# A Propositional Dynamic Logic for Instantial Neighborhood Semantics

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**Abstract**

We propose a new perspective on logics of computation by combining instantial neighborhood logic INL with bisimulation safe operations adapted from PDL. INL is a recently proposed modal logic, based on a richer extension of neighborhood semantics which permits both universal and existential quantification over individual neighborhoods. This language has a natural interpretation as a logic of computation in open systems. Motivated by this interpretation, we show that a number of familiar programs constructors can be adapted to the setting of instantial neighborhood semantics to preserve invariance for instantial neighborhood bisimulations, which give the appropriate bisimulation concept for INL. We also prove that our extended logic IPDL is a conservative extension of dual-free game logic, and its semantics generalizes the monotone neighborhood semantics of game logic. Finally, we provide a sound and complete system of axioms for IPDL, and establish its finite model property and decidability.

## 1 Introduction

In this paper, we introduce a new modal logic of computation, in the style of propositional dynamic logic, based on *instantial neighborhood logic* INL [6]. The logic INL is based on a recent variant of monotone neighborhood semantics for modal logics, called instantial neighborhood semantics. In the standard neighborhood semantics, the box operator has the interpretation: $\Box p$ is true at a point if *there exists* a neighborhood in which *all* the elements satisfy the proposition $p$. So the box operator has a built-in fixed existential-universal quantifier pattern. In instantial neighborhood logic, we allow both universal and existential quantification over individual neighborhoods, so the basic modality has the form $\Box(p_1, ..., p_n; q)$. This formula is true at a point if *there exists* a neighborhood $N$ in which *all* the elements satisfy the proposition $q$, and furthermore each of the propositions $p_1, ..., p_n$ are satisfied by *some* elements of $N$. INL is more expressive than monotone neighborhood logic, and comes with a natural associated notion of bisimulation together with a Hennessy-Milner theorem for finite models. It has a complete system of axioms, has the finite model property, is decidable and PSpace-complete.

Formally, our proposal is to consider an extension of the base language INL by bisimulation safe "program constructors", as in the standard propositional dynamic logic of sequential programs (PDL). The usual repertoire here consists of choice, test, sequential composition and a Kleene star for program iteration. Similar additions have already been studied extensively for the standard (monotone) neighborhood semantics, where the constructors are interpreted as methods of constructing complex *games* (this idea dates back to [18]). In the neighborhood setting, some additional operations are available, including the *dual* construction. This is a very powerful construction, and it is well known that dynamic game logic is not contained in any fixed level of the $\mu$-calculus alternation hierarchy [8].

We think of our extended logic, which we call instantial PDL (IPDL for short), as a dynamic logic for a richer notion of computation than sequential programs, which is sometimes referred to as *open systems* [2]. In open systems, a computational process is viewed as an agent acting in an uncertain environment that affects the outcome of each action. That is, each action by the agent is followed by a response from the environment, which is not uniquely determined. This is in contrast with *reactive systems*, where the behaviour of the system is non-deterministic but completely determined by the actions of the agent [1]. Many different logics for open systems have been proposed, perhaps the most well known being the alternating-time temporal logic ATL of Alur et al. Dynamic game logic can be interpreted in a similar way, thinking of processes as "games against the environment". Game logic is usually interpreted with a neighborhood semantics, in which neighborhoods of "worlds" in a model are taken to represent powers of some player, i.e. goals that can be enforced by some action or strategy. Instantial neighborhood semantics introduces a more fine-grained perspective to this setting, with a more expressive language and a finer bisimulation concept than standard neighborhood bisimilarity, namely the instantial neighborhood bisimulations of [6]. Since INL formulas allow existential quantification over individual neighborhoods, this language is suitable to describe not only what conditions an agent can enforce by some action, but allows more precise reasoning about exactly what possible outcomes may result from some action. Concretely, we introduce formulas of the following kind:

$$\langle a \rangle (\psi_1, ..., \psi_n; \varphi)$$

expressing the following property about the system/program $a$: "the agent can act so as to ensure that $\varphi$ holds, while allowing (for each $i \in \{1, ..., n\}$) the possibility that the property $\psi_i$ may hold". In other words, instantial neighborhood logic has a natural interpretation as a simple yet expressive modal logic for computation in open systems.

However, given a computational interpretation, it is a standard wisdom that one needs to extend the language to allow certain fixpoint constructions, since most specifications of systems that turn up in practice – safety, liveness, fairness etc. – involve fixpoints. There are many options available here, the most obvious one being to simply add unrestricted fixpoint operators as in the full modal $\mu$-calculus. This route can already be claimed to be quite well understood: it was

noted in [6] that INL is a *coalgebraic modal logic* in a completely standard sense, and so the $\mu$-calculus extension of INL is a coalgebraic modal $\mu$-calculus as in [21, 14]. Such coalgebraic $\mu$-calculi have been quite extensively studied, with generic results on decidability and complexity, [11] and completeness [12, 13]. But there are also other versions of modal fixpoint logics, often corresponding to fragments of $\mu$-calculi. Most notably these include propositional dynamic logics like PDL or game logic, and temporal logics like CTL or ATL. Thus an obvious point on the agenda, for further exploration of INL as a modal logic of computation, is to develop dynamic and temporal logic extensions of INL. This paper deals with the former, and sets up a propositional dynamic logic interpreted over instantial neighborhood semantics.

**Overview of the paper**

We first introduce syntax and semantics of instantial neighborhood logic, and extensions of it leading up to the full language IPDL, provide sound and complete systems of axioms, and establish bisimulation invariance and decidability. The latter amounts to bisimulation safety for our program constructors. The completeness proof for the language IPDL, including all the program constructors that we consider, is based on the standard completeness proof for PDL (see [9] for an exposition), but involves some non-trivial new features. In particular, the axiom system requires two distinct induction rules, corresponding to a nested least fixpoint induction, and the model construction makes heavy use of a normal form for INL-formulas established in [6]. Finally, we prove that our logic is a conservative extension of the dual free fragment of dynamic game logic.

The paper is an extended version of a conference paper presented at LORI VI 2017 [5].

# 2 Instantial neighborhood logic

## 2.1 Syntax and semantics

We start by reviewing the basic language for instantial neighborhood semantics. The only difference with [6] is that we are interpreting the language over *labelled* neighborhood structures, where the labels play the same role as "atomic programs" in PDL.

The syntax of INL is given by the following grammar:

$$\varphi := p \in \mathsf{Prop} \mid \varphi \wedge \varphi \mid \neg\varphi \mid \langle a\rangle(\Psi;\varphi)$$

where $a$ ranges over a fixed set $\mathcal{A}$ of *atomic labels*, and $\Psi$ ranges over finite sets of formulas of INL. We have deviated a bit from the syntax of [6] here in allowing $\Psi$ to be a finite *set* rather than a tuple of formulas. We shall sometimes write $\langle a\rangle(\psi_1, ..., \psi_n; \varphi)$ rather than $\langle a\rangle(\{\psi_1, ..., \psi_n\}; \varphi)$, in particular we write $\langle a\rangle(\psi;\varphi)$ rather than $\langle a\rangle(\{\psi\};\varphi)$, and $\langle a\rangle\varphi$ rather than $\langle a\rangle(\emptyset;\varphi)$.

The modalities of INL a number of possible interpretations. In the present setting, we interpret the formalism INL in terms of computation in open systems,

so that the formula $\langle a\rangle(\psi_1, ..., \psi_n; \varphi)$ is informally interpreted as saying: "in the system $a$, the agent has an action to enforce the condition $\varphi$ while simultaneously allowing possible outcomes satisfying each of the conditions $\psi_i$".

**Example 1.** Consider the following example: three separate servers are shared by a number of agents and protected by passwords available to the users. Each server can only be accessed by one user at a time. Taking the perspective of one of the agents, let $A_i$ stand for "the agent has access to server $S_i$", for $i \in \{1, 2, 3\}$, and let $O_i$ stand for "server $S_i$ is occupied". If we introduce a name $\sigma$ for the system so described, then the following is true for each given user, in each given state of the system $\sigma$:

$$\neg\langle\sigma\rangle(\neg O_1; A_1) \wedge \neg\langle\sigma\rangle(\neg O_2; A_2) \wedge \neg\langle\sigma\rangle(\neg O_3; A_3)$$

This expresses that the user cannot log in to a server without blocking the other users from having access to that server. The following also holds:

$$\neg O_3 \rightarrow \langle\sigma\rangle(\neg O_1, \neg O_2; A_3)$$

If server $S_3$ is available then the agent can access it while leaving servers $S_1$ and $S_2$ available to be occupied by other users. Note the distinction here: the user cannot *guarantee* that the servers $S_1$,$S_2$ will be available, they might be occupied by other users, but she can *allow* them to remain available. Finally, the following holds:

$$\neg\langle\sigma\rangle(\neg A_1, \neg A_2; (O_1 \rightarrow A_1) \vee (O_2 \rightarrow A_2))$$

This last example is perhaps less obvious: it says that the only way a user can make sure that at least one of the servers $S_1$ or $S_2$ will not be occupied by some other user is to log in to at least one of them herself.

For the formally precise semantics, formulas in $\mathsf{INL}$ will be interpreted over neighborhood structures.

**Definition 1.** A *neighborhood* frame is a structure $(W, R)$ where $W$ is a set and $R$ associates with each $a \in \mathcal{A}$ a binary relation $R_a \subseteq W \times \mathcal{P}W$. A *neighborhood model* $(W, R, V)$ is a neighborhood frame together with a valuation $V : \mathsf{Prop} \rightarrow \mathcal{P}W$.

**Definition 2.** We define the interpretations of all formulas in a neighborhood model $\mathfrak{M} = (W, R, V)$ as follows:

- $[\![p]\!] = V(p)$.

- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$.

- $[\![\neg\varphi]\!] = W \setminus [\![\varphi]\!]$.

- $u \in [\![\langle a \rangle(\psi_1, ..., \psi_k; \varphi)]\!]$ iff there is some $Z \subseteq W$ such that:

$(u, Z) \in R_a$ and $Z \subseteq [\![\varphi]\!]$, $Z \cap [\![\psi_i]\!] \neq \emptyset$ for $i \in \{1, ..., k\}$

We write $\mathfrak{M}, v \Vdash \varphi$ for $v \in [\![\varphi]\!]$, and we write $\Vdash \varphi$ and say that $\varphi$ is *valid* if, for every neighborhood model $\mathfrak{M}$ and $v \in W$, we have $\mathfrak{M}, v \Vdash \varphi$. We allow the notation $[\![-]\!]_{\mathfrak{M}}$ to make explicit reference to the model in the background.

Neighborhood models come with a natural notion of bisimulation, introduced in a more general setting in [6]. For this definition, the so called *Egli-Milner lifting* of a binary relation will play an important role:

**Definition 1.** The *Egli-Milner lifting* of a binary relation $R \subseteq X \times Y$, denoted $\overline{R}$, is a relation from $\mathcal{P}X$ to $\mathcal{P}Y$ defined by: $Z\overline{R}Z'$ iff:

1. For all $z \in Z$ there is some $z' \in Z'$ such that $zRz'$.
2. For all $z' \in Z'$ there is some $z \in Z$ such that $zRz'$.

We write $R; S$ for the composition of relations $R$ and $S$. It is well known that the Egli-Milner lifting preserves relation composition:

$$\overline{R; S} = \overline{R}; \overline{S}$$

**Definition 2.** Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ be any neighborhood models. The relation $B \subseteq W \times W'$ is said to be an *instantial neighborhood bisimulation* if for all $uBu'$ and all atomic labels $a$ we have:

**Atomic** For all $p$, $u \in V(p)$ iff $u' \in V'(p)$.

**Forth** For all $Z$ such that $uR_aZ$, there is some $Z'$ such that $u'R'_aZ'$ and $Z\overline{B}Z'$.

**Back** For all $Z'$ such that $u'R'_aZ'$ there is some $Z$ such that $uR_aZ$ and $Z\overline{B}Z'$.

We say that pointed models $\mathfrak{M}, w$ and $\mathfrak{N}, v$ are *bisimilar*, written $\mathfrak{M}, w \leftrightarroweq \mathfrak{N}, v$, if there is an instantial neighborhood bisimulation $B$ between $\mathfrak{M}$ and $\mathfrak{N}$ such that $wBv$.

It is easy to check that all formulas of $\mathsf{INL}$ are invariant for instantial neighborhood bisimilarity:

**Proposition 1.** *If $\mathfrak{M}, w \leftrightarroweq \mathfrak{N}, v$ then $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{N}, v \Vdash \varphi$, for each formula $\varphi$ of $\mathsf{INL}$.*

## 2.2 Axiomatization

We now turn to the task of axiomatizing the valid formulas of $\mathsf{INL}$. Our system of axioms is a gentle modification of the axiom system for instantial neighborhood logic presented in [6].

