Constructive Notions of Cofiniteness

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Abstract

In constructive mathematics, equivalences of various properties of cofinite subsets of natural numbers can no longer be proven. In this report, a number of these properties are investigated for subsets of natural numbers as well as binary relations on natural numbers.

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1 Introduction

> "Meaningful distinctions deserve to be maintained."
>
> — Errett Bishop[2]

This is a report on a project done under the supervision of Benno van den Berg in the spring of 2016.

1.1 Motivation

In mathematics, the question of how much should be assumed is a delicate one. On one hand, making fewer assumptions about our structures of interest enriches the study of these structures, providing a more challenging and perhaps more satisfying environment in which to establish results. On the other hand, we also want our structures to be tame enough so that we aren't forever lost and unable to understand anything about them. This is often reflected in a mathematician's own professional taste. For instance, a group theorist might find abelian groups boring, since the presence of commutativity seems to destroy many interesting problems and situations. However, they may at the same time find semigroups and monoids uninteresting as well, since the lack of invertibility limits the results they can prove and structure they can explore. To summarize: mathematicians want structure, but only just enough.

This same phenomenon presents itself in the foundations of mathematics. In the absence of classical principles, familiar notions can split into multiple inequivalent ones. The classical mathematician may find these differences uninspiring and consequently chooses to meld them back together with classical reasoning. On the contrary, the constructivist mathematician may welcome these newfound distinctions as enriching their area of study and providing new topics to explore.

Regardless of one's own preferences on these matters, it is difficult to justify destroying distinctions when said distinctions are natural. As a motivating example, consider the following theorem[1]:

Theorem 1.1 (Skolem-Mahler-Lech)**.** *Let $f : \mathbb{N} \to \mathbb{Z}$ be a linear recurrence. Then the zero set of f is, modulo some finite set of 'sporadic' zeroes, a union of residue classes.*

For a nice overview of this result and a discussion of some of the issues we are about to face, we direct the reader to [4].

We might say that the zero set of f is almost always periodic, where by 'almost always' we mean that it is cofinite.

But what exactly do we mean when we say that X is cofinite? Classically, we might say that its complement, the set of sporadic zeroes, is bound in size, i.e. it has size at most n for some $n \in \mathbb{N}$. Alternatively, we could say that this set of sporadic zeroes is bound above by some N. Yet another alternative is to explicitly list the complement:

$$\mathbb{N} \setminus X = \{x_0, \dots, x_{n-1}\}$$

[1] My thanks to Benno van den Berg for drawing my attention to this example.

for some n and elements x_i. Classically, all three are of course equivalent, and yet the third somehow seems to give us the most information: we have both the cardinality of the complement (n) as well as an upper bound on it ($\max(x_i)_{i=0}^{n-1}$) and on top of that, we fully understand what X looks like. The upper bound also seems more useful than the size bound, since an upper bound obviously doubles as a size bound, but not vice-versa.

Returning to our example, what can we say about our sporadic zeroes constructively? It is known that we may effectively find a bound on the number of sporadic zeroes of f.[4] However, it is still an open question as to whether or not we can effectively find an upper bound for these zeroes. We could say that the sporadic zeroes are effectively bound in size but not in space.

This example seems to suggest that there is a meaningful distinction between 'bound in size' and 'bound in space' which is hidden by classical logic. But are these actually two distinct notions? Thus far, all we have said is that it is not immediately obvious how to obtain the space bound from the size bound. In this report, we will tackle these sorts of questions and more by exploring various properties of subsets of natural numbers which all reduce to cofiniteness in a classical setting. In the specific case mentioned above, we will show that these two notions are indeed distinct, and through similar proofs we shall see that there are many more such notions which turn out to be in-equivalent in a constructive setting.

1.2 Overview of Original Contributions

We provide a proof that the so-called unavoidable and co-limited sets can be separated constructively. We introduce two new notions of cofiniteness, called repetition unavoidable and dense, and show that they have the filter properties. We prove a version of the binary Ramsey theorem for co-limited subsets.

