MODAL FRAME CORRESPONDENCES GENERALIZED

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Abstract

Taking Löb's Axiom in modal provability logic as a running thread, we discuss some general methods for extending modal frame correspondences, mainly by adding fixed-point operators to modal languages as well as their correspondence languages. Our suggestions are backed up by some new results – while we also refer to relevant work by earlier authors . But our main aim is advertizing the perspective, showing how modal languages with fixed-point operators are a natural medium to work with.

1 Introduction: easy and hard correspondences

The topic of this paper goes back to the mid 1970s, when a young Amsterdam logic circle including Wim Blok, Dick de Jongh and the present author, with visitors such as Craig Smorynski, was picking up an interest in modal logic. One special interest in those days has remained important since, viz.

Löb's axiom $[]([]p \to p) \to []p$

in the provability logic of arithmetic. This principle was discovered by Martin Löb, then one of our senior professors. At the time, I had just started working on modal correspondence theory for analyzing the relational frame content of modal axioms. This works quite neatly for the usual modal axioms such as the

K4-axiom $[]p \to [][]p$

Let us call a modal formula ϕ true at a point s in a frame $\boldsymbol{F} = (W, R)$ if it is true at s under all atomic valuations V on $\boldsymbol{F}$. Here is perhaps the most famous correspondence:

Fact 1 $\boldsymbol{F}, s \models []p \to [][]p$ iff $\boldsymbol{F}$'s accessibility relation R is *transitive* at s: i.e., $\boldsymbol{F}, s \models \forall y(Rxy \to \forall z(Ryz \to Rxz))$.

Proof If the relation is transitive, $[]p \to [][]p$ clearly holds under every valuation. Conversely, let $\boldsymbol{F}, s \models []p \to [][]p$. In particular, the *K4*-axiom will hold if we take $V(p)$ to be $\{y \mid Rsy\}$. But then, the antecedent $[]p$ holds at s, and hence so does the consequent $[][]p$. And the latter states the transitivity, by the definition of $V(p)$. ♣

Now Löb's Axiom was a challenge, as it does not fit this simple pattern of analysis. One day in 1973, I found a semantic argument for its correct correspondence:

Fact 2 Löb's Axiom is true at point s in a frame $\boldsymbol{F} = (W, R)$ iff

(a) $\boldsymbol{F}$ is upward R-well-founded starting from s, and also

(b) $\boldsymbol{F}$ is transitive at s.

Proof First, Löb's Axiom implies transitivity. Let Rsx and Rxy, but not Rsy. Setting $V(p) = W - (x, y\}$ makes Löb's Axiom false at s. Next, let (b) hold. If (a) fails, there is an ascending sequence $s = s_0\, R\, s_1\, R\, s_2\, \ldots$ and setting $V(p) = W - \{s_0, s_1, s_2, \ldots\}$ refutes Löb's Axiom at s. Conversely, if Löb's Axiom fails at s, there is an infinite upward sequence of $\neg p$-worlds. This arises by taking any successor of s where p fails, and repeatedly applying the truth of $[]([]p \rightarrow p)$ – using transitivity. ♣

Here the transitivity clause (b) was surprising, as the modal *K4*-axiom had always been postulated separately in provability logic. The next day, Dick de Jongh came up with a beautiful purely modal derivation of the transitivity axiom from Löb's Axiom. It revolved around one well-chosen substitution instance

$$[]([]([]p \wedge p) \rightarrow ([]p \wedge p)) \rightarrow []([]p \wedge p).$$

Remark 1 Consequence via substitution

In the preceding semantic frame argument deriving clause (b), there is a matching substitution of a first-order definable predicate for the proposition letter p:

$$V(p) := \{y \mid Rsy \wedge \forall z(Ryz \rightarrow Rsz)\}$$

With this choice of a predicate p, $[]([]p \rightarrow p)$ holds at s, and hence so does $[]p$ by the validity of the Löb Axiom – and transitivity then follows by the definition of $V(p)$. This theme of modal deduction via suitable set-based substitutions is pursued more systematically in d'Agostino, van Benthem, Montanari & Policriti 1997. ♣

Later, Wim Blok got into the game, and found the much more complex derivation of $[]p \rightarrow [][]p$ from Grzegorczyk's Axiom, the counterpart of Löb's Axiom on reflexive frames, using algebraic methods. (Cf. van Benthem & Blok 1978.) This was to have been one of many illustrations in a planned joint book on modal logic and universal algebra, commissioned by Anne Troelstra for "Studies in Logic" as a merge of our dissertations. The book never happened, though chapter drafts are still lying around. Our friendship remained, however, from those days until Wim's passing away.

The present paper collects a few observations about the behaviour of Löb's Axiom, as a stepping stone for exploring some generalizations of modal frame correspondence. My main concern is whether the usual correspondence arguments can deliver more than they do in their traditional formulation. I think they do.

2 Modal correspondence: from first-order to fixed-points

Let us look first at the general reason behind the above K4-example.

2.1 *Frame correspondence by first-order substitutions*

Here is a result from Sahlqvist 1975, discovered independently in van Benthem 1974:

Theorem 1 There is an algorithm computing first-order frame correspondents for modal formulas $\alpha \rightarrow \beta$ with an antecedent α constructed from atoms possibly prefixed by universal modalities, conjunction, disjunction, and existential modalities, and the consequent β any syntactically positive modal formula.

The translation algorithm for obtaining first-order frame properties works as follows:

(a) *Translate* the modal axiom into its standard first-order form, prefixed with monadic quantifiers for proposition letters: $\forall x\ \forall \mathbf{P}\ ST(\phi)(\mathbf{P}, x)$,

(b) Pull out all existential modalities occurring in the antecedent, and turn them into bounded universal quantifiers in the prefix,

(c) Compute a first-order *minimal valuation* for the proposition letters making the remaining portion of the antecedent true,

(d) *Substitute* this definable valuation for the proposition letters occurring in the body of the consequent – and if convenient,

(e) Perform some simplifications modulo logical equivalence.

