Corrections to some publications

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Preface

This report contains corrections to some of my books and one longer paper, and combines earlier lists posted on my profile page at the University of Amsterdam, with some more recent additions.

The most important correction concerns the recurion theory for continuous function application. Such a theory was outlined in Troelstra-van Dalen, Constructivism in Mathematics in the subsections 3.7.9-15, but the sketch was inadequate, since the strictness conditions for theories based on logic with partially defined terms had not been takern into account. An improved version was given in section 2.6 of my Handbook of Proof Theory article, 'Realizability', but as Joan Moschovakis found out, this version was still not general enough; jointly we worked out an improved version.

The report contains:

1. Corrections to volume 1 of A.S. Troelstra and D. van Dalen, *Constructivism in Mathematics*, Amsterdam: Elsevier 1988.

2. Corrections to volume 2 of A.S. Troelstra and D. van Dalen, *Constructivism in Mathematics*, Amsterdam: Elsevier 1988. The last part of these corrections is a renewed discussion of the theory of the 'creative subject'. I am not putting forward a new theory, but only correct some deficinecies in my earlier expositions.

3. Corrections to A.S. Troelstra (editor), *Metamathematical investigation of intuitionistic arithmetic and analysis*, Springer Lecture Notes 344, Heidelberg: Springer Verlag 1973.

4. Corrections to the second edition of A.S. Troelstra and H. Schwichtenberg, *Basic Proof Theory*, Cambridge, U.K.: Cambridge University Press 2000.

5. A few important corrections (list not complete) to the Chapter VI, 'Realizability', of S. Buss (editor), *Handbook of Proof Theory*, Amsterdam: Elsevier 1998.

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December 2018

A.S. Troelstra, D. van Dalen, Constructivism in Mathematics – Corrections

Corrections to Volume 1

compiled by Anne S. Troelstra and Joan Rand Moschovakis
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32 Add to E1.3.1: "*Hint.* A good notation helps in bookkeeping. For example, use (q, r) for the pair of proofs q, r; π_1, π_2 for the projections (unpairing functions); for a proof p of $A \vee B$ from a proof of A or a proof q of B we write $\langle 0, p\rangle$ and $\langle 1, q\rangle$ respectively. The lambda notation can be used to describe functions: $\lambda x.t$ is the function described by the term t as function of x."

32, line 1 of E1.3.3 read "... König's lemma in 2.4 for decidable trees"

32, E1.3.3 Change the hint to: "*Hint.* Use a decidable predicate A such that $\forall n(A(n) \vee \neg A(n)$, the n such that $A(n)$, if existing, is unique, but ?$(\exists n A(n) \to \exists k(A(2k) \vee \exists k A(2k+1))$."

38 in the first and second group of displayed prooftrees, interchange $\wedge\mathrm{E_r}$ and $\wedge\mathrm{E_l}$.

41^2 change "y not free in A." into "y not free in A; y must be free for x in A."

41^4 change "y not free in A." into "y not free in A; y must be free for x in A."

53_{12} change "(by (1))" to "(by (2))".

58^7 read "left to right" for "right to left".

60_{15} for "context" read "contexts".

64_1 read "I-isolating (I-spreading)".

65^2 read "**IQC**" for "**MQC**".

68_5 read "$\forall x(A \to B)$" in place of "$\exists x(A \to B)$".

68_1 "$x \notin \mathrm{FV}(B)$" in place of "$x \in \mathrm{FV}(B)$".

69_7 "$\forall$I" in place of "$\exists$I".

72_8 replace "or $y \notin \mathrm{FV}(B)$" by "(or $y \notin \mathrm{FV}(B)$ and y free for x in B)".

87_3 read "C-saturated" for "$\mathcal{L}$(C)-saturated".

88^{16} "$\Gamma^{k+1} = \Gamma^k$" in place of "$\Gamma_{k+1} = \Gamma_k$".

89_9 "C^*_{n+2}-saturated" instead of "C^*_{n+1}-saturated".

90 in the diagram on the right, replace "$\langle 0, 1, 2\rangle$" by "$\langle 0, 1, 3\rangle$".

91^2 read "of [finite] tree models".

91_2 read "$S(k) = S(k') \vee S(k') = S(t_{k,i})$".

91, line 3 of 6.11, insert "$k_0 \in K^*$", after "$K^* \subset K$".

$92^{8,9}$ read "$k \Vdash B_1 \to B_2$ iff $\forall k' \geq k(k' \Vdash B_1 \Rightarrow k' \Vdash B_2)$, which implies $\forall k' \geq^* k(k' \Vdash^* B_1 \Rightarrow k' \Vdash^* B_2)$, hence $k \Vdash$".

103_6 delete the first (.

108^{11} "$x \notin \mathrm{FV}(P)$" instead of "$x \notin \mathrm{FV}(A)$".

109_8 "$k_0 \nVdash' P$ for P prime" should be "for P prime, $k_0 \Vdash' P$ iff for all i $\mathcal{K}_i \Vdash P$".

$109_{6,5}$ delete "or an existential formula".

113^9 read "Π^0_2".

122_{14} "$y \dot{-} 0 = y$" instead of "$y \dot{-} 0 = 0$".

130 just above (4) read "so assume".

133^9 read "upper bound".

133_4 insert space between "$(\vec{z})$" and "(the".

138_8 drop "D" before "$\vdash$".

$141^{2,4}$ change "$\vdash$" to "$|$".

142_{11} read "**5.12.**" for "**5.1.2.**".

143 Correct the proof of 5.16, by replacing the lines 2–6 of the proof by:

PROOF. The proof of (i) follows, under the assumption $C \in \mathcal{RH}$, by showing that, if D is a s.p.p of C, then $C \vdash D \Rightarrow C|D$, using induction on D. It suffices to show:

(a) if $C \vdash A \wedge B$ and $C \vdash A \Rightarrow C|A$, $C \vdash B \Rightarrow C|B$, then $C|(A \wedge B)$;

(b) if $C \vdash A \to B$ and $C \vdash B \Rightarrow C|B$, then $C|(A \to B)$;

(c) if $C \vdash \forall x A$ and for all n, $C \vdash A[x/\bar{n}] \Rightarrow C|A[x/\bar{n}]$, then $C|\forall x A$;

(d) if P is prime and $C \vdash P$ then $C|P$.

157-158 Subsections 3.7.9 – 3.7.13 have to be replaced; see the corrected version at the end of the list of corrections for volume 1.

160^7 read $\Lambda^1 x.\varphi$ for $\Lambda^1 x.t$ and read $\Lambda^0 \alpha.t$ for $\Lambda^0 \alpha.\varphi$.

179^2 read "Let $\psi, \chi, \theta, \ldots$".

179_7 read "5.13" for "5.12".

179 replace in E3.5.3 "$\vdash$" everywhere by "$\Gamma \vdash$".

202^{10} for "$(x \simeq_1 x$" read "$\forall y(x \simeq_1 x'$", and for "$(\exists y\, A(x,y)$" read "$\exists\, y(A(x,y)$".

202^{17} for "$\approx_1$" read "$\simeq_1$".

210_{15} for "$\neg\exists y(\delta' y = 0)$" read "$\neg\exists x(\delta' x = 0)$" and for "$\neg\exists y(\delta'' y = 0)$" read "$\neg\exists x(\delta'' x = 0)$".

$202_{11,12,14}$ "$\mathrm{MP}_{\mathrm{PR}}$" instead of "$\mathrm{MR}_{\mathrm{PR}}$" (four times).

242, in second line of exercise **4.2.2** , replace "DC-D" by "DC-$\mathbb{N}\times$D".

245_3 read "$\beta \in \bar{\alpha}x$" for "$\beta \in \bar{\alpha}y$".

247^1 for "$\forall m \preceq n$" read "$\forall m \prec n$".

262^9 for "$\alpha k + 1$" read "αk" and read "$\leq$" for "$<$".

264_{11} read "$x \# 0$".

274, line 4 in 5.10 read "order-isomorphic".

274_{14} read "and ϕ leaves all rationals".

276^8 for "$\mathbb{R}$" read "$\mathbb{Q}$".

287_3 "$\exists$-PEM" should be "$\forall$-PEM".

293 Remark concerning the proof of 6.1.3. The property of covering is interpreted as: there is an operation which from a sufiiciently good approximation of a point d in the interval tells us either $d \in A$ or $d \in B$. (Both may be true, but the operation makes a choice.)

297^8 read "$m|x - y|$" for "m".

300^{11} read "$a_{n-1}x^{n-i-1}$" for "$a_{n-i}x^{n-i-1}$".

300^{13} read "2^{-1}" for "2^{-i}".

303, line 4 of 3.2, for "$U(2^{-n-1}$" read "$U(2^{-n}$".

306^1 replace "2^{-m}" by "2^{-m+1}" (twice).

306^{10} replace "2^{-m}" by "2^{-m+1}".

306^{12} replace "2^{-m}" by "2^{-m+1}".

308_5 read "$k + m + 1$ points".

309_{10} for "$\frac{1}{2}(r_n - s_n)$" read "$\frac{1}{2}(s_n - r_n)$".

310 in (i) of 4.5 read "$J_m \subset I_n^-$" for "$J_n \subset I_n^-$".

310_{13} delete "from $\langle I_n \rangle_n$".

311 in the picture the left end of I_n^- should coincide with the left end of J_{p+3}.

312 in the first proof of 6.4.8, the appeal to 6.4.5 is not needed; a singular cover by means of intervals with rational endpoints suffices.

312^1 for “quasi-order” read “quasi-cover”.

312^2 “I_2” should be “I_1”.

313_1 for “$\geq |I_n|$” read “$\geq |J_m|$”.

313_2 read “$J_m \subset I_n$ for some m”.

317^{13} read “$\neg\exists n'\forall mn$” for “$\neg\exists n'mn$”.

317_3 displayed formula (1) should end with “$< 2^{-n-1})$”.

319, 3 lines below (6), for “$-q_y|$” read “$-q_{\{z\}}(y)|$”, and “$q_{\{z\}\phi(y,\psi y)}|$” for “$q_{\phi(y,\psi y)}|$”.

320^1 read “From (9)” for “From (7)”.

338 correct Rasiowa (1954) , replacing “1, 229–231” by “2, 121–124”

pages I-IX from all page numbers of the preliminaries, in lower case roman numerals, one has to subtract 4 (“xv” becomes “xi” etc.) (the preliminaries ought to have been inserted after the table of contents and numbered accordingly).

