SET THEORY WITH AND WITHOUT URELEMENTS AND CATEGORIES OF INTERPRETATIONS

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Abstract. We show that the theories ZF and ZFU are synonymous, answering a question of A. Visser.

This paper is dedicated to Dick de Jongh on the occasion of his 65th birthday.

Albert Visser introduced five different categories of interpretations between theories INT_0 (the category of synonymy), INT_1 (the category of homotopy), INT_2 (the category of weak homotopy), INT_3 (the category of equivalence), and INT_4 (the category of mutual interpretability) [Vis04]. The objects in these categories are first order theories, the morphisms are interpretations up to some level of identification between interpretations. The category of synonymy has the strictest criteria for two interpretations to be the same, the category of mutual interpretability the weakest. Visser proved that $\mathsf{INT}_1 \neq \mathsf{INT}_4$ [Vis04, § 4.8.4.], but apart from that no separation results are known. One particular question is [Vis04, Open Question 4.16]:

$$\mathsf{INT}_0 \overset{?}{\neq} \mathsf{INT}_1.$$

Visser remarked that the theories ZF and ZFU are homotopic (*i.e.*, isomorphic in INT_1) and asked whether we can show that they are not synonymous.

In this note we produce a synonymy between ZF and ZFU. The result of this note is mentioned in [Vis04, p. 33*sq*].

1. Fixing the notation I. Categories of Interpretations

We basically follow [Vis04] in the definitions. Since only the categories INT_0 and INT_1 are relevant for our investigation, we shall only define those.

In both categories, the objects are first order theories in a countable language. A **signature** Σ is a triple $\langle P, \mathrm{ar}, \doteq\rangle$ where P is a finite

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set of predicates, $\mathrm{ar} : P \to \mathbb{N}$ is the arity function and $\doteq$ is a binary predicate representing the identity. Let Σ and Θ be signatures and $\Theta = \langle P_\Theta, \mathrm{ar}_\Theta, \doteq\rangle$ with $P_\Theta := \{p_0, ..., p_n\}$. We call τ a **translation** from Θ to Σ if τ is a sequence $\langle\delta, \langle p_0, \varphi_0\rangle, ..., \langle p_n, \varphi_n\rangle\rangle$ where the φ_i are Σ-formulas and φ_i has $\mathrm{ar}_\Theta(p_i)$ free variables. Using a relative translation τ, we can define translations of Θ-formulas into Σ-formulas by recursion. For a Θ-formula ψ, we denote its translation by τ with ψ^τ. If now S is a Σ-theory and T is a Θ-theory, we call $\langle T, \tau, S\rangle$ an **interpretation** of T in S if τ is a translation from Θ in Σ and for all Θ-formulas ψ, we have

$$T \vdash \psi \text{ implies } S \vdash \psi^\tau.$$

Now we define the morphisms in INT_0 as equivalence classes of interpretations with the equivalence relation $\equiv_0$ defined as follows: Let Σ and Θ be signatures, $\Theta = \langle P_\Theta, \mathrm{ar}_\Theta, \doteq\rangle$ with $P_\Theta := \{p_0, ..., p_n\}$, $\tau = \langle\delta, \langle p_0, \varphi_0\rangle, ..., \langle p_n, \varphi_n\rangle\rangle$ and $\tau' = \langle\delta', \langle p_0, \varphi'_0\rangle, ..., \langle p_n, \varphi'_n\rangle\rangle$ be two translations from Θ to Σ, T a Θ-theory and S a Σ-theory. Then we define $\langle T, \tau, S\rangle \equiv_0 \langle T, \tau', S\rangle$ to hold if and only if

$$\begin{array}{lll}
(\mathsf{s}_0) & S \vdash & \delta(\mathsf{v}_0) \leftrightarrow \delta'(\mathsf{v}_0), \text{ and} \\
(\mathsf{s}_1) & S \vdash & \delta(\mathsf{v}_0) \;\&\; ... \;\&\; \delta(\mathsf{v}_{\mathrm{ar}_\Theta(p_i)-1}) \\
 & & \to \varphi_i(\mathsf{v}_0, ..., \mathsf{v}_{\mathrm{ar}_\Theta(p_i)-1}) \leftrightarrow \varphi'_i(\mathsf{v}_0, ..., \mathsf{v}_{\mathrm{ar}_\Theta(p_i)-1}) \\
 & \multicolumn{2}{l}{(\text{for } 0 \leq i \leq n).}
\end{array}$$

We define an equivalence relation $\equiv_1$ on interpretations in terms of a morphism category $\mathsf{INT}^{\mathsf{morph}}$: two interpretations $\langle T, \tau, S\rangle$ and $\langle T, \tau', S\rangle$ are said to be $\equiv_1$-equivalent if they are isomorphic as objects in the category $\mathsf{INT}^{\mathsf{morph}}$ as defined in [Vis04, § 3.1]. The morphisms in INT_1 are now the $\equiv_1$-equivalence classes of interpretations.