1.3 Notation and Logical Principles

Let $\mathbf{2} := \{0, 1\}$, $\mathcal{S}$ denote the set of strictly increasing maps from $\mathbb{N}$ to $\mathbb{N}$, and $\mathcal{S}^*$ denote the (non-strictly) increasing maps from $\mathbb{N}$ to $\mathbb{N}$.

Throughout this work, we will be working in some system of constructive set theory which is neutral regarding classical principles (that is to say, it will be consistent to add the full Law of Excluded Middle). Consequently, we will not be able to show that two properties are distinct if they actually coincide in a classical setting. To get around this, we will instead separate properties by showing that their equivalence implies a certain constructive 'taboo', namely the Limited Principle of Omniscience.

We shall take **LPO** to denote the Limited Principle of Omniscience, that is,

$$\mathbf{LPO} :\equiv \forall f : \mathbb{N} \to \mathbf{2}, (\exists x, f(x) = 1) \vee (\forall x, f(x) = 0).$$

From a computational perspective, **LPO** tells us that the halting problem can be decided. Since the halting problem is not computable, and since we expect constructive proofs to have some computational content, we reasonably take **LPO** to be a non-constructive principle.

We shall let $\mathbf{AC}_{0,0}$ denote the following axiom scheme:

$$\forall m \in \mathbb{N}\ \exists n \in \mathbb{N}\ \varphi(m, n) \to \exists f : \mathbb{N} \to \mathbb{N}\ \forall m \in \mathbb{N}\ \varphi(m, f(m))$$

where φ is an arbitrary binary predicate on the natural numbers. Constructively, we may justify this axiom scheme by arguing that the strong constructive readings of the quantifiers on the left-hand side gives us an effective procedure, yielding the function from the right-hand side.

We shall let **M** denote Markov's Principle:

$$\mathbf{M} :\equiv \forall f : \mathbb{N} \to \mathbf{2}, \neg\neg\exists x.f(x) = 1 \to \exists x.f(x) = 1.$$

Constructively, we may justify this principle as the logical expression of unbounded search: If we know that f cannot be zero everywhere, check $f(0)$, $f(1)$, $f(2)$ and so on until we've encountered an i such that $f(i) = 1$.

By **CT** we will mean Church's Thesis:

$$\mathbf{CT} :\equiv \forall f : \mathbb{N} \to \mathbb{N}, \exists e \in \mathbb{N}, e \text{ encodes } f.$$

Church's thesis can be thought of as saying that every function f between natural numbers is computable in the general recursive sense, and furthermore that we have access to the "source code" for f (by means of the index e).

2 Notions of Cofiniteness

Let $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$. If $\mathcal{F}$ is to match our intuition behind cofinite subsets, it should satisfy a number of properties. In particular we expect that

1. $\mathbb{N} \in \mathcal{F}$.
2. For any A, B, if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.
3. For any $A, B \in F$, $A \cap B$ is also in $\mathcal{F}$.

Families of sets that satisfy the above three properties are called **filters**. Consequently, we shall refer to the above three properties as the filter properties. Typically, the first two properties are easy to verify, and throughout this work we will omit proofs of them.

2.1 Constructive Cofiniteness

Definition 2.1. $A \subseteq \mathbb{N}$ is said to be **cofinite** if there is a natural number n such that for all $k \geq n, k \in A$.

Let **Cof** denote the family of cofinite subsets of $\mathbb{N}$.

Lemma 2.2. **Cof** *is a filter.*

Proof. The proof is trivial.