Corrected version of subsections 3.7.9–15

3.7.9. Definition. In **EL** we introduce abbreviations

$$\begin{array}{rcl} \alpha(\beta) = x & := & \exists y(\alpha(\bar{\beta}y) = x + 1 \wedge \forall y'{<}y(\alpha(\bar{\beta}y') = 0) \\ \alpha|\beta = \gamma & := & \forall x(\lambda n.\alpha(\hat{x} * n)(\beta) = \gamma x) \wedge \alpha 0 = 0, \text{ or equivalently} \\ & & \forall x \exists y(\alpha(\hat{x} * \bar{\beta}y) = \gamma x + 1 \wedge \forall y'{<}y(\alpha(\hat{x} * \bar{\beta}y') = 0)) \wedge \alpha 0 = 0. \end{array}$$

We may introduce $|$, $\cdot(\cdot)$ as primitive operators in a conservative extension $\mathbf{EL}^*$ based on E^+-logic, also called LPT, the logic of partial terms. □

Definition. $\mathbf{EL}^*$ is a conservative extension of **EL** based on the logic of partial terms, to which $\lambda\alpha\beta.\alpha|\beta$ and $\lambda\alpha\beta.\alpha(\beta)$ have been added as primitive operations. Numerical lambda-abstraction satisfies:

$$s\downarrow \wedge (\lambda x.t)\downarrow \to (\lambda x.t)s = t[x/s], \quad (\lambda x.t)\downarrow \leftrightarrow \forall x(t\downarrow).$$

For function application we require strictness:

$$\phi t\downarrow \leftrightarrow \phi\downarrow \wedge\, t\downarrow .$$

(The implication from right to left must hold since $\phi\downarrow$ is supposed to imply totality of the function denoted by ϕ.) For Rec we have

$$\mathrm{Rec}(t, \phi)\downarrow \leftrightarrow t\downarrow \wedge\, \phi\downarrow .$$

We also require strictness for the operations $\cdot|\cdot$ and $\cdot(\cdot)$, that is to say

$$\phi|\psi\downarrow \to \psi\downarrow \wedge\, \phi\downarrow, \quad \phi(\psi)\downarrow \to \psi\downarrow \wedge\, \phi\downarrow$$

□

3.7.10. Definition. (*The class of neighbourhood functions*)

$$\alpha \in K^* \;:=\; \alpha 0 = 0 \wedge \forall nm(\alpha n > 0 \to \alpha n = \alpha(n * m)) \wedge \forall \beta \exists x(\alpha(\bar{\beta}x) > 0).$$

□

Crucial is the following

3.7.11. Proposition. To each function term ϕ of $\mathbf{EL}^*$, and each numerical term t of $\mathbf{EL}^*$ and free function variable α, we can construct function terms $\Phi^\alpha_\phi \in K^*$, $\Phi^\alpha_t \in K^*$ of **EL** such that if γ is free for α in ϕ or t respectively (and does not occur in ϕ or t unless γ is α):

(i) $\Phi^\alpha_\phi|\gamma \simeq \phi[\alpha/\gamma]$ and in particular $\Phi^\alpha_\phi|\alpha \simeq \phi$;

(ii) $(\Phi^\alpha_t|\gamma)\downarrow$ iff $t[\alpha/\gamma]\downarrow$;

(iii) $t[\alpha/\gamma]\downarrow \to (\Phi^\alpha_t|\gamma)0 = t[\alpha/\gamma]$ and in particular $t\downarrow \to (\Phi^\alpha_t|\alpha)0 = t$;

(iv) $\mathrm{FV}(\Phi^\alpha_t) \subset \mathrm{FV}(t) \setminus \{\alpha\}$, $\mathrm{FV}(\Phi^\alpha_\phi) \subset \mathrm{FV}(\phi) \setminus \{\alpha\}$, $\Phi^\alpha_t, \Phi^\alpha_\phi$ primitive recursive in their free variables.

PROOF. (i)–(iv) are proved by simultaneous induction on the construction of numerical and function terms. Instead of taking exactly the numerical and function terms as specified for $\mathbf{EL}^*$, we give the proof, for reasons of convenience, for a slightly different set interdefinable with the set generated by the primitives of $\mathbf{EL}^*$. In particular, instead of taking $\boldsymbol{r}$ as a primitive, we take its special case It ("iterator") satisfying

$$\mathrm{It}(t,\psi)0 = t, \quad \mathrm{It}(t,\psi)(Sz) = \psi(\mathrm{It}(t,\psi)z).$$

From It we can easily define Rec satisfying

$$\mathrm{Rec}(t,\phi)0 = t,\ \mathrm{Rec}(t,\phi)(Sz) = \phi(\mathrm{Rec}(t,\phi)z, z),$$

by taking

$$\mathrm{Rec}(t,\phi) := \lambda z.j_0(\mathrm{It}(j(t,0), \lambda u.j(\phi u, S(j_1 u)))z),$$

and from Rec we can readily define $\boldsymbol{r}$.

The reason that we need a function term with partial continuous application $|$ to represent a numerical term (instead of application $\cdot(\cdot)$) is that a numerical term t may contain function-terms as subterms, which all have to be defined by the strictness condition of the logic of partial terms; this is a Π^0_2-condition and cannot be expressed by definedness of a numerical term.

We consider a few typical cases. In all cases we put $\Phi^\alpha_\phi 0 = 0$, $\Phi^\alpha_t 0 = 0$. If $u \neq 0$ then $u = \hat{z} * \bar{\gamma}n$ for $z = (u)_0$ and every γ such that $(u)_{i+1} = \gamma(i)$ for all $i < n$, where $|u| = n+1$. For easier comprehension, for $u \neq 0$ we state the definitions of $\Phi^\alpha_\phi u$, $\Phi^\alpha_t u$ in terms of $\hat{z} * \bar{\gamma}n$ (for fresh variables z, n and γ) instead of u.

Case 1. $t \equiv x$. Take $\Phi^\alpha_t(\hat{z} * \bar{\gamma}n) = x + 1$. Similarly for $t \equiv 0$.

Case 2. $\phi \equiv \alpha$. Take

$$\Phi^\alpha_\alpha(\hat{z} * \bar{\gamma}n) = \begin{cases} \gamma z + 1 \text{ if } z < n, \\ 0 \text{ otherwise.} \end{cases}$$

Case 3. $\phi \equiv \beta$, $\beta \not\equiv \alpha$. Put

$$\Phi^\alpha_\beta(\hat{z} * \bar{\gamma}n) = \beta z + 1 \text{ for all } n.$$

Case 4. $\phi \equiv S$. Put

$$\Phi^\alpha_S(\hat{z} * \bar{\gamma}n) = Sz + 1 \text{ for all } n.$$

Case 5. $t \equiv \phi t'$. Put

$$\Phi^\alpha_t(\hat{z} * \bar{\gamma}n) = \begin{cases} \Phi^\alpha_\phi(\langle \Phi^\alpha_{t'}(\hat{0} * \bar{\gamma}n) \dot{-} 1\rangle * \bar{\gamma}n) \text{ if } \Phi^\alpha_{t'}(\hat{0} * \bar{\gamma}n) > 0 \\ \qquad \wedge\, \Phi^\alpha_\phi(\langle \Phi^\alpha_{t'}(\hat{0} * \bar{\gamma}n) \dot{-} 1\rangle * \bar{\gamma}n) > 0 \\ \qquad \wedge\, \Phi^\alpha_{t'}(\hat{z} * \bar{\gamma}n) > 0 \wedge \Phi^\alpha_\phi(\hat{z} * \bar{\gamma}n) > 0, \\ 0 \text{ otherwise.} \end{cases}$$

Case 6. $\phi \equiv \lambda x.t$. Put

$$\Phi^\alpha_\phi(\hat{z} * \bar{\gamma}n) = \begin{cases} \Phi^\alpha_{t[x/z]}(\hat{0} * \bar{\gamma}n) \text{ if } \Phi^\alpha_{t[x/z]}(\hat{0} * \bar{\gamma}n) > 0 \wedge \Phi^\alpha_{t[x/j_0 z]}(\langle j_1 z\rangle * \bar{\gamma}n) > 0, \\ 0 \text{ otherwise.} \end{cases}$$

Case 7. $\phi \equiv It(t, \psi)$. We put

$$\Phi^\alpha_\phi(\hat{0} * \bar{\gamma}n) = \begin{cases} \Phi^\alpha_t(\hat{0} * \bar{\gamma}n) \text{ if } \Phi^\alpha_t(\hat{0} * \bar{\gamma}n) > 0 \wedge \Phi^\alpha_\psi(\hat{0} * \bar{\gamma}n) > 0, \\ 0 \text{ otherwise.} \end{cases}$$

$$\Phi^\alpha_\phi(\langle Sz\rangle * \bar{\gamma}n) = \begin{cases} \phi^\alpha_\psi(\langle \Phi^\alpha_\phi(\hat{z} * \bar{\gamma}n) \dot{-} 1\rangle * \bar{\gamma}n) \text{ if} \\ \qquad \phi^\alpha_\psi(\langle \Phi^\alpha_\phi(\hat{z} * \bar{\gamma}n) \dot{-} 1\rangle * \bar{\gamma}n) > 0 \wedge \\ \qquad \Phi^\alpha_\phi(\hat{z} * \bar{\gamma}n) > 0 \wedge \Phi^\alpha_\psi(\langle Sz\rangle * \bar{\gamma}n) > 0 \wedge \\ \qquad \Phi^\alpha_t(\langle Sz\rangle * \bar{\gamma}n) > 0, \\ 0 \text{ otherwise.} \end{cases}$$

Case 8. $\phi \equiv j, j_1, j_2$. Easy and left to the reader.
Case 9. $t \equiv \phi(\psi)$. Take

$$\Phi^\alpha_{\phi(\psi)}(\hat{x} * \bar{\gamma}n) = \begin{cases} z + 1 \text{ if } \exists u{<}n \forall y{<}\mathrm{lth}(u)(\Phi^\alpha_\psi(\hat{y} * \bar{\gamma}n) = (u)_y + 1 \wedge \\ \qquad \Phi^\alpha_\phi(u) = z + 1) \wedge \Phi^\alpha_\phi(\hat{x} * \bar{\gamma}n) > 0 \wedge \Phi^\alpha_\psi(\hat{x} * \bar{\gamma}n) > 0, \\ 0 \text{ otherwise.} \end{cases}$$

Case 10. $\phi \equiv \psi|\xi$. Take

$$\Phi^\alpha_{\psi|\xi}(\hat{x} * \bar{\gamma}n) = \begin{cases} z + 1 \text{ if } \exists u{<}n \forall y{<}\mathrm{lth}(u)(\Phi^\alpha_\xi(\hat{y} * \bar{\gamma}n) = (u)_y + 1) \wedge \\ \qquad \Phi^\alpha_\psi(\hat{x} * u) = z + 1) \wedge \Phi^\alpha_\psi(\hat{x} * \bar{\gamma}n) > 0 \wedge \Phi^\alpha_\xi(\hat{x} * \bar{\gamma}n) > 0, \\ 0 \text{ otherwise .} \end{cases}$$

Note that in Cases 5, 7 (for Sz), 9, 10 of this definition the last two conjuncts, and in Case 6 the last conjunct, are needed to guarantee strictness. □

3.7.12. Definition. Let

$$\alpha|(\beta_0, \dots, \beta_{n-1}) := \alpha|\nu_n(\beta_0, \dots, \beta_{n-1}),$$

$$\alpha(\beta_0, \dots, \beta_{n-1}) := \alpha(\nu_n(\beta_0, \dots, \beta_{n-1})).$$

□

We are now ready to build an analogue of ordinary recursion theory with partial continuous function application instead of partial recursive application. The preceding lemma has as consequence a version of the smn-theorem:

3.7.13. Theorem. (smn-theorem)

(i) There is a primitive recursive binary functional $\wedge_n$ such that

$$(\alpha \wedge_n \beta_0)|(\beta_1, \ldots, \beta_n) \simeq \alpha|(\beta_0, \ldots, \beta_n).$$

(ii) There is a primitive recursive binary functional $\wedge'_n$ such that

$$(\alpha \wedge'_n \beta_0)(\beta_1, \ldots, \beta_n) \simeq \alpha(\beta_0, \ldots, \beta_n).$$

Proof. Straightforward by the preceding lemma. □

3.7.14. (No change from **3.7.14.**)

3.7.15. Notation ($\Lambda^0 x$, $\Lambda^1 x$, $\Lambda^0 \alpha$, $\Lambda^1 \alpha$). On the basis of Proposition 3.7.11, for each numerical term t and each function term ϕ of **EL*** we can now define function terms of **EL**:

$$\Lambda^0 \alpha.t = \Phi^\alpha_t,$$

$$\Lambda^1 \alpha.\phi = \Phi^\alpha_\phi,$$

$$\Lambda^1 x.\phi = \Phi^\alpha_{\phi'},$$

where $\phi'[\alpha] := \phi[x/\alpha 0]$, and with the properties $(\Lambda^0 \alpha.t)|\alpha)(0) \simeq t$, $(\Lambda^1 \alpha.\phi)|\alpha \simeq \phi$ and $(\Lambda^1 x.\phi)|\lambda y.x \simeq \phi$.