We concatenate morphisms as follows: If $\langle T, \tau, S\rangle$ and $\langle S, \tau', R\rangle$ are two interpretations with

$$\tau = \langle\delta, \langle p_0, \varphi_0\rangle, ..., \langle p_n, \varphi_n\rangle\rangle \text{ and } \tau' = \langle\delta', \langle q_0, \varphi'_0\rangle, ..., \langle q_m, \varphi'_m\rangle\rangle,$$

we define the concatenation to be the ($\equiv_i$-equivalence class of the) interpretation induced by

$$\hat{\tau} := \langle\hat{\delta}, \langle p_0, \hat{\varphi}_0\rangle, ..., \langle p_n, \hat{\varphi}_n\rangle\rangle$$

where

$$\begin{array}{rcl}
\hat{\delta}(\mathsf{v}_0) & \simeq & \delta'(\mathsf{v}_0) \;\&\; (\delta(\mathsf{v}_0))^{\tau'}, \text{ and} \\
\hat{\varphi}_i(\vec{\mathsf{v}}) & \simeq & (\varphi_i(\vec{\mathsf{v}}))^{\tau'} \text{ (for } 0 \leq i \leq n).
\end{array}$$

As usual in category theory, an isomorphism in a category is an invertible morphism, *i.e.*, a morphism $K : T \to S$ such that for some other morphism $L : S \to T$, we have $K \circ L = \mathrm{id}_S$ and $L \circ K = \mathrm{id}_T$.

For INT_0, this means that if T is a Θ-theory where

$$\Theta = \langle \{p_0, ..., p_n\}, \mathrm{ar}_\Theta, \dot{=} \rangle,$$
$$K = \langle T, \tau, S \rangle, \text{ and } \tau = \langle \delta, \langle p_0, \varphi_0 \rangle, ..., \langle p_n, \varphi_n \rangle \rangle,$$

then K is an INT_0-isomorphism (also called a **synonymy**) if there is another morphism

$$L = \langle S, \tau', T \rangle \text{ with } \tau' = \langle \delta', \langle q_0, \varphi'_0 \rangle, ..., \langle q_m, \varphi'_m \rangle \rangle$$

such that (for $0 \leq i \leq n$ and $0 \leq j \leq m$)

$$\begin{array}{ll} T \vdash \ \delta'(\mathsf{v}_0) \ \& \ (\delta(\mathsf{v}_0))^{\tau'}, & S \vdash \ \delta(\mathsf{v}_0) \ \& \ (\delta'(\mathsf{v}_0))^{\tau}, \\ T \vdash \ p_i(\vec{\mathsf{v}}) \leftrightarrow (\varphi_i(\vec{\mathsf{v}}))^{\tau'}, & S \vdash \ q_j(\vec{\mathsf{v}}) \leftrightarrow (\varphi'_j(\vec{\mathsf{v}}))^{\tau}; \end{array}$$

in particular, δ' must be T-provably equivalent to the trivial condition and δ must be S-provably equivalent to the trivial condition.

2. Fixing the notation II. ZF and ZFU

In the following, ZF will be the standard axiom system of Zermelo-Fraenkel set theory in a language with a binary predicate $\dot{\in}$, *i.e.*, the Axioms (or Axiom Schemes) of Extensionality, Pairing, Union, Power Set, *Aussonderung*, Infinity, Foundation and *Ersetzung*. We denote models of ZF by $\mathbf{V} = \langle V, \in \rangle$. We shall use the variables x, y and z for elements of a ZF-model. By the axiom of infinity, we have a set of natural numbers in each model of ZF which we shall denote by $\mathbb{N}^{\mathbf{V}}$. For technical reasons, we choose the Zermelo natural numbers, *i.e.*,

$$\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\{\{\varnothing\}\}\}, ...\}$$

By the axiom scheme of *Ersetzung*, we have a welldefined transitive closure operator in each model of ZF, and we write $\mathrm{tcl}^{\mathbf{V}}(x)$ for the $\subseteq$-smallest transitive set containing x as an element.

The language of ZFU will be a language with two binary relations $\dot{\in}$ and $\dot{F}$ and a unary relation $\dot{U}$. The unary relation describes the *urelements* (*i.e.*, u is an *urelement* if and only if $\dot{U}(u)$ holds). We shall denote models of ZFU by $\mathbf{W} = \langle W, \hat{\in}, \hat{F}, \hat{U} \rangle$. We shall use the variables u, v and w for elements of a ZFU-model. The theory ZFU consists of the standard axioms of ZF with the usual changes to Extensionality and Foundation due to the existence of *urelements* plus axioms governing the character of the *urelements* (see below). Note that the axioms of ZF give the existence of the set of natural numbers which is abbreviated by $\mathbb{N}$ in the formal language and denoted by $\mathbb{N}^{\mathbf{W}}$ in a given model $\mathbf{W}$. Again, we are using the set of Zermelo numbers. Now, using this notation, we can state the axioms governing the *urelements*:

$$\forall u \forall v(\dot{U}(u) \to \neg(v \mathrel{\dot{\in}} u)), \text{ and}$$

$$\begin{aligned}\forall u \forall v(\dot{F}(u,v) \to (u \in \mathbb{N} \ \&\ \dot{U}(v))) \ \&\\ \forall v(\dot{U}(v) \to \exists u(\dot{F}(u,v))) \ \&\\ \forall u \forall v \forall w((\dot{F}(u,v) \ \&\ \dot{F}(u,w)) \to v = w).\end{aligned}$$

(The latter states that $\dot{F}$ describes a bijection between $\mathbb{N}$ and the set of *urelements*.) We denote the (countable) set of *urelements* in $\mathbf{W}$ by $\mathbb{U}^{\mathbf{W}}$ and the ith *urelement* (*i.e.*, the value of i under the function described by $\dot{F}$) by $\mathbb{u}_i$.

