Decidability and finite model property of substructural logics

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1 Introduction

In this paper, we will give a short survey of results on decision problems and the finite model property of substructural logics. The paper is far from a complete list of these results, since a lot of results have been obtained already in some restricted classes of substructural logics, like relevant logics, and therefore it is impossible to cover all of them. (As for surveys of decision problems and the finite model property of relevant logics, see e.g. [1, 2, 7]. Also, see [16] for a survey of decision problems of logics related to linear logic.) Our aim of the present paper is to try to compare results from different classes of substructural logics with each other and discuss them as a whole, in order to get a perspective of them.[1]

2 Decidability and undecidability

As mentioned in Section 3.11 of [7], there exist two standard methods of proving the decidability of a given logic **L**. The first one is to formalize **L** in a Gentzen-type sequent calculus and then to prove the cut elimination theorem. The second is to prove the finite model property of **L**, i.e. to show that for any formula A which is not provable in **L** there exists a finite model for **L** in which A is not true. Then, by the well-known

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argument by Harrop, the decidability of **L** follows from the finite model property as long as **L** is finitely axiomatizable.

Apparently, the first method is much more constructive than the second. That is, we can usually get a concrete decision algorithm from a proof of the decidability by the first method, although it may not be feasible. On the other hand, we cannot get any practical algorithm from proofs by the second method.

But this remark may be misleading in the following point. ¿From practical purposes, it will be quite necessary to have a feasible decision algorithm for a given logic, as long as it is useful in the application. But, this will not be a sole goal of the study on the decidability. From theoretical interests, one may be concerned with the decidability as one of important logical properties. In such a case, to study relations of the decidability (or the finite model property) to other logical properties will be a central question. In fact, recently many interesting and strong results have been obtained on the decidability and the finite model property of modal, tense and intermediate logics in this direction (see e.g. [9, 31, 32, 30]) . (Here, we should mention the interesting decidability results given in [31]. They are obtained by using the Harrop style argument, although some of logics considered there don't have the finite model property.) This suggests us that the study of substructural logics in the similar direction might be promising also.

In the following, we will discuss the decidability and the undecidability of some basic substructural logics, which can be formalized by cut-free sequent calculi. In particular, we will pay our attention mainly to roles of several rules of inference, like the weakening rule, the contraction rule and the distributive law in decision problems. In the following, we assume the familiarity with the terminologies in [24]. Thus, **FL** denotes the logic obtained from the intuitionistic logic by deleting all the structural rules. Also, $\mathbf{FL_e}$, $\mathbf{FL_w}$ and $\mathbf{FL_{ew}}$ denote the logics obtained from **FL** by adding the exchange, the weakening and both of them, respectively, and **GL** denotes the classical linear predicate logic (without exponentials) etc..

In his proof of the decidability of the classical propositional logic, Gentzen proved the cut elimination theorem for **LK** first and then showed that it is enough to consider only *normal sequents*, i.e. sequents in which no formula appear more than three times either in their antecedents or in the succedents, when we decide the provability of a given sequent. Combining this remark with the subformula property of cut-free proofs, a decision algorithm for the classical propositional logic can be obtained. The similar argument works for the intuitionistic propositional logic and for some modal logics, once they can be formalized in cut-free systems. (*A digression* : The essential point in the above decidability proof is *not* the cut elimination, but the subformula property.

M. Takano showed the decidability of several modal logics, using sequent calculi for them which are not cut-free, but in which any provable formula has a proof with the subformula property. More precisely, any provable formula A has a proof in which every cut formula of any application of the cut rule is a subformula of A. See [26].)

We note here that any sequent can be reduced to a normal sequent, only when the logic under consideration has both the weakening and the contraction. Thus, it implies that the above method cannot be applied directly to logics which lack these structural rules.

So, in the following we will show what will happen if a logic lacks at least one of these rules. We will discuss also the role of the distributive law in the decision problem. The distributive law can be derived usually by the help of both weakening and contraction rules, and therefore it doesn't hold always in substructural logics. As the study of relevant logics shows, the existence of the distributive law in weak logics sometime causes a great complication.

