JANKOV FORMULAS AND AXIOMATIZATION TECHNIQUES FOR INTERMEDIATE LOGICS

G. BEZHANISHVILI AND N. BEZHANISHVILI

Abstract. We discuss some of Jankov's contributions to the study of intermediate logics, including the development of what have become known as Jankov formulas and a proof that there are continuum many intermediate logics. We also discuss how to generalize Jankov's technique to develop axiomatization methods for all intermediate logics. These considerations result in what we term subframe and stable canonical formulas. Subframe canonical formulas are obtained by working with the disjunction-free reduct of Heyting algebras, and are the algebraic counterpart of Zakharyaschev's canonical formulas. On the other hand, stable canonical formulas are obtained by working with the implication-free reduct of Heyting algebras, and are an alternative to subframe canonical formulas. We explain how to develop the standard and selective filtration methods algebraically to axiomatize intermediate logics by means of these formulas. Special cases of these formulas give rise to the classes of subframe and stable intermediate logics, and the algebraic account of filtration techniques can be used to prove that they all posses the finite model property (fmp). The fmp results about subframe and cofinal subframe logics yield algebraic proofs of the results of Fine and Zakharyaschev. We conclude by discussing the operations of subframization and stabilization of intermediate logics that this approach gives rise to.

Contents

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We are very happy to be able to contribute to this volume dedicated to V. A. Jankov. His work has been very influential for many generations of logicians, initially in the former Soviet Union, but eventually also abroad. In particular, it had a profound impact on our own research. While we have never met Professor Jankov in person, we have heard lots of interesting stories about him from our advisor Leo Esakia. Jankov is not only an outstanding logician, but also a role model citizen, who stood up against the Soviet regime. Because of this, he ended up in the Soviet political camps. A well-known Georgian dissident and human rights activist Levan Berdzenishvili spent several years there with Jankov. We refer to his memoirs [2] about the Soviet political camps of 1980s in general, and about Jankov in particular. One chapter of the book "Vadim" (pp. 127–141) is dedicated to Jankov, in which he is characterized as follows: "I can say with certainty that in our political prison, Vadim Yankov, omnipotent and always ready to help, embodied in the pre-Internet era the combined capabilities of Google, Yahoo, and Wikipedia".

1. Introduction

Intermediate logics are the logics that are situated between the intuitionistic propositional calculus IPC and the classical propositional calculus CPC. The study of intermediate logics was pioneered by Umezawa [77]. As was pointed out by Hosoi [48], such a study may be viewed as a study of the classification of classically valid principles in terms of their interdeducibility in intuitionistic logic.

Jankov belongs to the first wave of researchers (alongside Dummett, Lemmon, Kuznetsov, Medvedev, Hosoi, de Jongh, Troelstra, and others) who obtained fundamental results in the study of intermediate logics. He is best known for developing algebra-based formulas, which he called characteristic formulas, but are now commonly known as Jankov formulas. This allowed him to obtain deep results about the complicated structure of the lattice of intermediate logics. His first paper on these formulas dates back to 1963, and is one of the early jewels in the study of intermediate logics [51].

From the modern perspective, Jankov formulas axiomatize splittings and their joins in the lattice of intermediate logics. But this came to light later, after the fundamental work of McKenzie [62]. In 1980s Blok and Pigozzi [26] built on these results to develop a general theory of splittings in varieties with EDPC (equationally definable principal congruences). It should be pointed out that Jankov formulas were independently developed by de Jongh [37]. Because of this, Jankov formulas are also known as Jankov-de Jongh formulas [19, Rem. 3.3.5]. We point out that Jankov's technique was algebraic, while de Jongh mostly worked with Kripke frames.

Jankov [52] utilized his formulas to develop a method for generating continuum many intermediate logics, thus refuting an earlier erroneous attempt of Troelstra [76] that there are only countably many intermediate logics. Jankov's method also allowed him to construct the first intermediate logic without the finite model property (fmp). These results had major impact on the study of lattices of intermediate and modal logics.

We can associate the Jankov formula $\mathcal{J}(A)$ to any finite subdirectly irreducible Heyting algebra A. Then given an arbitrary Heyting algebra B, we can think of the validity of $\mathcal{J}(A)$ on B as forbidding A to be isomorphic to a subalgebra of a homomorphic image of B. This approach was adopted to modal logic by Rautenberg [70], and was further refined by Kracht [56] and Wolter [79]. An important result in this direction was obtained by Blok [25], who characterized splitting modal logics and described the degree of Kripke completeness for extensions of the basic modal logic K.

Independently of Jankov, Fine [44] developed similar formulas for the modal logic $\mathsf{S4}$ by utilizing its Kripke semantics. He associated a formula with each finite rooted $\mathsf{S4}$-frame $\mathfrak{F}$. The validity of such a formula on an $\mathsf{S4}$-frame $\mathfrak{G}$ forbids that $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{G}$. Because of this, these formulas are sometimes called Jankov-Fine formulas in the modal logic literature (see [24, p. 143] and [28, p. 332]).

In [45] Fine undertook a different approach by "forbidding" p-morphic images of arbitrary (not necessarily generated) subframes. This has resulted in the theory of subframe logics, which was further generalized by Zakharyaschev [83] to cofinal subframe logics. While Jankov and (cofinal) subframe formulas axiomatize large classes of logics, not every logic is axiomatized by them. This was addressed by Zakharyaschev [81, 82] who generalized these formulas to what he termed "canonical formulas" and proved that each intermediate logic and each extension of the modal logic $\mathsf{K4}$ is axiomatized by canonical formulas. Zakharyaschev's approach followed the path of Fine's and mainly utilized Kripke semantics. An algebraic approach to subframe and cofinal subframe logics via nuclei was developed for intermediate logics in [16], and was generalized to modal logics in [17].

In a series of papers [4, 5, 6], we developed an algebraic treatment of Zakharyaschev's canonical formulas, as well as of subframe and cofinal subframe formulas. This was done for intermediate logics, as well as for extensions of $\mathsf{K4}$ (and even for extensions of weak $\mathsf{K4}$). A somewhat similar approach was undertaken independently and slightly earlier by Tomaszewski [75]. The key idea of this approach for intermediate logics is that the $\vee$-free reduct of each Heyting algebra is locally finite. This is a consequence of a celebrated result of Diego [39] that the variety of (bounded) implicative semilattices is locally finite. Note that Heyting algebras have another locally finite reduct, which is even better known, namely the $\to$-free reduct. Indeed, it is a classic result that the variety of bounded distributive lattices is locally finite. Thus, it is possible to develop another kind of canonical formulas that also axiomatize all intermediate logics. This was done in [8] and generalized to modal logic in [11].[1]

To distinguish between these two types of canonical formulas, we call the algebraic counterpart of Zakharyaschev's canonical formulas *subframe canonical formulas*. This is motivated by the fact that dually subframe canonical formulas forbid p-morphic images from subframes (see Section 5.1). On the other hand, we call the other kind *stable canonical formulas* because they forbid stable images of generated subframes (see Section 5.2). In special cases, both types of canonical formulas yield Jankov formulas. An additional special case for subframe canonical formulas gives rise to subframe and cofinal subframe formulas of Fine [45] and Zakharyaschev [81, 83]. A similar special case for stable subframe formulas gives rise to new classes of stable and cofinal stable formulas studied in [8, 12] for intermediate logics[2] and in [11, 13] for modal logics. Our aim is to provide a uniform account of this line of research.

In this paper we only concentrate on the theory of canonical formulas for intermediate logics, which is closer to Jankov's original motivation and interests. We plan to discuss the theory of canonical formulas for modal logics elsewhere. As a rule of thumb, we supply sketches of proofs only for several central results. For the rest, we provide relevant references, so that it is easy for the interested reader to look up the details.

The paper is organized as follows. In Section 2 we recall the basic facts about intermediate logics, their algebraic and Kripke semantics, and outline Esakia duality for Heyting algebras. In Section 3 we overview the method of Jankov formulas and its main consequences, such as the Splitting Theorem and the cardinality of the lattice of intermediate logics. In Section 4 we extend the method of Jankov formulas to that of subframe and stable canonical formulas, and show that these formulas axiomatize all intermediate logics. Section 5 provides a dual approach to subframe and stable canonical formulas. In Section 6 we review the theory of subframe and cofinal subframe logics, and in Section 7 that of stable and cofinal stable logics. Finally, in Section 8 we discuss the operations of subframization and stabiliization for intermediate logics, and their characterization via subframe and stable formulas.

[1]The results of [8] were obtained earlier than those in [11]. However, the latter appeared in print earlier than the former.

[2]Again, the results of [8] were obtained earlier but appeared later than those in [12].

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2. Intermediate logics

In this preliminary section, to keep the paper self-contained, we briefly review intermediate logics and their algebraic and relational semantics.

2.1. **Intermediate logics.** As we pointed out in the Introduction, a propositional logic L (in the language of IPC) is an *intermediate logic* if $\mathsf{IPC} \subseteq \mathsf{L} \subseteq \mathsf{CPC}$. Intermediate logics are also called superintuitionistic logics (Kuznetsov's terminology). To be more precise, a propositional logic L is a *superintuitionistic logic* (or *si-logic* for short) if $\mathsf{IPC} \subseteq \mathsf{L}$. Since CPC is the largest consistent si-logic, we have that intermediate logics are precisely the consistent si-logics.

We identify each intermediate logic L with the set of theorems of L. It is well known that the collection of all intermediate logics, ordered by inclusion, is a complete lattice, which we denote by Λ. The meet in Λ is set-theoretic intersection, while the join $\bigvee\{\mathsf{L}_i \mid i \in I\}$ is the least intermediate logic containing $\bigcup\{\mathsf{L}_i \mid i \in I\}$. Clearly IPC is the least element and CPC the largest element of Λ.

For an intermediate logic L and a formula φ, we denote by $L + \varphi$ the least intermediate logic containing $L \cup \{\varphi\}$. As usual, if φ is provable in L, we write $\mathsf{L} \vdash \varphi$. For $\mathsf{L}, \mathsf{M} \in \Lambda$, if $\mathsf{L} \subseteq \mathsf{M}$, then we say that M is an *extension* of L.

As we pointed out in the Introduction, Jankov [52] proved that the cardinality of Λ is that of the continuum. Below we give a list of some well known intermediate logics (see, e.g., [28, p. 112, Table 4.1]).

(1) $\mathsf{KC} = \mathsf{IPC} + (\neg p \vee \neg\neg p)$ — the logic of the weak excluded middle.
(2) $\mathsf{LC} = \mathsf{IPC} + (p \to q) \vee (q \to p)$ — the Gödel-Dummett logic.
(3) $\mathsf{KP} = \mathsf{IPC} + (\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$ — the Kreisel-Putnam logic.
(4) $\mathsf{T}_n = \mathsf{IPC} + \mathsf{t}_n$ $(n \geq 1)$ — the Gabbay-de Jongh logics — where

$$\mathsf{t}_n = \bigwedge_{i=0}^{n} \Big((p_i \to \bigvee_{i \neq j} p_j) \to \bigvee_{i \neq j} p_j\Big) \to \bigvee_{i=0}^{n} p_i\Big).$$

(5) $\mathsf{BD}_n = \mathsf{IPC} + \mathsf{bd}_n$ $(n \geq 1)$, where

$$\begin{aligned} \mathsf{bd}_1 &= p_1 \vee \neg p_1, \\ \mathsf{bd}_{n+1} &= p_{n+1} \vee (p_{n+1} \to \mathsf{bd}_n). \end{aligned}$$

(6) $\mathsf{LC}_n = \mathsf{LC} + \mathsf{bd}_n$ $(n \geq 1)$ — the n-valued Gödel-Dummett logic.
(7) $\mathsf{BW}_n = \mathsf{IPC} + \mathsf{bw}_n$ $(n \geq 1)$, where

$$\mathsf{bw}_n = \bigvee_{i=0}^{n} (p_i \to \bigvee_{j \neq i} p_j).$$

(8) $\mathsf{BTW}_n = \mathsf{IPC} + \mathsf{btw}_n$ $(n \geq 1)$, where

$$\mathsf{btw}_n = \bigwedge_{0 \leq i < j \leq n}^{n} \neg(\neg p_i \wedge \neg p_j) \to \bigvee_{i=0}^{n} \Big(\neg p_i \to \bigvee_{j \neq i} \neg p_j\Big).$$

(9) $\mathsf{BC}_n = \mathsf{IPC} + \mathsf{bc}_n$ $(n \geq 1)$, where

$$\mathsf{bc}_n = p_0 \vee (p_0 \to p_1) \vee \cdots \vee (p_0 \wedge \cdots \wedge p_{n-1} \to p_n).$$

(10) $\mathsf{ND}_n = \mathsf{IPC} + (\neg p \to \bigvee_{1 \leq i \leq n} \neg q_i) \to \bigvee_{1 \leq i \leq n} (\neg p \to \neg q_i)$ $(n \geq 2)$ — Maksimova's logics.

2.2. **Heyting algebras.** We next recall the algebraic semantics of intermediate logics. A *Heyting algebra* is a bounded distributive lattice A with an additional binary operation $\to$ satisfying

$$a \wedge x \leq b \text{ iff } x \leq a \to b$$

for all $a, b, x \in A$. It is well known (see, e.g., [69, p. 124]) that the class of Heyting algebras is equationally definable. For Heyting algebras A and B, a *Heyting homomorphism* is a bounded lattice homomorphism $h : A \to B$ such that $h(a \to b) = h(a) \to h(b)$ for $a, b \in A$.

Definition 2.1. Let Heyt be the category (and the corresponding equational class) of Heyting algebras and Heyting homomorphisms.

A *valuation* v on a Heyting algebra A is a map from the set of propositional variables to A. It is extended to all formulas in an obvious way. A formula φ is *valid* on A if $v(\varphi) = 1$ for every valuation v on A. If φ is valid on A we write $A \models \varphi$. For a class $\mathcal{K}$ of Heyting algebras we write $\mathcal{K} \models \varphi$ if $A \models \varphi$ for each $A \in \mathcal{K}$.

For a Heyting algebra A and a class $\mathcal{K}$ of Heyting algebras, let

$$\mathsf{L}(A) = \{\varphi \mid A \models \varphi\} \text{ and } \mathsf{L}(\mathcal{K}) = \bigcap\{\mathsf{L}(A) \mid A \in \mathcal{K}\}.$$

It is well known that if A is a nontrivial Heyting algebra and $\mathcal{K}$ is a nonempty class of nontrivial Heyting algebras, then $\mathsf{L}(A)$ and $\mathsf{L}(\mathcal{K})$ are intermediate logics. We call $\mathsf{L}(A)$ the *logic* of A, and $\mathsf{L}(\mathcal{K})$ the *logic* of $\mathcal{K}$.

Definition 2.2. We say that an intermediate logic L is *sound and complete* with respect to a class $\mathcal{K}$ of Heyting algebras if $\mathsf{L} = \mathsf{L}(\mathcal{K})$; that is, $\mathsf{L} \vdash \varphi$ iff $\mathcal{K} \models \varphi$.

For a class $\mathcal{K}$ of Heyting algebras, we let $\mathbf{H}(\mathcal{K})$, $\mathbf{S}(\mathcal{K})$, $\mathbf{P}(\mathcal{K})$, and $\mathbf{I}(\mathcal{K})$ be the classes of homomorphic images, subalgebras, products, and isomorphic copies of algebras from $\mathcal{K}$. A *variety* is a class of algebras closed under $\mathbf{H}, \mathbf{S}$, and $\mathbf{P}$. By Birkoff's celebrated theorem (see, e.g., [27, Thm. 11.9]), varieties are precisely the equationally definable classes of algebras.

By the celebrated Lindenbaum algebra construction (see, e.g., [69, Ch. VI]), each intermediate logic is sound and complete with respect to the variety of Heyting algebras

$$\mathcal{V}(\mathsf{L}) := \{A \in \mathsf{Heyt} \mid A \models \mathsf{L}\}.$$

This variety is often called the variety *corresponding* to L. We call $A \in \mathcal{V}(\mathsf{L})$ an L-*algebra*.

We recall that a Heyting algebra A is *subdirectly irreducible* if it has a least nontrivial congruence. It is well known (see, e.g., [1, p. 179, Thm. 5]) that A is subdirectly irreducible iff $A \setminus \{1\}$ has the largest element s, called the *second largest element* of A.

Remark 2.3. This result also originates with Jankov, who referred to these algebras as Gödelean (see [51]).

A Heyting algebra A is *finitely subdirectly irreducible* or *well-connected* if

$$a \vee b = 1 \Rightarrow a = 1 \text{ or } b = 1$$

for each $a, b \in A$. Obviously each subdirectly irreducible Heyting algebra is well-connected, but there exist infinite well-connected Heyting algebras that are not subdirectly irreducible. On the other hand, a finite Heyting algebra is subdirectly irreducible iff it is well-connected.

By another celebrated result of Birkhoff (see, e.g., [27, Thm. 8.6]), each variety $\mathcal{V}$ is generated by subdirectly irreducible members of $\mathcal{V}$. Thus, each intermediate logic is complete with respect to the class of subdirectly irreducible algebras in $\mathcal{V}(\mathsf{L})$.

The next definition and theorem are well known, and go back to Kuznetsov.

Definition 2.4. Let L be an intermediate logic.

(1) Two formulas φ, ψ are L-*equivalent* if $\mathsf{L} \vdash \varphi \leftrightarrow \psi$.

(2) L is *locally tabular* if for each natural number n, there are only finitely many non-L-equivalent formulas in n-variables.
(3) L is *tabular* if L is the logic of a finite Heyting algebra.
(4) L has the *finite model property* (fmp for short) if $\mathsf{L} \nvdash \varphi$ implies that there is a finite Heyting algebra A such that $A \models \mathsf{L}$ and $A \not\models \varphi$.
(5) L has the *hereditary finite model property* (hfmp for short) if L and all its extensions have the fmp.

Theorem 2.5.
(1) L *is locally tabular iff* $\mathcal{V}(\mathsf{L})$ *is locally finite* (*each finitely generated* $\mathcal{V}(\mathsf{L})$*-algebra is finite*).
(2) L *is tabular iff* $\mathcal{V}(\mathsf{L})$ *is generated by a finite algebra.*
(3) L *has the fmp iff* $\mathcal{V}(\mathsf{L})$ *is generated by the class of finite* $\mathcal{V}(\mathsf{L})$*-algebras.*
(4) L *has the hfmp iff each subvariety of* $\mathcal{V}(\mathsf{L})$ *is generated by the class of its finite algebras.*

The next definition goes back to Kuznetsov [57].