Again, by the axiom scheme of *Ersetzung*, we have a welldefined transitive closure operator in each model of ZFU, and we write $\mathrm{tcl}^{\mathbf{W}}(u)$ for the $\subseteq$-smallest transitive set containing u as an element. Note that this allows the definition of a formula saying that a set is pure:

$$\Psi_{\mathrm{Pure}}(u) \mathrel{\hat{=}} \forall v(v \in \mathrm{tcl}(u) \to \neg(\dot{U}(v))).$$

3. Homotopy of ZF and ZFU

We remind the reader of the standard embeddings of ZF in ZFU and vice versa:

3.1. **Interpreting ZFU inside V.** Given a model $\mathbf{V} \models \mathsf{ZF}$, we can build a model of ZFU in it as follows: In the following, we work in $\mathbf{V}$, so all operations and sets (*e.g.*, the ordered pair, the natural numbers, the ordinals) are the operations and sets in $\mathbf{V}$. Let $U := \{\langle 0, n\rangle\ ;\ n \in \mathbb{N}\}$. Define a class W by transfinite recursion as follows:

$$\begin{aligned} W_0 &:= U, \\ W_{\alpha+1} &:= \{\langle 1, x\rangle\ ;\ x \subseteq W_\alpha\} \cup W_\alpha, \\ W_\lambda &:= \bigcup_{\alpha<\lambda} W_\alpha \text{ (for limit ordinals } \lambda). \end{aligned}$$

By the transfinite recursion theorem, there is a formula Φ_W defining the class $W := \bigcup_{\alpha\in\mathrm{Ord}} W_\alpha$. Now we define the following formulas:

$$\begin{aligned}
\Phi_{\dot\in}(x,y) &\;\hat{=}\; \exists z(\langle 1,z\rangle = y \;\&\; x\in z),\\
\Phi_{\dot U}(x) &\;\hat{=}\; \exists n(n\in\mathbb{N} \;\&\; x=\langle 0,n\rangle),\\
\Phi_{\mathbb{N}}(x) &\;\hat{=}\; \mathtt{function}(x) \;\&\; \mathrm{dom}(x)=\mathbb{N} \;\&\; x(0)=\langle 1,\varnothing\rangle\\
&\qquad \;\&\; \forall n(n\in\mathbb{N}\to x(n+1)=\langle 1,\{x(n)\}\rangle),\\
\Phi_{\dot F}(x,y) &\;\hat{=}\; \exists z(\Phi_{\mathbb{N}}(z) \;\&\; \exists n(n\in\mathbb{N} \;\&\; z(n)=x \;\&\; y=\langle 0,n\rangle)).
\end{aligned}$$

Then if you use the formulas $\Phi_{\dot\in}$, $\Phi_{\dot F}$ and $\Phi_{\dot U}$ to define binary and unary relations $\hat\in$, $\hat F$, and $\hat U$ respectively, then $\langle W,\hat\in,\hat F,\hat U\rangle \models \mathsf{ZFU}$. Consequently,

$$T_{\mathsf{ZFU},\mathsf{ZF}} := \langle \Phi_W, \langle\dot\in,\Phi_{\dot\in}\rangle, \langle\dot F,\Phi_{\dot F}\rangle, \langle\dot U,\Phi_{\dot U}\rangle\rangle$$

is translation that yields an interpretation of ZFU in ZF.

3.2. **Interpreting ZF inside $\mathbf{W}$.** Now assume that $\mathbf{W}=\langle W,\hat\in,\hat F,\hat U\rangle$ is a model of ZFU. As is well-known, the class of pure sets in a ZFU-model is a model of ZF, so we take the formula Ψ_{Pure} from above and the formula

$$\Psi_{\dot\in}(u,v) \;\hat{=}\; u\,\hat\in\, v,$$

and get that

$$T_{\mathsf{ZF},\mathsf{ZFU}} := \langle \Psi_{\mathrm{Pure}}, \langle\dot\in,\Psi_{\dot\in}\rangle\rangle$$

is a translation that yields an interpretation of ZF in ZFU. We denote the class of pure sets inside $\mathbf{W}$ with $V^{\mathbf{W}}$.

3.3. **Homotopy.** It is clear that neither $T_{\mathsf{ZFU},\mathsf{ZF}}$ nor $T_{\mathsf{ZF},\mathsf{ZFU}}$ can be INT_0-isomorphisms (synonymies) as neither Ψ_{Pure} nor Φ_W are the trivial condition (in fact, ZFU-provably, there are sets u such that $\neg\Psi_{\mathrm{Pure}}(u)$ and ZF-provably, there are sets x such that $\neg\Phi_W(x)$).

However, it is easy to see that they are INT_1-isomorphisms.[1]

4. Graphs representing sets

4.1. **Definitions.** A **pointed graph** is a triple $\langle G,E,\nu\rangle$ such that $\langle G,E\rangle$ is a directed graph, and $\nu\in G$, a **labelled pointed graph** is a quadrupel $\langle G,E,\nu,\ell\rangle$ such that $\langle G,E,\nu\rangle$ is a pointed graph and $\ell:\omega+1\to G$ is a function.