For details of this 'substitution algorithm' and a proof of its semantic correctness, cf. Blackburn, de Rijke &Venema 2000.

Example 1 For the modal transitivity formula $[]p \rightarrow [][]p$,

(a) yields $\forall x{:}\ \forall P{:}\ \forall xy\,(Rxy \rightarrow Py) \rightarrow \forall z\,(Rxz \rightarrow \forall u\,(Rzu \rightarrow Pu))$,
(b) is vacuous, while (c) yields the minimal valuation $Ps := Rxs$ – and then
(d) substitution yields $\forall x{:}\ \forall xy(Rxy \rightarrow Rxy) \rightarrow \forall z(Rxz \rightarrow \forall u(Rzu \rightarrow Rxu))$.
The latter simplifies to the usual form $\forall x{:}\ \forall z\,(Rxz \rightarrow \forall u\,(Rzu \rightarrow Rxu))$. ♣

Concrete modal principles not covered by the substitution method are Löb's Axiom – and also the following formula, whose antecedent typically has the wrong form:

McKinsey Axiom $[]<>p \rightarrow <>[]p$

The McKinsey Axiom is not first-order definable (van Benthem 1974, 1983).

2.2 An excursion on scattering

The substitution method is quite strong. In particular, the above procedure also works if all modalities are entirely *independent*, as in the following variant of the *K4*-axiom:

Fact 3 $[1]p \rightarrow [2][3]p$ also has a first-order frame correspondent, computed in exactly the same fashion, viz. $\forall x: \forall z\ (R_2xz \rightarrow \forall u\ (R_3zu \rightarrow R_1xu))$.

Here is the relevant general notion.

Definition 1 The *scattered version* of a modal formula ϕ arises by marking each modality in ϕ uniquely with an index for its own accessibility relation. ♣

The Sahlqvist Theorem applies to the scattered version of any implication of the above sort. The reason is that its conditions make statements about individual occurrences: they do not require pairwise co-ordination of occurrences. This sort of condition is frequent in logic, and hence many results have more general scattered versions. Scattering is of interest for several reasons. It suggests *most general versions* of modal results – and the interplay of many different modalities in a single formula fits with the current trend toward *combining logics*. E.g., in provability logic, different boxes could stand for the provability predicates of different arithmetical theories – not just that of Peano Arithmetic. Even so, scattering does not always apply:

Theorem 2 There are first-order frame-definable modal formulas whose scattered versions are not first-order frame-definable.

Proof Consider the first-order definable modal formula which conjoins the *K4* transitivity axiom with the McKinsey Axiom (cf. van Benthem 1983):

$$([]p \rightarrow [][]p)\ \&\ ([]<>p \rightarrow <>[]p)$$

Even its partly scattered version $([1]p \rightarrow [1][1]p)\ \&\ ([2]<2>p \rightarrow <2>[2]p)$ is not first-order definable. In any frame, taking the universal relation for R_1 will verify the left conjunct, and so, substituting these, the purported total first-order equivalent would become a first-order equivalent for the McKinsey axiom: *quod non*. ♣

Remark 2 Scattering proposition letters.

One can also make each occurrence of a *proposition letter* unique in modal formulas. But this sort of scattering makes any modal axiom first-order definable! First, propositionally scattered formulas are either upward or downward monotone in each proposition letter p, depending on the polarity of p's single occurrence. Now van Benthem 1983 shows that modal formulas $\phi(p)$ which are upward (downward) monotone in p are frame-equivalent to $\phi(\bot)$ $(\phi(T))$. So, propositionally scattered formulas are frame-equivalent to closed ones, and the latter are all first-order definable. ♣

2.3 *Frame correspondence and fixed-point logic*

Löb's Axiom is beyond the syntactic range of the Sahlqvist Theorem, as its antecedent has a modal box over an implication. But still, its frame-equivalent of transitivity plus well-foundedness, though not first-order, is definable in a natural extension – viz. *LFP(FO): first-order logic with fixed-point operators* (Ebbinghaus & Flum 1995).

Fact 4 The *well-founded part* of any binary relation R is definable as a smallest fixed-point of the monotone set operator $[](X) = \{y \mid \forall z(Ryz \to z \in X)\}$.

The simple proof is, e.g., in Aczel 1977. The well-founded part can be written in the language of *LFP(FO)* as the smallest-fixed-point formula $\mu P, x\bullet\ \forall y\ (Rxy \to Py)$.

How can we find modal frame equivalents of this extended *LFP(FO)*-definable form as systematically as first-order frame conditions? The following subsection presents some relevant results from van Benthem 2004A – while the idea of fixed-point -based correspondences has also been investigated by different methods in Nonnengart & Szalas 1999, Goranko & Vakarelov 2003. For a start, Löb's Axiom suggests a general principle, as the *minimal valuation* step in the substitution algorithm still works. Consider the antecedent $[]([]p \to p)$. If this modal formula holds anywhere in a model $\boldsymbol{M}, x$, then there must be a smallest predicate P for p making it true at $\boldsymbol{M}, x$ – because of this set-theoretic property guaranteeing a minimal verifying predicate:

Fact 5 If $[]([]p_i \to p_i)$ holds at a world x for all $i \in I$,

then $[]([]P \to P)$ holds at x for $P = \bigcap_{i \in I} [[p_i]]$.

This fact is easy to check. Here is the more general notion behind this observation.

Definition 2 A first-order formula $\phi(P, \boldsymbol{Q})$ has the *intersection property* if, in every model $\boldsymbol{M}$, whenever $\phi(P, \boldsymbol{Q})$ holds for all predicates in some family $\{\boldsymbol{P}_i \mid i \in I\}$, it also holds for the intersection, that is: $\boldsymbol{M}, \cap \boldsymbol{P}_i \models \phi(P, \boldsymbol{Q})$. ♣

Now, the Löb antecedent displays a typical syntactical format which ensures that the intersection property must hold. We can specify this more generally as follows.