2.1 Logics without contraction

Suppose that a given propositional logic **L** is formalized by a sequent calculus for which the cut elimination theorem holds. If the calculus doesn't have the contraction rules, then we can show the decidability of **L** without difficulties. To show this, take the propositional logic **FL** as an example. Let S be an arbitrary sequent. By the subformula property, it is enough to search for a cut-free proof of S consisting only of subformulas of formulas in S. To do so, we will construct *proof-search trees* of S. Since we don't have the contraction rule, (either of) the upper sequent of any rule of inference is simpler than the lower sequent. Therefore, the construction of each proof-search tree will terminate in finitely many steps. Then, it suffices to check whether there exists a proof of S among them or not. This gives us a decision algorithm. (As for the cut elimination theorem for several substructural logics, see e.g. [25, 23].)

This method works also for predicate logics. Rules which we must check are ($\exists$ right) and ($\forall$ left). For example, in the case of ($\forall$ left), $\forall x A(x), \Gamma \longrightarrow \Delta$ is inferred from $A(t), \Gamma \longrightarrow \Delta$. When the language doesn't contain any function symbol, any term of this language is either a constant or a variable and therefore (essentially) only a finite number of terms can be a possible candidate of this term t. Moreover, the latter sequent is simpler than the former one. In this way, we can show the decidability of predicate logics without contraction. As for the details, see [14]. Compare this with the case when we have both weakening and contraction. In this case, $\forall x A(x), \Gamma \longrightarrow \Delta$

is inferred from $A(t), \forall x A(x), \Gamma \longrightarrow \Delta$ and *vice versa*. Thus, we cannot make sure that the construction of proof-search trees will eventually terminates.

When the language contains function symbols, it is not so obvious whether we can restrict possible candidates of the term t to a finite number of terms or not. This difficulty will be overcome by combining the above method with the unification algorithm. As for the details, see [19, 13]. As a conclusion, we have the following.

Theorem 1 *Predicate logics* $\mathbf{FL}, \mathbf{FL_e}, \mathbf{FL_w}, \mathbf{FL_{ew}}, \mathbf{GL}$ *and* $\mathbf{GL_w}$ *are decidable (even if the language has function symbols).*

As for the complexity of the decision problem of **GL**, the following results are proved in [17, 18].

Theorem 2 *The decision problem of* **GL** *without function symbols is PSPACE-complete and of* **GL** *with function symbols is NEXPTIME-hard.*

It will be quite plausible that the similar results will hold also for other logics mentioned in Theorem 1. The above argument will work essentially also when a logic has the distributive law. We will mention the following result by Giambrone [10]. Here, $\mathbf{TW}_+$ and $\mathbf{RW}_+$ denote the positive fragments of the *contraction-free* subsystems of relevant logics **T** and **R**, respectively. Compare this with Theorem 6.

Theorem 3 *Relevant propositional logics* $\mathbf{TW}_+$ *and* $\mathbf{RW}_+$ *are decidable.*

For more information, see [2].

2.2 Logics with contraction

When a logic has the contraction rule but doesn't have the weakening rule, it will be hard to prove the decidability and in fact, sometimes it is shown to be undecidable. Suppose that a logic **L** is formalized by a cut-free sequent system. To show the decidability, let us try to apply the method of proof-search trees to it. But, then, we will face a certain difficulty in treating the contraction rule. To see this, let us suppose that a given sequent is obtained by applying the contraction rule as the last inference. Then the upper sequent is not simpler than the lower one. Moreover it seems to be unable to estimate in advance how many times the contraction rule will be applied in the proof. Thus, there would be no way of telling when the construction of proof-search trees eventually terminate.

To overcome this difficulty, Kripke in [15] introduced the notion of *cognate sequents* and showed that it is enough to search for only proof-search trees without *redundant* *branches* and under this restriction the above algorithm will eventually terminate. (See e.g. [7] for the details.) The key of the above proof is a result, called Kripke's Lemma. We will give here its abstract form. Let k be a positive integer and N_k be the set of all k-dimensional vectors, each of whose components is a non-negative integer. Define a partial order $\leq$ on N_k by

$$\langle m_1, \ldots, m_k \rangle \leq \langle n_1, \ldots, n_k \rangle \text{ if and only if } m_i \leq n_i \text{ for each } i.$$

Then, Kripke's Lemma says that the partially ordered set $\langle N_k, \leq \rangle$ contains no infinite antichain for each k.

