SPACE AND THE CONTINUUM FROM KANT TO POINCARÉ

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written by

Anna Franchini
(born June 7th, 1988 in Verona, Italy)

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Date of the public defense:	**Members of the Thesis Committee:**
July 4, 2019	Prof. Dr. Michiel van Lambalgen *(supervisor)*
	Dr. Nick Bezhanishvili
	Drs. Levin Hornischer
	Dr. Riccardo Pinosio
	Dr. Floris Roelofsen *(chair)*

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Alla nonna Pina e alla Luna

Abstract

The present thesis explores Kant's transcendental philosophy, focusing on the cognitive processes that bring to the formation of the concept of space. In light of the interpretation proposed by Pinosio and van Lambalgen [27], we provide an in-depth analysis of the fundamental passages of the *Critique of Pure Reason* expounding the synthetic activity that produces the consciousness of space as a formal intuition. We investigate Kant's constructive continuum and we compare it to the Aristotelian continuum, emphasising their similarities, in particular with respect to the notion of contact between regions. We then compare Kant's perspective to Poincaré's philosophy of space, suggesting some possible interactions between the two perspectives. We propose a parallel between the work of the figurative synthesis in Kant and the gradual process of abstraction that leads, in Poincaré's philosophy, to the formation of a mathematical continuum from a physical continuum. Finally, we produce a formal model for Kant's spatial continuum, adopting a mereological approach. Starting from a set of finite structures (Boolean algebras and their dual Stone spaces), which represent the spatial extent of possible experiences, we build a direct system and an inverse system. The limit of the inverse system, together with a relation of proximity, is the formal correlate to space as the formal intuition. The proximity relation (dual to a contact relation on Boolean algebras), is the key to obtain a continuum of points that are the emerging boundaries of regions.

Contents

Chapter 1

Philosophy of space from Kant to Poincaré

1.1 Introduction

The concept of space is a fundamental component of our experience of the world, and in everyday life we use terms and expressions coming from the semantic area of "space" so frequently that we tend to assume it as an established, unproblematic concept. The word "space" is taken to be so primitive that many disciplines, like physics and engineering often use it unproblematically to mean a generic concept of "physical space" without providing a formal definition of it. This approach works until one starts asking about the nature of such space. As of the beginning of the 20th century, the physical theories of space seek to apply to "the real structure" of the world in itself. But this has not always been the case and the issues with this conception are apparent as soon as one starts thinking of what we mean with "things in themselves". Even if we could come to an explicit and uncontroversial definition of this latter concept, many questions remain unanswered about the nature of space: is it a property of things - definable as spatial extension? Is it an external "container" with respect to which we can determinate the position of objects? Is it an actual object or does it depend on the constitution of our mind? What is its relation to the other fundamental component of our conscious experience, time? All these questions have generated a great deal of philosophical debates, dating back to antiquity (Greek philosophers like Plato and Aristotle already wondered about the nature and properties of space), and embracing a wide range of topics, from ontology to semantics, from epistemology to phenomenology. The physical theories describing the properties of space and objects moving in it have also developed considerably from the classical Aristotelian model, to Galilean and Newtonian physics, to the modern theory of relativity. For more than two thousands years, Euclidean geometry has been the solid mathematical base on which any physical theory of space could firmly rest, but the discovery, in the early 19th century, of many others self-consistent non-Euclidean geometries shook the foundations of centuries worth of physics to prepare the ground for a revolutionary approach to space that led Einstein, at the beginning of the 20th century, to his theory of relativity.

The present work is intended as an overview of the positions of two eminent philosophers in the debate about the nature and properties of space. Starting from an in-depth analysis of Kant's transcendental philosophy of space, based on the interpretation proposed by Pinosio in his recent work on Kant's temporal continuum ([27]), we move to the theory proposed by Henri Poincaré. There is, however, an apparent asymmetry in our treatment of the two positions, due to the fact that our work started as an investigation of Kant's theory of space aimed at the construction of a mathematical model of cognitive space. While building the formal model, we found similarities with some features of Poincaré's construction of the physical continuum, and we saw the opportunity for an interesting combination of the two perspectives. Although in the end we did not manage to incorporate the features of Poincaré's continuum into our model, we found that the observations produced during our comparison of the two points of view could motivate future research. Hence we included a synopsis of the work of the French mathematician on perceptual space, integrating it with some thoughts that proved relevant to our investigation.

Our research started from the study of the philosophical theory of space (and time) developed by the great German philosopher Immanuel Kant (1724-1804), who took a radical position in the philosophical debate originated by the Newton-Leibniz argument on the absolute or relational nature of space, refusing both solutions to propose a cognitive approach on the matter. In his most influential work, the *Critique of Pure Reason*, Kant argues that space and time are the "forms" of our sensibility, i.e. conditions of possibility of experience in general. His transcendental approach, however, cannot be comprised in the frame of classical idealism, as a *prima facie* interpretation of his solution could suggest. In truth, Kant engages in a long investigation on the cognitive processes that lead to the formation of the concepts of space and time where both *a priori* knowledge and experience play a decisive role. The result is that space and time depend on the structure of the human mind but we could never become conscious of them without a preliminary contact with the external world. The distinctive feature of Pinosio's interpretation is a clear analysis of the difference between space and time as forms of intuition and as formal intuitions. While the former is introduced in the Transcendental Aesthetic as the *a priori* condition to make empirical experience possible, before any elaboration of the understanding, the latter is described in the Transcendental Deduction[1] as the result of a process of synthesis which must involve the understanding. The tension between the two is solved by Pinosio in a clarifying exegesis that reveals how the two are complementary aspects of the same thing: the form of intuition is the passive form of receptivity that does not contain any determinate intuition (not even that of space and time), while the formal intuition is a unitary representation, accompanied by consciousness, produced by a process of synthesis. This distinction plays a key role in the formal model of cognitive space we propose.

The model seeks to capture the main features of space as expounded by Kant in (mainly the second edition of) the CPR, in particular in the TA and in the TD, and in his *Metaphysical Foundations of Natural Science*[2]. Moving from the observations and model of Kant's temporal continuum proposed by Pinosio, we provide an algebraic and topological formalization of space using a mereological approach. This perspective, which takes extended regions as primitives, is an alternative to the more classical point-based theories (as standard Cartesian geometries). This allows for an analysis which is more tied to perception of the world as we experience and process it. Alfred North Whitehead, who was one of the initiators of the region-based theories of space in the first decades of the twentieth century, argued, in his book *Concept of Nature* [40], that points are less basic than regions since they cannot be perceived directly, but they are the result of an abstraction. Kant's constructive treatment of the spatial continuum, in which points are only boundaries or limitations of regions, is close to the Whiteheadian construction of the continuum. Starting from a series of finite algebraic structures, representing the spatial form of possible experiences, we build an inverse system of topological spaces (dual to the algebras) as a formal correlate to the process of synthesis brought forth by the activity of the figurative synthesis. In our setting points are derived entities, representing the boundaries of regions in the inverse system, and can be constructed positing a natural relation of contact between regions. The topological counterpart of these relations on algebras are proximity relations. The final step in our formalization is to take the limit of the inverse system, together with a proximity relation inherited by the finite structures in the system. This is the formal correlate to space "as an object", i.e. the formal intuition obtained through the activity of the figurative synthesis. The quotient of the limit of the inverse system by the proximity relation gives us a continuum, in which we can recognize points defined as boundaries of regions.

During the construction of our model, we considered the notion of proximity relation and got struck by the similarity it bore with tolerance relations, introduced, over one

[1] Henceforth we will use the following abbreviations: CPR for the *Critique of Pure Reason*, TA for the Transcendental Aesthetic, TD for the Transcendental Deduction.

[2] Henceforth: MFNS.

hundred years after Kant's publication of the CPR, by the mathematician, physicist and philosopher Henri Poincaré. Starting from a completely different perspective, he sought a solution to the problem of determining the properties of perceptual space. His distinction between a sensible and a mathematical continuum (where the latter serves as a way of reasoning about the confused perceptions making up the former) is analogous, in spirit, to the activity of the figurative synthesis progressively structuring the manifold given in sensibility. Poincaré, however, well-aware of the existence of multiple possible mathematical descriptions of space (indeed, non-Euclidean geometries had been discovered at the beginning of the 19th century) maintains that there is no such thing as a "true" geometry of space , a point of view that became known as conventionalism. This position seems to be in strong contrast with the necessary *a priori* status of Euclidean geometry in Kant's system. Yet, we notice that Poincaré's conventionalism applies to theories of physical space, intended as the world of "things in themselves" to which Kant's transcendental idealism never sought to apply. We suggest that a form of reconciliation between the two philosophers can be attained, despite Kant's refusal of the possibility of conceiving non-Euclidean geometries and despite Poincarè's insistence in rejecting Kantian apriorism, especially in light of the interpretation suggested by Pinosio-van Lambalgen.

The present work is divided into two chapters. In the first chapter we introduce the philosophical groundwork, first trying to fathom the complex architecture of the mind expounded in the CPR, then presenting our analysis of Poincaré's philosophy of space. In the second chapter we produce the formal system, together with an account of the motivations and correlations between the cognitive system of the CPR and their mathematical counterparts.

1.2 Kant's philosophy of space

In the CPR Kant aims to explore the foundations and characterize the limits of human knowledge, establishing to which extent human reason can act independently from experience. He tackles the problem with a revolutionary approach, engaging in what he calls "transcendental idealism". The multitude of commentaries and exegetical works produced on Kant's First Critique prove how controversial his writings are and how hard it is to agree on an unambiguous interpretation of his philosophy. We are not going to dig into the vast literature surrounding his works (the interested reader can refer to [14] for an overview of the main interpretations and commentaries); instead we will focus on some passages of the TA and the TD that frame and motivate the current work.

One of the most prominent and controversial topics of the CPR is the discussion on space and time. Our analysis is based on the interpretation adopted by Pinosio [27], which is an extension and a refining of the work initiated by Achourioti and van Lambalgen in [1], where a first attempt to formalise Kant's transcendental logic is accomplished.

The innovative idea advanced by the above mentioned authors is that the semantics of Kant's logic does not pertain to classical logic, idea that led many commentators to claim Kantian logic is exceedingly reductive. This idea, first suggested by Longuenesse in her *Kant and the Capacity to Judge* [23] is made formal in [1], where a rigorous proof of the completeness of Kant's Table of Judgements for the semantics of geometric logic can be found. In the recent work by Pinosio [27] a detailed analysis of the distinction between formal intuition and form of intuition is provided to attain a clear interpretation which allows for a systematic account of the supposedly contradicting passages of the TA and the TD. The result is a surprisingly coherent reading, that clarifies the central role of the figurative synthesis as the fundamental process linking the way space and time are presented in the TA, as forms of outer and inner sense, to the full-fledged notions of space and time as objects, found in the TD.

Essential to the understanding of Kant's perspective is the context in which the First Critique was written, in particular the debate between Lebnizians' and Newtonians' conceptions of space and time, which motivated Kant to take a stand and conceive his very own approach to the matter, devising his famous theory of transcendental idealism. At the beginning of the TA Kant himself contrasts his own approach with the "absolutist" view, attributed to Newton, and the "relativist" view upheld by Leibniz' school. He asks:

> Now what are space and time? Are they actual entities? Are they only determinations or relations of things, yet ones that would pertain to them even if they were not intuited, or are they relations that only attach to the form of intuition alone, and thus to the subjective constitution of our mind, without which these predicates could not be ascribed to any thing at all? ([16] A23/B37-8).

In the above passage Kant introduces the dispute, but he refers to it throughout the Critique[3], and he claims that the clash between the two interpretations exposes a false dilemma: if, on the one hand, space and time cannot be the "eternal and infinite self-subsisting non-entities" needed by Newton "in order to contain all actuality within themselves", on the other hand they cannot merely be "relations between appearances"

[3]The following quotations refer to [16] A39-40/B56-7

that are completely "abstracted from experience" as the Leibnizians maintain. These accounts are too metaphysically involved, in that both of them assume the "absolute reality" of space and time, in the former case asserting they are "actual beings", in the latter taking them to be "determinations or even relations of things" [4]. In Kantian terms, the first point of view fails to capture the essence of space and time by assigning them the role of "conditions of all existence in general" [5] so that they would persist even without any object to be perceived in them, while Leibniz's account is confusing since it takes relations to be prior and detached from reality, missing the fact that, to be ordered, objects need first to be intuited.[6] A third way is not only possible, but necessary, following Kant: space and time are the pure forms of, respectively, outer and inner sense and are therefore subjective, in that they are "possible only insofar as the representational capacity of the subject is affected through [the existence of the object]". It is by means of sensibility (the faculty that gives us access to the manifold of appearances) that we can first acquire representations of the outer world, but these would be lacking any order if there were not *a priori* conditions structuring it spatiotemporally. It may seem contradictory to say that space and time are both dependent on experience and *a priori* conditions of its possibility, and it is here that Pinosio's analysis offers a truly insightful and coherent reading of some of the most controversial passages of the TD. As Pinosio correctly points out the notions of space and time appear in the CPR in different degrees of "formality", depending on the gradual involvement of the synthesis of the unity of apperception. In the following paragraphs I will summarize only the main passages of the interpretation: the interested reader is recommended a careful examination of chapter 3 of [27].

Unfolding the cognitive process making the manifold of appearances, passively acquired through the faculty of sensibility, into a proper cognition requires a careful handling of the language used by Kant. Indeed, his vocabulary has been the subject of a wide range of interpretations and discussions over the course of the past centuries (in particular after the development of contemporary cognitive science and logic) since the terms he uses are easily confused with their modern counterparts, despite, clearly, Kant could not have the background we now tend to accept as granted. So, let us start by clarifying some terminology, following the taxonomy given by Kant himself in the Transcendental Dialectic.[7]

A *representation* is a *determinatio mentis* referring to something external to the per-

[4] [16] A23/B37-8

[5] [16] B71

[6] For an extensive treatment of the contrast between Kant's view and the two mentioned positions see [10].

[7] [16] A320/B376-7

ceiving subject [8], and it is the root from which any possible cognition stems. Note that Kant never speaks of cognizing objects "in themselves" and he makes clear that we can have no cognition about the *noumenon*[9]: the passive manifold given to us through sensibility is that of appearances, which are mere representations, with no order to them, nor properties specific to objects in themselves. A *perception* is a representation with consciousness and if it affects the state of the subject it is a *sensation*. Finally, a *cognition* is an objective perception and as such it belongs to the faculty of the understanding. It is expedient here to make a quick detour on the different roles of the two faculties just mentioned. While *sensibility* is a passive faculty of receptivity, which acquires appearances as representations without consciousness, immediately related to the objects; the *understanding* is a faculty of thought and of rules (particular ones being concepts and judgements). This latter faculty relates to all objects by means of a synthesis in intuition and it can make connections among appearances even without objects. The link between the two faculties, the key making the understanding determining sensibility, to order the manifold of appearances in a way that it can be subsumed by concepts, is the *imagination*, a faculty whose activity is rooted in sensibility, but demands a spontaneous act of the understanding. It is by means of the imagination that our cognition proceeds from merely subjective impressions to objective knowledge, the source of which resides necessary in the understanding.

The dichotomy between understanding and sensibility, brings us to the fundamental distinction between *concepts* and *intuitions*. Both are forms of cognitions, but the former are universal and can be predicated of other representations, while the latter are singular and immediately related to the objects they represent. This is not the place to dig further in the formation of concepts: let just be said that they refer to objects only through the mediation of a synthesis which combines a multiplicity of representations into one cognition. Concepts, in turn, work as rules by means of which the understanding bestows a synthetic unity on a manifold of representations. Intuitions, on the other hand, refer immediately to the objects they represent and are spatio-temporally located. They may or may not be subsumed by concepts: we shall soon see more details about this process.

Time and space are here referred to as forms of, respectively, inner and outer sense. This, for the moment, simply means that intuitions are temporally organized as successions of data and that they are located in space. We will later see how we become aware of the properties of these forms of intuition (unity, infinity, continuity, and so forth).

[8] [19] R1676, (16:76)

[9] As he wrote some years later in response to Eberhard, the Critique "posits this ground of the matter of sensory representations not once again in things, as objects of the senses, but in something super-sensible, which grounds the latter, and of which we can have no cognition." ([2] *On a Discovery*, 8:205)

The process that makes the merely passive manifold of representations into a perception is due to the activity of the *empirical synthesis*, whose function is to bring together different representations combining them into a whole. This synthesis, that Kant calls the *synthesis of apprehension* in the second edition of the CPR, comprises what in the first edition were treated as logically separated, but otherwise inseparable, aspects of it: the action of "scanning" the manifold, and what he previously named the "reproduction in imagination" which is a productive activity subject to the categories. While the first moment of this synthesis is *subjective* and depends on shifts of attentional focus (so it is also always successive, and needs to be temporally extended), the second moment, i.e. the moment of *comprehension* (containing the reproduction) is what guarantees *objectivity* to apprehension itself. An example will help elucidating this process: to get the representation of a house, for instance, one needs to put together the representations of windows, walls, roof, and so on (which, in turn, are complex representations we assume, for now, as already elaborated through the process we are going to illustrate below) and order them in a particular order, reproducing them as a *unity* in imagination. To do so, the synthesis of apprehension must act in accordance to some objective rules which cannot belong to passive receptivity alone. The ground for this objectivity resides in the he *a priori* counterpart to this synthesis: the *figurative synthesis* or *synthesis speciosa*. Since this is a most delicate passage, we need to slow a bit this introduction to reflect upon the motivation and questions at stake here.

Up to now, it is not clear why and how the categories (the "pure concepts of the understanding") can apply to appearances by the mere fact that they are given in space and time, which, as *forms of intuition*, constrain sensibility forcing appearances to be spatio-temporally determined. Moreover, up to now we have been dealing only with empirically given manifolds of appearances, not really making clear how space and time, which are supposed to be *a priori* intuitions, are originally acquired and come to be *formal intuitions* (i.e. conscious representations). One may think that, for Kant, space and time are just given as *a priori* conditions of all experience, but this is a recurrent misunderstanding that have led too many commentators to misinterpret the CPR to the point that many passages become blatantly contradictory. Pinosio-van Lambalgen interpretation seems to us the only reading that brings back coherence to the ground-breaking theory of cognition expounded in the CPR.

The purpose of the TD is to justify the necessary application of the concepts of understanding (the categories) to any possible manifold of representations, to achieve objective knowledge that can be shared by any understanding. Indeed, without a solid *a priori* ground no combination of appearances could ever take place and "we could never have *a priori* neither the representations of space nor time" ([16] A100). Kant argues that, without a "*unity of consciousness*" accompanying them, representations would be juxtaposed to one another in a chaotic plurality of consciousness: it would be contingent to find any regularity in the appearances as they are given to us, and even if we could make associations, they would be subjective and not necessarily applicable to all possible appearances of the same kind. This analytic unity, to be objective, needs to be grounded on an *a priori* principle: the *synthetic unity of apperception*, under which any manifold can be brought thanks to the *synthesis intellectualis* [10]. It is by means of this synthesis that two instances of consciousness can be identified and recognized to belong to a single consciousness. As Kant makes explicit:

> The **possibility of experience** is therefore that which gives all of our cognitions *a priori* objective reality. Now experience rests on the synthetic unity of appearances, i.e., on a synthesis according to concepts of the object of appearances in general, without which it would not fit together in any context in accordance with rules of a thoroughly connected (possible) consciousness, thus not into the transcendental and necessary unity of apperception.([16] A156/B195-6)

The objective rules mentioned in the passage are the *categories*, which, as logical functions of judgements, make the combination of the manifold in one consciousness possible, through the activity of the intellectual synthesis. There are some differences between the A and the B edition of the CPR regarding the faculties assigned to carry out this synthesis (in the first edition the role of the imagination is emphasised over that of the understanding, while in the second edition it seems that this latter faculty is the only one capable of combining appearances into a whole), but since the imagination is strongly tied both to understanding and to sensibility, the argument is essentially the same. We will stick to the B edition version: for a critical treatment of these issues, see, again, [27] Chapter 3.

The problem remains as to *how* the categories do apply to the manifold of appearances. A key role here is played by the above mentioned *figurative synthesis*, which brings the manifold of representations provided by sensibility under the rule of the categories. To achieve this, it is necessary to determine the sensibility *a priori* so that any *possible* manifold of appearances is given already in the right "shape" for our understanding to bring it under the categories. This is to say that any manifold is given as spatio-temporally determined. This is a critical passage where Pinosio - van Lambalgen's interpretation - in the wake of Longuenesse's [23] insight - enlightens some of the most difficult passages of the TD. Space and time as forms of intuition are not, by themselves, related to the categories, since they are forms of the purely passive receptivity of our

[10] [16] B151

sensibility. The fact that we are conscious of their properties, however, indicates that the understanding must affect the sensibility in agreement with the forms of inner and outer sense and with the categories. To do so, the understanding needs to first provide sensibility with *a priori* sensory impressions. As Kant puts it,

> [...]inner sense contains the mere **form** of intuition, but without combination of the manifold in it, and thus it does not yet contain any **determinate** intuition at all, which is possible through the consciousness of the determination of the manifold through the transcendental action of the imagination (synthetic influence of the understanding on the inner sense), which I have named the figurative synthesis. ([16] B154)

The use of bold characters here is decisive to distinguish between the two notions of space and time as *forms of intuition*, purely passive and not yet containing any combination of appearances (not even unitary representations of space and time), from the *formal intuitions* of space and time, which are determined and accompanied by consciousness of their properties of unity, infinity, continuity, and so forth. This consciousness is produced by the actions of the figurative synthesis, which determines sensibility *a priori* and thus also assures homogeneity between the pure concepts of the understanding - the categories - and space and time as formal intuitions - "*a priori* consciousness of the necessary form of *any* act of apprehension"[11].

But how exactly do we become conscious of the properties of space and time in the first place? If they "precede all concepts" [12], how can they be even synthesised by the understanding? And if they "cannot be perceived in themselves" [13], how can they be originally intuited? This is probably where the interpretation we are following comes to its most fruitful insights. The activity of the figurative synthesis, as we have seen, grounds the synthesis of apprehension, which is the way we combine and become conscious of the manifold of appearances. It is thorough this synthesis that our first experiences of the world must have been processed and combined to produce consciousness in the first place. Thus it is only by making ourselves the first object of apprehension that we originally become aware of our sensible self. It is important here to distinguish the transcendental *I think* - the *intellectual self*, the apprehending subject - from the empirical, spatiotemporally structured, self - the *object of apprehension* [14]. It is by means of the former that we can acquire consciousness of the latter: by means of "acts of self-affection" we "construct" time as the form of representation of our inner state, and

[11] [27] p. 41)
[12] [16] B161n
[13] Ibidem B207
[14] Ibidem B155

we "institute" space as the form of representation of our outer state [15]. To guarantee objectivity, these acts of self-affection must be *a priori* movements; the TD reads:

> We cannot think of a line without **drawing** it in thought, we cannot think of a circle without **describing** it, we cannot represent the three dimensions of space at all without **placing** three lines perpendicular to each other at the same point, and we cannot even represent time without, in **drawing** a straight line (which is to be the external figurative representation of time), attending merely to the action of the synthesis of the manifold through which we successively determine the inner sense [...]. The understanding therefore does not does not **find** some sort of combination of the manifold already in inner sense, but **produces** it, by **affecting** inner sense.

