Proper Classes

Kees Doets

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Abstract

A formula $\Phi(x)$ in the one non-logical symbol ε with one free variable x is *Russell* if the sentence $\neg\exists r\forall x(x\,\varepsilon\, r \leftrightarrow \Phi(x))$ is logically valid. This note describes a pattern common to the classical examples of Russell formulas, adds a couple of new ones, and constructs many formulas that are near-Russell.

1 Introduction

Russell [8] discovered that the class $\mathbf{R} := \{x \mid x \notin x\}$ is not a set. Indeed, for all sets x we have that $x \in \mathbf{R} \Leftrightarrow x \notin x$; in particular if $\mathbf{R}$ were a set, this holds for $x := \mathbf{R}$, and we obtain $\mathbf{R} \in \mathbf{R} \Leftrightarrow \mathbf{R} \notin \mathbf{R}$, a propositional contradiction.

This argument does not use set theory at all; in fact, it shows that

Theorem 1.1 *For* $\mathbf{R}$ *equal to the formula* $\neg x\,\varepsilon\, x$*, the sentence*

$$\neg\exists r\forall x(x\,\varepsilon\, r \leftrightarrow \mathbf{R})$$

is logically valid.

Let A be any set. An example of a binary relation on A is the membership relation $\in_A := \{(x, y) \mid x, y \in A \wedge x \in y\}$. Application of Theorem 1.1 to the structure $(A, \in_A)$ yields the following result of Zermelo. (See [6] p. 264/5, Theorem 10. Zermelo's remarkable conclusion: $\mathbf{V}$ (the universe of all sets) *ist selbst keine Menge, — womit die Russellsche Antinomie für unseren Standpunkt beseitigt ist.*)

Corollary 1.2 *If A is a set, then* $\{u \in A \mid u \notin u\} \notin A$.

Again, let A be a set, let h be any function from A to its powerset $\wp(A) = \{X \mid X \subset A\}$, and let ε be the relation $\{(u, v) \mid u, v \in A \wedge u \in h(v)\}$. Application of Theorem 1.1 to the structure (A, ε) yields Cantor's result (which accounts for the existence of uncountable sets in set theory) that no $h : A \to \wp(A)$ can be surjective:

Corollary 1.3 *If* $h : A \to \wp(A)$*, then* $\{u \in A \mid u \notin h(u)\}$ *is not in the range of* h.

Finally, the Russell argument, under the name "diagonalisation", is crucial in showing that (say: first-order) satisfaction is not first-order definable over any structure, that non-recursive r.e. sets exist, etc.

Let us call a formula Φ whose only non-logical symbol is the binary relation symbol ε *Russell* if $\neg\exists r\forall x(x\,\varepsilon\,r \leftrightarrow \Phi)$ is logically valid. Thus, Theorem 1.1 states that $\mathbf{R}$ is Russell. Other well-known Russell formulas are summed up in Theorem 1.4. (Quine [3] p. 130 *181, Shen Yuting [5], Montague [2].)

Theorem 1.4 *The following formulas are Russell:*

1. *the first-order formula* $\mathbf{R}_{n+1}$*:*

   $\neg\exists x_1,\ldots,x_n(x\,\varepsilon\,x_n\,\varepsilon\,x_{n-1}\,\varepsilon\,\cdots\,\varepsilon\,x_1\,\varepsilon\,x)$,

2. *the* $(\omega_1\omega_1)$*-formula* $\mathbf{R}_\infty$*:*

   $\neg\exists x_1,x_2,x_3,\ldots(\ldots\,\varepsilon\,x_3\,\varepsilon\,x_2\,\varepsilon\,x_1\,\varepsilon\,x)$,

3. *the universal monadic second-order formula* $\mathbf{W}$*:*

   $\forall X(X(x)\rightarrow\exists y(X(y)\ \wedge\ \forall z(X(z)\rightarrow\neg z\,\varepsilon\,y)))$.

Note that $\mathbf{W}=\mathbf{R}_\infty\subset\mathbf{R}_n\subset\mathbf{R}_1=\mathbf{R}$; and $\mathbf{R}_m\subset\mathbf{R}_n$ is logically valid iff n divides m. The proof of all of this is straightforward and well-known. (That $\mathbf{R}_\infty\subset\mathbf{W}$ uses the Axiom of Dependent Choice.) The (class form of the) ZF Axiom of Regularity expresses that every set satisfies $\mathbf{W}$.

In the sequel, names of formulas like $\mathbf{R}_n$, $\mathbf{R}_\infty$ and $\mathbf{W}$ will be used also for the classes ($\{x\mid\mathbf{R}_n(x)\}$ etc.) these formulas define.

This note describes a pattern common to the classical examples of Russell formulas, adds a couple of new ones, and constructs many formulas that are near-Russell.

The context here may be thought of as Zermelo-Fraenkel set theory over a universe of sets $\mathbf{V}$, where classes are supposed to have a defining formula. However, the only set-theoretic axiom that is ever used is Separation. In particular, Extensionality does not come into play; therefore, atoms (memberless objects different, but otherwise indistinguishable from, the empty set $\emptyset$) are allowed. (Sometimes, $\emptyset$ is used as a generic name for memberless objects.) The topic is really purely logical.

Often, the first-order realm is left and a certain amount of set theory is imported into logic, allowing infinitary and second-order formulas (such as $\mathbf{R}_\infty$, resp., $\mathbf{W}$) or a game-quantifier.