By extending this argument, the following are proved in [20] and [13], where **LR** is the logic which is equal to the logic **R** without the distributive law.

Theorem 4 *Propositional logics* **LR** *and* $\mathbf{FL_{ec}}$ *are decidable.*

As a corollary, we can show the decidability of the implication-conjunction-fusion fragment of **R** (cf. Theorem 6). The above proof relies on Kripke's Lemma and König's Lemma and both of them are highly non-constructive. So, it may be supposed that the decision problems for these logics will have high complexity. In fact, Urquhart proved in [29] the following.

Theorem 5 *The decision problem of the implication-conjunction-fusion fragment of* **R** *is EXPSPACE-hard and the decision procedure obtained by Kripke's method is primitive recursive in Ackermann's function.*

Similarly, we can show that the decision procedure of $\mathbf{FL_{ec}}$ using Kripke's method is also primitive recursive in Ackermann's function. It might be an interesting problem to find the exact upper bound of the above decision problem, or to show that the above result is the best possible.

If a given logic lacks moreover the exchange rule, we don't know whether an substitute for Kripke's method holds on earth or not. Thus, the decision problems of the implicational fragment of $\mathbf{FL_c}$ and the implicational fragment of the relevant logic **T**, which is known also as $\mathbf{BB'IW}$, seem to remain open. For the former case, we don't know even its cut-free system. Here we remark that the cut-elimination theorem for $\mathbf{FL_c}$ was proved in some literatures. But the proofs in them contain some gaps. In fact, there is a counterexample. For instance, the following formula is provable in it, but it cannot be proved without cut (see [3]).

(1) $p \supset ((q \supset r) \supset s), (q \supset r) \supset p, t \supset r, q \supset t \longrightarrow s.$

To see this, consider the following sequent, which is provable without cut:

(2) $p \supset ((q \supset r) \supset s), (q \supset r) \supset p, q \supset r, q \supset r \longrightarrow s.$

By applying the contraction rule, we have the following.

(3) $p \supset ((q \supset r) \supset s), (q \supset r) \supset p, q \supset r \longrightarrow s.$

Then, taking a cut for $t \supset r, q \supset t \longrightarrow q \supset r$ with (3), we have (1). But, this cut is not interchangeable with the contraction rule just in the above. In fact, (1) cannot be obtained from the following sequent by applying the contraction, when we have no exchange rule.

(4) $p \supset ((q \supset r) \supset s), (q \supset r) \supset p, t \supset r, q \supset t, t \supset r, q \supset t \longrightarrow s$.

The above sequent will give also a counterexample for the cut elimination of $\mathbf{FL_{cw}}$. Notice here that if we add either the additive or the multiplicative conjunction to the implicational fragment of $\mathbf{FL_{cw}}$ then the exchange rule becomes an admissible rule in it. In other words, it becomes equal to a fragment of the intuitionistic logic, which ovbiously has a cut-free system.

When a logic with contraction has also the distributive law, Urquhart obtained interesting undecidability results in [28]. In fact, he succeeded to reduce these decision problems essentially to the word problem for finitely presented semigroups, which is known to be undecidable. Let $\mathbf{T}_+$, $\mathbf{E}_+$ and $\mathbf{R}_+$ be the positive fragments of relevant logics $\mathbf{T}$, $\mathbf{E}$ and $\mathbf{R}$, respectively. Then he proved the following.

Theorem 6 *Relevant propositional logics $\mathbf{T}_+$, $\mathbf{E}_+$ and $\mathbf{R}_+$ are undecidable.*

An example of an undecidable logic with the distributive law but without the contraction is given in [28].

¿From these results, it seems to be quite plausible that most of the predicate logics with contraction will be undecidable. For instance, it was shown in [13] that the predicate logic $\mathbf{FL_{ec}}$ is undecidable. Moreover, this holds even if we replace the contraction rule by some weaker rules. Let n be a natural number greater than 2. Let us call the following rule, $(n \rightsquigarrow 1)$ rule: ¿From $A, \ldots, A, \Gamma \longrightarrow \Delta$ (with n $A's$) infer $A, \Gamma \longrightarrow \Delta$. Then the predicate logic $\mathbf{FL_e}$ with $(n \rightsquigarrow 1)$ rule is undecidable. These results can be obtained by introducing an embedding of the intuitionistic predicate logic in to each of these logics. As for the details, see [13, 12].