**INL axioms**

**Mon:** $\langle a\rangle(\psi_1, ..., \psi_n; \varphi) \to \langle a\rangle(\psi_1 \vee \alpha_1, ..., \psi_n \vee \alpha_n; \varphi \vee \beta)$

**Weak:** $\langle a\rangle(\Psi; \varphi) \to \langle a\rangle(\Psi'; \varphi)$ for $\Psi' \subseteq \Psi$

**Un:** $\langle a\rangle(\psi_1, ..., \psi_n; \varphi) \to \langle a\rangle(\psi_1 \wedge \varphi, ..., \psi_n \wedge \varphi; \varphi)$

**Lem:** $\langle a\rangle(\Psi; \varphi) \to \langle a\rangle(\Psi \cup \{\gamma\}; \varphi) \vee \langle a\rangle(\Psi; \varphi \wedge \neg\gamma)$

**Bot:** $\neg\langle a\rangle(\bot; \varphi)$

**Rules**

**MP:**

$$\frac{\varphi \to \psi \qquad \varphi}{\psi}$$

**RE:**

$$\frac{\varphi \leftrightarrow \psi \qquad \theta}{\theta[\varphi/\psi]}$$

where $\theta[\varphi/\psi]$ is the result of substituting some occurrences of the formula $\psi$ by $\varphi$ in $\theta$.

We denote this system of axioms by Ax1 and write $\mathsf{Ax1} \vdash \varphi$ to say that the formula $\varphi$ is provable in this axiom system. We also write $\varphi \vdash_{\mathsf{Ax1}} \psi$ for $\mathsf{Ax1} \vdash \varphi \to \psi$, and say that $\varphi$ *provably entails* $\psi$.

**Theorem 1.** *The system* Ax1 *is sound and complete for validity on neighborhood models.*

The proof of this result is essentially the same as in [6], and will not be repeated here. Since the proof in [6] constructs a finite model for each consistent formula, we also get:

**Theorem 2.** *The logic* INL *has the finite model property and is decidable.*

**Example 2.** Continuing from Example 1, we recall the formula:

$$\neg\langle\sigma\rangle(\neg O_i; A_i)$$

expressing that a user cannot both log in to a server and leave it available to other users. This reduces, of course, to the fact that the formula $A_i \to O_i$ is true in every state: a server cannot be both accessed by a user and at the same time not occupied. So we can take this formula instead as an extra assumption. By replacing equivalent formulas we then get the implication:

$$\langle\sigma\rangle(\neg O_i; A_i) \to \langle\sigma\rangle(\neg O_i; A_i \wedge O_i)$$

We can now apply the axiom (Un) to get the implication:

$$\langle\sigma\rangle(\neg O_i; A_i \wedge O_i) \to \langle\sigma\rangle(\neg O_i \wedge A_i \wedge O_i; A_i \wedge O_i)$$

Replacing equivalents again we get:

$$\langle\sigma\rangle(\neg O_i; A_i \wedge O_i) \to \langle\sigma\rangle(\bot; A_i \wedge O_i)$$

But as an instance of (Bot) we have the implication:

$$\langle\sigma\rangle(\bot; A_i \wedge O_i) \to \bot$$

So we get:

$$\langle\sigma\rangle(\neg O_i; A_i) \to \bot$$

i.e. $\neg\langle\sigma\rangle(\neg O_i; A_i)$ as required.

# 3 Basic program operations

## 3.1 Semantics and basic model theory

In what follows we shall extend the language INL with program operations, corresponding to known operations from PDL. We also include the "dual choice" constructor from dynamic game logic. Of course, there are design choices to make here, and we need to set up some criteria for what counts as a correct definition of each program operation. We shall follow these three criteria:

1. The constructions should be as simple as possible.
2. Each operation should be a natural adaptation of the corresponding operation from PDL to the INL framework, with minimal modifications.
3. Most importantly: each operation should be *bisimulation safe*, i.e. the dynamic logic extending INL with all the program operations should remain invariant for instantial neighborhood bisimulations.

We first extend the language INL with four basic PDL-style operations: test, choice, parallel composition and sequential composition. The resulting language will be called *dynamic instantial neighborhood logic*, or (DINL). The syntax of DINL is defined by the following dual grammar.

$$\varphi := p \in \mathsf{Prop} \mid \varphi \wedge \varphi \mid \neg\varphi \mid \langle\pi\rangle(\Psi; \varphi)$$

$$\pi := a \in \mathcal{A} \mid \varphi? \mid \pi \cup \pi \mid \pi \cap \pi \mid \pi \circ \pi$$

The operation $\cup$ is interpreted as non-deterministic choice between two programs for the agent: $\pi_1 \cup \pi_2$ means "either do $\pi_1$ or do $\pi_2$". The operation $\cap$ is intepreted as a choice between two programs for the environment: $\pi_1 \cap \pi_2$ means "do $\pi_1$ and $\pi_2$ in parallell". Formally, the operation $\cap$ is similar to the parallell composition in concurrent PDL (see [16]). Finally, the operator $\circ$ is interpreted as sequential composition: $\pi_1 \circ \pi_2$ means "first do $\pi_1$ then do $\pi_2$". We define the formal interpretation $[\![o]\!]$ of each operation $o \in \{\cup, \cap, \circ\}$ in a neighborhood model $\mathfrak{M}$ as a binary map from pairs of neighborhood relations to neighborhood relations, as follows:

- $R_1[\![\cup]\!]R_2 = R_1 \cup R_2$
- $R_1[\![\cap]\!]R_2 = \{(w, Z_1 \cup Z_2) \mid (w, Z_1) \in R_1 \ \& \ (w, Z_2) \in R_2\}$
- $(w, Z) \in R_1[\![\circ]\!]R_2$ iff there is some set $Y$ and some family of sets $F$ such that $(w, Y) \in R_1$, $(Y, F) \in \overline{R}_2$ and $Z = \bigcup F$.

The interpretation $[\![?]\!]$ of the test operator will be a map $[\![?]\!]$ assigning a neighborhood relation to each subset $Z$ of $W$, defined by:

$$[\![?]\!]Z := \{(u, \{u\}) \mid u \in Z\}$$

We defer a more detailed discussion of the informal interpretation of the program operations to Section 3.2. Note that $[\![?]\!]$ is monotone in the sense that $Z \subseteq Z'$ implies $[\![?]\!]Z \subseteq [\![?]\!]Z'$. Each operator $o \in \{\cup, \cap, \circ\}$ is also monotone, in the sense that $R_1[\![o]\!]R_2 \subseteq R_1'[\![o]\!]R_2'$ whenever $R_1 \subseteq R_1'$ and $R_2 \subseteq R_2'$. For the sequential composition operator, this uses the well known fact that the Egli-Milner lifting is monotone, i.e. $\overline{R} \subseteq \overline{R'}$ whenever $R \subseteq R'$.

**Definition 3.** We define the semantic interpretations of all formulas, and the neighborhood relations corresponding to all complex labels $\pi$, by the following mutual recursion:

- $[\![p]\!] = V(p)$.

- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$.

- $[\![\neg\varphi]\!] = W \setminus [\![\varphi]\!]$.

- $u \in [\![\langle\pi\rangle(\psi_1, ..., \psi_k; \varphi)]\!]$ iff there is some $Z \subseteq W$ such that:

$(u, Z) \in R_\pi$ and $Z \subseteq [\![\varphi]\!]$, $Z \cap [\![\psi_i]\!] \neq \emptyset$ for $i \in \{1, ..., k\}$.

- $R_{\pi_1 o \pi_2} = R_{\pi_1}[\![o]\!]R_{\pi_2}$ for $o \in \{\cup, \cap, \circ\}$.

- $R_{\varphi?} = [\![?]\!][\![\varphi]\!]$

The definitions of the dynamic operations are tailored towards obtaining the following result:

**Proposition 2.** *All formulas of* DINL *are invariant for instantial neighborhood bisimulations.*

*Proof.* We first prove the following claim, expressing bisimulation safety of the operations that we have introduced:

**Claim 1.** Let $B$ be an instantial neighborhood bisimulation between models $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$. Then for any complex label $\pi$, such that every term of the form $\varphi$ appearing in $\pi$ the formula $\varphi$ is invariant for instantial neighborhood bisimulations, and any $u \in W$ and $u' \in W'$ such that $uBu'$, we have:

**Forth** For all $Z$ such that $uR_\pi Z$, there is some $Z'$ such that $u'R'_\pi Z'$ and $Z\overline{B}Z'$.

**Back** For all $Z'$ such that $u'R'_\pi Z'$ there is some $Z$ such that $uR_\pi Z$ and $Z\overline{B}Z'$.

We prove the Claim by induction on the complexity of labels. For atomic labels the result holds by definition. For the inductive steps, we only prove the "Forth" clause since the "Back" clause can be proved by a symmetric argument. For the test operator, the result follows immediately from the assumption that every formula appearing in a sub-term of $\pi$ is bisimulation invariant.

For choice, suppose $(u, Z) \in R_{\pi_1 \cup \pi_2}$. Then $(u, Z) \in R_{\pi_1}$ or $(u, Z) \in R_{\pi_2}$, say that the first case holds. Then by the Forth clause for $\pi_1$ there is some $Z'$ with $(u', Z') \in R'_{\pi_1}$ such that $Z\overline{B}Z'$. Since $(u', Z') \in R'_{\pi_1 \cup \pi_2}$ also, we are done.

For dual choice, suppose $(u, Z) \in R_{\pi_1 \cap \pi_2}$. Then $Z = Z_1 \cup Z_2$ where $(u, Z_1) \in R_{\pi_1}$ and $(u, Z_2) \in R_{\pi_2}$. By the Forth condition for $\pi_1$ and $\pi_2$ we find sets $Z'_1$ and $Z'_2$ such that $(u', Z'_1) \in R'_{\pi_1}$, $(u', Z'_2) \in R'_{\pi_2}$ and $Z_1\overline{B}Z'_1$, $Z_2\overline{B}Z'_2$. We leave it to the reader to check that:

$$(Z_1 \cup Z_2, Z'_1 \cup Z'_2) \in \overline{B}.$$

Since $(u', Z'_1 \cup Z'_2) \in R'_{\pi_1 \cap \pi_2}$, we are done.

Finally, for sequential composition, suppose there is a set $X$ such that $(u, X) \in R_{\pi_1 \circ \pi_2}$, witnessed by a set $Y$ such that $(u, Y) \in R_{\pi_1}$ and a family $F \subseteq \mathcal{P}(W)$ such that $(Y, F) \in \overline{R}_{\pi_2}$ and $X = \bigcup F$. By the Forth condition for $\pi_1$ there is a set $Y'$ such that $(u', Y') \in R'_{\pi_1}$ and $Y\overline{B}Y'$. We define a family $F' \subseteq \mathcal{P}(W')$ as follows: set $Z' \in F'$ iff there is some $v' \in Y'$, some $v \in Y$ and some $Z \in F$ such that: $(v', Z') \in R'_{\pi_2}$, $(v, Z) \in R_{\pi_2}$ and $Z\overline{B}Z'$.

First, we claim that $(Y', F') \in \overline{R'}_{\pi_2}$: first, if $Z' \in F'$ then it is immediate from the definition that $(v', Z') \in R'_{\pi_2}$ for some $v' \in Y'$. Conversely, given $v' \in Y'$, since $Y\overline{B}Y'$ there must be some $v \in Y$ with $vBv'$, and since $(Y, F) \in \overline{R}_{\pi_2}$ there is some $Z \in F$ with $(v, Z) \in R_{\pi_2}$. But then, by the Forth condition for $\pi_2$ there must be some $Z'$ with $(v', Z') \in R'_{\pi_2}$ and $Z\overline{B}Z'$. We immediately get $Z' \in F'$, as required.

We now show that:

$$(\bigcup F, \bigcup F') \in \overline{B}.$$

To see this, suppose first that $w \in \bigcup F$. Then $w \in Z$ for some $Z \in F$. Since $(Y, F) \in \overline{R}_{\pi_2}$ there is some $v \in Y$ with $(v, Z) \in R_{\pi_2}$. Since $Y\overline{B}Y'$ there is some $v' \in Y'$ such that $vBv'$. By the Forth condition for $\pi_2$ there is some $Z'$ with $Z\overline{B}Z'$ and $(v', Z') \in R'_{\pi_2}$. We get $Z' \in F'$, and there must be some $w' \in Z'$ with $wBw'$. But then $w' \in \bigcup F'$, as required.