Using Theorem 3.7.14, for each numerical term t of **EL*** there is a term t' (which we will denote by $\Lambda^0 x.t$) of **EL**, primitive recursive in the parameters of t minus x, such that $\{t'\}(x) \simeq t$ for all x.

Corrections to Volume 2

compiled by Anne S. Troelstra and Joan Rand Moschovakis
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357^{10} read "$\forall y \neg\neg A$" for "$\forall y\, A$".

357^{11} read "and so for $[x]_M$ a point of M, using Markov's principle" for "and so".

360 Delete formula (5) and replace the next line (line 11) by "(exercise).".

360^{16} read "(4)" for "(5)".

360_9 delete "For a ... see E7.3.2".

381 Delete exercise 7.3.2.

452 Due to an oversight, the axioms for $\mathbf{HA}_0^\omega$ as stated are too weak. Instead of the equality axioms as stated, one should use the formulation of in section 1.6.15 of A.S. Troelstra (editor), *Metamathematical Investigation of intuitionistic Arithmetic and Analysis*, Berlin 1973. The system $\mathbf{HA}_0^\omega$ is there called simply $\mathbf{HA}^\omega$. The equality axioms there required replacement in an arbitary context, for example $t[\mathbf{k}t_1t_2] = t[t_1]$ and $t[\mathbf{s}t_1t_2t_3] = t[t_1t_3(t_2t_3)]$. Another solution was proposed by Benno van den Berg in (*A note on arithmetic in finite types*, arXiv 1408.3557v2 [math.LO] 20 Sep 2016) namely adding to the axioms of $\mathbf{HA}_0^\omega$ one new congruence law $x =_0 y \to fx =_0 fy$, together with the axioms $\mathbf{b}xyz = x(yz)$ and $\mathbf{q}xyz = x(zy)$ for two new combinators $\mathbf{b}$ and $\mathbf{q}$, and defining equality at higher types in terms of equality at type 0 according to the principle of observational equivalence: $x =_\sigma y := \forall f^{\sigma\to 0}(fx =_0 fy)$. The congruence laws for equality at all finite types are provable in this version of $\mathbf{HA}_0^\omega$, correcting a circularity in the proof on pages 452-453. Then $\mathbf{HA}^\omega$ proves $\mathbf{b} = \mathbf{s}(\mathbf{ks})\mathbf{k}$ and $\mathbf{q} = \mathbf{b}(\mathbf{s}(\mathbf{bbs})(\mathbf{kk}))\mathbf{b}$.

476, 477 See at the end of this list.

478_{12} read "$h := \lambda\bar{x}.\mathbf{r}(f\bar{x})(\lambda uv.g\bar{x}(Pv)u)$".

542^7 read "$\Gamma \setminus \{A\} \vdash'$" for "$\Gamma \setminus \{A\} \vdash$".

543^{14} Read "S-successor" for "S-successor set".

543^{15} replace "$\theta \equiv$" by "The set $\theta \equiv$".

543_{17} delete "set".

543_1 read "$(P \to Q) \to P$" for "$P \to (Q \to P)$".

544 The first line of formulas should end with "$\ldots P \vdash P\}$".

547 line 7 of 5.6, read "$Et_0 \vee Et_1$" for "$Et_0 \wedge Et_1$".

565 last line of 9.4 read "maps" for "mappings".

566_1 Interchange "$\cap$" and "$\cup$".

583 line 5 of 2.11, read "FV(A_{i+j})" for "FV(A_n)".

656^{13} read "$\bar{\alpha}$" for "$\vec{\alpha}$".

660_{10} read "$(\#(\alpha_1, \dots, \alpha_p) \rightarrow$" for "$((\alpha_1, \dots, \alpha_p) \rightarrow$".

660_6 read "$A(\vec{\alpha}, x))$" for "$A(\vec{\alpha}, x)$".

660_3 read "$A(\vec{\alpha}, \vec{\beta})$" for "$A(\alpha, \beta)$".

661_6 read "$\dot{\forall}$" for "$\dot{\exists}$".

663^{10} read "$\vee$" for "$\wedge$".

663_3 read "$\vec{n} \circledast \vec{m}$" for "$\vec{n} * \vec{m}$".

680, line 1 of 1.5, insert after "*Beth model*": "The definition of Beth model obviously extends to the case where $(K, \leq_K)$ is a collection of spreads".

680, line 3 of 1.5, insert before "as follows": " ,where $(K', \preceq')$ is a set of spreads,".

680, line 4 of 1.5, read "finite, inhabited, nondecreasing".

680_2 read "$(\bar{\alpha}x \Vdash A$ or" for "$(\bar{\alpha}x \Vdash$ or".

681 remark concerning the proof of the theorem in 1.5. If the Kripke model has no root, the corresponding Beth model becomes rather a collection of spreads instead of a single spread, which is more general than permitted by our definition of Beth model. But this does not otherwise affect the proof.

681 add at the end of 1.5:
REMARK. The construction permits many slight variations. If we restrict attention to Kripke models with a root, and restrict K' to finite inhabited nondecreasing sequences starting with a root, the construction works equally well; this variant has been illustrated in fig. 13.1".

681^{17} read "k_n" for "k".

683_4 $\mathrm{LS_K}$, in a degenerate case, may consist of a single sequence, i.e., lawlike and lawless coincide. But this does not affect the argument.

687 replace line 4 of 2.4. by "$k \Vdash P := \exists z \forall k' \succeq_z k(\vdash \Gamma_{k'} \rightarrow A)$.".

687 replace line 4 of 2.5. by "**Case 1.** For A prime apply lemma 2.3."

689^6 $\vdash_{x+lth(k)}$ instead of $\vdash_x$ (twice).

689_9 for "lemma 2.3" read "the covering property (lemma 1.2(i))".

853^4 read "recovered" for "removed".

872 correct Rasiowa (1954) , replacing "1, 229–231" by "2, 121–124"

XXX under "Howard, W.A. 1980" read "565" for "564".

Corrections to pages 476, 477.

It is not generally true that if $x \notin \mathrm{FV}(t')$, $y \not\equiv x$, then

$$\lambda x.(t[y/t']) \simeq (\lambda x.t)[y/t'],$$

(consider e.g. $t \equiv y$, $t' \equiv \boldsymbol{kk}$), but if $x \notin \mathrm{FV}(t')$, $y \notin \mathrm{FV}(t'')$, $y \not\equiv x$, then

$$\boldsymbol{E}t'' \to ((\lambda x.t)[y/t'])t'' \simeq t[x/t''][y/t'] \simeq t[y/t'][x/t''].$$

The failure of the first equation is due to the fact that $\lambda x.t$ has been defined by induction on the complexity of t. This necessitates some repairs. For example, the argument in $476^{1,2}$ should read:

"$\chi\chi \simeq (\lambda zy.x(zz)y)\chi \simeq (\lambda y.x(zz)y)[z/\chi]$, and since an expression $\lambda x.\ldots$ always exists, uniformly in the parameters, i.e. remains "existing" if we substitute existing objects for the free variables, we see that $\boldsymbol{E}(\mathrm{fix}(x))$; also ...".

Corresponding corrections (i.e. postponement of substitution in a defined lambda-term) has to be made in $476_{6,4}$ and 477_{10}. Lines 476_{6-3} are to be replaced by:

$$\begin{aligned} \boldsymbol{r}tt'0 &\simeq \phi\rho 0 \simeq \rho(\phi\rho)0 \simeq \\ &\simeq \boldsymbol{d}(\boldsymbol{k}t)((\lambda z.y(\phi\rho(Pz))z)[y/t'])000 \\ &\simeq \boldsymbol{d}(\boldsymbol{k}t)(t'(\phi\rho(P0))00 \\ &\simeq \boldsymbol{k}t0 \simeq t. \end{aligned}$$

If $n \in \mathbb{N}$, $n \neq 0$, then

$$\begin{aligned} \boldsymbol{r}tt'n &\simeq \phi\rho n \simeq \rho(\phi\rho)n \\ &\simeq \boldsymbol{d}(\boldsymbol{k}t)((\lambda z.y(\phi\rho(Pz))z)[y/t'])n0n \\ &\simeq \boldsymbol{d}(\boldsymbol{k}t)(t'(\phi\rho(Pn))n)0n \\ &\simeq t'(\phi\rho(Pn))n \simeq t'(\boldsymbol{r}tt'(Pn))n \quad \Box \end{aligned}$$

Lines 477_{10-7} are to be replaced by

$$\mu f \simeq \phi M f \simeq M(\phi M)f \simeq M\mu f \simeq \boldsymbol{d}(\boldsymbol{k}0)((\lambda g.S(xg))[x/\mu])(f0)0f^+,$$

and this latter expression is equal to

$$\boldsymbol{k}0f^+ = 0 \text{ if } f0 = 0, \text{ and}$$

$$(\lambda g.S(xg)[x/\mu])f^+ \simeq S(\mu f^+) \text{ if } f0 > 0.$$

Some remarks on the theory of the 'creative subject'

The terms 'creative subject', or 'idealized mathematician', hitherto commonly used are perhaps not very felicitous. Henceforth, in this note, I will use 'creating mathematician', abbreviated as CM. The sole purpose of this note is to emend my earlier presentation of this topic in *Principles of Intuitionism* (1969) and *Constructivism in Mathematics* (1988). Hence I shall not go into the details of recent papers on the topic by J.M. Niekus and M. van Atten. For Niekus approach, see his papers 'Brouwer's incomplete objects', *History and Philosophy of Logic* 31 (2010), p.31–46; 'The method of the creative subject', *Proceedings of the KNAW* A90 (1967), 431-443; 'What is a choice sequence? How a solution of Troelstra's paradox shows the way to an anwer to this question' (*To appear*). It suffices here to say that it was Joop Niekus who convinced me that the theory of the creative mathematician as proposed by G.Kreisel assumed much more than is actually needed for L.E.J. Brouwer's counterexamples (at least the simpler ones), and that it was Mark van Atten who spotted a slight flaw in the original presentation of the so-called 'paradox' presented in Principles of Intuitionism and again in Constructivism in Mathematics. This flaw will be corrected in this note, along with other corrections.

The theory CM0–4. So let me first state (again) the theory of the CM as proposed by Kreisel. The starting point is the assumption CM0:

CM0 All actions of the CM are arranged in an ω-sequence of discrete stages

We write $[n]A$ for: 'at stage n the CM has evidence (proof, constructed as true) for statement A'. Three further principles are assumed:

CM1 $\forall nm([n]A \to [n+m]A)$

That is to say, the evidence is cumulative: what the CM knows at stage n, remains available at all later stages.

CM2 $\forall n([n]A \vee \neg[n]A)$

In words, at any stage the CM knows whether he has already obtained evidence for A or not.

