Canonical Pseudo-Correspondence

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Abstract

Generalizing an example from FINE [1] and inspired by a theorem in JÓNSSON [4], we prove that any modal formula of the form $\pi(p \vee q) \leftrightarrow \pi(p) \vee \pi(q)$ (with $\pi(p)$ a positive formula) is canonical. We also prove that any such formula is strongly sound and complete with respect to an elementary class of frames, definable by a first order formula which can be read off from π.

1 Introduction

For quite a while now, modal logicians have been interested in the relation between first order logic and canonical modal formulas; recall that the latter are formulas that are valid on the underlying frame of the canonical model. Some very interesting connections have been discovered, but there are also some intriguing open problems. Examples of important results are Fine's Theorem (cf. [1]) that the modal logic of an elementary class of frames is canonical, and Sahlqvist's Theorem (cf. [6]) identifying a class of modal formulas each of which is canonical and corresponds to a first order formula which can be read off from the modal formula. Probably the most important open problem in the area is whether any canonical modal logic is quasi-elementary, that is, complete with respect to some elementary frame class.

Besides the mentioned result, Fine's paper also contains a counterexample to the conjecture that canonical formulas are always elementary. Fine showed that the formula

$$(F) \qquad \Diamond\Box p \to \Diamond\Box(p \wedge q) \vee \Diamond\Box(p \wedge \neg q)$$

is canonical, while its class of frames is not even closed under elementary equivalence. Nevertheless, he proved that the logic KF is complete with respect to the class of frames satisfying the formula

$$(F') \qquad \forall xy\,(xRy \to \exists z\,(Rxz \wedge \forall uv\,((Rzu \wedge Rzv) \to (u = v \wedge Ryv)))).$$

A different (algebraic) proof for the canonicity of F was given in JÓNSSON [4]. Given a formula $\varphi(p)$, let $\mathsf{Add}(\varphi)$ ('additivity of φ') be the formula

$$\mathsf{Add}(\varphi) \;\equiv\; \varphi(p \vee q) \leftrightarrow \varphi(p) \vee \varphi(q).$$

It is easy to see that $\mathsf{Add}(\Diamond\Box p)$ is equivalent to F. Now Jónsson proves that if φ is a so-called *stable* formula, then $\mathsf{Add}(\varphi)$ is canonical (Theorem 6.1 in [4]). (We will say more about

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stable formulas in the next section). This is an interesting result which naturally leads to further questions. For instance, is there a syntactically definable class of formulas for which the formula $\mathsf{Add}(\varphi)$ is canonical?And second, are such formulas always pseudo-elementary?

Following Jónsson, in this paper we also look at Fine's example from a more general perspective, but we take the connection with first order logic into account as well. The formula F provides an example of what we will call *canonical pseudo-correspondence.*

Definition 1.1 *Let φ be a modal formula and α a first order sentence. We say that φ and α are canonical pseudo-correspondents if the following hold:*

1. *if $\mathfrak{g}$ is a descriptive general frame for φ, then α holds on the underlying (Kripke) frame of $\mathfrak{g}$,*
2. *if $\mathfrak{F}$ is a frame satisfying α then $\mathfrak{F} \models \varphi$.*

This definition is generalized to sets of formulas in the obvious way.

Observe that it follows immediately from the definition that if φ and α are canonical pseudo-correspondents, then the logic $\mathrm{K}\varphi$ is canonical and strongly sound and complete with respect to the class of frames satisfying α. Another immediate consequence of this definition is that canonical pseudo-correspondents are (ordinary) correspondents over the class of *finite* frames.

The main result of this paper (Theorem 1 below) states that for any *positive* formula $\pi(p)$, the formula $\mathsf{Add}(\pi)$ has a canonical pseudo-correspondent $C(\pi)$ that can be effectively obtained from π (precise definitions will be given in the next section). The basic observation underlying the proof of this result is that the axiom $\mathsf{Add}(\pi)$ forces the existence in the canonical frame of points with certain properties (in the case of F': the 'special' successor z of x).