Here Kant refers to the *a priori* motion of the subject positing itself as the original object of apprehension. To become conscious of the properties of space and time, we need to draw *trajectories*, thus affecting both inner and outer sense: the modifications produced by this original act of description of a space determine inner and outer sense. Shifting the focus of our attention from the former to the latter and comprehending the manifold so produced, we become aware of the properties of our forms of intuition, which must hold for *any* manifold given in sensibility. It is crucial to notice that the figurative synthesis does not produce the formal intuitions of space and time by composing particular times and spaces, but it is in the *act* of describing particular spaces that consciousness of the properties of the forms of intuition arises. In his 1790 letter to Kiesewetter [16], Kant writes

> The consciousness of space, however, is actually a consciousness of the synthesis by means of which we construct it, or, if you like, whereby we construct or draw the concept of something that has been synthesized in conformity with this form of outer sense

Finally, it is through *motion*, and in particular by *constructing* shapes in pure intuition and comprehending them, that the figurative synthesis makes possible the production of the schemata - transcendental rules - of pure sensible concepts. These are the concepts used in geometry, like the concept of a line or the concept of triangle, and they need to be objective - thus grounded on *a priori* intuitions, since geometry itself, as all mathematics, is a synthetic *a priori* discipline. This means that the principles and

[15] [27] p. 45
[16] [17] p.335-6

theorems of geometry are non-contingent, non-empirical truths that are necessarily true for any understanding (in contrast with analytical judgements, which have to be true in any logically possible world)[17]. Clearly the task of drawing shapes and figures even without the presence of the object in intuition cannot be carried out by anything else than the transcendental synthesis of the imagination, and, in particular, the productive imagination is responsible for the construction of such pure sensible concepts in pure intuition.

Now that the fundamental role of the figurative synthesis has been clarified, we can focus on the origin of our synthetic *a priori* knowledge of the principles and theorems of geometry, i.e., construction of objects in pure intuition. Since the concepts corresponding to the objects of geometry are pure *a priori*, the only thing the productive imagination needs to draw them is a rule, in the form of a constructive procedure, which - importantly - does not need to be actually realized: the mere existence of a possible procedure to construct an object makes it suitable to be thought as a pure sensible concept and, as such, subject to synthetic *a priori* judgements. In this way the understanding abstracts from the particular object - individual schema - represented in imagination, forgetting about all the properties that are not derived from the construction itself, to reason about it and base on it judgements that are valid for any possible intuition falling under the same concept (A713/B741).

As we pointed out above, our consciousness of the properties of unity, infinity and continuity of space as an 'object' (the formal intuition) derives from the aforementioned construction of objects in pure intuition[18]: the possibility of iterating the procedure of bisection of a line is the ground for infinity and continuity (we will see this better in the section on the continuum) and unity of space as a representation is based on the unity of apprehension, which, in the case of *a priori* concepts, is the unity of the act of the figurative synthesis. We will soon explore these properties in more detail, let us first make a quick detour to the debate mentioned at the beginning of this section.

[17]It must be noted that Kant leaves open the possibility that there may exist sentient beings with different forms of space and time (and so with a different understanding) from ours. This observation will prove crucial later on, when we consider the discovery of non-Euclidean geometries and Poincaré's philosophy of space. Euclidean geometry, in Kant's system, is the set of necessary principles ruling the construction in intuition, given the particular constitution of *our* understanding.

[18]Here Pinosio makes an important remark on the fundamental role played by the categories in constraining the action of the figurative synthesis, which already contains in itself the agreement of the manifold brought under the unity of apperception with the pure concepts of the understanding (see [27] p. 42). In particular the category of quantity and the category of community are involved, respectively, in the formation of a totality out of a series of homogeneous elements and in the coordination of parts through relations of reciprocal causality.

1.2.1 Kant's position in the Newton/Leibniz debate

To go back to the initial dispute between Newtonians and Leibnizians, we can now understand what Kant means when he claims that

> Space is merely the form of outer intuition (formal intuition), but not a real object that can be outwardly intuited. Space, prior to all things determining (filling or bounding) it, or which, rather, give an **empirical intuition** as to its form, is, under the name of absolute space, nothing more than the mere possibility of external appearances. Thus empirical intuition is not put together out of appearances and space (out of perception and empty intuition). The one is not to the other a correlate of its synthesis, but rather it is only bound up with it in one and the same empirical intuition, as matter and its form. If one would posit one of these two elements outside the other (space outside all appearances), then from this there would arise all sort of empty determinations. E.g., the world's movement or rest in infinite empty space is a determination of the relation of the two to one another that can never be perceived, and is therefore the predicate of a mere thought-entity. ([16] A429/B457n)

From this passage, we get a glimpse of how Newton's idea has been, to a certain extent, absorbed in Kant's theory of space and time. Indeed, Kant's conception of space shifted from a rejection *in toto* of absolute space, in his early writings[19], to a reconsideration, which has some interesting consequences on his works in the critical period.

Albeit failing altogether to capture the metaphysical nature of space, both the Newtonians' and the Leibnizians' positions partially capture the status of space and time, the former as "conditions of possibility" (of appearances, not of objects in the outer world - so tied to our sensibility, not to the world itself), the latter as relations (again, among appearances) subject to rules that order the manifold *in an objective way*. A further analysis of Kant's reception of these two conceptions of space and time will prove useful to understand better some details about his own conception of space.

As for Newtonian absolute space[20], Kant does, in many occasions, underline that

[19] See, in particular, Kant's position in the debate in the *New Elucidation* (1755)and in his *Physical Monadology* (1756), where he seems to embrace Leibniz's idea of relational space, albeit grounding it on the mutual causal dependence of substances and taking monads to be space-filling without being divisible (an idea borrowed from Crusius - see [12] for a review of these positions).

[20] Following Friedman's analysis in [9] we will refer, throughout the following paragraph, to "Newtonian absolute space" without problematizing the fact that our modern interpretation of Newtonian physics

> space taken absolutely (simply by itself) alone cannot occur as something determining the existence of things, because it is not an object at all, but only the form of possible objects. ([16], A431/B459)

but he does not discard absolute space as a regulative idea "which is to serve as a rule for considering all motion therein merely as relative" [21]. To properly address this topic, it is crucial to observe that Kant's view of space is deeply tied to the conception of space born after the Copernican revolution, which unsettled the keystones of previous theories of space. If the earth is not the centre of the universe and a privileged frame of reference for all motion, we are left with an infinite and homogeneous space with no distinguished points. The only possible ground of differentiation that makes us able to orient in space must, as Kant maintains in his essay "What does it mean to orient oneself in thinking"[22], be subjective. The fact that we feel a difference between our left and right sides, for instance, makes us able to locate objects in a familiar dark room even without outer benchmarks. Kant contends that the same holds for orientation in thinking: when we reason about pure concepts, without a reference to real objects, we need to apply a principle given by pure reason - "a subjective ground for presupposing and assuming something which reason may not presume to know through objective grounds". In the case of space, this subjective ground is our own body, which, generating three perpendicular lines as directions, gives us the means to determine the position of any other object, relatively to ourselves, proceeding outward along those lines (see also [20]2:379). It is through motion in this arbitrary, self-centred frame of reference that we become conscious of the properties of space. Now, it is apparent that, from this standpoint, space and motion cannot be absolute, since the frame of reference can always be broadened to one with respect to which the subject and all the objects at rest are in a different state of motion. Still, in his treatment of motion - but also in many other *loci* of the MFNS - Kant makes use of a privileged frame of reference, defined at the limit as the "common center of gravity of all matter", which functions as a surrogate of Newtonian absolute space. Friedman analyses these issues in great detail in his exegesis of the MFNS ([9]) and it would be impossible here to dig further into his work without losing track: let just be noticed that absolute space can still play a role in Kantian philosophy as a limiting idea of reason, to model relative motion without drawing upon Galilean relativity.

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differs from Kant's in that we have the concept of classes of inertial frames, while Kant - who never explicitly acknowledged Galilean relativity - sought to construct a single frame of reference to which every relative motion could be referred (see [9], note 28 p.21 for an overview of the issue, which is then discussed in detail throughout the book). We will stick to the Kantian interpretation, to get an idea of the motivation that led him not to banish absolute space entirely.

[21] [18] 4:560. For a masterful treatment of these issues, see [9], in particular Chapter 1.

[22] [41], 8:135

With regard to Leibnizians, Kant seems more determined to contrast their fallacies, and he criticises their views in many passages of his vast *oeuvre*, possibly to detach his transcendental idealism from the idealism attributed to the monadist school. While Leibniz (the "great man" cited at [18] 4: 507) is appreciated by Kant for rejecting the "metaphysical" view and holding that "space belongs only to the appearance of outer things", the same does not apply to his advocates, who, on Kant's account, lack the ground of an *a priori* rule to warrant the validity of geometry and mathematics with respect to the outer world. A major misconception uphold by the "monadists" (term used by Kant to refer to Leibniz's disciples), concerns the nature of the continuum: while for Kant continuity of space follows from infinite divisibility of the construction of space in pure intuition, the disciples of Leibniz hold that space is filled by physical indivisible points (monads), and in this

> [...] they would not allow even the clearest mathematical proofs to count as insights into the constitution of space, insofar as it is in fact the formal condition of the possibility of all matter, but would rather regard these proofs only as inferences from abstract but arbitrary concepts which could not be related to real things. ([16] A439/B467)

If the physical points of which space is composed were not extended, they could never make up an extended space when combined. So points must be tiny extended regions, but since Leibinizians claim that points are indivisible, they run contrary to the infinite divisibility of physical space (otherwise a finite body would be composed by infinitely many extended regions, contradiction), thereby challenging the *a priori* validity of the principles of mathematics, and geometry in particular, in relation to the outer world.

1.2.2 Kant's spatial continuum

The above considerations lead us to a better understanding of the properties Kant ascribes to space "as an object" (the formal intuition). The original intuition grounding all spatial representations must, in fact, be an infinite, three-dimensional, continuous magnitude, such that every part of it can be constructed as the sum of homogeneous parts. Unity of space - or, better, its *unicity* - as we have seen, is grounded in the necessary unity of apperception and obtained through the transcendental synthesis of the imagination. In the following paragraphs we are going to examine the other properties more closely.

The problem of the ultimate composition of matter drew some attention in the philosophical debate between the seventeenth and the eighteenth centuries. Leibniz, in his relationalist, ideal account of space, posited an infinity of simple substances ("monads") making up spatially extended composite bodies. His idea was commented and revised by his followers. Wolff, for instance, held that space is relational and that extended bodies must be made up of simple, unexteded parts, but he thought that they must be finite in number. He was aware of the problem of obtaining an extended continuum from the composition of unextended simples, and he concluded that our perception must be confused. Newton, in his "substantivalist" perspective on space, had been an atomist too, maintaining that matter is made up of homogeneous simple parts, running counter to Decartes' claim that space, equated with matter, is infinitely divisible, as geometry (the science of extension, and so of matter) proves.[23]. Kant's early position[24] on the matter was that space must be composed of simple, indivisible monads filling it through their mutual causal interaction. He claimed to have solved the problem of obtaining a continuum from the finite composition of simple substances by asserting that these relations of reciprocal determination among simples are real external relations on which the continuous divisibility of the space of appearance is grounded.

In the critical period, however, as we have seen above, Kant abandoned a purely relational view of space to embrace the idea that space is mind-dependent and it is the "necessary representation" which "is the ground of all outer intuitions"[25], and so it cannot be constituted by relations among real substances, nor it can be constructed by the composition of simples. As Kant puts it:

> [...] one can only represent a single space, and if one speaks of many spaces, one understands by that only parts of one and the same space. And these parts cannot as it were precede the single all-encompassing space as its components [...], but rather are only thought in it.[...]
> Space is represented as a given infinite magnitude. [...] if there were not boundlessness in the progress of intuition, no concept of relations could bring with it a principle of their infinity. ([16] A25)

This *given* infinite magnitude is what Kant glosses, in his comments on essays by the mathematician Abraham Kästner in 1790[26] as *metaphysical space*, which grounds the

[23]Refer to Hatfield's essay ([12] pp.61-69) for an introduction to the debate

[24]In particular in his *Physical Monadology* (1756).

[25][16], A24/B39

[26]The comments to Kästner's work are clarifying on the two types of infinity, potential and actual. See, for instance:

> [...] the geometrician expressly grounds the possibility of this task of infinitely increasing space (of which there are many) on the original representation of a single, infinite space, as

potential infinity of geometric spaces. This metaphysical space is none other than our space as an object, but in the TA, where space is defined as a "given infinite magnitude", Kant is referring to the form of intuition and the formal intuition has not yet been introduced. The fact that Kant feels the need to specify this property of metaphysical space reflects his urgency to remark that space is a necessary *a priori* representation that *precedes* all empirical perceptions and that it cannot be obtained by a mere abstraction from particular experiences. By acquiring the consciousness of the unboundedness of the progress of construction in intuition we become aware of this property of space, but it is only by virtue of this property that we can conceive of this infinite iteration of constructions in the first place. It is apparent now why space as the formal intuition cannot be the mere sum of its parts, since it is *actually* infinite, while the process of extending a line in intuition is only *potentially* so.

The potential unboundedness of the process of construction in intuition, as an iterated division of a given space, is also fundamental to understand Kant's conception of continuity. His approach is inherently different from the modern Dedekind-Cantor theory of the continuum, which has been called a "point-set" account since it takes unextended points as primitives and relies on set theory and first-order logic to be rigorously formulated. Kant obviously lacked these tools and his treatment of space is region-based, while points emerge only as derived entities. The very idea that a magnitude is infinite insofar every magnitude of the same kind is only a part of it is mereological in nature, and it is at the basis of Kant's conception of continuum. A magnitude is continuous if it does not have simple parts. This, to be sure, does not coincide with infinite divisibility (which is a property shared with the Dedekind-Cantor continuum), but establishes an epistemological subordination of points to regions, as they can only be defined as limitations of the spaces they bound. Indeed, points supervene on regions in the process of successive division of a given space, which cannot be a mere multitude, composed by discrete parts, but is instead a *quantum continuum*, that can only be obtained as an abstraction from homogeneous parts:

> a singular representation, in which alone the possibility of all spaces, proceeding to infinity, is given. [...] it is just here that the Critique proves that space is not at all something objective, existing apart from us, but rather consists merely in the *pure form of the mode of sensible representation of the subject as an a priori intuition.* This is also in perfect agreement with [the fact] that mathematicians have to do only with an *infinito potentiali*, and that an *actu infinitum* (the metaphysical given) *non datur a parte rei, sed a parte cogitantis.* This latter mode of representation, however, is not for this reason invented and false. On the contrary, it absolutely underlies the infinitely progressing construction of geometrical concepts, and leads metaphysics to the *subjective* ground of the possibility of space, i.e., to its *ideality* ([3] p. 176)

> Spatium est quantum, sed non compositum. For space does not arise through the positing of its parts, but the parts are only possible through space; likewise with time. The parts may well be considered *abstrahendo a caeteris*, but cannot be conceived *removendo caetera*.([19], R4425, 17:541)

This conception of the continuum comes on the heels of a long tradition commencing with Aristotle. The Greek philosopher rejected the atomistic hypothesis that there exist extended simples from which space is composed. A continuum, for him, must be composed of extended regions and for two of them to be continuous it means that between them there is nothing of the same kind. The key notion is that of *contact* between adjacent things. Until two objects retain their own boundaries, they are not in contact. Only when the two boundaries are "fused" together, and they become a single entity, they can be called contiguous (or in contact). This strict condition bestows a strong form of indecomposability on the Aristotelian continuum: every time a continuous object is cut, two new entities are created - the boundaries of the two parts obtained. Another fundamental distinction between the contemporary conception of continua and Aristotle's view is about the notion of actual infinity, which he rejected in favour of a potential infinity achievable through iterated procedures. Continuity is, in this frame, obtained by a potential procedure of division of an extended whole, coming before its parts, which are themselves continua that can be further divided *ad infinitum*. The process is reversed from the one applied by Dedekind and Cantor: while the latter start from dimensionless points and derive an extended continua as a result, in Aristotle's approach the continuum is given and points supervene as (potential) limits of line segments.[27] This "top-down" approach to the continuum - which refutes the idea of starting from an actual infinity of points and instead takes regions as primitive, recovering points as defined as boundaries of subcontinua or, in other formulations, as sequences of nested regions - had a long trail of successors, including Russell, Whitehead, Brouwer and, of course, Kant.

Kant's conception of the continuum bears remarkable resemblance to Aristotle's theory. The following passage contains several points in common with it:

> The property of magnitudes on account of which no part of them is the smallest (no part is simple) is called their continuity. Space and time are *quanta continua*, because no part of them can be given except as enclosed between boundaries (points and instants), thus only in such a way that this part is again a space or time. Space therefore consists only of spaces, time of times. Points and instants are only boundaries, i.e. mere places of

[27] See [13] for a detailed comparison between the two approaches.

> their limitation; but places always presuppose those intuitions that limit or determine them, and from mere places, as components that could be given prior to space and time, neither space nor time can be composed. ([16] A169-70/B211)

The only difference with Aristotle's construction is that the Kantian boundaries are, although infinitely small, somehow extended and contact between two regions of space is not defined as a point belonging to both. Instead, the point of contact is external to both regions, in that it belongs to their common boundary, as Kant's example makes clear:

> If earth and moon were to be in contact with one another, the point of contact would still be a place where neither the earth nor the moon is, for the two are distanced from one another by the sum of their radii. Moreover, no part of either the earth or the moon would be found at the point of contact, for this point lies at the boundary of the two filled spaces, which constitutes no part of either the one or the other.([18] [4:513])

In this sense, the relation of contact and the concept of boundary are essential to "glue" together regions in a continuous fashion.

Objectivity of continuity, i.e. its pertinence to the outer world, has not been made explicit yet. To do so, of course, we need to look back at the activity of the figurative synthesis, and in particular to the process of construction in intuition. In fact, only the existence of a constructive procedure guarantees objectivity to an *a priori* apodictic geometrical truth, for it permits to exhibit it in intuition, making it a *real* property. As we mentioned above, continuity is obtained in virtue of the infinite divisibility of space. It is by dividing, by means of a constructive procedure (of bisection, for instance) a given region in intuition that we become conscious of the infinite iterability of such process. Our mind being finite, we will never be able to obtain an actual continuity (i.e. a process actually showing that there are no smallest parts of space), but we can become conscious of the *potential* continuity of the construction.

Note that the epistemological heterogeneity of points and regions, as the latter are only derived as limitations of the former, is often addressed by Kant as a fundamental factor of continuity. Consider, for example, his note: "All parts of space are in turn parts. The point is not a part, but a boundary. Continuity."[28] The fact that points do not have independent reality is a mereological concept that will be of primary importance

[28] 4756 (17:699)

in our construction, where continuity is achieved only after synthesising the boundaries emerging, through the activity of the figurative synthesis, as limitations of the parts of space.

One final remark must be made on the properties of space. We said Kant is convinced that space is three dimensional, but this property has a different kind of *a priori* necessity than the above properties[29]. Indeed, to become aware of this property, we need to refer to experience, as Kant himself makes clear:

> What is borrowed from experience always has only comparative universality, namely through induction. One would therefore only be able to say that as far as has been observed to date, no space has been found that has more than three dimensions.

This, however, does not preclude that space is necessarily three dimensional. Euclidean constructions of pure geometry, as Friedman argues extensively in his book *Kant and the Exact Sciences* [7] are necessary for us to be able to think and represent geometrical ideas and they secure objective reality to spatial concepts, providing objects corresponding to them and the constructive procedures to obtain them, and so being of crucial importance for the activity of the productive synthesis. Thus, the principles of Euclidean geometry have to be synthetic *a priori* and Euclidean space is the necessary and unique model of our sensible intuition. We shall address this issue again later on, when comparing Kant's point of view with the theory of space of another sharp thinker, who explored the nature of space a century later: Henri Poincaré.

[29]On this topic, see the detailed account offered by Kitcher in his essay *A priori* ([21]), in which he distinguishes two types of *a priori* knowledge, one based on *tacit* knowledge, and one already *explicit*. The former is absolutely independent from experience and it is not articulated to begin with. The subject is not conscious of its properties unless he performs a process of "disclosure". The latter is dependent on experience, in the sense that it "commences **with** experience, yet it does not on that account all arise **from** experience" ([16], B1) . While the properties of unity, infinity and continuity of space derive from the necessary processes underlying our cognition, and so can be deemed to belong to the class of tacit knowledge, the principles of geometry, which can be proved by construction in intuition are *a priori* knowledge of the second type.

1.3 Space in Poincaré's philosophy

Little more than 100 years after the publication of the CPR, the brilliant scientist and philosopher Henri Poincaré found himself reflecting upon space and time, addressing some of the issues that bothered Kant from a totally new point of view. A century's worth of research in the fields of mathematics, physics and philosophy brought about many new discoveries, and time was mature for preparing the ground for the famous foundational crisis that shook the pillars of mathematics. Discovered at the beginning of the 19th century, non-Euclidean geometries had been developed by mathematicians the likes of Gauss, Bolyai, Lobachevski and Riemann to prove that Euclid's postulates were not the only set of axioms from which a coherent theory of space could be deduced. We refer to the critical review offered by Torretti ([38], in particular chapter 4) for the exposition below.

Poincaré's epistemology was based on the presupposition that science has bare facts and their mutual relationships as object, and that there is no unique way to describe them. Therefore, he held that scientists must agree on conventions to decide which system could better capture different aspects of physical behaviours: his school of thought came to be known as *conventionalism*. Poincaré believed that the only possible criterion to choose a geometry modelling physical space is convenience, since experience could not give us precise instructions on which the "true" geometry of physical space should be.

He agreed with Kant in maintaining that Newton's absolute space could not exist as an independent "container" coming prior to material objects. Thanks to the discussion that flourished in the last decades of the 19th century, concerning the issues raised by Newton's definition of motion as change of position in absolute space, critical positions about the existence of such a container were proliferating at Poincaré's time, after having rested in a state of limbo for nearly 200 years. Carl Neumann raised the problem in 1870,[30] and the question was taken up by scientists of the calibre of Mach and Lange:[31] since absolute space cannot be directly perceived, how can we be sure of which bodies really move given that their relative motions are the same whether the space that contains them is at rest or in uniform linear motion? The introduction of inertial systems offered an elegant way to tackle the problem and include Newton's laws of motion in the new framework in a satisfactory way, although the problem of how to tell a circular motion from a uniform one still created some concern. Poincaré was well aware that absolute positions and movements of bodies cannot be observed and he was convinced that absolute space could not have a place in scientific observation.

[30] [25]

[31] See [38] (pp. 322-325) for an exposition of the debate.

Since we cannot be sure about the choice of a geometry for the description of physical space, discerning a straight trajectory from a curved one depended on the geometry adopted. In his book *Science and Hypothesis* he extensively argues for the impossibility of observing absolute movements and positions. For instance, he contends that

> Experiments only teach us the relations of bodies to one another. They do not and cannot give us the relations of bodies and space, nor the mutual relations of the different parts of space.[32]

Now, if experience is incapable of teaching us anything but these relations, why would their geometrical description need to put them in connection to an immaterial absolute space? Experience does not suggest this necessity and so we should get rid of the spook of such an elusive, artificial entity.

The above quote is linked to the discussion on the "law of relativity", which he presents as follows:

> the state of the bodies and their mutual distances at any moment will solely depend on the state of the same bodies and on their mutual distances at the initial moment, but will in no way depend on the absolute initial position of the system and of its absolute initial orientation[33]

This law, he claims, must be applied to the entire universe, but our experiments cannot say anything about the position and absolute orientation of such a system. Thus he revises the law, enunciating it as:

> The readings that we can make with our instruments at any given moment will depend only on the readings that we were able to make on the same instruments at the initial moment[34]

This statement, he holds, is "independent of all interpretation by experiments" and it is verifiable by systems described both by Euclidean and non-Euclidean geometries. This makes us incapable of picking one of them as the true description of reality.