The plan of this paper is as follows. Section 2 reviews the classical examples in a uniform setting, using the notion of (partial) progressivity. Section 3 discusses some relations that are well-founded if the Axiom of Separation is assumed. Hurkens [1] has noted that it is well-foundedness of one of these relations that is responsible for Scott's smooth development of set theory in [4]. Another one can be used to give alternative definitions of the class of von Neumann ordinals. Section 4 presents a couple of alternative proofs for Zermelo's Corollary 1.2. Section 5 constructs some more Russell formulas that are dual to those of Section 2. Finally, using a simple type of game, many (finite and infinite) sequences of formulas $\Phi_1, \Phi_2, \Phi_3, \ldots$ are constructed such that the corresponding disjunction $\bigvee_i(\neg\exists r\forall x(x \in r \leftrightarrow \Phi_i))$ is logically valid. The classical examples are subsumed by this construction.

For the rest of this paper, assume that $\mathbf{V}$ is a class and $\in$ is a binary relation on $\mathbf{V}$. No set theoretic axioms are assumed to hold in the system $(\mathbf{V}, \in)$ unless the contrary is stated explicitly.

Elements of $\mathbf{V}$ are *sets*, subclasses of $\mathbf{V}$ are *classes*. Note that $y \in X$ signifies a true membership statement only if its right-hand argument X is a class; when its right-hand argument is in $\mathbf{V}$ then the relation on $\mathbf{V}$ is meant.

If $X \subset \mathbf{V}$ and $a \in \mathbf{V}$ are such that $X = \{x \in \mathbf{V} \mid x \in a\}$, then *a represents* X. Sets are sometimes identified with the classes they represent (although this should not be taken as an assumption of Extensionality). If $a \in \mathbf{V}$ and $X \subset \mathbf{V}$ then $a \subset X$ is synonymous with $\forall x \in \mathbf{V}(x \in a \to x \in X)$.

2 Progressivity

Definition 2.1 The *powerclass* $\wp(X)$ of a class X is the class $\{x \in \mathbf{V} \mid x \subset X\}$ of sub*sets* of X.
A class X is *progressive* if $\wp(X) \subset X$, that is, if $\forall x \in \mathbf{V}(\forall y \in x(y \in X) \to x \in X)$.

Example 2.2

1. $\mathbf{V}$, $\mathbf{R}$, $\mathbf{R}_n$, $\mathbf{R}_\infty$ and $\mathbf{W}$ are progressive.
2. Any intersection of progressive classes is progressive.

Proof. $\mathbf{R}$ is progressive, for assume that $\forall y \in x(y \notin y)$. Then, in particular, $x \in x$ implies that $x \notin x$. Hence, $x \notin x$.

$\mathbf{W}$ is progressive, for assume that $\forall y \in x(y \in \mathbf{W})$. To show that $x \in \mathbf{W}$, assume that $x \in X$. It must be shown that X has a *minimal element*, that is, an element $z \in X$ such that $z \cap X = \emptyset$. If x itself is not minimal in X, then there exists $y \in X$ such that $y \in x$. By assumption, $y \in \mathbf{W}$. By definition of $\mathbf{W}$, X has a minimal element.
Other cases are left to the reader. $\square$

Lemma 2.3 (Separation) *There is no progressive set.*

Proof. Immediate from Corollary 1.2. For what it's worth, here is a second proof.

Suppose that K is a progressive set. Applying the Separation Axiom to K, form the intersection of all progressive sets $M := \bigcap\{X \mid \wp(X) \subset X\} = \{x \in K \mid \forall X(X \text{ progressive } \to x \in X)\}$.

Claim 1. M is progressive.
Proof. By Example 2.2.2.

Claim 2. $M \subset \mathbf{R}$.
Proof. By Example 2.2.1, $\mathbf{R}$ is progressive. By 2.2.2, $M \cap \mathbf{R}$ is progressive. By definition of M, $M \subset M \cap \mathbf{R} \subset \mathbf{R}$.

By Claim 1, for all $x \in \mathbf{V}$, $x \subset M \rightarrow x \in M$. Since $M \in \mathbf{V}$, this holds in particular for $x := M$. Thus, $M \in M$. By Claim 2, $M \notin M$. Contradiction. $\square$

The only set-theoretic axiom used in this argument is Separation. Nevertheless, it is *logical* validity that is aimed at. From the previous proof, the following can be extracted.

Theorem 2.4 *If K is progressive and $K \subset \mathbf{R}$, then K is not a set.*

It follows that the classical examples all belong to pure logic.

Corollary 2.5 *The classes $\mathbf{R}$, $\mathbf{R}_n$ (and intersections of classes $\mathbf{R}_n$), $\mathbf{R}_\infty$ and $\mathbf{W}$ are not sets.*

Proof. By Example 2.2, these classes are progressive and are easily seen to be included in $\mathbf{R}$. $\square$

Note that the formula $\mathbf{W}$ is monadic Π^1_1, but it can also be conceived as a first-order ZF-formula, replacing the second-order quantification over classes by a first-order one over sets. The proof that $\mathbf{W}$ is progressive under either interpretation does not need set theory. Only under the first-order interpretation does the inclusion $\mathbf{W} \subset \mathbf{R}$ need sethood of singleton classes, alternatively, an extremely simple instance of Separation. (Note that if $x \in x$, then $\{x\}$ equals $\{y \in x \mid y = x\}$.) Viewing $\mathbf{W}$ as a Π^1_1-defined class, the result holds already under the condition that at least singletons are in the range of the second-order quantifier. (If this range consists of singletons only, then $\mathbf{W} = \mathbf{R}$.)

Definition 2.6
1. Let $\in^\star$ be the transitive closure of $\in$. I.e., $a \in^\star b$ iff for some $n \geq 0$ and $x_1, \ldots, x_n$, $a \in x_1 \in \cdots \in x_n \in b$,
2. a class X is *transitive* if $X \subset \wp(X)$, i.e.: $y \in x \in X \rightarrow y \in X$,
3. The elements of the *transitive part* $\mathbf{Her}(X) := \{b \in X \mid \forall a \in^\star b (a \in X)\}$ of X are called *hereditarily-X* sets.