Definition 3 A first-order formula is a *PIA condition* – this is a short-hand for: 'positive antecedent implies atom' – if it has the following syntactic form:

$$\forall \mathbf{x}(\phi(P, \mathbf{Q}, \mathbf{x}) \to P\mathbf{x}) \quad \text{with } P \text{ occurring only positively in } \phi(P, \mathbf{Q}, \mathbf{x}). \quad ♣$$

Example 2 Löb's Axiom.

The antecedent has the first-order *PIA* form $\forall y\ ((Rxy\ \&\ \forall z(Ryz \to Pz)) \to Py)$. ♣

Example 3 Horn clauses.

A simpler case of the *PIA* format is the universal Horn clause defining modal accessibility via the transitive closure of a relation *R*: $Px \wedge \forall y \forall z\ ((Py \wedge Ryz) \to Pz))$. The minimal predicate *P* satisfying this consists of all points *R*-reachable from *x*. ♣

It is easy to see that this syntactic format implies the preceding semantic property:

Fact 6 *PIA*-conditions imply the Intersection Property.

By way of background, here is a general model-theoretic preservation result:

Theorem 3 The following are equivalent for all first-order formulas $\phi(P, \mathbf{Q})$:

(a) $\phi(P, \mathbf{Q})$ has the Intersection Property w.r.t. predicate *P*

(b) $\phi(P, \mathbf{Q})$ is definable by a conjunction of *PIA* formulas.

For our purposes, we rather need to know what minimal predicates defined using the Intersection Property look like. Here, standard fixed-point logic provides an answer:

Fact 7 The minimal predicates for *PIA*-conditions are definable in *LFP(FO)*.

Example 3 already provided an illustration, as the transitive closure of an accessibility relation used there is typically definable in *LFP(FO)*. Here is another one:

Example 4 Computing the minimal valuation for Löb's Axiom.

Analyzing *[]([]p→p)* a bit more closely, the minimal predicate satisfying the antecedent of Löb's Axiom at a world *x* describes the following set of worlds:

$$\{y \mid \forall z\ (Ryz \to Rxz)\ \&\ \text{no infinite sequence of } R\text{-successors starts from } y\}.$$

Then, if we substitute this description into the Löb consequent *[]p*, precisely the usual, earlier-mentioned conjunctive frame condition will result automatically. ♣

Now, plugging these conditions into the above substitution algorithm yields an extension of the Sahlqvist Theorem with a broader class of frame correspondents:

Theorem 4 Modal axioms with *PIA* antecedents and syntactically positive consequents have their corresponding frame conditions definable in *LFP(FO)*.

2.4 Further illustrations, and limits

This extended correspondence method covers much more than the two examples so far. Here are a few more examples of *PIA*-conditions, in variants of Löb's Axiom.

Example 5 Two simple Löb variants.

(a) With the formula $[](<>p \to p) \to []p$, the relevant smallest fixed-point for p in the antecedent is defined by $\mu P, y\bullet\ Rxy \wedge \exists z\ (Ryz \wedge Pz))$, with x the current world. This evaluates to the Falsum $\bot$, and indeed the formula $[](<>p \to p) \to []p$ is frame-equivalent to $[]\bot$, as may also be checked directly. (b) The well-known frame-incomplete 'Henkin variant' of Löb's Axiom reads as follows: $[]([]p \leftrightarrow p) \to []p$. This may be rewritten equivalently as $([]([]p \to p) \wedge [](p \to []p)) \to []p$. Here, the antecedent is a conjunction of *PIA*-forms, and unpacking these as above yields the minimal fixed-point $\mu P, y\bullet\ (Rxy \wedge \forall z\ (Ryz \to Pz)) \vee \exists z\ (Rxz \wedge Pz \wedge Rzy))$. ♣

But also, *scattering* makes sense again to obtain greater generality:

Fact 8 The modal formula $[1]([2]p \to p) \to [3]p$ is equivalent on frames $\mathbf{F} = (W, R_1, R_2, R_3)$ to the conjunction of the two relational conditions

(a) $R_3 ; (R_2)^* \subseteq R_1$ (with $(R_2)^*$ the reflexive-transitive closure of R_2)

(b) upward well-foundedness in the following sense: no world x starts an infinite upward sequence of worlds $x\ R_3\ y_1\ R_2\ y_2\ R_2\ y_3$ ...

Proof Scattered Löb implies the generalized transitivity clause (a) much as it implied transitivity before. Next, assuming the truth of (a), it is easy to see that any failure of scattered Löb produce an infinite upward y–sequence as forbidden in (b), while conversely, any valuation making p false only on such an infinite y-sequence will falsify the scattered Löb Axiom at the world x. ♣

Remark 2 Fact 8 arose out of an email exchange with Chris Steinsvold (CUNY, New York), who had analyzed the partially scattered axiom $[1]([2]p \to p) \to [1]p$. The general correspondence was also found independently by Melvin Fitting. ♣

But we can also look at quite different modal principles in the same way.

Fact 9 The modal axiom $(\langle\rangle p \wedge [](p \to []p)) \to p$ has a *PIA* antecedent whose minimal valuation yields the *LFP(FO)*-frame-condition that, whenever *Rxy*, *x* can be reached from *y* by some finite sequence of successive *R*-steps.

The complexity of the required substitutions can still vary considerably here, depending on the complexity of reaching the smallest fixed-point for the antecedent via the usual bottom-up ordinal approximation procedure. E.g., obtaining the well-founded part of a relation may take any ordinal up to the size of the model. But for Horn clauses with just atomic antecedents, the approximation procedure will stabilize uniformly in any model by stage ω, and the definitions will be simpler.

Even so, there are limits. Not every modal axiom yields to the fixed-point approach!

Fact 10 The tense-logical axiom expressing Dedekind Continuity is not definable by a frame condition in *LFP(FO)*.

