Marginalia on sequent calculi

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For Ettore Casari on occasion of his 65th birthday

1 Introduction

In this note we wish to draw attention to certain points of detail, concerning the subtle differences between several possible versions of Gentzen systems and systems of natural deduction. In notation and terminology we conform to [TS96].

Gentzen([Gen35]) introduced the calculi LJ, LK with left- and right introduction rules, operating on sequents; systems of this kind are in the literature often called "sequent calculi". Contrary to what many people think, natural deduction did not originate with Gentzen (there is, for example, the earlier work by Jáskowski, [Jás34]) although Gentzen's work made it wellknown. Versions of natural deduction are sometimes also presented as calculi operating on sequents (as in this note). For these reasons I have rejected the widely used designation "sequent calculi" for systems of the LJ,LK-type and call them "Gentzen systems" instead. (One might object that "Gentzen systems" might be interpreted as referring to all the formalisms discussed in Gentzen's paper, and that only "Gentzen sequent calculi" would be unambiguous. But in practice I find this too long.)

On the side of the Gentzen systems, there are combinatorial differences between systems with term labels and systems without term labels attached; the customary presentation of Gentzen systems does not include term labels. Correspondingly, for natural deduction, there are the versions obeying the Complete Discharge Convention (the CDC): open assumptions are always discharged at the earliest opportunity, and versions with labelled assumption classes, isomorphic to (suitable extensions of) simple type theory. The correlation between natural deduction and Gentzen systems is usually described for Gentzen systems with term labels on the one hand, and extensions of the simple type theory on the other hand. In this note we discuss the systems and their relationships primarily for the unlabelled variants.

The deductions in the systems we want to consider are, with few exceptions, represented as prooftrees with the nodes labeled by sequents $\Gamma \Rightarrow A$, where Γ is a multiset, but there are no term labels. To keep things simple, the whole discussion is carried out for intuitionistic implication logic, although there is no problem extending the discussion to full first-order intuitionistic logic. Technically, the results are simple adaptations of work by [Min96, Her95].

Possible reasons for being interested in these systems are (a) pedagogical: by comparison we see what the advantages/disadvantages of term labels are, and (b)

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for theorem-proving purposes, we are more interested in derivability than in derivations; natural deductions with the CDC permit representation by less data. Maybe this is useful; in any case it might be worth looking into.

The version $\mathbf{N_3}$ of natural deduction considered below may be seen as a presentation of natural deduction under the CDC. Natural deduction under the CDC is considered to have bad combinatorial properties (no strong normalization), when compared with the standard version of natural deduction which is isomorphic to simple type theory $\boldsymbol{\lambda}_{\rightarrow}$. On the other hand, the results in this note show that there are natural counterparts to $\mathbf{N_3}$ on the side of the Gentzen systems ($\mathbf{G_2}$, $\mathbf{G_5}$) such that known correlations of a more static nature for the term-labeled versions in an obvious way "project down" onto $\mathbf{N_3}$, $\mathbf{G_2}$, $\mathbf{G_5}$. In particular we shall consider normal forms on the side of the Gentzen systems (in two versions) in 1-1-correspondence with normal deductions in $\mathbf{N_3}$. These results may be regarded as a limited rehabilitation of deductions under the CDC. Also, the results seem to point to a natural correlation between standard Gentzen systems with contraction absorbed into the rules (Kleene's G3) and natural deduction under the CDC.

2 Labeled and unlabeled Gentzen systems

The following system $\mathbf{G_1}$ is a Gentzen system of term-labeled sequents:

$$\text{(Axiom)}\ \vec{y}{:}\,\Gamma, x{:}\,A \Rightarrow x{:}\,A$$

$$\text{R}{\rightarrow}\ \frac{\vec{y}{:}\,\Gamma, x{:}\,A \Rightarrow t{:}\,B}{\vec{y}{:}\,\Gamma \Rightarrow \lambda x.t{:}\,A \rightarrow B} \qquad \text{L}{\rightarrow}\ \frac{\vec{y}{:}\,\Gamma, z{:}\,A{\rightarrow}B \Rightarrow t(z){:}\,A \qquad \vec{y}{:}\,\Gamma, x{:}\,B \Rightarrow s(x){:}\,C}{\vec{y}{:}\,\Gamma, z{:}\,A \rightarrow B \Rightarrow s(zt(z)){:}\,C}$$

The antecedents are *sets* of typing statements $y_1{:}\,A_1, \ldots, y_{n-1}{:}\,A_{n-1}$; without loss of generality we may assume all the y_i to be distinct.