Conversely, suppose $w' \in \bigcup F'$. Then $w' \in Z'$ for some $Z' \in F'$. By definition of $F'$ there must be some $Z \in F$ with $Z\overline{B}Z'$, and so there must be some $w \in Z$ with $wBw'$. But then $w \in \bigcup F$ as required, and the claim is proved.

The proposition now follows from the claim by a routine argument. □

## 3.2 Informal interpretation

The neighborhood relation $R_\pi$ associated with a program term $\pi$ in a neighborhood model $\mathfrak{M}$ should be understood as follows: at each point $w$ in a model, there is a certain family of available *actions of type* $\pi$ that the agent can perform. Each such action $\alpha$ corresponds to a neighborhood $Z \in R_\pi[w]$, and $Z$ represents the possible outcomes of the action $\alpha$, as determined by the response of the environment. The interpretations of choice $\cup$ and dual choice $\cap$ should thus be clear: an action of type $\pi_1 \cup \pi_2$ is simply an action of either type $\pi_1$ or of type $\pi_2$, and so the definition of $[\![\cup]\!]$ as union of neighborhood relations is the natural one. For dual choice, an action $\alpha$ of type $\pi_1 \cap \pi_2$ consists of an action $\beta_1$ of type $\pi_1$ and an action $\beta_2$ of type $\pi_2$, where the action actually performed is determined by the environment. So a possible outcome of the action $\alpha$ is either one of the possible outcomes of $\beta_1$ or an outcome of $\beta_2$. This directly leads to the formal interpretation $[\![\cap]\!]$ of $\cap$ as it has been defined. The interpretation of the test operator is a straightforward adaption of the usual PDL-definition, and motivated in the same manner.

The less straightforward case is the sequential composition operation. Intuitively it appears to be clear what an action of type $\pi_1 \circ \pi_2$ is at a given state $w$: it is simply an action $\beta_1$ of type $\pi_1$ followed by an action $\beta_2^v$ of type $\pi_2$ performed at each possible outcome state $v$ of the action $\beta_1$ at $w$. A possible outcome of such an action $\alpha$ at $w$ should then be an outcome of one of the actions $\beta_2^v$, where $v$ is a possible outcome of the first action $\beta_1$. With this interpretation, one would expect the following definition, setting $(w, Z) \in R_1[\![\circ]\!]R_2$ iff there is some set $Y$ and a *function* $S : Y \to \mathcal{P}W$ such that:

1. $(w, Y) \in R_1$,
2. $(v, S_v) \in R_2$ for each $v \in Y$, and
3. $Z = \bigcup_{v \in Y} S_v$.

The conditions used in our actual definition of $[\![\circ]\!]$ are weaker than this, essentially allowing the assigment $S$ to be a *relation* rather than a function. The reason for using the less strict version of the composition operation is due to the fact that the "functional" version of the sequential composition operation *violates bisimulation safety*! The example shown in Figure 1, displaying two bisimilar rooted models, explains this.

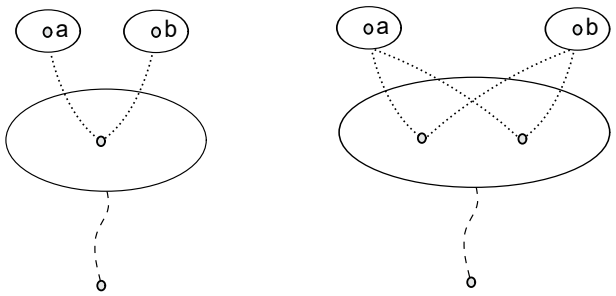


Figure 1: Failure of bisimulation safety

In the diagram, points are represented by bullets, neighborhoods are represented by ellipses, the dashed lines represent the neighborhood relation $R_1$ and the dotted lines represent $R_2$. In the model to the right, the root has a neighborhood $\{a, b\}$ according to the functional composition of $R_1$ and $R_2$, but not in the left model. Note that according to our "relational" definition of sequential composition, $\{a, b\}$ is a neighborhood in both models.

A possible response this would be to modify our notion of instantial neighborhood bisimulations in order to recover bisimulation safety. However, this route does not seem particularly attractive to us, as instantial neighborhood bisimulations do provide the natural bisimulation concept for INL, which forms the basis upon which our dynamic logic is built.

Rather, we suggest that our notion of sequential composition can be explained as follows: the behaviour of an agent interacting with a system may depend not only on the state of the system itself, but also on the internal state of the agent. For example, looking back to Example 1, the state of the system itself specifies which of the three servers are occupied by which agent. The internal state of each agent – which in this particular example is a human user – may for example involve the agent's current state of knowledge, preferences, intentions etc. So when the agent executes an action of some type $\pi$, the system and the agent both start in a given initial state which may change through the course of the computation, and as the internal state of the agent changes this may affect its later actions. In a computation corresponding to a composite program term of the form $\pi_1 \circ \pi_2$ executed at some state $w$, it then makes sense that the action of the agent in the computation $\pi_2$ at a later state $v$ resulting as the outcome of the computation $\pi_1$ might not be determined uniquely by the state $v$ of the system, since it may also depend on the internal state of the agent, which may change during the execution of $\pi_1$. This interpretation is thus consistent with our formal semantics of the sequential composition operator, as well as the other program operations.

### 3.3 Axiomatization

Our axiom system for DINL will take the sound and complete axioms for INL as its foundation, and extend it with reduction axioms for the test, choice, parallel composition and sequential composition operators. The axioms and rules are listed below; note that the INL axioms and the axioms for frame constraints are now stated for arbitrary complex labels $\pi$ rather than just atoms $a$.

**INL axioms:**

(Mon), (Weak), (Un), (Lem) and (Bot)

**Reduction axioms:**

**Test:** $\langle\gamma?\rangle(\Psi;\varphi) \leftrightarrow \gamma \wedge \bigwedge \Psi \wedge \varphi$

**Ch:** $\langle\pi_1 \cup \pi_2\rangle(\Psi;\varphi) \leftrightarrow \langle\pi_1\rangle(\Psi;\varphi) \vee \langle\pi_2\rangle(\Psi;\varphi)$

**Pa:** $\langle\pi_1 \cap \pi_2\rangle(\Psi;\varphi) \leftrightarrow \bigvee\{\langle\pi_1\rangle(\Theta_1;\varphi) \wedge \langle\pi_2\rangle(\Theta_2;\varphi) \mid \Psi = \Theta_1 \cup \Theta_2\}$

**Cmp:** $\langle\pi_1 \circ \pi_2\rangle(\psi_1, ..., \psi_n;\varphi) \leftrightarrow \langle\pi_1\rangle(\langle\pi_2\rangle(\psi_1;\varphi), ..., \langle\pi_2\rangle(\psi_n;\varphi);\langle\pi_2\rangle\varphi)$

**Rules:**

(MP) and (RE)

We denote this system of axioms by Ax2 and write $\mathsf{Ax2} \vdash \varphi$ to say that the formula $\varphi$ is provable in this axiom system. We also write $\varphi \vdash_{\mathsf{Ax2}} \psi$ for $\mathsf{Ax2} \vdash \varphi \to \psi$. We shall sometimes drop the reference to Ax2 to keep notation cleaner.

**Proposition 3** (Soundness)**.** *If* $\mathsf{Ax2} \vdash \varphi$*, then* $\varphi$ *is valid on all neighborhood models.*

*Proof.* We consider only soundness of the new reduction axioms. Soundness of (Ch) is almost immediate from the definition of $[\![\cup]\!]$, so we focus on (Test), (Pa) and (Cmp).

For (Test), suppose that $\mathfrak{M}, u \Vdash \langle\gamma?\rangle(\Psi;\varphi)$. Then $(u, \{u\}) \in R_{\gamma?}$, which means that $\mathfrak{M}, u \Vdash \gamma$, and $\{u\} \subseteq [\![\varphi]\!]$ and $\{u\} \cap [\![\psi]\!] \neq \emptyset$ for each $\psi \in \Psi$, which means that $\mathfrak{M}, u \Vdash \bigwedge \Psi \wedge \varphi$. So $\mathfrak{M}, u \Vdash \gamma \wedge \bigwedge \Psi \wedge \varphi$. The converse is similar.

For (Pa), suppose that $\mathfrak{M}, w \Vdash \langle\pi_1 \cap \pi_2\rangle(\Psi;\varphi)$. Then there is some set $Z$ such that $(w, Z) \in R_{\pi_1 \cap \pi_2}$, $Z \subseteq [\![\varphi]\!]$ and $Z \cap [\![\psi]\!] \neq \emptyset$ for all $\psi \in \Psi$. Hence $Z$ is of the form $Z_1 \cup Z_2$ where $(w, Z_1) \in R_{\pi_1}$ and $(w, Z_2) \in R_{\pi_2}$. Let $\Theta_1 = \{\psi \in \Psi \mid Z_1 \cap [\![\psi]\!] \neq \emptyset\}$, and let $\Theta_2 = \{\psi \in \Psi \mid Z_2 \cap [\![\psi]\!] \neq \emptyset\}$. Then, since $Z = Z_1 \cup Z_1$, we have $\Psi = \Theta_1 \cup \Theta_2$. Furthermore, we get

$$\mathfrak{M}, w \Vdash \langle\pi_1\rangle(\Theta_1;\varphi) \wedge \langle\pi_2\rangle(\Theta_2;\varphi)$$

as required. The converse direction of (Pa) is proved in a similar manner.

Next, we consider the case of sequential composition. For one direction of the equivalence, suppose that $\mathfrak{M}, w \Vdash \langle \pi_1 \circ \pi_2 \rangle(\psi_1, ..., \psi_n; \varphi)$. Then there is some set $Z$ with $(w, Z) \in R_{\langle \pi_1 \circ \pi_2 \rangle}$, $Z \subseteq [\![\varphi]\!]$ and $Z \cap [\![\psi_i]\!] \neq \emptyset$ for each $\psi_i$. By definition of the composition operator, we find a set $Y$ with $(w, Y) \in R_{\pi_1}$ and a family of sets $F$ such that $(Y, F) \in \overline{R}_{\pi_2}$ and $Z = \bigcup F$. So for each $v \in Y$ there is some $Z \in F$ with $(v, Z) \in R_{\pi_2}$, and we get $Z \subseteq [\![\varphi]\!]$ so $\mathfrak{M}, v \Vdash \langle \pi_2 \rangle \varphi$. Also, for each $\psi_i$ there is some $Z \in F$ with $Z \cap [\![\psi_i]\!] \neq \emptyset$, and there must be some $v \in Y$ with $(v, Z) \in R_{\pi_2}$, hence $\mathfrak{M}, v \Vdash \langle \pi_2 \rangle(\psi_i; \varphi)$. It follows that $\mathfrak{M}, w \Vdash \langle \pi_1 \rangle(\langle \pi_2 \rangle(\psi_1; \varphi), ..., \langle \pi_2 \rangle(\psi_n; \varphi); \langle \pi_2 \rangle \varphi)$ as required.

Conversely, suppose that $\mathfrak{M}, w \Vdash \langle \pi_1 \rangle(\langle \pi_2 \rangle(\psi_1; \varphi), ..., \langle \pi_2 \rangle(\psi_n; \varphi); \langle \pi_2 \rangle \varphi)$. Then there is some set $Y$ such that $(w, Y) \in R_{\pi_1}$, $Y \subseteq [\![\langle \pi_2 \rangle \varphi]\!]$ and $Y \cap [\![\langle \pi_2 \rangle(\psi_i; \varphi)]\!] \neq \emptyset$ for each $i \in \{1, ..., n\}$. Let:

$$F := \{Z \subseteq W \mid Z \subseteq [\![\varphi]\!] \ \& \ (v, Z) \in R_{\pi_2} \text{ for some } v \in Y\}$$

Since $Y \subseteq [\![\langle \pi_2 \rangle \varphi]\!]$ it follows that $(Y, F) \in \overline{R}_{\pi_2}$, so $(w, \bigcup F) \in R_{\pi_1 \circ \pi_2}$. Furthermore, since $Y \cap [\![\langle \pi_2 \rangle(\psi_i; \varphi)]\!] \neq \emptyset$ for each $i \in \{1, ..., n\}$ it follows that $\bigcup F \cap [\![\psi_i]\!] \neq \emptyset$ for each $i \in \{1, ..., n\}$. We get $\mathfrak{M}, w \Vdash \langle \pi_1 \circ \pi_2 \rangle(\psi_1, ..., \psi_n; \varphi)$ as required. □

By applying soundness of the reduction axioms, we can use a standard argument to obtain for every consistent formula $\varphi$ of DINL a provably (and hence semantically) equivalent formula $\varphi^t$ in INL, which is then satisfiable by Theorem 1. For example, the formula $\langle \gamma? \rangle(\psi_1, ..., \psi_n; \varphi)^t$ is defined to be $\gamma^t \wedge \psi_1^t \wedge ... \wedge \psi_n^t \wedge \varphi$.