CM3 $A \leftrightarrow \exists n[n]A$

In words, A is true if and only if the CM has evidence for A at some stage n. For this third principle also a weaker form has been proposed:

CM3* $(\exists n([n]A \to A) \wedge (A \to \neg\neg\exists n[n]A)$

Discussion. Principle CM1 seems unproblematic, once one has accepted CM0. Principle CM2 raises questions. At any given stage, which, and how many, statements have become evident? It is tempting, for example, to assume that whenever $[n]A$ and $[n]B$ one also has $[n](A \wedge B)$. But once on this track, can we assume that class of statements which is evident at stage n is closed under the deduction rules of intuitionistic first-order logic? That seems an unwarranted assumption, since we have no decision procedure for derivability in first-order logic. The only available way out, as far as I can see, is to assume the 'one-conclusion at a time' principle:

CM4 At any stage at most one new conclusion is drawn

Though weakening CM3 to CM3* does not block the paradox discussed below, there are other reasons to believe CM3* to be a more plausible principle than CM3. For the obvious justification of CM3 in the direction from left to right runs as follows. A proof of the impication $A \to \exists n([n]A)$ requires a construction that transforms any proof p of A into a proof $f(p)$ of $\exists n([n]A)$. But p occurs at some stage n, now take for $f(p)$ the pair(n, p). However,is it natural to consider the number of the stage at which proof p is found to be part of p? I feel the proofs and constructions in the well-known 'BHK-interpretation' should not be time-dependent in this way.

On the other hand, suppose that we have a proof of A, and that we know that $\neg\exists n([n]A)$, then the second assumption says in fact that the CM cannot find a proof of A. But then A must be false; contradiction.

The paradox. Let us now use α, β to denote arbitrary, not necessarily predeterminate, and not necessarily infinite sequences of natural numbers, and let us consider statements of the form 'α is a total sequence'. For example, if α is defined as a primitive recursive sequence, this conclusion is immediate as soon as α is defined. If α is initially given to us as a partial recursive function, we may at a later stage conclude that α is a total sequence, nameley if we have found a proof of this fact. A lawless sequence is from the moment it is initiated a total sequence.

The original idea for the paradox was as follows. Let α^n be the n-th total sequence the CM encounters when running through the stages of activity; then consider a sequence β defined by

$$\beta(n) = \alpha^n(n) + 1$$

β is total, and at some stage m β should appear as an α^n. But then $\beta(n) = \alpha^n(n) = \alpha^n(n) + 1$, a contradiction. This is just a classical diagonalization argument. Mark van Atten observed that perhaps β is not well-defined, because, having encountered α^n, we are not certain how long we have to wait before the next total sequence appears. This can be remedied as follows. At stage 0 we take α^0 to be the constant zero function. As long as no new total sequence is declared at stage $n+1$, we take α^{n+1} to be equal to α^n; and if at stage $n+1$ a new total sequence γ is found, we take α^{n+1} to be equal to γ. Then we can diagonalize as before.

Discussion. Originally, I used, instead of 'total sequence' the notion 'a total sequence determined by a recipe'. I used the word 'recipe' instead of 'lawlike', because I did not want to suggest that the sequence was recursive, only that it was fixed by a recipe relative to the activity of the CM in general. But in view of the fact that the CM is completely free in his actions, a 'sequence determined by a recipe' can be as un-predetermined as an arbitrary choice sequence.

There are several ways in which we may react to this paradox. One possibility is to note that L.E.J. Brouwer in his famous counterexamples actually assumed far less than the principles CM0–4. Another way out is to note that the notion of a total sequence contains an impredicativity, that is to say if we allow the construction of β in the paradox above, then we cannot make the universe of all total sequences into a species in the sense of Brouwer, because Brouwer requires that the individual elements of a species can be defined before the species itself; this is a kind of predicativity requirement. To me it is not clear whether and to what extent Brouwer tolerated impredicativity: does it make sense to speak of the power-species consisting of *all* subspecies of a given species? Anyway, this is the direction in which Van Atten thinks the solution is to be found: see his preprint *Predicativity and parametric polymorphism of Brouwerion mplication* (July 2017).

This would mean that in the argument for the paradox either the β cannot be an element of the species of total sequences, or that the notion of a total sequence does not describe a species. In *Principles of Intuitionism* I enforced predicativity in another way: I considered a stratified universe of constructions. Constructions of level 0 were defined without explicit reference to the activity of the CM in time; constructions involving such a reference belonged to level 1, and constructions defined relative to constructions of level 1 were to have level 2, and so on. Obviously, the construction of the paradox is blocked, because the diagonalizing β always is of a level one higher than the level of the species of total functions used in its construction.

Brouwer's argument. We now present one of Brouwer's counterexamples explicitly involving the actions of the CM, and argue that this example uses less than is assumed in the theory CM0-4. Let A be a statement which has not yet been tested, that is to say the CM does not have of a proof of either $\neg A$ or of $\neg\neg A$.

The CM now constructs a sequence α_A as follows. To determine the value of α_A at n, the CM asks whether he has already evidence for either $\neg A$ of for $\neg\neg A$. If not, the value of $\alpha_A(n)$ is taken to be zero. If the value of $\alpha_A(n-1)$ is zero, and when the CM decides to produce the value of $\alpha_A(n)$ and by then has proved either $\neg A$ or $\neg\neg A$, the value of $\alpha_A(n)$ is taken to be 1. If the value of $\alpha_A(n-1)$ is 1, $\alpha_A(n)$ =1.

This argument may easily varied in several ways. Brouwer's first examples did not construct a sequence of natural numbers, but a sequence of rationals instead, which he used to show that, intuitionistically, for reals x, y, $x \neq y$ does not imply $x \# y$.

In the example as presented here, $\neg(\alpha_A \neq \lambda x.0)$, that is to say α_A cannot be the constant zero function, for that would mean that the CM will never be able to find a proof of either $\neg A$ or $\neg\neg A$, but this can only mean that $\neg(\neg A \vee \neg\neg A)$ which is false. On the other hand, the CM cannot a priori predict when a proof of $\neg A$ or $\neg\neg A$ will be found, so the CM cannot indicate in advance a number n for which $\alpha_A(n) = 1$.

The argument does not rely on the principle CM0, which seems to say that the whole activity of the CM takes place along a single timeline; there is only a timeline associated with the generation of the values of α_A. On the other hand, statements CM1 and CM2 seem to be assumed here too as far as they relate to the timeline of α_A. Instead of CM3, only CM3* has been used.

A remark on proto-lawless sequences. The theory **LS** of lawless sequences has only a single existential axiom, stating that to every initial segment there exist some lawless sequence α with that initial segment. A lawless sequence is often compared with the sequence of the casts of a die, but in that case we cannot guarantee that there is a die which produces a particular finite initial sequence of casts, specified in advance. This leads to a slight modification, proposed by G. Kreisel, of the comparison: a lawless sequence behaves like the casts of a die, where a finite initial sequence of results may be specified in advance (by deliberate placings of the die so to speak). Then it seems natural to consider of lawless sequences without any specification of initial segment; these have been called 'proto-lawless' in the literature.

That this picture of lawless sequences suggested by comparison with the cast of a die is sound, seems to be confirmed by the result that from a single lawless sequence α we may construct a an enumerable universe of sequences $\mathbf{U} := \{n * \alpha^n | n \in N\}$, such that if we assume the variables for lawless sequences in the theory **LS** to range over **U**, then all the axioms of **LS** hold.

However, this is perhaps not the most natural way of looking at the existential axiom for lawless sequences. Let **U** be any notion of choice sequence (for example, **U** may be the universe of lawless sequences). If we think of an α in **U** as given at any particular stage n in its generation as specified by an initial segment σ, together with a tree of possible continuations, then we may say that an existential statement about choice sequences in **U** is satisfied if, starting from the empty segment, there is a continuation σ such that the statement holds for any element of **U** beginning with σ.

Metamathematical Investigation of Intuitionistic Arithmetic and Analysis – Corrections, 22 June 2009

This report contains corrections and additions to "Metamathematical Investigation of Intuitionistic Arithmetic and Analysis" which appeared in 1973 as number 344 in the "Springer Lecture Notes in Mathematics" series. The book has been out of print for several years now, but is at present available in elctronic form from the publisher. There the present list of corrections is also available.

This report has been typeset in Latex. Wavy underlining in the original text is now interpreted as boldface, underlining as italics. Double wavy underlining has been interpreted by a sans serif fount. However, we have retained double underlining and did not replace it by Fraktur.

A first list of Errata appeared in 1974 as a report of the Mathematical Institute of the University of Amsterdam; many more errata have been discovered since then. In particular I should like to thank Marc Bezem, Susumu Hayashi, Jane Bridge Kister, Jaap van Oosten and Jeffery Zucker for bringing corrections to my notice.

The counting of lines includes the lines in displayed formulas; for indications, e.g. a name or a number for a group of displayed lines, which are between lines so to speak, an ad hoc indication will be chosen.

Underlining in the original text has been rendered as italics in these correction; double underlining has been rendered as such, but a double wavy underlining corresponds to a sans serif letter in these corrections.

XIII Add below the summary of §6:

§7 *Applications: proof theoretic closure properties* 258
List of rules (3.7.1) — closure under ED, DP, CR_0, ECR_0, ED$'$, ACR, IPR'^{ω} (3.7.2–5) — closure under CR (3.7.6) — closure under ECR_1 (3.7.7–8) — extensions to analysis (3.7.9)

6 In 1.1.7, interchange "$\vee\mathrm{I_r}$" and "$\vee\mathrm{I_l}$".

7_{13} Read " $\exists$E " for " $\exists$I ".

8^5 Read "essentially".

8 In the first proof tree, "$B \to \curlywedge$" should be "$B \to \curlywedge$ (2)" and "(2)" should be repeated at the lowest horizontal line.

16^{13} Read " A " for " F ".

18 In 1.3.3 the axiom

$$x_i = x_i' \to \phi(x_1, \ldots, x_i, \ldots, x_n) = \phi(x_1, \ldots, x_i', \ldots, x_n)$$

(for any n-ary function constant ϕ, $1 \le i \le n$) can be replaced by the corresponding axiom for S only:

$$x = y \to Sx = Sy,$$

since the general case can be established by induction (since all ϕ except S are introduced by schemas for primitive recursive functions).

18_{11} Read "*Defining*" for "*Definining*".

19 Add at the bottom of the page rules expressing the functional character of the F_k:

$$\frac{F_k t_1 \dots t_{n-1} t_n \qquad F_k t_1 \dots t_{n-1} t'_n}{t_n = t'_n}.$$

20^8 Replace " F'_{k_m} " by " F_{k_m} ".

25 Addition to second paragraph of (D) : "Canonical" essentially means that the arithmetization provably satisfies the "same" inductive closure conditions as the predicate itself.

26^3 Read " $\ulcorner t \urcorner$ " for " t ".

26^5 Read "that $\ulcorner A(\bar{x}_1, \dots, \bar{x}_n) \urcorner$ stands for ...".

27^8 Read " $\simeq$ " for " $=$ ".

27 Add at the bottom of the page a paragraph:

We follow *Kleene 1952* and use $\Lambda x.t$, t a p-term, to indicate a gödelnumber for t as partial recursive function of x; if t contains, besides the free variables $x, x_1, \dots, x_n$, $\Lambda x.t$ is a (primitive) recursive function of $x_1, \dots, x_n$.

29_2 Read " y_0 " for " y ".

29_1 Read " y_1 " for " y ".

33_{10} Read "We put $\ulcorner \xi t_1 \dots t_n \urcorner = \dots$ ".

41_2 Read " $t' \not\equiv x^\sigma$ " for " $t' \neq x^\sigma$ ".

44^{17} Replace "slightly ... is" by "seemingly stronger (but in fact equivalent) variant is".

44^{19-21} Delete "In ... EXT-R′."

44_6–45^5 Replace these lines by the following:

The following two propositions are due to M. Bezem (Equivalence of Bar Recursors in the Theory of Functionals of Finite Type, *Archive for Mathematical Logic* 27 (1988), 149–160).

Proposition. The rule EXT-R′ is derivable in qf-$\mathbf{WE\text{-}HA}^\omega$.