2.3 Deducibility in substructural logics

There seem to be various ways of defining the notion of *deducibility* in a given substructural logic. In the following, we will give two definitions of deducibility, each of which will be obtained as a natural generalization of the deducibility for the standard logics.

If a given logic is formalized in a Hilbert style system, we will be led naturally to the following notion of deducibility. This definition was originally given by Urquhart in [28]. (In order to distinguish it from other definitions, we will call it U-deducibility in the present paper.)

Let **L** be a substructural logic, Γ a finite set of formulas and A a formula. We say that A is *U-deducible* from Γ in **L** if there exist formulas $B_1, \ldots, B_n$ such that

1. B_n is A,
2. each B_i satisfies one of the following conditions;
   - B_i is provable in **L**,
   - B_i belongs to Γ,
   - B_i is $B_j \wedge B_k$ for some $j, k < i$,
   - B_k is $B_j \supset B_i$ for some $j, k < i$.

For a given finite set of formulas Γ, define a set $\Gamma^\dagger$ of formulas by the smallest set Δ satisfying the following conditions;

1. every formula in Γ belongs to Δ,
2. if both A and B belong to Δ then both $A \wedge B$ and $A * B$ belong to Δ.

Here, $*$ denotes the fusion (or, the multiplicative conjunction). Now, suppose that a given logic **L** has the exchange rule or more restrictedly, the following inference is admissible in it;

from $A \supset B$ *and* $C \supset (B \supset D)$, *infer* $C \supset (A \supset D)$.

Then, we can show that a formula A is U-deducible from Γ in **L** if and only if either A is provable in **L** or $D \supset A$ is provable in **L** for some formula D in $\Gamma^\dagger$.

Next, we will introduce another notion of deducibility, which seems to be quite natural in sequent calculi.

For a given Γ, we will define a set $\Gamma^\star$ of formulas in the same way as $\Gamma^\dagger$, but we will replace its second condition by

3. if both A and B belong to Δ then $A * B$ belongs to Δ.

So, each element of $\Gamma^\star$ can be obtained by choosing formulas from Γ (possibly with repetitions) and combining them with fusions. We say that a formula A is *strongly deducible* from Γ in **L** if and only if either A is provable in **L** or $D \supset A$ is provable in **L** for some formula D in $\Gamma^\star$. In particular, when **L** has the exchange rule, we can easily show that A is strongly deducible from the set of formulas $\{C_1, \ldots, C_m\}$ if and only if the formula $({C_1}^{k_1} * \cdots * {C_m}^{k_m}) \supset A$ is provable in **L** for some nonnegative integers $k_1, \ldots, k_m$. Here, C^k denotes the formula $C * \cdots * C$ with k C's.

It is clear by the definition that the strong deducibility implies U-deducibility. The U- (and strong) deducibility problem for a logic **L** is the problem of determining whether A is U- (and strongly, respectively) deducible from Γ in **L** or not for any formula A and any finite set of formulas Γ. It is easy to see that when we have all structural rules, the above two notions of deducibility become the standard one, i.e., A is deducible from the set $\{C_1, \ldots, C_m\}$ in **L** if and only the formula $(C_1 \wedge \cdots \wedge C_m) \supset A$ is provable in **L**. Thus the deducibility problem can be reduced to the decision problem. Urquhart proved the following. (Recall that the decision problems of $\mathbf{TW}_+$ and $\mathbf{RW}_+$ are solvable as shown in Theorem 3.)

Theorem 7 *The U-deducibility problems of* $\mathbf{TW}_+$ *and* $\mathbf{RW}_+$ *are unsolvable.*

Using the notion of deducibility and checking the proof carefully, we can restate the well-known result on undecidability of the linear logic *with exponentials* by [17] in the following way (cf. Theorem 1). See also [27].

Theorem 8 *The decision problem of the linear logic* **GL** *is solvable, but its strong deducibility problem is unsolvable.*

The solvability of these deducibility problems depends apparently on what sort of deducibility we will take. Since there seem to be various candidates for the deducibility in substructural logics, it might be interesting to study them.