We call a pointed graph $\langle G,E,\nu\rangle$ a **ZF-graph** if it has the following properties:

[1] *Cf.* [Vis04, p. 33].

- the set G contains a subset $N := \{n_i\,;\, i \in \omega\}$ such that n_0 is the unique least element of $\langle G, E\rangle$ and for all $i \in \omega$, the following holds:
$$\forall x \in G\,(xEn_{i+1} \leftrightarrow x = n_i),$$
- $\langle G, E\rangle$ is wellfounded,
- $\langle G, E\rangle$ is extensional, and
- $G\backslash \mathrm{tcl}(\nu) \subseteq N$.

In analogy to the ZF-graphs, let's define the corresponding ZFU-graphs: Let $\langle G, E, \nu, \ell\rangle$ be a labelled pointed graph. We call it a **ZFU-graph** if it has the following properties:

- the function ℓ is a bijection between $\omega + 1$ and the minimal elements of $\langle G, E\rangle$ (let us denote the image of ℓ by A),
- the set G contains a subset $N := \{n_i\,;\, i \in \omega\}$ such that $\ell(\omega) = n_0$, and for all $i \in \omega$, the following holds:
$$\forall x \in G\,(xEn_{i+1} \leftrightarrow x = n_i),$$
- $\langle G, E\rangle$ is wellfounded,
- $\langle G\backslash A, E\rangle$ is extensional, and
- $G\backslash \mathrm{tcl}(\nu) \subseteq N \cup A$.

If now $\mathbf{V} = \langle V, \in\rangle \models \mathsf{ZF}$, and $x \in V$, then let $G_x := \mathrm{tcl}^{\mathbf{V}}(x) \cup \mathbb{N}^{\mathbf{V}}$ and $E_x := \in \cap\, G_x \times G_x$. Then $\langle G_x, E_x, x\rangle$ is a ZF-graph. If $\mathbf{W} = \langle W, \hat{\in}, \hat{F}, \hat{U}\rangle \models \mathsf{ZFU}$, and $u \in W$, then we define $H_u := \mathrm{tcl}^{\mathbf{W}}(u) \cup \mathbb{N}^{\mathbf{W}} \cup \mathbb{U}^{\mathbf{W}}$, $E_u := \hat{\in} \cap H_u \times H_u$ and the function ℓ by $\ell(\omega) := \varnothing^{\mathbf{W}}$ and $\ell(n) := \mathbb{u}_n^{\mathbf{W}}$. Then $\langle H_u, E_u, u, \ell\rangle$ is a ZFU-graph. Note that while we gave the definitions informally, they can be given within the models $\mathbf{V}$ and $\mathbf{W}$, respectively, and we denote by $\mathbf{G}_x^{\mathbf{V}}$ and $\mathbf{H}_u^{\mathbf{W}}$ the elements of $\mathbf{V}$ and $\mathbf{W}$ that are the ZF-graph associated to x and the ZFU-graph associated to u, respectively.

Proposition 1. Let $\mathbf{M} = \langle M, \in_0\rangle$ or $\mathbf{M} = \langle M, \in_0, F_0, U_0\rangle$ be a model of either ZF or ZFU, and let $V, \in, W, \hat{\in}, \hat{F}, \hat{U}$ be definable subclasses such that $\mathbf{V} := \langle V, \in\rangle \models \mathsf{ZF}$ and $\mathbf{W} := \langle W, \hat{\in}, \hat{F}, \hat{U}\rangle \models \mathsf{ZFU}$. Let $\mathbf{G} = \langle G, E, \nu\rangle \in M$ be a ZF-graph and $\mathbf{H} = \langle H, E, \nu, \ell\rangle \in M$ be a ZFU-graph.

(1) There are $\mathbf{M}$-definable operations $\mathrm{set}^{\mathbf{M},\mathbf{V}}$ and $\mathrm{iset}^{\mathbf{M},\mathbf{W}}$ such that $\mathrm{set}^{\mathbf{M},\mathbf{V}}(\mathbf{G}) \in V$ and $\mathrm{iset}^{\mathbf{M},\mathbf{W}}(\mathbf{H}) \in W$, $\mathbf{G}^{\mathbf{V}}_{\mathrm{set}^{\mathbf{M},\mathbf{V}}(\mathbf{G})}$ is isomorphic to $\mathbf{G}$ (as pointed graphs) and $\mathbf{H}^{\mathbf{W}}_{\mathrm{iset}^{\mathbf{M},\mathbf{W}}(\mathbf{H})}$ is isomorphic to $\mathbf{H}$ (as labelled pointed graphs).

(2) The operations $\mathrm{set}^{\mathbf{M},\mathbf{V}}$ and $\mathrm{iset}^{\mathbf{M},\mathbf{W}}$ are injective up to isomorphism, *i.e.*, if $\mathbf{G}_0$ and $\mathbf{G}_1$ are isomorphic as pointed graphs

and $\mathbf{H}_0$ and $\mathbf{H}_1$ are isomorphic as labelled pointed graphs, then $\mathrm{set}^{\mathbf{M},\mathbf{V}}(\mathbf{G}_0) = \mathrm{set}^{\mathbf{M},\mathbf{V}}(\mathbf{G}_1)$ and $\mathrm{iset}^{\mathbf{M},\mathbf{W}}(\mathbf{H}_0) = \mathrm{iset}^{\mathbf{M},\mathbf{W}}(\mathbf{H}_1)$.