For a transitive class X, $x \in^\star X$ and $x \in X$ are tantamount.
$\mathbf{Her}(X)$ is $(\omega_1\omega)$-defined from X. Alternatively, it can be monadically Σ^1_1-defined as the union of all transitive subclasses of X. (In fact, transitive sub*sets* suffice if transitive closures exist.)
$\mathbf{Her}(X)$ is transitive and $\mathbf{Her}(X) = X$ holds iff X is transitive.
For all X, Y, we have that $\mathbf{Her}(X \cap Y) = \mathbf{Her}(X) \cap \mathbf{Her}(Y)$.

Lemma 2.7 *If K is (partially, partially*, weakly, cf. below) progressive, then so is $\mathbf{Her}(K)$.*

Proof. Suppose that $x \subset \mathbf{Her}(K)$. By progressivity of K, $x \in K$. Thus, $x \in \mathbf{Her}(K)$. $\square$

Lemma 2.8 *1.* $\mathbf{W}$ *is transitive,*

2. $\mathbf{W}$ *is the least progressive class.*

Proof. 1. Suppose that $x \in y \in \mathbf{W}$. To show that $x \in \mathbf{W}$, assume that $x \in X$; we have to find an element in X that is minimal. Consider $Y := \{y\} \cup X$. As $y \in \mathbf{W}$, Y surely has such an element m. We cannot have $m = y$, since $x \in y$ and $x \in Y$. Thus, $m \in X$ is minimal in X.
2. If X is progressive but not $\mathbf{W} \subset X$, then $\mathbf{W} - X$ has a minimal element x. By 1, $x \subset \mathbf{W} \cap X$. By progressivity of X, $x \in X$, a contradiction. $\square$

By the first part of this lemma, $\mathbf{Her}(\mathbf{W}) = \mathbf{W}$. However, the classes $\mathbf{Her}(\mathbf{R}_n)$ are new examples of Russell classes. Again, the result does not involve set theory and belongs to (both infinitary and second-order) logic.

Corollary 2.9 *The classes* $\mathbf{Her}(\mathbf{R}_n)$ *(and intersections of these) are not sets.*

Proof. Use Lemma 2.7. The conditions of Theorem 2.4 are satisfied. $\square$

Next, we consider a weakened version of progressivity.

Definition 2.10 Define $\wp_U(X) := U \cap \wp(X)$. Let $\mathbf{Tr}$ be the class of transitive sets. A class K is *partially progressive* if $\wp_{\mathbf{Tr}}(K) \subset K$.

Although in general $\mathbf{Her}(X)$ is not first-order definable from X, $\mathbf{Her}(\mathbf{Tr})$ *is* first-order:

Lemma 2.11 $\mathbf{Her}(\mathbf{Tr}) = \mathbf{Tr} \cap \wp(\mathbf{Tr})$. *This class is partially progressive.*

Proof. For any class U, $\mathbf{Her}(U)$ is the greatest fixed point of the operation $\wp_U$. In particular, $\wp_{\mathbf{Tr}}(\mathbf{Her}(\mathbf{Tr})) = \mathbf{Her}(\mathbf{Tr})$; hence $\mathbf{Her}(\mathbf{Tr})$ is partially progressive. $\square$

The greatest fixed point of a (monotone) operator can be obtained by iterating application of the operator sufficiently many times, starting with the class on which it is defined. In the case of $\wp_{\mathbf{Tr}}$, this fixed point is obtained already by applying the operator to $\mathbf{V}$ only twice: $\wp_{\mathbf{Tr}}(\mathbf{V}) = \mathbf{Tr} \cap \wp(\mathbf{V}) = \mathbf{Tr} \cap \mathbf{V} = \mathbf{Tr}$; so $\wp_{\mathbf{Tr}}(\wp_{\mathbf{Tr}}(\mathbf{V})) = \wp_{\mathbf{Tr}}(\mathbf{Tr}) = \mathbf{Tr} \cap \wp(\mathbf{Tr})$.

Lemma 2.12 (Separation) *There is no partially progressive set.*

Proof. Much as before. Suppose that K is a partially progressive set. Using the Separation Axiom, form the intersection of all partially progressive sets $M := \{x \in K \mid \forall L(L \text{ partially progressive } \to x \in L)\}$.
Claim 1. M is partially progressive.
Proof. Intersections of partially progressive classes are partially progressive.
Claim 2. M is transitive.
Proof. Consider $L := \{x \in M \mid x \subset M\}$ $(= M \cap \wp(M))$. L is partially

progressive. For, if $y \subset L$ is transitive, then (as $L \subset M$) $y \subset M$, so (as M is partially progressive) $y \in M$, and $y \in L$. Thus, $M \subset L \subset \wp(M)$, as desired.

Claim 3. $M \subset \mathbf{R}$.

Proof. Consider $L := \{x \in M \mid x \notin x\} = M \cap \mathbf{R}$. Since M and $\mathbf{R}$ are both partially progressive, L is partially progressive. Thus, $M \subset L \subset \mathbf{R}$, as desired.