Proof. Assume EXT-R, and let $\vdash P \to s_1 = s_2$, $\vdash Q[x/t_1]$ Here $s_1 = s_2$ as usual is shorthand for an equation between terms of type 0 $s_1 x_1 x_2 \dots x_n = s_2 x_1 x_2 \dots x_n$, where $x_1, x_2, \dots, x_n$ are variables not free in P, s_1, s_2. Without loss of generality we can assume $P \equiv (t_1 = 0)$, $Q[x] \equiv (t[x] = 0)$ (x not free in P, s_1, s_2. Below we shall abbreviate $t[x/s]$, for arbitrary s, as $t[s]$. So we have

(1) $\quad \vdash t_1 = 0 \to s_1 = s_2, \quad \vdash t[s_1] = 0.$

Define

$$s_i' := \mathbf{R}_\sigma s_i 0^{(\sigma)(0)\sigma}, \quad s_i \in \sigma.$$

Then, with $x \notin \mathrm{FV}(s_i), i = 1, 2$:

$$\vdash x = 0 \to s_i' x = s_i, \qquad \vdash x \neq 0 \to s_i' x = 0^\sigma.$$

Applying EXT-R to $s_i' 0 = s_i$ yields $\vdash t[s_i] = t[s_i' 0]$. By replacement (i.e. $x = y \to t[x] = t[y]$) we obtain

$$\vdash t_1 = 0 \to t[s_i' 0] = t[s_i' t_1].$$

Since also (1) holds, and $t_1 = 0$ is decidable ,$\vdash s_1' t_1 = s_2' t_1$,, so again using EXT-R

$$\vdash t[s_1' t_1] = t[s_2' t_1],$$

hence

$$\vdash t_1 = 0 \to t[s_2] = t[s_2' 0] = t[s_2' t_1] = t[s_1' t_1] = t[s_1' 0] = t[s_1] = 0.$$

Q.e.d.

Proposition. The deduction theorem holds for qf-$\mathbf{WE\text{-}HA}^\omega$ + EXT-R$'$, hence also for qf-$\mathbf{WE\text{-}HA}^\omega$.

Proof. It suffices to prove the deduction theorem for the system with EXT-R$'$, and in this case the deduction theorem is easy.

$45_{16,17}$ Delete these lines.

55_1 Read " $z < x$ " for " $z < z$ ".

56_{12} Read " $Q(x, \underline{v})$ " for " $Q(0, \underline{v})$ ".

56_6 Read " T " for " T ".

58_{12} Read " T " for " T ".

59 Add at the bottom: "Cf. also *Luckhardt 73*, pp. 66–67."

63^{10} Delete the first equation.

67_{11} At end of line we add "assumed to be provably linear in **HA**".

71^{15} Read " $\sigma((Q_2 V_n)A) \equiv (Q_1 v_{m+n+1})\sigma(A), \ldots$ ".

74 In comparing section 1.9.14 with more recent literature (such as A.S. Troelstra, D. van Dalen, *Constructivism in Mathematics*, Amsterdam 1988), it is to be noted that definedness of a term containing functions and numbers with partial application is here supposed to be defined in the sense of *Kleene* 1969, that is to say a function applied to an argument is defined if we can sufficiently many values of the function to find its value at the argument; this convention does not agree with the logic of partial terms with its strictness condition.

75^4 Read " $U(\mathrm{lth}(u)) = y$ " for " $U(\mathrm{lth}(u) = y)$ ".

75_5 Read " x " for " α ".

80_{13} Addition to 1.9.23: "Cf. also *Kreisel 1967*, page 249, where the role of generalized bar induction in proving the continuous functionals to be a model of bar recursion is mentioned."

$83_{18,13}$ Read " 0_σ " for " 0^σ ".

83_{10} Read " $B_\sigma yzu(c * \hat{v}))c$ " for " $B_\sigma yzuc(c * \hat{v}))c$ ".

91^{12} Add " In *Friedman B* it is shown that for r.e. axiomatizable extensions of **HA**, DP implies ED. ".

$94_{4,5}$ Replace by: "In *Luckhardt A* it is shown that the principle is equivalent to M. "

95 Add at the end of 1.11.6:

It has been noted by C.A. Smorynski that, for theories with decidable prime formulas, IP + M together amount to the principle of the excluded third. E.g. for **HA**, **HA** + IP + M = $\mathbf{HA}^c$, which is seen as follows. Assume $A \vee \neg A$ to be proved already in **HA** + IP + M, and consider $\exists x Ax$. By M, $\forall x(Ax \vee \neg Ax) \& \neg\neg\exists x Ax \to \exists x Ax$; by the induction hypothesis and IP, this implies $\exists x Ax \vee \neg\exists x Ax$. Application of propositional operators preserves decidability, and $\forall x Ax \leftrightarrow \neg\exists\neg Ax$ by the decidability of A, hence $(\forall x Ax \vee \neg\forall x Ax) \leftrightarrow (\neg\exists x\neg Ax \vee \neg\neg\exists x\neg Ax) \leftrightarrow \neg\exists x\neg Ax \vee \exists x\neg Ax$, hence $\forall x Ax \vee \neg\forall x Ax$.

95_4 Replace " $\alpha x = Uz$ " by " $\alpha y = Uz$ ".

98^6 Read " $x_1^{(\sigma)\tau}$ " for " $x^{(\sigma)\tau}$ ".

111^9 Read " $t_1 \equiv t_1'$ " for " $t_1 \equiv t'$ ".

113_4 Read " $\mathbf{H} \vdash t = \bar{n}$ ".

$114^{21,22}$ Delete the sentence beginning "For yet another ...".

117_6 Read "and t is" for "and τ is".

119^4 Read "... representing αn).".

125_1 Read " $(\sigma)(\tau)\sigma$ " for " $(\sigma)(\tau), \sigma$ ".

126^{10} Read "of" for "if".

$128^{12,13}$ The open problem has been solved by M. Bezem, in the sense that the two structures are isomorphic: J.S.L. 50 (1985), pp. 359–371.

129 Add between lines 6 and 7:

If we replace in the right hand side of this equivalence A by a predicate letter X, we have the inductive condition $B(X, x, y)$ characterizing A.

$132^{18,20}$ Replace " $\mathbf{I\text{-}HA}^\omega$ " by " $\mathbf{I\text{-}HA}^\omega + \mathrm{IE}_0$ ".

133^4 Delete ", CTM′, CTNF′ ".

$133^{10,11}$ Replace final comma by stop in line 10 and delete " CTM′, CTNF′ " in line 11.

$133_{5,4}$ Read " CTNF′ " for " CTNF ".

141^{10} Read " W^1_σ " for " I^1_σ ".

144^8 Read " (4) " for " (1) ".

147_1,148^1 Read " p_q " for " p_k ".

148^1 Insert before comma "& $\{z\}(p_q) = \{z\}(y)$ (since $z \in E(V)$, $y \in V^*$) ".

158^{13} Read " $t =_\mathrm{e} s$ " for " $t = s$ ".

158^{16} Read " CTNF " for " CTNF′ ".

158_{11} Read " $x^2[\Sigma(\Pi x^2)\Pi]$ ".

158_9 Read ", that $\Sigma(\Pi x_1^2)\Pi$ and ".

158_8 Read " $\Sigma(\Pi x_2^2)\Pi$ in the model ".

159^{12} Read "…which $x^1\bar{n}_i$ contr $\overline{\alpha n_i}$ has".

159_{14} Read " $t' \in 2$ " for " $t' \in z$ ".

$159_{1,2}$ – 160^{1-4} Replace these lines by:

$$t^2\alpha = \begin{cases} 0 \text{ if } \exists i(1 \le i \le k \;\&\; \bar{\alpha}_1(k+1) = \bar{\alpha}(k+1) \\ m+1 \text{ otherwise, where } m = \max\{\alpha_i(y) \mid 1 \le i \le k,\ 0 \le y \le k\}. \end{cases}$$

Now $Ft^2 \neq 0$; for, $\overline{(\pi_{0,0}0)}(k+1), \ldots, \overline{(\pi_{0,0}k)}(k+1)$ are all distinct, hence one of them, say $\overline{(\pi_{0,0}k_0)}(k+1)$ $(0 \le k_0 \le k)$ is distinct from all $\bar{\alpha}_1(k+1), \ldots, \bar{\alpha}_k(k+1)$, and therefore $t^2(\pi_{0,0}k_0) = m+1$; but then $\overline{(\lambda x^0.t^2(\pi_{0,0}x))}(k+1)$ differs from all $\bar{\alpha}_1(k+1), \ldots, \bar{\alpha}_k(k+1)$, and thus Ft^2 takes the value $m+1$.

160_1 Read " $=$ " for " $\equiv$ ".

161^{12} Delete "not".

164 Remark to be added at the end of 2.8.5: J.M.E. Hyland showed in his thesis that Scarpellini's model coincides with the model ECF ".

167_7 Read "establishing" for "establish".

173 Remark to be added in 2.9.10:

If a coding by functions is given for the elements of $\sigma^{\smile}$, such that there are continuous Φ_0, Φ with $\Phi_0\xi$ the length of the sequence coded by ξ, $\Phi(n,\xi)$ the n-th component extracted from ξ, then one can construct a bijection between two codings of this kind."

179^{17} Read " $\mathbf{HA}^{c}$ + G_1.

180^{6} Read " $|\,F$ holds " for " $|$ holds ".

180_{13} read " P " for " D ".

183_2 Read " PCA). " for " P A). ".

184^{17} Add after " terms ": "satisfying ($t \in V$ and $t' = t \Rightarrow t' \in V$) ".

188^{17} Read "mathematical".

189^{12} Read " $\&\, P(B(j_1x))$ " for " $\&\, P(A(j_1x))$ ".

190_{17} Delete ", and $P(F_1^*)$, ..., $P(F_n^*)$ ".

190_{12} Delete " Also $\forall x P(Bx)$ ".

190_4 Delete "It follows that ... hence $P(C)$.", and replace "Also" by "Then".

192^1 Add after "hence": " $!t \,\&\, t\, \mathsf{r}_P\, A$ is an abbreviation for $(\exists x(t \simeq x \,\&\, x\, \mathsf{r}_P\, A)$.)".

192_{1-7} The argument given in the first edition is not correct. The result is a consequence of the unprovability in **HA** of the DP, which has been proved by J. Myhill (A note on indicator functions, Proc. Amer. math. Soc. 29 (1973), 181–183) and by Friedman in a stronger form (On the derivability of instantiation properties, J.S.L. 42 (1977), 506–514).

194_{10} Insert " $\mathbf{HA} \vdash$ " between " (ii) " and " $A(\underline{\underline{a}}) \to\, !\psi(\underline{\underline{a}}) \& \dots$ ".

194_9 Replace this line by

Proof. The "only if" part is established as follows. Assume $\vdash A\underline{\underline{a}} \leftrightarrow B\underline{\underline{a}}$, B almost negative. Then there is a recursive ϕ such that $\vdash \forall u(u\, \mathsf{r} A\underline{\underline{a}} \to\, !\{j_1\phi\underline{\underline{a}}\}(u)$ $\&$ $\{j_1\phi\underline{\underline{a}}\}(u)\, \mathsf{r} B\underline{\underline{a}})$, and $\vdash \forall u(u\, \mathsf{r} B\underline{\underline{a}} \to\, !\{j_2\phi\underline{\underline{a}}\}(u)$ $\&$ $\{j_2\phi\underline{\underline{a}}\}(u)\, \mathsf{r} A\underline{\underline{a}})$, which together with 3.2.11 for B readily yields the desired conclusion.

194_2 Read " $Uv\, \mathsf{r}\, A\underline{\underline{a}}$ " for " $v\, \mathsf{r}\, A\underline{\underline{a}}$ ".

198 Add after 3.2.22:

Remark. In the writings of the Russian constructivist school (cf. e.g. Dragalin 1969) one finds the following extension of CT_0:

CT' $\quad \forall x(\neg Ax \to \exists y Bxy) \to \exists u \forall x(\neg Ax \to \exists v(Tuxv \,\&\, B(x, Uv)))$.