We get:

**Theorem 3** (Completeness)**.** *A formula $\varphi$ of* DINL *is valid on all neighborhood models iff* Ax2 $\vdash \varphi$.

Furthermore, the finite model property and decidability clearly carry over from INL:

**Theorem 4.** *The logic* DINL *has the finite model property and is decidable.*

# 4 Program iteration and the language IPDL

We now introduce the final operation that we consider here, a Kleene star for finite iteration. This operation will be set up to generalize the game iteration operation from game logic. The corresponding language will be denoted by IPDL, read "instantial PDL", and is given by the following dual grammar:

$$\varphi := p \in \mathsf{Prop} \mid \varphi \wedge \varphi \mid \neg\varphi \mid \langle \pi \rangle(\Psi; \varphi)$$

$$\pi := a \in \mathcal{A} \mid \varphi? \mid \pi \cup \pi \mid \pi \cap \pi \mid \pi \circ \pi \mid \pi^*$$

The operation $(-)^*$ is interpreted as finite iteration: $\pi^*$ means "repeat $\pi$ a finite number of times". More specifically, we think consider *action of type* $\pi^*$

to be a long term strategy of the agent, such that each possible execution of this strategy consists of finitely many actions of type $\pi$.

For the formal semantic interpretation of the Kleene star, it will be useful to first define the relation $\mathsf{skip}$ by:

$$\mathsf{skip} := \{(w, \{w\}) \mid w \in W\}$$

**Definition 4.** We define a relation $R^{[\xi]}$ for each ordinal $\xi$ by induction as follows.

- $R^{[0]} = \emptyset$
- $R^{[\xi+1]} = \mathsf{skip}[\![\cup]\!](R[\![\circ]\!]R^{[\xi]})$
- $R^{\kappa} = \bigcup_{\xi<\kappa} R^{[\xi]}$ if $\kappa$ is a limit ordinal.

We define $[\![*]\!]R$ to be equal to $R^{[\xi]}$, where $\xi$ is the smallest ordinal satisfying $R^{[\xi]} = R^{[\xi+1]}$.

It is easy to see that this is a standard least fixpoint construction, in particular we have:

**Proposition 4.** *Let $W$ be a finite set and $R \subseteq W \times \mathcal{P}(W)$. Then:*

$$[\![*]\!]R = \bigcup_{n\in\omega} R^{[n]}$$

Proposition 4 does not hold for arbitrary models, unlike the case for PDL the closure ordinal of the least fixpoint corresponding to the Kleene star may appear above $\omega$. Note that this does not contradict the reading of the Kleene star as finite iteration. The situation is analogous to the case of the $\mu$-calculus formula:

$$\mu x.\Box x$$

which can be thought of as expressing that "all computations are finite". It is well known that the closure ordinal of the least fixpoint of this formula can be higher than $\omega$, which just means the formula may be true although the statement "all computations have length $\leq k$" is false for all $k$. Similarly, the formula $\langle \pi^* \rangle \varphi$ expresses that the condition $\varphi$ can be forced by an action that only ever produces finitely many computations of type $\pi$, while there may be no finite upper bound on the number of iterations of $\pi$ required.

**Definition 5.** Semantics of IPDL-formulas in a neighborhood model $\mathfrak{M} = (W, R, V)$ are given as follows:

- $[\![p]\!] = V(p)$.

- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$.

- $[\![\neg\varphi]\!] = W \setminus [\![\varphi]\!]$.

- $u \in [\![\langle\pi\rangle(\psi_1, ..., \psi_k; \varphi)]\!]$ iff there is some $Z \subseteq W$ such that:

$(u, Z) \in R_\pi$ and $Z \subseteq [\![\varphi]\!]$, $Z \cap [\![\psi_i]\!] \neq \emptyset$ for $i \in \{1, ..., k\}$.

- $R_{\pi_1 o \pi_2} = R_{\pi_1} [\![o]\!] R_{\pi_2}$ for $o \in \{\cup, \cap, \circ\}$.

- $R_{\varphi?} = [\![?]\!][\![\varphi]\!]$.

- $R_{\pi^*} = [\![*]\!] R_\pi$.

**Proposition 5.** *All formulas of* IPDL *are invariant for instantial neighborhood bisimulations.*

The proof of this rests on a bisimulation safety argument, and the step for the Kleene star involves using the bisimulation safety of union and sequential composition to prove the appropriate back-and-forth conditions for each approximant $R_\pi^{[\xi]}$ of the least fixpoint $R_{\pi^*} = [\![*]\!] R_\pi$. We omit the details.

## 4.1 Axiomatization

Our axiomatization for IPDL is given below.

**INL axioms:**

(Mon), (Weak), (Un), (Lem) and (Bot).

**Reduction axioms from DINL:**

(Test), (Ch), (Pa) and (Cmp).

**Basic rules:**

(MP) and (RE).

**Kleene star**

Finally we add axioms and rules for iteration. The Kleene star is a least fixpoint construction, and a standard approach to axiomatizing least fixpoints is to use one *fixpoint axiom* and one *induction rule* (see [17]). The fixpoint axiom **Fix** is stated as follows:

$$\langle\pi^*\rangle(\Psi; \varphi) \leftrightarrow (\bigwedge \Psi \wedge \varphi) \vee \langle\pi \circ \pi^*\rangle(\Psi; \varphi)$$

We will actually need *two* induction rules:

**Ind1:**

$$\frac{\varphi \to \gamma \qquad \langle\pi\rangle\gamma \to \gamma}{\langle\pi^*\rangle\varphi \to \gamma}$$

**Ind2:**

$$\frac{(\psi \wedge \varphi) \to \gamma \qquad \langle\pi\rangle(\gamma; \langle\pi^*\rangle\varphi) \to \gamma}{\langle\pi^*\rangle(\psi; \varphi) \to \gamma}$$

**Remark 1.** The reason that we require two distinct induction rules can be seen as follows: the reduction axioms for IPDL should be interpreted as encoding a recursive translation of the language IPDL into the modal $\mu$-calculus (interpeted on instantial neighborhood models). When we pass by formulas involving the Kleene-star in this translation, the translation will not surprisingly involve least fixpoint operators, and the induction rules then correspond to the Kozen-Park induction rules for least fixpoint operators. This step of the translation is trickier than the step for the Kleene star in a translation of PDL into the $\mu$-calculus (see [10]), and requires use of nested least fixpoint variables.

Note also that the second induction axiom only involves a single instantial formula $\psi$. This is because we can "pre-process" an arbitrary formula $\langle\pi^*\rangle(\psi_1, ..., \psi_n; \varphi)$ by applying the axiom **Fix**, and then applying the composition axiom (Cmp) to the formula $\langle\pi \circ \pi^*\rangle(\psi_1, ..., \psi_n; \varphi)$ to obtain the formula:

$$\langle\pi\rangle(\langle\pi^*\rangle(\psi_1; \varphi), ...., \langle\pi^*\rangle(\psi_n; \varphi); \langle\pi^*\rangle\varphi)$$

Here, each occurrence of the operator $\langle\pi^*\rangle$ is followed by at most one instantial formula.

We denote this axiom system as Ax3 and write $\varphi \vdash_{\mathsf{Ax3}} \psi$ to say that $\mathsf{Ax3} \vdash \varphi \to \psi$. We will also sometimes drop the explicit reference to the system Ax3, simply writing $\vdash \varphi$ or $\varphi \vdash \psi$.

**Theorem 5.** *The axiom system* Ax3 *is sound and complete for validity over neighborhood models.*

We begin by checking soundness:

**Proposition 6** (Soundness)**.** *If $\varphi$ is provable in* Ax3 *then it is valid over all neighborhood models.*

*Proof.* We focus on proving soundness of the two induction rules. For the first induction rule, suppose that the formulas $\varphi \to \gamma$ and $\langle\pi\rangle\gamma \to \gamma$ are valid. Suppose that $\mathfrak{M}, u \Vdash \langle\pi^*\rangle\varphi$. Then there is some $Z$ such that $(u, Z) \in R_{\pi^*}$ and $Z \subseteq [\![\varphi]\!]$. By definition of $R_{\pi^*}$ it suffices to prove, by induction on an ordinal $\xi$, that for all $u, Z$: if $(u, Z) \in R_\pi^{[\xi]}$ and $Z \subseteq [\![\varphi]\!]$ then $u \in [\![\gamma]\!]$. For $\xi = 0$ this is trivial, since $R_\pi^0 = \emptyset$. For a successor ordinal $\xi + 1$, if $(u, Z) \in R_\pi^{[\xi+1]}$ then either $Z = \{u\}$ or there is a set $Y$ and a family of sets $F$ such that

$(Y, F) \in \overline{R_\pi^{[\xi]}}$, $(u, Y) \in R_\pi$ and $\bigcup F \subseteq [\![\varphi]\!]$. In the first case we get $\mathfrak{M}, u \Vdash \varphi$, hence $\mathfrak{M}, u \Vdash \gamma$. In the second case it follows that there is, for each $v \in Y$, some $Z_v$ such that $(v, Z_v) \in R_\pi^{[\xi]}$ and $Z_v \subseteq \bigcup F \subseteq [\![\varphi]\!]$. By the induction hypothesis we get $Y \subseteq [\![\gamma]\!]$. But then $\mathfrak{M}, u \Vdash \langle\pi\rangle\gamma$, hence $\mathfrak{M}, u \Vdash \gamma$ as required. Finally, the induction step for limit ordinals is almost immediate, by the definition of $R_\pi^{[\xi]}$ as the union of all $R_\pi^{[\rho]}$ for $\rho < \xi$.

For the second induction rule, suppose that the formulas $(\psi \wedge \varphi) \to \gamma$ and $\langle\pi\rangle(\gamma; \varphi) \to \gamma$ are valid. Suppose that $\mathfrak{M}, u \Vdash \langle\pi^*\rangle(\psi; \varphi)$. Then there is some $Z$ such that $(u, Z) \in R_{\pi^*}$ and $Z \subseteq [\![\varphi]\!]$, $Z \cap [\![\psi]\!] \neq \emptyset$. By definition of $R_{\pi^*}$ it suffices to prove, by induction on an ordinal $\xi$, that for all $u, Z$: if $(u, Z) \in R_\pi^{[\xi]}$ and $Z \subseteq [\![\varphi]\!]$, $Z \cap [\![\psi]\!] \neq \emptyset$ then $u \in [\![\gamma]\!]$. For $\xi = 0$ this is trivial, since $R_\pi^0 = \emptyset$. For a successor ordinal $\xi + 1$, if $(u, Z) \in R_\pi^{[\xi+1]}$ then either $Z = \{u\}$ or there is a set $Y$ and a family of sets $F$ such that $(Y, F) \in \overline{R_\pi^{[\xi]}}$, $(u, Y) \in R_\pi$ and $\bigcup F \subseteq [\![\varphi]\!]$. In the first case we get $\mathfrak{M}, u \Vdash \psi \wedge \varphi$, hence $\mathfrak{M}, u \Vdash \gamma$. In the second case it follows that there is, for each $v \in Y$, some $Z_v \in F$ such that $(v, Z_v) \in R_\pi^{[\xi]}$ and $Z_v \subseteq \bigcup F \subseteq [\![\varphi]\!]$. Furthermore, there is some set $Z' \in F$ such that $Z' \cap [\![\psi]\!] \neq \emptyset$, and $Z' \subseteq \bigcup F \subseteq [\![\varphi]\!]$. Since $(Y, F) \in \overline{R_\pi^{[\xi]}}$ there must be some $w \in Y$ with $(w, Z') \in R_\pi^{[\xi]}$, and by the induction hypothesis we get $w \Vdash \gamma$. But then $Y \subseteq [\![\langle\pi^*\rangle\varphi]\!]$ (since $R_\pi^{[\xi]} \subseteq R_{\pi^*}$) and $Y \cap [\![\gamma]\!] \neq \emptyset$, so $\mathfrak{M}, u \Vdash \langle\pi\rangle(\gamma; \varphi$. Hence $\mathfrak{M}, u \Vdash \gamma$ as required. Finally, the induction step for limit ordinals is again immediate, by the definition of $R_\pi^{[\xi]}$ as the union of all $R_\pi^{[\rho]}$ for $\rho < \xi$. □

For the completeness proof, we shall rely heavily on the following lemma, which was proved (in a slightly different formulation) in [6]: fix a finite and subformula closed set of formulas $\Sigma$. An *atom* over $\Sigma$ is a maximal consistent subset of $\Sigma$, and we denote the set of atoms over $\Sigma$ by $\mathsf{At}(\Sigma)$. Given any atom $w \in \mathsf{At}(\Sigma)$, let $\widehat{w}$ be its conjunction, and let $\widehat{Z} = \{\widehat{w} \mid w \in Z\}$ for a set of atoms $Z$.