By Claim 1, if $x \in \mathbf{V}$ is transitive and $\forall y \in x (y \in M)$, then $x \in M$. Since $M \in \mathbf{V}$, this holds in particular for $x := M$. By Claim 2, it follows that $M \in M$, contradicting Claim 3. □

Again, the only set-theoretic axiom used in the argument is Separation. So in order to have a purely logical result we take the Claims of the proof as extra assumptions, obtaining:

Theorem 2.13 *If K is partially progressive, transitive and $K \subset \mathbf{R}$, then K is not a set.*

Corollary 2.14 *The classes*
$\mathbf{Her}(\mathbf{R} \cap \mathbf{Tr}) = \mathbf{Her}(\mathbf{R}) \cap \mathbf{Her}(\mathbf{Tr})$,
$\mathbf{Her}(\mathbf{R}_n \cap \mathbf{Tr}) = \mathbf{Her}(\mathbf{R}_n) \cap \mathbf{Her}(\mathbf{Tr})$, *and*
$\mathbf{Her}(\mathbf{W} \cap \mathbf{Tr}) = \mathbf{Her}(\mathbf{W}) \cap \mathbf{Her}(\mathbf{Tr}) = \mathbf{W} \cap \mathbf{Her}(\mathbf{Tr})$
are not sets.

Proof. Use Lemmas 2.7, 2.11 and Theorem 2.13. All of these intersect a transitive progressive class with the transitive partially progressive $\mathbf{Her}(\mathbf{Tr})$ and so are transitive and partially progressive themselves. □

The class $\mathbf{W} \cap \mathbf{Her}(\mathbf{Tr})$ coincides with the class OR of all von Neumann ordinals. This shows that Burali-Forti's argument belongs to pure (be it second-order) logic.

There is a notion of progressivity that is weaker still.

Definition 2.15 K is partially* progressive if $\wp_{\mathbf{Her}(\mathbf{Tr})}(K) \subset K$.

Lemma 2.16 (Separation) *There is no partially* progressive set.*

Proof. Let M be the least partially* progressive set.

Claim 1. $M \in \mathbf{Tr}$.

Claim 2. $M \subset \mathbf{Tr}$.

Claim 3. $M \subset \mathbf{R}$.

Since $\mathbf{Her}(\mathbf{Tr}) = \mathbf{Tr} \cap \wp_{\mathbf{Tr}}$, it follows from Claims 1 and 2 that $M \in M$, contradicting Claim 3. □

Lemma 2.16 transforms into the logical result 2.17, but this one and 2.21 below do not appear to yield more Russell formulas.

Theorem 2.17 *If K is partially* progressive, transitive, and $K \subset \mathbf{Tr} \cap \mathbf{R}$, then $K \notin \mathbf{V}$.*

Finally, intermediate between progressivity and partial progressivity is weak progressivity:

Definition 2.18 K is *weakly progressive* if $\forall y \in \mathbf{V}(\forall x \in^\star y(x \in K) \to y \in K)$.

Lemma 2.19 *1. Progressivity implies weak progressivity,*

2. weak progressivity implies partial progressivity,

3. if K is weakly progressive, then $\{y \in \mathbf{V} \mid \forall x \in^\star y(x \in K)\}$ is progressive.

Proof. 2. If $y \subset K$ is transitive, then obviously $\forall x \in^\star y(x \in K)$; hence $y \in K$ by weak progressivity.
3. Assume that (a) $\forall y(\forall x \in^\star y(x \in K) \to y \in K)$ and (b) $\forall y \in z \forall x \in^\star y(x \in K)$. We have to show that $\forall x \in^\star z(x \in K)$. Thus, assume that $x \in^\star z$. Then (i) $x \in z$, or (ii) for some $y \in z$, $x \in^\star y$. If $x \in z$ then by (b), $\forall x' \in^\star x(x' \in K)$, hence $x \in K$ by (a). If $x \in^\star y \in z$, then $x \in K$ by (b). □

The *Principle of $\in$-Induction* (an equivalent of the equality $\mathbf{W} = \mathbf{V}$) states that $\mathbf{V}$ is the only progressive class. Similarly, $\in^\star$-Induction states that $\mathbf{V}$ is the only weakly progressive class. Lemma 2.19.3 shows that the first principle implies the second one (the converse is trivial).

Lemma 2.20 (Separation) *There is no weakly progressive set.*

Proof. As before. Suppose that K is a weakly progressive set. Form the intersection of all weakly progressive sets $M := \{x \in K \mid \forall L(L$ weakly progressive $\to x \in L)\}$.
Claim 1. M is weakly progressive.
Claim 2. M is transitive.
Claim 3. $M \subset \mathbf{R}$.
By Claim 1, for all $y \in \mathbf{V}$, $\forall x \in^\star y(x \in M) \to y \in M$. For $y := M$, by Claim 2, $M \in M$; contradicting Claim 3. □

Theorem 2.21 *If K is weakly progressive, transitive and $K \subset \mathbf{R}$, then K is not a set.*

Definition 2.22 $prog(X) := \{x \in \mathbf{V} \mid \forall Y(X \subset Y \wedge Y$ progressive $\to x \in Y)\}$.
Similarly:
$wprog(X) := \{x \in \mathbf{V} \mid \forall Y(X \subset Y \wedge Y$ weakly progressive $\to x \in Y)\}$,
$pprog(X) := \{x \in \mathbf{V} \mid \forall Y(X \subset Y \wedge Y$ partially progressive $\to x \in Y)\}$,
$p^\star prog(X) := \{x \in \mathbf{V} \mid \forall Y(X \subset Y \wedge Y$ partially* progressive $\to x \in Y)\}$.

If X is a class of atoms, then $prog(X) = wprog(X)$ is the class of well-founded sets generated from these atoms.

It is straightforward that $prog(X)$ ($wprog(X)$, $pprog(X)$, $p^\star prog(X)$) is (resp., weakly, partially, partially*) progressive for any class X. Thus, the following may produce concrete examples of proper classes in the presence of atoms or an anti-foundation axiom.