□

At first glance, it might seem that there is a stronger notion of cofiniteness which we've already mentioned in the introduction. Consider the following proposed definition:

Definition 2.3. A is said to be **strongly cofinite** if there is a number n and numbers $b_0, \ldots b_{n-1}$ such that

$$\mathbb{N} \setminus A = \{b_0, \ldots, b_{n-1}\}.$$

As it turns out, this definition is too strong in a constructive setting. Specifically, it does not form a filter:

Lemma 2.4. *Suppose that the strongly cofinite sets are upward closed. Then the Law of Excluded Middle holds.*

Proof. Let φ be an arbitrary proposition. Define

$$A_\varphi := \{n \mid \varphi \vee n > 0\}.$$

Clearly, $\{1, 2, 3, \ldots\}$ is strongly cofinite and $\{1, 2, 3, \ldots\} \subseteq A_\varphi$. However, if A_φ is strongly cofinite, we can then decide φ by checking whether $0 \in \mathbb{N} \setminus A_\varphi$. □

As it turns out, **Cof** is the strongest notion of cofiniteness that satisfies the filter properties as well as other intuitive properties of cofiniteness. Therefore, we are justified in giving it the actual name "cofinite". To show this, we introduce the following property:

Definition 2.5. Let $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$. We shall say that $\mathcal{F}$ is **closed under finite difference** if for every $A \in F$ and $n \in \mathbb{N}$, $A \setminus \{n\} \in \mathcal{F}$ as well.

Lemma 2.6. **Cof** *is closed under finite difference.*

Proof. Suppose $A \in$ **Cof** with associated value n, and $m \in \mathbb{N}$. It is easy to see that $n' := \max(n, m + 1)$ does the trick for $A \setminus \{m\}$.

□

Clearly, being closed under finite difference is a property that we should expect a notion of cofiniteness to have. Therefore, we make our argument for **Cof** being the strongest constructive notion of cofiniteness through the following lemma:

Lemma 2.7. *Let $\mathcal{F}$ be a filter which is closed under finite difference. Then* **Cof** $\subseteq \mathcal{F}$.

Proof. It follows from finite difference and induction that for each n, we have that $\{x \in \mathbb{N} \mid x \geq n\} \in \mathcal{F}$. For arbitrary $A \in$ **Cof** it is the case that $\{x \in \mathbb{N} \mid x \geq n\} \subseteq A$ for some n. Thus $A \in \mathcal{F}$ by upward-closedness. □

2.2 Co-limitedness

Definition 2.8. $A \subseteq \mathbb{N}$ is said to be **co-limited** if there is a natural number n such that for all sets of numbers $a_0, \dots, a_n$ distinct, there is some a_i $(0 \leq i \leq n)$ such that $a_i \in A$.

Let **Col** denote the family of co-limited subsets of $\mathbb{N}$.

Lemma 2.9. *Let $A \in$* **Col** *with associated value k. For every n and $a_0, \dots, a_{k+n}$ distinct, there are $a_{i_0}, \dots, a_{i_n} \in A$ which are distinct.*

Proof. Induction on n. If $n = 0$ this is precisely the definition of co-limitedness. Suppose then that the statement holds for n and we have $a_0, \dots, a_{k+n+1}$ distinct. By the inductive hypothesis, we can find $a_{i_0}, \dots, a_{i_n} \in A$. There are $k + n + 2 - (n + 1) = k + 1$ remaining numbers, so we may select one more number in A amongst the numbers which haven't already been selected. □

Lemma 2.10. **Col** *is a filter.*

Proof. Let $A, B \in$ **Col** with associated values k and l, respectively. We claim that $k + l$ does the trick for $A \cap B$. Suppose that $a_0, \dots, a_{k+l}$ are distinct numbers. By the previous lemma, there are $a_{i_0}, \dots, a_{i_l} \in A$ which are distinct. It then follows from the co-limitedness of B that there is some $a_{i_j} \in B$ amongst these numbers. We then conclude that $a_{i_j} \in A \cap B$. □

Just as cofinite sets can be thought of as sets whose complement is explicitly bound, co-limited sets can be thought of as sets whose complement is bound in terms of cardinality. A bound on the complement of a set obviously gives a cardinality bound as well, but the cardinality bound will generally not tell us how to bound the complement spatially. These facts are expressed and proven below:

Lemma 2.11. **Cof** $\subseteq$ **Col**.