Indeed, the spatial features of any object or event could be satisfactorily described by different systems of geometry, and he contends, against Kant, that

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[32] [32], p.79

[33] Ibidem, p.76

[34] Ibidem, p. 77

> *The geometrical axioms are therefore neither synthetic a priori intuitions nor experimental facts.* They are conventions. Our choice among all possible conventions is *guided* by experimental facts; but it remains *free*[35]

Poincaré's argument that *a priori* geometry is not possible rests on the fact that, otherwise, one could not conceive a (still consistent) system, where one of the axioms of the original geometry is negated, and non-Euclidean geometries could not exist. Since they actually do, Euclid's system (or any other consistent system) cannot describe the necessary structure of physical space[36]. But this argument hinges on a misconception of Kantian apriorism. Poincaré rejects the logical necessity of a unique possible description of the actual geometrical structure of the physical world (this is clear when he says that we could not decide if the world was not Euclidean, since our instruments would not be sensitive to the curvature of a hyperbolic geometry). However, in Kant's system, the necessity of geometry is not a logical necessity in the modern sense and geometry is not supposed to capture the structure of the world of things in themselves as the argument would suggest; instead, Euclidean geometry gives us the synthetic *a priori* principles to apply our concept of space to *our sensible experience*, as conditions under which alone a concept of extended magnitude is possible. One could well *conceive* of a different, consistent set of axioms, but lacking the original intuition "it would be a thought as far as its form is concerned, but without any object, and by its means no cognition of anything at all would be possible".[37]

To be sure, the failure of this argument does not imply that Poincaré would have embraced the Kantian *a priori* intuition of space as the form of outer sense. He actually denies, later on, that we have an immediate, non-empirical intuition of space, claiming that while geometry is a pure, exact science, our sense perceptions are imprecise and cannot be more than an "inspiration" for the construction of mathematically coherent theories, which can be then used to model experience. Thus there is no unique transcendental condition of our spatial intuition, depending on the necessary structure of our understanding, but only a series of mathematically consistent theories, which are equally apt to describe phenomena, provided they're rich enough for the purpose. These issues are connected to Poincaré's distinction between *geometrical space* and *sensible space* ("*espace représentatif*"). While the former is our description of the space by mathematical models, the latter is the "sensible space" of phenomena, which can function as a guide for our mathematical constructions, but is totally independent from the theories themselves. This "sensible space", in Kantian terms, can be seen as the manifold given in sensibility without constraints, but it is not, in Poincaré's perspec-

[35]Ibidem, p.50
[36][32], p.48
[37][16], B146)

tive, subject to a necessary process of synthesis under universal conditions valid for any understanding. Instead, convention and education are the basis for the way we classify phenomena, as he remarks by way of examples - recurrent in his writings - of possible alternative worlds, where the inhabitants have different education from ours and receive external impressions that lead them to develop "a geometry different from ours, and better adapted to their impressions"[38]. This would not prevent us from representing their world as Euclidean:

> As for us, whose education has been made by our actual world, if we were suddenly transported into this new world, we should have no difficulty in referring phenomena to our Euclidean space. Perhaps somebody may appear on the scene some day who will devote his life to it, and be able to represent to himself the fourth dimension. ([32], p. 51)

A suggestive example of such an Euclidean representation of an hyperbolic world is Poincaré's disk model. We present it briefly to give an idea of how an interaction of two different geometrical perspectives can look like. Poincare's disk is a model of two-dimensional hyperbolic geometry in which all the points of the hyperbolic plane are mapped to points in the open Euclidean unit disk. Recall that the hyperbolic geometry (or Bolyai-Lobachevskian geometry) is obtained from Euclidean geometry by replacing the fifth postulate (the famous Parallel Postulate, asserting that given any line l and any point P not on l, there exists exactly one line through P that is parallel to l) with the following statement: given any line l and any point P not in l, there are at least two distinct lines through P that are parallel to l. The points of the model are all the points inside the open disk. Lines (called geodesics) are arcs obtained by intersecting the unit disk with (Euclidean) circles that are perpendicular to the unit circle (included the limit case of straight lines passing through the centre of the circle). Recall that circle inversion is a simple (Euclidean) straightedge and compass construction that maps a point A to the point A', laying on the line through A and the centre of the circle, such that the distance from A' to the centre is the reciprocal of the distance from A to the centre. Given two points A and B, the circle passing through A, B and A', where A' is the inverse of A through the unit circle, is a geodesic. Thereby the first postulate of Euclidean geometry is satisfied. It can be proved that also the other three are valid. Distance between points of the model is defined in such a way that the shortest path from A to B is a geodesic.
Now, an Euclidean observer looking at an inhabitant of the disk moving, with constant velocity, toward the edge of the disk, would notice that her speed decreases the closer she get to the edge. In fact, from a viewpoint inside the disk distances get longer and

[38] [32], p.71

longer the closer one gets to the edge, since the whole hyperbolic plane is mapped to the disk, which, from the inside, is then unbounded and limitless, with the edge infinitely far away from the centre. All triangles inside the disk have angles that sum to less than 180 degrees. It must be noticed that the larger a triangle is, the closer the sum of its internal angles approximates 0 degrees (limit "triangle" with vertices ideal points on the edge of the disk), the smaller the triangle is, approaching a single point, the closer the sum of its internal angles approximates 180 degrees. Hyperbolic circles in the disk are Euclidean circles observed from the outside, but their centre, in general, does *not* coincide with the Euclidean centre.

This example shows what Poincaré meant when he said that we could never be sure of the "true" geometry of physical space. The disk, *locally*, behaves like an Euclidean space. So, it could well be that we live in a Lobachevskian space, but the triangles we deal with are too small for our instruments to perceive the curvature. Kant, actually, leaves open the possibility of thinking beings having different forms of space and time from ours[39], but he says that we cannot know whether their intuitions "are bound to the same conditions that limit our intuition and that are universally valid for us"[40]. Hence, we can think of a being that has a non Euclidean faculty of imagination, but we cannot intuit it, since, as far as our sensibility is concerned, there would be no possible object instantiating this merely logical possibility[41] One might wonder what Kant would have thought of Poincaré's example of the disk. There, a straightedge and compass construction made a non-Euclidean world accessible to our representative capacity. Would Kant have weakened the claim that we cannot know anything about non-Euclidean beings and instead contend that, as far as a possible representation in intuition of their movements and geometric constructions is possible, we can know something about them? We think so. We would still lack a proper intuition of a non-Euclidean perspective of the world, but we would at least be able to produce an ostensive construction of some geometrical properties in our intuition. This is the case of all Euclidean models of non-Euclidean geometries. Indeed, the only way to imagine a non-Euclidean geometry is either a purely logical axiomatization, or the construction of an Euclidean model for it, to picture it in

[39] See, for instance, A286/B342-3.

[40] [16], A27/B43

[41] Indeed, Kant remarks that:

> To think of an object and to cognize an object are thus not the same. For two components belong to cognition: first, the concept, through which an object is thought at all (the category), and second, the intuition, through which it is given; for if an intuition corresponding to the concept could not be given at all, then it would be a thought as far as its form is concerned, but without any object, and by its means no cognition of anything at all would be possible, since, as far as I would know, nothing would be given nor could be given to which my thought could be applied. ([16], B146)

an intuitive way. Poincaré seems aware of this fact. In fact, even if he forcefully remarks that the choice of a geometry to describe our sensible space is a matter of conventions, he maintains that, in practice, it is more convenient to choose Euclidean geometry over non-Euclidean axiom systems, and in the last years of his life he even comes to affirm that there exists a sort of "geometrical intuition" grounding our conception of the continuum and which is the origin of Hilbert's axioms of order. In his essay *Why space has three dimensions*, he claims that these axioms are "true intuitive propositions, relating to *analysis situs*"[42], the discipline later known as topology that focuses only on *qualitative* properties of figures. Poincaré, who was one of the founding fathers of this discipline, contends that *analysis situs* is the source of all truly geometrical properties. He writes :

> [...] it is to facilitate this [geometric] intuition that the geometer needs to draw figures or at least form a mental image of them. Now, if he minimizes the importance of the metric or projective properties of these figures, if he concentrates only on their purely qualitative properties, it is because herein only does geometric intuition truly play a role. ([31], p.26)

The echoes of the Kantian notion of geometric intuition, based on the exhibition of concepts in pure intuition, is clear. However, two observations must be made. The first is that, as Pinosio suggested in his master thesis, the universal properties that are derivable by construction in intuition, for Kant, are not only topological, but also some metric properties that follow from the definition or from the established topological properties of the figure(see [26], pp.34-35, for a description of such properties). The second consideration concerns the role of sensible experience, which, in Kant, is the trigger that enables us to unfold our *a priori* forms of intuition[43]. The kind of *a priori* described by Poincaré is completely innate and he claims that the intuition of Hilbert's axioms would be sufficient to build "a geometry in which figures will not be needed, and which could be understood by a man who possesses neither sight, nor touch, nor muscular senses, and which would be reduced to pure understanding". This directly contradicts the Kantian *dictum* that "all our cognition begins with experience" [44], since without the original motion of the subject in space, positing himself as the object of apprehension, no notion of space, time and geometry would be possible. Not to mention that a geometry without figures, is impossible for Kant. Poincaré, actually, adds that the man above mentioned would not understand why one would prefer Hilbert's axioms

[42] [31], p. 43

[43] According to Kitcher's analysis ([21], p.41), before we have sensory experience, we do not have any explicit knowledge. The principles of geometry are *a priori* in this very sense and cannot be cognized if the subject is not previously spatiotemporally structured, acquiring empirical consciousness of itself.

[44] [16], B1

to any other possible collection of axioms, and that they retain a special status to us because they are "true intuitive propositions" relating to the qualitative properties of space. In this, and in his conclusion about the necessity of the intuition of the continuum to ground the possibility of experience, he has almost Kantian views, and his (late) perspective can thus be reconciled, at least partially, with Kant's point of view. We now turn to Poincaré's treatment of the continuum, which has a great importance in his philosophy, but a completely different structure from the Kantian continuum.

1.3.1 Poincaré's spatial continuum

We mentioned above the distinction between sensible (or representative) space and geometrical space in Poincaré's philosophy. The tension between the two is crucial to understand his conception of the continuum. The spatial properties of sensible experience are, for him, completely detached from the properties of the mathematical structures we use to model it, although our representation is motivated by our actual experience of the sensible space. The constructions of pure geometry, however, can be refined to suit multiple different contexts, different from the one to which we are accustomed. In *Science and Hypothesis* Poincaré lists the properties of geometrical space: it is continuous, infinite, of three dimensions, homogeneous and isotropic (i.e., all points in it and all directions are geometrically equivalent). On the other hand, representative space is given to us by many sensible impression, of three different characters: visual, tactile and motor. The features of this space are disappointingly weak: it is not homogeneous, nor isotropic, and we are not sure about its three-dimensionality (Poincaré holds that it has, in principle, as many dimensions as we have nerve fibres). Indeed, our sensations are multiple and partial: every point of sensible space is an aggregate of simultaneous sensations and we need to disregard some of their differences if we want to isolate certain stimuli and be able to make some sense of it. Poincaré assumes that muscular sensations can be clearly separated from the rest. The aim of this analysis of sensible sensations, is to obtain a mathematical structure that can somehow capture the essential spatial properties of our experience. To do so, we must be able to ignore all the stimuli that do not carry a geometric character. *Analysis situs* provides a set of pertinent properties that can be identified even in perceptual space. Poincaré is strongly convinced that if, on the one hand, we cannot identify geometrical space with sensible space, we can still *reason* about it as if it had the properties ascribed to geometrical space.

In his construction of a continuum[45], Poincaré is guided by the above idea: he dis-

[45] In the following exposition we will refer to Poincaré's formulation of his theory in his 1912 essay

tinguishes between a *mathematical continuum*, characterised by the individuality and indivisibility of its ultimate components (the points), and a *physical continuum*, which depends on our senses and is subject to Fechner's law. According to this principle there is a "threshold of consciousness" under which two stimuli cannot be differentiated. Fechner's law applies to the elements of a physical continuum (which are sets of sensations) in the following sense:

> There can be two sets of sensations which we can tell apart without being able to tell either one set or the other from a third set. ([31], p. 30)

That is, we can experience situations in which three sets of sensations A, B, C are such that

$$A = B, B = C, A < C$$

where the symbol "$=$" stands for a certain relation of indiscernibility, which is symmetric and reflexive, but not transitive. We will call this type of relation a *tolerance* relation. To provide an example, he takes three weights $A = 10$ grams, $B = 11$ grams, and $C = 12$ grams, and assumes our senses cannot perceive a difference less than 2 grams. A and B are indistinguishable up to tolerance, and the same holds of B and C, while A is distinct from C. Analogous examples can be found in vision or touch, where, for instance, two pin-points moved across our skin remain distinguishable until they come to affect the same neighbourhood of nerves. Through the combination of visual and tactile stimuli with motion (represented by sets of muscular sensations) Poincaré sought to construct a physical continuum and study its properties.

It would be out of the scope of this thesis to describe in similar detail the process through which Poincaré constructs the physical continuum and, through successive idealizations, makes it suitable to be subsumed by a mathematical continuum.[46] Nonetheless, there are some interesting aspects that should be taken into consideration. An important observation about the motivation that led Poincaré to his definition of physical continuum is contained in another of his late essays, *Space and time*:

> [Our intuitive idea of space] is reduced to a constant association between certain sensations and certain movements. This is the same as saying that the members with which we make these movements also play the role, so to speak, of measuring instruments. These instruments, which are less precise than those of the scientist, are sufficient for everyday life, and it is with

([31]), but his theory was formulated and expounded in several previous works, in particular see [28] and [29].

[46] A thorough analysis of these issues can be found in [38], pp. 340-352

> these that the child, like primitive man, had measured space or, to be more correct, has constructed a space which fulfils the needs of his daily life. ([30] p.17)

Here Poincaré echoes the Kantian notion of original motion through which the subject, affecting its inner and outer sense, becomes conscious of the properties its forms of sensibility. The crucial role played by motion is of utmost significance for our analysis. Indeed, even if Poincaré repeatedly claims that he could have chosen a different abstraction (and thus generated a different type of continuum), the most natural way he found to make the physical continuum into a mathematical object, given the the way our senses perceive space, was to generate an Euclidean group of motions (representing a continuous group of displacements). These transformations are precisely the ones generating any possible construction in intuition in Kant's system.

A second, brief, observation that needs to be done is about the nature of Poincaré's continuum. Although the properties of the mathematical continuum are precisely the ones Kant ascribes to his spatial continuum, there is an important difference in the construction. As we have seen, Kant's conception of the continuum is part of the tradition initiated by Aristotle, Poincaré's continuum is instead part of a modern conception of continuum, depending on the new mathematical tools that permitted, at the end of the 19th century, to deal with *actual* infinities. The primitive entities in this Cantorean continuum are dimensionless points and the construction is "bottom-up", starting from an actually infinite set of points to obtain an extended whole. Thus, in this case, the emergent property is extendedness. This already impairs the possibility that our model could capture the idea of spatial continuum Poincaré had in mind. Still, since we proved continuity of our model by providing an homeomorphism with the real line, there was a hope that the relation of proximity we introduced could be a formal correlate to tolerance, making a possible sensible continuum into the mathematical continuum Poincaré devised. Indeed, the ordered field $\mathbb{R}$ with the standard topology is the prototype of the mathematical continuum. As for the number of dimensions, Poincaré claimed that we have the intuition of a continuum of n dimensions, and repeatedly held that the choice of a three-dimensional space was purely conventional. The sensible continuum, as we have mentioned, has as many dimensions as we possess nerve-fibres. However, he believed that experience suggest us to choose a three-dimensional continuum. To support this idea he devised a convoluted proof relying on the combination of motion with visual and tactile stimuli, whose core intuition was the fact that we can always distinguish points in one, two and three dimensional spaces, but every time we try to add a source of additional dimensionality, the space generated is isomorphic to the 3-dimensional case.

This result has been proved in 1986[47] in one of the very few studies about tolerance relations. The theorem states that spaces with dimension number higher than three are locally "isomorphic" up to tolerance with the three-dimensional space, while this result does not hold for spaces of lower dimension. The reach of this result is great, if one considers that tolerance relations are defined to capture the concept of "perceptual indiscernibility" and that they may really represent a formal correlate to human sensory perception.

Poincaré was the first to talk about dimension as a topological property as opposed to a geometrical one. The key difference is that a topological space has no projective or metric properties (it is "amorphous"), which means that figures are not classified by their magnitude or form, but only by those properties that are "inherently spatial", as Poincaré would have said. By this, intuitively, we mean that however stretched or twisted, a figure will be recognized as equivalent to the starting one. All deformations are allowed that can preserve properties such as the presence (or absence) of holes inside the figure, the closedness (or openness) of lines and surfaces, the intersection of lines. These deformations are called "continuous functions". The notion of dimension rests upon the idea of cut. Poincaré was motivated in his definition by the fact that to divide a line into two separated parts it is enough to "cut" it in one point; to obtain two parts of a plane it is sufficient to "cut" it with a line; and a space is "cut" in two by a surface. An analogous notion was suggested by Euclid when he said that the boundaries of lines are points, the boundaries of surfaces are lines and the boundaries of bodies are surfaces. Cuts are seen as points (lines, surfaces) "through which we shall not pass": in topological terms a cut disconnects a connected continuum obtaining two subcontinua. Reasoning along these lines, Poincaré defines the topological notion of dimension of a continuum. A continuum has dimension number n if "it is possible to divide it into many regions by means of one or more cuts which are themselves continua of $n-1$ dimensions"[48]. This definition corresponds to the analytic idea that a continuum of n dimensions must have n coordinates, i.e. n independently variable quantities satisfying certain inequalities.

As we mentioned above, at the end of his essay, he admitted that there exists a kind of spatial intuition, but that it is not through it that we first come to the conclusion that space has three dimensions. The fundamental, truly *a priori* intuition, he believes, is that of a continuum of n dimensions and the fact that we reason better with a three-dimensional continuum depends on the habit we have to deal with the world provided

[47] See Sossinsky's paper for the definition of "toleomorphism" and for the proof that the 3-cube is toleomorphic to the n-cube for $n \geqslant 3$, and is not toleomorphic to the point, the line segment and the square ([36] p.152).

[48] [31], p.29

by our senses. His conclusion is strikingly Kantian:

> I shall conclude that there is in all of us an intuitive notion of the continuum of any number of dimensions whatever because we possess the capacity to construct a physical and mathematical continuum; and that this capacity exists in us before any experience because, *without it experience properly speaking would be impossible* and would be reduced to brute sensations, unsuitable for any organization; and because this intuition is merely the *awareness* that we possess this faculty. And yet this faculty could be used in different ways; it could enable us to construct a space of four just as well as a space of three dimensions. It is the exterior world, *it is experience which induces us to make use of it* in one sense rather than in the other. [49]

Here we can certainly draw a parallel: the intuition of the continuum (although constructed differently) is, for Poincaré as for Kant, a fundamental presupposition for the possibility of experience. The awareness of the intuition of the continuum is analogous to the consciousness raised by the activity of the figurative synthesis. The main difference lies in the fact that the intuition of the continuum, for Poincaré, is n-dimensional, while this would be plainly impossible for Kant. However, Poincarè claims that we have a purely "mathematical" intuition of the n-dimensional continuum, while experience (contingently) teaches us to reason in three dimensions. Recalling that the motivation that led Kant to declare that space is three-dimensional was "borrowed from experience", we can see how the two positions only diverge in the way they refer to experience, which, for Kant, was necessarily bounded by the structure of our mind.

In conclusion, looking at the development of Poincaré's philosophy of space, we can see how decisive a role Kant's theory of cognitive space played in shaping it. Despite his initial refusal to ascribe any determined geometrical structure to space, Poincaré admitted that, when it comes to sensory experience, Euclidean geometry is unexcelled in its descriptive power. This does not make it the necessary set of rules our imagination uses to construct space, but even when he describes possible beings with a different "education", Poincaré refers to an Euclidean model to picture the reality of an hyperbolic universe. True, he does not accept that there is a unique possible geometry to describe physical space, but Kant never argued we could know anything about the true structure of the world of things in themselves. Poincaré's conventionalism, in our opinion, has a different scope from Kant's theory of space. When Poincaré says that we *reason* about space in terms of a three-dimensional mathematical continuum, a possible parallel with the work of the figurative synthesis can be drawn. Noticeably, the fact that for

[49] [31], p. 44, emphases added

Poincaré this is a contingent fact, while for Kant space is necessarily Euclidean makes the two approaches clearly diverge. But the process of subsequent abstraction that lead us to the formation of a mathematical continuum in Poincaré's philosophy are very much akin, in essence, to the activity of the synthesis of the unity of apperception that gradually structures the passive manifold given in sensibility. The project of the CPR is not diminished in light of the new discoveries of the 19th century, rather it laid a solid base on which Poincaré's considerations could build to prepare the ground for the revolutionary developments brought forth by Einstein a few years later[50].

[50] A compelling theory of the role of the ("relativized") Kantian *a priori* through the successive stages of the evolution of physics in the 19th and 20th centuries can be found in [8]

Chapter 2

A formal model of Kant's spatial continuum

In the present chapter we propose a model for Kant's spatial continuum. In the first section we introduce the mathematical tools used, then we proceed to the construction of the model. Finally we provide a philosophical justification, linking the formal aspects to the cognitive processes expounded in the previous chapter.

2.1 Mathematical Preliminaries

In this section we present some definitions and results that will be of use to build the formal system. These have been separated from the body of the construction to simplify the reading. We assume the reader to be familiar with basic notions from order theory and topology.

2.1.1 Order theory and Boolean algebras

A *partially ordered set* or *poset* is a structure $(P, \leqslant)$, where P is a set and $\leqslant$ is a partial order, i.e. a reflexive, transitive and antisymmetric relation. Given $A \subseteq P$, sup$A$ and inf$A$ denote, respectively, the *supremum* of A and the *infimum* of A, if they exists. We call, respectively, *join* and *meet* of $a, b \in P$, $a \vee b :=$sup$\{a, b\}$ and $a \wedge b :=$inf$\{a, b\}$ (if they exist).

A poset $(P, \leqslant)$ is called a *lattice* if, for all $a, b \in P$, their meet and join exist.

To the above order-theoretic presentation of a lattice corresponds an equivalent algebraic definition:

Definition 2.1.1 (Lattice). A *lattice* $\mathbb{L} = (L, \wedge, \vee)$ is a structure where L is a nonempty set and $\vee, \wedge : L^2 \to L$ are binary operations satisfying, for all $a, b, c \in L$:

- Commutativity laws: $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$;
- Associativity laws: $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$;
- Idempotence laws: $a \vee a = a$ and $a \wedge a = a$
- Absorption laws: $a = a \vee (a \wedge b)$ and $a = a \wedge (a \vee b)$

A lattice $\mathbb{L} = (L, \wedge, \vee)$ can be made into a poset defining a partial order on it as follows: $a \leqslant b$ if $a = a \wedge b$, or equivalently, if $b = a \vee b$, for all $a, b \in L$.

A *bounded lattice* is a lattice which has a greatest element (or *top*, denoted by $\top$ or 1) and a least element (or *bottom*, denoted by $\bot$ or 0).

A bounded lattice is said to be *distributive* if, for all $a, b, c \in L$: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Definition 2.1.2 (Boolean algebra). A *Boolean algebra*[1] is a structure $\mathbb{B} = (B, \vee, \wedge, \neg, 0, 1)$, where B is a set equipped with two binary operations $\vee, \wedge : B^2 \to B$, one unary operation $\neg : B \to B$ and two distinguished elements $0, 1 \in B$ such that:

- $(B, \vee, \wedge)$ is a distributive lattice;
- for all $a \in B$, $a \wedge 1 = a$ and $a \vee 0 = a$;
- for all $a \in B$, $a \vee \neg a = 1$ and $a \wedge \neg a = 0$.

We denote by $\mathbb{B}^+$ the set of nonzero elements of B equipped with the Boolean structure, i.e. with $\vee, \wedge, \neg, 1$ defined as above.

An *atom* of a BA $\mathbb{B}$ is an element $a \in \mathbb{B}$ which is different from 0 and, for every $b \in \mathbb{B}$, $b \leqslant a$ implies $b = a$ or $b = 0$.

The following characterisations of an atom $a \in \mathbb{B}$ are equivalent:

1. for all $b \in \mathbb{B}$, either $a \leqslant b$ or $a \wedge b = 0$, but not both;

[1] Henceforth BA will stand for Boolean algebra

2. for all $b \in \mathbb{B}$, either $a \leqslant b$ or $a \leqslant \neg b$, but not both;

3. $a \neq 0$ and if $a \leqslant b \vee c$, then $a \leqslant b$ or $a \leqslant c$.

A BA $\mathbb{B}$ is said to be *atomic* if for every non-zero element $b \in \mathbb{B}$ there is at least one atom a s.t. $a \leqslant b$. It can be shown that every finite BA is atomic. In an atomic algebra every element is the supremum of the atoms below it.