Lemma 2.23 *If $X \subset \mathbf{R}$, then $prog(X)$, $wprog(X)$, $pprog(X)$, $p^\star prog(X) \subset \mathbf{R}$.*

Proof. $\mathbf{R}$ is a progressive class $\supset X$. $\square$

Although progressive *sets* of the various kinds do not exist, in ZF, the various least progressive *classes* are definable. $\mathbf{W}$, the class of well-founded sets, is the least progressive class; and this is also the least weakly progressive class. The least partially progressive class coincides with the least partially* progressive class; this is the class $\mathbf{W} \cap \mathbf{Her}(\mathbf{Tr})$ of von Neumann ordinals.

3 Some well-founded relations

The context of this section is set theory based on the Separation Axiom alone.

Theorem 3.1 *If Π is a monotone operation defined on $\mathbf{V}$ such that for no $a \in \mathbf{V}$, $\Pi(a) \subset a$, then the relation defined by the condition $\Pi(a) \subset b$ is well-founded on $\mathbf{V}$.*

Proof. Suppose that for no set a, $\Pi(a) \subset a$. Assume that K is a non-empty class that has no element minimal with respect to the relation described. By Separation, $\bigcap K \in \mathbf{V}$. By assumption, it suffices to show the
Claim. $\Pi(\bigcap K) \subset \bigcap K$.
Proof. Let $b \in K$ be arbitrary. Since b is not minimal, for some $a \in K$, $\Pi(a) \subset b$. Therefore, by monotonicity, $\Pi(\bigcap K) \subset \Pi(a) \subset b$. Thus, $\Pi(\bigcap K) \subset \bigcap K$. $\square$

Part 1 of the next Corollary is due to Hurkens [1] (as is the proof of the last item that is indicated below).

Corollary 3.2 *The relations defined by the following conditions are well-founded on $\mathbf{V}$:*

1. $\wp(a) \subset b$,
2. $\{x \subset a \mid \forall y \in^\star x (y \in a)\} \subset b$,
3. $\mathbf{Tr} \cap \wp(a) \subset b$,
4. $\mathbf{Her}(\mathbf{Tr}) \cap \wp(a) \subset b$, *and*
5. $\mathbf{W} \cap \mathbf{Her}(\mathbf{Tr}) \cap \wp(a) \subset b$.

Proof. Immediate from Theorem 3.1 and Lemmas 2.3, 2.12, 2.16 and 2.20.
A short proof for 1 can be obtained from Theorem 3.1 and Corollary 1.2; one for 3 from, e.g., Corollary 4.3.5 or 6. $\square$

Note that $\mathbf{W} \cap \mathbf{Her}(\mathbf{Tr}) \cap \wp(a) \subset \mathbf{Her}(\mathbf{Tr}) \cap \wp(a) \subset \mathbf{Tr} \cap \wp(a) \subset \{x \subset a \mid \forall y \in^\star x(y \in a)\} \subset \wp(a)$; hence, in Corollary 3.2, well-foundedness of a later relation implies well-foundedness of an earlier one.

By the way, $\mathbf{W} \cap \mathbf{Her}(\mathbf{Tr}) \cap \wp(a) = \{\alpha \in OR \mid \alpha \subset a\}$ (OR is the class of von Neumann ordinals) is the largest von Neumann ordinal that is subset of a, and if $\mathbf{W} \cap \mathbf{Her}(\mathbf{Tr}) \cap \wp(a) \subset b$, then $\{\alpha \in OR \mid \alpha \subset a\} \in \{\alpha \in OR \mid \alpha \subset b\}$.

Definition 3.3 B is *supertransitive* if $\forall x \in B \forall y \subset x (y \in B)$, i.e.: $\bigcup_{y\in B} \wp(y) \subset B$;
B is *partially supertransitive* if $\forall x \in B \forall y \subset x (y$ transitive $\rightarrow y \in B)$, i.e.: $\bigcup_{y\in \mathbf{Tr}\cap B} \wp(y) \subset B$.

Corollary 3.4 1. *If B is supertransitive, then for all A: $A \in B \leftrightarrow \wp(A) \subset B$;*

hence, $\in$ is well-founded on the class of supertransitive sets.

2. *If B is partially supertransitive, then for all A: $A \in B \rightarrow \mathbf{Tr} \cap \wp(A) \subset B$;*

hence, $\in$ is well-founded on the class of sets that are partially supertransitive.

The second part of this corollary yields some new definitions of the class $OR = \{x \in \mathbf{Her}(\mathbf{Tr}) \mid \in \text{ is well-founded on } x\}$ of von Neumann ordinals that are remarkable in that they do not seem to refer to well-foundedness.

Corollary 3.5 $OR = \{x \in \mathbf{Her}(\mathbf{Tr}) \mid x \textit{ is partially supertransitive } \}$
$= \{x \in \mathbf{Her}(\mathbf{Tr}) \mid \forall y \in x (y \textit{ is partially supertransitive})\}$.

As it were by pure magic, Scott in [4] develops set theory by using a primitive notion of *partial universe* together with the Accumulation Axiom $\forall x[x \in V \leftrightarrow \exists V' \in V(x \in V' \vee x \subset V')]$, where V, V' denote partial universes. Hurkens [1] has noted that well-foundedness of the sequence of partial universes is an immediate and natural consequence of his Corollary 3.2.1. For this, it just suffices to realize that if $V' \in V$, then Accumulation yields $\wp(V') \subset V$. Using induction along $\in$ it now follows that partial universes are transitive: if $y \in x \in V$, then $V' \in V$ exists such that $x \subset V'$ or $x \in V'$, in which case, by induction hypothesis on V', $x \subset V'$ as well. Thus, $y \in V'$ and $y \in V$. From Accumulation it is immediate that partial universes that contain the same partial universes as elements must be equal. From this, by induction along $\in$, it follows that the partial universes are linearly ordered by $\in$. (No set theory involved here: *every* well-founded extensional relation is a well-ordering.) Finally, the new primitive can be eliminated: the class of partial universes is the unique fixed point of the monotone operator $X \mapsto \{\bigcup_{x\in h} \wp(x) \mid h \in \wp(X)\}$.