If we drop the term labels, we obtain a well-known Gentzen system $\mathbf{G_2}$ (corresponding to system $\rightarrow$**G3i** of [TS96]):

$$\text{(Axiom)}\ \ \Gamma, A \Rightarrow A \qquad \text{R}{\rightarrow}\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \qquad \text{L}{\rightarrow}\ \frac{\Gamma, A{\rightarrow}B \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \rightarrow B \Rightarrow C}$$

Several $\mathbf{G_1}$-derivations correspond to a single $\mathbf{G_2}$-derivation, for example stripping term labels from the following two deductions results in the same deduction in $\mathbf{G_2}$.

$$\cfrac{\cfrac{x{:}\,A, y{:}\,A \Rightarrow y{:}\,A}{y{:}\,A \Rightarrow \lambda x.y{:}\,A{\rightarrow}A}}{\Rightarrow \lambda yx.y{:}\,A{\rightarrow}(A{\rightarrow}A)} \qquad \cfrac{\cfrac{x{:}\,A, y{:}\,A \Rightarrow y{:}\,A}{x{:}\,A \Rightarrow \lambda y.y{:}\,A{\rightarrow}A}}{\Rightarrow \lambda xy.y{:}\,A{\rightarrow}(A{\rightarrow}A)}$$

$\mathbf{G_1}$ is not just a sequent calculus, but at the same time describes a mapping from $\mathbf{G_1}$-deductions to natural deductions. Here we are thinking of the standard version $\mathbf{N_1}$ (= $\rightarrow$**Ni** of [TS96]), which is isomorphic to simple type theory $\boldsymbol{\lambda}_{\rightarrow}$. $\mathbf{G_2}$ is not a natural choice for describing the mapping, since the term corresponding to the natural deduction to be assigned is not uniquely determined, as may be seen from the example above.

The usual presentation of deductions for the standard version of the natural deduction system is as trees labeled with formulas, the assumptions appearing at the top nodes (leaves), the conclusion at the bottom. The assumptions get some marker (a variable symbol $x, y, z, \ldots$ say) which is repeated at the inference where the assumptions so marked are being discharged. An alternative but equivalent presentation of $\mathbf{N_1}$ is

$$\vec{y}{:}\,\Gamma, x{:}\,A \Rightarrow x{:}\,A \text{ (Axiom)}$$

$$\to\mathrm{I}\,\frac{\vec{y}{:}\,\Gamma, x{:}\,A \Rightarrow t{:}\,B}{\vec{y}{:}\,\Gamma \Rightarrow \lambda x.t{:}\,A \to B} \qquad \to\mathrm{E}\,\frac{\vec{y}{:}\,\Gamma \Rightarrow t{:}\,A \to B \qquad \vec{y}{:}\,\Gamma, \Rightarrow s{:}\,A}{\vec{y}{:}\,\Gamma \Rightarrow ts{:}\,B}$$

Another version $\mathbf{N_3}$ of natural deduction discharges assumptions according to the CDC mentioned in the introduction, that is to say, if assumption A is discharged at an $\to$I-application, all occurrences of A at top nodes above the $\to$I-application are discharged which have not been discharged higher up in the tree (cf. [TS96]). For the version based on CDC, the variable markers are redundant. However, the deductions of the CDC-version may also be represented by a subclass of terms of the calculus $\boldsymbol{\lambda}_\to$, namely the class of terms, whenever a variable x^A appears within the scope of a lambda operator λy^A, then $x^A \equiv y^A$. Hence, in any subterm, for each type there is at most one free variable of that type.

A simple representation of $\mathbf{N_3}$ without the use of term labels is the following:

$$\text{Axiom } \Gamma, A \Rightarrow A \qquad \to\mathrm{I}\,\frac{\Gamma \Rightarrow B}{\Gamma \setminus \{A\} \Rightarrow A \to B} \qquad \to\mathrm{E}\,\frac{\Gamma \Rightarrow A \to B \qquad \Gamma \Rightarrow A}{\Gamma \Rightarrow B}$$

Here the antecedents of the sequents are treated as *sets* of formulas. Weakening has been built into the axioms; in the standard tree presentations such as $\mathbf{N_1}$, with formulas instead of sequents at the nodes, this is regarded as implicit, in the sense that a deduction of A from assumptions Γ is tacitly assumed to represent not only a deduction of $\Gamma \Rightarrow A$, but also the deductions of $\Gamma, \Delta \Rightarrow A$ obtained by weakening the conclusion.