My original motivation for studying this topic was related to this observation, but came from the opposite direction. Suppose that we are trying to axiomatize an elementary frame class definable by some first order sentences that have no ordinary canonical correspondent. In such a case it would be nice if we could at least find canonical *pseudo*-correspondents. Since our theorem applies in the opposite direction, and only refers to a rather restricted class of modal formulas, it will not be of immediate use for this purpose. For instance, in the second part of the paper we will show that there is no single canonical pseudo-correspondent for the first order formula $\forall x \exists y\, (Rxy \wedge Ryy)$. On the other hand, in this case there is an *infinite* set of modal axioms doing the job (these results follow from HUGHES [3]). In general, we hope that a more general theory of canonical pseudo-correspondence could bring a better understanding of the connection between canonicity and first order logic.

2 Additivity

In this section we state and prove the main result of the paper. Although we work in the basic modal language with one unary diamond $\Diamond$, it is easy to see that all our results carry over to polymodal modal languages, possibly with polyadic operators. We first formulate and prove our result, we then give two examples and we finish with a brief comparison of our result to that of Jónsson.

A formula $\pi(p)$ in this language is *positive* if every occurrence of p is in the scope of an even number of negation symbols; without loss of generality we assume that a positive formula is built up from propositional variables, $\top$ and $\bot$, by applying $\Diamond$, $\Box$, $\wedge$ and $\vee$. For a formula $\pi(p)$, we let $\bar{\pi}$, the *dual* of π, denote the formula that is obtained from π by simultaneously replacing $\vee$ with $\wedge$, $\Diamond$ with $\Box$, and vice versa. Obviously, the dual of a positive formula is positive, and it holds that $\bar{\pi}(p)$ is equivalent to $\neg\pi(\neg p)$.

We assume familiarity with the *standard translation* of a modal formula; $ST_x(\varphi)$ denotes the standard translation of φ, with x as the single free variable. Let α and $\beta(x)$ be formulas in the first order frame language, and P a monadic predicate of the first order model language (corresponding to the propositional variable p); then $\alpha[\lambda v.\beta(v)/P]$ denotes the formula that is obtained from α by replacing each subformula of the form Pu (u some variable) by $\beta(u)$.

For a positive formula $\pi(p)$, $\rho_\pi(x,y)$ denotes the formula

$$\rho_\pi(x,y) \;\equiv\; ST_x(\pi)[\lambda v.y = v/P].$$

Finally, let $C(\pi)$ be the first order sentence

$$C(\pi) \;\equiv\; \forall x\,(ST_x(\pi)[\lambda v.v \neq v/P] \vee ST_x(\bar{\pi})[\lambda v.\rho_\pi(x,v)/P]).$$

The intuition behind these formulas will become clear in the proofs below, examples will be given later on.

We are now ready to formulate the main result of this paper.

Theorem 1 *Let $\pi(p)$ be a positive formula. Then the formula* $\mathsf{Add}(\pi)$ *canonically pseudo-corresponds to $C(\pi)$.*

As a corollary we have that the logic $\mathrm{K}\mathsf{Add}(\pi)$ (that is, the extension of the basic modal logic K with the axiom $\mathsf{Add}(\pi)$), is canonical and strongly sound and complete with respect to the class of frames satisfying $C(\pi)$. Also, it follows that over the class of *finite* frames, the formulas $C(\pi)$ and $\mathsf{Add}(\pi)$ are equivalent.

Note that Theorem 1 only applies to formulas π using *one* variable. It will become clear below that the proof method may be of use as well for formulas in which more variables occur, but I have not been able to formulate a general result in this direction. In particular, I could not prove that additivity of a positive formula $\pi(p,q)$ in *both* p and q is a canonical property.

Theorem 1 follows immediately from the Propositions 2.1 and 2.2 below.