A *BA-homomorphism* is a map $f : \mathbb{A} \to \mathbb{B}$ between two BAs $\mathbb{A}$ and $\mathbb{B}$ preserving the Boolean structure, i.e., for every elements $a, b \in \mathbb{A}$ $f(a \vee b) = f(a) \vee f(b)$, $f(a \wedge b) = f(a) \wedge f(b)$, $f(\neg a) = \neg f(a)$, $f(0_{\mathbb{A}}) = 0_{\mathbb{B}}$, $f(1_{\mathbb{A}}) = 1_{\mathbb{B}}$. When it is clear from the context we will drop the subscripts distinguishing elements of different algebras (e.g. we will write 0 for $0_{\mathbb{B}}$).

A particular kind of BA, which will be used throughout our construction, is the following:

Definition 2.1.3 (**Free Boolean algebra**)**.** The *free Boolean algebra* $\mathbb{B}$ *generated by* E is a Boolean algebra $\mathbb{B}$ with a distinguished subset of elements $E = \{p_i | i \in I\} \subseteq \mathbb{B}$ such that $\bigwedge_{i \in I} \pm p_i \neq 0$ and every other element of $\mathbb{B}$ is a finite Boolean combination of the generators.

A central notion in the theory of BAs, that will play an important role in the next sections, is the following:

Definition 2.1.4 (**Ultrafilter**)**.** An *ultrafilter* is a proper subset U of a Boolean algebra $\mathbb{B}$ s.t.

(U1) $a \in U \,\&\, a \leqslant b \Rightarrow b \in U$;

(U2) $a \in U \,\&\, b \in U \Rightarrow a \wedge b \in U$;

(U3) for every $a \in \mathbb{B}$, $a \in U$ or $\neg a \in \mathbb{B}$.

A subset $F \neq 0$ of $\mathbb{B}$ satisfying (U1) and (U2) (with F in place of U) is called a *filter*.

Note that, under (U1) and (U2), (U3) is equivalent to:

(U3') $a \vee b \in U \Rightarrow a \in U$ or $b \in U$ (i.e. U is a *prime filter*).

(U3") for any other proper filter F (i.e. $F \subsetneq \mathbb{B}$ satisfying (U1) and (U2)), $U \subseteq F \Rightarrow U = F$ (i.e. U is a *maximal filter*)

Given an element $a \in B$ the set $\{b \in B | a \leqslant b\}$ is a filter and is called the *principal filter* generated by a.

We have already noticed the central importance of a relation of contact for our model. We recall here the notion of closure and present the formal definition of a contact relation on BAs, following the approach of Düntsch and Winter ([5]).

A P *closure* R' of a binary relation R on a set X is the smallest extension of R satisfying property P. Formally:

1. $R \subseteq R'$;
2. R' satisfies P;
3. for all R'' for which 1. and 2. hold, $R' \subseteq R''$.

The reflexive and symmetric closure of a relation always exist. We will make use of one more closure which we will call the *upward closure* of R, where the property of being upward closed, for an arbitrary binary relation S, is defined as: $((x, y) \in S \wedge x' \geqslant x \wedge y' \geqslant y) \to (x', y') \in S$.

Definition 2.1.5 (Contact relation). A binary relation C on a Boolean algebra $\mathbb{B}$ is called a *contact relation* if it satisfies:

(C0) $(\forall a) 0 \not{C} a$;

(C1) $(\forall a)[a \neq 0 \Rightarrow aCa]$

(C2) $(\forall a)(\forall b)[aCb \Rightarrow bCa]$

(C3) $(\forall a)(\forall b)(\forall c)[(aCb \text{ and } b \leqslant c) \Rightarrow aCc]$

(C4) $(\forall a)(\forall b)(\forall c)[(aC(b \vee c) \Rightarrow (aCb \text{ or } aCc)]$

where $a \not{C} b$ means it is not true that aCb, i.e. $aCb \Rightarrow \perp$.

2.1.2 Category Theory

Some basic notions from category theory will be helpful to understand the constructions we are going to use and their mutual relations. The following account is only aimed at introducing the concepts which are strictly necessary to our purposes.The interested reader can start from [4] to get a gentle introduction to the field, or refer to [24] for a complete overview.

A *category* is a structure consisting of *objects* and *arrows* (or *morphisms*) between them. Every arrow f has two associated objects, the domain $dom(f)$ and the codomain $cod(f)$; for every two arrows $f : A \to B$ and $g : B \to C$ there exists an arrow (called their composite) $g \circ f : A \to C$; and each object A has an identity arrow $1_A : A \to A$. For all arrows $f : A \to B$, $g : B \to C$, $h : C \to D$ the following identities hold:

- $h \circ (g \circ f) = (h \circ g) \circ f$
- $f \circ 1_A = f = 1_B \circ f$

The *opposite* (or *dual*) category $\mathcal{C}^{op}$ of the category $\mathcal{C}$ is a category with the same objects as $\mathcal{C}$ and reversed arrows: for every arrow $f : B \to A$ in $\mathcal{C}$ we have the corresponding arrow $f^* : A \to B$ in $\mathcal{C}^{op}$.

An important advantage of the categorical approach is that every statement in the language of category theory has a dual which is equivalent to it. The dual is obtained replacing *dom* for *cod*, *cod* for *dom* and $f \circ g$ for $g \circ f$. If we have a statement holding in a category $\mathcal{C}$, the dual statement holds automatically in the category $\mathcal{C}^{op}$.

Definition 2.1.6 (**Mono- and epimorphisms**)**.** Given a category $\mathcal{C}$, an arrow $f : A \to B$ is a *monomorphism* if, given any $g, h : C \to A$, $f \circ g = f \circ h$ implies $g = h$, where A, B, C are any objects in $\mathcal{C}$. $f : A \to B$ is an *epimorphism* if given $i, j : B \to D$, $i \circ f = j \circ f$ implies $i = j$ for any object A, B, C in $\mathcal{C}$.

A *functor* between two categories $\mathcal{C}$ and $\mathcal{D}$ is an arrow in the category of all categories, i.e. a mapping associating objects of $\mathcal{D}$ to objects of $\mathcal{C}$ and arrows of $\mathcal{D}$ to arrows of $\mathcal{C}$, preserving composition and identities.

A *contravariant* functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor of the form $F : \mathcal{C}^{op} \to \mathcal{D}$. This kind of functor reverses arrows, i.e. it takes $f : A \to B$ to $F(f) : F(B) \to F(A)$ and $F(g \circ f) = F(f) \circ F(g)$.

Boolean algebras form a category together with Boolean homomorphisms, which we will denote **BA**. We will soon see that its dual category is a special class of topological spaces: the category **Stone** of Stone spaces and continuous maps between them.

Direct and inverse systems

A *directed poset* $\mathbb{I} = (I, \leqslant)$ is a set I together with a binary relation $\leqslant$, such that:

- $\leqslant$ is a partial order;
- $\leqslant$ is directed, i.e. for any $i, j \in \mathbb{I}$, there exists some $k \in \mathbb{I}$ such that $i, j \leqslant k$.

Definition 2.1.7 (Direct system). Let $\mathcal{C}$ be a category. A *direct system* in $\mathcal{C}$ consists of an indexed family $\{A_i | i \in I\}$ of objects in $\mathcal{C}$ whose index set is a directed poset I, together with a set of morphisms $\{f_{ij} : A_i \to A_j\}$ for any $i, j \in I$ with $i \leqslant j$, satisfying the following properties:

- $f_{ii} : A_i \to A_i$ is the identity map;
- For any $i, j, k \in I$ such that $i \leqslant j \leqslant k$ we have $f_{jk} \circ f_{ij} = f_{ik}$.

We will use the notation $\{A_i, f_{ij}\}$ to denote a direct system.

The dual structure is then obtained reversing the arrows, as follows:

Definition 2.1.8 (Inverse system). Let $\mathcal{C}$ be a category. An *inverse system* in $\mathcal{C}$ consists of an indexed family $\{A_i | i \in I\}$ of objects in $\mathcal{C}$ whose index set is a directed poset I, together with a set of morphisms $\{h_{ji} : A_j \to A_i\}$ for any $i, j \in I$ with $i \leqslant j$, satisfying the following properties:

- $h_{ii} : A_i \to A_i$ is the identity map;
- For any $i, j, k \in I$ such that $i \leqslant j \leqslant k$ we have $h_{ji} \circ h_{kj} = h_{ki}$.

We will use the notation $\{A_i, h_{ji}\}$ to denote such an inverse system.

We now introduce the notion, central to our construction, of inverse limit of an inverse system:

Definition 2.1.9 (Inverse limit). Let $\{A_i, h_{ji}\}$ be an inverse system, indexed by I, in a category $\mathcal{C}$. The *inverse limit* of $\{A_i, h_{ji}\}$ consists of an object L in $\mathcal{C}$ together with morphisms $p_i : L \to A_i$, for every $i \in I$, such that the following conditions are satisfied:

- $h_{ji} \circ p_j = p_i$, for all $i, j \in I$ with $i \leqslant j$;
- If there exists any object X equipped with morphisms $g_i : X \to A_i$ for any $i \in I$ and such that $h_{ji} \circ g_j = g_i$, for all $i, j \in I$ with $i \leqslant j$, then there must exist a unique morphism $f : X \to L$ such that $p_i \circ f = g_i$ for all $i \in I$.

We will write $\varprojlim A_i = \{L, p_i\}$ to indicate the limit just defined. The arrows p_i are called *projections*.

The second property listed in the definition is called *universality* (or *universal mapping property* , UMP) of the limit and is a fundamental way to characterise objects in category theory as unique up to isomorphism.

A picture will help figure out the situation. The dashed lines represent arrows that are unique.

$$
\begin{tikzcd}
& X \arrow[d, dashed, "f"] \arrow[ddl, bend right, "g_i"'] \arrow[ddr, bend left, "g_j"] & \\
& L \arrow[dl, "p_i"'] \arrow[dr, "p_j"] & \\
A_i & & A_j \arrow[ll, "h_{ji}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& X & \\
& C \arrow[u, dashed] & \\
A_i \arrow[ur, "\iota_i"] \arrow[rr, "f_{ij}"'] \arrow[uur, bend left, "g_i"] & & A_j \arrow[ul, "\iota_j"'] \arrow[uul, bend right, "g_j"']
\end{tikzcd}
$$

Figure 2.1: Inverse limit of an inverse system and colimit of a direct system

We are also interested in the dual notion of limit, to represent what happens in the dual category:

Definition 2.1.10 (**Colimit of a direct system**)**.** Let $\{A_i, f_{ij}\}$ be a direct system, indexed by I, in a category $\mathcal{C}$. The *colimit* of $\{A_i, f_{ij}\}$ consists of an object C in $\mathcal{C}$ together with morphisms $\iota_i : A_i \to C$, for every $i \in I$, such that the following conditions are satisfied:

- $\iota_j \circ f_{ij} = \iota_i$, for all $i, j \in I$ with $i \leqslant j$;
- If there exists any object X equipped with morphisms $g_i : A_i \to X$, for any $i \in I$, such that $g_j \circ f_{ij} = g_i$, for all $i, j \in I$ with $i \leqslant j$, then there must exist a unique morphism $u : C \to X$ such that $u \circ \iota_i = g_i$ for all $i \in I$.

We will write $\varinjlim A_i = \{C, \iota_i\}$ to indicate the colimit just defined. The arrows ι_i are called *immersions* in the colimit.

2.1.3 Topology

In this section we provide the topological groundwork necessary to define the dual category to Boolean algebras (the category of Stone spaces), in addition to some properties and results useful for our construction.

A *topological space* $\mathbb{X} = (X, \tau)$ is a set X together with a collection of subsets $\tau \subseteq \mathcal{P}(X)$ such that $\emptyset, X \in \tau$ and τ is closed under finite intersections and arbitrary unions. A *subspace* $\mathbb{Y}$ of a space topological space $\mathbb{X}$ is a subset Y of X equipped with the topology $\tau_Y = \{Y \cap \mathcal{U} \,|\, \mathcal{U} \in \tau\}$. $\mathbb{Y}$ is itself a topological space and its topology is called the *subspace topology* induced by Y.

The sets in τ are called *open* in (X, τ), and the subsets of X which have an open complement are called *closed* in (X, τ). A set can be neither closed nor open; moreover there can be *clopen* sets, that is subsets of X that are both closed and open in τ.

The interior $Int(\mathcal{U})$ of a set $\mathcal{U} \subseteq \mathbb{X}$ is the union of all the open sets contained in $\mathcal{U}$, which is again an open set, its closure $Cl(\mathcal{U})$ is the intersection of all the closed sets containing $\mathcal{U}$.

Definition 2.1.11 (**Regular open set**). A set $\mathcal{U} \in \mathbb{X}$ that is the interior of its own closure, i.e.$\mathcal{U} = Int(Cl(\mathcal{U}))$, is called a *regular open* set.

A *basis* B for a topology τ is a set of open subsets of X such that every open set in τ is a union of elements of B. A *subbasis* $S \subseteq \tau$ is such that every open in τ can be written as a union of finite intersections of elements of S (i.e. finite intersections of elements in S form a basis for τ).

A map $f : \mathbb{X} \to \mathbb{Y}$ between two topological spaces (X, τ) and (Y, τ') is called *continuous* if the preimage $f^{-1}(\mathcal{U})$ of any open set $\mathcal{U} \in \tau'$ is open in τ.

In our construction we will be interested in topologies induced on sets by spaces mapped to them. The notion of final topology, in particular, will recur.

Definition 2.1.12 (**Final topology**). Given a set X and a family of topological spaces $\{Y_i, \tau_i\}_{i \in I}$ such that for each $i \in I$ there is a map $f_i : Y_i \to X$, the *final topology* τ_f on X with respect to the family $\{f_i\}_{i \in I}$ is the finest topology on X that makes these maps continuous, i.e. for every other topology τ making all the maps in the family continuous, $\tau \subseteq \tau_f$.

It can be shown that $\mathcal{U} \in \tau_f$ iff $f_i^{-1}(\mathcal{U}) \in \tau_i$ for each $i \in I$.

Topological spaces can be characterised by the way two distinct points or subsets of the space are distinguishable in terms of disjoint sets separating them. In particular, we are interested in the following separation condition:

A topological space $\mathbb{X} = (X, \tau)$ is said to be T_2, or *Hausdorff* if for any two distinct points $x, y \in X$ there exist two disjoint open sets $\mathcal{U}, \mathcal{V}$ such that $x \in \mathcal{U}$ and $y \in \mathcal{V}$.

Another way to characterise topological spaces is to distinguish them on the basis of how connected they are, where a topological space $\mathbb{X}$ is *connected* if it cannot be obtained as the union of two disjoint open sets. This is the case of $\mathbb{R}$ with the Euclidean topology.

The spaces we are going to consider are, on the other hand, not only disconnected (not connected), but *totally disconnected* (they have no non-trivial connected subsets) or even *extremally disconnected*, i.e. spaces in which the closure of every open set is open.

The last important topological property we are going to use is compactness. There are many ways to characterise this property, which is, intuitively, a generalisation of the notion of closed and bounded region in an Euclidean space. The most typical definition, however, is the following: a space $\mathbb{X}$ is *compact* if every open cover has a finite subcover. That is, if F is a family of open subsets of $\mathbb{X}$ and $X = \bigcup_{\mathcal{U}\in F}\mathcal{U}$, then there exists a finite $G \subseteq F$ such that $X = \bigcup_{\mathcal{U}\in G}\mathcal{U}$.

Since the limit of an inverse system of topological spaces is a subspace of the product of such spaces, we are going to need one last definition:

Definition 2.1.13 (Product space). The Cartesian product $X = \prod_{i\in I} X_i$ of the underlying sets of a family of spaces $\mathbb{X}_i$ indexed by I, together with the *product topology*, defined as the coarsest topology such that all the canonical projections $\pi_i : \mathbb{X} \to \mathbb{X}_i$ are continuous, is called the *product space* of the spaces $\mathbb{X}_i$. A basis for this topology is

$$\Big\{\prod_{i\in I}\mathcal{U}_i \,|\, \mathcal{U}_i \in \tau_i \text{ and } \mathcal{U}_i \neq \mathbb{X}_i \text{ for finitely many } i\Big\}$$

As anticipated in the previous section, to obtain a continuum of boundaries, we will quotient the limit of our inverse system. To define a quotient, we first need the notion of *equivalence relation* on a set S, which is a *reflexive, symmetric* and *transitive* binary relation on elements of S. We call *equivalence class* of an element a with respect to the equivalence relation $\equiv$ the set $\{x \in S | x \equiv a\}$. The quotient set $X/\equiv$ is the set of equivalence classes of elements of X, i.e. $(X/\equiv) = \{[x] : x \in X\}$.

Definition 2.1.14 (Quotient space). Let $\mathbb{X} = (X, \tau)$ be a topological space, and let $\equiv$ be an equivalence relation on X. The *quotient space* $\mathbb{X}/\equiv$ consists of the quotient set $X/\equiv$ equipped with the topology $\tau_\equiv$ where the open sets are the sets of equivalence classes whose unions are open sets in $\mathbb{X}$, i.e.

$$\mathcal{U} \in \tau_\equiv \text{ iff } \bigcup_{[x]\in\mathcal{U}} \in \tau$$

Given the quotient map $q : \mathbb{X} \to \mathbb{X}/\equiv$ s.t. $q(x) = [x]$ we can give an equivalent definition of $\tau_\equiv$ as the final topology on $X/\equiv$ with respect to q.

A subset Y of a set X is said to be *saturated* with respect to an equivalence relation $\equiv$ if it is a union of equivalence classes, i.e. if $\exists T \subset X : Y = \bigcup_{t\in T}[t]$, or, equivalently, if it is the preimage of some set under q, i.e. $\exists V \subset X/\equiv : Y = q^{-1}(V)$.

A topological space can be endowed with a binary relation between its subsets which resembles the contact relation on BAs:

Definition 2.1.15 (**Proximity relation**)**.** A binary relation $\delta \subseteq \mathbb{X} \times \mathbb{X}$ is called a *proximity relation* if it satisfies the following axioms:

(P0) $A\delta B \;\Rightarrow\; A, B \neq \emptyset$

(P1) $A \cap B \neq \emptyset \;\Rightarrow\; A\delta B$

(P2) $A\delta B \;\Rightarrow\; B\delta A$

(P3) $A\delta(B \cup C)$ iff $A\delta B$ or $A\delta C$

(P4) $A\not\delta B \;\Rightarrow\;$ there is a $C \subseteq X$ s.t. $A\not\delta C$ and $B\not\delta(X \setminus C)$

where $A\not\delta B$ stands for $(A, B) \notin \delta$.

This definition is the most used in many region-based theories of space, and is called Efremovič proximity ([6]). A weaker notion of proximity has been studied by Lodato ([22]). This weaker formulation only satisfies the first four axioms (P0-P3) and has been characterised to widen the range of topological spaces interested by relations of "nearness". Even if our context should be in the scope of the stronger formulation (being all our spaces compact Hausdorff), we will see in the construction that the particular finite spaces we use are not well suited to the standard definition of proximity as non-empty intersection of closures. Only the proximity defined on the limit is a full Efremovič proximity, while the proximity defined on the spaces in the inverse system only satisfies (P0-P3) (we will add a remark about this in due course).

Construction of the real numbers as Cauchy sequences of rational numbers

To link our structure to a known continuum, we will map elements of our limit to the real line, making a substantial use of the following construction. For the details of the following passages see [35], chapter 3.

The set $\mathbb{R}$ of real numbers can be defined as a completion of the set of rational numbers $\mathbb{Q}$, defined using Cauchy sequences in $\mathbb{Q}$, where:

Definition 2.1.16 (**Cauchy sequence**)**.** A *Cauchy sequence* (of rational numbers) is a sequence $(x_1, x_2, x_3, ...)$ such that for every $\epsilon \in \mathbb{Q}_{>0}$ there exists $N_\epsilon \in \mathbb{N}_{>0}$ such that, for all $m, n > N_\epsilon$, $|x_n - x_m| < \epsilon$.

It can be shown that every Cauchy sequence is bounded and that if (x_n) and (y_n) are Cauchy sequences, then also $(x_n + y_n)$ and $(x_n \cdot y_n)$ are Cauchy sequences. Thus we can define sums and products of Cauchy sequences componentwise:

$$(x_n) + (y_n) = (x_n + y_n)$$

$$(x_n) \cdot (y_n) = (x_n \cdot y_n)$$

The constant sequences $0 = (0, 0, ...)$ and $1 = (1, 1, ...)$ are the additive and multiplicative identities. Every Cauchy sequence (x_n) has an additive inverse $(-x_n)$, but not every nonzero Cauchy sequence has a multiplicative inverse (consider any sequence containing at least one occurrence of 0).

Now, we can impose an equivalence relation between Cauchy sequences by:

$$(x_n) \sim (y_n) \text{ iff } \lim_{n\to\infty} |x_n - y_n| = 0$$

This equivalence relation is compatible with the operations defined above, moreover every nonzero equivalence class of Cauchy sequences has a multiplicative inverse.

So we get that the set of equivalence classes of Cauchy sequences forms a field, namely what we will call the field of real numbers $\mathbb{R}$.

We can embed $\mathbb{Q}$ into $\mathbb{R}$ via the map which assigns to $x \in \mathbb{Q}$ the class $[(x, x, x, ..)] \in \mathbb{R}$. Moreover, we can define the standard order on $\mathbb{R}$: given $s, t \in \mathbb{R}$ we say that $s > t$ if $s - t \neq 0$ and $s - t = [(x_n)]$ for some Cauchy sequence s.t. for some N, $x_n > 0$ for all $n > N$. We obtain that $\mathbb{R}$ is an ordered field whose order extends that of $\mathbb{Q}$. Also, $\mathbb{Q}$ is *dense* in $\mathbb{R}$, i.e. for all $x \in \mathbb{R}$ and $\epsilon \in \mathbb{Q}$ there exists $r \in \mathbb{Q}$ s.t. $|x - r| < \epsilon$.

Definition 2.1.17. A sequence (x_n) *converges to the limit* l if for every $\epsilon \in \mathbb{Q}_{>0}$ there is an $N_\epsilon \in \mathbb{N}_{>0}$ for which $|x_n - l| < \epsilon$ for all $n \geqslant N_\epsilon$.

The limit of a convergent sequence is unique in $\mathbb{R}$

Lemma 2.1.1. *A Cauchy sequence of rational numbers (x_n) converges to $[(x_n)]$.*

Finally, $\mathbb{R}$ is *complete* in the sense that every Cauchy sequence of real numbers converges to a real number.

Another useful theorem which will be used in our construction is the following:

Theorem 1 (**Nested intervals Thm.**). *Let $([l_n, r_n])_{n\in\mathbb{N}}$ be a sequence of closed bounded nested intervals in $\mathbb{R}$, i.e. $[l_{n+1}, r_{n+1}] \subseteq [l_n, r_n]$ where $[l_n, r_n] \subset \mathbb{R}$ for all $n \in N$. Then there exists a real number r such that $r = \bigcap_{n\in\mathbb{N}} [l_n, r_n]$.*

2.1.4 Stone duality

Algebras and topological spaces are deeply connected in a compelling way: the renowned representation theorems establish isomorphisms between such structures, which in some cases result in full dualities between categories. In particular, we are interested in the duality between Boolean algebras and Stone spaces. We present here only the key passages of Stone's representation theorem. For a complete treatment of these topics and for the proofs of the main theorems see [15].

Definition 2.1.18. A *Stone space* is a compact totally disconnected Hausdorff space, or, equivalently, a compact Hausdorff space with a basis of clopen sets.

Let $\mathbb{B}$ be a BA and call $X_{\mathbb{B}}$ the set of all ultrafilters of $\mathbb{B}$. The mapping

$$\phi : \mathbb{B} \to \mathcal{P}(X_{\mathbb{B}})$$

$$b \mapsto \{U \in X_{\mathbb{B}} | b \in U\}$$

is an embedding of $\mathbb{B}$ into $\mathcal{P}(X_{\mathbb{B}})$.