Definition 2.6.
(1) Let Λ_{t} be the subclass of Λ consisting of tabular intermediate logics.
(2) Let Λ_{lt} be the subclass of Λ consisting of locally tabular intermediate logics.
(3) Let Λ_{fmp} be the subclass of Λ consisting of intermediate logics with the fmp.
(4) Let Λ_{hfmp} be the subclass of Λ consisting of intermediate logics with the hfmp.

We then have the following hierarchy of Kuznetsov [57]:

$$\Lambda_{\mathsf{t}} \subsetneq \Lambda_{\mathsf{lt}} \subsetneq \Lambda_{\mathsf{hfmp}} \subsetneq \Lambda_{\mathsf{fmp}} \subsetneq \Lambda.$$

2.3. **Kripke frames and Esakia spaces.** We now turn to Kripke semantics for intermediate logics. In this case Kripke frames are simply posets (partially ordered sets). We denote the partial order of a poset P by $\leq$. For $S \subseteq P$, the *downset* of S is the set

$${\downarrow}S = \{x \in P \mid \exists s \in S \text{ with } x \leq s\}.$$

The *upset* of S is defined dually and is denoted by ${\uparrow}S$. If S is a singleton set $\{x\}$, then we write ${\downarrow}x$ and ${\uparrow}x$ instead of ${\downarrow}\{x\}$ and ${\uparrow}\{x\}$.

We call $U \subseteq P$ an *upset* if ${\uparrow}U = U$ (that is, $x \in U$ and $x \leq y$ imply $y \in U$). A *downset* of P is defined dually. Also, we let $\max(U)$ and $\min(U)$ be the sets of maximal and minimal points of U.

Let $\mathsf{Up}(P)$ and $\mathsf{Do}(P)$ be the sets of upsets and downsets of X, respectively. It is well known that $(\mathsf{Up}(P), \cap, \cup, \to, \varnothing, P)$ is a Heyting algebra, where for each $U, V \in \mathrm{Up}(X)$, we have:

$$U \to V = \{x \in P \mid {\uparrow}x \cap U \subseteq V\} = P \setminus {\downarrow}(U \setminus V).$$

Similarly, $(\mathsf{Do}(P), \cap, \cup, \to, \varnothing, P)$ is a Heyting algebra, but we will mainly work with the Heyting algebra of upsets of X.

Each Heyting algebra A is isomorphic to a subalgebra of the Heyting algbera of upsets of some poset. We call this representation the *Kripke representation* of Heyting algebras. Let X_A be the set of prime filters of A, ordered by inclusion. Then X_A is a poset, known as the *spectrum* of A. Define the *Stone map* $\zeta : A \to \mathsf{Up}(X_A)$ by

$$\zeta(a) = \{x \in X_A \mid a \in x\}.$$

Then ζ is a Heyting algebra embedding, and we arrive at the following well-known theorem.

Theorem 2.7 (Kripke representation)**.** *Each Heyting algebra is isomorphic to a subalgebra of* $\mathsf{Up}(X_A)$.

To recover the image of A in $\mathsf{Up}(X_A)$, we need to introduce a topology on X_A. We recall that a subset of a topological space X is *clopen* if it is both closed and open, and that X is *zero-dimensional* if clopen sets form a basis for X. A *Stone space* is a compact, Hausdorff, zero-dimensional space. By the celebrated Stone duality [74], the category of Boolean algebras and Boolean homomorphisms is dually equivalent to the category of Stone spaces and continuous maps. In particular, each Boolean algebra A is represented as the Boolean algebra of clopens of a Stone space (namely, of the prime spectrum of A) which is unique up to homeomorphism.

Stone duality for Boolean algebras was generalized to Heyting algebras by Esakia [40] (see also [42]).

Definition 2.8. An *Esakia space* is a Stone space X which in addition is a poset and the partial order $\leq$ is *continuous*, meaning that the following two conditions are satisfied:

(1) ${\uparrow}x$ is a closed set for each $x \in X$.
(2) U clopen implies that ${\downarrow}U$ is clopen.

We recall that a map $f : P \to Q$ between two posets is a *p-morphism* if ${\uparrow}f(x) = f[{\uparrow}x]$ for each $x \in P$. For Esakia spaces X and Y, a map $f : X \to Y$ is an *Esakia morphism* if it is a continuous p-morphism.

Definition 2.9. Let Esa be the category of Esakia spaces and Esakia morphisms.

Theorem 2.10 (Esakia duality)**.** Heyt *is dually equivalent to* Esa*.*

In particular, each Heyting algebra A is represented as the Heyting algebra of clopen upsets of the prime spectrum X_A of A, where the topology on X_A is defined by the basis

$$\{\zeta(a) \setminus \zeta(b) \mid a, b \in A\}.$$

We refer to this representation as the *Esakia representation* of Heyting algebras.

If we restrict Esakia duality to the finite case, we obtain that the category of finite Heyting algebras is dually equivalent to the category of finite posets. In particular, each finite Heyting algebra A is isomorphic to $\mathsf{Up}(X_A)$. We refer to this duality as *finite Esakia duality* (but point out that this finite duality has been known before Esakia; see e.g., [38]).

It follows from Esakia duality that onto Heyting homomorphisms dually correspond to one-to-one Esakia morphisms, and one-to-one Heyting homomorphisms to onto Esakia morphisms. In particular, homomorphic images of a Heyting algebra A correspond to closed upsets of X_A, while subalgebras of A to special quotients of X_A known as *Esakia quotients* (see, e.g., [10]).

Definition 2.11. Let X be an Esakia space.

(1) We call X *rooted* if there is $x \in X$, called the *root* of X, such that $X = {\uparrow}x$.
(2) We call X *strongly rooted* if X is rooted and the singleton $\{x\}$ is clopen.

It is well known (see, e.g., [41] or [3]) that a Heyting algebra A is well-connected iff X_A is rooted, and that A is subdirectly irreducible iff X_A is strongly rooted.

We evaluate formulas in a poset P by evaluating them in the Heyting algebra $\mathsf{Up}(P)$, and we evaluate formulas in an Esakia space X by evaluating them in the Heyting algebra of clopen upsets of X. These clopen upsets are known as *definable upsets* of X, so such valuations are called *definable valuations*.

Since each intermediate logic L is complete with respect to Heyting algebras, it follows from Esakia duality that L is complete with respect to Esakia spaces (but not necessarily with respect to posets as it is known [73] that there exist Kripke incomplete intermediate logics).

For a class $\mathcal{K}$ of posets or Esakia spaces, let $\mathcal{K}^*$ be the corresponding class of Heyting algebras (of all upsets or definable upsets of members of $\mathcal{K}$). We then say that an intermediate logic L is the *logic* of $\mathcal{K}$ if L is the logic of $\mathcal{K}^*$.

Definition 2.12. Let P be a finite poset and $n \geq 1$.

(1) The *length* of a chain in P is its cardinality.
(2) The *depth* of P is $\leq n$, denoted $d(P) \leq n$, if all chains in P have length $\leq n$.
(3) The *width* of $x \in P$ is $\leq n$ if the length of antichains in ${\uparrow}x$ is $\leq n$.
(4) The *cofinal width* (or *top width*) of $x \in P$ is $\leq n$ if $|\max({\uparrow}x)| \leq n$.
(5) The *width* of P is $\leq n$, denoted $w(P) \leq n$, if the width of each $x \in P$ is $\leq n$.
(6) The *cofinal width* (or *top width*) of P is $\leq n$, denoted $w_c(P) \leq n$, if the cofinal width of each $x \in P$ is $\leq n$.
(7) The *branching* of P is $\leq n$, denoted $b(P) \leq n$, if each $x \in P$ has at most n distinct immediate successors.
(8) The *divergence* of P is $\leq n$, denoted $div(P) \leq n$, if for each $x \in P$ and $Q \subseteq {\uparrow}x \cap \max(P)$ satisfying $|Q| \leq n$, there is $y \geq x$ with $\max({\uparrow}y) = Q$.

The next theorem is well known (see, e.g., [28]).

Theorem 2.13. *Let $n \geq 1$.*

(1) KC *is the logic of all finite rooted posets that have a largest element.*
(2) LC *is the logic of all finite chains.*
(3) KP *is the logic of all finite rooted posets satisfying*
$$\big(\forall x \forall y \forall z \big(xRy \wedge xRz \wedge \neg(yRz) \wedge \neg(zRy) \to \exists u((xRu \wedge uRy \wedge uRz) \wedge \\ \forall v(uRv \to \exists w(vRw \wedge (yRw \vee zRw))))\big)\big).$$
(4) T_n *is the logic of all finite rooted posets of branching $\leq n$.*
(5) BD_n *is the logic of all finite rooted posets of depth $\leq n$.*
(6) LC_n *is the logic of the chain of length n.*
(7) BW_n *is the logic of all finite rooted posets of width $\leq n$.*
(8) BTW_n *is the logic of all finite rooted posets of cofinal width $\leq n$.*
(9) BC_n *is the logic of all finite rooted posets of cardinality $\leq n$.*
(10) ND_n *is the logic of all finite rooted posets of divergence $\leq n$.*

Thus, each of these logics has the fmp. In fact, LC *as well as each* BD_n *is locally tabular, and each* LC_n *as well as each* BC_n *is tabular.*

3. Jankov formulas

As we pointed out in the Introduction, Jankov first introduced his formulas in the 1963 paper [51] under the name of characteristic formulas. They have since become a major tool in the study of intermediate and modal logics and are often referred to as Jankov formulas [28, p. 332], Jankov-de Jongh formulas [19, p. 59], or Jankov-Fine formulas [24, p. 143]. In this paper we will refer to them as Jankov formulas. First results about Jankov formulas were announced in [51]. Proofs of these results together with further properties of Jankov formulas were given in [53]. In [52] Jankov utilized his formulas to prove that there are continuum many intermediate logics (thus refuting an earlier erroneous claim of Troelstra [76] that there are only countably many intermediate logics). He also gave the first example of an intermediate logic without the fmp.

3.1. **Jankov Lemma.** The basic idea of Jankov formulas is closely related to the method of diagrams in model theory (see, e.g., [29, pp. 68–69]). Let A be a finite Heyting algebra.[3] We can encode the structure of A in our propositional language by describing what is true and what is false in A. This way we obtain two finite sets of formulas, Γ and Δ, where p_a is a new variable for each

[3]While the assumption that A is finite is not essential, it suffices for our purposes.

$a \in A$:

$$\begin{aligned}\Gamma = &\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b \mid a, b \in A\} \cup \\ &\{p_{a\vee b} \leftrightarrow p_a \vee p_b \mid a, b \in A\} \cup \\ &\{p_{a\to b} \leftrightarrow p_a \to p_b \mid a, b \in A\} \cup \\ &\{p_{\neg a} \leftrightarrow \neg p_a \mid a \in A\}\end{aligned}$$

and

$$\Delta = \{p_a \leftrightarrow p_b \mid a, b \in A \text{ with } a \neq b\}.$$

Thus, Γ describes what is true and Δ what is false in A. We can then work with the multiple-conclusion rule Γ/Δ and prove that this rule is characteristic for A in the following sense:

Lemma 3.1. [12] *Let A be a finite Heyting algebra and B an arbitrary Heyting algebra. Then*

$$B \not\models \Gamma/\Delta \text{ iff } A \in \mathbf{IS}(B).^4$$

However, at the time of Jankov, it was unusual to work with multiple-conclusion rules. Instead Jankov assumed that A is subdirectly irreducible. Then A has the second largest element s. Therefore, Δ can be replaced with p_s since everything that is falsified in A ends up underneath s. Thus, we arrive at the following notion of the Jankov formula of A:

Definition 3.2. Let A be a finite subdirectly irreducible Heyting algebra with the second largest element s. Then the *Jankov formula* of A is the formula

$$\mathcal{J}(A) = \bigwedge \Gamma \to p_s.$$

The defining property of Jankov formulas is presented in the following lemma, which we will refer to as the Jankov Lemma. Comparing the Jankov Lemma to Lemma 3.1, we see that the switch from the multiple-conclusion rule Γ/Δ to the formula $\mathcal{J}(A)$ requires on the one hand to assume that A is subdirectly irreducible, and on the other hand to also work with homomorphic images and not only with isomorphic copies of subalgebras of B as in Lemma 3.1.

Lemma 3.3 (Jankov Lemma)**.** *Let A be a finite subdirectly irreducible Heyting algebra and B an arbitrary Heyting algebra. Then*

$$B \not\models \mathcal{J}(A) \text{ iff } A \in \mathbf{ISH}(B).$$

Proof. (Sketch). First suppose that $A \in \mathbf{ISH}(B)$. By evaluating each p_a as a, it is easy to see that A refutes $\mathcal{J}(A)$. Therefore, since $A \in \mathbf{ISH}(B)$, we also have that $B \not\models \mathcal{J}(A)$.

Conversely, suppose that $B \not\models \mathcal{J}(A)$. By [80, Lem. 1], there is a subdrectly irreducible homomorphic image C of B such that $C \not\models \mathcal{J}(A)$. Since C is subdrectly irreducible, the valuation v on C refuting $\mathcal{J}(A)$ must be such that $v(\bigwedge \Gamma) = 1_C$ and $v(p_s) = s_C$, where s_C is the second largest element of C. Define $h : A \to C$ by setting $h(a) = v(p_a)$. That $v(\bigwedge \Gamma) = 1_C$ implies that h is a Heyting homomorphism, and that $h(p_s) = s_C$ yields that h is one-to-one. Thus, $A \in \mathbf{ISH}(B)$. ☐

Remark 3.4. Since the variety of Heyting algebras has the congruence extension property, we have $A \in \mathbf{ISH}(B)$ iff $A \in \mathbf{HS}(B)$. Therefore, the conclusion of the Jankov Lemma is often formulated as follows:

$$B \not\models \mathcal{J}(A) \text{ iff } A \in \mathbf{HS}(B).$$

Since A is a finite subdirectly irreducible Heyting algebra, by Esakia duality, A is isomorphic to the algebra P^* of upsets of a finite rooted poset P. To simplify notation, instead of $\mathcal{J}(A)$ we will often write $\mathcal{J}(P)$. Thus, we obtain the following dual reading of the Jankov Lemma.

Lemma 3.5. *Let P be a finite rooted poset and X an Esakia space. Then $X \not\models \mathcal{J}(P)$ iff P is isomorphic to an Esakia quotient of a closed upset of X.*

[4] This lemma is closely related to [29, Prop. 2.1.8].

3.2. **Splitting Theorem.** A very useful feature of Jankov formulas is that they axiomatize splittings in the lattice of intermediate logics. We recall that a pair (s, t) of elements of a lattice L *splits* L if L is the disjoint union of $\uparrow s$ and $\downarrow t$.

Definition 3.6. [28, Sec. 10.5] An intermediate logic L is a *splitting logic* if there is an intermediate logic M such that (L, M) split the lattice Λ of intermediate logics.

The next theorem is due to Jankov [51, 53], although not in the language of splitting logics.

Theorem 3.7 (Splitting Theorem)**.** *Let* L *be an intermediate logic. Then* L *is a splitting logic iff* $\mathsf{L} = \mathsf{IPC} + \mathcal{J}(\mathsf{A})$ *for some finite subdirectly irreducible Heyting algebra* A*.*

Proof. (Sketch). First suppose that L is a splitting logic. Then there is an intermediate logic M such that (L, M) splits Λ. Since the variety Heyt of Heting algebras is congruence-distributive and is generated by its finite members, a result of McKenzie [62, Thm. 4.3] yields that M is the logic of a finite subdirectly irreducible Heyting algebra A. But then for an arbitrary Heyting algebra B we have $B \models \mathcal{J}(A)$ iff $B \models \mathsf{L}$. Thus, $\mathsf{L} = \mathsf{IPC} + \mathcal{J}(\mathsf{A})$.

For the converse, suppose that $\mathsf{L} = \mathsf{IPC} + \mathcal{J}(\mathsf{A})$ for some finite subdirectly irreducible Heyting algebra A. Let $\mathsf{M} = \mathsf{L}(A)$. We claim that (L, M) splits Λ. To see this, first note that as $A \not\models \mathcal{J}(A)$, we have $\mathcal{J}(A) \notin \mathsf{M}$. Therefore, $\mathsf{L} \neq \mathsf{M}$. Next let N be an intermediate logic such that $\mathsf{L} \not\subseteq \mathsf{N}$. Then $\mathcal{J}(A) \notin \mathsf{N}$. Thus, there is a Heyting algebra B such that $B \models \mathsf{N}$ and $B \not\models \mathcal{J}(A)$. By the Jankov Lemma, $A \in \mathbf{ISH}(B)$. Therefore, $A \models \mathsf{N}$, and hence $\mathsf{N} \subseteq \mathsf{M}$. $\square$

The Splitting Theorem was generalized to varieties with EDPC by Blok and Pigozzi [26].

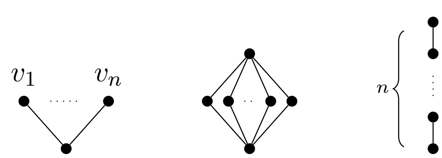
Definition 3.8. Let L be an intermediate logic.

(1) L is a *join-splitting logic* if L is a join in Λ of splitting logics.
(2) L is *axiomatizable by Jankov formulas* if there is a set Ω of finite subdirectly irreducible Heyting algebras such that $\mathsf{L} = \mathsf{IPC} + \{\mathcal{J}(A) \mid A \in \Omega\}$.

As an immediate consequence of the Splitting Theorem we obtain:

Theorem 3.9. *An intermediate logic* L *is a join-splitting logic iff* L *is axiomatizable by Jankov formulas.*

To give examples of intermediate logics that are join-splitting, for each $n \geq 1$, let $\mathfrak{F}_n$ be the n-fork, $\mathfrak{D}_n$ the n-diamond, and $\mathfrak{C}_n$ the n-chain.



$\mathfrak{F}_n$ $\mathfrak{D}_n$ $\mathfrak{C}_n$

n-fork n-diamond n-chain

FIGURE 1

The next theorem is well-known (although finding an exact reference is a challenge).

Theorem 3.10.