However, in the presence of M this is equivalent to ECT_0, i.e.

$$\mathbf{HA} + ECT_0 + M = \mathbf{HA} + CT' + M.$$

To see this, let us first assume CT', M, and let Ax be almost negative. then by M $Ax \leftrightarrow A'x$, A' negative, and hence $\neg\neg A'x \leftrightarrow Ax$ (1.10.8); thus an instance of ECT_0 can be interpreted as an instance of CT'.

Conversely, if ECT_0 and M are assumed, and we let $\forall x(\neg Ax \to \exists y Bxy)$, then by ECT_0, 3.2.8 $\neg Ax \leftrightarrow \exists z(z \,\mathsf{r}\, \neg Ax) \leftrightarrow \forall z(z \,\mathsf{r}\, \neg Ax) \leftrightarrow 0 \,\mathsf{r}\, \neg Ax$; $0 \,\mathsf{r}\, \neg Ax$ is almost negative. Replacing $\neg Ax$ by $0 \,\mathsf{r}\, \neg Ax$ we have $\forall x(0 \,\mathsf{r}\, \neg Ax \to \exists y Bxy)$ to which we can apply ECT_0 etc.

200–201 The claim of 3.2.26 is correct, but the proof given is incorrect. The attempted proof in list of Errata from 1974 is also flawed, but J. van Oosten presented me with a proof that IP_0 is derivable from ECT_0 and MP.

Here follows the proof: Assume

$$\forall x(Ax \vee \neg Ax), \ \ \forall x\, Ax \to \exists y\, B.$$

By ECT_0 there is a number n such that

$$\forall x(\{n\}x = 0 \leftrightarrow Ax), \ \ \text{hence} \ \ \forall x(\{n\}x = 0) \to \exists y\, B.$$

Again by ECT_0 there is a number m such that

$$\forall x(\{n\}x = 0) \to !\{m\}0 \wedge B(\{m\}0).$$

Let k be a number such that

$$\{k\}j(a, b) = \min_x[\{a\}x \neq 0 \vee Tb0x.]$$

Since

$$\begin{array}{l} \neg\neg(\exists x(\{n\}x \neq 0) \vee \forall x(\{n\}x = 0)), \ \ \text{it follows that} \\ \neg\neg(\exists x(\{n\}x \neq 0) \vee !\{m\}0, \ \ \text{therefore} \\ \neg\neg!\{k\}j(n, m); \ \ \text{hence with MP} \ \ !\{k\}j(n, m). \end{array}$$

From this we see, that by the definition of k

$$\begin{array}{l} \{n\}(\{k\}j(n, m)) \neq 0 \to \neg\forall x\, Ax, \\ T(m, 0, \{k\}j(n, m)) \to (\forall x\, Ax \to B(U(\{k\}j(n, m))). \end{array}$$

Hence

$$\exists y(\forall x\, Ax \to By).$$

203 Add after 3.2.29: "Friedman has shown (*Friedman B*) how to extend q-realizability by a similar trick.".

203_4 Read "*Cellucci*".

214_5 Replace "negative" by " $\exists$-free (i.e. not containing $\vee, \exists$) ".

$214_{4,3}$ Delete "on the convention ... omitted,".

215^1 Replace "negative" by " $\exists$-free".

215^9 Add " $\mathbf{N}$-$\mathbf{HA}^\omega$," after " $\mathbf{HA}^\omega$, ".

215^{11} Read "... sequence $\underline{t}$ of ...".

215_{15} Replace "negative" by " $\exists$-free".

216^{20} Read " $\underline{\underline{y}}\,\mathsf{mr}_P\,A$ " for " $y\,\mathsf{mr}_P\,A$ ".

$217^{1,2,17}$ Read "$\underline{\underline{T}}$" for "$T$" (4 times).

217^{13} Replace this line by:

$$\mathrm{IP}^{-} \qquad (A \to \exists y^\sigma B) \to \exists y^\sigma(\neg A \to B)$$

(y^σ not free in A, A $\exists$-free, i.e. not containing $\vee, \exists$)

$217_{16,15}$ Delete ", taking for ... into account". Add after 3.4.7:

Remark. The schema

$$\mathrm{IP}^{\omega} \qquad (\neg A \to \exists y^\sigma B) \to \exists y^\sigma(\neg A \to B) \ \ (y^\sigma \text{ not free in } B).$$

is readily seen to be modified-realizable, hence $\mathbf{H} + \mathrm{IP}^{-} + AC \vdash \mathrm{IP}^{\omega}$. Since in systems with decidable prime formulae negative and $\exists$-free formulas coincide, and for negative A $\neg\neg A \leftrightarrow A$, we have in such cases also that IP^{ω} implies IP^{-}.

$217_{10,9,4,2,1}$ Replace "IP^{ω} " by " IP^{-} ".

217_{10} Read " $\mathbf{H}\ +$ " for " $\mathbf{H} \vdash$ ".

217_{8} Add after line: "For $\mathbf{H} = \mathbf{HA}^{\omega}$, $\mathbf{I\text{-}HA}^{\omega}$, $\mathbf{HRO}^{-}$, $\mathbf{E\text{-}HA}^{\omega}$, IP^{-} may be replaced by IP^{ω}.".

217_{5} Read "$\exists$-free" for "negative".

221_{7} Read " M_{PR} " for " $\mathrm{MP}_{\mathrm{PR}}$ ".

221_{2} Read "3.4.14" for "3.4.4".

222 Add after 3.4.14:

Remark. V.A. Lifschitz has shown (Proceedings of the American Mathematical Society 73 (1979), 101–106) that also $\mathbf{HA} + \mathrm{CT}_0! \nvdash \mathrm{CT}_0$, where

$$\mathrm{CT}_0! \qquad \forall x \exists! y A(x,y) \to \exists u \forall x \exists v (Tuxv \,\&\, A(x, Uv)).\ "."$$

222_{2} Read " $\forall\alpha\neg\neg\exists x$ " for " $\forall\alpha\neg\neg\exists z$.

223^{1} Read "... was suggested by results contained in".

224_{1},225^{1} Read " IP^{-} " for " IP^{ω} ".

226_{16} Replace " for " by " . For ".

226_{8} Insert after "... numbers" "(provably linear in **HA**)".

227 Add "($\prec$ provably linear in **HA**)".

228_{16} Read " $U^1_{j(n,i)}x$ " for " $U^1_{j(n,i)}$ ".

$228_{7,6}$ These lines must read respectively " $\ldots \equiv \forall X^*\forall D_X(x\,\mathsf{mr}\,A(X))$ " and " $\ldots \equiv \exists X^*\exists D_X(x\,\mathsf{mr}\,A(X))$ ".

229_{11} Read " of s^1 in HRO ".

230^5 Read "eliminating" for elementary".

233_5 Read " Π^0_2 " for " Π^0_z ".

$238^{4,6,7}$ Delete " $]^{\mathrm{D}}$ ".

239_4 Read " $(\underline{\underline{x}}, \underline{\underline{v}}, \underline{\underline{\underline{Zv}}})$ " for " $(\underline{\underline{x}}, \underline{\underline{\underline{vZ}}}, \underline{\underline{v}})$ ".

240^3 Read " $\underline{\underline{y}}$ " for " $\underline{\underline{\underline{Yx}}}$ ".

242^{12} Read " $\vdash F^{\mathrm{D}}$ " for " $+F^{\mathrm{D}}$ ".

242^{17} Read "now" for "not".

$244_{7,6}$ Replace these lines by:

If we take everywhere X to be identically 1, we obtain the Dialectica interpretation.

245^{10} "Shoenfield" should be underlined.

251_4 Delete " $\mathbf{N\text{-}HA}^\omega$, " and add " ; $(\mathbf{N\text{-}HA}^\omega + \mathrm{IP}^- + \mathrm{AC}) \cap \Gamma_1 = \mathbf{N\text{-}HA}^\omega \cap \Gamma_1$ " (cf. the corrections to page 217).

255_6 Read " $\ldots \,\&\, A^*y)]$ ".

264^{16} Read " **HA** " for " HA ".

264_8 Read " $\to \exists x^\sigma A$ " for " $\to \neg\exists x^\sigma A$ ".

265^1 Read " $(z \neq 0 \to \neg A)$ " for " $(z \neq 0 \to A)$ ".

267_2 Read " $\exists y(y\,\mathsf{p}$ " for " $\exists(y\,\mathsf{p}$ ".

273_{17} Read " 3.9.13 " for " 3.9.11 ".

274^1 Read " 3.9.14 " for " 3.9.12 ".

274_4 Read " 3.9.15 " for " 3.9.13 ".

275_6 Delete " (".

278 Second proof tree under 4), read " Π " for the highest " Π_i ".

279 Replace in the first four proof trees exhibited the occurrences of "A" (but not the A in "A_1", "A_2", "Aa" or "$\exists x Ax$") by "B" (7 occurrences).

280 Replace under " 13) " " $A0$ " by " Aa ".

282^7 Read " $\Pi' \succ_1 \Pi''$, (without ...".

282^{23} Add " of $\curlywedge_\mathrm{I}$ " at the end.

282 In the display at the bottom of the page, the first two lines should be

$$\begin{array}{c}\Pi\\ \underline{A \quad 0=0}\end{array} \qquad \begin{array}{c}\Pi\\ \underline{A \quad sa=sa}\end{array}$$

283 In the displayed prooftree read " $\&_\mathrm{I}\mathrm{E}$ " for " &E ".

284^1 read "form (*Prawitz*" for "(from *Prawitz*".

285 Immediately above the paragraph starting with "This makes it ..." read

$$\begin{array}{c}\Pi'\\ [A]\\ \Pi\\ B\end{array} \quad \text{for} \quad \begin{array}{c}\Pi\\ [A]\\ \Pi\\ B\end{array}$$

286 Replace last paragraph of 4.1.7. by:

For applications, we need only a normalization theorem (not a strong normalization theorem) relative to $\mathcal{R}_{\mathrm{c}\curlywedge}$; so if the reader wishes, he may use the preceding remark and delete everything in the proof below referring to $\curlywedge$-contractions.

287^{16} Read " $\mathrm{Prd}_1(\Pi)$, $\mathrm{Prd}_2(\Pi)$, ... ".

287_5 Read " $\mathrm{SV}(\mathrm{Sub}(A, \Pi, \mathrm{Prd}(\Pi), \mathrm{Ass}(\Pi)))$ ".

287_2 Read " Π " for " Π_1 ".

288^9 Insert ", $\mathrm{Ass}(\Pi)$ " after " $\mathrm{Prd}_2(\Pi)$ ".

288 Directly above footnote, read

$$\frac{\Pi'_i}{A} \quad \text{for} \quad \frac{\Pi_i}{A}.$$

290^{17} Read " $\Pi' \in \mathrm{PRD}(\Pi)$ ".

290 In the first displayed prooftree, replace " At " by " At' ".

291 The second displayed prooftree should read:

$$\frac{\Pi'_1 \;\ldots\; \Pi'_n}{A}.$$

292^1 Read "condition IV " for " condition II ".

293 Second line of paragraph starting with "Condition IV for Π'...", read "for Π " instead of "for $\exists$ ".

294 Replace in the second display " Π_3 " by " Π_4 ", and in the line under this display, replace "Π_4" by "Π_5".

294 Replace in the third display "Π_3" by "Π_4". In the line under the third display, insert after "reduces to": "the left subdeduction of".

295^6 Read " Π_6 " for " Π_3 ".

295 In the third display, place in the second proof tree " Π_0 " above " $\exists x Bx$ ".

295_{12} Insert between "condition" and "I": "IV, and hence".