However, if we drop the term labels from $\mathbf{N_1}$, there results a system $\mathbf{N_2}$ where the antecedents are multisets, and which at first sight seems to be intermediate in character between the CDC-version $\mathbf{N_3}$ and the standard version $\mathbf{N_1}$:

$$\text{(Axiom) } \Gamma, A \Rightarrow A \qquad \to\mathrm{I}\,\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B} \qquad \to\mathrm{E}\,\frac{\Gamma \Rightarrow A \to B \qquad \Gamma \Rightarrow A}{\Gamma \Rightarrow B}$$

On closer scrutiny, however, we discover that $\mathbf{N_2}$ is in fact a variant of $\mathbf{N_3}$. **2.1.**

PROPOSITION. *There is a one-to-one correspondence between deductions in* $\mathbf{N_3}$ *and* $\mathbf{N_2}$*; the correspondence respects the size of the proofs and the type of the rules applied.*

PROOF. We construct a term representing a CDC-proof of $\mathbf{N_3}$, for each proof in $\mathbf{N_2}$.
Case 1. Assume the deduction is an axiom $\Gamma, A \Rightarrow A$. Take for the term $x{:}\,A$ (x arbitrary).
Case 2. Assume the last rule is

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

If at the node labeled with $\Gamma, A \Rightarrow B$ the associated term was $t{:}\,B$, we associate to the new bottom line of the proof $\lambda y{:}\,A.t$ (or $\lambda y^A.t$) with $y^A = x^A$ if t contains a free occurrence of x^A of a variable of type A, and y^A arbitrary otherwise.

Case 3. Assume the last inference is

$$\frac{\Gamma \Rightarrow A \to B \qquad \Gamma \Rightarrow A}{\Gamma \Rightarrow B}$$

Let $t\colon A{\to}B$ and $s\colon A$ be assigned to the left and right premise respectively. Since the terms assigned represent natural deductions under the CDC, t and s contain each at most one variable of each type free. If each actually contains a variable of type A, say x and y respectively, we identify the two by renaming one of them. The identification process is carried out for all relevant types. The result is the term to be assigned to the conclusion $\Gamma \Rightarrow B$.

Conversely, every proof in $\mathbf{N_3}$ is mapped to a proof in $\mathbf{N_2}$; it is easy to see from the construction that these maps are inverse to each other. □

Nonetheless, there is a combinatorial difference between these two variants. This becomes apparent if one realizes that there is only one obvious way of inserting term labels in $\mathbf{N_3}$, but that there may be several correct ways of term-labeling a deduction in $\mathbf{N_2}$.

What is to constitute a normalization step in $\mathbf{N_2}$? The most obvious solution is to regard any transformation as a correct normalization step which is obtained in the following way: first insert correctly a term-labeling into the $\mathbf{N_2}$-deduction, make a β-contraction, and delete the labels in the result afterwards.

On the other hand, the only correct way of reducing in $\mathbf{N_3}$ is to regard the deduction as a term in the subclass corresponding to the CDC, making a β-contraction, and then reinterpreting the result as a CDC-deduction (deduction in $\mathbf{N_3}$) by renaming some bound variables. As observed in [Lei79], $\mathbf{N_3}$ does not obey strong normalization; for a counterexample due to R. Statman see exercise 6.6.7D in [TS96].

Therefore the preceding proposition has the following corollary.

COROLLARY. *Deductions in* $\mathbf{N_2}$ *are normalizable, but* $\mathbf{N_2}$ *does not obey strong normalization.*

As observed above, the natural, unambiguous assignment of a deduction of $\mathbf{N_1}$ to a Gentzen style deduction requires the labeled version $\mathbf{G_1}$. Of course, we can also take a $\mathbf{G_2}$-deduction, insert in some deterministic way term labels, thus obtaining an associated $\mathbf{N_1}$-deduction. But is this procedure also "natural"? Instead of trying to answer this question, we may observe that one can use a direct assignment of *normal* $\mathbf{N_3}$-deductions (presented as proof trees say) to $\mathbf{G_2}$-deductions, by the method already given in [Pra65].

On the other hand, if we consider formulations with *sequences* (instead of multisets) of formulas in the antecedent, it is possible to write down a system $\mathbf{N_4}$ in 1-1-correspondence with $\mathbf{N_1}$, without using term labels. For $\mathbf{N_4}$ we can take for example:

$$\frac{\Gamma, A, A \Rightarrow B}{\Gamma, A \Rightarrow B} \qquad \frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, B, A, \Delta \Rightarrow C} \qquad \frac{\Gamma \Rightarrow B}{\Gamma, A \Rightarrow B}$$

$$A \Rightarrow A \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow (A \to B)} \qquad \frac{\Gamma \Rightarrow A \to B \qquad \Delta \Rightarrow B}{\Gamma\Delta \Rightarrow B}$$

We can unambiguously assign terms to deductions in this system (modulo the naming of variables) since each position in the antecedent of a sequent corresponds to a definite variable, and by explicit weakening, contraction and permutation we can keep track of those variables.