(1) $\mathsf{CPC} = \mathsf{IPC} + \mathcal{J}(\mathfrak{C}_2)$*, so* CPC *is a splitting logic.*
(2) $\mathsf{KC} = \mathsf{IPC} + \mathcal{J}(\mathfrak{F}_2)$*, so* KC *is a splitting logic.*
(3) $\mathsf{BD}_n = \mathsf{IPC} + \mathcal{J}(\mathfrak{C}_{n+1})$*, so each* BD_n *is a splitting logic.*
(4) $\mathsf{LC} = \mathsf{IPC} + \mathcal{J}(\mathfrak{F}_2) + \mathcal{J}(\mathfrak{D}_2)$*, so* LC *is a join-splitting logic.*

(5) $\mathsf{LC}_n = \mathsf{LC} + \mathcal{J}(\mathfrak{C}_{n+1})$, *so each* LC_n *is a join-splitting logic.*

This theorem shows that many well-known intermediate logics are indeed axiomatizable by Jankov formulas. However, not every intermediate logic is axiomatizable by Jankov formulas.

Theorem 3.11. [28, Prop. 9.50] BTW_3 *is not axiomatizable by Jankov formulas.*

We next give a criterion describing when an intermediate logic is axiomatizable by Jankov formulas. Define the following relation between Heyting algebras:

$$A \leq B \text{ iff } A \in \mathbf{ISH}(B).$$

Remark 3.12. As follows from Remark 3.4, $A \leq B$ iff $A \in \mathbf{HS}(B)$.

If A is finite and subdirectly irreducible, then by the Jankov Lemma, $A \leq B$ iff $B \not\models \mathcal{J}(A)$. It was noted already by Jankov [53] that $\leq$ is a quasi-order and that if A, B are finite and subdirectly irreducible, then $A \leq B$ and $B \leq A$ imply that A is isomorphic to B. Since the variety of Heyting algebras is congruence-distributive, this is also a consequence of Jónsson's Lemma [55]. The following result giving a criterion for an intermediate logic to be axiomatizable my Jankov formulas is well known. For a proof see, e.g., [19, Cor. 3.4.14].

Theorem 3.13 (Criterion of axiomatizability by Jankov formulas)**.** *Let* L *and* M *be intermediate logics such that* $\mathsf{L} \subseteq \mathsf{M}$*. Then* M *is axiomatizable over* L *by Jankov formulas iff for every Heyting algebra* B *such that* $B \models \mathsf{L}$ *and* $B \not\models \mathsf{M}$ *there is a finite Heyting algebra* A *such that* $A \leq B$, $A \models \mathsf{L}$, *and* $A \not\models \mathsf{M}$.

As a consequence of this criterion, we obtain the following result about axiomatizability for extensions of a locally tabular intermediate logic.

Theorem 3.14. *Let* L *be a locally tabular intermediate logic. Then every extension of* L *is axiomatizable by Jankov formulas over* L.

Proof. Let M be an extension of L and B a Heyting algebra such that $B \models \mathsf{L}$ and $B \not\models \mathsf{M}$. Then there is $\varphi(p_1, \ldots, p_n) \in \mathsf{M}$ such that $B \not\models \varphi(p_1, \ldots, p_n)$. Therefore, there is a valuation v on B refuting φ. Let A be the subalgebra of B generated by $\{v(p_1), \ldots, v(p_n)\}$. Then $A \leq B$ and A is finite since L is locally tabular. In addition, $A \models \mathsf{L}$ as A is a subalgebra of B and $A \not\models \varphi$ because $B \not\models \varphi$. Thus, $A \not\models \mathsf{M}$. By Theorem 3.13, M is axiomatizable over L by Jankov formulas. □

Theorem 3.14 can be generalized in two directions. Firstly we have that every locally tabular intermediate logic is axiomatizable by Jankov formulas, and that every tabular intermediate logic is axiomatizable by finitely many Jankov formulas; see [19, Thms. 3.4.24 and 3.4.27] (and also [32] and [75]).

Theorem 3.15.

(1) *Every locally tabular intermediate logic is axiomatizable by Jankov formulas.*
(2) *Every tabular intermediate logic is finitely axiomatizable by Jankov formulas.*

Secondly the assumption in Theorem 3.14 that L is locally tabular can be weakened to L having the hereditary finite model property.

Theorem 3.16. *Let* L *be an intermediate logic with the hereditary fmp. Then every extension of* L *is axiomatizable over* L *by Jankov formulas.*

Proof. (Sketch). Let M be an extension of L. We let $\mathcal{X}$ be the set of all finite non-isomorphic subdirectly irreducible L-algebras A such that $A \not\models \mathsf{M}$ and consider

$$\mathsf{N} = \mathsf{L} + \{\mathcal{J}(A) \mid A \in \mathcal{X}\}.$$

Let B be a finite subdirectly irreducible Heyting algebra such that $B \models \mathsf{L}$. By definition of $\mathcal{X}$,

$$B \models \mathsf{N} \text{ iff } B \models \mathcal{J}(A) \text{ for each } A \in \mathcal{X} \text{ iff } B \models \mathsf{M}.$$

Since L has the hereditary fmp, both M and N have the fmp. Thus, $\mathsf{M} = \mathsf{N}$, and hence every extension of L is axiomatizable over L by Jankov formulas. □

3.3. **Cardinality of the lattice of intermediate logics.** Jankov formulas are also instrumental in determining cardinalities of different classes of intermediate logics. We call a set Ω of $\leq$-incomparable Heyting algebras an $\leq$-*antichain*. The next theorem is well known (see [52] or [19, Thm. 3.4.18]).

Theorem 3.17. *Let Ω be a countably infinite $\leq$-antichain of finite subdirectly irreducible Heyting algebras. Then for $\Omega_1, \Omega_2 \subseteq \Omega$ with $\Omega_1 \neq \Omega_2$, we have*

(1) $\mathsf{IPC} + \{\mathcal{J}(A) \mid A \in \Omega_1\} \neq \mathsf{IPC} + \{\mathcal{J}(A) \mid A \in \Omega_2\}$.
(2) $\mathsf{L}(\Omega_1) \neq \mathsf{L}(\Omega_2)$.

Proof. (1). Without loss of generality we may assume that $\Omega_1 \not\subseteq \Omega_2$. Therefore, there is $B \in \Omega_1$ with $B \notin \Omega_2$. Since $B \not\models \mathcal{J}(B)$, by the Jankov Lemma, $B \not\models \mathsf{IPC} + \{\mathcal{J}(A) \mid A \in \Omega_1\}$. On the other hand, if $B \not\models \mathsf{IPC} + \{\mathcal{J}(A) \mid A \in \Omega_2\}$, then there is $A \in \Omega_2$ with $B \not\models \mathcal{J}(A)$. By the Jankov Lemma, $A \leq B$. However, since $A, B \in \Omega$, this contradicts the assumption that Ω is an $\leq$-antichain. Thus, $B \models \mathsf{IPC} + \{\mathcal{J}(A) \mid A \in \Omega_2\}$, and hence $\mathsf{IPC} + \{\mathcal{J}(A) \mid A \in \Omega_1\} \neq \mathsf{IPC} + \{\mathcal{J}(A) \mid A \in \Omega_2\}$.

(2). This is proved similarly to (1). □

To construct countable $\leq$-antichains, it is more convenient to use finite Esakia duality and work with finite rooted posets. For this it is convenient to dualize the definition of $\leq$. Let P and Q be finite posets. We set

$$P \leq Q \text{ iff } P \text{ is isomorphic to an Esakia quotient of an upset of } Q.$$

The following lemma is an immediate consequence of finite Esakia duality.

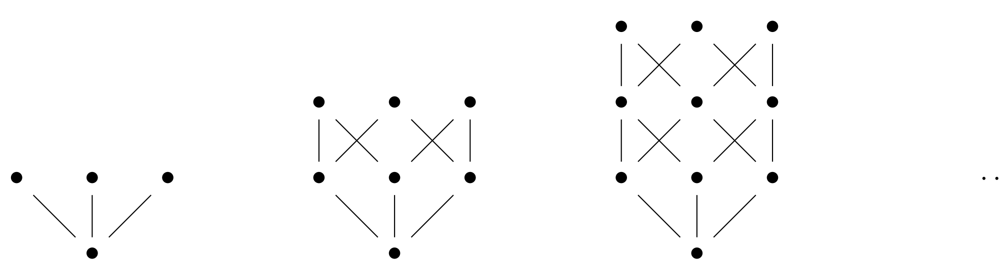
Lemma 3.18. *Let P and Q be finite posets. Then*

$$P \leq Q \textit{ iff } P^* \leq Q^*.$$

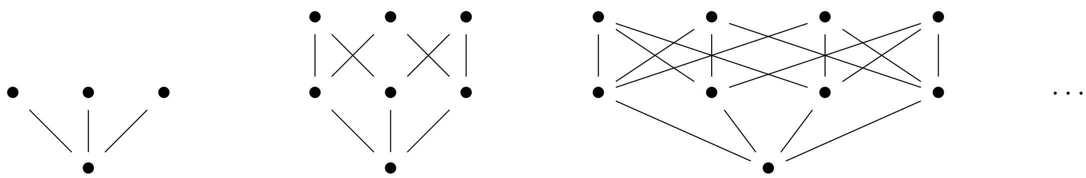
In the next lemma we describe two infinite $\leq$-antichains of finite rooted posets. The antichain Ω_1 is the dual version of Jankov's original antichain [52] of finite subdirectly irreducible Heyting algebras, while the antichain Ω_2 goes back to Kuznetsov [57].

Lemma 3.19. *There exist countably infinite $\leq$-antichains of finite rooted posets.*

Proof. (Sketch). We consider two countably infinite sets Ω_1 and Ω_2 of finite rooted posets. The set Ω_1 has infinite depth but width 3, while the set Ω_2 has infinite width but depth 3.



The antichain Ω_1



The antichain Ω_2

It is a tedious calculation to show that Ω_1 and Ω_2 are indeed $\leq$-antichains. □

As an immediate consequence of Theorem 3.17 and Lemma 3.19 we obtain the cardinality bound for the lattice of intermediate logics.

Theorem 3.20. [52] *There are continuum many intermediate logics.*

In fact, Lemma 3.19 implies a stronger result that the cardinality of intermediate logics of width 3 is that of the continuum, and that the cardinality of intermediate logics of depth 3 is that of the continuum.

Remark 3.21. We conclude this section by mentioning several more applications of Jankov formulas.

(1) Jankov formulas play an important role for obtaining intermediate logics that lack the fmp and are Kripke incomplete. First intermediate logic without the fmp was constructed by Jankov himself in [52]. Further examples of intermediate logics without the fmp were given by Kuznetsov and Gerčiu [58]. In fact, there are continuum many intermediate logics without the fmp (see, e.g., [28, Thm. 6,3] and [9, Cor. 5.41]). In [60] Litak used Jankov formulas to construct continuum many Kripke incomplete intermediate logics.
(2) In [80] Wronski utilized Jankov formulas to construct continuum many intermediate logics with the disjunction property. (We recall that an intermediate logic L has the *disjunction property* if $\mathsf{L} \vdash \varphi \vee \psi$ implies $\mathsf{L} \vdash \varphi$ or $\mathsf{L} \vdash \psi$.)
(3) Jankov formulas are essential in the study of HSC logics (hereditarily structurally complete intermediate logics). As was shown by Citkin [31, 33], there is a least HSC logic which is axiomatized by the Jankov formulas of five finite subdirectly irreducible Heyting algebras (see [23] for a proof using Esakia duality). This was extended to extensions of $\mathsf{K4}$ by Rybakov [72].
(4) A recent result [15] shows that there is a largest variety $\mathcal{V}$ of Heyting algebras in which every profinite algebra is isomorphic to the profinite completion of some algebra in $\mathcal{V}$. Again, $\mathcal{V}$ is axiomatized by the Jankov formulas of four finite subdirectly irreducible Heyting algebras.
(5) Jankov formulas are instrumental in the study of the refutation systems of [35] (these are formalisms that carry the information about what is not valid in a given logic).

4. Canonical formulas

As follows from Theorem 3.11, Jankov formulas do not axiomatize all intermediate logics. However, by Theorem 3.15(1), they do axiomatize every locally tabular intermediate logic. By Theorem 2.5(1), these correspond to locally finite varieties of Heyting algebras. Although the variety Heyt of all Heyting algebras is not locally finite, both the $\vee$-free and $\to$-free reducts of Heyt generate locally finite varieties. Indeed, the $\vee$-free reducts of Heyt generate the variety of bounded implicative semilattices, which is locally finite by Diego's theorem [39]. Also, the $\to$-free reducts of Heyt generate the variety of bounded distributive lattices, which is well known to be locally finite (see, e.g., [47, p. 68, Thm. 1]). On the one hand, these locally finite reducts can be used to prove the fmp of IPC and many other intermediate logics. On the other hand, they allow to develop powerful methods of uniform axiomatization of all intermediate logics.

The key idea is to refine further the Jankov method discussed in the previous section. As we pointed out, the Jankov formula of a finite subdirectly irreducible Heyting algebra A encodes the structure of A in the full signature of Heyting algebras. The refinement of the method consists of encoding fully only the structure of locally finite reduct of A. Then the embedding of A into a homomorphic image of B discussed in the proof of the Jankov Lemma only preserves the operations of the reduct. Yet, the embedding may preserve the remaining operation ($\vee$ or $\to$ depending on which locally finite reduct we work with) only on some elements of A. These constitute what we call the "closed domain" of A. Thus, this new "generalized Jankov formula," which following Zakharyaschev [81, 82, 28] we call the "canonical formula" of A, encodes fully the locally finite reduct of A that we work with, plus the remaining operation only partially, on the closed domain of A. Since we will mainly be working with two locally finite reducts of Heyting algebras, the $\vee$-free and $\to$-free reducts, we obtain two different types of canonical formulas. Based on the dual description of the homomorphisms involved (see Section 5), we call the canonical formulas associated with the $\vee$-free reduct "subframe canonical formulas," and the ones associated with the $\to$-free reduct "stable canonical formulas."

4.1. **Subframe canonical formulas.** In this section we survey the theory of subframe canonical formulas developed in [4] (under the name of $(\wedge, \to, 0)$-canonical formulas). For this we will work with bounded implicative semilattices.

In Section 3, with each finite subdirectly irreducible Heyting algebra A we associated the Jankov formula $\mathcal{J}(A)$ of A which encodes the structure of A in the full signature of Heyting algebras. The subframe canonical formula of A encodes the bounded implicative semilattice structure of A fully, but the behavior of $\vee$ only partially on some specified subset $D \subseteq A^2$.

Definition 4.1. Let A be a finite subdirectly irreducible Heyting algebra, s the second largest element of A, and D a subset of A^2. For each $a \in A$ we introduce a new variable p_a and define the *subframe canonical formula* $\alpha(A, D)$ associated with A and D as

$$\begin{aligned}\alpha(A, D) = \Big(& \bigwedge\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b \mid a, b \in A\} \wedge \\ & \bigwedge\{p_{a\to b} \leftrightarrow p_a \to p_b \mid a, b \in A\} \wedge \\ & \bigwedge\{p_{\neg a} \leftrightarrow \neg p_a \mid a \in A\} \wedge \\ & \bigwedge\{p_{a\vee b} \leftrightarrow p_a \vee p_b \mid (a, b) \in D\}\Big) \to p_s\end{aligned}$$

Remark 4.2. If $D = A^2$, then $\alpha(A, D) = \mathcal{J}(A)$.

Let A and B be Heyting algebras. We recall that a map $h : A \to B$ is an *implicative semilattice homomorphism* if

$$h(a \wedge b) = h(a) \wedge h(b) \text{ and } h(a \to b) = h(a) \to h(b)$$

for each $a, b \in A$. It is easy to see that implicative semilattice homomorphisms preserve the top element, but they may not preserve the bottom element. Thus, we call h *bounded* if $h(0) = 0$.

Definition 4.3. Let A, B be Heyting algebras, $D \subseteq A^2$, and $h : A \to B$ a bounded implicative semilattice homomorphism. We call D a $\vee$-*closed domain* of A and say that h satisfies the $\vee$-*closed domain condition* for D if $h(a \vee b) = h(a) \vee h(b)$ for $(a, b) \in D$.

To simplify notation, we abbreviate the $\vee$-closed domain condition by $\mathrm{CDC}_\vee$. An appropriate modification of the Jankov Lemma yields:

Lemma 4.4 (Subframe Jankov Lemma)**.** [4, Thm. 5.3] *Let A be a finite subdirectly irreducible Heyting algebra, $D \subseteq A^2$, and B an arbitrary Heyting algebra. Then $B \not\models \alpha(A, D)$ iff there is a homomorphic image C of B and a bounded implicative semilattice embedding $h : A \rightarrowtail C$ satisfying $\mathrm{CDC}_\vee$ for D.*

Our second key tool for the desired uniform axiomatization of intermediate logics is what we call the Selective Filtration Lemma. The name is motivated by the fact that it provides an algebraic account of the Fine-Zakharyaschev method of selective filtration for intermediate logics (see, e.g., [28, Thm. 9.34]). For a detailed comparison of the algebraic and frame-theoretic methods of selective filtration we refer to [7]. To formulate the Selective Filtration Lemma we require the following definition.

Definition 4.5. Let A, B be Heyting algebras with $A \subseteq B$. We say that A is a $(\wedge, \to)$-*subalgebra* of B if A is closed under $\wedge$ and $\to$, and we say that A is a $(\wedge, \to, 0)$-*subalgebra* of B if in addition $0 \in A$.

Lemma 4.6 (Selective Filtration Lemma). *Let B be a Heyting algebra such that $B \not\models \varphi$. Then there is a finite Heyting algebra A such that A is a $(\wedge, \to, 0)$-subalgebra of B and $A \not\models \varphi$. In addition, if B is subdirectly irreducible, then A can be chosen to be subdirectly irreducible as well.*

Proof. Since $B \not\models \varphi$, there is a valuation v on B such that $v(\varphi) \neq 1_B$. Let $\mathrm{Sub}(\varphi)$ be the set of subformulas of φ and let A be the $(\wedge, \to, 0)$-subalgebra of B generated by $v[\mathrm{Sub}(\varphi)]$. If B is subdirectly irreducible, then it has the second largest element s, and we generate A by $\{s\} \cup v[\mathrm{Sub}(\varphi)]$. By Diego's theorem, A is finite. Therefore, A is a finite Heyting algebra, where

$$a \vee_A b = \bigwedge \{c \in A \mid a, b \leq c\}$$

for each $a, b \in A$. It is easy to see that $a \vee b \leq a \vee_A b$ and that $a \vee_A b = a \vee b$ whenever $a \vee b \in A$. Since for $a, b \in v[\mathrm{Sub}(\varphi)]$, if $a \vee b \in v[\mathrm{Sub}(\varphi)]$, then $a \vee_A b = a \vee b$, we see that the value of φ in A is the same as the value of φ in B. As $v(\varphi) \neq 1_B$, we conclude that $v(\varphi) \neq 1_A$. Thus, A is a finite Heyting algebra that is a $(\wedge, \to, 0)$-subalgebra of B and refutes φ. Finally, if B is subdirectly irreducible, then s is also the second largest element of A, so A is subdirectly irreducible as well. $\square$

Now suppose that $\mathsf{IPC} \nvdash \varphi$ and $n = |\mathrm{Sub}(\varphi)|$. Since the variety of bounded implicative semilattices is locally finite, there is a bound $c(\varphi)$ on the number of n-generated bounded implicative semilattices. Let $A_1, \ldots, A_{m(n)}$ be the list of finite subdirectly irreducible Heyting algebras such that $|A_i| \leq c(\varphi)$ and $A_i \not\models \varphi$.