**Lemma 1.** *Let $\langle\pi\rangle(\Psi; \varphi)$ be any formula such that each formula in $\Psi \cup \{\varphi\}$ is a boolean combination of formulas in $\Sigma$. Then $\langle\pi\rangle(\Psi; \varphi)$ is provably equivalent to a disjunction of formulas of the form $\langle\pi\rangle(\widehat{Z}; \bigvee \widehat{Z})$ for $Z \subseteq \mathsf{At}(\Sigma)$ being some set of atoms with $w \vdash \varphi$ for each $w \in Z$ and for all $\psi \in \Psi$ there is some $v \in Z$ with $v \vdash \psi$.*

*Proof.* Very similar to [6]. □

We shall also need an adapted concept of Fischer-Ladner closure:

**Definition 3.** A set $\Sigma$ of formulas is said to be *Fischer-Ladner closed* if the following clauses hold:

- If $\varphi \in \Sigma$, and the main connective of $\varphi$ is not $\neg$, then the formula $\neg\varphi$ is in $\Sigma$.
- Any subformula of a formula in $\Sigma$ is in $\Sigma$.

– If $\langle\gamma?\rangle(\Psi;\varphi)$ is in $\Sigma$ then so is $\gamma \wedge \bigwedge \Psi \wedge \varphi$.

– If $\langle\pi_1 \circ \pi_2\rangle(\psi_1, ..., \psi_n; \varphi) \in \Sigma$, then $\langle\pi_1\rangle(\langle\pi_2\rangle(\psi_1;\varphi), ..., \langle\pi_1\rangle(\psi_n;\varphi); \langle\pi_2\rangle\varphi)$ is in $\Sigma$ too.

– If $\langle\pi_1 \cup \pi_2\rangle(\Psi;\varphi) \in \Sigma$ then $\langle\pi_1\rangle(\Psi;\varphi) \vee \langle\pi_2\rangle(\Psi;\varphi) \in \Sigma$ too.

– If $\langle\pi_1 \cap \pi_2\rangle(\Psi;\varphi) \in \Sigma$ then the formula:

$$\bigvee\{\langle\pi_1\rangle(\Theta_1;\varphi) \wedge \langle\pi_2\rangle(\Theta_2;\varphi) \mid \Psi = \Theta_1 \cup \Theta_2\}$$

is in $\Sigma$ too.

– If $\langle\pi^*\rangle(\Psi;\varphi) \in \Sigma$ then $(\bigwedge \Psi \wedge \varphi) \vee \langle\pi \circ \pi^*\rangle(\Psi;\varphi)$ is in $\Sigma$ too.

**Lemma 2.** *Every formula $\varphi$ is a member of some finite Fischer-Ladner closed set of formulas.*

*Proof.* Standard, see for example [9]. □

Fix a finite and Fischer-Ladner closed set of formulas $\Sigma$. An *atom* over $\Sigma$ is a maximal consistent subset of $\Sigma$, and we denote the set of atoms over $\Sigma$ by $\mathsf{At}(\Sigma)$. Given any atom $w \in \mathsf{At}(\Sigma)$, let $\widehat{w}$ be its conjunction, and let $\widehat{Z} = \{\widehat{w} \mid w \in Z\}$ for a set of atoms $Z$.

**Definition 4.** Given any label $\pi$, we define the relation $S^\Sigma_\pi \subseteq \mathsf{At}(\Sigma) \times \mathcal{P}(\mathsf{At}(\Sigma))$ by setting $(w, Z) \in S^\Sigma_\pi$ iff $\widehat{w} \wedge \langle\pi\rangle(\widehat{Z}; \bigvee \widehat{Z})$ is consistent with respect to the system $\mathsf{Ax3}$.

The *canonical neighborhood model* over $\Sigma$ denoted $\mathfrak{C}^\Sigma$ is defined as the triple $(W^\Sigma, R^\Sigma, V^\Sigma)$ where $W^\Sigma$ is the set of atoms over $\Sigma$, $R^\Sigma_a = S^\Sigma_a$ for each atomic label $a$, and $V^\Sigma(p) = \{w \in W^\Sigma \mid p \in w\}$.

The key lemma in the completeness proof, which is proved using the induction rules for the Kleene star, is the following:

**Lemma 3.** *For each label $\pi$, we have $S^\Sigma_{\pi^*} \subseteq [\![*]\!](S^\Sigma_\pi)$.*

*Proof.* Since the set of atoms is finite, we can make use of the characterization of the Kleene star operation on finite models given by Proposition 4.

Suppose that $(w, Z) \in S^\Sigma_{\pi^*}$, meaning that $\nvdash \neg(\widehat{w} \wedge \langle\pi^*\rangle(\widehat{Z}; \bigvee \widehat{Z})$. Let $\gamma[Z]$ be the disjunction of all formulas $\widehat{v}$ for $(v, Z) \in [\![*]\!](S^\Sigma_\pi)$. We want to show that $\langle\pi^*\rangle(\widehat{Z}; \bigvee \widehat{Z}) \vdash \gamma[Z]$. It will then follow that $\widehat{w} \wedge \gamma[Z]$ is consistent, and clearly since $w$ is an atom this can only happen if $\widehat{w}$ is already a disjunct of $\gamma[Z]$ which means that $(w, Z) \in [\![*]\!](S^\Sigma_\pi)$ as desired.

More generally, for $Z' \subseteq Z$ let $\gamma[Z', Z]$ be the disjunction of all formulas $\widehat{v}$ where $v$ is an atom such that $(v, Z'') \in [\![*]\!](S^\Sigma_\pi)$ and $Z' \subseteq Z'' \subseteq Z$ for some set $Z''$. We will show that $\langle\pi^*\rangle(\widehat{Z'}; \bigvee \widehat{Z}) \vdash \gamma[Z', Z]$. The special case for the formula $\gamma[Z, Z] = \gamma[Z]$ then yields the desired result.

We first prove the claim for the case of $Z' = \emptyset$. We have

$$\langle \pi^* \rangle (\widehat{\emptyset}; \bigvee \widehat{Z}) = \langle \pi^* \rangle (\emptyset; \bigvee \widehat{Z}) = \langle \pi^* \rangle \bigvee \widehat{Z}$$

So we want to show that $\langle \pi^* \rangle \bigvee \widehat{Z} \vdash \gamma[\emptyset, Z]$, and by the first induction rule it suffices to prove that $\bigvee \widehat{Z} \vdash \gamma[\emptyset, Z]$ and $\langle \pi \rangle \gamma[\emptyset, Z] \vdash \gamma[\emptyset, Z]$. Since $\gamma[\emptyset, Z]$ is a disjunction of conjunctions of atoms, it is routine to show that for any formula $\theta$ we have $\theta \vdash \gamma[\emptyset, Z]$ if and only if every atom that is consistent with $\theta$ is also consistent with $\gamma[\emptyset, Z]$.

So suppose first that $w$ is consistent with $\bigvee \widehat{Z}$. Then $w$ must be in $Z$, and since $(w, \{w\}) \in \mathsf{skip} \subseteq [\![*]\!](S^\Sigma_\pi)$ we get that $w$ is consistent with $\gamma[\emptyset, Z]$ as required.

Next, suppose that $w$ is consistent with $\langle \pi \rangle \gamma[\emptyset, Z]$. By Lemma 1 there must be some set $Z'$ such that $w$ is consistent with $\langle \pi \rangle (\widehat{Z'}; \bigvee \widehat{Z'})$ and $u \vdash \gamma[\emptyset, Z]$ for each $u \in Z'$. We get that $(w, Z') \in S^\Sigma_\pi$, and furthermore for each $u \in Z'$ there must be some $Z_u \subseteq Z$ with $(u, Z_u) \in [\![*]\!](S^\Sigma_\pi)$. We get:

$$(Z', \{Z_u \mid u \in Z'\}) \in \overline{[\![*]\!](S^\Sigma_\pi)}$$

and hence we obtain:

$$(w, \bigcup_{u \in Z'} Z_u) \in S^\Sigma_\pi [\![\circ]\!]([\![*]\!](S^\Sigma_\pi)) \subseteq [\![*]\!](S^\Sigma_\pi)$$

and so since $\emptyset \subseteq \bigcup_{u \in Z'} Z_u \subseteq Z$ we see that $w$ is consistent with $\gamma[\emptyset, Z]$ as required.

Next, we consider the case where $Z' \subseteq Z$ is a singleton $\{s\}$. We write $\gamma[s, Z]$ rather than $\gamma[\{s\}, Z]$. We want to show that $\langle \pi^* \rangle (\widehat{s}; \bigvee \widehat{Z}) \vdash \gamma[s, Z]$. By the second inducton rule, it suffices to prove that

$$\widehat{s} \wedge \bigvee \widehat{Z} \vdash \gamma[s, Z]$$

and

$$\langle \pi^* \rangle (\gamma[s, Z]; \langle \pi^* \rangle \bigvee \widehat{Z}) \vdash \gamma[s, Z]$$

The first statement is similar to the proof that $\bigvee \widehat{Z} \vdash \gamma[\emptyset, Z]$ so we leave it out. For the second part, suppose that the atom $w$ is consistent with the formula $\langle \pi \rangle (\gamma[s, Z]; \langle \pi^* \rangle \bigvee \widehat{Z})$. By the previous argument we get

$$\langle \pi^* \rangle \bigvee \widehat{Z}) \vdash \gamma[\emptyset, Z]$$

so by (Mon) we find that $w$ is consistent with $\langle \pi \rangle (\gamma[s, Z]; \gamma[\emptyset, Z])$. By Lemma 1 there must be some set $Y$ such that $w$ is consistent with $\langle \pi \rangle (\widehat{Y}; \bigvee \widehat{Y})$, $u \vdash \gamma[\emptyset, Z]$ for each $u \in Y$, and $v \vdash \gamma[s, Z]$ for some $v \in Y$. We get that $(w, Y) \in S^\Sigma_\pi$. Furthermore there is some set $Z_v$ such that $s \in Z_v \subseteq Z$ and $(v, Z_v) \in [\![*]\!](S^\Sigma_\pi)$, and for each $u \neq v$ in $Y$ there is some $Z_u \subseteq Z$ such that $(u, Z_u) \in [\![*]\!](S^\Sigma_\pi)$. If we set:

$$F = \{Z_v\} \cup \{Z_u \mid u \in Y \setminus \{v\}\}$$

then we get $\{s\} \subseteq \bigcup F \subseteq Z$. Furthermore, we get

$$(Y, F) \in \overline{[\![*]\!](S^{\Sigma}_{\pi})}$$

and hence we obtain:

$$(w, \bigcup F) \in S^{\Sigma}_{\pi}[\![\circ]\!]([\![*]\!](S^{\Sigma}_{\pi})) \subseteq [\![*]\!](S^{\Sigma}_{\pi})$$

as required.

