Intuitionistic propositional logic with only equivalence has no interpolation

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Abstract

We will show in this paper that there is no interpolation theorem for the fragment of pure equivalence in intuitionistic propositional logic. The computer program that was used to calculate the counterexample by computations on a finite Kripke model is briefly sketched.

1 Introduction

In classical propositional logic all fragments satisfy interpolation (proved by F. Ville, Chapter 1, excercise 2 [KK 71]): if ϕ is a formula in the fragment $F(\vec{p},\vec{q})$, ψ is a formula in the fragment $F(\vec{p},\vec{r})$ (and hence the propositional variables that both have in common are in $\vec{p}$) and $\phi \vdash \psi$, then there is a formula $\chi \in F(\vec{p})$ such that $\phi \vdash \chi \vdash \psi$. For intuitionistic propositional logic this is not the case. As J. Zucker first proved (see [Renardel 81]), fragments with defined connectives can be found that do not satisfy interpolation. On the other hand, [Renardel 89] proved that for most natural fragments of **IpL** the interpolation theorem does hold.

Contrary to the conjecture in [Renardel 89] however, the **IpL** fragment $[\leftrightarrow]$, containing all pure equivalential formulas does not have the interpolation property. A counterexample was found in a computer assisted search.

2 $[\leftrightarrow]$ lacks interpolation

The following fact can be established by a direct proof, or using some decision method like semantic tableaux (for example in a computer program).

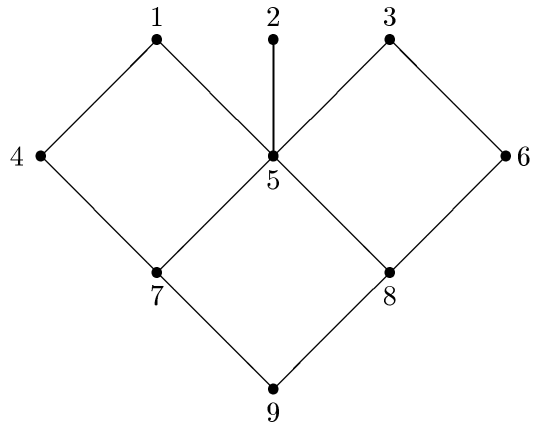
Fact 1

$$p \leftrightarrow (r \leftrightarrow (q \leftrightarrow r)) \vdash (q \leftrightarrow s) \leftrightarrow ((p \leftrightarrow q) \leftrightarrow (p \leftrightarrow s))$$

An interpolant in this case, if it would exist, should be in the fragment $[p, q, \leftrightarrow]$.

The *diagram* of $[p, q, \leftrightarrow]$, that is the ordered set of equivalence classes of the fragment, is known.

Lemma 2 *The* diagram *of* $[p, q, \leftrightarrow]$ *is*



where

1. p
2. $p \leftrightarrow q$
3. q
4. $q \leftrightarrow (p \leftrightarrow q)$
5. $(q \leftrightarrow (p \leftrightarrow (p \leftrightarrow q))) \leftrightarrow (p \leftrightarrow (q \leftrightarrow (p \leftrightarrow q)))$
6. $p \leftrightarrow (p \leftrightarrow q)$
7. $q \leftrightarrow (p \leftrightarrow (p \leftrightarrow q))$
8. $p \leftrightarrow (q \leftrightarrow (p \leftrightarrow q))$
9. $p \leftrightarrow p$

Proof. The construction of the diagram is straightforward and can be done by hand. For computer assistance see [Hendriks 96a]. $\dashv$

Theorem 3 *The interpolation property does not hold for* $[\leftrightarrow]$*.*

Proof. In the diagram above the formula $p \leftrightarrow (r \leftrightarrow (q \leftrightarrow r))$ implies exactly the formulas $5, 7, 8$ and of course 9. However, it can be checked that

$$(q \leftrightarrow (p \leftrightarrow (p \leftrightarrow q))) \leftrightarrow (p \leftrightarrow (q \leftrightarrow (p \leftrightarrow q))) \nvdash (q \leftrightarrow s) \leftrightarrow ((p \leftrightarrow q) \leftrightarrow (p \leftrightarrow s))$$

and hence neither formula 5 nor one of the other consequences of $p{\leftrightarrow}(r{\leftrightarrow}(q{\leftrightarrow}r))$ in $[p, q, \leftrightarrow]$ can act as an interpolant between the formulas $p{\leftrightarrow}(r{\leftrightarrow}(q{\leftrightarrow}r))$ and $(q{\leftrightarrow}s){\leftrightarrow}((p{\leftrightarrow}q){\leftrightarrow}(p{\leftrightarrow}s))$. $\dashv$