For an algebra A refuting φ via a valuation v, let $\Theta = v[\mathrm{Sub}(\varphi)]$ and let

$$D^\vee = \{(a, b) \in \Theta^2 \mid a \vee b \in \Theta\}.$$

Consider a new list $(A_1, D_1^\vee), \ldots, (A_{k(n)}, D_{k(n)}^\vee)$, and note that in general $k(n)$ can be greater than $m(n)$ since each A_i may refute φ via different valuations.

We have the following characterization of refutability:

Theorem 4.7. [4, Thm. 5.7 and Cor. 5.10] *Let B be a Heyting algebra.*

(1) *$B \not\models \varphi$ iff there is $i \leq k(n)$, a homomorphic image C of B, and a bounded implicative semilattice embedding $h : A_i \rightarrowtail C$ satisfying* CDC$_\vee$ *for $D_i^\vee$.*

(2) *$B \models \varphi$ iff $B \models \bigwedge_{i=1}^{k(n)} \alpha(A_i, D_i^\vee)$.*

Proof. Since (2) follows from (1) and the Subframe Jankov Lemma, we only sketch the proof of (1). The right to left implication of (1) is straightforward. For the left to right implication, let $B \not\models \varphi$. By the Selective Filtration Lemma, there is a finite Heyting algebra A such that $A \not\models \varphi$ and A is a $(\wedge, \to, 0)$-subalgebra of B. If v is a valuation on B refuting φ, then as follows from the proof of the Selective Filtration Lemma, v restricts to a valuation on A refuting φ. Since by Birkhoff's theorem (see, e.g., [27, Thm. 8.6]) A is isomorphic to a subdirect product of its subdirectly irreducible homomorphic images, there is a subdirectly irreducible homomorphic image A' of A such that $A' \not\models \varphi$. The valuation refuting φ on A' can be taken to be the composition

$\pi \circ v$ where $\pi : A \to A'$ is the onto homomorphism. Because homomorphic images are determined by filters, A' is the quotient A/F by some filter $F \subseteq A$. Let G be the filter of B generated by F and let C be the quotient B/G. Then we have the following commutative diagram, and a direct verification shows that the embedding $A' \rightarrowtail C$ satisfies $\mathrm{CDC}_\vee$ for $D^\vee$.

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, two heads] & B \arrow[d, two heads] \\ A' \arrow[r, tail] & C \end{tikzcd}$$

From this we conclude that the pair $(A', D^\vee)$ is one of the $(A_i, D_i^\vee)$ from the list, and the embedding of A' into a homomorphic image C of B satisfies $\mathrm{CDC}_\vee$ for $D_i^\vee$. □

Remark 4.8. The above sketch of the proof of Theorem 4.7(1) is simpler than the original proof given in [4, Thm. 5.7], where free algebras were used to obtain the list $(A_1, D_1^\vee), \dots, (A_{k(n)}, D_{k(n)}^\vee)$.

As an immediate consequence, we arrive at the following uniform axiomatization of all intermediate logics by subframe canonical formulas.

Theorem 4.9. [4, Cor. 5.13] *Each intermediate logic* L *is axiomatizable by subframe canonical formulas. Moreover, if* L *is finitely axiomatizable, then* L *is axiomatizable by finitely many subframe canonical formulas.*

Proof. Let $\mathsf{L} = \mathsf{IPC} + \{\varphi_i \mid i \in I\}$. Then $\mathsf{IPC} \nvdash \varphi_i$ for each $i \in I$. By Theorem 4.7, for each $i \in I$, there are $(A_{i1}, D_{i1}^\vee), \dots, (A_{ik_i}, D_{ik_i}^\vee)$ such that $\mathsf{IPC} + \varphi_i = \mathsf{IPC} + \bigwedge_{i=1}^{k_i} \alpha(A_i, D_i^\vee)$. Thus, $\mathsf{L} = \mathsf{IPC} + \left\{ \bigwedge_{i=1}^{k_i} \alpha(A_i, D_i^\vee) \mid i \in I \right\}$. □

Remark 4.10.

(1) As we pointed out in the Introduction, canonical formulas were first introduced by Zakharyaschev [81] where Theorem 4.9 was proved using relational semantics.

(2) The notion of subframe canonical formulas can be generalized to that of multiple-conclusion subframe canonical rules along the lines of Lemma 3.1. This was done by Jeřábek [54] whose approach was similar to that of Zakharyaschev [81]. In particular, Jeřábek proved that every intuitionistic multiple-conclusion consequence relation is axiomatizable by canonical rules. Jeřábek also gave an alternative proof of obtaining bases of admissible rules via these canonical rules, and gave an alternative proof of Rybakov's decidability of the admissibility problem in IPC [71].

(3) An alternate approach to canonical formulas and rules using partial algebras was undertaken by Citkin [34, 36].

4.2. **Negation-free subframe canonical formulas.**

Definition 4.11. We call a propositional formula φ *negation-free* if φ does not contain $\neg$.

For those intermediate logics that are axiomatized by negation-free formulas, we can simplify subframe canonical formulas by dropping the conjunct

$$\bigwedge\{p_{\neg a} \leftrightarrow \neg p_a \mid a \in A\}$$

in the antecedent. The resulting formulas we call negation-free subframe canonical formulas:

Definition 4.12. Let A be a finite subdirectly irreducible Heyting algebra, s the second largest element of A, and $D \subseteq A^2$ a $\vee$-closed domain of A. For each $a \in A$ we introduce a new variable p_a

and define the *negation-free subframe canonical formula* $\beta(A, D)$ associated with A and D by

$$\begin{aligned}\beta(A, D) = \Big(& \bigwedge\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b \mid a, b \in A\} \wedge \\ & \bigwedge\{p_{a\to b} \leftrightarrow p_a \to p_b \mid a, b \in A\} \wedge \\ & \bigwedge\{p_{a\vee b} \leftrightarrow p_a \vee p_b \mid (a, b) \in D\}\Big) \to p_s\end{aligned}$$

We can then prove analogs of the results obtained in Section 4.1 and axiomatize each intermediate logic that is axiomatized by negation-free formulas by negation-free canonical formulas. The difference is that everywhere in Theorems 4.4–4.7 "bounded" needs to be dropped and we need to work with not necessarily bounded implicative semilattice embeddings. Because of this, we only state the results without proofs.

Theorem 4.13. [4, Cor. 5.16 and 5.17] *Let φ be a negation-free formula such that $\mathsf{IPC} \nvdash \varphi$ and $n = |\mathrm{Sub}(\varphi)|$. Then there is a list $(A_1, D_1^\vee), \dots, (A_{k(n)}, D_{k(n)}^\vee)$ such that each A_i is a finite subdirectly irreducible Heyting algebra, $D_i \subseteq A_i^2$ is a $\vee$-closed domain of A_i, and for an arbitrary Heyting algebra B we have:*

1. *$B \not\models \varphi$ iff there is $i \leq k(n)$, a homomorphic image C of B, and an implicative semilattice embedding $h : A_i \rightarrowtail C$ satisfying* CDC$_\vee$ *for $D_i^\vee$.*
2. *$B \models \varphi$ iff $B \models \bigwedge_{i=1}^{k(n)} \beta(A_i, D_i^\vee)$.*

As a corollary, we obtain that each intermediate logic L that is axiomatized by negation-free formulas is axiomatizable by negation-free canonical formulas.

Corollary 4.14. [4, Cor. 5.19] *Each intermediate logic L that is axiomatized by negation-free formulas is axiomatizable by negation-free canonical formulas. Moreover, if L is axiomatized by finitely many negation-free formulas, then L is axiomatizable by finitely many negation-free canonical formulas.*

Remark 4.15.

1. Negation-free canonical formulas were first introduced by Zakharyaschev [81] where Theorem 4.14 was proved using relational semantics.
2. The notion of negation-free subframe canonical formulas can be generalized to that of negation-free multiple-conclusion subframe canonical rules. This was done by Jeřábek [54] who showed that every intuitionistic negation-free multiple-conclusion consequence relation is axiomatizable by negation-free multiple-conclusion canonical rules. Jeřábek's approach was similar to that of Zakharyaschev [81].

4.3. **Stable canonical formulas.** In this section we survey the theory of stable canonical formulas of [8] (where they were called $(\wedge, \vee)$-canonical formulas). The theory is developed along the same lines as the theory of subframe canonical formulas, with the difference that stable canonical formulas require to work with the $\to$-free reduct of Heyting algebras instead of the $\vee$-free reduct. We outline the similarities and differences between these two approaches.

We start by the following simple observation which will be useful throughout. Let A and B be Heyting algebras. If B is subdirectly irreducible and A is a subalgebra of B, then A does not have to be subdirectly irreducible. However, it is elementary to see that if B is well-connected and A is a bounded sublattice of B, then A is also well-connected. In particular, since a finite Heyting algebra is subdirectly irreducible iff it is well-connected, if B is well-connected and A is a finite bounded sublattice of B, then A is subdirectly irreducible.

We next define the stable canonical formula associated with a finite subdirectly irreducible Heyting algebra A and a subset D of A^2. This formula encodes the bounded lattice structure of A fully and the behavior of $\to$ partially, only on the elements of D.

Definition 4.16. Let A be a finite subdirectly irreducible Heyting algebra, s the second largest element of A, and $D \subseteq A^2$. For each $a \in A$ introduce a new variable p_a and set

$$\begin{aligned}\Gamma = &\{p_0 \leftrightarrow \bot\} \cup \{p_1 \leftrightarrow \top\} \cup \\ &\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b \mid a, b \in A\} \cup \\ &\{p_{a\vee b} \leftrightarrow p_a \vee p_b \mid a, b \in A\} \cup \\ &\{p_{a\to b} \leftrightarrow p_a \to p_b \mid a, b \in D\}\end{aligned}$$

and

$$\Delta = \{p_a \leftrightarrow p_b \mid a, b \in A \text{ with } a \neq b\}.$$

Then define the *stable canonical formula* $\gamma(A, D)$ associated with A and D as

$$\gamma(A, D) = \bigwedge \Gamma \to \bigvee \Delta.$$

Remark 4.17. In [8, Def. 3.1] Δ is defined as $\{p_a \to p_b \mid a, b \in A \text{ with } a \not\leq b\}$.

Remark 4.18. Comparing $\gamma(A, D)$ and $\alpha(A, D)$, we see that the antecedent of $\gamma(A, D)$ encodes the bounded lattice structure of A and the implications in D, while the antecedent of $\alpha(A, D)$ encodes the bounded implicative semilattice structure of A and the joins in D.

The consequent of $\gamma(A, D)$ is more complicated than that of $\alpha(A, D)$. The intention in both cases is that the canonical formula is "pre-true" on the algebra. For $\alpha(A, D)$, since the formula encodes implications of entire A, this can simply be expressed by introducing a variable for the second largest element s of A. For $\gamma(A, D)$ however we need a more complicated consequent because the formula encodes implications only from the designated subset D of A^2.

Remark 4.19. If $D = A^2$, then $\gamma(A, D)$ is equivalent to $\mathcal{J}(A)$ (see [8, Thm. 5.1]).

Definition 4.20. Let A, B be Heyting algebras, $D \subseteq A^2$, and $h : A \to B$ a bounded lattice homomorphism. We call D a *$\to$-closed domain* of A and say that h satisfies the *$\to$-closed domain condition* for D if $h(a \to b) = h(a) \to h(b)$ for all $(a, b) \in D$.

We abbreviate the $\to$-closed domain condition by $\mathrm{CDC}_\to$. The next lemma is a version of the Jankov Lemma for stable canonical formulas.

Lemma 4.21 (Stable Jankov Lemma)**.** [8, Thm. 3.4] *Let A be a finite subdirectly irreducible Heyting algebra, $D \subseteq A^2$ a $\to$-closed domain of A, and B a Heyting algebra. Then $B \not\models \gamma(A, D)$ iff there is a subdirectly irreducible homomorphic image C of B and a bounded lattice embedding $h : A \rightarrowtail C$ satisfying* $\mathrm{CDC}_\to$ *for D.*

Remark 4.22. The Stable Jankov Lemma plays the same role in the theory of stable canonical formulas as the Subframe Jankov Lemma in the theory of subframe canonical formulas, but it is weaker in that the C in the lemma is required to be subdirectly irreducible, while in the Subframe Jankov Lemma it is not. As is shown in [8, Rem. 3.5], this assumption is necessary.

The second main ingredient for obtaining uniform axiomatization of intermediate logics by means of stable canonical formulas is the following Filtration Lemma, which goes back to [65] (and for modal logics even further back to [63, 64]). The name is motivated by the fact that it provides an algebraic account of the method of standard filtration for intermediate logics (see, e.g., [28, Sec. 5.3]). For a detailed comparison of the algebraic and frame-theoretic methods of standard filtration we refer to [7].

Lemma 4.23 (Filtration Lemma)**.** *Let B be a Heyting algebra such that $B \not\models \varphi$. Then there is a finite Heyting algebra A such that A is a bounded sublattice of B and $A \not\models \varphi$. In addition, if B is well-connected, then A is subdirectly irreducible.*

Proof. Since $B \not\models \varphi$, there is a valuation v on B such that $v(\varphi) \neq 1_B$. Let A be the bounded sublattice of B generated by $v[\mathrm{Sub}(\varphi)]$. Since the variety of bounded distributive lattices is locally finite, A is finite. Therefore, A is a finite Heyting algebra, where

$$a \to_A b = \bigvee \{c \in A \mid a \wedge c \leq b\}$$

for each $a, b \in A$. As $a \to b = \bigvee\{d \in B \mid a \wedge d \leq b\}$, it is easy to see that $a \to_A b \leq a \to b$ and that $a \to_A b = a \to b$ whenever $a \to b \in A$. Since for $\psi, \chi \in \mathrm{Sub}(\varphi)$, if $\psi \to \chi \in \mathrm{Sub}(\varphi)$, then $v(\psi) \to_A v(\chi) = v(\psi) \to v(\chi)$, we see that the value of φ in A is the same as the value of φ in B. As φ is refuted on B, we conclude that φ is refuted on A. Thus, A is a finite Heyting algebra that is a bounded sublattice of B and refutes φ. Finally, if B is well-connected, then so is A, and as A is finite, A is subdirectly irreducible. □

Now suppose that $\mathsf{IPC} \not\vdash \varphi$ and $n = |\mathrm{Sub}(\varphi)|$. Since the variety of bounded distributive lattices is locally finite, there is a bound $c(\varphi)$ on the number of n-generated bounded distributive lattices. Let $A_1, \dots, A_{m(n)}$ be the list of all finite subdirectly irreducible Heyting algebras such that $|A_i| \leq c(\varphi)$ and $A_i \not\models \varphi$.

For an algebra A refuting φ via a valuation v, let $\Theta = v[\mathrm{Sub}(\varphi)]$ and let

$$D^{\to} = \{(a, b) \in \Theta^2 \mid a \to b \in \Theta\}.$$

Consider a new list $(A_1, D_1^{\to}), \dots, (A_{k(n)}, D_{k(n)}^{\to})$, and note that in general $k(n)$ can be greater than $m(n)$ since each A_i may refute φ via different valuations.

The next theorem provides an alternate characterization of refutability to that given in Theorem 4.7. The proof is along similar lines of the proof of Theorem 4.7, but with appropriate adjustments since here we work with a different reduct of Heyting algebras.

Theorem 4.24. [8, Thm 3.7 and Cor. 3.9] *Let B be a subdirectly irreducible Heyting algebra.*

(1) *The following conditions are equivalent:*
   (a) $B \not\models \varphi$.
   (b) *There is $i \leq k(n)$ and a bounded lattice embedding $h : A_i \rightarrowtail C$ satisfying* $\mathrm{CDC}_{\to}$ *for* $D_i^{\to}$.
   (c) *There is $i \leq k(n)$, a subdirectly irreducible homomorphic image C of B, and a bounded lattice embedding $h : A_i \rightarrowtail C$ satisfying* $\mathrm{CDC}_{\to}$ *for* $D_i^{\to}$.

(2) $B \models \varphi$ *iff* $B \models \bigwedge_{i=1}^{k(n)} \gamma(A_i, D_i^{\to})$.

Proof. Since (2) follows from (1) and the Stable Jankov Lemma, we only sketch the proof of (1). The implications (1b)⇒(1c)⇒(1a) are straightforward. We prove the implication (1a)⇒(1b). Suppose that $B \not\models \varphi$. As B is subdirectly irreducible, it is well-connected. By the Filtration Lemma, there is a finite subdirectly irreducible Heyting algebra A such that $A \not\models \varphi$ and A is a bounded sublattice of B. Moreover, it follows from the proof of the Filtration Lemma that for each $a, b \in B$ such that $a \to b \in v[\mathrm{Sub}(\varphi)]$ we have $a \to_A b = a \to b$. From this we conclude that the pair $(A, D^{\to})$ is one of the $(A_i, D_i^{\to})$ from the list and the embedding of A into B satisfies $\mathrm{CDC}_{\to}$ for $D_i^{\to}$. □

Remark 4.25. The above sketch of the proof of Theorem 4.24(1) is simpler than the original proof given in [8, Thm. 3.7], where free algebras were used to obtain the list $(A_1, D_1^{\to}), \dots (A_{k(n)}, D_{k(n)}^{\to})$.

Remark 4.26. Theorem 4.24 plays the same role in the theory of stable canonical formulas as Theorem 4.7 in the theory of subframe canonical formulas, but it is weaker in that the B in the theorem is required to be subdirectly irreducible, while in Theorem 4.7 it is arbitrary.