If we do not absorb weakening in the axioms, the following variant $\mathbf{N_5}$ of $\mathbf{N_2}$ suggests itself.

$$A \Rightarrow A \qquad \frac{\Gamma, A, A \Rightarrow B}{\Gamma, A \Rightarrow B} \qquad \frac{\Gamma \Rightarrow B}{\Gamma, A \Rightarrow B}$$

$$\frac{\Gamma, [A] \Rightarrow B}{\Gamma \Rightarrow (A \to B)} \qquad \frac{\Gamma \Rightarrow A \to B \qquad \Delta \Rightarrow A}{\Gamma, \Delta \Rightarrow B}$$

Here $[A]$ means that A may be absent or present. We can now distinguish between introduction of $\to$ by weakening and essential $\to$I, and thus we have two distinct deductions of $A \to (A \to A)$:

$$\cfrac{\cfrac{A \Rightarrow A}{A \Rightarrow A \to A}}{\Rightarrow A \to (A \to A)} \qquad\qquad \cfrac{\cfrac{A \Rightarrow A}{\Rightarrow A \to A}}{\Rightarrow A \to (A \to A)}$$

But nevertheless, in the absence of labeling, the one-to-one correspondence with typed terms is lost. Consider for example:

$$\cfrac{\cfrac{\cfrac{A \to A^{w} \qquad \cfrac{A \to A^{v} \qquad \cfrac{A \to A^{u} \qquad A^{x}}{A}}{A}}{A}}{(A \to A) \to A}\, u \text{ or } v \text{ or } w \qquad \cfrac{(A{\to}A){\to}(A{\to}A)^{z} \qquad A \to A^{y}}{A \to A}}{A}$$

If we rewrite this proof according to the system $\mathbf{N_5}$, and want to remove the cut formula, we have to decide at the $\to$E-inference at the bottom whether we are going to identify assumption y with u, v or w. Each decision leads to a distinct result when the cut is contracted. In $\mathbf{N_5}$ these differences are lost.

For $\mathbf{N_5}$ we may define normalization in the same way as for $\mathbf{N_2}$; I do not know whether strong normalization holds for $\mathbf{N_5}$.

3 Normal forms in G-systems

In this section we want to demonstrate that the correspondence between normal natural deductions and *normal* Gentzen deductions, first mentioned in [How80], and discussed at some length in [Min96], can also be established in a fairly direct way for Gentzen systems with structural rules absorbed, and moreover also holds in a straightforward way between suitable Gentzen systems without term labels on the one hand, and natural deduction under the complete discharge convention on the other hand. The proofs are simple adaptations of proofs in the literature.

3.1. DEFINITION. A deduction in $\mathbf{G_2}$ is *normal* if the antecedent active formula of an L$\to$-application is itself principal formula (of an axiom or of another L$\to$-application). A branch in a deduction of $\mathbf{G_2}$ is *normal* if in the branch this condition is met.

Let us write $\mathcal{D} \vdash_n \Gamma \Rightarrow A$ if the sequent $\Gamma \Rightarrow A$ is derivable by a deduction $\mathcal{D}$ of size $\leq n$, where the *size* of a deduction is the number of its nodes. □

To make life slightly easier, we shall below restrict attention to proofs in $\mathbf{G_2}$ with the active formulas in axioms atomic. The following lemma holds for this system:

LEMMA. *In $\mathbf{G_2}$, with active formulas in axioms atomic, the operations of the inversion lemma, weakening and contraction respect normality. More precisely, there are operations ϕ_0, ϕ_1, ϕ_2 such that*

(i) If $\mathcal{D} \vdash_n \Gamma, A \to B \Rightarrow C$ then $\phi_0(\mathcal{D}) \vdash_n \Gamma, B \Rightarrow C$;

(ii) If $\mathcal{D} \vdash_n \Gamma \Rightarrow C$, then $\phi_1(\mathcal{D}) \vdash_n \Gamma\Delta \Rightarrow C$;

(iii) If $\mathcal{D} \vdash_n \Gamma, A, A \Rightarrow C$, then $\phi_2(\mathcal{D}) \vdash_n \Gamma, A \Rightarrow C$.

The ϕ_i do not increase the number of violations of the normality condition. In particular they preserve normality.

For what follows, we need not so much the lemma as stated, but rather the definitions of the operations ϕ_0, ϕ_1, ϕ_2. So let us briefly review these constructions from the proofs of 3.4.3, 3.4.4 (vii), 3.4.5 in [TS96].