(3) If $x \in_0 y$, then $\mathbf{G}_x$ is a subgraph of $\mathbf{G}_y$, and if $\mathbf{G} = \langle G, E, \nu\rangle$ is a ZF-graph and a subgraph of $\mathbf{G}^{\mathbf{V}}_x$ for some $x \in \mathbf{V}$, then $\mathrm{set}^{\mathbf{M},\mathbf{V}}(\mathbf{G}) \in x$.

(4) Similarly, if $u \in_0 v$, then $\mathbf{H}_u$ is a subgraph of $\mathbf{H}_v$, and if $\mathbf{H} = \langle H, E, \nu, \ell\rangle$ is a ZFU-graph and a subgraph of $\mathbf{H}^{\mathbf{W}}_u$ for some $u \in \mathbf{W}$, then $\mathrm{iset}^{\mathbf{M},\mathbf{W}}(\mathbf{H}) \mathrel{\hat{\in}} u$.

Proof. The operations $\mathrm{set}^{\mathbf{M},\mathbf{V}}$ and $\mathrm{iset}^{\mathbf{M},\mathbf{W}}$ are defined by transfinite recursion along the wellfounded relations $\in$ and $\hat{\in}$ in the models $\mathbf{V}$ and $\mathbf{W}$ in the obvious way by translating the elements of the graph into elements of V or W and finally reading off the value by looking at the value of ν (in the ZFU-case, we are assigning $\mathbb{u}^{\mathbf{W}}_i$ to the node $n \in H$ with $\ell(i) = n$ and $\varnothing^{\mathbf{W}}$ to the node n with $\ell(\omega) = n$). The assignment function produced during this process serves as an isomorphism between $\mathbf{G}$ and $\mathbf{G}^{\mathbf{V}}_{\mathrm{set}^{\mathbf{M},\mathbf{V}}(\mathbf{G})}$, and $\mathbf{H}$ and $\mathbf{H}^{\mathbf{W}}_{\mathrm{iset}^{\mathbf{M},\mathbf{W}}(\mathbf{H})}$.

The injectivity up to isomorphism follows immediately from the isomorphy of the original graph with the associated ZF- or ZFU-graph. $\square$

4.2. **Transforming graphs.** Now we shall describe operations that link ZF- and ZFU-graphs. We work in a model $\mathbf{M}$ of either ZF or ZFU.

Let $\mathbf{G} = \langle G, E, \nu\rangle$ be a ZF-graph with special subset $N = \{n_i\,;\, i \in \mathbb{N}\} \subseteq G$. We split up the set N into an even part $N_0 := \{n_{2i}\,;\, i \in \mathbb{N}\}$ and an odd part $N_1 := \{n_{2i+1}\,;\, i \in \mathbb{N}\}$ and use N_0 as the natural numbers and N_1 as the *urelements* in the definition of a ZFU-graph.

Define

$$nE^*n' \iff (n = n_{2i}\ \&\ n' = n_{2i+2}) \text{ or } (n' \notin N\ \&\ nEn'),$$

$$\ell(\omega) = n_0, \text{ and } \ell(i) = n_{2i+1}.$$

The following is obvious:

Proposition 2. If $\langle G, E, \nu\rangle$ is a ZF-graph and E^* and ℓ are defined as above, then $\langle G, E^*, \nu, \ell\rangle$ is a ZFU-graph. We denote it by $\mathbf{zfu}(\mathbf{G})$.

In words: In a ZF-graph, n_0 takes the rôle of $0 = \varnothing$ and n_{i+1} takes the rôle of $i+1 = \{i\}$. In order to make a ZFU-graph out of it, we have to designate nodes as the natural numbers and others as the *urelements*. The node n_{2i} will take the rôle of $\{i\}$ and n_{2i+1} will take the rôle of $\mathbb{u}_i$. All other edges stay the same, so, for instance, a node that was representing $\{1, 2, 7, \{3, 10\}\}$ in a ZF-graph $\mathbf{G}$, will be representing $\{\mathbb{u}_0, 1, \mathbb{u}_3, \{\mathbb{u}_1, 5\}\}$ in $\mathbf{zfu}(\mathbf{G})$.

For the other direction, let $\mathbf{H} = \langle H, E, \nu, \ell\rangle$ be a ZFU-graph with special subsets $A = \{a_i\,;\, i \in \mathbb{N}\}$ and $N = \{n_i\,;\, i \in \mathbb{N}\}$. If we define

$$\begin{aligned} nE^*n' \quad \iff \quad & (n = a_i \ \&\ n' = n_{i+1}) \text{ or } (n = n_i \ \&\ n' = a_i) \text{ or} \\ & (n' \notin N \ \&\ nEn'), \end{aligned}$$

then again, the following is obvious:

Proposition 3. If $\langle H, E, \nu, \ell\rangle$ is a ZFU-graph and E^* is defined as above, then $\langle H, E^*, \nu\rangle$ is a ZF-graph. We denote it by $\mathbf{zf}(\mathbf{H})$.

Note that, clearly, the two operations are inverses of each other, and so $\mathbf{G} = \mathbf{zf}(\mathbf{zfu}(\mathbf{G}))$ and $\mathbf{H} = \mathbf{zfu}(\mathbf{zf}(\mathbf{H}))$.

