Concepts and axioms

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Abstract

The paper discusses the transition from informal concepts to mathematically precise notions; examples are given, and in some detail the case of lawless sequences, a concept of intuitionistic mathematics, is discussed. A final section comments on philosophical discussions concerning intuitionistic logic in connection with a "theory of meaning".

What I have to tell here is not a new story, and it does not contain any really new ideas. The main difference with my earlier discussions of the same topics ([TD88, chapter16],[Tro91]) is in the emphasis. This paper starts with some examples of the transition from informal concepts to mathematically precise notions, followed by a more detailed discussion of one of these examples, the intuitionistic notion of a choice sequence, arguing for the lasting interest of this notion for the philosophy of mathematics.

In a final section, I describe my own position relative to some of the philosophical discussions concerning intuitionistic logic in the writings of Dummett, George, Prawitz and Wright. The discussion in the earlier sections provides illustrations to some of the points made in this section.

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Mathematical concepts

How does mathematics arrive at its concepts, and discover the principles holding for those concepts? This is a typically philosophical question, more easily posed than answered. A procedure which certainly has played a role and still plays a role might be described as *informally rigorous analysis of a concept.*

That is to say,

- given an informally described, but intuitively clear concept,

- one analyzes the concept as carefully as possible, and attempts to formulate formally precise principles characterizing the concept to a greater or lesser extent.

This is full of vague words such as "intuitively clear", "informal", etc.; the worst case of vagueness is the word "characterizing", as we shall see. Instead of plunging myself deeper into the morass of vagueness and imprecision by trying to explain in general terms what I mean by these words, I shall give some examples below.

But before doing this, let me point out that usually an informal concept serving as basis for such an analysis is not found in nature but suggests itself only on the basis of a good deal of *mathematical* experience.

Also it is good to point out that many mathematicians, in the course of their work, are never called upon to make such a concept analysis, since there is always a vast amount of work to be done based on sharply delimited, well-understood mathematical notions. Nevertheless, the study of mathematical-concept analysis is of obvious interest for the philosophy and history of mathematics.

Examples

Example 1. The notion of *natural number*, formally characterized by induction. Through over-familiarity this example may look deceptively simple to us. But although natural numbers are historically at the basis of mathematics, it seems that the principle of induction was not explicitly formulated before the 17th century. The analysis of the concept culminated in the work of Dedekind and Peano, formally characterizing them by the validity of induction in full generality.

Classical as well as intuitionist mathematicians agree in regarding the natural numbers as a well-understood, unproblematic notion. Nevertheless it is possible to raise some doubts.

Full second-order induction characterizes the natural numbers in the sense of categoricity. On the other hand, full second-order induction does not provide us with a simple axiomatic characterization of the natural numbers, since the incompleteness theorem remains in force.

From a classical predicativistic viewpoint, E. Nelson ([Nel86]) attacked full first-order induction, arguing that we have no good reason to assume that the tally-numbers are closed under exponentiation; a point made before in a somewhat different way by Parikh ([Par71]).

Within a constructivistic setting one can also criticize unrestrained induction. Already with the acceptation of full first-order induction in intuitionistic mathematics the idealization, or if you want schematization sets in. For, on the one hand, one pictures natural numbers as obtained by successively adjoining units (counting or tallying); on the other hand, admitting exponentiation as an everywhere defined function, we can write down $100^{100^{100}}$ which ought to be a natural number if exponentiation is always defined, but which certainly cannot be generated as a tally-number by any human being. So the mathematician creating intuitionistic mathematics is an idealized mathematician, a point Brouwer was certainly aware of, but did not choose to emphasize.

At this point the actualist (or ultra-finitist, strict finitist) attacks intuitionism: an actualist wants to restrict attention to numbers which can actually be counted as tally-numbers by humans. Hence the actualist may discuss $100^{100^{100}}$, perhaps even regard it as having some kind of meaning, but certainly it does not represent a natural number to him. One might say that the actualist has a different concept of natural number[1].