4 Subclasses that are not elements

Although it was suggested that Corollaries 1.2 and 1.3 are immediate from Theorem 1.1, a little cheating must be admitted now: this is not *exactly* the case. For instance, Theorem 1.1 applied to $(A, \in_A)$ only yields that for no $r \in A$, $A \cap r = \{x \in A \mid x \notin x\}$, and not that $\{x \in A \mid x \notin x\} \notin A$. The desired result, Corollary 4.3.1, can also be obtained from the following. Here, Φ^U denotes the interpretation of Φ w.r.t. the structure $(U, \in)$.

Lemma 4.1 *If $\Phi(x)$ is Russell, then, for every class U, $\{x \in U \mid \Phi^U(x)\} \notin U$.*

Proof. Suppose that $a := \{x \in U \mid \Phi^U(x)\} \in U$. For all x, $x \in a$ iff $x \in U \wedge \Phi^U(x)$. A fortiori, $\forall x \in U\,(x \in a \leftrightarrow \Phi^U(x))$ holds; therefore, the structure $(U, \in)$ satisfies the sentence $\exists a\,\forall x\,(x \in a \leftrightarrow \Phi(x))$. This contradicts Φ being Russell. □

Also the next result produces subclasses of a class U that are not members of U.

Theorem 4.2 *Suppose that K and U are classes and $K \subset \mathbf{R}$.*

1. *If K is progressive, then $U \cap K \notin U$, and $\mathbf{Her}(U) \cap K \notin U$,*
2. *if K is partially progressive and transitive, then $\mathbf{Her}(U) \cap K \notin U$.*

Proof. $U \cap K$: Suppose that $h := U \cap K \in U$. By progressivity, $h \in K$. Thus, $h \in U \cap K = h$, contradicting $K \subset \mathbf{R}$.
$\mathbf{Her}(U) \cap K$: Suppose that $h := \mathbf{Her}(U) \cap K \in U$. Then $\{h\} \cup \mathbf{Her}(U) \subset U$ is transitive, hence $h \in \mathbf{Her}(U)$. Furthermore, $h \subset K$ (and if K is transitive, then so is h), hence $h \in K$. Thus, $h \in \mathbf{Her}(U) \cap K = h$, again contradicting $K \subset \mathbf{R}$.
□

Corollary 4.3 *Let U be an arbitrary class. The following are subclasses of U that are not element of U:*
1. *$U \cap \mathbf{R}$ (cf. Corollary 1.2),*
2. *$U \cap \mathbf{R}_n$,*
3. *$U \cap \mathbf{W}$,*
4. *$\mathbf{Her}(U) \cap \mathbf{R}_n$,*
5. *$\mathbf{Her}(U) \cap \mathbf{W}$ (this is the least fixed point of $\wp_U$),*
6. *$\mathbf{Her}(U \cap \mathbf{R}_n)$,*
7. *$\mathbf{Her}(U \cap \mathbf{R}_n \cap \mathbf{Tr})$,*
8. *$\mathbf{Her}(U \cap \mathbf{W} \cap \mathbf{Tr}) = \mathbf{W} \cap \mathbf{Her}(U \cap \mathbf{Tr})$.*

Proof. For 1, 2 and 3, use the first part of Theorem 4.2.1, for 4, 5 and 6, the second part. For 7 and 8, use 4.2.2.

Note that $\mathbf{W} \cap \mathbf{Her}(U \cap \mathbf{Tr}) = \bigcup\{\alpha \in OR \mid \alpha \subset U\}$ is either the greatest ordinal that is subset of U, or (in case all ordinals are in U), the class OR of all ordinals. □

5 Games

This section's examples are defined using games. (For another game-theoretic paradox, see [7].)

Definition 5.1 Every set a determines a two-person *game of Pick and Choose* that is played as follows. Players *Pick* and *Choose* move alternately. *Pick* starts by picking some $a_1 \in a$. *Choose* follows by choosing some $a_2 \in a_1$. Then *Pick* picks $a_3 \in a_2$, etc.

The first player that picks/chooses an elementless object *wins* and the other player *loses*.

$\mathbf{P}$ is the class of sets such that *Pick* has a winning strategy in the corresponding game and $\mathbf{C}$ is the class of sets for which *Choose* has a winning strategy.

If $\mathbf{V}$ is non-well-founded, a play of the game may go on for ever and neither player wins or loses.

A player loses if he has no admissible move. Therefore the game that starts at $a = \emptyset$ is lost by *Pick* and won by *Choose* right at the start.

Lemma 5.2 *1.* $\exists x \in a(x \in \mathbf{C}) \; \rightarrow \; a \in \mathbf{P}$,

2. $\forall x \in a(x \in \mathbf{P}) \; \rightarrow \; a \in \mathbf{C}$.

Proof. Immediate from Definition 5.1. □

Consider the monotone operator $\wp$ ($\wp(X) = \{x \in \mathbf{V} \mid x \subset X\}$) and its dual $\wp^d$ ($\wp^d(X) := \{x \in \mathbf{V} \mid x \cap X \neq \emptyset\} = \mathbf{V} - \wp(\mathbf{V} - X)$). Lemma 5.2 states that $\mathbf{P}$ and $\mathbf{C}$ satisfy the (pre-) fixed point conditions $\wp^d(\mathbf{C}) \subset \mathbf{P}$ and $\wp(\mathbf{P}) \subset \mathbf{C}$. It is also true that $\mathbf{P}$ and $\mathbf{C}$ are the *least* classes satisfying these conditions and, consequently, that they are fixed points: $\wp^d(\mathbf{C}) = \mathbf{P}$ and $\wp(\mathbf{P}) = \mathbf{C}$ — but this is also immediate from Definition 5.1.