4.3. **Graphs in submodels.** For the following, suppose that $\mathbf{V} = \langle V, \in\rangle$ is a model of ZF, and that $\mathbf{W}^{\mathbf{V}}$ is the model of ZFU inside $\mathbf{V}$ defined in Section 3.1. We shall be working with the usual Kuratowski pairing function, so

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\},$$

and, consequently, in $\mathbf{W}^{\mathbf{V}}$, we have

$$\langle u, v\rangle^{\mathbf{W}^{\mathbf{V}}} = \langle 1, \{\langle 1, \{u\}\rangle, \langle 1, \{u, v\}\rangle\}\rangle.$$

Suppose that $\mathbf{W}^{\mathbf{V}} \models$ "$\mathbf{H} = \langle H, E, \nu, \ell\rangle$ is a ZFU-graph". Then we can define an isomorphic ZFU-graph in $\mathbf{V}$ as follows. Let $H = \langle 1, x\rangle$ and $E = \langle 1, y\rangle$. Since $\mathbf{W}^{\mathbf{V}}$ thinks that $\langle H, E\rangle^{\mathbf{W}^{\mathbf{V}}}$ is a graph, we know that the ($\in$-)elements of E are of the form

$$\langle 1, \{\langle 1, \{u\}\rangle, \langle 1, \{u, v\}\rangle\}\rangle$$

for some u and v such that $u \mathrel{\hat\in} H$ and $v \mathrel{\hat\in} H$.

We work in $\mathbf{V}$ and define a $\mathbf{V}$-graph $\mathbf{H}^\natural$. Let $H^\natural := \{u\,;\, u \mathrel{\hat\in} H\}$ and for $u, v \in H^\natural$, we define

$$u\, E^\natural\, v :\iff \langle 1, \{\langle 1, \{u\}\rangle, \langle 1, \{u, v\}\rangle\}\rangle \mathrel{\hat\in} E.$$

For the definition of $\ell^\natural$, let Z be the $\mathbf{V}$-function with $\mathrm{dom}(Z) = \omega + 1$ such that $Z(x)$ is the unique element of $\mathbf{W}^{\mathbf{V}}$ representing x. Then

$$\begin{aligned} \ell^\natural(x) = u \quad :\iff & \quad \langle 1, \{\langle 1, \{Z(x)\}\rangle, \langle 1, \{Z(x), u\}\rangle\}\rangle \mathrel{\hat\in} \ell \\ \iff & \quad \mathbf{W}^{\mathbf{V}} \models \ell(Z(x)) = u. \end{aligned}$$

Proposition 4. Work inside $\mathbf{V}$. If $\mathbf{W}^{\mathbf{V}} \models$ "$\mathbf{H} = \langle H, E, \nu, \ell\rangle$ is a ZFU-graph" and $H^\natural$, $E^\natural$ and $\ell^\natural$ are defined as above, then $\mathbf{H}^\natural = \langle H^\natural, E^\natural, \nu, \ell^\natural\rangle$ is a ZFU-graph.

Moreover, $\mathrm{iset}^{\mathbf{V}, \mathbf{W}^{\mathbf{V}}}(\mathbf{H}^\natural) = \mathrm{iset}^{\mathbf{W}^{\mathbf{V}}, \mathbf{W}^{\mathbf{V}}}(\mathbf{H})$.

Of course, there is no need for a similar retraction between $\mathbf{W}$ and $\mathbf{V}^{\mathbf{W}}$, as the element relation stays the same when you move from $\mathbf{W}$ to $\mathbf{V}^{\mathbf{W}}$, so if $\mathbf{V}^{\mathbf{W}} \models$ "$\mathbf{G}$ is a ZF-graph", then $\mathbf{G}$ literally is a ZF-graph in $\mathbf{W}$.

5. The synonymy of ZF and ZFU

In the following, we shall use the operations $x \mapsto \mathbf{G}_x$, $u \mapsto \mathbf{H}_u$, $\mathrm{set}^{\mathbf{W},\mathbf{V}^{\mathbf{W}}}$, $\mathrm{iset}^{\mathbf{V},\mathbf{W}^{\mathbf{V}}}$, $\mathbf{zf}$, and $\mathbf{zfu}$ to define an interpretation of ZFU in ZF which is a synonymy.

5.1. **Interpreting ZFU inside $\mathbf{V}$ (second version).** We start with a model $\mathbf{V} = \langle V, \in \rangle$ of ZF. By the work from Section 3.1 and Proposition 2, the operation

$$I : x \mapsto \mathbf{G}_x \mapsto \mathbf{zfu}(\mathbf{G}_x) \mapsto \mathrm{iset}^{\mathbf{V},\mathbf{W}^{\mathbf{V}}}(\mathbf{zfu}(\mathbf{G}_x))$$

is definable in $\mathbf{V}$. We define a translation

$$T^*_{\mathsf{ZFU},\mathsf{ZF}} = \langle \delta, \langle \dot{\in}, \Xi_{\dot{\in}} \rangle, \langle \dot{F}, \Xi_{\dot{F}} \rangle, \langle \dot{U}, \Xi_{\dot{U}} \rangle \rangle$$

with

$$\begin{aligned}
\delta(\mathsf{v}_0) &\doteq \mathsf{v}_0 \dot{=} \mathsf{v}_0,\\
\Xi_{\dot{\in}}(\mathsf{v}_0, \mathsf{v}_1) &\doteq \Phi_{\dot{\in}}(I(\mathsf{v}_0), I(\mathsf{v}_1)),\\
\Xi_{\dot{F}}(\mathsf{v}_0, \mathsf{v}_1) &\doteq \Phi_{\dot{F}}(I(\mathsf{v}_0), I(\mathsf{v}_1)), \text{ and}\\
\Xi_{\dot{U}}(\mathsf{v}_0) &\doteq \Phi_{\dot{U}}(I(\mathsf{v}_0)).
\end{aligned}$$