Proposition 2.1 *Let $\pi(p)$ be a positive formula, and let $\mathfrak{g}$ be a descriptive general frame for* $\mathsf{Add}(\pi)$. *Then $C(\pi)$ holds on the underlying (Kripke) frame of $\mathfrak{g}$.*

PROOF. Let $\mathfrak{F} = (W,R)$ be a frame and A a collection of subsets of W such that $\mathfrak{g} = (W,R,A)$ is a general frame on which $\mathsf{Add}(\pi)$ is valid.

It is useful to see formulas in p as operations on the power set of W, defined as follows. For a subset $X \subseteq W$, let V_X denote the valuation given by $V_X(p) = X$. Then we see the modal formula $\varphi(p)$ as the following operation on $\mathcal{P}(W)$:

$$\varphi(X) = \{s \mid \mathfrak{F}, V_X, s \Vdash \varphi(p)\}.$$

(Of course, we could also have given an inductive definition for this operation.)

We define R_π as the extension of the formula ρ_π; that is,

(1) $$R_\pi := \{(s,t) \in W^2 \mid \mathfrak{F} \models \rho_\pi(s,t)\}.$$

It follows immediately from the definitions and the correspondence property of the standard translation on the level of models that

(2) $$\text{for all states } s,t\text{: } R_\pi st \text{ iff } s \in \pi(\{t\}).$$

The basic idea underlying the proof of Proposition 2.1 is to pretend that $\pi(p)$ is in fact a formula of the kind $\Diamond_\pi p$, where $\Diamond_\pi$ is a second diamond in the language, having R_π as its accessibility relation. Note that we cannot derive the formula $\neg\pi(\bot)$ from the axiom $\mathsf{Add}(\pi)$; hence, in general $\Diamond_\pi$ will not be a normal modality. That R_π behaves like an accessibility relation for $\pi(p)$ is reflected by the following:

(3) $$\text{for all states } s,t \text{ in } W\text{: } R_\pi st \text{ iff } s \in \pi(a) \text{ for all admissible } a \text{ with } t \in a.$$

For the direction from left to right, assume that s and t are states such that Rst, and that a is admissible subset with $t \in a$. We have to show that $s \in \pi(a)$. But this is immediate from (2) and the fact that π is positive and hence, monotone: $\pi(a) \subseteq \pi(b)$ whenever $a \subseteq b$.

For the other direction, recall that $\mathfrak{g}$ is a descriptive frame; in particular, we have that singletons are *closed* sets in the induced topology: $t = \bigcap_{t\in a} a$. But then it follows from the Intersection Lemma in Sambin & Vaccaro [7] that

$$\pi(\{t\}) = \bigcap_{t\in a} \pi(a).$$

This proves (3). In accordance with our 'diamond perspective' on $\pi(p)$, we now introduce a new operation on the power set of W:

$$\Diamond_\pi X := \{s \in W \mid R_\pi sx \text{ for some } x \in X\}.$$

The dual operation of $\Diamond_\pi$ is denoted by $\Box_\pi$. Our next observation is that

(4) $$\text{for all } b \in A\text{: } \pi(b) = \pi(\varnothing) \cup \Diamond_\pi(b).$$

(This clearly reveals the diamond character of $\pi(p)$, modulo normality).

The inclusion '$\subseteq$' is proved in a standard way: assume that $s \in \pi(b) \setminus \pi(\varnothing)$; we have to show that $s \in \Diamond_\pi(b)$. In other words, we have to prove the existence of a state t in b such that $R_\pi st$. Define B to be the set $B := \{a \in A \mid s \in \bar{\pi}(a)\}$. From $s \notin \pi(\varnothing)$ it follows that $s \in \bar{\pi}(W)$, so B is not empty. Using the fact that $\bar{\pi}(c \cap d) = \bar{\pi}(c) \cap \bar{\pi}(d)$ it is easy to show that B is closed under taking intersections. We claim that the set $B \cup \{b\}$ has the finite intersection property. For otherwise we may assume that $b \cap a = \varnothing$ for some $a \in A$ with $s \in \bar{\pi}(a)$. But if $b \cap a = \varnothing$, then b is a subset of the complement a^c of a, whence $\pi(b) \subseteq \pi(a^c)$ by monotonicity of π. This gives $\pi(b) \cap \bar{\pi}(a) = \varnothing$, contradicting the fact that $s \in \pi(b) \cap \bar{\pi}(a)$.