295 In the line below the third display, insert before "is SV":
"and also

$$\begin{array}{c} \Pi_4 \\ [B_1 t'] \\ \Pi'_1(t') \\ [Dt] \\ \Pi_1(t) \\ A \end{array}$$

"

297^{17} Read " 2.2.25 " for " 2.2.31 ".

299^{15} Add " (major) " at the end.

300^6 Add after the comma "which may be empty,".

$300^{9,10}$ Replace "preceding" by "succeeding".

301^2 Read "were" for "would be".

301^{10} Add before ") ": " ; also, A_i cannot be discharged by IND, since no application of IND lies below A_1 ".

301_6 read " 4.2.8 " for " 2.8 ".

302^5 Read "normal" for "formal".

302^{14} Add " σ " at the end.

304^3 Insert "(by 4.2.7)" before " ; ".

304^{10} Read "were" for "would be".

304^{11} Read "or IND-application occurring" for "occurring".

304^{14} Replace " IND-application " by " begin with a conclusion of an IND-application".

304_{19} Read "Let Φ denote the ".

304 The proof in subsection 4.2.16 in the first edition contains a gap. Much simpler is the following argument:

PROOF. Note that Ψ is equivalent to a set of Harrop formulas: if $\exists x Px \in \Psi$, then we may replace this formula by $P\bar{n}$ for some $\bar{n}$ such that $\mathbf{HA} \vdash P\bar{n}$. Then we can apply 4.2.12.

305_1 delete " , or A_1 is a basic axiom ".

305^5 Read " $A_1 \in \Psi$, i.e. B is prime ".

305 Remark at 4.2.17: instead of referring to 3.6.7(ii), it suffices to note that only true closed Σ^0_1-formulae are provable in **HA** and $\mathbf{HA}^c$.

306, proof of 4.2.18. This proof is incorrect as it stands, since the conclusion of an IND-application is not necessarily atomic, only quantifier-free. The proof is correct if we replace in the statement of the theorem **H**, qf-**HA** by the corresponding systems with induction for atomic formulas only.

To establish the theorem as stated, we can e.g. proceed as follows: define a path of order 0 to be a path $A_1, \ldots, A_n$ with A_n conclusion of the deduction, and define a path of order $m+1$ to be a path $A_1, \ldots, A_n$ such that either A_n is minor premiss of an $\to$E-application the major premiss of which belongs to a path of order m, or premiss of an IND-application the conclusion of which belongs to a path of order m.

In a strictly normal derivation, every formula occurrence belongs to some path of order m, for suitable m (since redundant applications of $\vee$E, $\exists$E have been removed). Then one readily proves, by induction on m, that for a strictly normal derivation of a quantifier-free formula in **H** all formula occurrences on a path of order m are quantifier-free. (Note that here normalization also w.r.t. permutative reductions is necessary, in contrast to other applications. This could have been avoided by reduction of qf-**HA** to a logic-free calculus, which is not a very elegant solution, however.)

306_{16} Read " 2.5.7 " for " 2.5.6 ".

307, line 2 below second display. Read " $\mathcal{R}_\mathrm{c}$ " for " $\mathcal{R}_\mathrm{C}$ ".

307_2 Read " 4.1.16 " for " 4.1.15 ".

308 Last line of first display, replace in proof tree Π'' " $\neg A \to Bx$ " by " $\neg A \to \exists x Bx$ ".

309 Replace second sentence of the statement of 4.3.5 by:

Then a spine of Π not ending with an introduction does not contain IP-applications, and ends with an application of a basic rule or an atomic instance of $\curlywedge_\mathrm{I}$.

Delete the third sentence.

The proof of 4.3.5. should be reformulated as follows:

Let $A_1, \ldots, A_n$ be a spine of Π not ending with an introduction. Then, by 4.3.4(iii) there are two cases:
(1°) A_1 is a basic axiom. So the spine coincides with its minimum part, hence A_n is atomic.
(2°) A_1 is of the form $\neg B$, to be discharged by $\to$I, followed by IP. this case is excluded, for the sort of inference following A_1 can be (not IP, or $\to$I + IP, but) $\to$E only, leaving us with $A_2 \equiv \curlywedge$, and a minimum part $A_2, \ldots, A_n$.

311^8 read "any one".

311^{17} Read "Red_1" for "Red".

311_{19} Read "of" for "from".

311_{12} Read " $\vee\text{I}_\text{r}$ " for " $\vee\text{I}_\text{l}$ ".

311_{11} Read " $\vee\text{I}_\text{l}$ " for " $\vee\text{I}_\text{r}$ ".

311_7 Read " $\text{SV}_{\text{d}-1}(\text{Subst}(\text{Param}(\Pi), x, \text{Prd}_1(\Pi)))]$ ".

313^1 Read " (ii) " for " (iii) ".

313_{12} Read " 4.4.3 " for " 4.4.2 ".

313_{10} Insert "(1.5.6)" before " : ".

314^4 Read " Proof_n " for " Proof ".

314_5 Read " 4.4.3 " for " 4.4.2 ".

314_3 Read " $\mathbf{HA} \vdash \forall x \exists yz(\text{Proof}_n(y, \ulcorner A(\bar{x}, \bar{z})\urcorner \,\&\, A(x, z))$ ".

321 As observed by S.Hayashi, (On derived rules of intuitionistic second order arithmetic, Commentarii Mathematici Universitatis Sancti Pauli 26 (1977), 77–103), the proof of 4.5.8 indicated in the text of the first edition establishes a result which is too weak, e.g.

$$\forall n \forall A \in \text{Fm}^{(n)}(\vdash \text{Sat}^{(n)}(X, \ulcorner \forall x A x \urcorner) \leftrightarrow \forall x \text{Sat}^{(n)}(X, \ulcorner A(\bar{x})\urcorner))$$

instead of

$$\forall n(\vdash \forall A \in \text{Fm}^{(n)}(\text{Sat}^{(n)}(X, \ulcorner \forall x A x \urcorner) \leftrightarrow \forall x \text{Sat}^{(n)}(X, \ulcorner A(\bar{x})\urcorner))).$$

Following Hayashi, the desired stronger conclusion can be established as follows.

We first define the notion of a formation sequence of a formula A in $\text{Fm}^{(n)}$.

Definition. A *formation sequence* (fs) of $A \in \text{Fm}^{(n)}$ is a finite sequence of quadruples $\langle a_0, b_0, c_0, t_0\rangle, \ldots, \langle a_m, b_m, c_m, t_m\rangle$ such that

(1) $t_m = \ulcorner A \urcorner$; t_0, and c_i for $1 \le i \le m$ are codes of formulas of complexity $\le n$.

(2) $a_i \in \mathbb{N}$ for $0 \leq i \leq m$, $a_{i+1} \leq i$ for $0 \leq i < m$.

(3) $b_i, c_i \in \mathbb{N}$ for $0 \leq i \leq m$; t_{i+1} is the code of the term which is the result of substituting the term with code $t_{a_{i+1}}$ for the second-order variable $V^p_{b_{i+1}}$ in the formula (with index) c_{i+1} and logical complexity $\leq n$, where p is the number of free variables in $t_{a_{i+1}}$ (end of definition).

Now $\mathrm{Sat}_n(X, \ulcorner A \urcorner)$ is constructed as before. Let f, g, h range over formation sequences. We then define, similar to $\mathrm{Sat}^{(n)}(X, \ulcorner A \urcorner)$ of the text, and with help of Sat_n, the formula $\mathrm{Sat}^{(n)}_f(X, \ulcorner A \urcorner)$, where f is an fs for A with $t_m = \ulcorner A \urcorner$, and $\mathrm{Sat}^{(n)}_f$ is constructed parallel to the substitutions of f. Then one proves

Lemma. In **HAS**

(i) $\forall f \forall A, B \in \mathrm{Fm}^{(n)} \exists g, h \forall X (\mathrm{Sat}^{(n)}_f(X, \ulcorner A \circ B \urcorner)$
$$\leftrightarrow \mathrm{Sat}^{(n)}_g(X, \ulcorner A \urcorner) \circ \mathrm{Sat}^{(n)}_h(X, \ulcorner B \urcorner))$$
for $\circ \in \{\rightarrow, \&, \vee\}$.

(ii) $\forall f \forall A \in \mathrm{Fm}^{(n)} \exists g \forall X (\mathrm{Sat}^{(n)}_f(X, \ulcorner Q v_i A(v_i) \urcorner) \leftrightarrow (Q v_i) \mathrm{Sat}^{(n)}_g(X, \ulcorner A(\bar{v}_i) \urcorner))$ for $Q \in \{\forall_1, \exists_1\}$.

(iii) $\forall f \forall A \in \mathrm{Fm}^{(n)} \exists g \forall X (\mathrm{Sat}^{(n)}_f(X, \ulcorner Q V^p_i A(V^p_i) \urcorner) \leftrightarrow$ $(QY^p) \forall Z^1 (\forall y_1, y_2 (j(y_1, y_2) \neq j(p, i) \rightarrow Z^1_{(y_1,y_2)} = X_{(y_1,y_2)}) \wedge Z^1_{(p,i)} = Y \rightarrow$
$$\mathrm{Sat}^{(n)}_g(Z, \ulcorner A(V^p_i) \urcorner)$$
for $Q \in \{\forall_2, \exists_2\}$.

(iv) $\forall X, f, g, n (\mathrm{FS}(f, n) \wedge \mathrm{FS}(g, n) \rightarrow \mathrm{Sat}^{(n)}_f(X, n) \leftrightarrow \mathrm{Sat}^{(n)}_g(X, n))$, where $\mathrm{FS}(f, n)$ expresses "f is a formation sequence of a formula A with $\ulcorner A \urcorner = n$".

Proof. The proof of (i)–(iii) by induction on the length of f; the proof of (iv) uses (i)–(iii) and induction on n.

We may then put

$$\mathrm{Sat}^{(n)}(X, \ulcorner A \urcorner) \leftrightarrow \exists f \mathrm{Sat}^{(n)}_f(X, \ulcorner A \urcorner)$$

and can then establish a stronger version of 4.5.8, namely

$$\forall n (\mathbf{HAS} \vdash \forall A \in \mathrm{Fm}^{(n)} (\mathrm{Sat}^{(n)}(X, \ulcorner \forall x A x \urcorner) \leftrightarrow \forall x \mathrm{Sat}^{(n)}(X, \ulcorner A \bar{x} \urcorner)))$$

etc. etc.

322^{13} read "choice" for "hoice".

375_{17} Read "$\alpha > \alpha_0$ &" for "$\alpha > \alpha_0$".

389 Subsection 5.7.3: more information about Kripke models for second-order intuitionistic arithmetic may be found in: D.H.J. de Jongh, C.A. Smoryński, Kripke models and the intuitionistic theory of species, Annals of Mathematical Logic 9 (1977), 157–186.

391_3 read "$(\alpha x \neq 0)$" for "$(\alpha x \neq 0$".

391_1 Add after ")" "in the presence of continuity axioms".

398_{12} Read "$\mathcal{L}[x]$" for "$\mathcal{L}$".

414_{11} Read "$S_2 f \in Y$" for "$S_1 f \in Y$".

422_{10} Read "$\mathbf{ID}^{\mathrm{c}}_{\nu}$" for "$\mathbf{ID}^{\mathrm{C}}_{\nu}$".

435^{10} Read "$|\mathbf{ID}^{\mathrm{c}}_{\nu}|$" for "$|\mathbf{ID}^{\mathrm{c}}|$".

438^5 Read "$P_1(Xn, n)]$" for "$P_1(Xn, x)]$".

439_{16} Read "type-0-valued".