Proof Dedekind Continuity holds in $(IR, <)$ and fails in $(Q, <)$. But these two relational structures are equivalent w.r.t. *LFP(FO)*-sentences, as there is a *potential isomorphism* between them, for which such sentences are invariant. ♣

Returning to the modal language, one often views the Löb' antecedent $[]([]p \to p)$ and the McKinsey antecedent $[]\langle\rangle p$ as lying at the same level of complexity, beyond Sahlqvist forms. But in the present generalized analysis of minimizable predicates, the latter seems much more complicated than the former!

Conjecture 1 The McKinsey Axiom has no *LFP(FO)*-definable correspondent.

3 Modal fixed-point languages

A conspicuous trend in modal logic is the strengthening of modal languages to remove expressive deficits of the base with just *[]*, <>. This reflects a desire for logic design with optimal expressive power, without being hampered by the peculiarities of weaker languages bequeathed to us by our ancestors.

3.1 The modal μ-calculus

One such extended language fits very well with Section 2. It is the modal *μ–calculus* – the natural modal fragment of *LFP(FO)*, and a natural extension of propositional dynamic logic. Harel, Kozen & Tiuryn 2000 has a quick tour of its syntax, semantics, and axiomatics. This formalism can define smallest fixed-points in the format

$\mu p\bullet\ \phi(p)$, provided that p occurs only positively in ϕ.

This adds general syntactic recursion, with no assumption on the accessibility order.

Definition 4 Fixed-point semantics.

In any model ***M***, the formula $\phi(p)$ defines an inclusion-monotone set transformation

$$F_\phi(X) = \{s \in X \mid (\boldsymbol{M}, p{:=}X), s \models \phi\}$$

By the Tarski-Knaster Theorem, the operation F_ϕ must have a smallest fixed-point. This can be reached bottom-up by ordinal approximation stages

$$\phi^0, \quad \ldots, \quad \phi^\alpha, \phi^{\alpha+1}, \quad \ldots, \quad \phi^\lambda, \qquad \ldots$$

with $\phi^0 = \varnothing$, $\phi^{\alpha+1} = F_\phi(\phi^\alpha)$, and $\phi^\lambda = \bigcup_{\alpha<\lambda} \phi^\alpha$

The smallest fixed-point formula $\mu p\bullet\ \phi(p)$ denotes the first stage where $\phi^\alpha = \phi^{\alpha+1}$. ♣

Example 6 Transitive closure and dynamic logic.

The μ-calculus can define a typical transitive closure modality from dynamic logic like 'some ϕ-world is reachable in finitely many R_a-steps':

$$\langle a^*\rangle\phi = \mu p\bullet\ \phi \vee \langle a\rangle p.$$ ♣

Example 7 Well-foundedness again.

The modal import of Fact 4 is this. The smallest fixed-point formula $\mu p\bullet\ []p$ defines the well-founded part of the accessibility relation for *[]* in any modal model. ♣

The μ–calculus also includes *greatest fixed points* $\nu p\bullet\ \phi(p)$, defined as $\neg\mu p\bullet\ \neg\phi(\neg p)$. Smallest and greatest fixed-points need not coincide, and others may be in between.

Finally, we recall that the μ–calculus is decidable, and that its validities are effectively axiomatized by the following two simple proof rules on top of the minimal system K:

(i) $\mu p\bullet\ \phi(p) \leftrightarrow \phi(\mu p\bullet\ \phi(p))$ *Fixed-Point Axiom*

(ii) if $\vdash \phi(\alpha) \to \alpha$, then $\vdash \mu p\bullet\ \phi(p) \to \alpha$ *Closure Rule*

3.2 Working with fixed-points in modal logic

Given that the frame correspondence language for natural modal axioms involves *LFP(FO)* with fixed-point operators, it makes sense to also extend the modal language itself to a complete μ–calculus, restoring the balance between the two. This extended formalism is quite workable, as we will show by a few examples.

For convenience, we dualize the above $<a^*>\phi$ to a dynamic logic-style modality $[]^*\phi$ saying that ϕ is true at all worlds reachable in the transitive closure of the accessibility relation R for single $[]$. The resulting language formalizes earlier correspondence arguments, and it also suggests new variations on modal axioms.

Fact 11 $[]^*([]p \to p) \to []^*p$ defines just upward well-foundedness of R.

Thus, transitivity needs an additional explicit *K4*-axiom, separating the two aspects of Löb's provability logic explicitly. We will return to this way of stating things later.

Next, here is a formal correspondence argument recast as a modal deduction.

Example 8 Scattered Löb Revisited.

The scattered Löb Axiom of Fact 8 implied the frame condition that $R_3 ; (R_2)^* \subseteq R_1$, which corresponds to the modal axiom

$$[1]p \to [3][2^*]p$$

In the right dynamic language this is indeed derivable from a scattered Löb Axiom:

(a) $[1]([2][2^*]p \to [2^*]p) \to [3]\, [2^*]p)$ scattered Löb axiom with $[2^*]p$ for p

(b) $[2^*]p \leftrightarrow p \,\&\, [2][2^*]p$ fixed-point axiom for *

(c) $p \to ([2][2^*]p \to [2^*]p)$ consequence of (b)

(d) $[1]p \to [1]([2][2^*]p \to [2^*]p)$ consequence of (c)

(e) $[1]p \to [3][2^*]p$ from (a), (d) ♣

Another illustration of this modal formalization is the original Fact 2 itself. It says that Löb's Axiom is equivalent to the *K4*-axiom plus the μ--calculus axiom $\mu p \bullet []p$ for upward well-foundedness. But this can also be shown by pure modal deduction!

Theorem 6 Löb's Logic is equivalently axiomatized by the two principles

(a) $[]p \to [][]p$, (b) $\mu p \bullet []p$.