If we endow $X_{\mathbb{B}}$ with the topology generated by the basis $\{\phi(b) | b \in \mathbb{B}\}$, we get a Stone space $\mathbb{X}_{\mathbb{B}}$ called the *Stone space dual to* $\mathbb{B}$. Every clopen set in this space is of the form $\phi(b)$ for some $b \in \mathbb{B}$, thus ϕ is surjective onto $Clop(\mathbb{X}_{\mathbb{B}})$. It turns out that $(Clop(\mathbb{X}_{\mathbb{B}}), \cup, \cap, \setminus, \emptyset, X)$ is a BA. Thus we obtained the first part of the duality.

Theorem 2 (**Stone**)**.** *Every Boolean algebra* $\mathbb{B}$ *is isomorphic to the Boolean algebra of clopen subsets* $Clop(\mathbb{X}_{\mathbb{B}})$ *of its dual space.*

On the other hand, given any Stone space $\mathbb{X}$, we can define its *dual algebra* of clopens, with the Boolean operations listed above and show that $\mathbb{X}$ is isomorphic to the dual space of this BA.

In fact, this is a full categorical duality:

Theorem 3. *There are two contravariant functors* **Spec** : **BA** $\to$ **Stone** *and* **Clop** : **Stone** $\to$ **BA***, the composite of which is is naturally isomorphic to the identity functor respectively on the category* **BA** *of Boolean algebras and Boolean homomorphisms and on the category* **Stone** *of Stone spaces and continuous maps between them.*

In particular, given a BA-homomorphism $f : \mathbb{A} \to \mathbb{B}$, the associated $\mathbf{Spec}(f) : \mathbb{X}_{\mathbb{B}} \to \mathbb{X}_{\mathbb{A}}$, defined as $\mathbf{Spec}(f)(x) = \{a \in \mathbb{A} | f(a) \in x\} = f^{-1}(x)$ is a continuous function.

Conversely, given a continuous function $h : \mathbb{X} \to \mathbb{Y}$, the associated morphism $\mathbf{Clop}(h) : Clop(\mathbb{X}) \to Clop(\mathbb{Y})$, defined by $\mathbf{Clop}(h)(Q) = h^{-1}(Q)$ for all $Q \in Clop(\mathbb{Y})$ Is a BA-homomorphism.

2.2 The construction

Direct system of Boolean algebras

To make the formal representation more compact, we separated the justification of the model from the formalisation. In what follows we will only add cursory comments about the connections with Kant's cognitive theory. The reader can refer to the next section for a complete overview of the construction.
The starting point of our model are finite Boolean algebras whose elements represent regions in space - the spatial extent of possible experiences. Indeed, following Kant's analysis, a finite subject can only have finite experiences, hence beginning with a given infinite set of regions would contravene Kant's insistence that we can never have the intuition of an actual infinity. As we will soon show, the elements of these algebras can be seen as regular open subsets of the real line: we will be guided by this intuition throughout our construction. From the merely passive form of outer sense, embodied in these finite BAs, we will move to the construction of a direct system, representing the action of the figurative synthesis, and we will see how the manifold gets gradually structured to finally yield a continuum.

Consider the family of Boolean algebras $\{\mathbb{B}_n\}_{n \in \mathbb{N}}$, where $\mathbb{B}_n$ is the BA freely generated by n propositional letters. Note that these algebras are atomic. To look at this in a formal way, let us first introduce some notation. Let $\mathbb{B}_n$ be a free Boolean algebra generated by the set $E_n = \{p_i | i \in \{1, ..., n\}\}$, and let $At(\mathbb{B}_n)$ be the set of its atoms. By a well-known theorem (see, for instance, Theorem 11.2 of [11], p. 81), we have, for each $p_i \in E_n$:

$$a \in At(\mathbb{B}_n) \text{ iff } a = \bigwedge_{i \leqslant n} \pm p_i$$

where $p_i \in E_n$, and $\pm p_i$ indicates either p_i or $\neg p_i$. Note that each p_i appears either positively (p_i) or negatively ($\neg p_i$), but not both, in each atom. Moreover, the representation is unique. Defining a *literal* to be an (either positive or negative) occurrence of $\pm p_i$, we have that atoms are given precisely by maximal conjunctions of literals.

To obtain a direct system, we equip this family of BAs with Boolean homomorphism

$$h_{nm} = h_{m-1} \circ ... \circ h_n : \mathbb{B}_n \to \mathbb{B}_m$$

given by composing the canonical embeddings

$$h_n : \mathbb{B}_n \hookrightarrow \mathbb{B}_{n+1}$$

$$b \mapsto b$$

where we are identifying the generators $p_1, ..., p_n \in E_n$ with their correspondents in E_{n+1}. So the h_n's are injective BA-homomorphisms, and hence monomorphisms in the category **BA**.

Remark. Notice that given $b \in \mathbb{B}_n$, $h_{nm}(b) = b \in \mathbb{B}_m$, where we are giving the same name to the elements of different Boolean algebras, since they correspond to the same Boolean combination of generators. Now, for every $n \in \mathbb{N}$, given a BA-homomorphism $f : \mathbb{B}_n \to \mathbb{A}$, where $\mathbb{A}$ is any BA, commuting via $g : \mathbb{B}_{n+1} \to \mathbb{A}$ with h_n, we have that f commutes also with h_{nm}. This can be obtained defining an extension of g to $\mathbb{B}_m$, which restricted to $\mathbb{B}_n$ (as subalgebra of $\mathbb{B}_m$) gives f, since the codomain of f and g must coincide.

The following observation is essential to construct the dual spaces of the $\mathbb{B}_n$'s:

Proposition 2.2.1. *The ultrafilters of $\mathbb{B}_n$ are all and only the principal filters generated by atoms of $\mathbb{B}_n$, i.e.*

$$Ult(\mathbb{B}_n) = \{\uparrow a_i | a_i \in At(\mathbb{B}_n)\}$$

Proof.

- (⇒) Let $a_i \in At(\mathbb{B}_n)$. (U1) and (U2) are satisfied since $\uparrow a_i$ is a filter. Moreover, since for every element b of B_n we have $b = \bigvee_{i \in I} a_i$ and $\neg b = \bigvee_{j \in J} a_j$, with $\{a_i\}_{i \in I} \cap \{a_j\}_{j \in J} = \emptyset$ and $\{a_i\}_{i \in I} \cup \{a_j\}_{j \in J} = At(\mathbb{B}_n)$, we have that (U3) is satisfied: if $a \notin \uparrow a_i$, then $\neg a \in \uparrow a_i$ and vice versa.

- (⇐) Let $U \in Ult(\mathbb{B}_n)$. Then $\uparrow a_i \subseteq U$ for some $a_i \in At(\mathbb{B}_n)$. In fact, for all $a \in \mathbb{B}_n$, by (U3), either a or $\neg a$ is in U; wlog, say $a \in U$. Then, being $a = \bigvee_{j \in J} a_j$ for some finite set of atoms J, by (U3'), at least one of the a_j is in U, say a_i, so, by (U1), $\uparrow a_i \subseteq U$.

  Now suppose $\uparrow a_i \subsetneq U$ Then there is at least one $a \in U$ s.t. $a \notin \uparrow a_i$. So, by (U2), $a \wedge a_i \in U$. But, since $a_i \in At(\mathbb{B}_n)$ and $a_i \nleq a$, we must have $a \wedge a_i = 0$. By (U1), this means that $U = \mathbb{B}_n$. But we assumed U to be proper. Contradiction. Thus, $U = \uparrow a_i$.

□

The family $\{\mathbb{B}_n\}_{n \in \mathbb{N}}$ together with the morphisms h_{nm} form a *direct system* of finite BAs.

Let us now inspect a construction that will be useful to study the diagram in the category of Stone spaces dual to the ongoing construction. A picture of this direct system of BAs together with its *colimit* in **BA** will help the reader following our explanation.

$$\begin{tikzcd}
 & \mathbb{A} & \\
 & \mathbb{B}_\omega \arrow[u, dashed, "f"'] & \\
\mathbb{B}_n \arrow[rr, "h_{nm}"] \arrow[ur, "e_n"] \arrow[uur, bend left, "g_n"] & & \mathbb{B}_m \arrow[ul, "e_m"'] \arrow[uul, bend right, "g_m"']
\end{tikzcd}$$

Figure 2.2: Colimit of the direct system of Boolean algebras

Claim. The colimit of the diagram formed by the finite free Boolean algebras $\mathbb{B}_n$ is the free Boolean algebra $\mathbb{B}_\omega$ generated by ω generators.

Proof. First, there are evident embeddings e_n of each BA $\mathbb{B}_n$ into $\mathbb{B}_\omega$. Indeed, let the set of generators of $\mathbb{B}_\omega$ be $E_\omega = \{q_1, ..., q_n, ...\}$; then, given any finite BA $\mathbb{B}_n$, we can map its generators to $\mathbb{B}_\omega$ by: $e_n(p_1) = q_1, ..., e_n(p_n) = q_n$. These injections make the wanted triangles commute: $e_n = e_m \circ h_{nm}$ by the above remark. So the first property of the colimit is satisfied.
To check that it has the UMP of the colimit, take any BA $\mathbb{A}$ and injections $g_n : \mathbb{B}_n \to \mathbb{A}$ such that the triangles commute with the h_{nm}'s. This means that $g_n(b) = g_m(b)$ for all $b \in B_n$ and all $m \geqslant n$. Following the diagram above, there is only one possible choice of a homomorphism f from $\mathbb{B}_\omega$ to $\mathbb{A}$ s.t. $g_n = f \circ e_n$, for all $n \in \mathbb{N}$. f is defined by $f(a) = g_n(b)$, for an arbitrary n such that $a \in e_n(\mathbb{B}_n)$ and where $b = e_n^{-1}(a)$. Notice that b does not depend on n. Given that e_n and g_n are BA-homomorphisms, it is easy to show that f is a BA-homomorphism too.

□

Inverse system of Stone spaces

The dual structures of the finite BAs in the category of Stone spaces form an inverse system, which represents the action of the figurative synthesis "running thorugh" and "holding together" the manifold of intuition. The limit of this system is an object encoding the structure of all these finite Stone spaces and retracting to all of them in a way that is consistent with the mappings between them. This limit represents the intuition produced by the consciousness of the activity of the figurative synthesis, through which we become aware of the necessary spatial properties of any possible experience.

Consider the family $\{\mathbb{X}_n\}_{n\in\mathbb{N}}$, where $\mathbb{X}_n$ is the Stone space dual to $\mathbb{B}_n$. We recall that $\mathbb{X}_n$ is the space of ultrafilters of $\mathbb{B}_n$, equipped with the topology with basis $\{\phi(b)|b \in \mathbb{B}\}$, where $\phi(b) = \{U \in \mathbb{X}_n \,|\, b \in U\}$.

Note that, being $\mathbb{X}_n$ finite, for each $n \in \mathbb{N}$, for every $x \in X_n$, the singleton $\{x\}$ is closed, every set is closed and so every set is open, i.e. every set is clopen. Thus, in our case,

$$\mathbb{B}_n \cong Clop(\mathbb{X}_n) = \mathcal{P}(\mathbb{X}_n)$$

so the topology on $\mathbb{X}_n$ is the discrete topology. Each $\mathbb{X}_n$ is easily seen to be a compact, Hausdorff, extremally disconnected space.

From now on, we will identify each $\mathbb{B}_n$ with the dual algebra of $\mathbb{X}_n$, namely the powerset algebra $Clop(\mathbb{X}_n)$. The morphisms will be identified as well: the map $Clop(\mathbb{X}_n) \to Clop(\mathbb{X}_{n+1})$ will be called h_n as the correspondent map between $\mathbb{B}_n$ and $\mathbb{B}_{n+1}$. The following observation illustrates the central role of atoms and their duals - isolated points (which in our case are all the points in $\mathbb{X}_n$).

Claim. Atoms of $\mathbb{B}_n$ correspond to singletons of $\mathbb{X}_n$.

Proof. Recall that an atom in the dual algebra of $\mathbb{X}_n$ is defined to be a non-empty $\mathcal{U} \in Clop(\mathbb{X}_n)$ such that, for all non-empty $\mathcal{V} \in Clop(\mathbb{X}_n)$, $\mathcal{V} \subseteq \mathcal{U} \Rightarrow \mathcal{V} = \mathcal{U}$. Suppose, for reductio, there exist two distinct $x, y \in \mathcal{U}$, and take $\mathcal{V}$ to be a clopen of $\mathbb{X}_n$ such that $x \in \mathcal{V}, y \notin \mathcal{V}$. $\mathcal{V}$ exists since $Clop(\mathbb{X}_n) = \mathcal{P}(\mathbb{X}_n)$. Then $\mathcal{U} \cap \mathcal{V}$ is a clopen, it is non-empty, and it is a proper subset of $\mathcal{U}$, contradiction. On the other hand, given a singleton $\{x\}$ in $\mathbb{X}_n$, for any non-empty $\mathcal{V} \in Clop(\mathbb{X}_n)$ s.t. $\mathcal{V} \subseteq \{x\}$ we have $\mathcal{V} = \{x\}$ and so $\{x\}$ is an atom of $Clop(\mathbb{X}_n) = \mathbb{B}_n$.

□

Note that this means that each atom $a = \bigwedge_{i\leqslant n} \pm p_i$ in $\mathbb{B}_n$ has a dual $x \in \mathbb{X}$ which is the only point lying in the intersection of all clopens of the form $\phi(\pm p_i)$. Indeed, for each $\pm p_i$ there must be an ultrafilter U of $\mathbb{B}_n$ containing both $\pm p_i$ and a. Since every ultrafilter is a principal filter, this must be the principal filter generated by a and so it is in the intersection of $\phi(\pm p_i)$. It is also the only one since if there was another one, it would contain another atom (being principal) and so also all $\pm p_j$'s above it. But since atoms are uniquely determined by the generators above them, one of these would be the complement of one of the $\pm p_i$'s. Contradiction.
Conversely, each point $x \in \mathbb{X}_n$ is dual to an atom, being x an ultrafilter of $\mathbb{B}_n$ and being each ultrafilter generated by an atom.

It should be clear now that each $\mathbb{X}_n$ has cardinality 2^n and that the point $x \in \mathbb{X}_n$ corresponding to the atom $a = \bigwedge_{i \leqslant n} \pm p_i$, belongs to all the clopens (i.e. all the subsets of X_n) of the form $\phi(p_i)$, and is the only point belonging to the intersection of all such clopens. Now, to obtain duals to the morphisms h_{nm}'s in **BA**, we would like to map x to the only $y \in \mathbb{X}_{n-1}$ belonging to the intersection of the clopens of the form $\phi(p_i)$, $i \leqslant (n-1)$. This way each $x \in \mathbb{X}_n$ is the image of exactly two points: the ones corresponding to the two atoms $a' = a \wedge p_{n+1}$ and $a'' = a \pm p_i \wedge \neg p_{n+1}$ of $\mathbb{B}_{n+1}$. Since **Spec** is a contravariant functor, we know that a morphism dual to $h_n : \mathbb{B}_n \to \mathbb{B}_{n+1}$ in the category *Stone* is a continuous function

$$\psi_n : \mathbb{X}_{n+1} \to \mathbb{X}_n$$

Also, being h_n a monomorphism, its dual ψ_n is an epimorphism. By duality on homomorphisms (see [11], p. 348), for every $h_n : \mathbb{B}_n \to \mathbb{B}_{n+1}$, we have that $\psi_n : \mathbb{X}_{n+1} \to \mathbb{X}_n$ is uniquely determined by:

$$\psi_n(x) \in Q \text{ iff } x \in h_n^{-1}(Q)$$

for all clopen sets Q of $\mathbb{X}_n$, and all $x \in \mathbb{X}_{n+1}$.
It follows that ψ_n defined by

$$\psi_n(x) \in Q \text{ iff } x \in Q \cap Clop(\mathbb{X}_{n+1})$$

for all $Q \in Clop(\mathbb{X}_n)$, will do.
These are trivially continuous maps (the domain is a discrete space); well-defined (imagine x had two images, then they must coincide); and surjective (every point in $\mathbb{X}_n$ is the image of exactly two points, as noticed above).

Composing such morphisms we obtain, for every $n < m$,

$$\psi_{nm} = \psi_n \circ ... \circ \psi_{m-1} : \mathbb{X}_m \to \mathbb{X}_n$$

Thus we finally have our inverse system of Stone spaces representing finite spatial experiences.

The inverse limit in **Stone** of such an inverse system exists, it is non empty, and is a closed subspace of the product of the $\mathbb{X}_n$'s ([34], p. 4).

It is well known (see, for instance, [11], p.432) that the product $\mathbb{P} = \prod_{n \in \mathbb{N}} \mathbb{X}_n$ with the product topology τ_π and the canonical projections $\pi_n : \mathbb{P} \to \mathbb{X}_n$ is the categorical product of the Stone spaces.

Now, the limit $\hat{\mathbb{X}}$ in **Stone** of the spaces $\mathbb{X}_n$'s is the subspace of $\mathbb{P}$ which has as underlying set

$$X = \{\hat{x} = (x_1, x_2, ..., x_n, ...) \in \mathbb{P} |\, x_n = \psi_{nm}(x_m)\}$$

endowed with canonical projections $\chi_n : \hat{\mathbb{X}} \to \mathbb{X}_n$, defined by $\chi_n(\hat{x}) = x_n$, and the subspace topology $\tau \subset \tau_\pi$. $\hat{x} = (x_1, x_2, ..., x_n, ...)$ is called a *thread* in the limit, to indicate that the x_i in it "behave well" with respect to the morphisms of the inverse system.
The situation is represented in the following diagram:

$$\begin{tikzcd}
& \mathbb{Y} \arrow[dl, dashed] \arrow[dr, dashed, "f"] \arrow[ddl, bend right, "\phi_n"'] \arrow[ddr, bend left, "\phi_m"] & \\
\hat{\mathbb{X}} \arrow[rr, "e"] \arrow[d, "\chi_n"'] \arrow[drr, "\chi_m" near start] & & \prod \mathbb{X}_n \arrow[dll, "\Pi_n" near start] \arrow[d, "\Pi_m"] \\
\mathbb{X}_n & & \mathbb{X}_m \arrow[ll, "\psi_{mn}"]
\end{tikzcd}$$

Figure 2.3: Inverse limit of the inverse system of Stone spaces

Clearly $\hat{X}$ is a subset of the product of the X_n's, and the embedding $e : \hat{\mathbb{X}} \to \mathbb{P}$ makes the triangles formed with the projections from the limit and the projections from the product commute: $\pi_n \circ e = \chi_n$ for all $n \in \mathbb{N}$.

Let us take a closer look at the topology τ. Recall that τ_π is the topology on the product $\mathbb{P}$ generated by the subbasis $\{\pi_n^{-1}(\mathcal{U}_n) |\, \mathcal{U}_n \in \tau_n\}$, i.e. every open set in $\mathbb{P}$ is a union of finite intersections of sets of the form $\pi_n^{-1}(\mathcal{U}_n)$, for $\mathcal{U}_n$ open in $\mathbb{X}_n$. Note that, in our specific case, a basis for τ_π is the set $\{\chi_n^{-1}(\{x\}) |\, x \in \mathbb{X}_n\}$, since each finite intersection of preimages of subsets of $\mathbb{X}_n$'s is a union of preimages of singletons in $\mathbb{X}_n$'s.
The subspace topology on $\hat{\mathbb{X}}$ is $\tau = \{\hat{X} \cap \mathcal{U} \mid \mathcal{U} \in \tau_\pi\}$, and, again, we have a basis of preimages of singletons: $\{\chi_n^{-1}(\{x\}) |\, x \in \mathbb{X}_n, n \in \mathbb{N}\}$.

Remark. Note that the form of a basic open is then determined by the the singleton of which it is the preimage. A basic open $\mathcal{U} = \chi_n^{-1}(\{x_n\})$ contains all the threads that have a fixed initial segment (determined by x_n and its images via ψ_i for $i \leqslant n$), and any final segment compatible with the morphisms ψ_i, for $i > n$.

So the general form of an open set in $\hat{\mathbb{X}}$ is

$$\bigcup_{j\in J}\Big(\bigcap_{i\in n_j}\chi_i^{-1}(x)\Big),\ x\in\mathbb{X}_i$$

The last thing we need to check to see that $\hat{\mathbb{X}}$ is indeed our limit is that it has the UMP of the limit. Recall the UMP of the product $\mathbb{P}$ of the $\mathbb{X}_n$'s:

> for every Stone space $\mathbb{Y}$ with projections $\phi_n : \mathbb{Y} \to \mathbb{X}_n$ for each $\mathbb{X}_n$, there exists a unique arrow (continuous mapping) f from $\mathbb{Y}$ to $\mathbb{P}$ such that $\pi_n \circ f = \phi_n$.

So, to prove the UMP of the limit, it is sufficient to show that:

Proposition 2.2.2. *Given a Stone space $\mathbb{Y}$ equipped with projections $\phi_n : \mathbb{Y} \to \mathbb{X}_n$ to each $\mathbb{X}_n$, such that $\psi_n \circ \phi_{n+1} = \phi_n$, the image of $f : \mathbb{Y} \to \mathbb{P}$, the unique function given by the UMP of the product, is contained in $\hat{X}$.*

Proof. We know, by the UMP of the product, that f exists and that it is unique. We want to show that $f(\mathbb{Y}) \subseteq \hat{\mathbb{X}}$. So, let $x = f(y)$ for some $y \in \mathbb{Y}$ and suppose,that $x \notin \hat{\mathbb{X}}$. Then, there exists $n \in \mathbb{N}$ such that $\psi_n(x_{n+1}) \neq x_n$, i.e. $\psi_n(\pi_{n+1}(x)) \neq \pi_n(x)$. Hence $\psi_n(\pi_{n+1}(f(y)) \neq \pi_n(f(y))$. But, by the UMP of the product, $\phi_m = \pi_m \circ f_m$ for all $m \in \mathbb{N}$, so we get $\psi_n(\phi_{n+1}(y)) \neq \phi_n$.

□

We are now in the condition to prove that the clopen sets of the limit are exactly the preimages of clopen sets in $\mathbb{X}_n$ along the projections $\chi_n : \hat{\mathbb{X}} \to \mathbb{X}_n$:

Proposition 2.2.3. $\mathcal{U} \in Clop(\hat{\mathbb{X}})$ iff $(\exists \mathcal{U} \in \mathbb{X}_n)(\chi_n^{-1}(\mathcal{U}_n)) = \mathcal{U}$

Proof. ($\Leftarrow$) Let $\mathcal{U}_n$ be a clopen subset of $\mathbb{X}_n$ (i.e. any subset of $\mathbb{X}_n$). Then, its preimage along χ_n is the set $\chi_n^{-1}(\mathcal{U}_n) = \{(x_1, ..., x_n, ...)|x_n \in \mathcal{U}_i\}$, which is open in τ. Now, the complement of this set is $\hat{\mathbb{X}} \setminus \chi_n^{-1}(\mathcal{U}_n) = \chi_n^{-1}(\mathbb{X}_n) \setminus \chi_n^{-1}(\mathcal{U}_n) = \chi_n^{-1}(\mathbb{X}_n \setminus \mathcal{U}_n)$. This is an open set in τ, since it is the preimage of an open set in $\mathbb{X}_n$. Thus, $\chi_n^{-1}(\mathcal{U}_n)$ is clopen in $\hat{\mathbb{X}}$.
($\Rightarrow$) Let $\mathcal{U}$ be a clopen set in $\hat{\mathbb{X}}$. Since it is open, it must be the union of clopen sets in the basis: $\mathcal{U} = \bigcup_{i\in I} \chi_i^{-1}(\{x\})$ for some $x \in \mathbb{X}_i$, $I \subseteq \mathbb{N}$. If this union is finite, we are done, since there must be a $n \in \mathbb{N}$ such that $x_i \in \mathbb{X}_n$ for all $i \in I$ and so $\{x_i | i \in I\} \subseteq \mathbb{X}_n$ is the image of $\mathcal{U}$ along χ_n. If I is infinite, $\mathcal{U}$ is a closed subset of a compact space,

covered by an infinite union of clopen subsets of $\hat{\mathbb{X}}$. Then there must be a finite set of opens in the basis of τ covering it, i.e. it is the union of finitely many clopen sets and we are back to the finite case.

□

Finally, note that the projections are trivially open and closed maps (the image of an open set is open and the image of a closed set is closed).