Finally, let $Z' \subseteq Z$ be an arbitrary non-empty set, and suppose $w$ is consistent with $\langle \pi^* \rangle(\widehat{Z'}; \bigvee \widehat{Z})$, where $Z' = \{s_1, ..., s_n\}$. Then by the axiom (Fix), $w$ is consistent with the formula

$$(\bigwedge \widehat{Z'} \wedge \bigvee \widehat{Z}) \vee \langle \pi \circ \pi^* \rangle(\widehat{Z'}; \bigvee \widehat{Z})$$

So it now suffices to prove that:

$$(\bigwedge \widehat{Z'} \wedge \bigvee \widehat{Z}) \vdash \gamma[Z', Z]$$

and

$$\langle \pi \circ \pi^* \rangle(\widehat{Z'}; \bigvee \widehat{Z}) \vdash \gamma[Z', Z]$$

Once again, the first claim follows by a familiar argument using $\mathsf{skip} \subseteq [\![*]\!](S^{\Sigma}_{\pi})$, so we omit it. For the second claim, it suffices by the axiom (Cmp) to prove that:

$$\langle \pi \rangle(\langle \pi^* \rangle(\widehat{s}_1; \bigvee \widehat{Z}), ..., \langle \pi^* \rangle(\widehat{s}_n; \bigvee \widehat{Z}); \langle \pi^* \rangle \bigvee \widehat{Z}) \vdash \gamma[Z', Z]$$

But, using the previous arguments together with the axiom (Mon), we find that it suffices to prove:

$$\langle \pi \rangle(\gamma[s_1; Z], ..., \gamma[s_n; Z]; \gamma[\emptyset, Z]) \vdash \gamma[Z', Z]$$

We show that every atom consistent with the formula on the left-hand side is also consistent with the formula on the right-hand side. Suppose that $w$ is consistent with the formula $\langle \pi \rangle(\gamma[s_1; Z], ..., \gamma[s_n; Z]; \gamma[\emptyset, Z])$. By Lemma 1 there must be some set $Y$ such that $w$ is consistent with $\langle \pi \rangle(\widehat{Y}; \bigvee \widehat{Y})$, $u \vdash \gamma[\emptyset, Z]$ for each $u \in Y$, and for each $i \in \{1, ..., n\}$ we have $v_i \vdash \gamma[s_i, Z]$ for some $v_i \in Y$. We get that $(w, Y) \in S^{\Sigma}_{\pi}$. Furthermore for each $i \in \{1, ..., n\}$ there is some set $S_i$ such that $s_i \in S_i \subseteq Z$ and $(v_i, S_i) \in [\![*]\!](S^{\Sigma}_{\pi})$, and for each $u \notin \{v_1, ..., v_n\}$, $u \in Y$, there is some $Z_u \subseteq Z$ such that $(u, Z_u) \in [\![*]\!](S^{\Sigma}_{\pi})$. If we set:

$$F = \{S_1, ..., S_n\} \cup \{Z_u \mid u \in Y \setminus \{v_1, ..., v_n\}\}$$

then we get $\{s_1, ..., s_n\} \subseteq \bigcup F \subseteq Z$. Furthermore, we get

$$(Y, F) \in \overline{[\![*]\!](S^{\Sigma}_{\pi})}$$

and hence we obtain:

$$(w, \bigcup F) \in S^{\Sigma}_{\pi}[\![\circ]\!]([\![*]\!](S^{\Sigma}_{\pi})) \subseteq [\![*]\!](S^{\Sigma}_{\pi})$$

as required. □

Lemma 3 is needed to prove Lemma 4 below, by induction on the complexity of program terms. Say that a label $\pi$ is *safe* if, for every formula $\gamma$ such that the term $\gamma?$ appears in $\pi$, we have $\gamma \in \Sigma$ and furthermore, $\gamma \in w$ iff $\mathfrak{C}^\Sigma, w \Vdash \gamma$ for each $w \in \mathsf{At}(\Sigma)$.

**Lemma 4.** *For every safe label $\pi$, we have $S^\Sigma_\pi \subseteq R^\Sigma_\pi$.*

*Proof.* By induction on the complexity of safe labels. For $\gamma?$, the result follows from the safety assumption and the observation that

$$S^\Sigma_{\gamma?} = [\![?]\!]\{w \mid \mathsf{At}(\Sigma) \mid \gamma \in w\}$$

This observation can be proved as follows: since $\gamma$ is safe we have $\gamma \in \Sigma$, so $\widehat{w} \wedge \langle \gamma? \rangle (\widehat{Z}, \bigvee \widehat{Z})$ is consistent $\widehat{w} \wedge \bigwedge \widehat{Z} \wedge \bigvee \widehat{Z}$ is consistent, iff $\gamma \in w$ and $\widehat{Z} = \{w\}$ since $w$ is an atom and $\widehat{Z}$ a set of atoms. Hence $S^\Sigma_{\gamma?} = \{(w, \{w\}) \mid \gamma \in w\}$ and the result follows from the definition of $[\![?]\!]$.

For the Kleene star, by Lemma 3 we have $S^\Sigma_{\pi^*} \subseteq [\![*]\!](S^\Sigma_\pi)$ for each label $\pi$. Similarly we may prove: $S^\Sigma_{\pi_1 \cup \pi_2} \subseteq S^\Sigma_{\pi_1} [\![\cup]\!] S^\Sigma_{\pi_2}$ and $S^\Sigma_{\pi_1 \circ \pi_2} \subseteq S^\Sigma_{\pi_1} [\![\circ]\!] S^\Sigma_{\pi_2}$. We omit the easy argument for $\cup$. For $\cap$, suppose that $\widehat{w} \wedge \langle \pi_1 \cap \pi_2 \rangle (\widehat{Z}; \bigvee \widehat{Z})$ is consistent. Then there are sets $Z_1, Z_2$ such that $Z = Z_1 \cup Z_2$ such that:

$$\widehat{w} \wedge \langle \pi_1 \rangle (\widehat{Z}_1; \bigvee \widehat{Z}) \wedge \langle \pi_2 \rangle (\widehat{Z}_2; \bigvee \widehat{Z})$$

is consistent. Hence both $\widehat{w} \wedge \langle \pi_1 \rangle (\widehat{Z}_1; \bigvee \widehat{Z})$ and $\widehat{w} \wedge \langle \pi_2 \rangle (\widehat{Z}_2; \bigvee \widehat{Z})$ are consistent, and using Lemma 1 we find sets $Y_1, Y_2 \subseteq \mathsf{At}(\Sigma)$ (corresponding to disjuncts of the normal form) such that $Z_1 \subseteq Y_1 \subseteq Z$ and $Z_2 \subseteq Y_2 \subseteq Z$ and such that both $\widehat{w} \wedge \langle \pi_1 \rangle (\widehat{Y_1}; \bigvee \widehat{Y_1})$ and $\widehat{w} \wedge \langle \pi_2 \rangle (\widehat{Y_2}; \bigvee \widehat{Y_2})$ are consistent. Hence $(w, Y_1) \in S^\Sigma_{\pi_1}$ and $(w, Y_2) \in S^\Sigma_{\pi_2}$, hence $(w, Y_1 \cup Y_2) \in S^\Sigma_{\pi_1} [\![\cap]\!] S^\Sigma_{\pi_2}$. The result now follows since clearly $Y_1 \cup Y_2 = Z$.

For composition, suppose that $w$ is consistent with the formula $\langle \pi_1 \circ \pi_2 \rangle (\widehat{Z}; \bigvee \widehat{Z})$, where $Z = \{v_1, ..., v_n\}$. Then $w$ is consistent with the formula

$$\langle \pi_1 \rangle (\langle \pi_2 \rangle (\widehat{v}_1; \bigvee \widehat{Z}), ..., \langle \pi_2 \rangle (\widehat{v}_n; \bigvee \widehat{Z}); \langle \pi_2 \rangle \bigvee \widehat{Z})$$

by the axiom (Cmp). For each $i \in \{1, ..., m\}$ let $\delta_i$ be the disjunction of the set of all formulas $\widehat{u}$ such that $u$ is an atom with $(u, U) \in S^\Sigma_{\pi_2}$ for some set of atoms $U$ with $v_i \in U$ and $U \subseteq Z$, and let $\theta$ be the disjunction of all formulas $\widehat{u}$ such that $u$ is an atom with $(u, U) \in S^\Sigma_{\pi_2}$ for some $U \subseteq Z$. We first claim that:

$$\langle \pi_1 \rangle (\langle \pi_2 \rangle (\widehat{v}_1; \bigvee \widehat{Z}), ..., \langle \pi_2 \rangle (\widehat{v}_n; \bigvee \widehat{Z}); \langle \pi_2 \rangle \bigvee \widehat{Z}) \vdash \langle \pi_1 \rangle (\delta_1, ..., \delta_n; \theta)$$

To see this, let the maximum modal depth of formulas in $\Sigma$ be $k$, and let $F^{2+k}_\Sigma$ be the set of all formulas of modal depth at most $2 + k$, such that only labels appearing appearing in formulas in $\Sigma$ may appear in formulas in $F^{2+k}_\Sigma$. Let an *extended atom* be a maximal consistent subset of $F^{2+k}_\Sigma$. Since there are only finitely many formulas in $F^{2+k}_\Sigma$ up to provable equivalence, there are at most finitely many extended atom, and for each extended atom $e$ we

can form the conjunction $\widehat{e}$ of all formulas in $e$ "up to logical equivalence", picking one conjunct from each logical equivalence class. Since both formulas $\langle\pi_1\rangle(\langle\pi_2\rangle(\widehat{v}_1; \bigvee \widehat{Z}), ..., \langle\pi_2\rangle(\widehat{v}_n; \bigvee \widehat{Z}); \langle\pi_2\rangle \bigvee \widehat{Z})$ and $\langle\pi_1\rangle(\delta_1, ..., \delta_n; \theta)$ are of modal depth $\leq 2 + k$, it suffices to prove that every extended atom $e$ containing the formula:

$$\langle\pi_1\rangle(\langle\pi_2\rangle(\widehat{v}_1; \bigvee \widehat{Z}), ..., \langle\pi_2\rangle(\widehat{v}_n; \bigvee \widehat{Z}); \langle\pi_2\rangle \bigvee \widehat{Z})$$

also contains:

$$\langle\pi_1\rangle(\delta_1, ..., \delta_n; \theta).$$

So let $e$ be an extended atom containing the first of these two formulas. Once again, by a proof similar to that of Lemma 1, we can prove that the formula

$$\langle\pi_1\rangle(\langle\pi_2\rangle(\widehat{v}_1; \bigvee \widehat{Z}), ..., \langle\pi_2\rangle(\widehat{v}_n; \bigvee \widehat{Z}); \langle\pi_2\rangle \bigvee \widehat{Z})$$

is equivalent to a disjunction of formulas of the form $\langle\pi_1\rangle(\widehat{E}, \bigvee \widehat{E})$ where $E$ is a set of extended atoms such that $\langle\pi_2\rangle \bigvee \widehat{Z} \in \bigcap E$ and $\langle\pi_2\rangle(\widehat{v}_i; \bigvee \widehat{Z}) \in \bigcup E$ for each $i \in \{1, ..., n\}$. So one of these disjuncts $\langle\pi_1\rangle(\widehat{E}, \bigvee \widehat{E})$ belongs to $e$ Furthermore, it is not hard to show that $\vdash \widehat{e'} \to \theta$ for each $e' \in E$, and similarly one can show that $\vdash \widehat{e'} \to \delta_i$ for each $e' \in E$ such that $\langle\pi_2\rangle(\widehat{v}_1; \bigvee \widehat{Z}) \in e'$ (since $e' \cap \Sigma$ is an atom consistent with $\langle\pi_2\rangle(\widehat{v}_1; \bigvee \widehat{Z})$). So we get:

$$\langle\pi_1\rangle(\widehat{E}, \bigvee \widehat{E}) \vdash \langle\pi_1\rangle(\delta_1, ..., \delta_n; \theta)$$

by (Mon), hence $\langle\pi_1\rangle(\delta_1, ..., \delta_n; \theta)$ belongs to $e$ as well.

So $w$ is consistent with the formula $\langle\pi_1\rangle(\delta_1, ..., \delta_n; \theta)$, and by Lemma 1 there is a set $Q$ of atoms such that $w$ is consistent with $\langle\pi_1\rangle(\widehat{Q}; \bigvee \widehat{Q})$, $s \vdash \theta$ for each $s \in Q$ and for each $i \in \{1, ..., n\}$ there is $t_i \in Q$ such that $t_i \vdash \delta_i$. It follows from this that for each $s \in Q$ there is some $U_s \subseteq Z$ such that $(s, U_s) \in S^\Sigma_{\pi_2}$, and for each $i \in \{1, ..., n\}$ there is some $P_i \subseteq Z$ such that $v_i \in P_i$ and $(t_i, P_i) \in S^\Sigma_{\pi_2}$. If we set

$$F = \{U_s \mid s \in Q\} \cup \{P_i \mid i \in \{1, ..., n\}\}$$

then we get $(Q, F) \in \overline{S^\Sigma_{\pi_2}}$, and so $(w, \bigcup F) \in S^\Sigma_{\pi_1} [\![\circ]\!] S^\Sigma_{\pi_2}$. But $\bigcup F = Z$, so we get $(w, Z) \in S^\Sigma_{\pi_1} [\![\circ]\!] S^\Sigma_{\pi_2}$ as required.

Finally, a straightforward induction now shows that $S^\Sigma_\pi \subseteq R^\Sigma_\pi$ for each safe label $\pi$, using monotonicity of each of the operations $[\![\cup]\!], [\![\cap]\!], [\![\circ]\!], [\![*]\!]$. For atomic labels the claim holds by definition of $R^\Sigma_a = S^\Sigma_a$. For the case of iteration, as an example, we have:

$$\begin{aligned} S^\Sigma_{\pi^*} &\subseteq [\![*]\!](S^\Sigma_\pi) \\ &\subseteq [\![*]\!](R^\Sigma_\pi) \\ &= R^\Sigma_{\pi^*} \end{aligned}$$

The other cases are similar. □

Using Lemma 4 we can prove a truth lemma for the canonical model:

**Lemma 5.** *For every atom $w$ and any $\psi \in \Sigma$, we have $(\mathfrak{C}^\Sigma, w) \Vdash \psi$ if and only if $\psi \in w$.*

*Proof.* By induction on the complexity of $\psi$. Note that the induction hypothesis for subformulas of $\psi$ guarantees that every label appearing in $\psi$ is safe. The only interesting cases are formulas of the form $\langle\pi\rangle(\Psi;\varphi)$.