Proof From Löb's Logic to (a) was an earlier-mentioned purely modal deduction. Next, (b) is derived as follows. By the fixed-point axiom of the μ-calculus, we have that $[]\mu p \bullet []p \to \mu p \bullet []p$. So it suffices to get $[]\mu p \bullet []p$. Now Löb's Axiom implies:

$$[]([]\mu p \bullet []p \to \mu p \bullet []p) \to []\mu p \bullet []p$$

and the antecedent of this is derivable by modal Necessitation from the converse of the μ-calculus fixed-point axiom. Next, assume (a) and (b). We show that, in *K4*,

$$\mu p \bullet []p \to ([]([]q \to q) \to []q).$$

By the earlier derivation rule for smallest fixed-points, $\mu p \bullet []p \to \alpha$ can proved for any formula α if $[]\alpha \to \alpha$ can be proved. But we can prove

$$[]([]([]q \to q) \to []q) \to ([]([]q \to q) \to []q)$$

by means of a straightforward derivation in *K4*. ♣

Next, Löb's Axiom implies upward well-foundedness, and hence a form of inductive proof over this well-founded order. Thus, there must also be a direct connection between Löb's Axiom and the *induction axiom* of propositional dynamic logic :

$$([]\phi \ \& \ [*](\phi \to []\phi)) \to [*]\phi \qquad IND$$

Fact 12 Löb's Axiom plus the Fixed-Point Axiom $[*]\phi \leftrightarrow ([]\phi \ \& \ [][*]\phi)$ (*FIX*) derive the Induction Axiom of propositional dynamic logic.

Proof This can be shown using the above analysis of Löb's Axiom, since the Induction Axiom expresses the greatest fixed-point character of $[*]$. An explicit modal derivation is in van Benthem 2004B, which points out that earlier published logics of finite trees have a redundant axiom set of full *PDL* plus Löb's Axiom. ♣

But one can also recast the link between provability logic and fixed-point logics:

Theorem 7 Löb's Logic can be faithfully embedded into the μ–calculus.

Proof The translation doing this works as follows:

(a) replace every *[]* in a formula ϕ by its transitive closure version $[]^*$

(b) for the resulting formula $(\phi)^*$, take the implication $\mu p \bullet [] p \rightarrow (\phi)^*$.

It is straightforward to check that a plain modal formula ϕ is valid on transitive upward well-founded models iff $\mu p \bullet [] p \rightarrow (\phi)^*$ is valid on all models. ♣

As a consequence, decidability of Löb's Logic follows from that of the μ–calculus. Other features may have applications, too, such as the latter's strong interpolation properties (d'Agostino & Hollenberg 2000). Now, the μ–calculus is much more expressive than the usual modal language of provability logic. But this extended setting also raises interesting new issues in the latter area – for instance:

Question 1 Can the usual arithmetical interpretation of provability logic be extended to provability logic with a full _–calculus?

3.3 Frame correspondence in extended modal languages

The _–calculus is just one in a spectrum of extensions of the basic modal language.

Fragments of the _–calculus A useful weaker language is propositional dynamic logic (PDL) with modalities [_] for program expressions _ constructed out of atomic accessibility relations a, b, ... and tests ?__onarbitrary formulas _, using composition ;, union (, and iteration * on binary relations. PDL can deal with most of the preceding examples, witness Fact 11, which says that a PDL-variant of Löb's Axiom defines _p• []p. Further examples of its expressive power will follow in Section 3.4. Example 6 already showed how *PDL* is contained in the μ–calculus. Harel, Kozen & Tiuryn 2000 shows that it is strictly weaker, though.

Fact 13 The fixed-point formula $\neg\mu p \bullet []p$ (or alternatively, $\nu p \bullet <>p$) is not *PDL*- definable.

Proof This formula defines the set of worlds where some infinite *R*-sequence starts, and this set is not *PDL*-definable – by a simple semantic argument. ♣

Looking top-down, this observation shows that the μ–calculus has natural fragments restricting its powers of recursion. One of these already occurred in Fact 9:

Definition 5 The ω–μ–calculus.

The ω–μ–*calculus* only allows smallest fixed-point operators in the following existential format, then approximation sequences always stabilize by stage ω:

$\mu p \bullet \phi(p)$ with ϕ constructed according to the syntax

p | p-free formulas | $\wedge$ | $\vee$ | existential modalities. ♣

Van Benthem 1996 proves a preservation theorem showing the adequacy of this format for the required property of 'finite distributivity' for the approximation maps. Clearly, *PDL* is contained in the ω–μ–calculus. But there is a genuine hierarchy:

Fact 14 The μ–ω–formula $\mu p \bullet ([1]\bot \wedge [2]\bot) \vee (\langle 1\rangle p \wedge \langle 2\rangle p)$ is not definable in *PDL*.

Proof (Sketch) This formula expresses that there is a finite binary tree-like submodel starting from the current world, with both R_1- and R_2-daughters at each non-terminal node. Now *PDL*-formulas only describe reachability along finite traces belonging to some regular language over tests and transitions. This tree property is not like that. ♣

Still, *PDL* is closed under smallest simultaneous fixed-points of a yet more special type of recursion, consisting of disjunctions of existential formulas $\langle\pi\rangle p$ where the propositional recursion variables p occurs only in the end position. We omit details here (cf. van Benthem, van Eijck & Kooi 2005).

Propositional quantifiers But there are further relevant extended modal languages. In particular, the μ–calculus is related to the much stronger system *SOML* of modal logic with second-order quantifiers over proposition letters. Cf. ten Cate 2005 for a recent model-theoretic study of *SOML*. Fact 11 and Theorem 6 suggest the following.

Fact 15 The μ–calculus is definable in *SOML* plus a *PDL*-style iteration modality $[]^*$ referring to all worlds accessible from the current one.

Proof A smallest fixed-point formula $\mu p \bullet \phi(p)$ denotes the intersection of all 'pre-fixed points' of the map $F_\phi(X)$ of Definition 4, where $F_\phi(X) \subseteq X$. But the latter set is also defined by the *SOML*-formula $\forall p: []^*(\phi(p) \to p) \to p$. ♣

The *PDL*-addition is necessary here, since *SOML*-formulas by themselves have a finite modal depth to which they are insensitive, just like basic modal formulas.

