H^{\sharp} \urcorner)),$$
i.e.
$$\mathbf{PA} \vdash \text{“}l = j\text{”} \to \neg (\Box H)^{\sharp}).$$

If $j \succ k$, then $\mathbf{PA} \vdash H^{\sharp} \to \text{“}l \neq j\text{”}$,
$$\mathbf{PA} \vdash Provable(\ulcorner H^{\sharp} \urcorner) \to Provable(\ulcorner \text{“}l \neq j\text{”} \urcorner),$$

$$\mathbf{PA} \vdash \neg Provable(\ulcorner \text{“}l \neq j\text{”} \urcorner) \to \neg Provable(\ulcorner H^{\sharp} \urcorner),$$
but, by 4.6(4),
$$\mathbf{PA} \vdash \text{“}l = k\text{”} \to \neg Provable(\ulcorner \text{“}l \neq j\text{”} \urcorner),$$
whence $\mathbf{PA} \vdash \text{“}l = k\text{”} \to \neg Provable(\ulcorner H^{\sharp} \urcorner)$.

However, $Pr(y)$ may differ from $Provable(y)$ only on the Gödel numbers of the quasi-atomic formulas from T; every formula from T is valid at each node of the model $\mathcal{K}'$, thus $H \notin T$, and
$$\mathbf{PA} \vdash Pr(\ulcorner H^{\sharp} \urcorner) \to Provable(\ulcorner H^{\sharp} \urcorner).$$

Finally, we have got

$$\mathbf{PA} \vdash \text{“}l = k\text{”} \to \neg Pr(\ulcorner H^\sharp \urcorner),$$

and again

$$\mathbf{PA} \vdash \text{“}l = j\text{”} \to \neg(H^\sharp \& Pr(H^\sharp)),$$

i.e.

$$\mathbf{PA} \vdash \text{“}l = j\text{”} \to \neg(\boxdot H)^\sharp.$$

Now let $k = 0$. Again, the basis of the induction on formulas is done, and the case of “$\to$” is trivial. Let $G = \boxdot H$.

If $0 \Vdash \boxdot H$, then for all $j = 1, \ldots, n \quad j \Vdash H$. As above, for all $j = 1, \ldots, n \quad \mathbf{PA} \vdash \text{“}l = j\text{”} \to H^\sharp$, thus

$$\mathbf{PA} \vdash \bigvee_{1 \le j \le n} \text{“}l = j\text{”} \to H^\sharp.$$

Also, $1 \Vdash H \ \Rightarrow \ 0 \Vdash H$, and, by the induction hypothesis, $\mathbf{PA} \vdash \text{“}l = 0\text{”} \to H^\sharp$. By 4.6(1), $\mathbf{PA} \vdash \text{“}0 \le j \le n\text{”}$ and thus

$$\mathbf{PA} \vdash H^\sharp, \ \mathbf{PA} \vdash \mathit{Provable}(\ulcorner H^\sharp \urcorner),$$

and $\mathbf{PA} \vdash \text{“}l = 0\text{”} \to G^\sharp$. If $0 \nVdash \boxdot H$, then for some $j = 0, \ldots, n \ \ j \nVdash H$. Again consider two cases.

If $j = 0$, then, by the induction hypothesis, $\mathbf{PA} \vdash \text{“}l = 0\text{”} \to \neg H^\sharp$ and trivially $\mathbf{PA} \vdash \text{“}l = 0\text{”} \to \neg(\boxdot H)^\sharp$.

If $j > 0$, then, as above,

$$\mathbf{PA} \vdash \text{“}l = j\text{”} \to \neg H^\sharp,$$

$$\mathbf{PA} \vdash H^\sharp \to \text{“}l \neq j\text{”},$$

$$\mathbf{PA} \vdash \mathit{Provable}(\ulcorner H^\sharp \urcorner) \to \mathit{Provable}(\ulcorner \text{“}l \neq j\text{”} \urcorner),$$

$$\mathbf{PA} \vdash \neg \mathit{Provable}(\ulcorner \text{“}l \neq j\text{”} \urcorner) \to \neg \mathit{Provable}(\ulcorner H^\sharp \urcorner).$$

By 4.6(4),

$$\mathbf{PA} \vdash \text{“}l = 0\text{”} \to \neg \mathit{Provable}(\ulcorner \text{“}l \neq j\text{”} \urcorner),$$

thus

$$\mathbf{PA} \vdash \text{“}l = 0\text{”} \to \neg \mathit{Provable}(\ulcorner H^\sharp \urcorner).$$

The same argument as above shows that

$$\mathbf{PA} \vdash \ Pr(\ulcorner H^\sharp \urcorner) \to Provable(\ulcorner H^\sharp \urcorner),$$

and we again have $\mathbf{PA} \vdash$ "$l = 0$" $\to \neg G^\sharp$.
■

4.9 Corollary. *Prf is a standard proof predicate.*

Proof. We can prove even more: $\mathbf{PA} \vdash \forall y(Pr(y) \ \leftrightarrow \ Provable(y))$.

Indeed, $\mathbf{PA} \vdash \forall y(Pr(y) \ \to \ Provable(y))$ because $\mathbf{PA}$ proves FPE, and $Pr(y)$ may differ from $Provable(y)$ only when $y = \ulcorner G^\sharp \urcorner$ for some $G \in T$. Every such G is a quasi-atomic formula, which is true at all nodes of the model $\mathcal{K}'$. By 4.7,

$$\mathbf{PA} \vdash \ [\bigvee_{k=0}^{k=n} \text{“}l = k\text{”}] \to G^\sharp,$$

but, by lemma 4.6(1),

$$\mathbf{PA} \vdash \ \bigvee_{k=0}^{k=n} \text{“}l = k\text{”},$$

whence $\mathbf{PA} \vdash \ G^\sharp$, and $\mathbf{PA} \vdash \ Provable(\ulcorner G^\sharp \urcorner)$. Therefore

$$\mathbf{PA} \vdash \ y = \ulcorner G^\sharp \urcorner \to Provable(y),$$

which implies

$$\mathbf{PA} \vdash \forall y(Pr(y) \ \leftrightarrow \ Provable(y)).$$

■

We will complete now the proof of the Theorem 4.5: as $0 \not\Vdash A\tau$ we have

$$\mathbf{PA} \vdash \text{“}l = 0\text{”} \to \neg(A\tau)^\sharp,$$

and

$$\mathbf{PA} \vdash (A\tau)^\sharp \to \text{“}l \neq 0\text{”}.$$

Suppose $\mathbf{PA} \vdash (A\tau)^\sharp$, then $\mathbf{PA} \vdash$ "$l \neq 0$", which contradicts lemma 4.6(2); so $\mathbf{PA} \nvdash (A\tau)^\sharp$. Define $* := \tau \circ \sharp$ and get $\mathbf{PA} \nvdash A^*$.
■

Finally, here is the arithmetical completeness theorem for **MGrz** with respect to the usual Gödel proof predicate.

4.10 Theorem. $\mathbf{MGrz} \vdash A$ *iff for every realization* $*$ *based on the usual Gödel proof predicate* $\mathbf{PA} \vdash A^*$.

The proof of this theorem is an easy variation on that of the corresponding theorem for **M** from [1], where the induction steps for the $\boxdot$ case should be treated as in lemma 4.5 above.

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