The difficulty with the actualist viewpoint is a kind of sorites-paradox (called "Wang's paradox" in [Dum75]): if n can be tallied, so can $n+1$, hence by induction all numbers can be tallied; so there is no least number which cannot be tallied, though intuitively it looks as if there ought to be at least some such number. This is not a formal contradiction, but an indication of the conceptual difficulty of a consistent adherence to the actualist point of view. Since we do not want $100^{100^{100}}$ to be countable, exponentiation may be definable, but should not be provably total. (Dummett, in [Dum75], discusses Wang's paradox at length and concludes that actualism is untenable, since incoherent.) As an approximation to the idea of an actualist arithmetic one might restrict attention to polynomial-time computable functions. Still this is a schematization of the actualist position: it is quite conceivable that already certain polynomials should be regarded as too fast-growing.

Example 2. The ε–δ–definition of *continuity of functions* may be obtained by reflecting on the idea of a smooth curve, a curve which makes no "jumps". The mathematical definition approximates our idea of a smooth function, but cannot be said to be co-extensive with it. The idea of a "smooth" function contains elements of differentiability, and the possibility of a continuous, nowhere differentiable function is not obviously falling under the naive idea of a smooth function.

Example 3. The notion of *area* for a wide class of point-sets in the plane. A few intuitively obvious properties for the notion of area lead to a complete and unique characterization, modulo the theory of the reals; and this applies in a classical as well as an intuitionistic setting. ("Area" should conform to the standard euclidean definition of the area of a rectangle, should be additive, monotone and take only positive values.)

[1] The actualist criticism of the classical and intuitionistic concept of natural number has an old history, going back to Mannoury ([Man09]) at least.

Example 4. Another nice example taken from analysis is the notion of a *distribution*, where the raw intuitive idea of certain functions with infinite values at certain points (delta-functions) can be made precise by treating such functions as *functionals* operating on other functions.

Example 5. The notion of *function calculable by a human being following a fixed routine* (short: humanly computable (partial) function), or *algorithm*, has been analyzed by Turing and leads to the mathematically precise notion of a recursive function (= Turing-computable function, λ-definable function, etc.). A detailed discussion of Turing's analysis is in [Gan88, Sie94]. A similar analysis is possible for the notion of *mechanically computable function*, leading to the same formal concept ([Gan80]).

An objection one might raise against Turing's analysis, is that it relies on computability "in principle", that is to say, limitations on memory size are disregarded. However, Turing's analysis is perfectly adequate for what he set out to do: to find a mathematically precise notion, modelling the notion of humanly computable function, such that it would at least embrace all humanly computable functions (possibly more), and to use this notion to show the existence of non-computable functions. See [Par96].

Example 6. The notions of a *random sequence* and *independency* of random sequences. An analysis in our sense (not completely satisfactory) of this concept has been carried out by the probability theorist R. von Mises ([Mis36]; in recent years M. van Lambalgen has published on this ([Lam87, Lam90, Lam92]. I shall not discuss any details here, since this would certainly carry us too far. Van Lambalgen argues that even if we can develop probability theory on an axiomatic basis, starting from the notion of a probability measure, instead of starting from the notion of random sequence, we still have to face the question "what is a random sequence" when confronted with the applications of probability theory.

This example shows some similarities with our next example:

Example 7. Choice sequences. The details of this example are reserved for the next section.

Example 8. It is debatable in what sense the *infinitesimals* in analysis represent an example; see the discussion at the end of [Rob66]. Appropriate use of infinitesimals is justified in modern non-standard analysis; but it is to be noted that there is no *standard* model for analysis with infinitesimals.

For contrast, let us briefly discuss a "non-example": the notion of "set", as it figures in the modern practice of axiomatic set theory. Basis for my discussion are some papers by Penelope Maddy ([Mad88, Mad93]) and Michael Hallett's book ([Hal84]).

The justifications for axioms of set theory are of two kinds, in Maddy's terminology "intrinsic" and "extrinsic" ones. The intrinsic arguments argue from the intuitive understanding of the concept of set; the extrinsic (pragmatic, heuristic) arguments are based on, e.g., the desirability of certain mathematical consequences, intertheoretic consequences, explanatory power. In the use of extrinsic justifications there is an analogy with the natural sciences: experimenting with the consequences helps us to choose the "right" axioms. Extrinsic arguments are brought forward, not only for fancy axioms stating the existence of certain very large cardinals, but also for the basic axioms of the well-known Zermelo–Fraenkel axiomatization. This happens even for an axiom looking so uncontroversial as the extensionality axiom; most authors regard it as intrinsically justified, but some give also extrinsic reasons: the extensional notion of set is simpler, clearer and more convenient than its non-extensional ("intensional") counterpart (sets as properties). Clearly, "external" justifications have little to do with concept analysis.