2.2.1 Representation in the unit interval

To picture how the clopen sets in the limit of this inverse system capture the idea of regions in space, we show how how to put it in correspondence with th unit cube. To simplify the process, we establish a mapping from $\hat{\mathbb{X}}$ to the unit interval $[0,1]$ with the standard topology, to be then extended to the unit cube. Note that, at the current stage, the two topologies prevent us from building a homeomorphisms, since $\hat{\mathbb{X}}$ is totally disconnected, while $[0,1]$ with the subspace topology inherited from $\mathbb{R}$ is connected.

We will see later that adding a contact relation to the finite Boolean algebras $\mathbb{B}_n$'s (then extended to a proximity on the discrete Stone spaces $\mathbb{X}_n$'s) will allow us to give more structure to the limit and to define a quotient homeomorphic to $[0,1]$. It is convenient to first have a look at the correspondence we would like to trace between threads in the limit and points on the unit interval, to fix some ideas and give a flavor of the geometrical intuition behind our construction.

We introduced earlier the construction of the line of real numbers using classes of Cauchy sequences. What we want to show is that every thread in the inverse limit is in correspondence with a class of Cauchy sequences converging on a real in $[0,1]$.

Recall that a thread is a tuple $\hat{x} = (x_1, ..., x_n, ...)$ such that $x_i \in X_i$ for all $i \in \mathbb{N}$ and $x_n = \psi(x_{n+1})$. Every $x_n \in X_n$ is an isolated point, so it corresponds univocally to an atom in the dual algebra $a_n = \bigwedge_{i \leqslant n} \pm p_i$ of $\mathbb{B}_n$. In the following we will always refer to $a_n, a_{n+1}, ...$ to denote the atoms corresponding to $x_n, x_{n+1}, ...$ in the way just described, and $b_n, b_{n+1}, ...$ to denote the atoms corresponding to $y_n, y_{n+1}, ...$, if necessary.

We would like to assign to every regular open (or clopen, since they coincide) set in $\mathbb{X}_n$ a region in space (or an interval in $[0,1]$, in our simplified example). To do so we start from singletons of points in $\mathbb{X}_n$, i.e. basic clopens of $\mathbb{X}_n$, and the region assigned to every other clopen subset of X_n can be computed from this assignment. Each x_n in a thread $\hat{x}$ is sent, through this assignment to the smallest region in $Clop(\mathbb{X}_n)$ containing x_n. To do so we partition the unit interval into subsets that correspond to atoms of

$\mathbb{B}_n$, i.e to maximal conjunctions of literals thereof. We will then map each thread in the limit to a real number, defined as the limit of a Cauchy sequence. Clearly, the more we climb the hierarchy of structures in the inverse system, following a thread $\hat{x}$, the smaller the area corresponding to the singleton $\{x_n\}$ becomes.

It must be noted that the construction of the reals via Cauchy sequences needs a much stronger notion of infinity that the one Kant had in mind, requiring first an *actual* infinity of points to be given. Therefore we use this method only as a useful tool to map our construction to the real line, keeping in mind that this cannot be a way to capture the Kantian continuum, but just a way to represent it in the modern topological setting.

So, let us define a map assigning to each clopen in $\mathbb{X}_n$ a regular open set in $[0, 1]$ with the Euclidean topology. Defining it on singletons will suffice for our purposes, but we must keep in mind that the regular open subsets of $[0, 1]$ are open intervals and interiors of closures of unions of open intervals. So, if we assign open intervals to our singleton sets, when we take subsets of X_n that are unions of singletons we must map them to the interior of the closure of the union of the images of the singletons composing them.

$$e_n : \mathcal{P}(X_n) \to RegOp([0, 1])$$
$$\{x\} \mapsto (l^x_n, r^x_n)$$

where (l^x_n, r^x_n) is recursively defined by $(l^x_0, r^x_0) = (0, 1)$ and

$$(l^x_{k+1}, r^x_{k+1}) = \begin{cases} \left(l^x_k, \frac{l^x_k + r^x_k}{2}\right) & \text{if } p_{k+1} \geqslant a \\ \left(\frac{l^x_k + r^x_k}{2}, r^x_k\right) & \text{if } \neg p_{k+1} \geqslant a \end{cases}$$

where a is the atom of $\mathbb{B}_n$ corresponding to x. A subset $\{x_1, ..., x_m\} \in \mathcal{P}(X_n)$ is mapped to $Int\big(Cl\big(\bigcup_{i \leqslant m}(l^x_m, r^x_n)\big)\big)$.

From this point forward we will drop the superscripts when no confusion can arise about the element of $\mathbb{X}_n$ to which a certain interval corresponds.

Notice that, for each n, we are dividing the unit interval into 2^n pieces, so that each of them corresponds univocally to an atom, and so to a singleton in $\mathcal{P}(X_n)$: if $a = \bigwedge_{i \leqslant n} \pm p_i$ contains a positive occurrence of p_1, $e_n(\{x\})$ is a region in $(0, 1/2)$, while if the sign of p_1 is negative $e_n(\{x\}) \subseteq (1/2, 1)$; if p_2 occurs positively, then $e_n(\{x\})$ is going to be in the left half of the interval previously chosen, while if p_2 occurs negatively, $e_n(\{x\})$ is going to be in the right half, and so on.

Since each atom is univocally determined as a maximal conjunction of literals, and since e_n partitions $[0, 1]$ into intervals corresponding to these maximal conjunctions, we have that e_n is injective for all $n \in \mathbb{N}$.

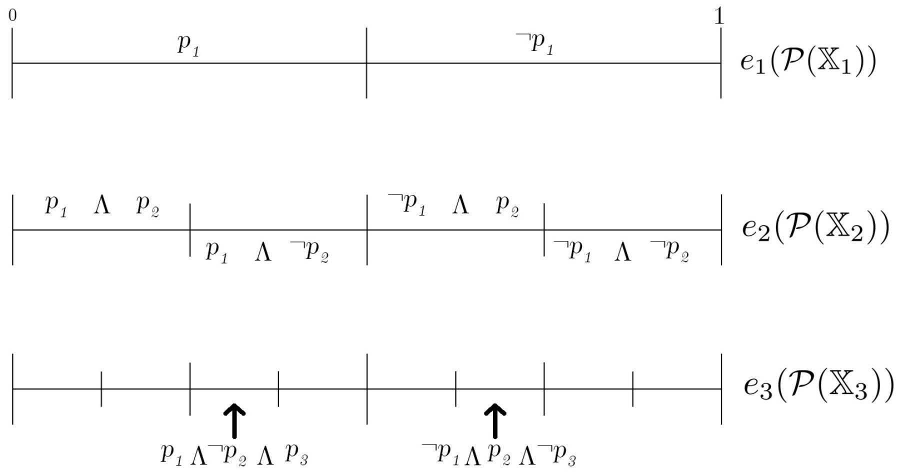


Figure 2.4: Representation of the $\mathbb{X}_n$'s in the unit interval

Consider now a thread $\hat{x} = (x_1, ...x_n, ...)$ in the limit. We want to assign to it a Cauchy sequence of rational numbers. To do so we can map each x_n in the thread to some representative of the interval $e_n(x_n)$. The two natural choices are the boundaries of our interval, l_n and r_n. Call l and r respectively, the two functions assigning to each interval its left and its right boundary, i.e. $l((l_n^{x_n}, r_n^{x_n})) = l_n^{x_n}$ and $r((l_n^{x_n}, r_n^{x_n})_{x_n}) = r_n^{x_n}$. Consider, then, for each component of the thread $l(e_n(\{x_n\}))$ and $r(e_n(\{x_n\}))$ and obtain two sequences (l_n) and (r_n). In this way we can associate each thread $\hat{x}$ with the class $[(l_n)] = [(r_n)]$.

What we need to show to get the desired correspondence between threads and real numbers is that the sequences of left and right boundaries of such intervals are Cauchy sequences of rational numbers converging to the same real number.
To see this, recall that our threads are sequences of points $x_n \in \mathbb{X}_n$ such that $x_n = \psi_n(x_{n+1})$. Thus we have that if a_n and a_{n+1} are the atoms corresponding to x_n and x_{n+1} respectively, then $\pm p_i \geqslant a_n$, implies $\pm p_i \geqslant a_{n+1}$, for any $i \leqslant n$. This implies that the intervals corresponding to $\{x_n\}$ and $\{x_{n+1}\}$ are nested, i.e. $l_n \leqslant l_{n+1}$ and $r_{n+1} \leqslant r_n$. To show that (l_n) and (r_n) are Cauchy sequences, it is sufficient to take, for any arbitrarily small ϵ, as N_ϵ the first positive integer larger than $-log_2\epsilon$ and both $|l_n - l_m| \leqslant \epsilon$ and $|r_n - r_m| \leqslant \epsilon$ for all $n, m \geqslant N_\epsilon$.
Being Cauchy, these two sequences are convergent in $\mathbb{R}$ (and thus in $[0, 1]$, since, for all n, l_n and r_n are both in $[0, 1]$, which is closed and so contains all its limit points). So, consider the limit of (l_n), say l, and suppose, for reductio, that (r_n) converges to $r \neq l$. But then $|r - l| > 0$, so take $\epsilon = \frac{|r-l|}{4}$. By definition of limit, we have that there exists an M such that for all $n \geqslant M$, $|l_n - l| \leqslant \epsilon$. Now, since $|r_n - l_n| = \frac{|r_{m+1} - l_{n+1}|}{2} = \frac{1}{2^n}$, and given that $\frac{1}{2^n} < \frac{|l-r|}{4}$ iff $n > \log_2\left(\frac{|l-r|}{4}\right)$, we have that for any $n > \max\{M, \log_2\left(\frac{|l-r|}{4}\right)\}$,

$r_n = l_n + |r_n - l_n| \leqslant l + 2\epsilon = \frac{l+r}{2} < r$, i.e. taken $\epsilon' > 2\epsilon$ there is no $N_{\epsilon'}$ such that from that index on $|r - r_n| < \epsilon$. Contradiction.

□

Thus we have shown that each thread can be mapped to a class of Cauchy sequences. In our proof we have used the classes of (l_n) and (r_n), but keep in mind that any other choice of representative would have been fine since at least one of the boundaries is the furthest point on the interval chosen from the limit point, which will be contained in the interval itself (since the intervals are nested). To obtain our mapping to the real numbers, we assign to each thread in $\hat{\mathbb{X}}$ the limit $\lim_{n\to\infty} l_n = \lim_{n\to\infty} r_n$:

$$\theta : \hat{\mathbb{X}} \to [0,1]$$

$$\hat{x} \mapsto \theta(\hat{x})$$

where $\theta(\hat{x}) = \lim_{n\to\infty} l_n = \lim_{n\to\infty} r_n$.

This function, despite not being an isomorphism, as we will soon see, gives an insight on how open sets in $\hat{\mathbb{X}}$ are mapped to $[0,1]$, according to the e_n's. It is easy to see that a basic open $\chi_n^{-1}(\{x_n\})$ in $\hat{\mathbb{X}}$ is mapped by θ to the open interval $e_n(\{x_n\})$, since it contains all the threads with fixed initial segment up to x_n and any final segment compatible with ψ_i for $i > n$. By definition of e_n, these threads are all the possible threads converging to points inside the interval $[l_n^{x_n}, r_n^{x_n}]$. This fact is a consequence of the nested *closed* intervals theorem, closing all the intervals that are images of x_n. We don't need the limit to be *inside* the regions assigned to each x_n; on the contrary, we are especially interested in these limit points that emerge only in the limit, as we shall see soon.

So θ is an open map, and it is continuous. In fact, consider any basic open in $[0,1]$, say (l, r). This is the image of an infinite union of basic open subsets of $\hat{\mathbb{X}}$, namely all the sets of the form $\chi_n^{-1}(\{x_n\})$ with $l_n^{x_n} \geqslant l$ and $r_n^{x_n} \leqslant r$. The same holds for arbitrary open sets in $[0,1]$. Note that the regular open subsets of the limit must be mapped to regular open subsets of the reals.

Note that θ maps some threads to the same real number. Think, for instance, to the threads $\hat{x} = (x_1, ..., x_n, ...)$ and $\hat{y} = (y_1, ..., y_n, ...)$, such that

$$a_1 = p_1 \text{ and, for n>1, } a_n = a_{n-1} \wedge \neg p_n$$

$$b_1 = \neg p_1 \text{ and, for n>1, } b_n = b_{n-1} \wedge p_n$$

In this case, x_n and y_n are both assigned sequences converging to 1/2 “from the left side” and “from the right side”.

This can be generalised to any pair of sequences for which there is an index $m \in \mathbb{N}$ such that $a_i = b_i$ for $i < m$ and

$$a_m = a_{m-1} \wedge p_m \text{ and } a_i = a_{i-1} \wedge \neg p_i \text{ for } i > m$$

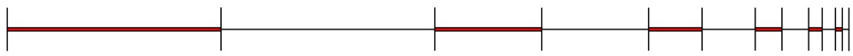
$$b_m = b_{m-1} \wedge \neg p_m \text{ and } b_i = b_{i-1} \wedge p_i \text{ for } i > m$$

Therefore θ is not injective.

At this point, having fixed our intuitions about the shape of open sets in $\hat{\mathbb{X}}$, we can make some remarks on the structure of the limit. One interesting characteristic of the limit is that it contains some regular open subsets that are not "lifted" from any discrete Stone space, meaning that there is no finite space containing a region that has the same extension (the same image on the reals) as these new ones. Think, for instance, to the set $\mathcal{V}$ whose image is

$$\theta(\mathcal{V}) = \bigcup_{i \in \mathbb{N}} \Big(1 - \frac{1}{2^n}, 1 - \frac{1}{2^n} + \frac{1}{2^{n+2}}\Big)$$

$\mathcal{V}$ contains all the threads of the form $(\neg p_1, ..., \neg p_{m-1}, p_m, p_{m+1}, \pm p_{m+2} ..., \pm p_n, ...)$ for $m \geqslant 1$. So, we get that $\theta(\mathcal{V}) \neq e_n(\chi_n(\mathcal{V}))$. This is an open set, generated by the basis of preimages of clopen sets in the finite spaces, which is *not* the preimage of a clopen in any of the $\mathbb{X}_n$'s.



This fact is the formal correlate to Kant's claim that "though all the parts are contained in the intuition of the whole, the whole division is not contained in it", since "parts of space" (the clopen subsets of our finite spaces) do not exhaust the whole represented by the limit.

Our construction, however, is not complete yet, since we now have a space of regions that are totally disconnected and they do not form a continuum, as the intuition of space should be. These regions of space are therefore a model of the spatial extent of possible experiences when they are "separated" and the contact is "destroyed". But what is this contact that has been "broken"? As we have noticed in the representation of the limit on the real line, there are points, emerging from our construction, that are not contained in the regions assigned to the clopen subsets of the $\mathbb{X}_n$'s, points that can represent their boundaries in the Aristotelian sense. To obtain a continuum, we need to "melt" them together (for instance, in the example above, the boundary of the region on the left and the boundary of that on the right). In the next section we are going to tackle the problem and study the last, fundamental, feature of space as the formal intuition.

2.2.2 Contact

The considerations above suggest a natural way to impose a contact relation on the inverse limit: two regions of space (two regular open sets of $[0,1]$ in our simplified model) are in contact if and only if their closures have nonempty intersection. This clearly cannot reflect what happens in the discrete Stone spaces of our system, assigning open intervals of $[0,1]$ to clopen sets in $\mathbb{X}_n$. Think, for instance, to the singletons of the two points $x, y \in \mathbb{X}_1$, corresponding respectively to the two atoms p_1 and $\neg p_1$ in $\mathbb{B}_1$. They are represented, respectively, by the two intervals $(0, 1/2)$ and $(1/2, 1)$. We would obviously like these two regions to be in contact, so we cannot impose a standard proximity on $\mathbb{X}_n$, requiring the closures of the two sets to have nonempty intersection to be proximal. Still, it makes sense to say that the two regions represented by the singletons (a particular "slice" of the universe and its complement) are in contact, and we are going to present an alternative contact relation that captures this intuition.

To do so, we can start from atoms in the algebras $\mathbb{B}_n$'s, by imposing a contact relation, which will be mirrored by a proximity on the Stone spaces. This establishes the conditions for two minimal regions in the discrete system to be proximal.

Consider two atoms a, b in $\mathbb{B}_n$ and the following relation between them:

$$aC^*b \quad \text{iff} \quad a = \Big(\bigwedge_{i\leqslant k-1} q_i\Big) \wedge p_k \wedge \Big(\bigwedge_{k<i\leqslant n} \neg p_i\Big) \ \text{ and } \ b = \Big(\bigwedge_{i\leqslant k-1} q_i\Big) \wedge \neg p_k \wedge \Big(\bigwedge_{k<i\leqslant n} p_i\Big)$$

where $q_i = \pm p_i$, for $p_i \in E_n$, and and k is any $k \in \mathbb{N}_{>0}$ s.t. $k \leqslant n$ (in the case of $k = 1$ the first conjunct is removed, while in the case of $k = n$ the third conjunct is removed). Note that each atom is in relation with at most two other atoms, represented by the two regions (one region if $a = \bigwedge_{i\leqslant n} p_i$ or $a = \bigwedge_{i\leqslant n} \neg p_i$) adjacent to it in $[0,1]$.

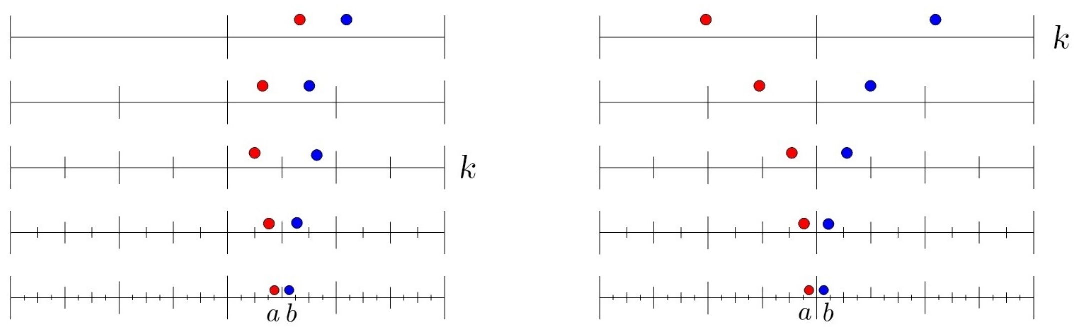


Figure 2.5: Two cases of contact between atoms in $\mathbb{B}_5$

Proposition 2.2.4. *The reflexive, symmetric and upward closure C on $\mathbb{B}_n^+$ of this relation is a contact relation on $\mathbb{B}_n$.*

Proof. (C0)-(C3) follow immediately from the definition of C.
As for (C4): let $aC(b \vee c)$. We know that every element of $\mathbb{B}_n$ can be written in a unique way as a finite join of atoms, say $a = \bigvee_{i \in I} a_i$, $b \vee c = \bigvee_{j \in J} a_j$, $b = \bigvee_{k \in K} a_k$ and $c = \bigvee_{l \in L} a_l$, and that $\{a_j | j \in J\} = \{a_k | k \in K\} \cup \{a_l | l \in L\}$. By our definition, if a is in contact with $b \vee c$, then there must be $i \in I, j \in J$ such that $a_i C a_j$. But then there must be either a $k \in K$ or an $l \in L$ such that $a_i C a_k$ or $a_i C a_l$. By symmetry and upward closure we get the desired result.

□

On the dual inverse system of Stone spaces, we would like to have a relation mirroring this contact. Note that we are looking for some contact between regions, represented by clopen sets, not between points. However, there is a natural way to induce a "closeness" relation between points in the discrete Stone spaces from the contact relation on the finite BAs :

$$x \delta_n^* y \text{ iff } aCb$$

where a, b are the atoms of $\mathbb{B}_n$ corresponding, respectively, to the points x, y in $\mathbb{X}_n$.
From this we can derive a relation between clopen sets of $\mathbb{X}_n$ in the following way:

$$P \delta_n Q \text{ iff } (\exists x \in P)(\exists y \in Q)(x \delta_n^* y)$$

Claim. This is a Lodato proximity relation on $\mathbb{X}_n$.

Proof. This fact is easily verified, but we provide the proof for the sake of completeness. Recall that a Lodato proximity is a relation satisfying (P0-P3) of Definition 2.1.15. Suppose $P \delta_n Q$. Then $(\exists x \in P)(\exists y \in Q)(x \delta_n^* y)$, thus (P0) is immediately satisfied. Now suppose $P \cap Q \neq \emptyset$. Then there exist $x \in P, Q$ and by (C1) we have $x \delta_n^* x$, so (P1) is satisfied. (P2) descend immediately from the definition and from (C2). Now suppose $P \delta (Q \cup R)$ then $(\exists x \in P)(\exists y \in (Q \cup R))(x \delta_n^* y)$. Then $y \in Q$ or $y \in P$, i.e. $P \delta Q$ or $P \delta R$. The other way around is equally easy, since if there is $y \in Q$ or $y \in P$ such that $(\exists x \in P)(x \delta_n^* y)$, then $y \in (Q \cup R)$ and so $P \delta (Q \cup R)$. Hence, (P3) is satisfied.

□

This definition of proximity clarifies why we can't use an Efremovič proximity on the finite spaces. This depends on the fact that the spaces $\mathbb{X}_n$ are not dense. In fact, the $\mathbb{X}_n$ are constructed to represent regular open subsets of the reals, and we can see how the additional axiom ($A \not\delta B \Rightarrow$ there is a $C \subseteq X$ s.t. $A \not\delta C$ and $B \not\delta (X \setminus C)$) is already ruled out in the simple case of $\mathbb{X}_2$, taking A to be the singleton whose only element corresponds to $p_1 \wedge p_2$ and B to be the singleton whose only element corresponds to $\neg p_1 \wedge p_2$. The only regions whose complements are not proximal to B must contain the element corresponding to $p_1 \wedge \neg p_2$, which is "close" to the only element in A.

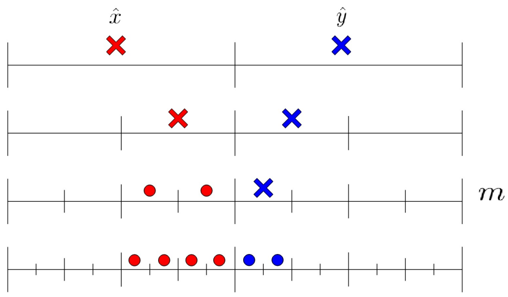
Remark. Note that if two threads $\hat{x}, \hat{y} \in \hat{X}$ are such that $x_n \delta_n^* y_n$ for some $n \in \mathbb{N}$, it follows that, for all $m \leqslant n$, $x_m \delta_m^* y_m$, since for all atoms $a_m \in At(\mathbb{B}_m)$: $h_m(a_m) = a_m \wedge \pm p_{m+1}$ and so, by our definition of contact, if $a_m = a_{m-1} \wedge \pm p_m$ is in contact with b_m, also a_{m-1} (corresponding to x_{m-1}) is in contact with b_{m-1} (corresponding to y_{m-1}).
On the other hand, if two threads $\hat{x}, \hat{y} \in \hat{X}$ are such that $x_n \not\delta_n^* y_n$ for some $n \in \mathbb{N}$, we have that for all $m \geqslant n$, $x_m \not\delta_m^* y_m$, since if a_m is not in contact with b_m, $a_{m+1} = a_m \wedge \pm p_{m+1}$ cannot be in contact with $b_{m+1} = b_m \wedge \pm p_{m+1}$.