As a consequence, we arrive at the following axiomatization of intermediate logics by means of stable canonical formulas, which is an alternative to Theorem 4.9. The proof is along the same lines as that of Theorem 4.9, the only difference being that we have to work with subdirectly irreducible Heyting algebras instead of arbitrary Heyting algebras. Since each variety of Heyting algebras is generated by its subdirectly irreducible members, the end result is the same.

Theorem 4.27. [8, Cor. 3.10] *Each intermediate logic* L *is axiomatizable by stable canonical formulas. Moreover, if* L *is finitely axiomatizable, then* L *is axiomatizable by finitely many stable canonical formulas.*

Remark 4.28.

(1) The same way subsframe canonical formulas can be generalized to subframe canonical rules (see Remark 4.10(2)), in [12] stable canonical formulas were generalized to stable canonical rules and it was shown that every intuitionistic multiple-conclusion consequence relation is axiomatizable by stable canonical rules. These rules were used in [21] to give an alternative proof of the existence of bases of admissible rules and the decidability of the admissibility problem for IPC, thus providing an analogue of Jeřábek's result [54] via stable canonical rules.

(2) Stable canonical formulas were generalized to substructural logics in [22].

5. Canonical formulas dually

In this section we discuss the dual reading of both subframe and stable canonical formulas. For subframe canonical formulas this requires a dual description of bounded implicative semilattice homomorphisms, and for stable canonical formulas a dual description of bounded lattice homomorphisms. For the former we will work with the generalized Esakia duality of [4], and for the latter with Priestley duality for bounded distributive lattices [66, 67].

5.1. **Subframe canonical formulas dually.** As was shown in [4], implicative semilattice homomorphisms are dually described by means of special partial maps between Esakia spaces.

Definition 5.1. Let X and Y be Esakia spaces, $f : X \to Y$ a partial map, and $\mathrm{dom}(f)$ the domain of f. We call f a *partial Esakia morphism* if the following conditions are satisfied:

(1) If $x, z \in \mathrm{dom}(f)$ and $x \le z$, then $f(x) \le f(z)$.
(2) If $x \in \mathrm{dom}(f)$, $y \in Y$, and $f(x) \le y$, then there is $z \in \mathrm{dom}(f)$ such that $x \le z$ and $f(z) = y$.
(3) $x \in \mathrm{dom}(f)$ iff there is $y \in Y$ such that $f[\uparrow x] = \uparrow y$.
(4) $f[\uparrow x]$ is closed for each $x \in X$.
(5) If U is a clopen upset of Y, then $X \setminus {\downarrow} f^{-1}(Y \setminus U)$ is a clopen upset of X.

Remark 5.2. If $\mathrm{dom}(f) = X$ and hence the partial Esakia morphism $f : X \to Y$ is total, then f is an Esakia morphism (see Lemma 5.4(2)).

The next result describes the topological properties of the domain of a partial Esakia morphism that will be used subsequently.

Lemma 5.3. *Let* $f : X \to Y$ *be a partial Esakia morphism.*

(1) $\mathrm{dom}(f)$ *is a closed subset of* X.
(2) *If* Y *is finite, then* $\mathrm{dom}(f)$ *is a clopen subset of* X.

Proof. For a proof of (1) see [4, Lem. 3.7]. For (2), in view of (1), it is sufficient to show that $\mathrm{dom}(f)$ is open. Let $x \in \mathrm{dom}(f)$. We set

$$\begin{aligned} D_1 &= Y \setminus {\uparrow} f(x), \\ D_2 &= Y \setminus ({\uparrow} f(x) \setminus \{f(x)\}), \\ U &= {\downarrow} f^{-1}(D_2) \setminus {\downarrow} f^{-1}(D_1). \end{aligned}$$

Since Y is finite and D_1, D_2 are downsets of Y, Definition 5.1(5) yields that $\downarrow f^{-1}(D_2)$ and $\downarrow f^{-1}(D_1)$ are clopen downsets of X. Therefore, U is clopen in X. Since $f(x) \in D_2$, we have $x \in \downarrow f^{-1}(D_2)$. Also, $f(x) \notin D_1$ and D_1 a downset of Y implies that $f^{-1}(D_1)$ is a downset of $\mathrm{dom}(f)$ by Definition 5.1(1). Thus, $x \notin \downarrow f^{-1}(D_1)$, and so $x \in U$. Therefore, it is sufficient to show that $U \subseteq \mathrm{dom}(f)$. Let $y \in U$. Then there is $z \in \mathrm{dom}(f)$ such that $y \leq z$, $f(z) \notin (\uparrow f(x) \setminus \{f(x)\})$, and $f(z) \in \uparrow f(x)$. Thus, $f(z) = f(x)$. Since $y \leq z$ and $z \in \mathrm{dom}(f)$, we have

$$\uparrow f(x) = \uparrow f(z) = f[\uparrow z] \subseteq f[\uparrow y],$$

where the second equality follows from Definition 5.1(1,2). For the reverse inclusion, let $u \in \mathrm{dom}(f)$ and $y \leq u$. Since $y \notin \downarrow f^{-1}(D_1)$, we have that $f(u) \in \uparrow f(x)$. Therefore, $f[\uparrow y] = \uparrow f(x)$, and we conclude by Definition 5.1(3) that $y \in \mathrm{dom}(f)$. Thus, $U \subseteq \mathrm{dom}(f)$, and hence $\mathrm{dom}(f)$ is clopen in X. ☐

Lemma 5.4. *Let X, Y be Esakia spaces with Y finite and let $f : X \to Y$ be a partial Esakia morphism.*

(1) $\mathrm{dom}(f)$ *is an Esakia space in the induced topology and order.*
(2) f *restricted to* $\mathrm{dom}(f)$ *is an Esakia morphism.*

Proof. (1). It is well known (see, e.g., [42, Thm. 3.2.6]) that a clopen subset of an Esakia space is an Esakia space in the induced topology and order. Thus, the result is immediate from Lemma 5.3(2).

(2). That f is a p-morphism follows from Definition 5.1(1,2) and that f is continuous is proved in [4, Lem. 3.9]. ☐

Let A, B be Heyting algebras and X_A, X_B their Esakia spaces. Given an implicative semilattice homomorphism $h : A \to B$, define $h_* : B_* \to A_*$ by setting

$$\mathrm{dom}(h_*) = \{x \in B_* \mid h^{-1}(x) \in A_*\}$$

and for $x \in \mathrm{dom}(h_*)$ by putting $h_*(x) = h^{-1}(x)$.

Lemma 5.5. [4, Thm. 3.14] *$h_* : X_B \to X_A$ is a partial Esakia morphism.*

Conversely, let X, Y be Esakia spaces and $f : X \to Y$ a partial Esakia morphism. Let X^*, Y^* be the Heyting algebras of clopen upsets of X, Y and define $f^* : Y^* \to X^*$ by

$$f^*(U) = X \setminus \downarrow f^{-1}(Y \setminus U)$$

for each $U \in Y^*$.

Lemma 5.6. [4, Thm. 3.15] *$f^* : Y^* \to X^*$ is an implicative semilattice homomorphism.*

As was shown in [4, Thm. 3.27], this correspondence extends to a categorical duality between the category of Heyting algebras and implicative semilattice homomorphisms and the category of Esakia spaces and partial Esakia morphisms.

Definition 5.7. [4, Def. 3.30] Let X and Y be Esakia spaces. We call a partial Esakia morphism $f : X \to Y$ *cofinal* if for each $x \in X$ there is $z \in \mathrm{dom}(f)$ such that $x \leq z$.[5]

By [4, Sec. 3.5], bounded implicative semilattice homomorphisms $h : A \to B$ dually correspond to cofinal partial Esakia morphisms $f : X_B \to X_A$.

We next connect the $\vee$-closed domain condition we discussed in Section 4.1 with Zakharyaschev's closed domain condition, which is one of the main tools in Zakharyaschev's frame-theoretic development of canonical formulas [81].

Let X, Y be Esakia spaces and $f : X \to Y$ a partial Esakia morphism. For $x \in X$ let $\min f[\uparrow x]$ be the set of minimal elements of $f[\uparrow x]$. Since $f[\uparrow x]$ is closed, $f[\uparrow x] \subseteq \uparrow \min f[\uparrow x]$ (see [42, Thm. 3.2.1]).

[5] In [4, Def. 3.30] these morphisms were called well partial Esakia morphisms.

Definition 5.8. Let X, Y be Esakia spaces, $f : X \to Y$ a partial Esakia morphism, and $\mathfrak{D}$ a (possibly empty) set of antichains in Y. We say that f satisfies *Zakharyaschev's closed domain condition* (ZCDC for short) for $\mathfrak{D}$ if $x \notin \mathrm{dom}(f)$ implies $\min f[\uparrow x] \notin \mathfrak{D}$.

Let A, B be Heyting algebras, $h : A \to B$ an implicative semilattice homomorphism, and $a, b \in A$. Let also X_A, X_B be the Esakia spaces of A, B, $f : X_B \to X_A$ the partial Esakia morphism corresponding to h, and $\zeta(a), \zeta(b)$ the clopen upsets of X_A corresponding to a, b. We let

$$\mathfrak{D}_{\zeta(a),\zeta(b)} = \{\text{antichains } \mathfrak{d} \text{ in } \zeta(a) \cup \zeta(b) \mid \mathfrak{d} \cap (\zeta(a) \setminus \zeta(b)) \neq \varnothing \text{ and } \mathfrak{d} \cap (\zeta(b) \setminus \zeta(a)) \neq \varnothing\}.$$

Lemma 5.9. [4, Lem. 3.40] *Let A, B be Heyting algebras, $h : A \to B$ an implicative semilattice homomorphism, and $a, b \in A$. Let also X_A, X_B be the Esakia spaces of A, B and $f : X_B \to X_A$ the partial Esakia morphism corresponding to h. Then $h(a \vee b) = h(a) \vee h(b)$ iff f satisfies* ZCDC *for* $\mathfrak{D}_{\zeta(a),\zeta(b)}$.

Proof. Using duality, it is sufficient to prove that for any clopen upsets U, V we have $f^*(U \cup V) = f^*(U) \cup f^*(V)$ iff f satisfies ZCDC for $\mathfrak{D}_{U,V}$.

$\Rightarrow$: Let $x \notin \mathrm{dom}(f)$. If $\min f[\uparrow x] \in \mathfrak{D}_{U,V}$, then $f[\uparrow x] = \uparrow \min f[\uparrow x] \subseteq U \cup V$, but neither $f[\uparrow x] \subseteq U$ nor $f[\uparrow x] \subseteq V$. Therefore, $x \in f^*(U \cup V)$, but $x \notin f^*(U)$ and $x \notin f^*(V)$. This contradicts $f^*(U \cup V) = f^*(U) \cup f^*(V)$. Consequently, $\min f[\uparrow x] \notin \mathfrak{D}_{U,V}$, and so f satisfies ZCDC for $\mathfrak{D}_{U,V}$.

$\Leftarrow$: It is sufficient to show that $f^*(U \cup V) \subseteq f^*(U) \cup f^*(V)$ since the other inclusion always holds. Let $x \in f^*(U \cup V)$. Then $f[\uparrow x] \subseteq U \cup V$. We have that $x \in \mathrm{dom}(f)$ or $x \notin \mathrm{dom}(f)$. If $x \in \mathrm{dom}(f)$, then $f[\uparrow x] = \uparrow f(x)$. Therefore, $f[\uparrow x] \subseteq U \cup V$ implies $\uparrow f(x) \subseteq U \cup V$, hence $\uparrow f(x) \subseteq U$ or $\uparrow f(x) \subseteq V$. Thus, $x \in f^*(U)$ or $x \in f^*(V)$, and so $x \in f^*(U) \cup f^*(V)$. On the other hand, if $x \notin \mathrm{dom}(f)$, then as f satisfies ZCDC for $\mathfrak{D}_{U,V}$, we obtain that $\min f[\uparrow x] \notin \mathfrak{D}_{U,V}$. Therefore, $\min f[\uparrow x] \subseteq U$ or $\min f[\uparrow x] \subseteq V$. Thus, $f[\uparrow x] \subseteq \uparrow \min f[\uparrow x] \subseteq U$ or $f[\uparrow x] \subseteq \uparrow \min f[\uparrow x] \subseteq V$, which yields that $x \in f^*(U)$ or $x \in f^*(V)$. Consequently, $x \in f^*(U) \cup f^*(V)$, and so $f^*(U \cup V) \subseteq f^*(U) \cup f^*(V)$. $\square$

We are ready to give the dual reading of subframe canonical formulas of Section 4.1. Let A be a finite subdirectly irreducible Heyting algebra. By finite Esakia duality, its dual is a finite rooted poset P. Let $D \subseteq A^2$. We call the set $\mathfrak{D} = \{\mathfrak{D}_{\zeta(a),\zeta(b)} \mid (a, b) \in D\}$ *the set of antichains of P associated with D*. The following theorem is a consequence of the Subframe Jankov Lemma, Lemma 5.9, and generalized Esakia duality.

Theorem 5.10. [4, Cor. 5.5] *Let A be a finite subdirectly irreducible Heyting algebra and P its dual finite rooted poset. Let $D \subseteq A^2$ and $\mathfrak{D}$ be the set of antichains of P associated with D. Then for each Esakia space X, we have $X \not\models \alpha(A, D)$ iff there is a closed upset Y of X and an onto cofinal partial Esakia morphism $f : Y \twoheadrightarrow P$ such that f satisfies* ZCDC *for $\mathfrak{D}$.*

From this we derive the following dual reading of Theorem 4.7.

Theorem 5.11. [4, Cor. 5.9 and 5.11] *Suppose $\mathsf{IPC} \nvdash \varphi$ and $(A_1, D_1^{\vee}), \dots, (A_{k(n)}, D_{k(n)}^{\vee})$ is the corresponding list of finite refutation patterns of φ. For each $i \leq k(n)$ let P_i be the dual finite rooted poset of A_i and $\mathfrak{D}_i$ the set of antichains of P_i associated with $D_i^{\vee}$. Then for an arbitrary Esakia space X, we have:*

(1) *$X \not\models \varphi$ iff there is $i \leq k(n)$, a closed upset Y of X, and an onto cofinal partial Esakia morphism $f : Y \to P_i$ satisfying* ZCDC *for $\mathfrak{D}_i$.*

(2) *$X \models \varphi$ iff $X \models \displaystyle\bigwedge_{i=1}^{k(n)} \alpha(A_i, D_i^{\vee})$.*

We have a parallel situation with negation-free canonical formulas, the main difference being that "cofinal" has to be dropped from the consideration. We thus arrive at the following negation-free analogue of Theorem 5.11.

Theorem 5.12. [4, Cor. 5.16 and 5.17] *Suppose φ is a negation-free formula,* $\mathsf{IPC} \not\vdash \varphi$, *and* $(A_1, D_1^\vee), \ldots, (A_{k(n)}, D_{k(n)}^\vee)$ *is the corresponding list of finite refutation patterns of φ. For each $i \le k(n)$ let P_i be the dual finite rooted poset of A_i and $\mathfrak{D}_i$ the set of antichains of P_i associated with $D_i^\vee$. Then for an arbitrary Esakia space X, we have:*

(1) *$X \not\models \varphi$ iff there is $i \le k(n)$, a closed upset Y of X, and an onto partial Esakia morphism $f : Y \to P_i$ satisfying* ZCDC *for $\mathfrak{D}_i$.*

(2) *$X \models \varphi$ iff* $X \models \displaystyle\bigwedge_{i=1}^{k(n)} \beta(A_i, D_i^\vee)$.

Remark 5.13. Zakharyschev's canonical formulas [81, 28] are different but equivalent to subframe canonical formulas (see [4, Rem. 5.6]).

5.2. **Stable canonical formulas dually.** Let A, B be Heyting algebras. We recall that a map $h : A \to B$ is a *lattice homomorphism* if

$$h(a \wedge b) = h(a) \wedge h(b) \text{ and } h(a \vee b) = h(a) \vee h(b)$$

for each $a, b \in A$. A lattice homomorphism $h : A \to B$ is *bounded* if $h(0) = 0$ and $h(1) = 1$. It is a consequence of Priestley duality for bounded distributive lattices [66, 67] that bounded lattice homomorphisms $h : A \to B$ dually correspond to continuous order-preserving maps $f : X_B \to X_A$.

Definition 5.14. Let X, Y be Esakia spaces. We call a map $f : X \to Y$ a *stable morphism* if f is continuous and order-preserving.

Remark 5.15. The name "stable morphism" comes from modal logic, where it is used for continuous maps that preserve relation (see [11, 13, 50]). In Priestley duality for bounded distributive lattices these maps are known as Priestley morphisms.

Definition 5.16. [8, Def. 4.1] Let X, Y be Esakia spaces and $f : X \to Y$ a stable morphism.

(1) Let D be a clopen subset of Y. We say that f satisfies the *stable domain condition* (SDC for short) for D if

$$\uparrow f(x) \cap D \neq \varnothing \Rightarrow f[\uparrow x] \cap D \neq \varnothing.$$

(2) Let $\mathfrak{D}$ be a collection of clopen subsets of Y. We say that $f : X \to Y$ satisfies the *stable domain condition* (SDC for short) for $\mathfrak{D}$ if f satisfies SDC for each $D \in \mathfrak{D}$.

Lemma 5.17. [8, Lem. 4.3] *Let A, B be Heyting algebras, $h : A \to B$ a bounded lattice homomorphism, and $a, b \in A$. Let also X_A, X_B be the Esakia spaces of A, B, $f : X_B \to X_A$ the stable morphism corresponding to h, and $D_{\zeta(a),\zeta(b)} = \zeta(a) \setminus \zeta(b)$. Then $h(a \to b) = h(a) \to h(b)$ iff f satisfies* SDC *for $D_{\zeta(a),\zeta(b)}$.*

Proof. Using duality it is sufficient to show that for any clopen upsets U, V we have $f^{-1}(U) \to f^{-1}(V) = f^{-1}(U \to V)$ iff f satisfies SDC for $D_{U,V}$.

$\Rightarrow$: Suppose that $\uparrow f(x) \cap D_{U,V} \neq \varnothing$. Then $\uparrow f(x) \cap U \not\subseteq V$. Therefore, $f(x) \notin U \to V$, so $x \notin f^{-1}(U \to V)$. Thus, $x \notin f^{-1}(U) \to f^{-1}(V)$, and so $\uparrow x \cap f^{-1}(U) \not\subseteq f^{-1}(V)$. This implies $f[\uparrow x] \cap U \not\subseteq V$, and hence $f[\uparrow x] \cap D_{U,V} \neq \varnothing$. Consequently, f satisfies SDC for $D_{U,V}$.