We conclude with a concrete example that the new formalisms really extend the old.

Fact 16 Well-foundedness is not definable in basic modal logic.

Proof Suppose that a modal formula ϕ defined well-foundedness. Then it fails at 0 in the frame (N, S), with S the relation of immediate successor. But then, by the finite depth property of basic modal formulas, ϕ would also fail at 0 in some finite frame $(\{0, .., n\}, S)$, which is well-founded. A similar argument works for the above formula $p \to \langle\rangle^* p$, observing that the frames with a partial function R where it holds are just the collections of disjoint finite loops. ♣

The same proof shows that well-foundedness is not even definable in *SOML*.

Frame correspondences in different fixed-point languages Compared with the basic theory, languages with modal fixed-points support interesting new frame correspondences. Some of these occur inside propositional dynamic logic, others crucially involve the μ–calculus, and eventually, one could look at *SOML* as well.

Example 9 Cyclic return simplified.

The formula $p \to \langle\rangle^* p$ says that every point x is part of some finite R-loop. ♣

Example 10 Term rewriting.

The formula $\langle\rangle^*[b]\ ^*p \to [b]\ \langle\rangle^* p$ expresses the Weak Confluence property that points diverging from a common root have a common successor in the transitive closure of the relation. Basic laws of term rewriting (cf. Bezem, Klop & de Vrijer 2003) amount to implications between such modally definable graph properties .

♣

These results are subsumed under the following generalization of Theorem 1. It is by no means the best possible result, but it shows one easy generalization.

Theorem 8 There is an algorithm finding frame-correspondents in *LFP(FO)* for all modal implications $\alpha \to \beta$ whose consequent β is wholly positive, and whose antecedent α is constructed using (a) proposition letters possibly prefixed by universal modalities $[\pi]$ in whose *PDL*-program π all proposition letters occur positively, and over these (b) $\vee$, $\wedge$, as well as existential modalities $<\sigma>$ with a test-free *PDL*-program σ.

Proof (Outline) The main algorithm extracts universal prefixes for the $<\sigma>$ as in Section 2. Next, the dynamic logic operators $[\pi]$ express modal *PIA*-conditions which can be used a basis for minimization inside *LFP(FO)*. ♣

Still, this version seems sub-optimal, as a genuine fixed-point version might describe the relevant syntax very differently.

Example 11 Re-describing modalities.

From a μ–calculus perspective, a universal modality $[a^*]p$ is a greatest fixed-point operator $\nu q \bullet\ p \wedge [a]q$. So, minimizing for p here would compute the formidable-looking iterated fixed-point formula $\mu p \bullet\ \nu q \bullet\ p \wedge [a]q$. One then sees that this is equivalent to the set of worlds a^*-reachable from the current world – which can also be described by one μ-type fixed-point in *LFP(FO)*. ♣

On the other hand, moving to weaker languages, one might drop the universal modalities in Theorem 8, and work inside just the ω–μ–calculus or *PDL*.

Question 2 What is the best possible formulation of the Sahlqvist Theorem in propositional dynamic logic? And in the modal μ–calculus?

Cf. Goranko & Vakarelov 2003 for best results on fixed-point correspondence so far. Nonnengart & Szalas 1999 also provide a very general correspondence method *DLS* going back to Ackermann's Lemma in second-order logic. Finally, the *SCAN* algorithm of Gabbay & Ohlbach 1992 also covers both first- and higher-order cases.

In addition to correspondence issues, there is also modal definability. Many modal formulas in our examples still satisfy the usual semantic properties of basic modal formulas: they are preserved under generated subframes, disjoint unions, *p*-morphic images, and anti-preserved under ultrafilter extensions. The first three hold for all μ–calculus formulas, by their bisimulation invariance. As for anti-preservation under ultrafilter extensions, it is easy to see that the usual proof for the basic modal language does not go through, as some sort of infinite disjunction splitting would be needed. But we have not been able to find a counter-example to the property as such. The typical difference with basic modal formulas might lie really in the finite evaluation bound of the former, as opposed to even *PDL*-formulas involving $<>^*$.

These observations suggest various newe questions. Here we merely state one:

Question 3 Is there a Goldblatt-Thomason Theorem for modal logic with fixed-points, saying that the modally definable *LFP(FO)* frame classes are just those satisfying the stated four semantic preservation properties?

Remark 3 Extended languages and expressive completeness.
Sometimes, a language extension to *tense logic* makes sense to express earlier correspondences compactly. Consider the modal axiom $(\langle a\rangle p \wedge [a](p \rightarrow [a]p)) \rightarrow p$ of Fact 9, expressing a variant of Cyclic Return. This frame property can also be expressed in propositional dynamic logic with a past tense operator as follows:

$$p \rightarrow [a]\langle((PAST\ p)?\ ;\ a)^*\rangle\langle a\rangle p$$

Venema 1991 shows the naturalness of 'versatile' formalisms with converse modalities for the purpose of defining the substitutions of Section 2 inside the modal language. The general point here is that languages with nominals naming specific worlds and backward-looking tense operators define minimal predicate substitutions – making the modal language expressively complete for its own Sahlqvist correspondences. ♣

4 An Excursion into Provability Logic

The μ–calculus is perhaps the most natural modal fixed-point logic. But there are other, and older, modal fixed-point results! This section, which can be skipped without loss of continuity, discusses the linkage between the two perspectives.

4.1 The De Jongh-Sambin fixed-point theorem

A celebrated result in provability logic is the following modal version of the arithmetical Fixed-Point Lemma underlying the proof of Gödel's Theorem:

Theorem 9 (De Jongh-Sambin) Consider any modal formula equivalence $\phi(p, \boldsymbol{q})$ in which proposition letters p only occur in the scope of at least one modality, while $\boldsymbol{q}$ is some sequence of other proposition letters. There exists a formula $\psi(\boldsymbol{q})$ such that $\psi(q) \leftrightarrow \phi(\psi(\boldsymbol{q})), \boldsymbol{q})$ is provable in Löb's Logic, and moreover, any two solutions to this fixed-point equation w.r.t. ϕ are provably equivalent.