Now let U be any ultrafilter (in the Boolean algebra of admissible sets) extending $B \cup \{b\}$. It is immediate that for any admissible a, $a \in U$ implies $s \in \pi(a)$. But since $\mathfrak{g}$ is descriptive there is some t in W with $\{t\} = \bigcap U$. It follows from (3) that $R_\pi st$ and from $b \in U$ that $t \in b$.

For the other inclusion, first assume that $s \in \pi(\varnothing)$. It then follows by monotonicity of π that $s \in \pi(a)$ for *any* set a, and in particular, $s \in \pi(a)$. On the other hand, if there is some state t with $R_\pi st$ and $t \in b$, the result follows from (3). This proves (4).

Now fix some element $s \in W$. It follows from (4) that

$$\text{for all } b \in A\text{: } s \notin \Diamond_\pi b \text{ only if } s \in \pi(\varnothing) \text{ or } s \notin \pi(b).$$

Since A is closed under complementation, by our definition of $\bar{\pi}$ this is equivalent to

$$\text{for all } a \in A\text{: } s \in \Box_\pi a \text{ only if } s \in \pi(\varnothing) \cup \bar{\pi}(a). \tag{5}$$

Let $R_\pi[s]$ denote the set of 'R_π-successors' of s; that is, $R_\pi[s] := \{t \in W \mid R_\pi st\}$. Then (5) can be rewritten to

$$R_\pi[s] \subset a \text{ only if } s \in \pi(\varnothing) \cup \bar{\pi}(a),$$

which in its turn is equivalent to

$$s \in \ \pi(\varnothing) \ \cup \bigcap_{R_\pi[s] \subseteq a} \bar{\pi}(a).$$

Now the crucial observation is that the set $R_\pi[s]$ is *closed* in the topology induced by the general frame: it follows from (3) that $R_\pi[s] = \bigcap_{s \in \bar{\pi}(a)} a$. Hence, by the fact that $\bar{\pi}$ is a positive formula, it follows again from the Intersection Lemma in Sambin & Vaccaro [7] that

$$\bigcap_{R_\pi[s] \subseteq a} \bar{\pi}(a) \ = \ \bar{\pi}(R_\pi[s]).$$

But then, putting the previous two observation together we obtain that

$$s \in \pi(\varnothing) \cup \bar{\pi}(R_\pi[s]). \tag{6}$$

Since s was arbitrary, we have that (6) in fact holds for all s in $\mathfrak{F}$. But this forms an elementary condition which is expressed by the first order formula $C(\pi)$. QED

Proposition 2.2 *Let $\pi(p)$ be a positive formula, and suppose that $\mathfrak{F}$ is a frame satisfying $C(\pi)$. Then $\mathfrak{F} \models \mathsf{Add}(\pi)$.*

Proof. Assume that $\mathfrak{F} \models C(\pi)$. The easiest way to prove this Proposition is to pretend again that $\mathfrak{F}$ is a frame for a language with, besides $\Diamond$, a second diamond $\Diamond_\pi$ having the relation R_π (defined as in (1)) as its accessibility relation. For, observe that the formula $C(\pi)$ is the Sahlqvist correspondent of the formula $\pi(p) \to \pi(\bot) \vee \Diamond_\pi p$ (a short proof for this equivalence goes via the earlier mentioned fact that $C(\pi)$ is equivalent with the condition that (6) holds for all s in $\mathfrak{F}$).