The axioms of Zermelo-Fraenkel set theory, must be regarded as basic in determining what sets are. Several authors have attempted to justify the Zermelo–Fraenkel axioms by analyzing the intuitive picture of the cumulative hierarchy, hence by giving "intrinsic" justifications. As argued at length by Hallett ([Hal84, 5.3, 6.1]) and Maddy, these "intrinsic" justifications encounter certain definite obstacles, in particular in connection with the power-set axiom. As Hallett puts it (page 221) "These presentations of the iterative conception of set go wrong for much the same reason that the limitation of size argument (in its original form) goes wrong. Namely, because they attempt to explain the power-set axiom by reference to concepts which do not at all capture, and indeed contradict, its nature." Therefore these justifications of ZF cannot be taken as convincing examples of concept analysis.

From the examples, and the non-example above, it will be clear that on the one hand concept analysis has played, and may be expected to continue playing, an important role in the development of mathematics, but on the other hand that it is by no means the only way of introducing new concepts.

Choice sequences

Choice sequences were introduced by L.E.J. Brouwer around 1916, in order to give an intuitively more satisfactory account (than hitherto) of the continuum in the context of his intuitionistic mathematics ([Tro82]). Before that time, Brouwer, just like the french "empiristes" such as E. Borel, had viewed the continuum as a primitive concept which could not be understood as the "set of its elements". The introduction of choice sequences restored this possibility (of having an "arithmetic" theory of the continuum, in contemporary terminology).

A very interesting aspect of Brouwer's notion was that choice sequences, when taken seriously as mathematical objects, *enforced* the use of intuitionistic logic, since some of the principles valid for choice sequences contradicted classical logic. Thus they provided an alternative mathematical universe with a deviant logic. Nowadays this is commonplace, in particular in topos theory, but it was certainly not commonplace in Brouwer's time. The deviant logic is connected with a particular domain; it is quite possible to have intuitionistic logic for properties containing choice sequences as parameters, while adopting classical logic, or assuming "Church's thesis", for all properties not depending on choice parameters.

Brouwer's notion of choice sequence is of lasting interest for the *philosophy* of mathematics, more[2] than for mathematics. Here I want to look at a special aspect of choice sequences, namely their interest as an important "case-study" in the philosophy of mathematics.

A choice sequence in Brouwer's sense is a sequence of natural numbers (to keep it simple), which is not a priori given by a law or recipe for the elements, but which is created step by step by the ideal mathematician (anyone of us may serve as an approximation of the ideal mathematician); the process will go on indefinitely, but is never finished. Brouwer's discovery was that one could use such "unfinished objects" in a meaningful way in (intuitionistic) mathematics. It is not the individual choice sequence which matters, but the principles which hold for all of them. Since continuous operations on sequences can be carried out by operating on initial segments, choice sequences are closed under continuous operations.

Logicians such as S.C. Kleene ([KV65]) encoded the mathematical practice involving choice sequences, and proved consistency relative to a system in the intersection of intuitionistic and classical mathematics.

Over the years Brouwer repeatedly indicated how one should think of the individual choice sequence, but his detailed views seemed to oscillate.

Some intuitionists are of the opinion that further analysis of the individual choice sequence is unnecessary, nay, even impossible — there is a single concept, and no room for meaningful subuniverses ([GVS81, Swa92]). I call this the *holistic*[3] approach. I follow here another, *analytical* approach, which highlights the intensional aspects of choice sequences.

A property of sequences, or an operation on sequences is said to be

[2]In particular, the work of Bishop ([BB85]) and his school has taught us that constructive mathematics can be developed successfully without choice sequences. Also, results on the elimination of choice sequences in certain formal theories ([TD88]) indicate that choice sequences are not essential. On the other hand, the possibility of discovering areas of application where choice sequences have definite advantages cannot be excluded.