For a proof, cf. Smorynski 1984. This author also gives a simple algorithm for computing the fixed-point $\psi(\boldsymbol{q})$. Typical outcomes are the following fixed points:

Example 12 Solving fixed-point equations in provability logic.
Here are a few typical cases:

equation:	$p \leftrightarrow []p$	*solution:*	$p = T$
	$p \leftrightarrow \neg[]p$		$p = \neg[]\bot$
	$p \leftrightarrow ([]p \rightarrow q)$		$p = ([]q \rightarrow q)$

More complex recursions arise when the body of the modal equation has multiple occurrences of p. Solutions are then obtained by iterating the single-step case. ♣

There are two aspects to Theorem 9: (a) existence and uniqueness of the new predicate defined, and (b) explicit definability of that predicate in the modal base language. Here, existence and uniqueness of the predicate p is just a general property of all recursive definitions over well-founded orderings. But we also get the concrete information that this recursive predicate can be defined *inside the original modal language*, without explicit μ- or ν-operators. Let's compare this with the μ–calculus.

4.2 Provability fixed-points and μ-calculus

We can obviously compare the general approximation procedure of Section 3 and the special-purpose algorithm mentioned just now. For a start, evidently, definitions $\mu p \bullet \phi(p)$ with only positive boxed occurrences of p in ϕ fall under both approaches.

Example 13 The fixed-point for the modal equation $p \leftrightarrow []p$.

$\mu p \bullet []p$ defined the well-founded part of the order R. Thus, in well-founded models, it defines the whole universe – which explains Smorynski's solution T ('true'). ♣

But the De Jongh-Sambin Theorem also allows for negative occurrences of p in the defining equation. These fall outside of general fixed-point logics.

Example 14 The fixed-point for the modal equation $p \leftrightarrow \neg[]p$.

Here, the approximation sequence for the set operator $F_{\neg[]p}$ can fail to yield a fixed point, oscillating all the way. E.g., in the model $(N, <)$, one gets $\emptyset, N, \emptyset, N, \ldots$ ♣

Actually, the situation in general fixed-point logic is a bit more complex. Formulas with mixed positive and negative occurrence can sometimes be admissible after all.

Example 15 The mixed-occurrence formula $p \leftrightarrow (p \vee \neg[]p)$.

In this case, the approximation sequence will be monotonically non-decreasing, because of the initial disjunct p. So, in any model, there must be a smallest fixed-point. With our formula $p \vee \neg[]p$, the sequence stabilizes at stage 2, yielding $<>T$. There is also a greatest fixed-point, which is the whole set defined by T. ♣

This case is beyond Theorem 9, as the first occurrence of p in $p \vee \neg[]p$ is not boxed. Indeed, there is no *unique* definability in this extended format, as the smallest and greatest fixed-points are different here. In fixed-point logic, this example motivates an extension of the monotonic case (Ebbinghaus & Flum 1995).

Definition 6 *Inflationary fixed-points* for arbitrary formulas $\phi(p, \mathbf{q})$ without syntactic restrictions on the occurrences of p are computed using an ordinal approximation sequence which forces upward cumulation at successor steps:

$\phi^{\alpha+1} = \phi^{\alpha} \cup \phi(\phi^{\alpha})$, taking unions again at limit ordinals. ♣

There is no guarantee that a set P where this stabilizes is a fixed-point for the modal formula $\phi(p, \mathbf{q})$. It is rather a fixed-point for the modified formula $p \vee \phi(p, \mathbf{q})$.

4.3 Combining the two sorts of fixed-point

Comparison may also mean combination. Would *adding* general monotone fixed-points extend the scope of the De Jongh-Sambin result? The answer is no.

Fact 17 Any p-positive formula $\mu p\bullet \phi(p)$ with $\phi(p)$ having unboxed occurrences of p is equivalent to one in which all occurrences of p occur boxed.

Proof Without loss of generality, we can take the formula to be of the form

$\mu p \bullet (p\&A) \vee B$ with only boxed occurrences of p in A, B

Let ϕ^α be the approximation sequence for $\phi = (p\&A) \vee B$, and let B^α be such a sequence executed separately for the formula B. We have the following collapse:

Lemma 1 $\phi^\alpha = B^\alpha$ for all ordinals α

This is proved by induction. The zero and limit cases are obvious. Next,

$$\phi^{\alpha+1} = (\phi^\alpha \ \& \ A(\phi^\alpha)) \vee B(\phi^\alpha)$$

$$= (B^\alpha \ \& \ A(B^\alpha)) \vee B(B^\alpha)$$

where, by the fact that F_B is monotone: $B^\alpha \subseteq B(B^\alpha)$, and hence $B^\alpha \cap A(B^\alpha)) \subseteq B(B^\alpha)$

$$= B(B^\alpha)$$

$$= B^{\alpha+1}$$

Thus, the same fixed-point is computed by the boxed formula $\mu p \bullet B$. ♣

Next, can we fit De Jongh-Sambin recursions into general fixed-point logic? Recall that well-founded relations have an inductive character: their domains are smallest fixed-points defined by $\mu p \bullet []p$. On such orders, the whole universe is eventually computed through the monotonically increasing ordinal approximation stages

$$D^0, D^1, .., D^\alpha, ...$$

of the modal formula $p \leftrightarrow []p$. Now we cannot compute similar cumulative stages for the fixed-point formula $\phi(p, \mathbf{q})$ in Theorem 9, as ϕ may have both positive and negative occurrences of the proposition letter p. But we can define the related monotonic sequence of *inflationary* fixed-points, defined above. As we noted, this inflationary process need not lead to a fixed-point for $\phi(p, \mathbf{q})$ per se. But this time, we do have monotone growth within the D-hierarchy, as the ϕ's stabilize inside its stages:

Fact 18 $\phi^{\alpha+1} \cap D^\alpha = \phi^\alpha \cap D^\alpha$

Thus a general fixed-point procedure for solving De Jongh-Sambin equations runs monotonically when *restricted* to approximation stages for a well-founded universe. This prediction pans out for the above modal examples $[]p$, $\neg[]p$, and $[]p \rightarrow q$. We will not prove this here, as we will re-describe the situation now.