$\Leftarrow$: It is sufficient to show that $f^{-1}(U) \to f^{-1}(V) \subseteq f^{-1}(U \to V)$ since the other inclusion always holds. Suppose that $x \notin f^{-1}(U \to V)$. Then $f(x) \notin U \to V$. Therefore, $\uparrow f(x) \cap U \not\subseteq V$, which means that $\uparrow f(x) \cap D_{U,V} \neq \varnothing$. Thus, $f[\uparrow x] \cap D_{U,V} \neq \varnothing$. This means that $\uparrow x \cap (f^{-1}(U) \setminus f^{-1}(V)) \neq \varnothing$. Consequently, $\uparrow x \cap f^{-1}(U) \not\subseteq f^{-1}(V)$, implying that $x \notin f^{-1}(U) \to f^{-1}(V)$. ☐

We recall from Section 2.3 that the Esakia dual of a subdirectly irreducible Heyting algebra is a strongly rooted Esakia space, the Esakia dual of a finite subdirectly irreducible Heyting algebra is a finite rooted poset, and the Esakia dual of a subdirectly irreducible homomorphic image of a

Heyting algebra A is a strongly rooted closed upset of the Esakia dual of A. We also recall [66, Thm. 3] that bounded sublattices of A dually correspond to onto stable morphisms from the Esakia dual of A. Thus, Lemma 5.17 yields the following dual reading of the Stable Jankov Lemma and Theorem 4.24.

Theorem 5.18. [8, Thm. 4.4]

(1) *Let A be a finite subdirectly irreducible Heyting algebra and P its dual finite rooted poset. For $D \subseteq A^2$, let $\mathfrak{D} = \{D_{\zeta(a),\zeta(b)} \mid (a,b) \in D\}$. Then for each Esakia space X, we have $X \not\models \gamma(A,D)$ iff there is a strongly rooted closed upset Y of X and an onto stable morphism $f : Y \twoheadrightarrow P$ such that f satisfies* SDC *for $\mathfrak{D}$.*

(2) *Suppose* $\mathsf{IPC} \not\vdash \varphi$ *and $(A_1, D_1^{\to}), \dots, (A_{k(n)}, D_{k(n)}^{\to})$ is the corresponding list of finite refutation patterns of φ. For each $i \le k(n)$, let P_i be the dual finite rooted poset of A_i and $\mathfrak{D}_i = \{D_{\zeta(a),\zeta(b)} \mid (a,b) \in D_i^{\to}\}$. Then for each strongly rooted Esakia space X, the following conditions are equivalent:*
   (a) $X \not\models \varphi$.
   (b) *There is $i \le k(n)$ and an onto stable morphism $f : Y \twoheadrightarrow P_i$ such that f satisfies* SDC *for $\mathfrak{D}_i$.*
   (c) *There is $i \le k(n)$, a strongly rooted closed upset Y of X, and an onto stable morphism $f : Y \twoheadrightarrow P_i$ such that f satisfies* SDC *for $\mathfrak{D}_i$.*

(3) *For each strongly rooted Esakia space X, we have $X \models \varphi$ iff $X \models \bigwedge_{i=1}^{k(n)} \gamma(A_i, D_i^{\to})$.*

Remark 5.19. When comparing the dual approaches to these two types of canonical formulas, we see that in the case of subframe canonical formulas we work with cofinal partial Esakia morphisms whose duals are bounded implicative semilattice homomorphisms, and Zakharyaschev's closed domain condition ZCDC provides means for the dual to also preserve $\vee$. On the other hand, in the case of stable canonical formulas we work with stable morphisms whose duals are bounded lattice homomorphisms, and the stable domain condition SDC provides means for the dual to also preserve $\to$. In the end, both approaches provide the same result, that all intermediate logics are axiomatizable either by subframe canonical formulas or by stable canonical formulas. However, both the algebra and geometry of the two approaches are different.

6. Subframe and cofinal subframe formulas

As we saw in Remark 4.2, when the closed domain D of a subframe canonical formula $\alpha(A,D)$ is the entire A^2, then $\alpha(A,D)$ coincides with the Jankov formula $\mathcal{J}(A)$. Another extreme case is when $D = \varnothing$. In this case, we simply drop D and write $\alpha(A)$ or $\beta(A)$ depending on whether we work with $\alpha(A,D)$ (in the full signature of subframe canonical formulas) or with $\beta(A,D)$ (in the negation-free signature). As a result, we arrive at subframe formulas (when working with $\beta(A)$) or cofinal subframe formulas (when working with $\alpha(A)$), and the corresponding subframe and cofinal subframe logics. We briefly recall that subframe logics were first studied by Fine [45] and cofinal subframe logics by Zakharyaschev [83] for extensions of $\mathsf{K4}$. The study of subframe and cofinal subframe intermediate logics was initiated by Zakharyaschev [81]. Both Fine and Zakharyaschev utilized relational semantics. In this section we survey the theory of these logics utilizing algebraic semantics.

Definition 6.1. Let A be a finite subdirectly irreducible Heyting algebra and $D = \varnothing$.

(1) We call $\alpha(A,D)$ the *cofinal subframe formula* of A and denote it by $\alpha(A)$.
(2) We call $\beta(A,D)$ the *subframe formula* of A and denote it by $\beta(A)$.

The names "subframe formula" and "cofinal subframe formula" are justified by their connection to subframes and cofinal subframes of Esakia spaces discussed below (see Theorems 6.15 and 6.16).

Let A, B be Heyting algebras with A finite and subdirectly irreducible. As an immediate consequence of the Subframe Jankov Lemma we obtain that $B \not\models \alpha(A)$ iff there is a homomorphic image C of B and a bounded implicative semilattice embedding $h : A \rightarrowtail C$, and similarly for $\beta(A)$. However, this result can be improved by dropping homomorphic images from the consideration. For this we require the following lemma.

Lemma 6.2. *Let A, B, C be finite Heyting algebras and $h : B \twoheadrightarrow C$ an onto Heyting homomorphism.*

1. *If $e : A \rightarrowtail C$ is an implicative semilattice embedding, then there is an implicative semilattice embedding $k : A \rightarrowtail B$ such that $h \circ k = e$.*
2. *If in addition e is bounded, then so is k.*

$$
\begin{tikzcd}
C & B \arrow[l, two heads, "h"'] \\
& A \arrow[ul, "e"] \arrow[u, "k"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X_C \arrow[r, hook, "id"] \arrow[dr, "f"'] & X_B \arrow[d, "g"] \\
& X_A
\end{tikzcd}
$$

Proof. (1). Let X_A, X_B, and X_C be the dual finite posets of A, B, and C. We identify A with the upsets of X_A, B with the upsets of X_B, and C with the upsets of X_C. Since C is a homomorphic image of B, we have that X_C is (isomorphic to) an upset of X_B. Also, since $e : A \to C$ is an implicative lattice embedding, there is an onto partial Esakia morphism $f : X_C \to X_A$ (see [4, Lem. 3.29]). Viewing f also as a partial map $f : X_B \to X_A$, it is straightforward that f satisfies conditions (1), (2), and (5) of Definition 5.1. Therefore, by [28, Thm. 9.7], $f^* : \mathsf{Up}(X_A) \to \mathsf{Up}(X_B)$ is an implicative semilattice embedding, and it is clear that $h \circ f^* = e$.

(2). Suppose in addition that e is bounded. Then $f : X_C \to X_A$ is cofinal, so $\max(X_C) \subseteq \mathrm{dom}(f)$. Let $y \in \max(X_A)$ and define a partial map $g : X_B \to X_A$ by setting $\mathrm{dom}(g) = \mathrm{dom}(f) \cup \max(X_B)$ and for $x \in \mathrm{dom}(g)$ letting

$$
g(x) = \begin{cases} f(x) & \text{if } x \in \mathrm{dom}(f) \\ y & \text{if } x \in \max(X_B) \setminus \mathrm{dom}(f) \end{cases}
$$

It is then straightforward that g satisfies conditions (1), (2), and (5) of Definition 5.1. Therefore, $g^* : \mathsf{Up}(X_A) \to \mathsf{Up}(X_B)$ is an implicative semilattice embedding. It follows from the definition of g that ${\downarrow}\mathrm{dom}(g) = X_B$ and $g|_{X_C} = f$. Thus, g^* is bounded (see [4, Lem. 3.32]) and $h \circ g^* = f^*$. ☐

Theorem 6.3.

1. *Let A be a finite subdirectly irreducible Heyting algebra and B an arbitrary Heyting algebra.*
   (a) *$B \not\models \beta(A)$ iff there is an implicative semilattice embedding $h : A \rightarrowtail B$.*
   (b) *$B \not\models \alpha(A)$ iff there is a bounded implicative semilattice embedding $h : A \rightarrowtail B$.*
2. *Let A be a finite subdirectly irreducible Heyting algebra, P its dual finite rooted poset, and X an arbitrary Esakia space.*
   (a) *$X \not\models \beta(A)$ iff there is an onto partial Esakia morphism $f : X \twoheadrightarrow P$.*
   (b) *$X \not\models \alpha(A)$ iff there is an onto cofinal partial Esakia morphism $f : X \twoheadrightarrow P$.*

Proof. Since (2) is the dual statement of (1), it is sufficient to prove (1). The right to left implication is obvious. For the left to right implication, suppose $B \not\models \beta(A)$. By the Selective Filtration Lemma, there is a finite $(\wedge, \to)$-subalgebra D of B such that $D \not\models \beta(A)$. By the Subframe Jankov Lemma, there is a homomorphic image C of D and an implicative semilattice embedding of A into C. By Lemma 6.2(1), there is an implicative semilattice embedding of A into D, and hence there is an implicative semilattice embedding of A into B. ☐

Remark 6.4. Theorem 6.3 improves [4, Cor. 5.24] in that $B \not\models \beta(A)$ is equivalent to the existence of an implicative semilattice embedding of A directly into B, rather than a homomorphic image of B (and the same for $\alpha(A)$).

Definition 6.5. Let L be an intermediate logic.

(1) We call L a *subframe logic* if there is a family $\{A_i \mid i \in I\}$ of finite subdirectly irreducible Heyting algebras such that $\mathsf{L} = \mathsf{IPC} + \{\beta(A_i) \mid i \in I\}$.
(2) We call L a *cofinal subframe logic* if there is a family $\{A_i \mid i \in I\}$ of finite subdirectly irreducible Heyting algebras such that $\mathsf{L} = \mathsf{IPC} + \{\alpha(A_i) \mid i \in I\}$.
(3) Let Λ_{Subf} be the set of subframe logics and Λ_{CSubf} the set of cofinal subframe logics.

Theorem 6.6. [28, Sec. 11.3]

(1) Λ_{Subf} *is a complete sublattice of* Λ_{CSubf} *and* Λ_{CSubf} *is a complete sublattice of* Λ.
(2) *The cardinalities of both* Λ_{Subf} *and* $\Lambda_{\mathsf{CSubf}} \setminus \Lambda_{\mathsf{Subf}}$ *are that of the continuum.*

Definition 6.7. Let φ be a propositional formula.

(1) Call φ a *DN-free formula* if φ does not contain disjunction and negation.
(2) Call φ a *disjunction-free formula* if φ does not contain disjunction.

Since $\alpha(A)$ encodes the bounded implicative semilattice structure and $\beta(A)$ the implicative semilattice structure of A, one would expect that subframe logics are exactly those intermediate logics that are axiomatizable by DN-free formulas and cofinal subframe logics are those that are axiomatizable by disjunction-free formulas. This indeed turns out to be the case, as was shown by Zakharyaschev [81] using relational semantics. To give an algebraic proof and obtain other equivalent conditions for an intermediate logic to be a subframe or cofinal subframe logic, we introduce the following notation.

Definition 6.8.

(1) Let A, B be Heyting algebras with $A \subseteq B$. We say that A is a $(\wedge, \to)$-*subalgebra* of B if A is closed under $\wedge$ and $\to$, and we say that A is a $(\wedge, \to, 0)$-*subalgebra* of B if in addition $0 \in A$.
(2) We say that a class $\mathcal{K}$ of Heyting algebras is *closed under* $(\wedge, \to)$-*subalgebras* if from $B \in \mathcal{K}$ and A being isomorphic to a $(\wedge, \to)$-subalgebra of B it follows that $A \in \mathcal{K}$.
(3) We say that a class $\mathcal{K}$ of Heyting algebras is *closed under* $(\wedge, \to, 0)$-*subalgebras* if from $B \in \mathcal{K}$ and A being isomorphic to a $(\wedge, \to, 0)$-subalgebra of B it follows that $A \in \mathcal{K}$.

Theorem 6.9. *For an intermediate logic* L*, the following conditions are equivalent.*

(1) L *is a subframe logic.*
(2) L *is axiomatizable by DN-free formulas.*
(3) *The variety* $\mathcal{V}(\mathsf{L})$ *is closed under* $(\wedge, \to)$-*subalgebras.*
(4) *There is a class* $\mathcal{K}$ *of* L-*algebras closed under* $(\wedge, \to)$-*subalgebras that generates* $\mathcal{V}(\mathsf{L})$.

Proof. (1)⇒(2). If L is a subframe logic, then L is axiomatizable by subframe formulas. But subframe formulas are DN-free formulas by definition. Thus, L is axiomatizable by DN-free formulas.

(2)⇒(3). Suppose $\mathsf{L} = \mathsf{IPC} + \{\varphi_i \mid i \in I\}$ where each φ_i is a DN-formula. Let A, B be Heyting algebras with $B \in \mathcal{V}(\mathsf{L})$ and A isomorphic to a $(\wedge, \to)$-subalgebra of B. From $B \in \mathcal{V}(\mathsf{L})$ it follows that each φ_i is valid on B. Since A is isomorphic to a $(\wedge, \to)$-subalgebra of B, each φ_i is also valid on A. Thus, $A \in \mathcal{V}(\mathsf{L})$.

(3)⇒(4). This is obvious.

(4)⇒(1). Let $\mathcal{X}$ be the set of all finite non-isomorphic subdirectly irreducible Heyting algebras such that $A \not\models \mathsf{L}$, and let

$$\mathsf{M} = \mathsf{IPC} + \{\beta(A) \mid A \in \mathcal{X}\}.$$

It is sufficient to show that $\mathsf{L} = \mathsf{M}$. Let B be a subdirectly irreducible Heyting algebra. It is enough to prove that $B \models \mathsf{L}$ iff $B \models \mathsf{M}$. First suppose that $B \not\models \mathsf{L}$. Then there is $\varphi \in \mathsf{L}$ such that $B \not\models \varphi$. By the Selective Filtration Lemma, there is a finite subdirectly irreducible Heyting algebra A such that $A \not\models \varphi$ and A is a $(\wedge, \to)$-subalgebra of B. By Theorem 6.3(1a), $B \not\models \beta(A)$. Therefore, $B \not\models \mathsf{M}$. Thus, $\mathsf{L} \subseteq \mathsf{M}$.

For the reverse inclusion, since L is the logic of $\mathcal{K}$, it is sufficient to show that if $B \in \mathcal{K}$, then $B \models \mathsf{M}$. If $B \not\models \mathsf{M}$, then $B \not\models \beta(A)$ for some $A \in \mathcal{X}$. By Theorem 6.3(1a), A is isomorphic to a $(\wedge, \to)$-subalgebra of B. Since $B \in \mathcal{K}$ and $\mathcal{K}$ is closed under $(\wedge, \to)$-subalgebras, $A \in \mathcal{K}$. Thus, $A \models \mathsf{L}$, a contradiction. Consequently, $B \models \mathsf{M}$, finishing the proof. □

Theorem 6.9 directly generalizes to cofinal subframe logics.

Theorem 6.10. *For an intermediate logic* L*, the following conditions are equivalent.*

1. L *is a cofinal subframe logic.*
2. L *is axiomatizable by disjunction-free formulas.*
3. *The variety* $\mathcal{V}(\mathsf{L})$ *is closed under* $(\wedge, \to, 0)$*-subalgebras.*
4. *There is a class* $\mathcal{K}$ *of* L*-algebras closed under* $(\wedge, \to, 0)$*-subalgebras that generates* $\mathcal{V}(\mathsf{L})$*.*

Theorems 6.9 and 6.10 allow us to give a simple proof that each subframe and cofinal subframe logic has the fmp. This result for subframe modal logics above $\mathsf{K4}$ was first established by Fine [45], for cofinal subframe intermediate logics by Zakharyaschev [81], and for cofinal subframe modal logics above $\mathsf{K4}$ by Zakharyaschev [83]. Both Fine and Zakharyaschev used relational semantics. We will instead prove this result by utilizing the Selective Filtration Lemma. This is closely related to the work of McKay [61].

Theorem 6.11.

1. *Each subframe logic has the fmp.*
2. *Each cofinal subframe logic has the fmp.*

Proof. Since $\Lambda_{\mathsf{Subf}} \subseteq \Lambda_{\mathsf{CSubf}}$, it is sufficient to prove (2). Let L be a cofinal subframe logic. By Theorem 6.10, L is axiomatized by a set of disjunction-free formulas $\{\chi_i \mid i \in I\}$. Suppose $\mathsf{L} \not\vdash \varphi$. By algebraic completeness, there is an L-algebra B such that $B \not\models \varphi$. By the Selective Filtration Lemma, there is a finite $(\wedge, \to, 0)$-subalgebra A of B such that $A \not\models \varphi$. Since each χ_i is disjunction-free, we have that $B \models \chi_i$ implies $A \models \chi_i$ for each $i \in I$. Thus, A is an L-algebra, and hence L has the fmp. □

We next justify the name "subframe logic" by connecting these logics to subframes of Esakia spaces.

Definition 6.12. [28, p. 289] Let X be an Esakia space. We call $Y \subseteq X$ a *subframe* of X if Y is an Esakia space in the induced topology and order and the partial identity map $X \to Y$ satisfies conditions (1), (2), and (5) of Definition 5.1.

The following is a convenient characterization of subframes of Esakia spaces.

Theorem 6.13. [16, Lem. 2] *Let X be an Esakia space. Then $Y \subseteq X$ is a subframe of X iff Y is a closed subset of X and U a clopen subset of Y* (*in the induced topology*) *implies ${\downarrow}U$ is a clopen subset of X.*

We call a subframe Y of X *cofinal* if ${\downarrow}Y = X$. In [28, p. 295] a weaker notion of cofinality was used, that ${\uparrow}Y \subseteq {\downarrow}Y$. Recall that Esakia spaces X, Y are *isomorphic* in Esa if they are homeomorphic and order-isomorphic.