[3]Not to be confused with "holistic" as applied to a theory of meaning, see below!

extensional, if it only depends on the "course–of–values" of the sequence, that is to say, for operations Φ and properties P:

$$\forall x(\alpha x = \beta x) \to \Phi\alpha = \Phi\beta, \text{ and } \forall x(\alpha x = \beta x) \to (P(\alpha) \leftrightarrow P(\beta)).$$

In this paper, intensional properties are simply non-extensional properties.

For example, suppose we consider sequences given by algorithms. The operation which assigns to a sequence its algorithm is non-extensional. Usually this introduction of non-extensionality is avoided by talking about the algorithms themselves, instead of the sequences. But this way of avoiding non-extensionality is not feasible in the case of choice sequences.

On the analytical approach, we try to analyze what it means to say that a choice sequence is "given", and what it means to assert that something holds for all choice sequences. An algorithmic sequence is obviously "given" by describing the algorithm, but how is a choice sequence "given"?

I shall endeavour to illustrate the analytical approach by one example, for a rather extreme variant of choice sequence, the so-called *lawless sequences* ([TD88, chapter 12]).

Informally, we think of a lawless sequence of natural numbers as a process of choosing values in $\mathbf{N}$, started by the ideal mathematician (in the mind of which, in Brouwer's view, all mathematics is constructed); the process is started under the a priori restriction that

- at any stage of the construction never more than an initial segment has been determined, and that *no* restrictions have been imposed on future choices;
- there is a commitment to determine more and more values (so the sequence is infinite).

How does one recognize the choice of a number as a step in the ongoing construction of a particular lawless[4] sequence? Simply by deciding that the value chosen will be part of a particular process we have started before, so to speak by pointing to that process.

If we further reflect on the stipulations for lawless sequences, the following principle seems to be evident.

The principle of open data

[4]The terminology "lawless" is to remind us of the fact that *no* restrictions ("laws") are to be put on future choices at any stage. Instead of "lawless" we might also have used "restriction-free". One does not need to be very bright to realize that "no restrictions" amounts to a restriction or law of a higher order, and this is reflected in some of the principles valid for such sequences. But this seems to be insufficient reason to reject the term "lawless sequences" as a misnomer ([McC96, p.328]).

If P is a property of lawless sequences (α, β will be used for lawless sequences), and $P(\alpha)$, then there is an initial segment $\overline{\alpha}n = \langle \alpha(0), \ldots, \alpha(n-1) \rangle$ such that

$$\forall \beta(\overline{\alpha}n = \overline{\beta}n \to P(\beta)) \text{ or } \forall \beta(\forall m{<}n(\alpha m = \beta m) \to P\beta).$$

The reason is that if we can assert at some stage in our activity $P(\alpha)$ for a lawless sequence α, then the basis for our assertion can only be an initial segment of α, and therefore any lawless sequence starting with the same initial segment ought to have the same property.

Of course, this would not hold if we could identify the various lawless processes by numbers, say, and the property of lawless sequences considered would explicitly refer to those numbers. So there is certainly a linguistic restriction on P involved.

A lawless sequence may be compared to the sequence of casts of a die. There too, at any given stage in the generation of the sequence never more than an initial segment is known. But in discussing random sequences, we are usually interested in another notion of truth: instead of absolute truth, we have "true with probability 1". A random sequence of zero's and one's contains a zero with probability one, but there is no absolute certainty that we will encounter a one before this actually happens.

All this is very nice, but what can we say about the existence of lawless sequences? Certainly we can assume that we can start creating some lawless sequences, even as many as we like, but some experimenting soon reveals that what we need to get a mathematical theory at all, is that all possible initial segments occur:

Density principle

For each initial segment there is a lawless sequence starting with that segment.

In order to guarantee this, we have to modify our notion of lawless sequence a little: at the start, when generating a lawless sequence, we permit the stipulation of a finite initial segment in advance. After that, things go on as before. In the "model" of the casts of a die, this would amount to permitting initially a number of deliberate placings of the die.