Ad (i). Given a proof $\mathcal{D}$ of $\Gamma, A \to B \Rightarrow C$, one replaces, beginning at the bottom node (conclusion) of $\mathcal{D}$, occurrences of $A \to B$ in the antecedent by B, going upwards until one arrives at an L$\to$-application where $A \to B$ is introduced:

$$\frac{\Gamma', A{\to}B \Rightarrow A \qquad \Gamma', B \Rightarrow C'}{\Gamma', A{\to}B \Rightarrow C'}$$

Here one simply drops the subdeduction of $\Gamma', A{\to}B \Rightarrow A$ and the L$\to$-application shown and leave the subdeduction on the right unchanged.

Ad (ii). This is a completely trivial transformation.

Ad (iii). The construction is defined by recursion on the depth of trees. We consider the most interesting step. Suppose we have a deduction $\mathcal{D}$ of $\Gamma, A, A \Rightarrow C$ terminating in

$$\frac{\begin{matrix}\mathcal{D}_0 \\ \Gamma, B{\to}D, B{\to}D \Rightarrow B\end{matrix} \qquad \begin{matrix}\mathcal{D}_1 \\ \Gamma, B{\to}D, D \Rightarrow C\end{matrix}}{\Gamma, B{\to}D, B{\to}D \Rightarrow C}$$

where $A \equiv B \to D$. We may apply ϕ_2 to $\mathcal{D}_0$ to obtain a proof of $\Gamma, B{\to}D \Rightarrow B$; for $\mathcal{D}_1$ we first appeal to the construction ϕ_0 to find a proof of $\Gamma, D, D \Rightarrow C$; then we apply ϕ_2 to obtain a proof of $\Gamma, D \Rightarrow C$.

3.2. THEOREM. *Let $\mathcal{D}$ be any proof of $\Gamma \Rightarrow A$ in $\mathbf{G_2}$; then by permuting applications of* R$\to$ *and* L$\to$ *downwards over* L$\to$*-applications we can obtain a normal proof $\mathcal{D}^n$ of $\Gamma \Rightarrow A$ in $\mathbf{G_2}$.*

PROOF. Permutation of R$\to$ downwards over L$\to$ transforms a (sub)deduction

$$\frac{\begin{matrix}\mathcal{D}_0 \\ \Gamma, A{\to}B \Rightarrow A\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma, B, C_1 \Rightarrow C_2\end{matrix}}{\Gamma, B \Rightarrow C_1{\to}C_2}\,\mathrm{R}{\to}}{\Gamma, A{\to}B \Rightarrow C_1{\to}C_2}\,\mathrm{L}{\to}$$

into

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_0[C_1] & \mathcal{D}_1\\ \Gamma, C_1, A{\to}B \Rightarrow A & \Gamma, B, C_1 \Rightarrow C_2\end{array}}{\Gamma, C_1, A{\to}B \Rightarrow C_2}\,\mathrm{L}{\to}}{\Gamma, A{\to}B, \Rightarrow C_1{\to}C_2}\,\mathrm{R}{\to}$$

Here $\mathcal{D}_0[C_1]$ is the deduction obtained from $\mathcal{D}_0$ by weakening with C_1 at every node, i.e. every sequent $\Delta \Rightarrow D$ is replaced by $\Delta, C_1 \Rightarrow D$. The new deduction has the same size as the original one.

Clearly, after finitely many permutations applied to a deduction $\mathcal{D}$ we have found a deduction $\mathcal{D}^*$ where no right premise of L→ is a conclusion of R→. Then a deduction ending with L→ is of the following form

$$\dfrac{\mathcal{D}_1 \quad \Gamma_0 \Rightarrow C}{\Gamma_1 \Rightarrow C}\,\mathrm{L}{\to}$$

$$\vdots$$

$$\dfrac{\mathcal{D}_n \qquad \dfrac{\mathcal{D}_{n-1} \qquad \Gamma_{n-2} \Rightarrow C}{\Gamma_{n-1} \Rightarrow C}\,\mathrm{L}{\to}}{\Gamma_n \Rightarrow C}\,\mathrm{L}{\to}$$

where $\Gamma_0 \Rightarrow C$ is an axiom.

It is possible to permute L→ applications as follows. If $\mathcal{D}'$ is a deduction of the following shape ($AB, A'B'$ abbreviating $A \to B, A' \to B'$ respectively):

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ A'B', AB, \Gamma \Rightarrow A\end{array} \qquad \dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ A'B', B, \Gamma \Rightarrow A' & B, B', \Gamma \Rightarrow C\end{array}}{A'B', B, \Gamma \Rightarrow C}}{A'B', AB, \Gamma \Rightarrow C}$$

We can transform this into a deduction $\mathcal{D}^{**}$ ($\mathcal{D}[\Gamma]$ denoting the deduction obtained from $\mathcal{D}$ by weakening with the multiset Γ at all nodes, as before, and $2A'B'$ stands for two copies of $A'B'$)