Lemma 5.3 *If*

1. $\wp^d(C) \subset P$, *and*

2. $\wp(P) \subset C$,

then $\mathbf{P} \subset P$ *and* $\mathbf{C} \subset C$.

Proof. $\mathbf{P} \subset P$: Suppose that $a \notin P$. Consider the strategy for *Choose* that consists in choosing her moves (if possible) outside P. As a matter of fact, she can always choose her moves that way: by 5.3.1, $a \cap C = \emptyset$, therefore, a first move $a_1 \in a$ of *Pick* will not be in C. By 5.3.2, $a_1 - P \neq \emptyset$, and so *Choose* has a first move outside P, etc. Since by using this strategy *Choose* will never get out of moves, *Pick* cannot win. This shows that $a \notin \mathbf{P}$.
$\mathbf{C} \subset C$: Suppose that $a \notin C$. Consider the strategy for *Pick* that consists in picking his moves (if possible) outside P. *Pick* can always do so: by 5.3.2, $a - P \neq \emptyset$, and so he has a first move a_1 outside P. By 5.3.1, $a_1 \cap C = \emptyset$, therefore *Choose* cannot help choosing her first move outside C, etc. Since *Pick* never is out of moves, *Choose* cannot win. This shows that $a \notin \mathbf{C}$. □

Example 5.4 Here are three simple models. The first two are well-founded, the third one is not. In the first one, $\mathbf{P}$ is a set but $\mathbf{C}$ is not, in the second one, conversely, $\mathbf{C}$ is a set but $\mathbf{P}$ is not, whereas in the last model both $\mathbf{P}$ and $\mathbf{C}$ are sets.

1. Consider the one-element model $M = \{P\}$ where $P = \emptyset$. Here, $\mathbf{P} = \emptyset$, $\mathbf{C} = \{P\}$. The class $\mathbf{P}$ is represented by the set $P \in M$: for all $m \in M$ we have that $m \in P \leftrightarrow m \in \mathbf{P}$.

2. $M = \{\emptyset, C\}$ has two elements, $C = \{\emptyset\}$. Now $\mathbf{P} = \{C\}$, $\mathbf{C} = \{\emptyset\}$. The class $\mathbf{C}$ is represented by C: for all $m \in M$ it holds that $m \in C \leftrightarrow m \in \mathbf{C}$.

3. $M = \{\emptyset, P, C\}$ has three elements, $P = \{C\}$, $C = \{\emptyset, P\}$. Here $\mathbf{P} = \{C\}$, $\mathbf{C} = \{\emptyset, P\}$. $\mathbf{P}$ is represented by P and $\mathbf{C}$ is represented by C.

Theorem 5.5 *In a well-founded model,*

1. $\mathbf{P}$ *and* $\mathbf{C}$ *partition the universe,*

2. $\mathbf{P}$ *and* $\mathbf{C}$ *cannot both be sets.*

Proof. 1. This is easy to show using induction but is also immediate from the well-known fact that finite games are determined.
2. Suppose that both $\mathbf{P}$ and $\mathbf{C}$ are sets. Since $\mathbf{P} \subset \mathbf{P}$, by 5.2.2, $\mathbf{P} \in \mathbf{C}$. Since $\mathbf{P} \in \mathbf{C} \cap \mathbf{C}$, by 5.2.1, $\mathbf{C} \in \mathbf{P}$. Therefore, $\in$ cannot be well-founded. □

Theorem 5.6 *If* 2 *divides* m *or* $m = \infty$*, then* $\mathbf{R}_m \cap \mathbf{P}$ *and* $\mathbf{R}_m \cap \mathbf{C}$ *cannot both be sets.*

Proof. If not, by Theorem 5.5 and progressivity of $\mathbf{R}_m$, it follows that $\mathbf{R}_m \cap \mathbf{P} \in \mathbf{R}_m \cap \mathbf{C} \in \mathbf{R}_m \cap \mathbf{P} \in \mathbf{R}_2$, a contradiction. □

The "paradox" that is hidden here deviates from the classical examples in that from *two* classes at least *one* must be proper. The examples of 5.4.1/2 show we cannot predict which one. (In the model of 5.4.3, $\mathbf{R}_2 \cap \mathbf{P} = \emptyset$ is represented, but $\mathbf{R}_2 \cap \mathbf{C} = \{\emptyset\}$ is not.) Of course, in full set theory *both* classes are proper.

This example can be generalized to games where the players do not have to move alternatingly. For simplicity, I do not describe the games but instead give the relevant pre-fixed point conditions.