We have completed all the necessary groundwork to define the resulting proximity relation between clopen sets P, Q in the limit $\hat{\mathbb{X}}$:

$$\begin{aligned} P \delta Q \text{ iff } & (\forall n \in \mathbb{N})(\chi_n(P)\, \delta_n\, \chi_n(Q)) && (2.1)\\ \text{iff } & (\forall n \in \mathbb{N})[(\exists x_n \in \chi_n(P))(\exists y_n \in \chi_n(Q))(x_n \delta_n^* y_n)] && (2.2)\\ \text{iff } & (\exists \hat{x} \in P)(\exists \hat{y} \in Q)[(\forall n \in \mathbb{N})(x_n \delta_n^* y_n)] && (2.3) \end{aligned}$$

The only non-trivial equivalence above is the last one. The direction (3) $\Rightarrow$ (2) is obvious, while for (2) $\Rightarrow$ (3) we need to examine the clopen sets in the limit topology. Consider first two sets in the basis, $P = \chi_n^{-1}(P_n)$ and $Q = \chi_m^{-1}(Q_m)$, where P_n and Q_m are singletons in $\mathbb{X}_n$ and $\mathbb{X}_m$ respectively. Assume (2) holds and suppose wlog $m \geqslant n$. By the fact that, for every n, there exist two points respectively in $\chi_n(P)$ and $\chi_n(Q)$ s.t. $x_n \delta_n^* y_n$, there must be two threads, in P and Q respectively, to whom x_n and y_n belong, so we can pick $\hat{x} \in P$ and $\hat{y} \in Q$ such that $x_m \delta_m^* y_m$. Now, either $\hat{x} = \hat{y}$ (in which case we are done), or, by definition of δ_n^* and of C, there exists a $j \in \mathbb{N}$ s.t. $a_m = \left(\bigwedge_{i<j} q_i\right) \wedge p_j \wedge \left(\bigwedge_{j<i\leqslant m} \neg p_i\right)$ and $b_m = \left(\bigwedge_{i<j} q_i\right) \wedge \neg p_j \wedge \left(\bigwedge_{j<i\leqslant m} p_i\right)$ or vice versa. By the above remark, for all $k \leqslant m$, $x_k \delta_k^* y_k$. This holds for all the threads with the same initial segment as $\hat{x}$ and $\hat{y}$, up to m. Now, since $m \geqslant n$, all the possible threads in $\hat{\mathbb{X}}$ with such initial segments are elements respectively of P and Q (which are preimages of singletons in $\mathbb{X}_n$ and $\mathbb{X}_m$ respectively, and so they contain all threads with initial segment fixed by the element in P_n and Q_n respectively, and any possible final segment) and they must include the two threads, say $\hat{x'}$ and $\hat{y'}$ with corresponding sequences of atoms $a'_r = \left(\bigwedge_{i<j} q_i\right) \wedge p_j \wedge \left(\bigwedge_{j<i\leqslant r} \neg p_i\right)$ and $b'_r = \left(\bigwedge_{i<j} q_i\right) and \neg p_j \wedge \left(\bigwedge_{<i\leqslant r} p_i\right)$, for all $r \geqslant m$. These are the two threads that satisfy (3). Finally, since this holds for sets in the basis we can extend the result to every clopen in $\hat{\mathbb{X}}$.

$\square$

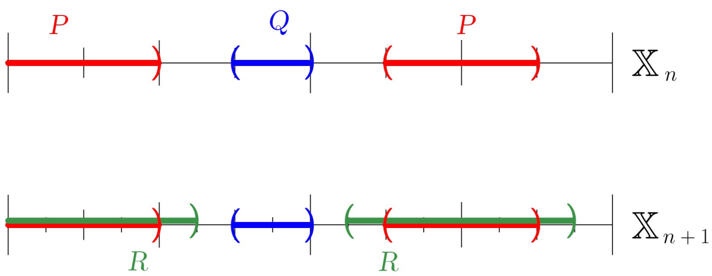


Claim. The relation δ on $Clop(\hat{\mathbb{X}}) \times Clop(\hat{\mathbb{X}})$, defined by

$$P\delta Q \text{ iff } (\exists \hat{x} \in P)(\exists \hat{y} \in Q)[(\forall n \in \mathbb{N})(x_n \delta_n^* y_n)]$$

is an Efremovič proximity relation.

Sketch of the proof. The axioms (P0)-(P3) are easily verified. As for (P4), it can be shown that δ satisfies it, but the proof is tedious and it involves awkward notation. However, looking at the behaviour of δ with respect to the finite $\mathbb{X}_n$ we can get an intuitive idea of how the proof should proceed. A picture will help fixing our intuitions:



We want to show that, given two clopen sets $P, Q \in \hat{\mathbb{X}}$ s.t. $P \not\delta Q$, there exists a clopen set $R \in \hat{\mathbb{X}}$ s.t. $P \not\delta R$ and $Q \not\delta (\hat{\mathbb{X}} \setminus R)$. In the picture we abuse the notation to have a neat image of how P and Q get mapped to $e_n(\mathcal{P}(\mathbb{X}_n))$ by $e_n \circ \chi_n$. By equivalence (2) above, we have that $P \not\delta Q$ implies that $(\exists n \in \mathbb{N})[(\forall x_n \in \chi_n(P))(\forall y_n \in \chi_n(Q)(x_n \not\delta_n^* y_n)]$. The intuitive meaning of this fact is that there is an $n \in \mathbb{N}$ such that the projections of P and Q are assigned intervals of $[0, 1]$ that are disjoint, and there is a "gap" separating them (they are not proximal in $\mathbb{X}_n$). Now, if we look at $\mathbb{X}_{n+1}$, we see that this "gap" gets bisected into two smaller intervals. Thus, we can "enlarge" P of one such interval and obtain the image in $\mathbb{X}_{n+1}$ of the desired R. This can be done to every "piece" of P, as the figure shows. The preimage along e_n and χ_n of this interval is a clopen of $\hat{\mathbb{X}}$ that contains P and is not proximal to Q.

It is useful to define a closeness relation for threads, too:

$$\hat{x}\delta^*\hat{y} \text{ iff } (\forall n \in \mathbb{N})(x_n \delta_n^* y_n)$$

Thus, we get, for P, Q clopen subsets of $\hat{X}$:

$$P\delta\, Q \text{ iff } (\exists \hat{x} \in P)(\exists \hat{y} \in Q)[\hat{x}\delta^*\hat{y}]$$

Two threads in relation δ^* are "points of contact" between two regions and will be the key to make the totally disconnected limit into a connected continuum. The following lemma is decisive to show that these are exactly the points that need to be "fused together" to obtain the real line:

Lemma 2.2.1. *Given two threads $\hat{x}, \hat{y} \in \hat{\mathbb{X}}$, we have*

$$\hat{x}\delta^*\hat{y} \text{ iff } \theta(\hat{x}) = \theta(\hat{y})$$

Proof.
($\Rightarrow$) Assume $\hat{x}, \hat{y} \in \hat{\mathbb{X}}$ are close to each other, i.e. $a_n C b_n$ for all $n \in \mathbb{N}$. Then either $\hat{x} = \hat{y}$ (in which case we are done); or, for all $n \in \mathbb{N}$, there exists a $m \leqslant n$ such that $a_k = b_k$ for all $k < m$ and, wlog, $a_j = \bigwedge_{i<m} q_i \wedge p_m \wedge \bigwedge_{m<i\leqslant j} \neg p_i$ and $b_j = \bigwedge_{i<m} q_i \wedge \neg p_m \wedge \bigwedge_{m<i\leqslant j} p_i$ for all $j \geqslant m$. This means that the intervals assigned to x_n and y_n are the same up to the index $m-1$; that $\left[l_m^{x_m}, r_m^{x_m}\right] = \left[l_{m-1}, \frac{l_{m-1}+r_{m-1}}{2}\right]$ and $\left[l_m^{y_m}, r_m^{y_m}\right] = \left[\frac{l_{m-1}+r_{m-1}}{2}, r_{m-1}\right]$; and that for all indexes from x_{m+1} onward, the right boundary of the interval assigned to x_n is fixed and the left boundary of the interval assigned to y_n is fixed. Considering the limit of the left boundaries, we get:

$$\theta(\hat{x}) = \lim_{n\to\infty}(l_n^{x_n}) = l_{m-1} + \lim_{q\to\infty}\left(\frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + \dots + \frac{1}{2^{m+q}} + \dots\right)$$

$$= l_{m-1} + \lim_{q\to\infty}\left(\frac{1}{2^m} - \frac{1}{2^q}\right) = l_{m-1} + \frac{1}{2^m}$$

$$\theta(\hat{y}) = \lim_{n\to\infty}(l_n^{y_n}) = l_m^{y_m} = l_{m-1} + \frac{1}{2^m}$$

Thus, $\theta(\hat{x}) = \theta(\hat{y})$.
($\Leftarrow$) Suppose $\theta(\hat{x}) = \theta(\hat{y})$, then $\bigcap_{n\in\mathbb{N}}[l_n^{x_n}, r_n^{x_n}] = \bigcap_{n\in\mathbb{N}}[l_n^{y_n}, r_n^{y_n}]$, i.e. for all $n \in \mathbb{N}$ there is a z which belongs both to $[l_n^{x_n}, r_n^{x_n}]$ and to $[l_n^{y_n}, r_n^{y_n}]$. By the nested interval theorem, we have that both $\bigcap_{n\in\mathbb{N}}[l_n^{x_n}, r_n^{x_n}]$ and $\bigcap_{n\in\mathbb{N}}[l_n^{y_n}, r_n^{y_n}]$ contain exactly one point and so there are two cases: either they are equal, i.e. $\hat{x} = \hat{y}$, in which case we are done; or there is an $m \in \mathbb{N}$ such that $[l_i^{x_i}, r_i^{x_i}] = [l_i^{y_i}, r_i^{y_i}]$ for all $i < m$, $[l_j, r_j]_{x_j} \neq [l_j, r_j]_{y_j}$ for all $j \geqslant m$, and either $r_j^{x_j} = l_j^{y_j}$ for all $j \geqslant m$, or $r_j^{y_j} = l_j^{x_j}$ for all $j \geqslant m$. Wlog, suppose $r_j^{x_j} = l_j^{y_j}$ for all $j \geqslant m$. Then the only possibility is $r_m^{x_m} = \frac{l_{m-1}+r_{m-1}}{2} = l_m^{y_m}$ (i.e. $p_m \geqslant a_m$ and $\neg p_m \geqslant b_m$) and, for all $j > m$, $r_j^{x_j} = r_{j-1}^{x_{j-1}}$ and $l_j^{y_j} = l_{j-1}^{y_{j-1}}$ (i.e. $\neg p_j \geqslant a_j$ and $p_j \geqslant b_j$).
□

Claim. The relation δ^* is an equivalence relation on $\hat{\mathbb{X}}$.

Proof. Reflexivity and symmetry follow immediately from reflexivity and symmetry of C. As for transitivity, let $\hat{x}\delta^*\hat{y}$ and $\hat{y}\delta^*\hat{z}$. Let a_n, b_n and c_n be, respectively, the atoms associated to x_n, y_n and z_n. By our assumption we have that for all $n \in \mathbb{N}$, $a_n C b_n$ and $b_n C c_n$. Recall that each atom $a = \bigwedge_{i \leqslant n} \pm p_i$ is in contact with at most two other atoms, namely the one having the same initial segment and differing only in the sign of p_n, and, if it exists, the one having the same initial segment up to the j-th conjunct, and s.t. , from $\pm p_j$ on, all the other conjuncts have opposite sign to p_j. It should be clear that, for each thread $\hat{y}$, there can be only one other thread such that $b_n C a_n$ for all $n \in \mathbb{N}$, since $\hat{y}$ is an infinite sequence of y_n. So, $a_n = c_n$ for all n, i.e. $\hat{x}\delta^*\hat{z}$.

□

Finally, "glueing" together the points of contact of our regions in the limit, we obtain the desired continuum. This operation is done, formally, by quotienting $\hat{\mathbb{X}}$ by the relation δ^*

The quotient space $\hat{\mathbb{X}}/\delta^*$ has the final topology τ_q with respect to the standard inclusion:

$$q : \hat{\mathbb{X}} \to \hat{\mathbb{X}}/\delta^*$$

$$\hat{x} \mapsto [\hat{x}]$$

So, $\mathcal{U} \in \tau_q$ iff $q^{-1}(\mathcal{U}) \in \tau$.

Homeomorphism with the unit interval

We are now ready to see that this final operation of synthesis yields a continuum. To show this, we trace a homeomorphism between the quotient $\hat{\mathbb{X}}/\delta^*$ and $[0, 1]$.

Define the map:

$$\iota : \hat{\mathbb{X}}/\delta^* \to [0, 1]$$

$$[\hat{x}] \mapsto \theta(\hat{x}) = \lim_{n\to\infty} l_n = \lim_{n\to\infty} r_n$$

where $\hat{x} = (x_1, ..., x_n, ...)$ and l_n, r_n are as defined above.

Define the map inverse to ι:

$$\kappa : [0, 1] \to \hat{\mathbb{X}}/\delta^*$$

$$r \mapsto [(x_n)]$$

where $x_n =\uparrow \bigwedge_{i\leqslant n} q_i$ and

$$q_i = \begin{cases} p_i & \text{if } r \in \left[l_{i-1}, \frac{l_{i-1}+r_{i-1}}{2}\right) \\ \neg p_i & \text{if } r \in \left[\frac{l_{i-1}+r_{i-1}}{2}, r_{i-1}\right] \end{cases}$$

with $[l_0, r_0] = [0, 1]$.

Proof. By lemma 2.2.1 ι is well-defined and injective. As for κ, it assigns to each r the only sequence of x_n s.t. $r \in e(\{x_n\})$. Note that if r is a "border point", to get a well-defined function, we need to choose to assign it to one of the two sequences converging to it (which belong to the same class modulo δ^*). It is clear by construction that $\kappa \circ \iota = id_{\hat{\mathbb{X}}/\delta^*}$ and $\iota \circ \kappa = id_{[0,1]}$, thus κ is indeed the map inverse to ι.
Note that we also have that $\iota \circ q = \theta$ and $\kappa \circ \theta = q$. So, to prove that ι and κ are continuous let us first examine θ. Note that the preimage of a basic open set $\mathcal{U} = (l, r) \in [0, 1]$ is saturated in $\hat{\mathbb{X}}$ with respect to δ^*. Indeed, as we already noted, this preimage is the infinite union of all the basic opens of the form $\chi_n^{-1}(\{x_n\})$ with $l_n^{x_n} > l$ and $r_n^{x_n} < r$. This means that all the threads that converge to reals between l and r are in $\theta^{-1}(\mathcal{U})$, and since the class $[\hat{x}]$ is given by all the threads converging to the same real as $\hat{x}$, $\theta^{-1}(\mathcal{U})$ is saturated with respect to δ^*. This result can immediately be extended to any open in $[0, 1]$.
Now, since θ is continuous and since a set is open in the quotient $\hat{\mathbb{X}}/\delta^*$ iff its preimage via the standard embedding $q : \hat{\mathbb{X}} \to \hat{\mathbb{X}}/\delta^*$ is open, we have that ι is continuous.
To prove continuity of κ, consider an open $\mathcal{U} \in \tau_q$. We want to prove that for every $y \in \mathcal{U}$ there is an open $\mathcal{O}$ in $[0, 1]$ s.t. $\kappa^{-1}(y) \in \mathcal{O}$ and $\mathcal{O} \subseteq \kappa^{-1}(\mathcal{U})$. To see that it is always possible to find such an $\mathcal{O}$, recall that $q^{-1}(\mathcal{U}) = \bigcup\{\mathcal{A}_i | i \in I\}$ where $\mathcal{A}_i = \chi_i^{-1}(\{x\})$ for some $x \in \mathbb{X}_n$, $n \in \mathbb{N}$. This means, in our representation on the unit interval, that each $\mathcal{A}_i$ contains all the threads that are mapped by θ to real numbers inside the interval $e_n(\{x\})$, included the ones assigned to Cauchy sequences converging to the borders "from the inside" of the interval (i.e., given the interval $\theta(\mathcal{A}_i) = [a, b] = e_n(\{x\})$, the two threads $(y_j)_{j\in\mathbb{N}}$ and $(z_j)_{j\in\mathbb{N}}$ such that, for all $m > n$, $e_m(\{y_m\}) = [a, k_m]$ and $e_m(\{z_m\}) = [l_m, b]$, where $a < k_m < b$ and $a < l_m < b$). Now, given any $y \in \mathcal{U}$, it must be the case that its preimage via q is contained in $q^{-1}(\mathcal{U})$, i.e. if y is the class containing the two distinct threads $y', y'' \in \hat{\mathbb{X}}$, both y' and y'' must be in $q^{-1}(\mathcal{U})$. So, if y' is one of the two threads "converging from the inside" to one of the borders of $\mathcal{A}_i$, there must exist $\mathcal{A}_j \subseteq q^{-1}(\mathcal{U})$ that contains y''. This way we can rule out any "bad case", by iterating this process of "fattening" the $\mathcal{A}_i$'s so to include every $\kappa^{-1}(y) \in \kappa^{-1}(\mathcal{U})$ in an open $\mathcal{O}$ of the desired form. It is possible to iterate this process ω times, since $q^{-1}(\mathcal{U}) = \bigcup\{\mathcal{A}_i | i \in I\}$ is open, bounded, and does not contain its borders.

□

2.2.3 Boundaries

Now that we have defined the last salient relation between regions, obtained the quotient of the limit and proved continuity of the quotient, one might ask what do the elements of this quotient represent. It is time to introduce the notion of boundary between regions, emerging as a result of the activity of the figurative synthesis as limitations of regions in space.

The intuition behind the definition below is that the boundary of a region in space is nothing more than what is left if we take away from the whole of space the two spaces that are uniquely determined as the "inside" and the "outside" of the region. In Kantian terms, we are considering "the differences between things only as limitations arising through the negations attaching to them".[2] In our model regions are clopen subsets of the $\mathbb{X}_n$'s and their boundaries can be retrieved in the quotient of the limit. In this passage from $\mathbb{X}_n$ to $\hat{\mathbb{X}}/\delta^*$ we must be careful since, in $\mathbb{X}_n$, P and its complement are open regions, corresponding, via e_n, to regular open subsets of the reals. Hence, to find the boundary of P, we need to take the interior of the projection of $X_n \setminus P$ on the quotient to obtain what is "outside" P.

Before giving the formal definition of boundary, it is convenient to adopt some abbreviations. For the sake of simplicity, we will denote the complement $X \setminus P$ of a set $P \in X$ as $\overline{P}$ and, given a clopen $P \subseteq \mathbb{X}_n$, its image on the quotient obtained by "lifting" it to the limit and then projecting it on the quotient $q(\chi^{-1}(P))$ will be denoted by P^*.

Definition 2.2.1 (Boundary). The *boundary* of a clopen $P \in Clop(\mathbb{X}_n)$, or the *boundary between P and its complement* $\overline{P}$, is the triplet $(P, \overline{P}, \beta P)$, where βP is defined as $\beta P = \overline{P^* \cup Int(\overline{P^*})}$.

This definition of boundary can be extended to clopen subsets of the limit: given $P \in Clop(\mathbb{X})$, its boundary is $(P, \overline{P}, \beta P)$, where $\beta P = \overline{q(P) \cup Int(\overline{q(P)})}$

Note that βP contains two elements of $\hat{\mathbb{X}}/\delta^*$ (represented, on $[0,1]$, by the left and right boundaries of an interval). The two elements of $\hat{\mathbb{X}}$ mapped by ι to 0 an 1 are considered as "formal boundaries" [3] and function as formal correlates to the infinity of space. Indeed, recall that $[0,1]$ is homeomorphic to the *extended* real line, which is obtained by adding to $\mathbb{R}$ two elements $-\infty$ and $+\infty$ making it a compact Hausdorff space.

This definition captures the idea of "common boundary of two spaces, which is therefore

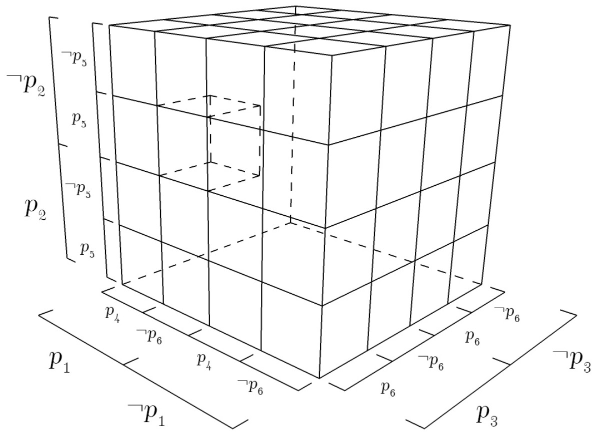
[2] [41] 8:138n

[3] Where the term and role in the construction are borrowed from [27] (pp.70,117-118)

within neither the one nor the other space"[4]. Indeed, if we look at the images through e_n of a clopen and its complement on $RegOp(\mathbb{R})$, we have that the third element of the triplet is the intersection of their closures, which does not belong to either region. We will soon see that this definition applies, unchanged, to the three dimensional case, but the boundary of a region will not be a set of point, but a set of surfaces.

2.2.4 Representation in the unit cube

We are now ready to generalise the above example to the three dimensional case.
Consider the unit cube $[0,1]^3 \subseteq \mathbb{R}^3$. To split the space in a way that is analogous to the above construction, we need to apply the bisection method to each edge and obtain 2^{3n} atoms at the n-th step. Hence we are not going to consider all the algebras $\mathbb{B}_n$'s, but a subset of them, namely the ones with index a multiple of 3. For instance, the first cut will split the space in 8 mini-cubes, corresponding to all the possible combinations of the free generators p_1, p_2, p_3, assigned respectively to the the first halves of the x, y, z edges, and their negations, assigned to the second halves of such segments; the second cut will split each half in two, giving rise to 64 mini cubes, each corresponding to an atom in the algebra $\mathbb{B}_6$, and so on.



Fixing the three axes x, y, z as in figure 2.2.4, a cuboid in $[0,1]^3$ is completely determined by the coordinates of its projection on the axes. In the following we will refer to a cuboid C by giving such coordinates as intervals on the axes, indicated as (l_C, r_C) (projection on the x axis), (d_C, u_C) (projection on the y axis) and (f_C, b_C) (projection on the z axis).

[4]([18] [4:512])

Our functions e_n's can be generalised to:

$$\epsilon_n : \mathcal{P}(\mathbb{X}_n) \to RegOp[0,1]^3$$
$$\{x\} \mapsto C_n^x$$

where $(l_{C_n^x}, r_{C_n^x}, (d_{C_n^x}, u_{C_n^x})$ and $(f_{C_n^x}, b_{C_n^x})$ will be denoted, respectively, (l_n^x, r_n^x), (d_n^x, u_n^x) and (f_n^x, b_n^x); and are computed analogously to the $e_n(\{x\})$'s above, but modulo 3, in the following way:

- (l_n^x, r_n^x) is defined, for $m = \max\{k \leqslant n | k \mod 3 = 1\}$, as (l_m^x, r_m^x) where $(l_1^x, r_1^x) = [0,1]$ and
$$(l_{k+3}^x, r_{k+3}^x) = \begin{cases} (l_k^x, \frac{l_k^x + r_k^x}{2}) & \text{if } p_{k+3} \geqslant a \\ (\frac{l_k^x + r_k^x}{2}, r_k^x) & \text{if } \neg p_{k+3} \geqslant a \end{cases}$$
- (d_n^x, u_n^x) is defined, for $m = \max\{k \leqslant n | k \mod 3 = 2\}$, as (d_m^x, u_m^x) where $(d_2^x, u_2^x) = [0,1]$ and
$$(d_{k+3}^x, u_{k+3}^x) = \begin{cases} (d_k^x, \frac{d_k^x + u_k^x}{2}) & \text{if } p_{k+3} \geqslant a \\ (\frac{d_k^x + u_k^x}{2}, u_k^x) & \text{if } \neg p_{k+3} \geqslant a \end{cases}$$
- (f_n^x, b_n^x) is defined, for $m = \max\{k \leqslant n | k \mod 3 = 0\}$, as (f_m^x, b_m^x) where $(f_0^x, b_0^x) = [0,1]$ and
$$(f_{k+3}^x, b_{k+3}^x) = \begin{cases} (f_k^x, \frac{f_k^x + b_k^x}{2}) & \text{if } p_{k+3} \geqslant a \\ (\frac{f_k^x + b_k^x}{2}, b_k^x) & \text{if } \neg p_{k+3} \geqslant a \end{cases}$$

Now, as it can be seen comparing the construction on the real line, we have that a thread in mapped, according to the ϵ_n's of the singletons of its components, to the intersection of ω nested *closed* cubes, namely the closures of the open intervals assigned by the ϵ_n's. Define an analogous function to θ by:

$$\Theta : \hat{\mathbb{X}} \to [0,1]^3$$
$$\hat{x} \mapsto \bigcap_n \epsilon(\{x_n\})$$

$\Theta(\hat{x})$ is well-defined since the $\epsilon(\{x_n\}$'s are nested cubes with vertices corresponding to Cauchy sequences of reals.
Again, we have that this function is not injective, since there are now at most six threads corresponding to the same point in $[0,1]^3$

So, we are going to define a contact relation on the $\mathbb{B}_n$'s such that the final quotient will give us the wanted result. The new C^* relation will be defined on $\mathbb{B}_n$ in the following way:

Definition 2.2.2. Let $a, b \in \mathbb{B}_n$, then aC^*b iff there exists a j such that:

- for all $k < j$, if $p_k \geqslant a$, then $p_k \geqslant b$;
- $p_j \geqslant a$ and $\neg p_j \geqslant b$
- for all $k > j$ $\begin{cases} \neg p_k \geqslant a \text{ and } p_k \geqslant b & \text{if } k \in [j]_3 \\ p_k \geqslant a \Rightarrow p_k \geqslant b & \text{if } k \notin [j]_3 \end{cases}$

The new contact relation C on $\mathbb{B}_n$ is the reflexive, symmetric and upward closure of C^* on $\mathbb{B}_n^+$. The proof of this fact is the same as the one in the one dimensional case.