Proof. Suppose $A \in$ **Cof** with associated value n. If $a_0, \dots, a_n$ are distinct, then there is some $a_i \geq n$ by the pigeonhole principle. Therefore, $a_i \in A$. □

Lemma 2.12. **Col** $\subseteq$ **Cof** $\Rightarrow$ **LPO**

Proof. Suppose **Col** $\subseteq$ **Cof** and let $f : \mathbb{N} \to \mathbf{2}$ be arbitrary. Define $\hat{f} : \mathbb{N} \to \mathbf{2}$ so that $\hat{f}(n) = 1$ if and only if n is the least value such that $f(n) = 1$. It is easy to see that $\hat{f}$ is primitive-recursively definable from f, and so its existence is justified constructively.

Let $A_f := \{n \mid \hat{f}(n) = 0\}$. A_f is co-limited using the value $n = 1$: If a_0, a_1 are distinct, then at most one of them can correspond to the first occurrence of a 1 in f. Therefore, at least one of them is in A_f. Thus, we have that A_f is also cofinite, by our assumption that all co-limited sets are cofinite. Let n then be such that for all $k \geq n, k \in A_f$. If a 1 occurs in f, then clearly the first such occurrence happens before n, by the definition of A_f. It then suffices to check only the values $f(0), \ldots, f(n-1)$: if one of them is 1, we have found that f achieves the value 1 somewhere; and if they are all 0, we are assured that f is 0 everywhere. $\square$

2.3 Unavoidable Sets

Definition 2.13. $A \subseteq \mathbb{N}$ is said to be **unavoidable** or **almost full** if for every $f \in \mathcal{S}$, there is some n such that $f(n) \in A$.

Definition 2.14. $A \subseteq \mathbb{N}$ is said to be **repetition unavoidable** if for every $f \in \mathcal{S}^*$, there is either some n such that $f(n) \in A$, or $i \neq j$ such that $f(i) = f(j)$.

Let **Una** and **Una*** denote the families of unavoidable and repetition unavoidable sets, respectively.

Lemma 2.15. *Let $A \in$ **Una**. Then for every $f \in \mathcal{S}$, there is some $g \in \mathcal{S}$ such that $(f \circ g)(\mathbb{N}) \subseteq A$.*

Proof. See Lemma 5.2 in [5]. Note that this proof makes use of $\mathbf{AC}_{0,0}$. $\square$

Lemma 2.16. *Let $A \in$ **Una***. Then for every $f \in \mathcal{S}$, there is some $g \in \mathcal{S}^*$ such that for all n, $f(g(n))$ is either in A or occurs twice in the image of f.*

Proof. The proof is almost identical to the proof of Lemma 5.2 of [5]. The only difference is that instead of proving that

$$\forall m \exists n.(n > m \land n \in A)$$

we prove that

$$\forall m \exists n.(n \geq m \land (n \in A \lor \exists n'.n' \neq n \land f(n) = f(n')))$$

in much the same way as in the original proof. We then obtain a sequence by apply $\mathbf{AC}_{0,0}$ as before.

$\square$

Lemma 2.17. **Una** *is a filter.*

Proof. Let A, B be unavoidable, and let $f \in \mathcal{S}$. Let $g \in \mathcal{S}$ be as in 2.15. Since $\mathcal{S}$ is closed under composition, $f \circ g \in \mathcal{S}$. Since B is unavoidable, there is an n such that $f(g(n)) \in B$. By the previous lemma, $f(g(n)) \in A$, so $f(g(n)) \in A \cap B$. $\square$

Lemma 2.18. $\mathbf{Una}^*$ *is a filter.*

Proof. Similar to the above proof. $\square$

Lemma 2.19. $\mathbf{Una}^* \subseteq \mathbf{Una}$.

Proof. Obvious. $\square$

Lemma 2.20. $\mathbf{Col} \subseteq \mathbf{Una}^*$

Proof. Suppose A is co-limited by the value n and suppose $f \in \mathcal{S}^*$. Check the values of $f(0), \ldots, f(n)$. This is a finite list of numbers, so of course we may check for repetition. If there is repetition, we are done. If not, then $f(0), \ldots, f(n)$ constitute $n+1$ distinct numbers, and so $f(i) \in A$ for some $i \leq n$ by co-limitedness of A. $\square$

Lemma 2.21. $\mathbf{Una}^* \subseteq \mathbf{Col} \Rightarrow \mathbf{LPO}$.