3 Finite model property

Until now, we have only a few results on the finite model property of substructural logics. An interesting attempt was made by Fine in [8] on the finite model property of some weak relevant logics. While his method has a certain generalities, it doesn't work for a logic containing either the axiom scheme

$$\mathbf{B}: \quad (B \supset C) \supset ((A \supset B) \supset (A \supset C))$$

or the axiom scheme

$\mathbf{B}'$: $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$

as pointed in the paper. Notice that in most cases we consider substructural logics in which the above axiom **B** holds.

To prove the finite model property, usually we prove the completeness theorem first and get a frame in which A is not valid, where A is any unprovable formula. Then we *extract* a finite frame from that frame, in which A is not valid either. But, Kripke-type frames for these substructural logics have semigroup structures in general. Therefore, it is necessary to introduce a semigroup operation in this finite frame, which should be *compatible* with the original one. Here, we will have the main difficulty. In fact, any standard filtration technique doesn't work well at this point.

In [21] Meyer proved the finite model property of $\mathbf{R}_{\rightarrow}$ (or equivalently **BCIW**) which is the implicational fragment of the relevant logic **R**. He used Kripke's Lemma in the proof to get a finite frame. The same technique works also for the implicational fragment of $\mathbf{FL_{ew}}$, i.e. **BCK** (see [22]).

Theorem 9 *The implicational logics* **BCIW** *and* **BCK** *have the finite model property.*

Here we will give a rough sketch of the proof for **BCK**. First, we prove the completeness theorem with respect a certain class $\mathcal{F}$ of Kripke structures determined by finitely generated free commutative monoids with partial orders. Therefore, for each formula A which is not provable in **BCK** there is a Kripke structure **M** in $\mathcal{F}$ with the unit element 0 and a valuation $\models$ such that $0 \models A$ doesn't hold. Then, by using Kripke's Lemma mentioned before, it is possible to eliminate an *upper* part of **M** without changing the non-validity of A. That is, there exists such a *lower* part of **M**, which is enough large but still finite, that forms a Kripke structure in which $0 \models^{*} A$ doesn't hold either.

Some attempts have been made to extend the method to some other implicational logics. See [12]. (We remark that adding the conjunction as a logical connective doesn't cause essential difficulties in these cases. On the other hand, it doesn't seem so easy to add other logical connectives. In fact, the finite model property of these logics with additional logical connectives seems to remain open.)

It is unlikely that the method mentioned in the above will not be applicable to logics with neither the weakening nor the contraction rule. On the other hand, Buszkowski obtained the following results in [4, 5]. Here, we let $\mathbf{BCI^r}$ and $(\mathbf{FL}_{\supset})^{\mathbf{r}}$ be subsystems of sequent calculi **BCI**, and the implicational fragment of **FL** with two implications ($\mathbf{FL}_{\supset}$, in symbol), respectively, each of which is obtained by restricting sequents with those with nonempty antecedents. They are called *product-free Lambek calculi.*

Theorem 10 *Both* $\mathbf{BCI^r}$ *and* $(\mathbf{FL}_{\supset})^{\mathbf{r}}$ *have the finite model property.*

To show this, he introduced an interesting technique, called the method of *barriers*. Using this, we can construct a Kripke structure $\mathbf{M}$ with a valuation $\models$ such that $0 \not\models A$, for a given formula A which is not provable. The essential feature of this model is that the valuation $\models$ satisfies moreover that $a \models p$ holds for all but finitely many pairs of an element a in $\mathbf{M}$ and a propositional variable p appearing in A. Such a model is called a *cofinite model*. This structure $\mathbf{M}$ may not be finite. But by using the above condition on $\models$, it can be shown that $\mathbf{M}$ is essentially finite. That is, from $\mathbf{M}$ and $\models$ we can get a *finite* Kripke structure $\mathbf{N}$ and a valuation $\models^*$ such that the same formulas are true in $\mathbf{M}$ with $\models$ and $\mathbf{N}$ with $\models^*$. From this, the above theorem follows. Quite recently, he succeeded to extend his method and prove the following (see [6]).

Theorem 11 *Both* **BCI** *and* $\mathbf{FL}_{\supset}$ *have the finite model property.*

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