Now assume that for some valuation V and state s, we have $\mathfrak{F}, V, s \Vdash \pi(p \vee q)$. If $\mathfrak{F}, V, s \Vdash \pi(\bot)$ then $\mathfrak{F}, V, s \Vdash \pi(p)$ because of monotonicity. Otherwise, there must be a state t with $R_\pi st$ and $\mathfrak{F}, V, t \Vdash p \vee q$. From $R_\pi st$ it follows, as in (2), that $s \in \pi(\{t\})$, while by the second fact we must have either $\mathfrak{F}, V, t \Vdash p$ or $\mathfrak{F}, V, t \Vdash q$. Now we use monotonicity of π again, in the first case to show that $\mathfrak{F}, V, s \Vdash \pi(p)$, in the second case that $\mathfrak{F}, V, s \Vdash \pi(q)$. QED

Let us now consider two examples.

Example 2.3 Let π be the formula $\Diamond\Box p$. Then $\mathsf{Add}(\pi)$ is equivalent to Fine's formula F. A simple calculation reveals that $\rho_\pi(x,y)$ is the formula

$$\rho_\pi(x,y) \;\equiv\; \exists z_0\,(Rxz_0 \wedge \forall z_1\,(Rz_0z_1 \to z_1 = y)).$$

Since the dual of $\Diamond\Box p$ is the formula $\Box\Diamond p$, some more involved calculations show that the first order condition 'pseudo-corresponding' to $\mathsf{Add}(\pi)$ is the formula

$$\begin{array}{ll}\forall x\,[& \exists x_0\,(Rxx_0 \wedge \forall x_1(Rx_0x_1 \to x_1 \neq x_1)) \vee \\ & \forall x_0\,(Rxx_0 \to \exists x_1(Rx_0x_1 \wedge \rho_\pi(x,x_1))) \quad],\end{array}$$

giving, after replacing $\rho_\pi(x,x_1)$ with its definition and performing some simplifications:

$$\begin{array}{ll}\forall x\,[& \exists x_0\,(Rxx_0 \wedge \neg\exists x_1 Rx_0x_1) \vee \\ & \forall x_0\,(Rxx_0 \to \exists x_1(Rx_0x_1 \wedge \exists z_0\,(Rxz_0 \wedge \forall z_1\,(Rz_0z_1 \to z_1 = x_1)))) \quad],\end{array}$$

We leave it to the reader to verify that this formula is indeed equivalent to the first order formula F' mentioned in the introduction.

Example 2.4 Now let π be the formula $\Box\Diamond p$, whence $\mathsf{Add}(\pi)$ is the formula $\Box\Diamond(p \vee q) \leftrightarrow \Box\Diamond p \vee \Box\Diamond q$, and $\rho_\pi(x,y)$ the first order formula

$$\rho_\pi(x,y) \;\equiv\; \forall z_0\,(Rxz_0 \to \exists z_1\,(Rz_0z_1 \wedge z_1 = y)),$$

or shorter:

$$\rho_\pi(x,y) \;\equiv\; \forall z_0\,(Rxz_0 \to Rz_0y).$$

The dual of π is the formula $\Diamond\Box p$, so the first order pseudo-correspondent of $\mathsf{Add}(\pi)$ is the formula

$$\begin{array}{ll}\forall x\,[& \forall x_0\,(Rxx_0 \to \exists x_1(Rx_0x_1 \wedge x_1 \neq x_1)) \vee \\ & \exists x_0\,(Rxx_0 \wedge \forall x_1(Rx_0x_1 \to \rho_\pi(x,x_1))) \quad],\end{array}$$

which is equivalent to

$$\begin{array}{ll}\forall x\,[& \forall x_0\,(Rxx_0 \to \neg\exists x_1 Rx_0x_1) \vee \\ & \exists x_0\,(Rxx_0 \wedge \forall x_1(Rx_0x_1 \to \forall z_0\,(Rxz_0 \to Rz_0x_1))) \quad].\end{array}$$

Note that $\mathsf{Add}(\pi)$ is equivalent to the Sahlqvist formula

$$\Diamond\Box p \wedge \Diamond\Box q \to \Diamond\Box(p \wedge q).$$

The elementary condition corresponding to this formula is given by the first order formula

$$\forall xy_0y_1\,[\,(Rxy_0 \wedge Rxy_1) \to \exists z_0\,(Rxz_0 \wedge \forall z_1\,(Rz_0z_1 \to (Ry_0z_1 \wedge Ry_1z_1)))\,]$$

This condition is strictly weaker than the first order pseudo-correspondent $C(\pi)$ of $\mathsf{Add}(\pi)$!