For right to left, suppose $\langle\pi\rangle(\Psi;\varphi) \in w$. By Lemma 1 we find a set $Z$ of atoms such that $\langle\pi\rangle(\widehat{Z}, \bigvee \widehat{Z})$ is consistent with $w$, hence $(w, Z) \in S^\Sigma_\pi$, and such that $\Psi \subseteq \bigcup Z$ and $\varphi \in \bigcap Z$. By Lemma 4 we get $(w, Z) \in R^\Sigma_\pi$, and the induction hypothesis applied to the formulas in $\Psi \cup \{\varphi\}$ now readily yields $\mathfrak{C}^\Sigma, w \Vdash \langle\pi\rangle(\Psi;\varphi)$ as required.

For left to right, it suffices to show that for all formulas $\langle\pi\rangle(\Psi;\varphi) \in \Sigma$, all sets of atoms $Z$ and all atoms $w$ such that $(w, Z) \in R^\Sigma_\pi$, $\varphi \in \bigcap Z$ and $\Psi \subseteq \bigcup Z$, we have $\langle\pi\rangle(\Psi;\varphi) \in w$. The required result then follows by applying the induction hypothesis to $\Psi, \varphi$. We prove the claim by induction on the complexity of the label $\pi$, under the assumption that $\pi$ is a safe label

If $\pi$ is an atomic label $a$ then we have $R^\Sigma_a = S^\Sigma_a$. So if $(w, Z) \in R^\Sigma_a$ then $(w, Z) \in S^\Sigma_a$, so $w$ is consistent with $\langle a\rangle(\widehat{Z}; \bigvee \widehat{Z})$. From this we can easily derive that $w$ is consistent with $\langle a\rangle(\Psi;\varphi)$ by an argument combining axioms (Mon) and (Weak), given that $\varphi \in \bigcap Z$ and $\Psi \subseteq \bigcup Z$. Since $\langle a\rangle(\Psi;\varphi) \in \Sigma$ and $w$ is an atom it follows that $\langle a\rangle(\Psi;\varphi) \in w$ as required.

The induction steps for test, choice, parallel composition and sequential composition are easy, making use of Fischer-Ladner closure of $\Sigma$ at each step.

We now focus on the case of the Kleene star. Suppose that there is some $Z$ such that $(w, Z) \in R^\Sigma_{\pi^*}$, $\varphi \in \bigcap Z$ and $\Psi \subseteq \bigcup Z$. By Proposition 4 there is some natural number $n$ with $(w, Z) \in (R^\Sigma_\pi)^{[n]}$, so we reason by induction on $n$. That is, we show that for all $w, Z, \Psi, \varphi$ and all $n \in \omega$, if $(w, Z) \in (R^\Sigma_\pi)^{[n]}$, $\varphi \in \bigcap Z$ and $\Psi \subseteq \bigcup Z$, then $\langle\pi^*\rangle(\Psi;\varphi) \in w$.

For $n = 0$ the result holds trivially since $(R^\Sigma_\pi)^{[n]} = \emptyset$. Supposing that the induction hypothesis holds for $n$, if $(w, Z) \in (R^\Sigma_\pi)^{[n+1]}$ then either $(w, Z) \in \mathsf{skip}$, or:

$$(w, Z) \in R^\Sigma_\pi[\![\circ]\!](R^\Sigma_\pi)^{[n]}$$

In the first case, we have $Z = \{w\}$ so it immediately follows (using Fischer-Ladner closure of $\Sigma$ and $\langle\pi^*\rangle(\Psi;\varphi) \in \Sigma$ ) that $\bigwedge \Psi \wedge \varphi \in w$. By the axiom (Fix) we must have $\langle\pi^*\rangle(\Psi;\varphi) \in w$ as required.

Otherwise, if $(w, Z) \in R^\Sigma_\pi[\![\circ]\!](R^\Sigma_\pi)^{[n]}$ then there is some set $Y$ and a family of sets $F$ such that $(w, Y) \in R^\Sigma_\pi$, $(Y, F) \in \overline{(R^\Sigma_\pi)^{[n]}}$, $\varphi \in \bigcap X$ for each $X \in F$ and for each $\psi \in \Psi$ there exists some $X_\psi \in F$ with $\psi \in \bigcup X_\psi$. By Fischer-Ladner closure we get $\langle\pi \circ \pi^*\rangle(\Psi;\varphi) \in \Sigma$ and hence:

$$\langle\pi\rangle(\langle\pi^*\rangle(\psi_1;\varphi), ..., \langle\pi^*\rangle(\psi_n;\varphi); \langle\pi^*\rangle\varphi) \in \Sigma$$

where $\Psi = \{\psi_1, ..., \psi_n\}$. By applying the induction hypothesis to the label $\pi$ and the "inner" induction hypothesis to $n$, we now find that:

$$\langle\pi\rangle(\langle\pi^*\rangle(\psi_1;\varphi), ..., \langle\pi^*\rangle(\psi_n;\varphi); \langle\pi^*\rangle\varphi) \in w$$

By applying the axiom (Cmp) we get $\langle \pi \circ \pi^* \rangle(\Psi; \varphi) \in w$, hence by the axiom (Fix) we get $\langle \pi^* \rangle(\Psi; \varphi) \in w$ as required. □

*Proof of Theorem 5.* suppose the formula $\varphi$ is not provable, so that $\neg\varphi$ is consistent. By Lemma 2, $\neg\varphi$ belongs to some finite Fischer-Ladner closed set $\Sigma$ and since $\neg\varphi$ is consistent it belongs to some atom $w$. Hence $\varphi \notin w$ and by Lemma 5 we have $\mathfrak{C}^\Sigma, w \nVdash \varphi$. So $\varphi$ is not valid. □

We note that as a corollary to the completeness proof, which produces a finite model of effectively bounded size for a consistent formula, we get:

**Theorem 6.** IPDL *has the finite model property and is decidable.*

# 5 Comparison with game logic

We now show that IPDL can, in a precise sense, be viewed as a language extension of dual-free game logic. We shall denote this language simply by GL, for "game logic", although the full dynamic game logic also includes a dual constructor. Formally, formulas of GL and game terms are defined by the following dual grammar:

$$\varphi := p \in \mathsf{Prop} \mid \varphi \wedge \varphi \mid \neg\varphi \mid \langle \pi \rangle \varphi$$

$$\pi := a \in \mathcal{A} \mid \varphi? \mid \pi \circ \pi \mid \pi \cup \pi \mid \pi \cap \pi \mid \pi^*$$

where Prop is a fixed set of propositional variables and $\mathcal{A}$ is a set of atomic games, both assumed to be countably infinite. Note that GL is a syntactic fragment of IPDL. Here, $\cup$ is interpreted as "angelic choice" (choice for Player I), $\cap$ is interpreted as "demonic choice" (choice for Player II), $\circ$ is sequential game composition and $*$ is finite game iteration (controlled by Player I).

Semantics of game logic formulas are given by neighborhood frames, with the extra constraint that neighborhoods associated with a world are upwards closed under subsethood:

**Definition 5.** A neighborhood frame $(W, R)$ is said to be a *monotonic power frame* if the following condition holds for each $a \in \mathcal{A}$:

(Monotonicity) For all $u \in W$, if $(u, Z) \in R_a$ and $Z \subseteq Z'$ then $(u, Z') \in R_a$.

A monotonic power model is a neighborhood model whose underlying frame is a monotonic power frame.

In order to provide the semantic interpretations of formulas in a model, we need to provide semantic interpretations of the game constructors. We shall use double vertical lines $\|-\|$ to refer to semantic interpretations of formulas in GL and game constructors in monotonic neighborhood models, in order to distinguish it from the semantics given for PDL, where we use square brackets $[\![-]\!]$. We follow the definitions in [3]. Formally, we define operations on the lattice $\mathcal{N}W = \mathcal{P}(W \times \mathcal{P}(W))$ of *neighborhood relations* over $W$ as follows:

- $R\|\cup\|R' = R \cup R'$

- $R\|\cap\|R' = R \cap R'$

- $(u, Z') \in R\|\circ\|R'$ iff there is some $Z \subseteq W$ with $(u, Z) \in R$ and $(v, Z') \in R'$ for all $v \in Z$.

- $\|?\|(Z) = \{(w, Z') \in W \times \mathcal{P}(W) \mid w \in Z \cap Z'\}$

Finally, we define $\|*\|R$ to be the least fixpoint in the lattice $\mathcal{N}W$ of the monotone map $F$ defined by:

$$FS = \mathsf{skip}^{\uparrow}\|\cup\|(R\|\circ\|S)$$

where $\mathsf{skip}^{\uparrow} = \{(w, Z) \in W \times \mathcal{P}(W) \mid w \in Z\}$. We can now set up the semantics of GL. Fixing a monotonic power model $\mathfrak{M}$, we define the interpretation of every formula $\varphi$ and the neighborhood relations $R_\pi$ corresponding to each game term $\pi$ in the obvious way, so that in particular we have -$R_{\pi_1 \cup \pi_2} = R_{\pi_1}\|\cup\|R_{\pi_2}$, $R_{\pi_1 \cap \pi_2} = R_{\pi_1}\|\cap\|R_{\pi_2}$ etc., and $u \in \|\langle\pi\rangle\varphi\|$ iff $(u, \|\varphi\|) \in R_a$. For a monotonic power model $\mathfrak{M} = (W, R, V)$ and $u \in W$ we shall also write $\mathfrak{M}, u \vDash \varphi$ for $u \in \|\varphi\|$. Since semantic interpretations are always defined relative to a model, if necessary we shall use the notation $\|-\|_{\mathfrak{M}}$ rather than $\|-\|$ to make it clear which model $\mathfrak{M}$ is being referred to. We write $\vDash \varphi$ if $\mathfrak{M}, u \vDash \varphi$ for every pointed monotone power model $(\mathfrak{M}, u)$. We get the following result, showing in what sense IPDL indeed generalizes the semantics of GL:

**Proposition 7.** *For any* GL*-formula $\varphi$, and any monotonic power model $\mathfrak{M}$, we have $\|\varphi\|_{\mathfrak{M}} = [\![\varphi]\!]_{\mathfrak{M}}$.*

*Proof.* The proof is an induction on the complexity of formulas, with the interesting step being formulas of the form $\langle\pi\rangle\varphi$. We leave out the test operator in this proof, leaving this as an easy exercise. We first prove the following claim, and the rest of the proof is then easy:

**Claim 2.** Let $R, S \in \mathcal{N}W$ be neighborhood relations that are both closed upwards under subsethood, in the sense that $(w, Z) \in R$ and $Z \subseteq Z'$ implies $(w, Z') \in R$. Then:

1. $R[\![\cup]\!]S = R\|\cup\|S$
2. $R[\![\cap]\!]S = R\|\cap\|S$
3. $R[\![\circ]\!]S = R\|\circ\|S$
4. $[\![*]\!]R = \|*\|R$

The rest of the proof is devoted to establishing this claim. Item (1) is immediate from the definitions. For item (2), suppose first that $(w, Z) \in R[\![\cap]\!]S$.

Then there are $Z', Z''$ such that $Z = Z' \cup Z''$ and $(w, Z') \in R$, $(w, Z'') \in S$. By closure under subsethood we have $(w, Z) \in R$ and $(w, Z) \in S$, so $(w, Z) \in R \cap S = R\|\cap\|S$. Conversely, if $(w, Z) \in R\|\cap\|S$ then $(w, Z) \in R$ and $(w, Z) \in S$. So $(w, Z \cup Z) = (w, Z) \in R[\![\cap]\!]S$. Item (3) is proved in a fairly similar manner, so we leave it as an exercise.

For item (4), we note that $[\![*]\!]R$ is the least fixpoint of the map:

$$\lambda Z.\mathsf{skip}[\![\cup]\!](R[\![\circ]\!]Z)$$

which is equal to the least fixpoint of $\lambda Z.\mathsf{skip}\|\cup\|(R\|\circ\|Z)$ by items (1) and (3) (and noting that all the relations appearing in the approximating sequence for the fixpoint are upwards closed under subsethood). But the latter fixpoint is by definition equal to $\|*\|R$.

□

From this proposition, we get the following result:

**Theorem 7.** IPDL *is a conservative extension of* GL. *That is, for every* GL*-formula $\varphi$, we have*

$$\vDash \varphi \textit{ iff } \Vdash \varphi$$

*Proof.* For every neighborhood model $\mathfrak{M}$, we define a monotonic power model $\mathfrak{M}^{\uparrow}$ as follows: let $\mathfrak{M} = (W, R, V)$. We define the monotonic power model $\mathfrak{M}^{\uparrow} = (W, \underline{R}, V)$ as follows: set $(u, Z) \in \underline{R}_a$ iff there is some $Z' \subseteq Z$ with $(u, Z') \in R_a$.