Proof. Suppose $\mathbf{Una}^* \subseteq \mathbf{Col}$ and let $f : \mathbb{N} \to \mathbf{2}$ be arbitrary. Define $\hat{f} : \mathbb{N} \to \mathbf{2}$ so that $\hat{f}(n) = 1$ if and only if $s \leq n \leq 2s$ where s is the first occurrence of a 1 in f (if it exists), and 0 otherwise. Once more, $\hat{f}$ is primitive-recursively definable from f.

Let $A_f := \{n \mid \hat{f}(n) = 0\}$. A_f is repetition unavoidable: if $g \in \mathcal{S}^*$, then first check the value that $\hat{f}$ takes at $g(0)$. If $\hat{f}(g(0)) = 0$, we are done. If $\hat{f}(g(0)) = 1$, then we have that $s \leq g(0) \leq 2s$ where s is the first occurrence of a 1 in f. Then check the values of $g(0), g(1), \ldots, g(g(0)), g(g(0) + 1)$. If there is repetition in this list, we are done. Since $g \in \mathcal{S}^*$, we may then suppose that this list is strictly increasing. Thus, $g(g(0)+1) > 2g(0) \geq 2s$ and so $g(g(0)+1) \in A_f$.

By assumption, this means that A_f is co-limited. Suppose then it is co-limited by the value k. If f does achieve the value one at some point, then the first such occurrence s must occur before k: $s, \ldots, 2s$ are $s+1$ distinct numbers not in A_f, so $s < k$. Therefore, we may decide as before whether or not f achieves the value 1 by only looking at the finitely many values $f(0), \ldots, f(k-1)$. $\square$

3 Cofiniteness for Decidable Subsets

We briefly turn our attention to notions of cofiniteness for decidable subsets. Let $\mathcal{P}_{\text{dec}}(\mathbb{N})$ denote the collection of decidable subsets of $\mathbb{N}$. All of the classes of subsets we have introduced can obviously be recast in terms of $\mathcal{P}_{\text{dec}}(\mathbb{N})$ instead of $\mathcal{P}(\mathbb{N})$.

As it turns out, **Cof** and **Col** as well as **Col** and $\mathbf{Una}^*$ remain separated, even after restrict them to decidable subsets:

Lemma 3.1. **Col** $\subseteq$ **Cof** $\Rightarrow$ **LPO** *and* **Una*** $\subseteq$ **Col** $\Rightarrow$ **LPO** *still hold, even if we take those families to range over* $\mathcal{P}_{\text{dec}}(\mathbb{N})$ *instead of* $\mathcal{P}(\mathbb{N})$*.*

Proof. Examination of the proofs of 2.12 and 2.21 shows that the counterexamples that were constructed were always decidable. □

However, we have good evidence that **Una** and **Una*** cannot be separated within $\mathcal{P}_{\text{dec}}(\mathbb{N})$:

Lemma 3.2.

$$\mathbf{M} \Rightarrow \mathbf{Una} \cap \mathcal{P}_{\text{dec}}(\mathbb{N}) \subseteq \mathbf{Una}^* \cap \mathcal{P}_{\text{dec}}(\mathbb{N}).$$

Proof. Let A be unavoidable and decidable, and let $f \in \mathcal{S}^*$. Our claim is that

$$\exists i, j \in \mathbb{N}.(f(i) \in A \vee (i \neq j \wedge f(i) = f(j))$$

which would complete the proof. Since A is decidable, the entire matrix of this formula is decidable with respect to i, j. Thus, by Markov's principle, it suffices to show

$$\begin{aligned} &\neg\neg\exists i, j \in \mathbb{N}.(f(i) \in A \vee (i \neq j \wedge f(i) = f(j)) \\ \equiv &\neg\forall i, j \in \mathbb{N}.(f(i) \notin A \wedge (i = j \vee f(i) \neq f(j)). \end{aligned}$$

This last formula says that it can't be the case that f never hits A while also being injective. Since f is increasing, this means that it can't be that f never hits A while also being strictly increasing. This of course follows from the unavoidability of A.