Definition 6.14. Let $\mathcal{K}$ be a class of Esakia spaces and X, Y Esakia spaces.

1. We call $\mathcal{K}$ *closed under subframes* if from $X \in \mathcal{K}$ and Y being isomorphic to a subframe of X it follows that $Y \in \mathcal{K}$.
2. We call $\mathcal{K}$ *closed under cofinal subframes* if from $X \in \mathcal{K}$ and Y being isomorphic to a cofinal subframe of X it follows that $Y \in \mathcal{K}$.

Dualizing Theorem 6.9 we obtain:

Theorem 6.15. *For an intermediate logic* L*, the following conditions are equivalent.*

1. L *is a subframe logic.*
2. L *is axiomatizable by DN-free formulas.*
3. *The class of all Esakia spaces validating* L *is closed under subframes.*
4. L *is sound and complete with respect to a class* $\mathcal{K}$ *of Esakia spaces that is closed under subframes.*

Proof. (1)⇒(2). This is proved in Theorem 6.9.

(2)⇒(3). Let $X \models \mathsf{L}$ and Y be isomorphic to a subframe of X. Then $X^* \models \mathsf{L}$ and Y^* is isomorphic to a $(\wedge, \to)$-subalgebra of X^*. By Theorem 6.9, $Y^* \models \mathsf{L}$, and hence $Y \models \mathsf{L}$.

(3)⇒(4). This is straightforward.

(4)⇒(1). Let $\mathcal{K}^* = \{X^* \mid X \in \mathcal{K}\}$. We proceed as in the proof of Theorem 6.9 by showing that $\mathsf{L} = \mathsf{IPC} + \{\beta(A) \mid A \in \mathcal{X}\}$ (where we recall from the proof of Theorem 6.9 that $\mathcal{X}$ is the set of all finite non-isomorphic subdirectly irreducible Heyting algebras A such that $A \not\models \mathsf{L}$). Let B be a subdirectly irreducible Heyting algebra. That $B \not\models \mathsf{L}$ implies $B \not\models \{\beta(A) \mid A \in \mathcal{X}\}$ is proved as in Theorem 6.9. For the converse, since L is the logic of $\mathcal{K}$, it is sufficient to assume that $B = X^*$ for some $X \in \mathcal{K}$. If $B \not\models \beta(A)$ for some $A \in \mathcal{X}$, then by Theorem 6.3(1a), A is isomorphic to a $(\wedge, \to)$-subalgebra of B. Therefore, there is an onto partial Esakia morphism $f : X \to X_A$. Since X_A is finite, $\mathrm{dom}(f)$ is a clopen subset of X by Lemma 5.3(2). Therefore, it follows from Theorem 6.13 that $\mathrm{dom}(f)$ is a subframe of X. Thus, $\mathrm{dom}(f) \in \mathcal{K}$, so $\mathrm{dom}(f) \models \mathsf{L}$. But $f : \mathrm{dom}(f) \to X_A$ is an onto Esakia morphism by Lemma 5.4(2). Therefore, $X_A \models \mathsf{L}$, and hence $A \models \mathsf{L}$. The obtained contradiction proves that $B \models \{\beta(A) \mid A \in \mathcal{X}\}$, finishing the proof. □

We also have the following dual version of Theorem 6.10, the proof of which is analogous to that of Theorem 6.15 and we skip it.

Theorem 6.16. *For an intermediate logic* L*, the following conditions are equivalent.*

1. L *is a cofinal subframe logic.*
2. L *is axiomatizable by disjunction-free formulas.*
3. *The class of all Esakia spaces validating* L *is closed under cofinal subframes.*
4. L *is sound and complete with respect to a class* $\mathcal{K}$ *of Esakia spaces that is closed under cofinal subframes.*

Remark 6.17. There are several other interesting characterizations of subframe and cofinal subframe logics.

1. In [16] it is shown that subframes of Esakia spaces correspond to nuclei on Heyting algebras, and that cofinal subframes to dense nuclei. From this it follows that an intermediate logic L is a subframe logic iff its corresponding variety $\mathcal{V}(\mathsf{L})$ is a nuclear variety, and that L is a cofinal subframe logic iff $\mathcal{V}(\mathsf{L})$ is a dense nuclear variety.
2. A different description of subframe and cofinal subframe formulas is given in [19, Sec. 3.3.3] and [20] (see also [50]), where it is shown that subframe formulas are equivalent to the NNIL-formulas of [78].
3. Many important properties of logics (such as, e.g., canonicity and strong Kripke completeness) coincide for subframe and cofinal suframe logics (see, e.g., [28, Thms. 11.26 and 11.28]).

We finish this section with some examples of subframe and cofinal subframe logics. To simplify notation, we write $\beta(\mathfrak{F})$ instead of $\beta(\mathfrak{F}^*)$ and $\alpha(\mathfrak{F})$ instead of $\alpha(\mathfrak{F}^*)$. Then, recalling Figure 1, we have:

Theorem 6.18. [28, p. 317, Table 9.7]

1. $\mathsf{CPC} = \mathsf{LC} + \beta(\mathfrak{C}_2)$*, hence* CPC *is a subframe logic.*

(2) $\mathsf{LC} = \mathsf{IPC} + \beta(\mathfrak{F}_2)$, *hence* LC *is a subframe logic.*
(3) $\mathsf{BD}_n = \mathsf{IPC} + \beta(\mathfrak{C}_{n+1})$, *hence* BD_n *is a subframe logic.*
(4) $\mathsf{LC}_n = \mathsf{LC} + \beta(\mathfrak{C}_{n+1})$, *hence* LC_n *is a subframe logic.*
(5) $\mathsf{BW}_n = \mathsf{IPC} + \beta(\mathfrak{F}_{n+1})$, *hence* BW_n *is a subframe logic.*
(6) $\mathsf{KC} = \mathsf{IPC} + \alpha(\mathfrak{F}_2)$, *hence* KC *is a cofinal subframe logic.*
(7) $\mathsf{BTW}_n = \mathsf{IPC} + \alpha(\mathfrak{F}_{n+1})$, *hence* BTW_n *is a cofinal subframe logic.*

On the other hand, there exist intermediate logics that are not subframe (e.g., KC) and also ones that are not cofinal subframe (e.g., KP).

7. Stable formulas

We can develop the theory of stable formulas which is parallel to that of subframe formulas. As we pointed out in Remark 4.19, if $D = A^2$, then $\gamma(A, D)$ is equivalent to the Jankov formula $\mathcal{J}(A)$. As with subframe formulas, we can consider the second extreme case when $D = \varnothing$. We call the resulting formulas stable formulas and denote them by $\gamma(A)$. Then the theory of cofinal subframe formulas can be developed in parallel to the theory of cofinal subframe formulas. In Section 7.1 we survey the theory of stable formulas and the resulting stable logics, and in Section 7.2 that of cofinal stable formulas and the resulting cofinal stable logics. We also compare these new classes of logics to subframe and cofinal subframe logics.

7.1. **Stable formulas.**

Definition 7.1. Let A be a finite subdirectly irreducible Heyting algebra and $D = \varnothing$. We call $\gamma(A, D)$ the *stable formula* of A and denote it by $\gamma(A)$.

Let A, B be Heyting algebras with A finite and subdirectly irreducible. As an immediate consequence of the Stable Jankov Lemma we have $B \not\models \gamma(A)$ iff there is a subdirectly irreducible homomorphic image C of B and a bounded lattice embedding $h : A \rightarrowtail C$. In analogy with what happened in Section 6, we can improve this by dropping homomorphic images from the consideration. For this we require the following lemma, which is an analogue of Lemma 6.2.

Lemma 7.2. [8, Lem. 6.2] *Let A, B, C be finite Heyting algebras with A subdirectly irreducible, $h : B \twoheadrightarrow C$ an onto Heyting homomorphism, and $e : A \rightarrowtail C$ a bounded lattice embedding. Then there is a bounded lattice embedding $k : A \rightarrowtail B$ such that $h \circ k = e$.*

$$\begin{tikzcd}
C & B \arrow[l, two heads, "h"'] \\
 & A \arrow[ul, "e"] \arrow[u, "k"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X_C \arrow[r, hook, "id"] \arrow[dr, "f"'] & X_B \arrow[d, "g"] \\
 & X_A
\end{tikzcd}$$

Proof. Let X_A, X_B, and X_C be the dual finite posets of A, B, and C. We identify A, B, and C with the upsets of X_A, X_B, and X_C. Since A is subdirectly irreducible, X_A is rooted. As C is a homomorphic image of B, we have that X_C is (isomorphic to) an upset of X_B, and because $e : A \to C$ is a bounded lattice embedding, there is an onto stable map $f : X_C \twoheadrightarrow X_A$.

Let x be the root of X_A. Define $g : X_B \to X_A$ by

$$g(y) = \begin{cases} f(y) & \text{if } y \in X_C \\ x & \text{otherwise} \end{cases}$$

Clearly g is a well-defined map extending f, and it is onto since f is onto. To see that g is stable, let $y, z \in X_B$ with $y \le z$. First suppose that $y \in X_C$. Then $z \in X_C$ as X_C is an upset of X_B. Since f is stable, $f(y) \le f(z)$. Therefore, by the definition of g, we have $g(y) \le g(z)$. On the other hand, if $y \in X_B \setminus X_C$, then $g(y) = x$. As x is the root of X_A, we have $x \le u$ for each $u \in X_A$.

Thus, $x \le g(z)$ for each $z \in X_B$, which implies that $g(y) \le g(z)$. Consequently, g is an onto stable map extending f.

From this we conclude that $g^{-1} : \mathsf{Up}(X_A) \to \mathsf{Up}(X_B)$ is a bounded lattice embedding such that $h \circ g^{-1} = f^{-1}$. □

Theorem 7.3. [8, Thm. 6.3] *Let A, B be subdirectly irreducible Heyting algebras with A finite. Then $B \not\models \gamma(A)$ iff there is a bounded lattice embedding of A into B.*

Proof. The right to left implication follows directly from the Stable Jankov Lemma. For the left to right implication, suppose $B \not\models \gamma(A)$. By the Filtration Lemma, there is a finite bounded sublattice D of B such that $D \not\models \gamma(A)$. By the Stable Jankov Lemma, there is a subdirectly irreducible homomorphic image C of D and a bounded lattice embedding of A into C. By Lemma 7.2, there is a bounded lattice embedding of A into D, and hence a bounded lattice embedding of A into B. □

The dual reading of Theorem 7.3 is as follows.

Theorem 7.4. [8, Thm. 6.5] *Let A be a finite subdirectly irreducible Heyting algebra and P its dual finite rooted poset. For a strongly rooted Esakia space X, we have $X \not\models \gamma(A)$ iff there is an onto stable morphism $f : X \twoheadrightarrow P$.*

Definition 7.5.

(1) We call an intermediate logic L *stable* if there is a family $\{A_i \mid i \in I\}$ of finite subdirectly irreducible algebras such that $\mathsf{L} = \mathsf{IPC} + \{\gamma(A_i) \mid i \in I\}$.
(2) We say that a class $\mathcal{K}$ of subdirectly irreducible Heyting algebras is *closed under bounded sublattices* if for any subdirectly irreducible Heyting algebras A, B from $B \in \mathcal{K}$ and A being isomorphic to a bounded sublattice of B it follows that $A \in \mathcal{K}$.

In the next theorem, the equivalence of (1) and (2) is given in [8, Thm. 6.11]. For further equivalent conditions for an intermediate logic to be stable see [12, Thm. 5.3].

Theorem 7.6. *For an intermediate logic L, the following conditions are equivalent.*

(1) *L is a stable logic.*
(2) *The class $\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ of subdirectly irreducible L-algebras is closed under bounded sublattices.*
(3) *There is a class $\mathcal{K}$ of subdirectly irreducible L-algebras that is closed under bounded sublattices and generates $\mathcal{V}(\mathsf{L})$.*

Proof. (1)⇒(2). Let A, B be subdirectly irreducible Heyting algebras such that B is an L-algebra and A is isomorphic to a bounded sublattice of B. Suppose that A is not an L-algebra. Since L is stable, there is a finite subdirectly irreducible Heyting algebra C such that $\gamma(C) \in \mathsf{L}$ and $A \not\models \gamma(C)$. By Theorem 7.3, C is isomorphic to a bounded sublattice of A, and hence to a bounded sublattice of B. Therefore, $B \not\models \gamma(C)$, a contradiction. Thus, A is an L-algebra.

(2)⇒(3). This is obvious.

(3)⇒(1). Let $\mathcal{X}$ be the set of finite non-isomorphic subdirectly irreducible Heyting algebras A such that $A \not\models \mathsf{L}$. Let

$$\mathsf{M} = \mathsf{IPC} + \{\gamma(A) \mid A \in \mathcal{X}\}.$$

It is sufficient to show that $\mathsf{L} = \mathsf{M}$. Let B be a subdirectly irreducible Heyting algebra. First suppose that $B \not\models \mathsf{L}$. Then there is $\varphi \in \mathsf{L}$ such that $B \not\models \varphi$. By the Filtration Lemma, there is a finite subdirectly irreducible Heyting algebra A such that $A \not\models \varphi$ and A is a bounded sublattice of B. By Theorem 7.3, $B \not\models \gamma(A)$. Therefore, $B \not\models \mathsf{M}$. Thus, $\mathsf{L} \subseteq \mathsf{M}$.

For the reverse inclusion, since L is the logic of $\mathcal{K}$, it is sufficient to show that if $B \in \mathcal{K}$, then $B \models \mathsf{M}$. If $B \not\models \mathsf{M}$, then $B \not\models \gamma(A)$ for some $A \in \mathcal{X}$. By Theorem 7.3, A is isomorphic to a bounded sublattice of B. Since $B \in \mathcal{K}$ and $\mathcal{K}$ is closed under bounded sublattices, $A \in \mathcal{K}$. Thus, $A \models \mathsf{L}$, a contradiction. Consequently, $B \models \mathsf{M}$, finishing the proof. □

Remark 7.7. In [20] a new class of formulas, called $\mathsf{ONNILLI}$, was described syntactically. It was shown that each formula in this class is preserved under stable images of posets, and that for a finite subdirectly irreducible Heyting algebra A, the formula $\gamma(A)$ is equivalent to a formula in $\mathsf{ONNILLI}$. This provides another description of stable formulas.

Remark 7.8. Various subclasses of stable logics that are closed under MacNeille completions were studied in [18]. In [59] a subclass of stable logics was identified and it was shown that an intermediate logic is axiomatizable by $\mathcal{P}_3$-formulas of the substructural hierarchy of Cibattoni, Galatos, and Terui [30] iff it belongs to this subclass.

Definition 7.9. We say that a class $\mathcal{K}$ of strongly rooted Esakia spaces is *closed under stable images* if for any strongly rooted Esakia spaces X, Y from $X \in \mathcal{K}$ and Y being an onto stable image of X it follows that $Y \in \mathcal{K}$.

For a class $\mathcal{K}$ of Esakis spaces, let $\mathcal{K}^* = \{X^* \mid X \in \mathcal{K}\}$. It is easy to see that $\mathcal{K}^*$ is closed under bounded sublattices iff $\mathcal{K}$ is closed under stable images. Thus, as an immediate consequence of Theorem 7.6, we obtain:

Theorem 7.10. *For an intermediate logic* L *the following conditions are equivalent.*

(1) L *is stable.*
(2) *The class of strongly rooted Esakia spaces of* L *is closed under stable images.*
(3) L *is sound and complete with respect to a class* $\mathcal{K}$ *of strongly rooted Esakia spaces that is closed under stable images.*

Theorem 7.10 can be thought of as a motivation for the name "stable logic." We next show that all stable logics have the fmp. This is an easy consequence of Theorem 7.6 and the Filtration Lemma.

Theorem 7.11. *Each stable logic has the fmp.*

Proof. Let L be a stable logic and let $\mathsf{L} \nvdash \varphi$. Then there is a subdirectly irreducible Heyting algebra B such that $B \models \mathsf{L}$ and $B \not\models \varphi$. By the Filtration Lemma, there is a finite Heyting algebra A such that A is a bounded sublattice of B and $A \not\models \varphi$. Since B is subdirectly irreducible, so is A. As L is stable and $B \models \mathsf{L}$, it follows from Theorem 7.6 that $A \models \mathsf{L}$. Because A is finite and $A \not\models \varphi$, we conclude that L has the fmp. ☐

Definition 7.12. Let Λ_{Stab} be the set of all stable logics.

Theorem 7.13.

(1) [14, Thm. 3.7] Λ_{Stab} *is a complete sublattice of* Λ.
(2) [8, Thm. 6.13] *The cardinality of* Λ_{Stab} *is that of the continuum.*

We conclude with some examples of stable logics. Recall that $\mathfrak{F}_n, \mathfrak{D}_n$, and $\mathfrak{C}_n$ denote the n-fork, n-diamond, and n-chain (see Figure 1). For a rooted poset P we abbreviate $\gamma(P^*)$ with $\gamma(P)$.

Theorem 7.14. [8, Thm.7.5]

(1) $\mathsf{CPC} = \mathsf{LC} + \gamma(\mathfrak{C}_2)$.
(2) $\mathsf{KC} = \mathsf{IPC} + \gamma(\mathfrak{F}_2)$.
(3) $\mathsf{LC} = \mathsf{IPC} + \gamma(\mathfrak{F}_2) + \gamma(\mathfrak{D}_2)$.
(4) $\mathsf{LC}_n = \mathsf{LC} + \gamma(\mathfrak{C}_{n+1})$.
(5) $\mathsf{BW}_n = \mathsf{IPC} + \gamma(\mathfrak{F}_{n+1}) + \gamma(\mathfrak{D}_{n+1})$.
(6) $\mathsf{BTW}_n = \mathsf{IPC} + \gamma(\mathfrak{F}_{n+1})$.

On the other hand, there are intermediate logics that are not stable; e.g., BD_n for $n \geq 2$ (see [8, Thm 7.4]).

7.2. **Cofinal stable rules and formulas.** As we have seen, cofinal subframe logics are axiomatizable by formulas of the form $\alpha(A)$, while subframe logics by formulas of the form $\beta(A)$. In analogy, stable logics are axiomatizable by formulas of the form $\gamma(A)$. These can be thought of to be parallel to $\beta(A)$ because both $\beta(A)$ and $\gamma(A)$ do not encode the behavior of negation. On the other hand, $\alpha(A)$ does encode it. It is natural to seek a stable analogue of $\alpha(A)$.