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_0[A'B'] & \mathcal{D}_1[A'B']\\ 2A'B', AB, \Gamma \Rightarrow A & 2A'B', B, \Gamma \Rightarrow A'\end{array}}{2A'B', AB, \Gamma \Rightarrow A'} \qquad \dfrac{\begin{array}{cc}\mathcal{D}_0[B'] & \mathcal{D}_2[A'B']\\ B', A'B', AB, \Gamma \Rightarrow A & B, B', A'B', \Gamma \Rightarrow C\end{array}}{B', A'B', AB, \Gamma \Rightarrow C}}{2A'B', AB, \Gamma \Rightarrow C}$$

and then applying the contraction operation. Call the result $\mathcal{D}^*$. Note that the size of the subdeductions is less than the size of $\mathcal{D}'$. We restrict now attention to deductions in which R→ has already been permuted downwards over L→ as far as possible.

Suppose now that in such a proof we encounter an application of L→

$$\dfrac{\Gamma, AB \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, AB \Rightarrow C}$$

where the active occurrence of B in the premise on the right is not principal, but derives from a non-active occurrence B in the axiom at the top of the rightmost branch above the right-hand premise. That is to say, there is a sequence of occurrences of B, from the non-active occurrence in the antecedent of the axiom at the top, down to the active antecedent occurrence in the right-hand premise of the application of L$\to$ considered. Such an application of L$\to$ we call *redundant*.

If $\mathcal{D}'$ is a deduction with a *single final* redundant L$\to$-inference, we can apply the permutation of L$\to$ over L$\to$ as shown above, resulting in a deduction $\mathcal{D}^{**}$ (as above), followed by the contraction operation. This introduces no new redundancies. In the resulting $\mathcal{D}^{*}$ the immediate subdeductions have, after permuting R$\to$ downwards when necessary, at most one final redundant L$\to$-application and are of smaller size, so we can apply induction on the size. If the offending L$\to$ occurs directly below an axiom:

$$\dfrac{\begin{matrix} \cdots \\ \Gamma', AB, C \Rightarrow A \end{matrix} \qquad \Gamma', B, C \Rightarrow C \text{ (axiom)}}{\Gamma', AB, C \Rightarrow C}$$

this subdeduction may be replaced by the axiom $\Gamma', AB, C \Rightarrow C$. (The offending L$\to$ is therefore just pushed upwards along the rightmost branch until it occurs directly below an axiom; then it disappears.) After eliminating redundant L$\to$-applications in this way, and with all R$\to$-applications pushed downwards as far as they will go, normality holds.

To see this, it suffices to consider a rightmost branch; if we start from the axiom at the top, with principal formula C, we see that the first L$\to$ must have C as active antecedent formula, resulting in, say, $\Gamma, AC \Rightarrow C$; at the next L$\to$, AC must be active, since choice of an active formula in Γ would result in a redundant L$\to$-application, etc. etc. $\square$

We may now streamline a normal deduction still further by the following operations:

(i) apply the contraction lemma, preserving normality, until the conclusion is of the form $\Gamma \Rightarrow A$ with Γ a set, i.e. each formula in Γ occurs with multiplicity 1;

(ii) remove global weakening throughout the deduction, i.e. delete in the antecedents of sequents, throughout the deduction, occurrences of a formula A, provided (a) A occurs in the antecedent of every sequent, and (b) is not active in any of the axioms (another occurrence of A may be counted as active in the axioms, perhaps).

Let us call such a deduction *pruned* (Mints' terminology, but with different application). Then we have the following:

PROPOSITION. *There is a one-to-one correspondence between pruned deductions in* $\mathbf{G_2}$ *(with axioms restricted to atomic active formulas) and deductions in standard natural deduction (i.e. implicational* **Ni** *of [TS96] restricted to deductions under the CDC) under the complete discharge convention in long normal form.*

PROOF. For the notion of "long normal form" see section 6.5 of [TS96]. In a long normal form, the minimal formulas along a branch in the natural deduction proof are always atomic. The proof is straightforward, cf. the argument in [Min96]. It will suffice to describe the construction of a $\phi(\mathcal{D})$ in long normal form from a pruned normal deduction in $\mathbf{G_2}$. This can be done by induction on the size (number of nodes) of $\mathcal{D}$. In applications of $\to$E, the major premise is assumed to be the left one.

If $\mathcal{D}$ ends with R$\to$, we transform according to

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ A,\Gamma\Rightarrow B\end{array}}{\Gamma\Rightarrow A\to B}\qquad\text{is mapped to}\qquad\dfrac{\begin{array}{c}\phi(\mathcal{D}_0)\\ B\end{array}}{A\to B}$$

If a normal $\mathcal{D}_1$ ends with conclusion $\Gamma, B \Rightarrow C$ and B is the active formula of the conclusion, then we find at the top of the *leftmost* branch of $\phi(\mathcal{D}_1)$ the formula B; along this branch there are only $\to$E-applications.