□

3.1 Dense Sets

We now introduce a unique notion of cofiniteness for decidable subsets.

Let $A \subseteq \mathbb{N}$ be decidable. For any number n and function $f : \mathbb{N} \to \mathbb{N}$, let $A_{f,n} := \{k \mid k \leq n \wedge f(k) \in A\}$.

We shall say that A is **dense** if for every $f \in \mathcal{S}$ we have that

$$\lim_{n\to\infty} \frac{|A_{f,n}|}{n+1} = 1.$$

Let **Dense** $\subseteq \mathcal{P}_{\text{dec}}(\mathbb{N})$ denote the family of dense subsets.

Notice that decidability of A is required so that $|A_{f,n}|$ can be computed. It is then not obvious how this definition could be extended to arbitrary (possibly undecidable) subsets of $\mathbb{N}$.

It is fairly obvious that $\mathbb{N}$ is dense, and that dense sets are upward closed. Showing that they are closed under intersection is a bit more interesting. To do so, keep the following basic combinatorial lemma in mind:

Lemma 3.3. *Let A, B be finite sets. Then $|A \cup B| = |A| + |B| - |A \cap B|$.*

We now proceed with a proof that dense sets are closed under intersection:

Lemma 3.4. Dense *is a filter.*

Proof. Suppose A, B are dense. Let f be a strictly increasing function and let $\varepsilon > 0$ be arbitrary. Since A and B are dense, there are N_0, N_1, respectively, such that for every $n \geq N_0$ we have that

$$\frac{|A_{f,n}|}{n+1} > 1 - \frac{\varepsilon}{2}$$

and for every $n \geq N_1$ we have that

$$\frac{|B_{f,n}|}{n+1} > 1 - \frac{\varepsilon}{2}$$

Take $N := \max(N_0, N_1)$, and let $n \geq N$. By the above lemma, we have that

$$\begin{aligned} |(A \cap B)_{f,n}| &= |A_{f,n} \cap B_{f,n}| \\ &= |A_{f,n}| + |B_{f,n}| - |(A \cup B)_{f,n}| \\ &\geq |A_{f,n}| + |B_{f,n}| - (n+1). \end{aligned}$$

Dividing out by $n + 1$ yields

$$\begin{aligned} \frac{|(A \cap B)_{f,n}|}{n+1} &\geq \frac{|A_{f,n}|}{n+1} + \frac{|B_{f,n}|}{n+1} - 1 \\ &> (1 - \frac{\varepsilon}{2}) + (1 - \frac{\varepsilon}{2}) - 1 \\ &= 1 - \varepsilon. \end{aligned}$$

□

Claim 3.5. Col $\subseteq$ Dense.

Proof. Let A be co-limited via the value k, and let f be strictly increasing. Given $\varepsilon > 0$, pick N such that $\frac{N}{N+k} > 1 - \varepsilon$. By co-limitedness, among $f(0), \ldots, f(k)$ there is some $f(i) \in A$. Likewise, among $f(0), \ldots f(i-1), f(i+1), \ldots, f(k), f(k+1)$ we may find some $f(j) \in A$. More generally, for any n we may find n elements of A among $f(0), \ldots, f(k+n-1)$. Thus, $|A_{f,k+n-1}| \geq n$. If $n \geq N$, we then have that

$$\frac{|A_{f,k+n-1}|}{k+n} \geq \frac{n}{k+n} \geq \frac{N}{k+N} > 1 - \varepsilon.$$

□

Claim 3.6. **Dense** $\subseteq$ **Una**.

Proof. Let A be dense and $f \in \mathcal{S}$. By density, there is some N such that $\frac{A_{f,N}}{N+1} \geq \frac{1}{2}$. Thus, $A_{f,n}$ contains at least one element. $\square$

Claim 3.7. (**Una** $\subseteq$ **Dense**) $\Rightarrow$ **LPO**.