Theorem 5.7 *Suppose that* $Q_i = \wp$ *or* $Q_i = \wp^d$ *(*$i = 2, \ldots, n$*), let* n *divide* m *or* $m = \infty$*, and let* X *be a class. If (i)* $\wp Q_{n-1} \cdots Q_1 X \subset X$ *or (ii)* $\{\emptyset\} \cup \wp^d Q_{n-1} \cdots Q_1 X \subset X$*, then not everyone of the* n *classes* $\mathbf{R}_m \cap X$, $\mathbf{R}_m \cap Q_1 X$, $\mathbf{R}_m \cap Q_2 Q_1 X$, $\ldots$, $\mathbf{R}_m \cap Q_{n-1} \cdots Q_1 X$ *is a set.*

Proof. Suppose that $X_0 := \mathbf{R}_m \cap X$, $X_1 := \mathbf{R}_m \cap Q_1 X, \ldots$, $X_n := \mathbf{R}_m \cap Q_{n-1} \cdots Q_1 X$ are sets. Then since $\mathbf{R}_m$ is progressive, $X_0, \ldots, X_n \in \mathbf{R}_m$.
(i) $\wp Q_{n-1} \cdots Q_1 X \subset X$.
We have that $X_{n-1} \subset Q_{n-1} \cdots Q_1 X$. Therefore, $X_{n-1} \in \wp Q_{n-1} \cdots Q_1 X \subset X$ and $X_{n-1} \in \mathbf{R}_m \cap X = X_0$.
Next, if $Q_1 = \wp$, then $X_0 \subset X$, $X_0 \in Q_1 X$ and $X_0 \in X_1$. Also, if $Q_1 = \wp^d$, then $X_{n-1} \in X_0 \cap X$, therefore $X_0 \in Q_1 X$ and $X_0 \in X_1$ in this case as well.
Continuing this argument, we obtain that $X_{n-1} \in X_0 \in X_1 \in \cdots \in X_{n-1}$. Thus, $X_0 \notin \mathbf{R}_n$ and $X_0 \notin \mathbf{R}_m$, a contradiction.

(ii) $\{\emptyset\} \cup \wp^d Q_{n-1} \cdots Q_1 X \subset X$.
If $X_{n-1} = \emptyset$, then $X_{n-1} \in X$ and $X_{n-1} \in X_0$. But also if $X_{n-1} \neq \emptyset$, $x \in X_{n-1}$ exists such that $x \in Q_{n-1} \cdots Q_1 X$ and it follows that $X_{n-1} \in \wp^d Q_{n-1} \cdots Q_1 X \subset X$ and $X_{n-1} \in X_0$.

As before, we obtain that $X_{n-1} \in X_0 \in \cdots \in X_{n-1}$. □

Example 5.8 Special cases of Theorem 5.7 are the following.

1. 5.7(i), $n = 0$. A game-theoretic interpretation of *well-foundedness*.

   Pick does not participate in this game: *Choose* is the only one to move. *Pick* wins if *Choose* hits $\emptyset$.

   This special case coincides with Theorem 2.4. The winning class is $\mathbf{R}_\infty$, the least fixed point of $\wp$.

2. 5.7(i), $n = 1$, $Q_1 = \wp^d$.

   Defining $Y := \wp^d X$, this coincides with Theorem 5.6.

3. 5.7(ii), $n = 0$. $\wp^d X \subset X$ iff for some B, $X = \{x \mid x \in B \vee \exists y \in B(y \in^\star x)\}$.

   Choose does not participate in this game: *Pick* is the only one to move and wins if he hits $\emptyset$.

   The winning class here is $\{a \mid a = \emptyset \vee \emptyset \in^\star a\}$, which is the least fixed point of $\wp^d$ that contains $\emptyset$ as an element. This produces examples of Russell-like formulas such as $\mathbf{R} \cap \{a \mid a = \emptyset \vee \emptyset \in^\star a\}$.

   Note that if $\{\emptyset\} \cup \wp^d X \subset X$, then X is progressive. Therefore, this example falls under Theorem 2.4.

As has been noted above, the Axiom of Dependent Choice (DC) is needed to see that $\mathbf{W}$ and $\mathbf{R}_\infty$ are equivalent. This phenomenon also permeates the game context and appears to be intimately connected with the usual view of playing games and the notion of a winning strategy.

Instead of iterating some fixed finite pattern in which *Pick* and *Choose* move (e.g., corresponding to the condition $\wp\wp\wp^d X \subset X$, the pattern in which the participants move is: *Choose–Choose–Pick–Choose–Choose–Pick* and so on, ad infinitum), one can impose arbitrary (non-iterating) patterns. The next result produces continuum many examples, one for each such pattern : $\{1, 2, 3, \ldots\} \to \{Pick, Choose\}$.

Theorem 5.9 *Suppose that for $i = 1, 2, 3, \ldots$, $Q_i = \wp$ or $Q_i = \{\emptyset\} \cup \wp^d$ (where $(\{\emptyset\} \cup \wp^d)(X) := \{\emptyset\} \cup \wp^d(X)$). If for all j ($j = 0, 1, 2, \ldots$), $Q_{j+1} W_{j+1} \subset W_j$, then not every class $\mathbf{R}_\infty \cap W_j$ is a set.*

Proof. Suppose they are all sets. Then by progressivity of $\mathbf{R}_\infty$, $\mathbf{R}_\infty \cap W_j \in \mathbf{R}_\infty$. If $Q_{j+1} = \wp$, then, since $\mathbf{R}_\infty \cap W_{j+1} \subset W_{j+1}$, $\mathbf{R}_\infty \cap W_{j+1} \in W_j$, and $\mathbf{R}_\infty \cap W_{j+1} \in \mathbf{R}_\infty \cap W_j$.
If $Q_{j+1} = \{\emptyset\} \cup \wp^d$, then the same conclusion can be drawn: if $\mathbf{R}_\infty \cap W_{j+1} = \emptyset$, then obviously $\mathbf{R}_\infty \cap W_{j+1} \in W_j$ and $\mathbf{R}_\infty \cap W_{j+1} \in \mathbf{R}_\infty \cap W_j$; and if $\mathbf{R}_\infty \cap W_{j+1} \neq \emptyset$, this follows as well.

However, we cannot have that $\cdots \in W_2 \in W_1 \in W_0 \in \mathbf{R}_\infty$, by virtue of the definition of $\mathbf{R}_\infty$. □

Problem. Is there a sense in which the above classification captures every Russell-like formula (say, in a language of one of the types considered here)?

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