To finish off the section, we compare our Theorem 1 with the earlier mentioned result in Jónsson [4] stating that $\mathsf{Add}(()\pi)$ is canonical if π is stable. Let us first explain the notion of stability, which plays a crucial role in the algebraic approach towards canonicity. Since not every reader of this paper may be familiar with this algebraic perspective, we rephrase the notion in terms of general frames.

Earlier in this section we already mentioned that given a frame $\mathfrak{F} = (W, R)$, we can see a formula $\varphi(p)$ as an operation on the power set of W:

$$\varphi(X) = \{s \mid \mathfrak{F}, V_X, s \Vdash \varphi(p)\}.$$

Now, if $\mathfrak{g} = (\mathfrak{F}, A)$ is a descriptive frame on $\mathfrak{F}$, we can associate a second kind of operation φ^+ with φ, defined (in stages) as follows. First, we define, for a *closed* set $c \subseteq W$:

$$\varphi'(c) \ = \ \bigcap\{\varphi(a) \mid c \subseteq a \in A\}.$$

Then, for an arbitrary set $X \subseteq W$, define

$$\varphi^+(X) \ = \ \bigcup\{\varphi'(c) \mid X \supseteq c, \ c \text{ closed}\}.$$

Now a formula is called *stable* if it is monotone and satisfies $\varphi = \varphi^+$ on every descriptive frame. (In fact, Jónsson defines the notion of stability for formulas in arbitrary many variables, but his Theorem 6.1 only refers to unary formulas). Stability is thus a semantic concept, and one may wonder what the relation is between stable and positive formulas.

First, it follows from monotonicity that every stable formula is equivalent to a positive one. In the other direction, Jónsson proves (see the Theorems 5.7 and 4.4) that formulas built up from $\bot$, $\top$, $\Box^n p$ $(n \geq 0)$ by applying $\Diamond$, $\wedge$ and $\vee$ are stable. On the other hand, not *every* positive formula is stable, as the following example shows. The property that motivates the definition of stability is that formula $\varphi \leftrightarrow \psi$ is canonical if φ and ψ are stable formulas. Now consider the positive formulas $\varphi = \Diamond\Box p \wedge \Box\Diamond p$ and $\psi = \Box\Diamond p$. The formula $\varphi \leftrightarrow \psi$ is equivalent to McKinsey's axiom $\Box\Diamond p \to \Diamond\Box p$ which was proved *not* to be canonical in GOLDBLATT [2].

Thus, Theorem 1 can be seen as a strengthening of Jónsson's result (apart from the connection with first-order logic, which is not mentioned in Jónsson's article). I do not know whether there is a syntactic characterization of the class of stable formulas.

3 Reflexive successors

In this section we show that is not straightforward to apply the notion of pseudo-correspondence in order to axiomatize a given class of frames. As an example we treat the class $\mathcal{RS}$ of frames in which every state has a reflexive successor. $\mathcal{RS}$ is the standard example of a class that is closed under taking disjoint unions, generated subframes and bounded morphic images, but does *not* reflect ultrafilter extensions.

Since $\mathcal{RS}$ is defined by an elementary condition, it follows from Fine's Theorem that the logic $\Lambda_{\mathcal{RS}}$ is canonical. It would be interesting if there were a modal formula canonically pseudo-corresponding to the first order formula $\forall z \exists y\, (Rxy \wedge Ryy)$. That this is not the case follows from the fact that the modal logic of the class $\mathcal{RS}$ is not finitely axiomatizable (cf. Theorem 10 in HUGHES [3]).