If $\mathcal{D}$ ends with L$\to$, with conclusion $\Gamma, A \to B \Rightarrow C$, and $\mathcal{D}_1$ is the right subdeduction with conclusion $\Gamma, B \Rightarrow C$ and B the active formula, and $\mathcal{D}_0$ the left subdeduction with conclusion $\Gamma, A \to B \Rightarrow A$, we obtain $\phi(\mathcal{D})$ by replacing the top occurrence B in the leftmost branch of $\phi(\mathcal{D}_1)$ by

$$\dfrac{A\to B\qquad\begin{array}{c}\phi(\mathcal{D}_0)\\ A\end{array}}{B}$$

Obviously, the resulting proof is in long normal form. □

Remark. This result is similar to the result on the correspondence between normal Gentzen deductions and normal natural deductions sketched in [Min96]. However, here no term labels are involved, and the use of a system with built-in contraction saves on the proof (in the case of [Min96], the proof given there needs to be supplemented with a discussion of contraction).

If we drop the pruning step (ii) we obtain a one-to-one correspondence with $\mathbf{N_2}$ or $\mathbf{N_3}$.

[Her95] describes a simple transformation of LJ-proofs into proofs of his LJT + Cut (where LJT corresponds to system $\mathbf{G_5}$ defined below) which however introduces cuts. Combined with cut elimination for LJT, one of course also obtains a transformation of LJ-proofs into (cutfree) LJT.

[Sch97] proves strong normalization for permutation of inferences in an intuitionistic Gentzen system, with deductions represented as terms, and without the operators $\vee, \exists$ (although the results may be extended to include these).

3.3. Definition. Let $\mathbf{G_5}$ be the following system with at most one privileged formula appearing into the antecedent to the right of $|$.

$$\text{Ax}\ \ \vec{y}{:}\,\Gamma \mid x{:}\,A \Rightarrow x{:}\,A \qquad \text{C}\ \dfrac{\vec{x}{:}\,\Gamma, y{:}\,A \mid z{:}\,A \Rightarrow t{:}\,B}{\vec{x}{:}\,\Gamma, y{:}\,A \mid\ \Rightarrow t{:}\,B}$$

$$\text{L}\!\to \dfrac{\vec{x}{:}\,\Gamma \mid\ \Rightarrow t{:}\,A \qquad \vec{x}{:}\,\Gamma \mid y{:}\,B \Rightarrow s(y){:}\,C}{\vec{x}{:}\,\Gamma \mid z{:}\,A \to B \Rightarrow s(zt){:}\,C} \qquad \text{R}\!\to \dfrac{\vec{x}{:}\,\Gamma, y{:}\,A \mid\ \Rightarrow t{:}\,B}{\vec{x}{:}\,\Gamma \mid\ \Rightarrow \lambda y.t{:}\,A \to B}$$

C is called the contraction rule. The system $\mathbf{G_4}$ is obtained from $\mathbf{G_5}$ by dropping the term labels:

$$\Gamma \mid A \Rightarrow A \qquad \dfrac{\Gamma, A \mid A \Rightarrow B}{\Gamma, A \mid\ \Rightarrow B} \qquad \dfrac{\Gamma \mid\ \Rightarrow A \qquad \Gamma \mid B \Rightarrow C}{\Gamma \mid A \to B \Rightarrow C} \qquad \dfrac{\Gamma, A \mid\ \Rightarrow B}{\Gamma \mid\ \Rightarrow A \to B}$$

□

3.4. PROPOSITION. *Deductions of* $\Gamma \mid\; \Rightarrow A$ *in* $\mathbf{G_4}$ *correspond one-to-one with normal deductions of* $\Gamma \Rightarrow A$ *in* $\mathbf{N_2}$ *or* $\mathbf{N_3}$.

PROOF. Basically, the proof is nothing but a verification that the argument of [Her95] also applies in the absence of term labels. Given a normal proof $\mathcal{D}$ in $\mathbf{N_2}$, we construct the corresponding proof $\mathcal{D}^*$ in $\mathbf{G_4}$ by induction on the size of $\mathcal{D}$.
Case 1. $\mathcal{D}$ is an axiom $\Gamma, A \Rightarrow A$. Take

$$\mathcal{D}^* \equiv \frac{A, \Gamma \mid A \Rightarrow A}{A, \Gamma \mid\; \Rightarrow A}$$

Case 2. $\mathcal{D}$ ends with →I, as on the left; $\mathcal{D}^*$ is shown on the right.