For this we need to work with the pseudocomplemented lattice reduct of Heyting algebras, instead of just the bounded lattice reduct like in the case of stable logics. Fortunately, the corresponding variety of pseudocomplemented distributive lattices remains locally finite (see e..g., [12, Thm. 6.1]), and hence the algebraic approach is applicable. This allows us to develop the theory of cofinal stable logics, which generalizes the theory of stable logics. However, there is a key difference, which is due to the fact that an analogue of Lemma 7.2 fails for the pseudocomplemented lattice reduct (see [12, Exmp. 7.5]). This, in particular, forces us to work with cofinal stable rules, rather than formulas.

Definition 7.15. Let A be a finite Heyting algebra, and for $a \in A$ let p_a be a new variable.

(1) The *cofinal stable rule* of A is the multiple-conclusion rule $\sigma(A) = \Gamma/\Delta$, where

$$\begin{aligned}\Gamma = \{p_0 \leftrightarrow 0\} \cup \\ &\{p_{a\vee b} \leftrightarrow p_a \vee p_b \mid a, b \in A\} \cup \\ &\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b \mid a, b \in A\} \cup \\ &\{p_{\neg a} \leftrightarrow \neg p_a \mid a \in A\}\end{aligned}$$

and

$$\Delta = \{p_a \leftrightarrow p_b \mid a, b \in A \text{ with } a \neq b\}.$$

(2) If in addition A is subdirectly irreducible, then the *cofinal stable formula* of A is

$$\delta(A) = \bigwedge \Gamma \to \bigvee \Delta.$$

Remark 7.16. There is no need to add $p_1 \leftrightarrow 1$ to Γ because $p_{\neg 0} \leftrightarrow \neg 0$ is contained in $\{p_{\neg a} \leftrightarrow \neg p_a \mid a \in A\}$.

We then have the following generalization of the Stable Jankov Lemma.

Theorem 7.17 (Cofinal Stable Jankov Lemma)**.** [12, Prop. 6.4 and 7.1] *Let A, B be Heyting algebras with A finite.*

(1) *$B \not\models \sigma(A)$ iff A is isomorphic to a pseudocomplemented sublattice of B.*
(2) *If A is subdirectly irreducible, then $B \not\models \delta(A)$ iff there is a subdirectly irreducible homomorphic image C of B such that A is isomorphic to a pseudocomplemented sublattice of C.*

Similarly, we have the following generalization of the Filtration Lemma.

Lemma 7.18 (Cofinal Filtration Lemma)**.** *Let B be a Heyting algebra such that $B \not\models \varphi$. Then there is a finite Heyting algebra A such that A is a pseudocomplemented sublattice of B and $A \not\models \varphi$. In addition, if B is well-connected, then A is subdirectly irreducible.*

However, we no longer have an analogue of Theorem 7.6. To see why, we need the following definition.

Definition 7.19. We say that a class $\mathcal{K}$ of subdirectly irreducible Heyting algebras is *closed under pseudocomplemented sublattices* if for any subdirectly irreducible Heyting algebras A, B from $B \in \mathcal{K}$ and A being isomorphic to a pseudocomplemented sublattice of B it follows that $A \in \mathcal{K}$.

As follows from [12, Exmp. 7.9] it is no longer the case that an intermediate logic L is axiomatizable by cofinal stable formulas iff the class $\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ of subdirectly irreducible L-algebras is closed under pseudocomplemented sublattices. Because of this we define cofinal subframe logics as those intermediate logics that are axiomatizable by stable canonical rules. We recall that an intermediate logic L is *axiomatizable* by a set $\mathcal{R}$ of multiple-conclusion rules if $\mathsf{L} \vdash \varphi$ iff the rule $/\varphi$ is derivable from $\mathcal{R}$. For more details about multiple-conclusion consequence relations we refer to [54, 49].

Definition 7.20. An intermediate logic L is a *cofinal subframe logic* if it is axiomatizable by cofinal subframe rules.

We can utilize the Cofinal Stable Jankov Lemma and the Cofinal Filtration Lemma to prove the following analogue of Theorem 7.6.

Theorem 7.21. *For an intermediate logic* L *the following are equivalent.*

(1) *L is a cofinal stable logic.*
(2) *$\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ is closed under pseudocomplemented sublattices.*
(3) *There is a class $\mathcal{K}$ of subdirectly ireducible Heyting algebras that is closed under pseudocomplemented sublattices and generates $\mathcal{V}(\mathsf{L})$.*

Further characterizations of cofinal stable logics can be found in [12, Thm. 7.11].

Definition 7.22. Let Λ_{CStab} be the set of cofinal stable logics.

Clearly $\Lambda_{\mathsf{Stab}} \subseteq \Lambda_{\mathsf{CStab}} \subseteq \Lambda$.

Theorem 7.23.

(1) [12, Rem. 7.8] *Each cofinal stable logic has the fmp.*
(2) [12, Prop. 8.3] *The cardinality of $\Lambda_{\mathsf{CStab}} \setminus \Lambda_{\mathsf{Stab}}$ is that of the continuum.*

In particular, the Maksimova logics ND_n, for $n \geq 2$, are examples of cofinal stable logics that are not stable logics (see [12, Lem. 9.4 and 9.5]).

As follows from [12, Prop. 7.7], each cofinal stable logic is axiomatizable by cofinal stable formulas. However, as we have already pointed out, the converse is not true in general. As far as we know, the problem mentioned in [12, Rem. 7.10]—whether all intermediate logics that are axiomatizable by cofinal stable formulas have the fmp—remains open, as does the problem of a convenient characterization of this class of intermediate logics.

Dual spaces of pseudocomplemented distributive lattices were described by Priestley [68]. Since pseudocomplemented distributive lattices are situated between bounded distributive lattices and Heyting algebras, their dual spaces are situated between Esakia spaces and Priestley spaces. Of interest to our considerations is the dual description of pseudocomplemented lattice homomophisms between Esakia spaces. These are special stable maps $f : X \to Y$ that in addition satisfy the following *cofinality condition*:

$$\max{\uparrow} f(x) = f\left(\max{\uparrow} x\right).$$

for each $x \in X$. Utilizing this, cofinal stable logics can be characterized as those intermediate logics for which the class of strongly rooted Esakia spaces is closed under cofinal stable images (meaning that, if X, Y are strongly rooted Esakia spaces such that $X \models \mathsf{L}$ and Y is a cofinal stable image of Y, then $Y \models \mathsf{L}$). We skip the details and refer the interested reader to [12].

8. Subframization and stabilization

As we pointed out in Theorems 6.6(1) and 7.13(1), the lattices of subframe and stable logics form complete sublattices of the lattice of all intermediate logics. Therefore, for each intermdiate logic L, there is a greatest subframe logic contained in L and a least subframe logic containing L, called the downward and upward subframizations of L. Similarly, there is a greatest stable logic contained in L and a least stable logic containing L, called the downward and upward stabilizations of L. These are closest subframe and stable "neighbors" of L, and were studied in [14], where connections with the Lax Logic and the intuitionistic $\mathsf{S4}$ were also explored. The operation of subframization in modal logic was first studied by Wolter [79].

8.1. **Subframization.**

Definition 8.1. For an intermediate logic L, define the *downward subframization of* L as

$$\mathsf{Subf}_{\downarrow}(\mathsf{L}) = \bigvee\{\mathsf{M} \in \Lambda_{\mathsf{Subf}} \mid \mathsf{M} \subseteq \mathsf{L}\}$$

and the *upward subframization of* L as

$$\mathsf{Subf}_{\uparrow}(\mathsf{L}) = \bigwedge\{\mathsf{M} \in \Lambda_{\mathsf{Subf}} \mid \mathsf{L} \subseteq \mathsf{M}\}.$$

Lemma 8.2. [14, Lem. 4.2] $\mathsf{Subf}_{\downarrow}$ *is an interior operator and* $\mathsf{Subf}_{\uparrow}$ *a closure operator on* Λ.

A semantic characterization of the downward and upward subframizations was given in [14, Prop. 4.3]. We next use subframe canonical formulas to give a syntactic characterization. For this we require the following lemma.

Lemma 8.3. *Let A, B be finite Heyting algebras with A subdirectly irreducible and $D \subseteq A^2$. If $B \models \beta(A)$, then $B \models \alpha(A, D)$.*

Proof. Suppose $B \not\models \alpha(A, D)$. By the Subframe Jankov Lemma, there is a homomorphic image C of B and a bounded implicative semilattice embedding $h : A \to C$ satisfying $\mathrm{CDC}_{\vee}$ for D. Since B is finite, Lemma 6.2 yields an implicative semilattice embedding of A into B. Therefore, $B \not\models \beta(A)$ by Theorem 6.3(1). □

Let L be an intermediate logic. By Theorem 4.9, there are suframe canonical formulas $\alpha(A_i, D_i)$, $i \in I$, such that $\mathsf{L} = \mathsf{IPC} + \{\alpha(A_i, D_i) \mid i \in I\}$.

Theorem 8.4. [14, Thm. 4.4] *Let $\mathsf{L} = \mathsf{IPC} + \{\alpha(A_i, D_i) \mid i \in I\}$ be an arbitrary intermediate logic.*
(1) $\mathsf{Subf}_{\downarrow}(\mathsf{L}) = \mathsf{IPC} + \{\beta(A) \mid \mathsf{L} \vdash \beta(A)\}$.
(2) $\mathsf{Subf}_{\uparrow}(\mathsf{L}) = \mathsf{IPC} + \{\beta(A_i) \mid i \in I)\}$.

Proof. (1). By Definition 6.5(1), every subframe logic is axiomatizable by subframe formulas. Therefore, every subframe logic contained in L is axiomatizable by a set of subframe formulas that are provable in L. Thus, $\mathsf{IPC} + \{\beta(A) \mid \mathsf{L} \vdash \beta(A)\}$ is the largest subframe logic contained in L.

(2). Let $\mathsf{M} = \mathsf{IPC} + \{\beta(A_i) \mid i \in I\}$. Then M is a subframe logic by definition. Let B be a finite Heyting algebra. If $B \models \mathsf{M}$, then $B \models \beta(A_i)$ for all $i \in I$. By Lemma 8.3, $B \models \beta(A_i, D_i)$ for all $i \in I$. Thus, $B \models \mathsf{L}$, and so $\mathsf{L} \subseteq \mathsf{M}$ because M has the fmp (see Theorem 6.11). It remains to show that M is the least subframe logic containing L. If not, then there is a subframe logic $\mathsf{N} \supseteq \mathsf{L}$ and a Heyting algebra B such that $B \models \mathsf{N}$ and $B \not\models \mathsf{M}$. Therefore, $B \not\models \beta(A_i)$ for some $i \in I$. By Theorem 6.3(1), A_i is isomorphic to a $(\wedge, \to)$-subalgebra of B. Since N is a subframe logic, $A_i \models \mathsf{N}$ by Theorem 6.9. But $A_i \not\models \beta(A_i, D_i)$. Consequently, $A_i \not\models \mathsf{L}$, which is a contradiction since $\mathsf{N} \supseteq \mathsf{L}$. □

Remark 8.5.
(1) The above proof of Theorem 8.4(2) is different form the one given in [14].
(2) As was pointed out in [14, Rem. 4.6], Theorem 8.4 can also be derived from the theory of describable operations of Wolter [79].

In the next theorem we axiomatize the downward and upward subframizations for many well-known intermediate logics.

Theorem 8.6. [14, Prop. 4.7]
(1) $\mathsf{Subf}_{\downarrow}(\mathsf{KC}) = \mathsf{IPC}$ *and* $\mathsf{Subf}_{\uparrow}(\mathsf{KC}) = \mathsf{LC}$.

(2) $\mathsf{Subf}_\downarrow(\mathsf{KP}) = \mathsf{IPC}$ *and* $\mathsf{Subf}_\uparrow(\mathsf{KP}) = \mathsf{BW}_2$.
(3) $\mathsf{Subf}_\downarrow(\mathsf{T}_n) = \mathsf{IPC}$ *and* $\mathsf{Subf}_\uparrow(\mathsf{T}_n) = \mathsf{BW}_n$ *for every* $n \geq 2$.
(4) $\mathsf{Subf}_\downarrow(\mathsf{BTW}_n) = \mathsf{IPC}$ *and* $\mathsf{Subf}_\uparrow(\mathsf{BTW}_n) = \mathsf{BW}_n$ *for every* $n \geq 2$.
(5) $\mathsf{Subf}_\downarrow(\mathsf{ND}_n) = \mathsf{IPC}$ *and* $\mathsf{Subf}_\uparrow(\mathsf{ND}_n) = \mathsf{BW}_2$ *for every* $n \geq 2$.

Remark 8.7. Since subframes are closely related to nuclei [16], they are also connected to the propositional lax logic PLL [43, 46], which is an intuitionistic modal logic whose algebraic models are nuclear Heyting algebras. There is a nucleic Gödel-Gentzen translation of IPC into PLL which is extended to an embedding of the lattice of intermediate logics into the lattice of extensions of PLL. This yields a new characterization of subframe logics in terms of the propositional lax logic. We refer to [14, Sec. 6] for details.

8.2. **Stabilization.** In this section we obtain similar results for the downward and upward stabilizations.

Definition 8.8. For an intermediate logic L, define the *downward stabilization of* L as

$$\mathsf{Stab}_\downarrow(\mathsf{L}) = \bigvee\{\mathsf{M} \in \Lambda_{\mathsf{Stab}} \mid \mathsf{M} \subseteq \mathsf{L}\}$$

and the *upward stabilization of* L as

$$\mathsf{Stab}_\uparrow(\mathsf{L}) = \bigwedge\{\mathsf{M} \in \Lambda_{\mathsf{Stab}} \mid \mathsf{L} \subseteq \mathsf{M}\}.$$

Lemma 8.9. [14, Lem. 7.2] $\mathsf{Stab}_\downarrow$ *is an interior operator and* $\mathrm{Stab}_\uparrow$ *a closure operator on* Λ.

A semantic characterization of the downward and upward stabilizations was given in [14, Prop. 7.3]. We use stable canonical formulas to give a syntactic characterization, which requires the following lemma.

Lemma 8.10. *Let A, B be finite subdirectly irreducible Heyting algebras and $D \subseteq A^2$. If $B \models \gamma(A)$, then $B \models \gamma(A, D)$.*

Proof. Suppose $B \not\models \gamma(A, D)$. By the Stable Jankov Lemma, there is a subdirectly irreducible homomorphic image C of B and a bounded lattice embedding $h : A \to C$ satisfying $\mathrm{CDC}_\to$ for D. Since B is finite, Lemma 7.2 yields a bounded lattice embedding of A into B. Therefore, $B \not\models \gamma(A)$ by Theorem 7.3. ☐

Let L be an intermediate logic. By Theorem 4.27, there are stable canonical formulas $\gamma(A_i, D_i)$, $i \in I$, such that $\mathsf{L} = \mathsf{IPC} + \{\gamma(A_i, D_i) \mid i \in I\}$.

Theorem 8.11. [14, Thm. 7.4] *Let $\mathsf{L} = \mathsf{IPC} + \{\gamma(A_i, D_i) \mid i \in I\}$ be an arbitrary intermediate logic.*
(1) $\mathsf{Stab}_\downarrow(\mathsf{L}) = \mathsf{IPC} + \{\gamma(A) \mid \mathsf{L} \vdash \gamma(A)\}$.
(2) $\mathsf{Stab}_\uparrow(\mathsf{L}) = \mathsf{IPC} + \{\gamma(A_i) \mid i \in I\}$.

Proof. (1). By Definition 7.5, $\mathsf{IPC} + \{\gamma(A) \mid \mathsf{L} \vdash \gamma(A)\}$ is a stable logic, and clearly it is the largest stable logic contained in L. Therefore, $\mathsf{Stab}_\downarrow(\mathsf{L}) = \mathsf{IPC} + \{\gamma(A) \mid \mathsf{L} \vdash \gamma(A)\}$.

(2). Let $\mathsf{M} = \mathsf{IPC} + \{\gamma(A_i) \mid i \in I\}$. Then M is a stabe logic by definition. Let B be a finite subdirectly irreducible Heyting algebra such that $B \models \mathsf{M}$. Then $B \models \gamma(A_i)$ for all $i \in I$. By Lemma 8.10, $B \models \gamma(A_i, D_i)$ for all $i \in I$. Thus, $B \models \mathsf{L}$, and so $\mathsf{L} \subseteq \mathsf{M}$ since M has the finite model property (see Theorem 7.11). Suppose N is a stable extension of L, and B is a subdirectly irreducible Heyting algebra such that $B \models \mathsf{N}$. If $B \not\models \gamma(A_i)$ for some $i \in I$, then A_i is isomorphic to a bounded sublattice of B by Theorem 7.3. Therefore, $A_i \models \mathsf{N}$ by Theorem 7.6. But $A_i \not\models \gamma(A_1, D_i)$. So $A_i \not\models \mathsf{L}$, which contradicts N being an extension of L. Thus, $B \models \gamma(A_i)$ for all $i \in I$, and so $\mathsf{M} \subseteq \mathsf{N}$. Consequently, M is the least stable extension of L, and hence $\mathsf{Stab}_\uparrow(\mathsf{L}) = \mathsf{M}$. ☐

Remark 8.12.
(1) The above proof of Theorem 8.11(2) is different form the one given in [14].

(2) As was pointed out in [14, Rem. 7.6], an alternate proof of Theorem 8.11 can be obtained using Wolter's describable operations.

The next theorem axiomatizes the downward and upward stabilizations of several intermediate logics.

Theorem 8.13. [14, Prop. 7.7]

(1) $\mathsf{Stab}_{\downarrow}(\mathsf{BD}_n) = \mathsf{IPC}$ *and* $\mathsf{Stab}_{\uparrow}(\mathsf{BD}_n) = \mathsf{BC}_n$ *for all* $n \geq 2$.
(2) $\mathsf{Stab}_{\downarrow}(\mathsf{T}_n) = \mathsf{IPC}$ *and* $\mathsf{Stab}_{\uparrow}(\mathsf{T}_n) = \mathsf{BW}_n$ *for all* $n \geq 2$.
(3) *If* L *has the disjunction property, then* $\mathsf{Stab}_{\downarrow}(\mathsf{L}) = \mathsf{IPC}$.

Remark 8.14. In [14, Sec. 8] the Gödel translation was utilized to embed the lattice of intermediate logics into the lattice of multiple-conclusion consequence relations over the intuitionistic $\mathsf{S4}$, and it was shown that this provides a new characterization of stable logics.

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New Mexico State University
Email address: guram@nmsu.edu

University of Amsterdam
Email address: N.Bezhanishvili@uva.nl