The dual on the inverse system of Stone spaces is the same as before: let us briefly summarise the main passages.

Closeness between two points $x, y \in \mathbb{X}_n$, with corresponding atoms $a, b \in \mathbb{B}_n$ is defined as:

$$x\delta_n^* y \text{ iff } aCb$$

Proximity between two clopen sets $P, Q \in \mathbb{X}_n$ is then:

$$P\delta_n Q \text{ iff } (\exists x \in P)(\exists y \in Q)(x\delta_n^* y)$$

The remark holds for the same reasons to the one dimensional case, so if $x_n\delta_n^* y_n$ for some threads $\hat{x}, \hat{y} \in \hat{\mathbb{X}}$, we have $x_m\delta_m^* x_m$ for all $m \leqslant n$.

This, together with the fact that any clopen set in $\hat{\mathbb{X}}$ contains all the threads beginning with a particular initial segment, up to a n index m and continuing with any sequence of combinations of $p_k, k \geqslant m$, gives us, with a proof analogous to the one above, the equivalence between the component-wise definition of proximity on the limit and the more handy:

$$P\delta Q \text{ iff } (\exists \hat{x} \in P)(\exists \hat{y} \in Q)(\hat{x}\delta^* \hat{y})$$

where $(\hat{x}\delta^*\hat{y})$ iff $(\forall n \in \mathbb{N})(x_n\delta_n^* y_n)$. The proof of Lemma 2.2.1 is totally analogous to the one in the one dimensional case, except that it needs to make use of the three-parted definition of ϵ_n, therefore requiring more involved calculations. The result still holds: $\hat{x}\delta^*\hat{y}$ iff $\Theta(\hat{x}) = \Theta(\hat{y})$.

The relation $\hat{x}\delta^*\hat{y}$ is an equivalence relation. The proof is analogous to the one for δ^*, except that we now have that $\hat{x}$ is close to at most six other threads (the ones converging to $\lim\limits_{n\to\infty} \bigcap \epsilon_n(\{x_n\})$).

Finally, the homeomorphism between $\hat{\mathbb{X}}/\delta^*$ with the final topology wrt the standard inclusion and $[0,1]^3$ is:

$$\eta : \hat{\mathbb{X}}/\delta^* \to [0,1]^3$$
$$[\hat{x}] \mapsto \bigcap_{n\in\mathbb{N}} \epsilon_n(\{x_n\})$$

with inverse:

$$\zeta : [0,1]^3 \to \hat{\mathbb{X}}/\delta^*$$
$$r \mapsto [(x_n)]$$

where $x_n = \uparrow \bigwedge_{0<i\leqslant n} q_i$ and q_i is defined as follows:

- $\cdot$ if $i \mod 3 = 1$, $q_i = \begin{cases} p_i & \text{if } r_{\mathsf{x}} \in \left[l_{i-1}, \frac{l_{i-1}+r_{i-1}}{2}\right) \\ \neg p_i & \text{if } r_{\mathsf{x}} \in \left[\frac{l_{i-1}+r_{i-1}}{2}, r_{i-1}\right] \end{cases}$

- $\cdot$ if $i \mod 3 = 2$, $q_i = \begin{cases} p_i & \text{if } r_{\mathsf{y}} \in \left[d_{i-1}, \frac{d_{i-1}+u_{i-1}}{2}\right) \\ \neg p_i & \text{if } r_{\mathsf{y}} \in \left[\frac{d_{i-1}+u_{i-1}}{2}, u_{i-1}\right] \end{cases}$

- $\cdot$ if $i \mod 3 = 0$, $q_i = \begin{cases} p_i & \text{if } r_{\mathsf{z}} \in \left[f_{i-1}, \frac{f_{i-1}+b_{i-1}}{2}\right) \\ \neg p_i & \text{if } r_{\mathsf{z}} \in \left[\frac{f_{i-1}+b_{i-1}}{2}, b_{i-1}\right] \end{cases}$

where $r_{\mathsf{x}}, r_{\mathsf{y}}, r_{\mathsf{z}}$ are, respectively the $\mathsf{x}, \mathsf{y}, \mathsf{z}$ coordinates of r and $[l_0, r_0] = [d_0, u_0] = [f_0, b_0] = [0, 1]$.

Proof. The proof that this is a homeomorphism is analogous to the one on the unit interval, with a few observations on the specifics of this case. Well-definedness and bijectivity are easy as before.
For continuity let us examine Θ. Note that the preimage of a basic open C in $[0,1]^3$, with edges $[l_C, r_C], [d_C, u_C], [f_C, b_C]$, is again saturated in $\hat{\mathbb{X}}$ wrt δ^*, since it is the infinite union of all the basic opens of the form $\chi^{-1}(\{x_n\})$ such that $\epsilon(\{x_n\}) \subset C$. By continuity of Θ and the definition of topology on the quotient, we get, as before, that η is continuous.The proof for continuity of ζ is totally analogous to the one for κ above, again considering each edge as an interval on the real line.

□

It is now clear why the definition of boundaries needs not be changed in the case of three dimensions, since nothing relevant changes in the relationships between the clopen subsets of $\mathbb{X}_n$ and the quotient. The representation on the unit interval is again, that of a triplet, consisting of a regular open set, together with the interior of its complement and the set obtained subtracting these two regions from the totality of space (i.e. $[0,1]^3$, in our example). Note that, in this case, boundaries are surfaces, according to Euclid's *dictum* and to Poincaré's definition of dimension number. Orientation of the three axes is dependent on the subject (as remarked by Kant in the Orientation essay([41]) and $-\infty, +\infty$ on each axis are formal boundaries representing infinity of space in every direction.

2.3 Justification of the construction

Region-based theories of space have been developed in the last century starting from the seminal works of Whitehead ([39]) and Tarski ([37]) in 1929. The fact that reasoning and speaking of spatial relations among regions is more natural than referring to points and relations among them, attracted researches that wanted to model cognitive processes and the mereological approach to theories of spatial representation flourished as an alternative to the standard point-based approaches. To have the same expressive power as other theories of space, the region-based models needed to be able to recover points as definable entities. In a very Kantian way, points were derived as "limitations" of regions, through the definition of a *contact relation* between regions. Whitehead named this relation "connection", echoing the Aristotelian notion of boundaries serving to "link" regions in a continuum.

Despite Kant's eminently constructive approach to geometry, trying to formalize Kant's spatial continuum in a constructive algebraic setting from the beginning seemed to us less appealing than using much more manageable objects such as the classical free Boolean algebras. The reason behind this choice was initially one of convenience - for, as we will soon see, Boolean algebras of regular open sets are an ideal frame for our purposes - and the idea was to be able to then see if the whole construction could be adapted to the context of Heyting algebras. Moreover, a notion of contact on Boolean algebras has been studied far more extensively than a similar notion on Heyting algebras, so we decided to start there. Unfortunately, it turned out that the type of contact resulting from our construction (the only possible one, given the space we obtained) was shaped to capture the specific way regions are related in our Boolean model, and it is certainly not extensible to a properly constructive setting. Still, the construction bore some significance, and after trials and errors led to some interesting observations, so we present it both as an exercise to study the matter of the spatial continuum more closely, and as a warning about the limits of our method.

The very first step in building a formal model of space is to make a choice about the collection of basic entities which make up our interpretation. Several mathematical settings have been devised to provide mereological models of space; in particular, the topological interpretation based on regular open sets has the benefit of providing a natural setting for the definition of a proximity relation (the topological counterpart of the contact relation defined on algebras). There are many good reasons to choose regular open sets as models for regions in space [5]; many intuitions come from considerations on the real space $\mathbb{R}^3$ and we can reason easily on $\mathbb{R}^2$ to fix some basic ideas. We would like a region to be somewhat "uniform" - in the sense that it does not have points or lines "breaking" it - and we would like points to supervene to regions, as their boundaries or limitations. A regular open set in $\mathbb{R}^2$ is indeed a region with no "pin-holes" or "cracks"[6], in the form of lines, since they are the interior of their own closure. Moreover, taking the open unit square as our model of space and dividing it into two regions, say the left and right rectangles with base, respectively, $(0, 1/2)$ and $(1/2, 1)$, their boundary can be defined as the intersection of their closures. This boundary is a line, which comes into being only as a limit of regions. The same happens, of course, in the case of a one dimensional continuum, where a point is obtained as the boundary of two adjacent regions (regular open subsets of $\mathbb{R}$), or in $\mathbb{R}^3$, where a surface is the boundary of two three-dimensional regions. Every point is a boundary (but not every boundary is a point, in particular in the models with more than one dimension). The Euclidean idea of dimension number, successively formalised by Poincaré, guided us in our intuitions about the possible extension of the model from the real line to the Euclidean three-dimensional space.

These considerations bear remarkable similarities with the observations Kant makes in the second chapter of the MFNS about contact:

> Contact in the mathematical sense is the common boundary of two spaces, which is therefore within neither the one nor the other space. [...] A circle and a straight line, or two circles, are in contact at a point, surfaces at a line, and bodies at surfaces[...] ([18] [4:512])

Regular open sets represent precisely this fact: the two regions we considered in the above example are in contact and the line corresponding to their common boundary is not contained in either of them. Now that we have an idea of the result we want to obtain, let us turn to the relations captured by our formalization.

Our model is constructed starting from finite sets of regions, representing the spatial extent of possible experiences. The formal correlate to the form of intuition is represented

[5] For a detailed account of why these are good candidates for representing regions see [33].

[6] See, again, [33], p. 15, where these concepts are illustrated with clarifying pictures.

by the finite Boolean algebras and their dual topological spaces. The successive action of the figurative synthesis determines the manifold, ordering it according to the categories. Since Kant's writings lack a clear list of primitive relations, our model is built to capture only the most general aspects we could ascribe to the Kantian conception of spatial experience. The main relevant relations are, of course, parthood relations, which are at the core of the mereological approach. The procedure adopted is one of bisection[7], carried out in agreement with the category of community, which is the concept of mutual determination of substances. It is by means of this category that two parts of space are coordinated in partitioning a whole, being cause of the determinations of each other, insofar as their existence is concerned[8]. The dual process that permits to obtain a whole from the composition of two regions, derives from the application of the category of quantity, which guarantees that a whole (totality) can result from the combination of a multitude (multiplicity) of given homogeneous elements (unities).

The natural algebraic structures representing complementary regions in space generated by a process of bisection are free Boolean algebras. The elements of these algebras can be easily visualised as regular open subsets of the real line (only in the end we will generalize our system to be isomorphic to the real three-dimensional space, with due care).

To provide a formal correlate to the activity of the figurative synthesis "running through and holding together" the manifold of intuition, we constructed, as first proposed in [1], an inverse system of finite structures mapped to each other via retractions. This construction has proved successful in modelling time in Pinosio's work ([27]) and, since the action of the figurative synthesis is analogous in the case of space, we adopted the

[7]Hellman and Shapiro proposed a formalization of Aristotle's notion of continuum through a construction which bears clear resemblance to ours, in the adoption of Cauchy sequences of regions obtained by successive operations of bisection. We came up with our model independently, having discovered Hellman-Shapiro's solution only in the final phase of the present work. This can be taken as a sign that our approach is among the most natural ones to the matter. Incidentally, note that our construction solves the problem of capturing Aristotle's notion of boundaries, which is recognised by the two academics as a flaw of their theory.

[8]Thus Kant:

> Now a similar connection is thought of in **an entirety of things**, since one is not **subordinated**, as effect, under another, as the cause of its existence, but is rather **coordinated** with the other simultaneously and reciprocally as cause with regard to its determination [...]. The understanding follows the same procedure when it represents the divided sphere of a concept as when it thinks of a thing as divisible, and just as in the first case the members of the division exclude each other and yet are connected in one sphere, so in the latter case the parts are represented as ones to which existence (as substances) pertains to each exclusively of the others, and which are yet connected in one whole. ([16], B112-113)

same approach, adapting it to our context of finite Boolean algebras (and their dual topological spaces). This setting has been shown to be convenient for several reasons. As we have seen in the previous chapter, a precondition for the possibility of cognition is for our diverse experiences to be recognized to belong to the same consciousness. Thus, our model should be able to relate any two distinct representations, as parts, to an encompassing whole representation. This principle of the unity of apperception is embodied in the directedness condition imposed on the set indexing the finite Boolean algebras. The index set itself represents the fact that the figurative synthesis is a process that takes place in time, where each index can be viewed as an interval of time necessary for the synthesis to produce an ordered manifold out of the multiplicity without constraints given by sensibility. In fact, what each Boolean algebra really represents is a set of acts of self-affection, produced by the figurative synthesis and thus spatiotemporally structured, since they are the result of the synthesis of a space in time.

The link between the figurative synthesis and the formal intuition of space is attained by the use of retraction maps. An observation on the duality between Boolean algebras and Stone spaces is due here, to prevent confusion. The use of two different perspectives (the order-theoretic one, on the one side, and the topological one, on the other) allows for a much richer analysis of the structures taken into consideration, not to mention the benefits this approach produces on the visualisation of this abstract setting on the real line. The key result that links the algebraic setting to the topological side of the model is the Stone representation theorem, which, in layman's terms, asserts the equivalence between Boolean algebras and Stone spaces, with "reversed arrows" (i.e. upon switching domain and codomains of the morphisms involved). This is an extremely convenient framework for our purposes, since, thanks to the finiteness of our Boolean algebras, we get that their atoms (representing minimal regions) correspond exactly to the elements of the dual Stone spaces, and so we can shift freely between the two, taking care only of the reversed morphisms between structures. Up to this point, we could consider the Stone space dual to each finite Boolean algebra in our system as behaving in the exact same way as the algebra, given the duality just mentioned. When the retraction maps come into play, however, we need to be careful to remember that, to obtain an inverse system of Stone spaces (which is our ultimate goal, since we are concerned with the topological properties of its inverse limit), the corresponding mappings between Boolean algebras have to be reversed. The retractions are particular continuous maps between Stone spaces, constructed to relate the two parts of a bisected region to the whole region combining them, and they impose a requirement of consistency throughout the system. On the side of Boolean algebras, the construction obtained is called a direct system.

If, on the one hand, the activity of the figurative synthesis finds its formal correlate in the inverse system of Stone spaces, capturing the "boundlessness in the progress of intuition"(A25); the dual system, on the other hand, models the potential infinite divisibility of space. The clopen sets in each finite space correspond to elements of the algebras, and represent "parts of space", synthesised through acts of self-affection, which *can* be iterated progressively *ad infinitum* (note the use of the modal formulation). The consciousness of this synthetic activity - space as the formal intuition, or "space as an object" - is an intuition produced by this process, but it is also the ground for it: in fact, it is through it that the subject becomes aware of the necessary form of any possible spatial experience. Hence, the model should comprise a structure that is both generated by the inverse system, and that encapsulates all the information contained in the system. This is the inverse limit of the system, which is the smallest structure that retracts to every Stone space consistently with the morphisms. The limit is itself a Stone space, with the crucial property of being infinite, capturing the potential unboundedness of the process of construction in intuition, but still not resulting from the composition of its parts. Indeed, we shall see that the projections of this limit on the inverse system are such that each clopen in the finite Stone spaces is the image of a clopen in the limit, but there are open subsets of the limit that are not the preimage of any open in the finite spaces. This property is the formal correlate to the idea that space as a whole is not the sum of its parts, as remarked by Kant in the last section dedicated to the Antinomies of Pure Reason:

> [...] it is by no means permitted to say of such a whole, which is divisible to infinity, that **it consists of infinitely many parts.** For though all the parts are contained in the intuition of the whole, the **whole division** is **not** contained in it; this division consists only in the progressive decomposition, or in the regress itself, which first makes the series actual. Now since this regress is infinite, all its members (parts) to which it has attained are of course contained in the whole as an **aggregate**, but the whole **series** of the **division** is not, since it is infinite successively and never is **as a whole**; consequently, the regress cannot exhibit any infinite multiplicity or the taking together of this multiplicity into one whole. (A524/B552)

The properties of space as the formal intuition are embodied in the limit in a very pleasant way (as shown in [27], p. 136-153). As we have just seen, infinity corresponds to the notion used by Kant. Unity of space descends from the fact that every "part of space" can be embedded in the "whole" represented by the limit through standard embeddings of the finite Stone spaces in it. Uniqueness of the limit up to isomorphism enforces this notion of unity. Finally, continuity is obtained in the succession of boundaries, as we will soon clarify.

The clopen sets of the limit correspond to regular open sets on the real line. We found convenient to define a relation of *contact* (which in topology is called a *proximity*) between regions that are adjacent in the representation on the real line (this relation is then generalized to the three-dimensional case). Since contact is several times glossed by Kant as "an infinitely small distance"[9] and since "Parts, and thus also matters, are separated, when the contact is [...] destroyed or reduced in quantity"[10] it becomes clear why our Stone spaces, without taking into account contact, only represent separated regions, and so are totally (or even extremally, in the finite case) disconnected. It is only by quotienting the limit by this proximity relation that we obtain, finally, the desired continuum, homeomorphic to the real line (or to the Euclidean three-dimensional space in the generalisation).

The elements of this continuum are the desired boundary points, which emerge as the infinitely small limitations of regions "glueing" them together to form a whole. They will be defined as triplets consisting of two regions complementary to each other and a set in the quotient obtained by intersecting the closures of their images on the continuum. The union of the three gives the whole space. This definition is in agreement with the idea that points are derived entities; in particular, the use of complementary regions seems appropriate in light of the following observation found in Kant's 1786 essay "What does it mean to orient oneself in thinking?" :

> [...]reason needs to presuppose reality as given for the possibility of all things, and considers the differences between things only as limitations arising through the negations attaching to them ([41] 8:138n)

The quotient obtained is then the last, fundamental, component of our model of space "as an object". Completing the inverse limit with boundaries, we obtain a continuum such that its parts "can be distinguished, but not separated, and the *divisio non est realis, sed logica*".

The following table provides a summary of the interpretation of the philosophical notions in our formalization.

[9] [18] 4:505, but also, for instance 4:521-2, where we find:

> [...] since the adjacent parts of a continuous matter are in contact with one another, [...] one then thinks these distances as infinitely small [...]. But the infinitely small intervening space is not at all different from contact.

[10] [18] 4:527]

Philosophical notion	**Formal correlate**
Form of intuition	Finite Stone spaces and finite Boolean algebras
Part of space	Clopen sets in the $\mathbb{X}_n$'s and atoms in the BAs
Figurative synthesis	Inverse system of Stone spaces and direct system of BAs
Synthesis of the unity of apperception	Directedness of the inverse system of Stone spaces under retractions and directedness of the direct system of BAs under homomorphisms
Space as an object (formal intuition)	Limit $\hat{\mathbb{X}}$ of the inverse system equipped with the proximity relation δ on clopen sets
Unity of space as an object	Inverse limit $\hat{\mathbb{X}}$ and its universality up to isomorphism
Kantian-Aristotelian boundaries	Points in the quotient $\hat{\mathbb{X}}/\delta^*$
Common boundary of two regions	Triplet $(P, \overline{P}, \beta P)$ where P and $\overline{P}$ are complementary regions in $\mathbb{X}_n$
Infinity of space	Infinity of the limit (capturing the potential infinity of the inverse system) and "formal boundaries" representing $-\infty$ and $+\infty$
Infinite divisibility of space	Infinite process of bisection, embodied in the direct system of BAs
The spatial continuum	Quotient $\hat{\mathbb{X}}/\delta^*$ of the limit by the proximity relation

2.3.1 Conclusions and ideas for further research

The present work provides an account of Kant's peculiar solution to the problem of the nature of space, focusing on the cognitive processes that enable us to acquire the concepts of space and time. Our analysis tries to unfold the passages of the CPR that expound the philosophical foundations of his transcendental theory, tracing back the synthetic processes underlying the formation of the concept of space. We believe that Kant's theory of space (and time) can be of interest not only for the philosopher, but, in general, for anyone who seeks to investigate the cognitive basis of our perception of space-time. As Pinosio points out[11], the importance of Kant's contributions for contemporary debates in cognitive science, philosophy of physics and mathematics is of

[11] See [27], pp. 50-54, for an exposition of the illuminating perspectives of Kant's architecture of the mind for contemporary cognitive science.

no small import. Our formal model captures the idea of a continuum generated by finite structures in a potential infinite succession, which can be elaborated to support different axiomatisations. A key role is reserved to the relation of contact, and the emergence of points as boundaries of regions through this relation is the distinctive feature that makes our model object of interest for future developments. As for every model, this formal representation does not presume to be an exhaustive description of our perception of space, instead, we hope it can suggest some insights which can turn out to be useful in future developments of cognitive models. An interesting suggestion for further research would be to connect our representation of space to Pinosio's model of time, to get a full-fledged formalization of cognitive space-time and capture the mutual dependence of the two.

The suggestive idea that brought us to analyse Poincaré's conception of space was to connect our construction with his concept of the continuum through the relation of proximity, which can be seen as the 20th century heir of tolerance. However the way we defined contact (and thus proximity) on the limit did not permit such a rendition. We tried, unsuccessfully, to define a different tolerance relation on the limit that could capture the passage from a physical continuum to a mathematical continuum, but due to time restrictions we could not produce any significant result. However, we believe that a development of our formal system in this direction could be object of further investigation, especially in light of the possible determination of a notion of spatial dimension connected to the result obtained by Sossinsky ([36]).

Although our formal attempts to capture Poincaré's point of view failed, we produced a set of interesting observations about the connection between Kant's and Poincaré's philosophies of space. Notwithstanding the clear divergences in their ideas of the necessity or contingency of our spatial representation, we found some elements of continuity in their conceptions of phenomenal space. On the one hand, we hypothesised that Kant would have revised his idea of the absolute inconceivability of a non-Euclidean being, if an Euclidean model of its reasoning could be provided. On the other hand, we claimed that Poincaré's conventionalism applied to theories of physical space, but when it comes to phenomenal space he relied on Euclidean spaces to represent space in a way that everyone would understand. The mathematical interchangeability of Euclidean and non-Euclidean geometries does not imply that they are equally good models of our sensory experience. Moreover, we saw how the late Poincaré renounced his strong belief in the impossibility of *a priori* spatial intuitions. His conception of the intuition of a continuum as condition of possibility of experience has impressive Kantian resonances, although he never accepted any form of transcendental idealism.

In sum, we think our work provided an interesting account of the possibilities Kant's transcendental philosophy has offered and could offer in the future, through the development of logical and cognitive models of space. Our analysis brought to light some aspects of Kant's influence on the philosophy of one of the most brilliant mathematicians of the 19th century, and several studies are being published to prove the significance of his theory of cognition to the contemporary philosophical and scientific debates. We believe that any contribution to the subject can be helpful to generate new, stimulating ideas . With this work, we hope to have offered some useful observations and comments to be added to the increasingly evolving literature on Kant.

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