We have the following result:

**Claim 3.** For any GL-formula $\varphi$ and any neighborhood model $\mathfrak{M}$, we have $\|\varphi\|_{\mathfrak{M}^{\uparrow}} = [\![\varphi]\!]_{\mathfrak{M}}$.

*Proof of Claim 3.* The result follows easily once we have established the following claim:

**Claim 4.** Given a neighborhood model $\mathfrak{M}$ and a term $\pi$, let $R_\pi$ denote the neighborhood relation corresponding to $\pi$ in $\mathfrak{M}$ computed by applying the operations $[\![?]\!]$, $[\![\cup]\!]$, $[\![\cap]\!]$, $[\![\circ]\!]$, $[\![*]\!]$, and let $S_\pi$ denote the neighborhood relation corresponding to $\pi$ in $\mathfrak{M}^{\uparrow}$ computed by applying the operations $\|?\|$, $\|\cup\|$, $\|\cap\|$, $\|\circ\|$, $\|*\|$. Then for all $w, Z$, we have $(w, Z) \in S_\pi$ iff there is some $Z' \subseteq Z$ with $(w, Z') \in R_\pi$.

We devote the rest of the proof to establishing this claim. The claim is immediate for atomic games, and the step for the test operator follows trivially from the definitions. The direction from right to left is easy in each case, so we focus on the converse implication.

The induction step for $\cup$ is entirely straightforward. For $\cap$, if $(w, Z) \in S_{\pi_1 \cap \pi_2}$ then $(w, Z) \in S_{\pi_1}$ and $(w, Z) \in S_{\pi_2}$. By the induction hypothesis, there are sets $Y, Y' \subseteq Z$ such that $(w, Y) \in R_{\pi_1}$ and $(w, Y') \in R_{\pi_2}$. So $(w, Y \cup Y') \in R_{\pi_1 \cap \pi_2}$. Since $Y \cup Y' \subseteq Z$, we are done.

For the sequential composition operator, suppose $(w, Z) \in S_{\pi_1 \circ \pi_2}$. Then there exists a set $Y$ such that $(w, Y) \in S_{\pi_1}$ and $(v, Z) \in S_{\pi_2}$ for each $v \in Y$. By the induction hypothesis, there are sets $\{Z'_v\}_{v \in Y}$ with $(v, Z'_v) \in R_{\pi_2}$ and $Z'_v \subseteq Z$, and there is $Y' \subseteq Y$ with $(w, Y') \in R_{\pi_1}$. We get $(w, \bigcup\{Z'_v \mid v \in Y'\}) \in R_{\pi_1 \circ \pi_2}$, and since $\bigcup\{Z'_v \mid v \in Y'\} \subseteq \bigcup\{Z'_v \mid v \in Y\} \subseteq Z$, we are done.

Finally, we consider the case of game iteration. First, we recall that $\mathsf{skip}$ denotes the neighborhood relation $\{(w, \{w\}) \mid w \in W\}$, and $\mathsf{skip}^\uparrow$ denotes the relation $\{(w, Z) \mid w \in Z\}$.

Suppose the induction hypothesis holds for $R_\pi$. Let:

$$F := \lambda Z.\mathsf{skip}^\uparrow \| \cup \| (S_\pi \| \circ \| Z)$$

so that $S_{\pi^*}$ is equal to the least fixpoint for $F$. Alternatively, we can describe $S_{\pi^*}$ as the least fixpoint of the map $F$ restricted to the complete sub-lattice of $\mathcal{N}W$ given by $\{R \in \mathcal{N}W \mid \mathsf{skip}^\uparrow \subseteq R\}$. The bottom element of this sub-lattice is $\mathsf{skip}^\uparrow$, so we can write the approximating sequence for the least fixpoint as:

$$\mathsf{skip}^\uparrow \subseteq F\mathsf{skip}^\uparrow \subseteq F^2\mathsf{skip}^\uparrow \subseteq F^3\mathsf{skip}^\uparrow \ldots \quad F^\omega\mathsf{skip}^\uparrow \subseteq F^{\omega+1}\mathsf{skip}^\uparrow \ldots$$

We denote the first two entries in the series as $F^0\mathsf{skip}^\uparrow$ and $F^1\mathsf{skip}^\uparrow$. We show, by transfinite induction, that $(w, Z) \in F^\xi\mathsf{skip}^\uparrow$ iff there is some $Z' \subseteq Z$ such that $(w, Z') \in (R_\pi)^{[\xi]}$. The result then follows by considering $\xi$ such that $R_{\pi^*} = (R_\pi)^{[\xi]}$ and $\gamma$ such that $S_{\pi^*} = F^\gamma\emptyset$. Then, pick some $\rho$ greater than both $\gamma$ and $\xi$. The result then follows since $R_{\pi^*} = R_\pi^{[\rho]}$ and $S_{\pi^*} = F^\rho\emptyset$.

To establish the claim, the case for $\xi = 0$ is trivial since by definition $F^0\mathsf{skip}^\uparrow = \mathsf{skip}^\uparrow$ and $R_\pi^{[0]} = \mathsf{skip}$. Successor ordinals $\xi + 1$ are handled by unfolding and comparing the definitions of $R_\pi^{[\xi+1]}$ and $F^{\xi+1}\mathsf{skip}^\uparrow$, applying the "inner" induction hypothesis to $F^\xi\mathsf{skip}^\uparrow$, applying the "outer" induction hypothesis to $S_\pi$, and then repeating and combining the previous arguments for $\cup$ and $\circ$. Finally, limit ordinals $\kappa$ are handled by simply noting that $R_\pi^{[\kappa]} = \bigcup_{\xi<\kappa}(R_\pi)^{[\xi]}$ and $F^\kappa\mathsf{skip}^\uparrow = \bigcup_{\xi<\kappa} F^\xi\mathsf{skip}^\uparrow$. □

We can now prove Theorem 7 as follows. Suppose $\varphi$ is a formula of $\mathsf{GL}$ and $\Vdash \varphi$. Then since every monotonic power frame is a neighborhood frame, it follows by Proposition 7 that $\vDash \varphi$ as well. Conversely, suppose $\vDash \varphi$, so that $\varphi$ is valid on every monotonic power frame. Then for any neighborhood model $\mathfrak{M}$ and every state $w$ in $W$, we have $\mathfrak{M}^\uparrow, w \vDash \varphi$, so $\mathfrak{M}, w \Vdash \varphi$ by Proposition 3. Hence $\Vdash \varphi$ as required. □

In other words: the formulas of $\mathsf{IPDL}$ that are valid on arbitrary neighborhood frames form a conservative extension of the $\mathsf{GL}$-formulas that are valid over monotonic power frames.

# 6 Concluding remarks

In this paper, we have introduced a new propositional dynamic logic $\mathsf{IPDL}$ defined over instantial neighborhood logic, as a tool for exploring a new open

systems perspective on computation. We found program operations that respect a natural notion of bisimulation in this setting, and we axiomatized the complete logic, which presented some non-trivial and interesting deviations from the usual proof format for PDL. Finally, we positioned our logic with respect to related views of computation by completely clarifying its relation to current game logics.

Our system fits in a broader technical context. Various extensions of our base language would make sense, notably, the addition of least and greatest fixpoint operators. Just as standard PDL can be translated into the modal $\mu$-calculus, our logic IPDL can be translated into the extension of INL with fixpoints, a translation that is implicit in our axiom system for IPDL. The fixpoint extension of INL is very well behaved from a co-algebraic perspective. As shown in [6], INL is a coalgebraic modal logic corresponding to a weak pullback preserving functor - the double covariant powerset functor - that additionally preserves finite sets. This means that the $\mu$-calculus extension of INL inherits a number of properties that hold in much wider generality. In particular, it has the finite model property and it is decidable [21], and a sound and complete system of axioms is available [12]. However, as usual, such general results need not transfer to natural fragments that zoom in more closely on computation. Examples are Reynold's highly non-trivial completeness proof for $\mathsf{CTL}^*$ [20], or Parikh's game logic, which still lacks a complete system of axioms. A closer comparison for our system would be coalgebraic PDL, [15], but there, unlike in INL, the coalgebraic type functor is a monad. Still, there is more work to be done here. For instance, our sequential program composition resembles the standard Kleisli composition for the powerset function – but we leave these issues to future investigation.

These are not the only connections to be clarified. In follow-up work, we intend to show that IPDL can also thow new light on other logical systems for computation, such as concurrent PDL ([19, 7, 16]), and that it can contribute to a more fine-structured analysis of game equivalence and powers of players, linking up with game theory (see [4], for which an extended follow-up manuscript is currently in preparation).

# References


[1] L. Aceto, A. Ingólfsdóttir, K. G. Larsen, and J. Srba. *Reactive systems: modelling, specification and verification*. Cambridge University Press, 2007.

[2] R. Alur, T. A. Henzinger, and O. Kupferman. Alternating-time temporal logic. *Journal of the ACM (JACM)*, 49(5):672–713, 2002.

[3] J. van Benthem. *Logic in games*. MIT Press, Cambridge, MA, 2014.

[4] J. van Benthem, N. Bezhanishvili, and S. Enqvist. A new game equivalence and its modal logic. In *Proceedings Sixteenth Conference on Theoretical Aspects of Rationality and Knowledge, TARK 2017, Liverpool, UK, 24-26 July 2017.*, pages 57–74, 2017.

[5] J. van Benthem, N. Bezhanishvili, and S. Enqvist. A propositional dynamic logic for instantial neighborhood models. In A. Baltag, J. Seligman, and T. Yamada, editors, *Logic, Rationality, and Interaction, LORI 2017, Proceedings*, volume 10455 of *Lecture Notes in Computer Science*, pages 137–150. Springer, 2017.

[6] J. van Benthem, N. Bezhanishvili, S. Enqvist, and J. Yu. Instantial neighborhood logic. *The Review of Symbolic Logic*, 10(1):116–144, 2017.

[7] J. van Benthem, S. Ghosh, and F. Liu. Modelling simultaneous games in dynamic logic. *Synthese*, 165(2):247–268, 2008.

[8] D. Berwanger. Game logic is strong enough for parity games. *Studia Logica*, 75(2):205–219, 2003.

[9] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Number 53 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2001.

[10] F. Carreiro and Y. Venema. PDL inside the $\mu$-calculus: A syntactic and an automata-theoretic characterization. *Advances in Modal Logic*, 10:74–93, 2014.

[11] C. Cîrstea, C. Kupke, and D. Pattinson. EXPTIME tableaux for the coalgebraic $\mu$-calculus. In E. Grädel and R. Kahle, editors, *Computer Science Logic (CSL 2009)*, volume 5771 of *Lecture Notes in Computer Science*, pages 179–193. Springer, 2009.

[12] S. Enqvist, F. Seifan, and Y. Venema. Completeness for coalgebraic fixpoint logic. In *Proceedings of the 25th EACSL Annual Conference on Computer Science Logic (CSL 2016)*, volume 62 of *LIPIcs*, pages 7:1–7:19, 2016.

[13] S. Enqvist, F. Seifan, and Y. Venema. Completeness for $\mu$-calculi: a coalgebraic approach. Technical Report PP-2017-04, Institute for Logic, Language and Computation, Universiteit van Amsterdam, 2017.

[14] G. Fontaine, R. Leal, and Y. Venema. Automata for coalgebras: An approach using predicate liftings. In *Automata, Languages and Programming: 37th International Colloquium ICALP'10*, volume 6199 of *LNCS*, pages 381–392. Springer, 2010.

[15] H. H. Hansen and C. Kupke. Weak completeness of coalgebraic dynamic logics. *arXiv preprint arXiv:1509.03017*, 2015.

[16] D. Harel, D. Kozen, and J. Tiuryn. *Dynamic logic*. MIT press, 2000.

[17] D. Kozen. Results on the propositional $\mu$-calculus. *Theoretical Computer Science*, 27:333–354, 1983.

[18] R. Parikh. The logic of games and its applications. *Annals of Discrete Mathematics*, 24:111–139, 1985.

[19] D. Peleg. Concurrent dynamic logic. *Journal of the ACM (JACM)*, 34(2):450–479, 1987.

[20] M. Reynolds. An axiomatization of full computation tree logic. *Journal of Symbolic Logic*, pages 1011–1057, 2001.

[21] Y. Venema. Automata and fixed point logic: a coalgebraic perspective. *Information and Computation*, 204:637–678, 2006.