3 Other fragments with $\leftrightarrow$

Restricting ourselves to the *elementary* connectives $\wedge, \vee, \rightarrow$ and $\neg$, there are not many fragments with 'pure' equivalence. Using

- $A{\leftrightarrow}B \;\equiv\; (A{\rightarrow}B) \wedge (B{\rightarrow}A)$;
- $A \wedge B \;\equiv\; (A{\leftrightarrow}B){\leftrightarrow}(A \vee B)$;
- $A \wedge B \;\equiv\; A{\leftrightarrow}(A{\rightarrow}B)$;
- $A{\rightarrow}B \;\equiv\; (A \wedge B){\leftrightarrow}A$;
- $A{\rightarrow}B \;\equiv\; (A \vee B){\leftrightarrow}B$;

one can prove that

1. $[\wedge, \leftrightarrow] \;\equiv\; [\wedge, \rightarrow]$;
2. $[\rightarrow, \leftrightarrow] \;\equiv\; [\wedge, \rightarrow]$;
3. $[\vee, \leftrightarrow] \;\equiv\; [\wedge, \vee, \rightarrow]$;

For the fragments $[\leftrightarrow, \neg]$ and $[\leftrightarrow, \neg\neg]$ with 'pure' equivalence the counterexample of the previous section still holds. (In the second fragment $\neg\neg$ is considered as a single connective.) Hence neither of these fragments has the interpolation property. This can be checked using the diagrams of these fragments. Here some computer assistance is welcome, as the diagram of $[\leftrightarrow, \neg\neg]$ has 169 elements and the size of the diagram of $[\leftrightarrow, \neg]$ is 538. In contrast to these results the author has been able to show that $[\rightarrow, \neg\neg]$ does satisfy interpolation [Hendriks 96b].

4 Carriers of interpolation

The counterexample used in the previous sections has been constructed with a computer program building a diagram and calculating for each class the so called *carriers of interpolation*. The notion of a carrier of interpolation may have an interest of its own. It is related to the notion of *uniform interpolation* (see [Pitts 92]): the *left uniform interpolant* of a formula ϕ can be regarded as the singleton left carrier of ϕ [1].

The partial order $\vdash$ on equivalence classes can be used to define a left and a right carrier of interpolants of a formula.

Definition 4 *Let ϕ be a formula in the fragment $F(\vec{p},\vec{q})$. The* left carrier of interpolation *of ϕ in $F(\vec{p})$, $L\vec{q}\phi$ is the set of* maximal *formulas in $F(\vec{p})$ implying ϕ. The* right carrier of interpolation *of ϕ in $F(\vec{p})$, $R\vec{q}\phi$ is the set of* minimal *formulas in $F(\vec{p})$ implied by ϕ.*

Using the carriers of interpolation for $F(\vec{p},\vec{q})$ in $F(\vec{p})$ one can single out candidates for counterexamples against interpolation in F. The situation we are looking for is:

Lemma 5 *If for formula ϕ and ψ in $F(\vec{p},\vec{q})$:*

1. $\phi \vdash \psi$;
2. *for every formula $\chi \in R_{\vec{q}}(\phi)$ and every formula $\theta \in L_{\vec{q}}(\psi)$ we have* $\chi \not\vdash \theta$;
3. $\phi \vdash \psi[\vec{q} := \vec{r}]$.

then ϕ implies $\psi[\vec{q} := \vec{r}]$ without an interpolant in $F(\vec{p})$.

Proof. Obvious. $\dashv$

Note that the third of the constraints in the lemma implies the first. However, the first two conditions can be tested within a complete model for the fragment $F(\vec{p},\vec{q})$ [Hendriks 96a]. In case we have no program for model checking in $F(\vec{p},\vec{q},\vec{r})$, we have to use a relatively slow theorem tester. By only applying the test for the third condition on candidates ϕ and ψ that already passed the first two tests, we have designed a more efficient test for counterexamples against interpolation in F.