Proof. Suppose **Una** $\subseteq$ **Dense** and let $f : \mathbb{N} \to \mathbf{2}$ be arbitrary. Define $\hat{f}$ and A_f as in 2.21. As before, A_f is repetition unavoidable and thus unavoidable. Therefore, A_f is dense by assumption, and so there is some N such that for all $n \geq N$ we have that

$$\frac{|(A_f)_{\mathrm{id},n}|}{n+1} \geq \frac{2}{3}.$$

If s is the first occurrence of a 1 in f, then clearly, $2s \leq N$, since the proportion of 1's below $2s$ in A_f is exactly $\frac{1}{2}$. Therefore, we may decide whether or not f achieves the value 1 by checking f up to $\frac{N}{2}$.

$\square$

We end with section with a couple of open questions:

1. Can the notion of density be reasonably defined on all subsets of $\mathbb{N}$?
2. Can **Dense** and **Col** be separated constructively?
3. Call A **weakly dense** if for every strictly increasing f, we have that

$$\lim_{n\to\infty} \frac{|A_{f,n}|}{n+1} > 0.$$

   Call A **uniformly weakly dense** if there is some $p \in (0,1]$ such that for every f strictly increasing, we have that

$$\lim_{n\to\infty} \frac{|A_{f,n}|}{n+1} > p.$$

   How do the notions of dense, weakly dense, and uniformly weakly dense relate?

4 Binary Ramsey Theorems

In this section we will concern ourselves with questions about binary relations on $\mathbb{N}$. We shall extend our notions of cofiniteness to cover binary relations as well. We will recall the Finite Ramsey Theorem and show that binary co-limited relations are closed under intersection.

Throughout this section we shall take a graph to be simple and undirected, i.e. an irreflexive and symmetric binary relation.

Recall the following constructively valid result, due to Ramsey:

Theorem 4.1 (Finite Ramsey Theorem)**.** *For any natural numbers n, k, there is a natural number N such that for every graph G of size N with edges colored using k colors, there is a n-sized clique in G of homogeneous coloring.*

Letting $k = 2$, we obtain the following corollary:

Corollary 4.2. *For any natural number n, there is a natural number N such that for any decidable, symmetric and irreflexive binary relation R on $\mathbb{N}$, we have that for every set $A = \{a_0, \ldots, a_{N-1}\}$, there is either an R-n-clique in A or an $\overline{R}$-n-clique in A.*

We now define some classes of binary relations which in some sense correspond with the notions of cofiniteness which we discussed earlier.

Definition 4.3. Let R be a binary relation on $\mathbb{N}$. We say that R is **unavoidable** if for every $f \in \mathcal{S}$ there are m, n such that $R((f(m), f(n))$.

Lemma 4.4. *Let R, S be unavoidable. Then $R \cap S$ is unavoidable.*

Proof. See Theorem 6.5 of [5]. □

It is worth noting that the proof in Veldman and Bezem makes use of monotone bar induction, a rather strong intuitionistic principle in Brouwerian analysis. A testament to its strength is that it is inconsistent with other constructive principles, namely Church's thesis (see 8.6.1 of [5]). Veldman and Bezem further show that **CT** is able to prove the negation of 4.4, showing that the use of bar induction is necessary for their proof. Nonetheless, a proof avoiding the use of bar induction is given by Coquand within the alternative framework of inductive definitions.[3]

We shall prove similar results in the cases of cofinite, dense, and co-limited binary relations which avoids the use of this principle.

Definition 4.5. Let R be a binary relation on $\mathbb{N}$. We say that R is **cofinite** if there is an N such that for all $m \neq n \geq N$, $R(m, n)$.

Lemma 4.6. *Let R, S be cofinite. Then $R \cap S$ is cofinite.*

Proof. Trivial. □

Definition 4.7. Let R be a decidable binary relation on $\mathbb{N}$. We say that R is **dense** if for every $f \in \mathcal{S}$ the proportion of edges among $f(0), \ldots, f(n)$ in R tends to 1 as n tends to infinity.