However, this modification of our notion weakens the justification for the principle of open data. After this modification it may happen that, for two lawless sequences α, β, at some stage the same initial segment may have been determined, while nevertheless there is a piece of information about these sequences which cannot be read off from these initial segments alone, namely *which* parts of these initial segments had been fixed at the start. So we may initially stipulate that α begins with 2,1 and β with 2,1,4, but in a later stage for both sequences we know the initial segment 2,1,4,2,3 and nothing more. One may well feel that we do not want to distinguish mathematically between these two situations at later stages, or in other words, these differences should not enter in the properties of sequences we are interested in. In short, we want to *abstract* from these distinctions. If we do, we expect "open data" to hold. In this way several more principles may be formulated which yield a formally consistent non-classical theory, which, up to a point, completely characterizes lawless sequences.

In this example, mathematical desiderata led us to introduce a complication into the intuitive picture, and to weaken the link between the informal concept and the formal principles obtained by analyzing the concept. We have weakened the intuitive basis in order to gain in mathematical manageability. Of course this happens elsewhere too.

Similarly, some intuitionists adopt a "holistic" view of the continuum, and maintain that it is neither desirable nor even possible to distinguish subdomains such as the lawless sequences ([GVS81, Swa92]).

However, if one does distinguish subdomains, according to how precisely a sequence is given, there is a wealth of possibilities giving rise to consistent and interesting formalisms. From a mathematical point of view, the main problem is to discover which notions lead to a nice theory.

As examples we mention two possibilities, more or less arbitrarily chosen.

(A) Instead of stipulating, as for lawless sequences, that at every stage future choices are completely unrestricted, we now permit at any stage either to leave future choices completely free, or to restrict henceforth choices to a law (definite recipe), at least if this option has not already earlier been chosen.

(B) We permit at any stage restriction of the continuation of the sequence to lie with a definite (i.e., not depending on choice parameters) tree of possibilities; later restrictions should be compatible with the restrictions chosen before. As an extreme case the choices may be narrowed down to a law, as a single-branch tree.

Lack of space prevents me from discussing the properties of the sequences in these examples — not in themselves important — therefore I only mention the result of the analysis: (A) and (B) have quite distinct properties, which distinguish them from each other and from the lawless sequences.

But I expect my discussion suffices to show that choice sequences provide very interesting examples for experimentation with concept analysis, perhaps precisely because these examples are far less familiar to us than say, "natural number", "area" or "continuous function".

Theory of meaning

Dummett has presented, in a number of publications ([Dum73, Dum76, Dum77]), a theory of meaning, in which the meaning of a statement is taken to be fully manifest in its use, and he argues that such a theory of meaning inevitably leads to intuitionistic, not to classical logic. Dummett's papers on this topic are not easy reading; the most accessible, if not the most complete account may be found in [Dum77]. A readable exposition and discussion of Dummett's view is given in [Pra77]. A thorough critical discussion of Dummett's views is found, a.o., in [Nie97, Gai96]. Gaifman's paper came to my notice only in the very last stage of polishing this paper; it seems to me to contain the best account of Dummett's views to date.

Personally, I am of the opinion that philosophy of mathematics should at all times remain in contact with examples from mathematics. Dummett's top-down approach represents a radically opposite strategy, thereby risking irrelevance in its application to mathematics.

In [Wri82, Geo88] and [Gai96] (and very briefly in [TD88, p.850–851]) Dummett's defense of intuitionistic logic is criticized, on the grounds that Dummett's arguments lead automatically to the acceptance of the the notion of natural number as used in existing intuitionistic mathematics and to the acceptance of computability "in principle". Thereby Dummett becomes vulnerable to criticism by the actualists (usually called ultra-finitists), who maintain that also the intuitionistic concept of natural number, involving as it does acceptance of say $100^{100^{100}}$ as a natural number, is too remote from actual human capabilities; or to put it in another way, why stop at intuitionism, why not interpreting human cognitive powers in the actualist sense?

To this, it might be objected that actualism lacks coherence, as witnessed by the sorites-type paradox mentioned above ([Dum75], cf. also [Gan82]). However, the objections of the actualists to Dummett's theory may be rephrased as follows (cf. [Par96]): even if we have to admit some idealization, some schematization of our actual cognitive powers, why not used a more restricted, refined interpretation. Intuitionism is committed to all the primitive recursive functions as being computable "in principle"; but we might restrict attention to say functions computable in polynomial time, which are commonly regarded as a much better approximation of what is actually computable.