[1] The uniform left (right) interpolant will exist if the disjunction (conjunction) of the formulas in the right (left) carrier of interpolation is (equivalent to a formula) in the fragment.

5 Computer programs

The diagram of $[\leftrightarrow]^2$ that was used in the counterexample of theorem 3 is not included in [Hendriks 96a], but can be computed with the methods described there (and also in [JHR 91]).

In the program implementing the three tests of lemma 5 the diagram of $[\leftrightarrow]^2$ was calculated using the (61 element) exact Kripke model of $[\wedge, \rightarrow]^3$. The program then computed part of the diagram of $[\leftrightarrow]^3$. For each equivalence class ϕ the program calculated the corresponding set $[\![\phi]\!]$ of elements in the model where the formula ϕ holds. Also the carriers $L_r(\phi)$ and $R_r(\phi)$ where computed using the (sets of the) formulas representing the classes of $[\leftrightarrow]^2$.

Each time distinct classes ϕ and ψ were found such that $\phi \vdash \psi$ (or equivalently $[\![\phi]\!] \subset [\![\psi]\!]$), a test was performed whether for some $\chi \in R_r\phi$ and some $\theta \in L_r\psi$ it was the case that $[\![\chi]\!] \subseteq [\![\psi]\!]$ (and hence $\chi \vdash \psi$). If not, the pair ϕ and ψ passed the second test of lemma 5.

After passing test 2, a formula ξ was made by substituting s for r in ψ. To test whether $\phi \vdash \xi$, a call was made to a theorem tester based on semantic tableaux.

The counterexample of theorem 3 was the first one generated by the computer program (modulo some renaming of variables).

We used another computer program to check whether the counterexample against interpolation would also hold in $[\leftrightarrow, \neg]$ (and hence also in $[\leftrightarrow, \neg\neg]$). This program constructed the diagram of the fragment $[\leftrightarrow, \neg]^2$ using the 61 element exact Kripke model of $[\wedge, \rightarrow]^3$ mentioned above. For each generated formula ϕ it compared the set of worlds forcing ϕ (in the model mentioned above) with the sets corresponding to $p \leftrightarrow (r \leftrightarrow (q \leftrightarrow r))$ and $(q \leftrightarrow s) \leftrightarrow ((p \leftrightarrow q) \leftrightarrow (p \leftrightarrow s))$. It was found that the sets of the following formulas are in between those of the formulas of theorem 3:

1. $\neg\neg q \leftrightarrow p$
2. $((q \leftrightarrow p) \leftrightarrow q) \leftrightarrow \neg\neg q$
3. $((\neg\neg q \leftrightarrow p) \leftrightarrow ((q \leftrightarrow p) \leftrightarrow q)) \leftrightarrow ((\neg\neg p \leftrightarrow q) \leftrightarrow ((q \leftrightarrow p) \leftrightarrow p))$
4. $((\neg q \leftrightarrow \neg p) \leftrightarrow q) \leftrightarrow ((q \leftrightarrow p) \leftrightarrow p)$
5. $((\neg q \leftrightarrow \neg p) \leftrightarrow p) \leftrightarrow ((q \leftrightarrow p) \leftrightarrow q)$
6. $\neg q \leftrightarrow \neg p$

So, in the Kripke model used by the program, it is true for only these formulas ϕ that

$$p\leftrightarrow(r\leftrightarrow(q\leftrightarrow r) \models \phi \models (q\leftrightarrow s)\leftrightarrow((p\leftrightarrow q)\leftrightarrow(p\leftrightarrow s)).$$

However, either by hand or using a formula tester, one can prove that $p\leftrightarrow(r\leftrightarrow(q\leftrightarrow r))$ does not imply the first two formulas, and that the others do not imply $(q\leftrightarrow s)\leftrightarrow((p\leftrightarrow q)\leftrightarrow(p\leftrightarrow s))$. Which proves that there is no interpolant between $p\leftrightarrow(r\leftrightarrow(q\leftrightarrow r))$ and $(q\leftrightarrow s)\leftrightarrow((p\leftrightarrow q)\leftrightarrow(p\leftrightarrow s))$ in $[\leftrightarrow,\neg]^2$.

The same method can be used to find the two interpolants between these formulas that do exist in $[\rightarrow]^2$: $q\rightarrow p$ and $((p\rightarrow q)\rightarrow q)\rightarrow p$.

References

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