Lemma 4.8. *Let R, S be dense. The $R \cap S$ is dense.*

Proof. Similar to the proof of 3.4. □

Much more interesting is the proof that co-limited relations are closed under intersection:

Definition 4.9. Let R be a binary relation on $\mathbb{N}$. We say that R is **co-limited** if there is a k such that for all $a_0, \ldots, a_{k-1}$ distinct, there are $i \neq j < k$ such that $R(a_i, a_j)$.

Using 4.2, we immediately obtain the desired result for decidable relations:

Lemma 4.10. *Let R, S be decidable and co-limited. Then $R \cap S$ is co-limited.*

Proof. Let R, S be co-limited via the values n, k respectively, and without loss of generality, suppose $n \geq k$. By 4.2, there is an N such that for every $A = \{a_0, \ldots, a_{N-1}\}$ we have that either there is an R-n-clique in A or an $\overline{R}$-n-clique in A. The second case is ruled out, since R is co-limited by n. Thus, there is an R-n-clique in A, whose elements we will denote by $a_{i_1}, \ldots, a_{i_n}$. Since $k \leq n$, it follows from the co-limitedness of S that there are $a_{i_j} \neq a_{i_k}$ such that $S(a_{i_j}, a_{i_k})$. Since they are elements of the R-clique, we also have that $R(a_{i_j}, a_{i_k})$. Thus, $(R \cap S)(a_{i_j}, a_{i_k})$.

This is true for arbitrary A, so we see that $R \cap S$ is co-limited by N. □

Can we extend this result to arbitrary relations? First, we need the following lemma:

Lemma 4.11. *Let R be co-limited. For every natural number l, there is a number N such that for all $a_0, \ldots, a_{N-1} \in \mathbb{N}$ distinct, there are $a_{i_1}, \ldots, a_{i_l}$ distinct such that they are a clique in R.*

Proof. Suppose that R is co-limited via the number k. Let $n := \max(k, l)$ and let N be the number corresponding to n from 4.2. Let $A = \{a_0, \ldots, a_{N-1}\}$ be an arbitrary set of N distinct numbers.

Our strategy now is to construct an n-clique in A by iteratively building up a decidable sub-relation of R in A by repeatedly applying Corollary 4.2.

Let $S_0 := \varnothing$, the empty relation. From S_i, use the corollary to see if there is an S_i-n-clique or a $\overline{S_i}$-n-clique in A. In the former case, we are done, since $S_i \subseteq R$. In the latter case, use the co-limitedness of R to find some $(a, b) \in R$ which is in the $\overline{S_i}$-n-clique. Then let $S_{i+1} := S_i \cup \{(a, b)\}$.

It is easy to see by induction that each S_i is indeed contained in R and is decidable. Therefore, this algorithm is well-defined. Furthermore, the number of edges in S_i is strictly increasing in i, since we are always adding a new edge from an $\overline{S_i}$-clique. Therefore, we know this algorithm must terminate and give us an R-n-clique in A in at most $\frac{N(N-1)}{2}$ stages. [2] □

[2]For the reader that prefers a tighter bound, Turán's Theorem tells us that a graph on N vertices without a n-clique can have at most $\lfloor \frac{N^2(n-2)}{2(n-1)} \rfloor$ edges.

We are finally ready to show that co-limited relations are closed under intersection:

Lemma 4.12. *Let R, S be co-limited binary relations on $\mathbb{N}$. Then $R \cap S$ is co-limited.*

Proof. Let l be the constant corresponding to the co-limitedness of S. By the previous lemma, there is some N such that every set of N elements has an R-l-clique.

For any N distinct elements in $\mathbb{N}$, we may then find an R-l-clique within these elements. We may then use the co-limitedness of S to find a, b within this clique such that $S(a, b)$. Since a, b are in an R-l-clique, $R(a, b)$. Therefore, $(R \cap S)(a, b)$. □

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