$$\begin{array}{c}\mathcal{D}_0\\ \Gamma, A \Rightarrow B \\ \hline \Gamma \Rightarrow A \to B\end{array} \qquad\qquad \begin{array}{c}\mathcal{D}_0^*\\ \Gamma, A \mid\; \Rightarrow B\\ \hline \Gamma \mid\; \Rightarrow A \to B\end{array}$$

Case 3. $\mathcal{D}$ ends with →E. Then $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\Gamma \Rightarrow B_1 \to \ldots \to B_m \to C \qquad \begin{matrix}\mathcal{D}_1\\ \Gamma \Rightarrow B_1\end{matrix}}{\Gamma \Rightarrow B_2 \to \ldots \to C} \qquad \begin{matrix}\mathcal{D}_2\\ \Gamma \Rightarrow B_2\end{matrix}}{\begin{matrix}\vdots\\ \dfrac{\Gamma \Rightarrow B_m \to C \qquad\qquad \begin{matrix}\mathcal{D}_m\\ \Gamma \Rightarrow B_m\end{matrix}}{\Gamma \Rightarrow C}\end{matrix}}$$

where the leftmost top sequent is an axiom, so $\Gamma \equiv \Gamma', B_1 \to \ldots \to B_n \to C$. By IH we have proofs $\mathcal{D}_i^*$ with conclusion $\Gamma \mid\; \Rightarrow B_i$ and hence we can take for $\mathcal{D}^*$

$$\dfrac{\begin{matrix}\dfrac{\Gamma \mid\; \Rightarrow B_{m-1} \qquad \dfrac{\Gamma \mid\; \Rightarrow B_m \qquad \Gamma \mid C \Rightarrow C}{\Gamma \mid B_m \to C \Rightarrow C}}{\Gamma \mid B_{m-1} \to B_m \to C \Rightarrow C}\\ \vdots\\ \Gamma \mid B_1 \to \ldots \to B_m \to C \Rightarrow C\end{matrix}}{\Gamma \mid\; \Rightarrow C}\,\mathrm{C}$$

It is easy to construct the inverse mapping $^+$ from proofs $\mathcal{D}$ of $\Gamma \mid\; \Rightarrow A$ in $\mathbf{G_4}$ to proofs $\mathcal{D}^+$ of $\Gamma \Rightarrow A$ in $\mathbf{N_2}$; note that $\mathcal{D}$ must end with R→ or with C preceded by an axiom or a sequence of L→-inferences. □

An alternative method of proof is the construction of a correspondence between $\mathbf{G_4}$-proofs and normal $\mathbf{G_2}$-proofs.

References

[Gen35] G. Gentzen. Untersuchungen über das logische Schliessen I, II. *Mathematische Zeitschrift*, 39:176–210, 405–431, 1935. Translation in *[Gen69]*, 68–131.

[Gen69] G. Gentzen. *The Collected Papers of Gerhard Gentzen.* North-Holland Publ. Co., Amsterdam, 1969. English translation of Gentzen's papers, edited and introduced by M. E. Szabo.

[Her95] H. Herbelin. A λ-calculus structure isomorphic to Gentzen-style sequent calculus structure. In L. Pacholski and J. Tiuryn, editors, *Computer Science Logic. 8th Workshop, CSL'94. Kazimierz, Poland, September 1994*, Lecture Notes in Computer Science 933, pages 61–75. Springer-Verlag, Berlin, Heidelberg, New York, 1995.

[How80] W. A. Howard. The formulae-as-types notion of construction. In *[SH80]*, pages 480–490, 1980. Circulated as preprint since 1969.

[Jás34] S. Jáskowski. On the rules of supposition in formal logic (Polish). *Studia Logica (old series)*, 1:5–32, 1934. Translation *Polish Logic 1920–39*, S. McCall, ed., Clarendon Press, Oxford, 1967, 232–258.

[Lei79] D. Leivant. Assumption classes in natural deduction. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 25:1–4, 1979.

[Min96] G. E. Mints. Normal forms for sequent derivations. In P. Odifreddi, editor, *Kreiseliana*, pages 469–492. A. K. Peters, Wellesley, Mass., 1996.

[Pra65] D. Prawitz. *Natural deduction. A Proof-theoretical Study.* Almquist and Wiksell, Stockholm, 1965.

[Sch97] H. Schwichtenberg. Termination of permutative conversions in intuitionistic Gentzen calculi, 1997. Draft from June 2, 1997, 11 pages.

[SH80] J. P. Seldin and J. R. Hindley, editors. *To H. B. Curry: Essays on Combinatory logic, Lambda Calculus and Formalism.* Academic Press, New York, 1980.

[TS96] A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory.* Cambridge University Press, Cambridge U.K., 1996.