Nevertheless, in order to axiomatize the class $\mathcal{RS}$, we may still apply the method of using formulas that force the canonical frame to contain points with certain nice properties; the difference is that for $\mathcal{RS}$ we need an infinite collection of axioms. Our logic KΞ below can be seen as a variation of the system KMT proved to be complete by Hughes (Theorem 1).

Definition 3.1 *For $i \in \{0,1\}$ and φ a formula, we define*

$$\pm_i\varphi = \begin{cases} \varphi & \textit{if } i = 1, \\ \neg\varphi & \textit{if } i = 0. \end{cases}$$

Let $|\sigma|$ denote the length of a finite string σ of 0s and 1s. Now we define the following formulas.

$$\begin{aligned} \pi_\sigma &:= \bigwedge_{i<n} \pm_{\sigma i} p_i \\ \xi_\sigma &:= \Diamond(\pi_\sigma \wedge \Diamond\pi_\sigma), \\ \xi_n &= \bigvee_{|\sigma|=n} \xi_\sigma. \end{aligned}$$

Finally, let KΞ *be the basic modal logic* K *extended with the axioms* $\{\xi_n \mid n \in \omega\}$.

We adopt the convention that a conjunction $\bigwedge_{i\in\varnothing} \varphi_i$ is the formula $\top$, so ξ_0 denotes the formula $\Diamond(\top \wedge \Diamond\top)$. The idea underlying the axiom set Ξ is the following. The formula $\xi_n(\varphi_0, \ldots, \varphi_{n-1})$ holds in a point s iff it has a successor t with a successor u such that each formula φ_i $(i < n)$ has the same truth value in t as in u. This means that in the canonical model for KΞ, each maximal consistent set Σ will have a successor Δ with a successor Θ such that for *each* formula φ, φ has the has the same truth value in Δ as in Θ; then by the truth lemma this implies that Δ and Θ must be identical.

Theorem 2 KΞ *is strongly sound and complete with respect to the class $\mathcal{RS}$.*

Proof. We give a direct proof for this Theorem, via the canonical model method: it suffices to show that the canonical frame $\mathfrak{F} = (W, R)$ is in $\mathcal{RS}$. Hence, let Σ be an arbitrary maximal KΞ-consistent set (an MCS). We have to prove the existence of an MCS Δ such that $R\Sigma\Delta$ and $R\Delta\Delta$.

Enumerate the formulas of the language $\top = \varphi_0, \varphi_1, \ldots,$ and let π'_σ be the formula $\pi_\sigma(\varphi_0, \ldots, \varphi_{n-1})$; that is, substitute uniformly φ_i for p_i in π_σ. We use analogous convention for ξ'_σ and ξ'_n. Observe that ξ'_n, being a substitution instance of an axiom, belongs to Σ.

Now let B be the infinite binary tree of finite sequences of 0s and 1s. Paint a node σ of B *black* if the formula ξ'_σ belongs to Σ. It follows from $\xi'_n \in \Sigma$ that for each n there is at least one node σ of length n such that ψ_σ belongs to Σ, and from $\vdash \xi_{\sigma*i} \to \xi_\sigma$ that the collection of black sequences is closed under taking initial segments.

It thus follows that the black nodes form an infinite subtree of B. By Königs Lemma this subtree must contain an infinite branch $\beta \in \{0,1\}^\omega$. Define

$$\Delta = \{\pm_{\beta(i)}\varphi_i \mid i \in \omega\}.$$

It is straightforward to prove that Δ is the required reflexive successor of Σ. QED

In terms of canonical pseudocanonicity, a slight generalization of the above proof (working with arbitrary descriptive general frames instead of with the canonical frame) shows that Ξ and $\forall x\exists y\,(Rxy \wedge Ryy)$ are canonical pseudo-correspondents.

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