In order to show that this translation induces an interpretation, define relations $\in^*$, F^* and U^* on $\mathbf{V}$, defined via the mentioned formulas: $x \in^* y :\iff \Xi_{\dot{\in}}(x, y)$, $F^*(x, y) :\iff \Xi_{\dot{F}}(x, y)$, and $x \in U^* :\iff \Xi_{\dot{U}}(x)$.

Proposition 5. The operation $I : \langle V, \in^*, F^*, U^* \rangle \prec \langle W^{\mathbf{V}}, \hat{\in}, \hat{F}, \hat{U} \rangle$ is an elementary embedding.

Proof. This is proved by induction on the formula complexity. The only interesting step is the universal quantifier. Suppose that u witnesses that $\mathbf{W}^{\mathbf{V}} \models \neg\forall \mathsf{v}_0\, \psi$, *i.e.*, $\mathbf{W}^{\mathbf{V}} \models \neg\psi[u]$. Let $\mathbf{H}_u^{\mathbf{W}^{\mathbf{V}}}$ be the ZFU-graph of u as defined in $W^{\mathbf{V}}$. We use the $\natural$-operation defined from Proposition 4 and get a graph $\mathbf{H}^\natural := (\mathbf{H}_u^{\mathbf{W}^{\mathbf{V}}})^\natural$ in $\mathbf{V}$ such that

$$\mathrm{iset}^{\mathbf{V},\mathbf{W}^{\mathbf{V}}}(\mathbf{H}^\natural) = \mathrm{iset}^{\mathbf{W}^{\mathbf{V}},\mathbf{W}^{\mathbf{V}}}(\mathbf{H}_u^{\mathbf{W}^{\mathbf{V}}}) = u.$$

Now let $x := \mathrm{set}^{\mathbf{V},\mathbf{V}}(\mathbf{zf}(\mathbf{H}^\natural))$. Then $I(x) = u$, and thus by the induction hypothesis $\langle V, \in^*, F^*, U^* \rangle \models \neg\psi[x]$, whence $\langle V, \in^*, F^*, U^* \rangle \models \neg\forall \mathsf{v}_0\, \psi$. $\square$

Corollary 6. The translation $T^*_{\mathsf{ZFU},\mathsf{ZF}}$ induces an interpretation from ZFU in ZF.

5.2. **Interpreting ZF inside $\mathbf{W}$ (second version).** Using the ideas from Section 5.1, we do the same for a ZFU-model $\mathbf{W}$:

We start with a model $\mathbf{W} = \langle W, \hat{\in}, \hat{F}, \hat{U}\rangle$ of ZFU. By the work from Section 3.2 and Proposition 3, the operation

$$J : u \mapsto \mathbf{H}_u \mapsto \mathbf{zf}(\mathbf{H}_u) \mapsto \mathrm{set}^{\mathbf{W},\mathbf{V}^{\mathbf{W}}}(\mathbf{zf}(\mathbf{H}_u))$$

is definable in $\mathbf{W}$. We define a translation

$$T^*_{\mathsf{ZF},\mathsf{ZFU}} = \langle \delta', \langle \dot{\in}, \Upsilon_{\dot{\in}}\rangle\rangle$$

with

$$\begin{array}{rcl} \delta'(\mathsf{v}_0) & \simeq & \mathsf{v}_0 \dot{=} \mathsf{v}_0, \text{ and} \\ \Upsilon_{\dot{\in}}(\mathsf{v}_0, \mathsf{v}_1) & \simeq & \Psi_{\dot{\in}}(J(\mathsf{v}_0), J(\mathsf{v}_1)). \end{array}$$

We define a relation E^* on W by $u\, E^*\, v :\Longleftrightarrow \Upsilon_{\dot{\in}}(u, v)$ and prove in analogy to Proposition 5:

Proposition 7. The operation $I : \langle V, \in^*, F^*, U^*\rangle \prec \langle W^{\mathbf{V}}, \hat{\in}, \hat{F}, \hat{U}\rangle$ is an elementary embedding.

Corollary 8. The translation $T^*_{\mathsf{ZF},\mathsf{ZFU}}$ induces an interpretation from ZF in ZFU.

5.3. **Synonymy.** Everything is prepared to state the main result of this note:

Theorem 9. The theories ZF and ZFU are synonymous (*i.e.*, isomorphic in INT_0).