440_6 Read "$\mathrm{Max}_1(\alpha, 0) = \alpha$" for "$\mathrm{Max}_1(\alpha, 0)$".

$448_{13,14}$ The equality in (7) of 6.9.1 was proved for all recursive ν by 1977, independently by Buchholz, Pohlers and Sieg, using various sophisticated proof-theoretic techniques (see W. Buchholz, S. Feferman, W. Pohlers, and W. Sieg, Iterated Inductive Definitions and Subsystems of Analysis: Recent Proof-Theoretical Studies. Springer Verlag, Berlin 1977). Hence the equalities

$$|\mathbf{ID}^{\mathrm{c}}_2| = |\mathbf{ID}_2| = |\mathbf{T}_2|$$

hold (end of 6.8.9). Hence also the equalities (5) and (6) of 6.9.1 are true.

451^{12-17} See the remark to page 448.

457_{14} Read "$(\lambda n.X_1 \ldots X_k)$" for "$(\lambda X_1 \ldots X_k)$".

462^1 Insert before the second line "Corrections in the bibliography consist sometimes in replacements, sometimes in added information between square brackets."

$462^{8,9}$ Replace by "LMPS IV: P.Suppes, L.Henkin, Gr.C. Moisil, A. Joja (eds.), North-Holland Publ. Co., Amsterdam 1973."

462^{17} Read "Cambridge Summer School in Mathematical Logic"

462^{18} Read: H.Rogers, A.R.D. Mathias (eds.), 1973.

462^{19} Add at the end ", 1973".

462^{20} Delete.

462_{15} Add "[Zeitschrift für mathematische Logik und Grundlagen der Mathematik 20 (1974), 289–306.]"

462_{14} Add: "[cf. H.P. Barendregt, Combinatory logic and the axiom of choice, Indagationes Mathematicae 35(1973), 203–221.]"

$462_{10,11}$ Replace by: " Theoretical Computer Science 3 (1977), 225–242.

462_5 Add "[J.S.L. 41 (1976), 328– 336]".

462_3 Add "[J.S.L. 40 (1975), 321– 346]".

462_1 Add "[J.S.L. 41 (1976), 18–24]".

463^1 Read " Cellucci".

463^{15} Read "Cambr. Proc. 1–94."

463_{25} Add "[Did not appear]"

463_{21} Read "Archiv für mathematische Logik 16 (1974), 49–66."

464^5 Read "Cambr. Proc. 113–170."

$464_2$3 Read "Schliessen."

464_1 Add ", 232–252".

465^{21} Read "Cambr. Proc. 274–298."

465^{24} Read "J.S.L. 38 (1973), 453–459."

465^{26} Add "[Did not appear]"

465^{28} Add "[Did not appear]"

465^{30} Read "J.S.L. 41 (1976), 574–582."

466^{26} Read "Section VI" for "Section IV".

466_{14} Add: "[Appeared in: J.P.Seldin, J.R.Hindley (eds.), To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism. Academic Press, New York 1980, 480–490.]"

467^{22} Add: "[Never published]"

467^{24} Add: "[Appeared in: A.S. Troelstra, D. van Dalen (eds.), The L.E.J Brouwer Centenary Symposium. North-Holland Publ. Co., Amsterdam 1982, 51–64.]"

468^8 Read "Philosophica".

469^9 Read "in" for "in:".

469^{12} Read "Zentralblatt".

469^{26} Read "IPT" for Oslo Proc."

469_{24} Read " Π^1_1" for "Π".

469_8 Read "1" for "I".

470_{12} Add: "[Cf. paper under this title in: S. Kanger (ed.), Proceedings of the 3rd Scandinavian Logic Symposium, North-Holland Publ. Co., Amsterdam 1975, 81–109.]"

471^{10} Read: "Compositio Mathematica 26 (1973), 261–275."

472^{5} Insert before stop: ", 225–250".

472_{1} Replace by "Archiv für Mathematische Logik 16 (1974), 147–158."

473^{11} Add "[Unpublished]".

474_{11} Read "1970" for "170".

475^{13} Read "Cambr. Proc. 171–205."

Basic Proof Theory, 2nd edition
Corrections, 31 August 2009

The list includes a list of corrections brought to our notice by Yiorgios Stavrinos of the University of Athens, Greece, July 2003.

9^{11} replace "linearly" by "well-ordered".

11_5 replace "$\mathrm{FV}(x^A) = x^A$" by "$\mathrm{FV}(x^A) = \{x^A\}$"

15_6 replace the first "=" by "$\succeq$".

$19^{3,6}$ the rules for **k**, **s** should have been given for terms instead of variables.

19_6 add "$\lambda^* x^A.c^B := \mathbf{k}^{B,A}c^B$ for c a constant"

39 in the first proof tree the second rule is →E, not →I.

42 remove * after exercise **2.1.8B♠**; add * after **2.1.8D♠**.

47 in the case of the intuitionistic absurdity rule $\perp_{\mathrm{i}}$ read $\mathrm{FV}_{\mathrm{a}}(\mathrm{E}^{\perp}_{A}(t)) := \mathrm{FV}_{\mathrm{a}}(t)$.

52^{12} read for "and →I and →E" instead "→I, ∀I and →E".

$52_{1,2}$ read: "and the rules →I, ∀I and →E only".

63 in the last line above the second group of prooftrees delete ", in the right deduction $x \notin \mathrm{FV}(B)$".

$69^{5,6}$ read "$\mathrm{Set}(\Gamma') \subset \mathrm{Set}(\Gamma)$" for "$\mathrm{Set}(\Gamma') \subset \Gamma$".

69^{10} read "$\mathrm{Set}(\Gamma') \subset \mathrm{Set}(\Gamma)$" for "$\Gamma' \subset \mathrm{Set}(\Gamma)$".

70 the last inference in the second prooftree, a series of contractions marked by a double horizontal line, is in fact redundant. Similarly in the case dealing with →E.

71^{10} read "we have $\vdash \neg\neg A \to A, \Gamma \Rightarrow A$ if $\vdash \Gamma, \neg A \Rightarrow$,".

71 in the second proof tree in the proof of the lemma, a double line should appear below the axiom $A \Rightarrow A$.

71 replace in the fourth prooftree in the proof of the lemma the axiom $\perp \Rightarrow$ by

$$\frac{\perp \Rightarrow}{A, \perp \Rightarrow}$$

88_{12} replace Γ by $\Gamma \Rightarrow \Delta$.

88_9 replace Γ' by $\Gamma' \Rightarrow \Delta'$.

92_5 read "$\vdash A\Gamma' \Rightarrow \Delta'$ " for "$A\Gamma' \Rightarrow \Delta'$ ".

96^{20} middle of page, read "produces a deduction $\mathcal{D}''$ such that".

97^{19} "This is transformed into a deduction $\mathcal{D}^*$:".

97 in the lines above the second prooftree from below, delete the first sentence "The new cut ... remove this cut."

100^{21} read "rank" for "degree".

100 in the prooftrees at the bottom read "$\mathcal{D}_{00}$" for "$\mathcal{D}_{00}[y]$".

101 delete B in the conclusion of the rule Multicut.

103^8 read "R$\forall$" for "R$\exists$".

103^8 read "*dp-invertible*" for "*invertible*".

103^9 add "and under "depth-preserving weakening".

103_7 read "Γ, Γ'" for "Γ".

104^2 delete Δ in the second line of the first displayed prooftree.

$105_{6,5}$ read "in $\bigwedge \Gamma \to \bigvee \Delta$ where $\bigwedge$ and $\bigvee$ are used for iterated conjunction and disjunction respectively.".

$116_{4,3}$ the negation sign is actually redundant in these lines.

117^8 read "$\perp$" for "$\perp \to \perp$".

117_2 read "$\Gamma A \Rightarrow \Delta BC$" for "$\Gamma A \Rightarrow \Delta C$".

118_9 read "$\Gamma, B, C \Rightarrow D$" for "$\Gamma, A \to B, C \Rightarrow D$".

$137^{13,15,18,20,21}$ replace $\mathcal{S}$ everywhere by S, $\mathcal{N}$ by F and $\mathcal{P}$ by G. This change brings the notation here in line with the notational conventions of subsection 1.1.4 (pages 5–6).

158_{16} add "$A_1 := \neg P_1$, $B_1 := \neg Q_1$".

182^{25} read "and is the only critical cut in $\mathcal{D}$" for "and is the only maximal segment in $\mathcal{D}$"

200_{15} replace "At" by "$A(t)$".

200_{14} replace "As" by "$A(s)$".

210_{11} the first line of the lemma should end with "Comp_A:".

211^9 read "We prove $ts \in \mathrm{Comp}_C$".

211_4 delete "strongly".

212^7 read " Let $\mathrm{FV}(t_1) \subset \{y{:}B, x_1{:}A_1, \ldots, x_n{:}A_n\}$ ".

212_12 read " $\psi(\forall x A) := (Q \to Q) \to \psi A$ " for " $\psi(\forall x A) := (Q \to Q) \to A$ ".

214^2 read " $\mathcal{D}_n(P^n, Q^n)$ " for " $\mathcal{D}_n(P, Q)$ ".

$261_{7,6}$ types do not match in definitions of evcur, curev: switch A and B.

$294^{5,6}$ interchange "(L∧,R∧)" with "(L∧′,R∧′)".

$300^{12,17}$ read "$\mathbf{G1i} \vdash !\Gamma \Rightarrow A$" for "$\mathbf{G1i} \vdash \Gamma \Rightarrow A$".

300^{18} read "$\mathbf{S4} \vdash \Box GG^{\circ} \Rightarrow A^{\circ}$" for "$\mathbf{S4} \vdash \Gamma^{\circ} \Rightarrow A^{\circ}$".

317 The definition of $\mathrm{Ind}(A,x)^*$ should read

$$\mathrm{Ind}(A,x)^* \qquad \forall x(\forall y{<}x\, A[x/y] \to A) \to \forall x A.$$

341 $\overline{\alpha}(x)$ is defined as usual, as $\langle \alpha 0, \alpha 1, \alpha 2, \ldots, \alpha(x-1)\rangle$.

341^{15} for "$\{\alpha | Q(\overline{\alpha}x)\}$" read "$\{\alpha | \forall x Q(\overline{\alpha}x)\}$".

341^{23} for "$\leq$" read "$<$".

367 read "**2.1.8D**" for "**2.1.8B**".

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410^{15} This is not literally what one gets by spelling out the clause for disjunction using the definition in terms of $\exists$, $\wedge$ and $\rightarrow$, but it is equivalent in the sense specified under (ii) below.

411_6 As its stands, it is not obvious that $\forall x(x = x)$ holds. This is remedied by correcting EQ as follows:

$$\text{EQ} \qquad \begin{cases} \forall xy(x = y \rightarrow y = x), \quad \forall xyz(x = y \wedge y = z \rightarrow x = z), \\ \forall \vec{x}\vec{y}(\vec{x} = \vec{y} \wedge F\vec{x}{\downarrow} \rightarrow F\vec{x} = F\vec{y}), \quad \forall \vec{x}\vec{y}(R\vec{x} \wedge \vec{x} = \vec{y} \rightarrow R\vec{y}) \end{cases}$$

By taking in the transitivity clause y for x, we see that $\forall xy(x = y \rightarrow y = x)$ becomes derivable.

412_4 Instead of looking at Troelstra-van Dalen [1988, 3.6, 3.7], see the emended and expanded version in the present report, headed **Corrected version of subsections 3.7.9–15**.

413^{15-23} The definition of $\underline{\texttt{rn}}$, $\underline{\texttt{rnt}}$ is better formulated for an arbitrary term t instead of the variable x.

426, section 2.6 This contains an early, not yet quite adequate version of the corrections to subsections 3.7.9–15 mentioned before; for a better version see the present report.