Theorem 10 De Jongh-Sambin fixed-points can be found by the following *simultaneous inflationary inductive definition:*

$r \quad \leftrightarrow \quad []r$

$p \quad \leftrightarrow \quad []r \,\&\, \phi(p, \boldsymbol{q})$

Proof We compute the approximation stages for *p, r* simultaneously:

$(r^{\alpha+1}, p^{\alpha+1}) \quad = \quad ([]r^{\alpha}, []r^{\alpha} \wedge \phi(p^{\alpha}))$ *successors*

$(r^{\lambda}, p^{\lambda}) \quad = \quad (\cup_{\alpha<\lambda} r^{\alpha}, \cup_{\alpha<\lambda} p^{\alpha})$ *limits*

Here the conjunct *[]r* (rather than *'r'*) for *p* makes sure that the next stage of *p* is computed by reference to the new value of *r*. Now it suffices to prove the following relation between the approximation stages – written here with some abuse of notation:

Lemma 2 If $\beta<\alpha$, then $p^{\alpha} \wedge r^{\beta} = r^{\alpha}$

Note that this implies monotonicity: if $\beta<\alpha$, then $p^{\beta} \rightarrow p^{\alpha}$.

Here, the main induction is best done on α, with an auxiliary one on β. The cases of *0* and limit ordinals are straightforward. For the successor step, we need two auxiliary facts. The first expresses the invariance of modal formulas for generated submodels, and the second is an immediate consequence of the approximation procedure for *r*:

(i) $\boldsymbol{M}, P, x \models \phi(p)$ iff $\boldsymbol{M}, P \cap R^{*}[x], x \models \phi(p)$

(ii) Let $R^{*}[x]$ be all points reachable from *x* by some finite number of *R*-steps. If $x \in r^{\alpha}$, then $R^{*}[x] \subseteq \cup_{\beta<\alpha} r^{\beta}$

Now we compute - again with some beneficial abuse of notation:

$x \models p^{\alpha+1} \wedge r^{\beta+1}$ iff

$x \models r^{\alpha+1} \wedge \phi(p^{\alpha}) \wedge r^{\beta+1}$ iff

$x \models \phi(p^{\alpha}) \wedge r^{\beta+1}$ iff (by (i), (ii))

$x \models \phi(p^{\alpha} \wedge r^{\beta}) \wedge r^{\beta+1}$ iff (ind. hyp.)

$x \models \phi(p^{\beta}) \wedge r^{\beta+1}$ iff

$x \models p^{\beta+1}$ ♣

4.4 Why the explicit definability?

Our μ–calculus analysis does not explain why provability fixed-points are *explicitly definable* in the modal base language. Indeed, the general reason seems unknown. We do know that this explicit definability is not specific to the modal language:

Theorem 11 Explicit definability for fixed-point equations with all occurrences of p in the scope of some operator holds for all propositional languages with generalized quantifiers $\boldsymbol{Q}p$ over sets of worlds satisfying

(a) = (i) above $\boldsymbol{Q}(P)$ is true at x iff $\boldsymbol{Q}(P \cap R_x)$ is true at x *Locality*

(b) $\boldsymbol{Q}p \to [\,]\boldsymbol{Q}p$ *Heredity*

This covers quantifiers $\boldsymbol{Q}$ like the modal "in some successor", the true first-order "in at most five successors", or the second-order "in most successors of each successor". Van Benthem 1987 has a proof for Theorem 11, in joint work with Dick de Jongh.

But the general rationale of explicit definability still eludes us. One factor besides appropriate base quantifiers $\boldsymbol{Q}$ is transitivity of accessibility. E.g., the Gödel equation $p \leftrightarrow \neg[\,]p$ has no explicit modal solution on finite trees with the immediate successor relation. But there may be still deeper model-theoretic reasons for the success of Theorem 9 in provability logic in terms of general fixed-point logic. Here is a suggestive observation. Smallest and greatest fixed points for a *first-order* formula $\phi(P)$ coincide if $\phi(P)$ implies an explicit definition for P. But the converse is true as well, by a straightforward appeal to Beth's Theorem (cf. Smorynski 1984). Such explicit first-order definitions for unique first-order fixed-points even arise uniformly by some fixed finite approximation stage in every model where they are computed, by the Barwise-Moschovakis Theorem (I owe this reference to Martin Otto).

Remark 4 Alternative modal formalisms for solving fixed-point equations.
Visser and d'Agostino have suggested analyzing explicit definability in provability logic with ideas from Hollenberg 1998, using uniform interpolation properties of the μ-calculus and its associated languages with so-called *bisimulation quantifiers*. ♣

5 Higher-order perspectives

Many topics in the preceding sections suggest a further extension into second-order logic, which is the natural habitat of frame truth of modal formulas interpreted as monadic Π^1_1-sentences. For instance, the Sahlqvist Theorem for basic modal logic also works with positive antecedents in any higher-order logic (van Benthem 1999). But as is well-known, our fixed-point extensions are also fragments of second-order logic. In particular, there might be Beth Theorems for suitable fragments of second-order logic behind the modal fixed-point results discussed in Section 4. Van Benthem 1983, ten Cate 2005 study modal logic partly as a way of finding well-behaved fragments of second-order logic. This seems another interesting way to go.

6 Conclusion

This note has shown how various aspects of provability logic, all high-lighted by Löb's Axiom, suggest a much broader background in modal and classical logic, with fixed-point languages as a running thread. 30 years after our student days, the content of our modal boxes, even in very familiar settings, has not yet been exhausted!

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