Proof. We claim that the interpretations $\langle \mathsf{ZFU}, T^*_{\mathsf{ZFU},\mathsf{ZF}}, \mathsf{ZF}\rangle$ and $\langle \mathsf{ZF}, T^*_{\mathsf{ZF},\mathsf{ZFU}}, \mathsf{ZFU}\rangle$ are inverses of each other. For this, let us look at their concatenations

$$K := \langle \mathsf{ZFU}, T^*_{\mathsf{ZFU},\mathsf{ZF}}, \mathsf{ZF}\rangle \circ \langle \mathsf{ZF}, T^*_{\mathsf{ZF},\mathsf{ZFU}}, \mathsf{ZFU}\rangle$$

and

$$L := \langle \mathsf{ZF}, T^*_{\mathsf{ZF},\mathsf{ZFU}}, \mathsf{ZFU}\rangle \circ \langle \mathsf{ZFU}, T^*_{\mathsf{ZFU},\mathsf{ZF}}, \mathsf{ZF}\rangle.$$

Let $\tau_K = \langle \Delta^L, \langle \dot{\in}, \Delta^K_{\dot{\in}}\rangle, \langle \dot{F}, \Delta^K_{\dot{F}}\rangle, \langle \dot{U}, \Delta^K_{\dot{U}}\rangle\rangle$ and $\tau_L = \langle \Delta^L, \langle \dot{\in}, \Delta^L_{\dot{\in}}\rangle\rangle$ be the translations defining K and L. It is obvious that both Δ^L and Δ^K are the trivial condition, so we have to show the following:

(1) $\mathsf{ZF} \vdash \mathsf{v}_0 \mathrel{\dot{\in}} \mathsf{v}_1 \leftrightarrow J(I(\mathsf{v}_0)) \mathrel{\dot{\in}} J(I(\mathsf{v}_1))$,

(2) $\mathsf{ZFU} \vdash \mathsf{v}_0 \mathrel{\dot{\in}} \mathsf{v}_1 \leftrightarrow I(J(\mathsf{v}_0)) \mathrel{\dot{\in}} I(J(\mathsf{v}_1))$,

(3) $\mathsf{ZFU} \vdash \dot{F}(\mathsf{v}_0, \mathsf{v}_1) \leftrightarrow \dot{F}(I(J(\mathsf{v}_0)), I(J(\mathsf{v}_1)))$, and

(4) $\mathsf{ZFU} \vdash \dot{U}(\mathsf{v}_0) \leftrightarrow \dot{U}(I(J(\mathsf{v}_0)))$.

As all of these proofs are rather similar, let us focus on the proof of (1): Let us work in some model $\mathbf{V} = \langle V, \in\rangle$ of ZF. Then

$$J(I(x)) = \mathrm{set}^{\mathbf{W}^{\mathbf{V}},\mathbf{V}^{\mathbf{W}^{\mathbf{V}}}}\left(\mathbf{zf}\left(\mathbf{H}^{\mathbf{W}^{\mathbf{V}}}_{\mathrm{iset}^{\mathbf{V},\mathbf{W}^{\mathbf{V}}}(\mathsf{zfu}(\mathbf{G}_x))}\right)\right).$$

To reduce notation, let's write

$$☿_x := \mathbf{H}^{\mathbf{W}^{\mathbf{V}}}_{\mathrm{iset}^{\mathbf{V},\mathbf{W}^{\mathbf{V}}}(\mathsf{zfu}(\mathbf{G}_x))}$$

(note that this is a labelled pointed graph in $\mathbf{W}^{\mathbf{V}}$). By Proposition 1, we have that $☿_x$ is isomorphic as a labelled pointed graph (in $\mathbf{V}$) to $\mathbf{zfu}(\mathbf{G}_x)$, so that $\mathbf{zf}(☿_x)$ is isomorphic as a pointed graph to $\mathbf{G}_x$ (again, in $\mathbf{V}$).

Now if $\mathbf{V} \models y \mathrel{\dot\in} z$, then by iterated applications of Proposition 1 (3) and (4), $\mathbf{zf}(☿_y)$ is a subgraph of $\mathbf{zf}(☿_z)$, and thus

$$\mathbf{V}^{\mathbf{W}^{\mathbf{V}}} \models J(I(y)) \mathrel{\dot\in} J(I(z)).$$

For the other direction, we write

$$♄_x := \mathbf{G}^{\mathbf{V}^{\mathbf{W}^{\mathbf{V}}}}_{J(I(x))} = \mathbf{G}^{\mathbf{V}^{\mathbf{W}^{\mathbf{V}}}}_{\mathrm{set}^{\mathbf{W}^{\mathbf{V}},\mathbf{V}^{\mathbf{W}^{\mathbf{V}}}}\left(\mathbf{zf}\left(\mathbf{H}^{\mathbf{W}^{\mathbf{V}}}_{\mathrm{iset}^{\mathbf{V},\mathbf{W}^{\mathbf{V}}}(\mathsf{zfu}(\mathbf{G}_x))}\right)\right)}.$$

We assume that $\mathbf{V}^{\mathbf{W}^{\mathbf{V}}} \models J(I(y)) \mathrel{\dot\in} J(I(z))$, and apply Proposition 1 (1) to see that (in $\mathbf{V}$), $♄_y$ is isomorphic to $\mathbf{zf}(☿_y)$ and thus (again by Proposition 1 (1) isomorphic to $\mathbf{G}_y$. Similarly, $♄_z$ is isomorphic to $\mathbf{G}_z$. But by our assumption, $♄_y$ is a subgraph of $♄_z$, and so $\mathbf{G}_y$ is isomorphic to a subgraph of $\mathbf{G}_z$. This yields that $y \in z$. □

References

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