A. George points to the possible role of side-constraints in grasping meanings; e.g. we understand something, say notion X, because the structure of our mind predisposes us to understanding X. Elsewhere, in [Tro91], I mentioned the possibility that meaning of mathematical statements is perhaps not completely molecular, without being all-out holistic: is it really obvious that the continuity axioms for choice sequences, which tell us something about the meaning of $\forall\alpha\exists x$-statements can be understood simply by first explaining $\forall\alpha$ and $\exists x$ separately? H. Gaifman ([Gai96]) presented arguments in favour of a moderately holistic theory of meaning, which to my mind carry conviction.

George also argues at some length that all explanations of the intuitionistic number concept are of necessity elucidatory (that is to say, explanations which rely on the notion to be explained, hence containing some circularity) and suggests that elucidations may nevertheless be informative to the learner. (From a holistic approach as defended by Gaifman, there is certainly nothing strange in elucidations being informative.)

In this connection it is to be noted that Martin-Löf's type theory ([Mar84]) with its strict adherence to the scheme of introduction and elimination rules, at first sight seems to conform to a theory of meaning of the type advocated by Dummett. Nevertheless, in the informal semantics of these type theories (cf. [TD88, 11.5.9–10, 11.7.6]) rules are justified by elucidations: in justifying the elimination rule for N, the type of the natural numbers, we use induction informally, and the justification of the elimination rules for the tree classes requires a (generalized) form of bar induction ([TD88, 11.7.6]).

Let me also point out that if Dummett's account of meaning is the right one, classical mathematics has to get its meaning in an indirect way, e.g. via a translation of classical mathematics into intuitionistic theories. But this certainly is not the way mathematicians understand their mathematics in everyday practice. In this understanding they agree perfectly; difference of opinion sets only in when it comes to admitting fundamentally new principles as as axioms, as in the frontier of axiomatic set theory. How should one picture the relation between a theory of meaning as envisaged by Dummett, and the practical understanding of the classical mathematician?

But if intuitionism cannot be defended along the lines proposed by Dummett, what other kind of justification are we to offer? To me, intuitionistic mathematics may be viewed as a mathematical model of certain informal ideas on constructivity and constructive reasoning, in much the same way that Turing described a mathematical model of "humanly computable" ([Par96]).

However, intuitionism is more complex than calculability by human beings: we do not have a single concept we want to model, we are trying to find models for several concepts which are to serve as constructive analogues of classical notions. Hence we repeatedly have a choice of idealizing assumptions to make in our modeling. A first decision is to accept full induction and to treat natural numbers as closed under all primitive recursive operators. In the discussion of sets ("species" in traditional intuitionistic lingo) we have to face the question whether we want to allow arbitrary impredicative definitions or not? Or perhaps we do accept generalized inductive definitions, but not full comprehension?

And when we come to choice sequences, first we have to face the question, whether they are to be admitted as legitimate at all, and if we answer yes, whether we want to look at them as a notion capable of further analysis, permitting distinctions between various kinds of choice sequences, or as a primitive notion not to be analyzed any further. (Obviously, the existence of many decision-points was realized by Heyting, when he distinguished between "degrees of evidence" for the intuitionistic notions, cf. [Hey62].)

My problems with Dummett's meaning theory are in short the following: I am not quite sure that I have understood it correctly (but after reading Gaifman's paper, I understand it a little better). To the extent that I understand it, I share some of the criticism voiced by George, Gaifman and others. But even when accepting Dummett's conclusions, I do not quite see how this theory helps me to understand the position of intuitionistic mathematics in the whole of mathematics, or how it helps me to understand the role of mathematics in human thinking.

I have no theory of meaning of my own to offer. But I believe that no theory of meaning for mathematics can be satisfactory, which does not give full weight to the schematic, idealizing, "approximative" character of mathematical theories and concepts. Achieving understanding is not a clear-cut, simple process, it is much of the time a tentative, exploring, experimenting type of activity with hazy edges. From a theory of meaning I expect that it somehow reflects this fact of life, at least if one does not want to dissociate *meaning* completely from *understanding*.

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