Quota Rules for Incomplete Judgments

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Abstract

Suppose that a group of individuals are asked to aggregate their judgments on different—possibly logically interconnected—propositions in order to reach a collective decision. Quota rules are natural aggregation rules requiring that a proposition be collectively accepted if and only if the number of individuals that agree with it exceeds a given threshold. In cases where the individuals may also abstain on some of the issues at stake and report incomplete judgments, there are several ways for determining the relevant threshold, depending on the number of abstentions or the margin between those that agree and those that disagree with a given proposition. In this paper I systematically design quota rules for incomplete inputs, within the framework of judgment aggregation, and explore their formal properties. In particular, I characterise axiomatically three distinct classes of quota rules, extending known results of the literature that so far only applied to complete inputs.

Keywords: Judgment Aggregation, Quota Rules, Incompleteness, Axioms

JEL Classification Code: D71. Microeconomics – Analysis of Collective Decision Making – Social Choice

1. Introduction

Collective decision making takes place at smaller or larger scale, supporting the democratic foundations of our society. Various issues, often logically interconnected, are at stake in different decision contexts, ranging from important political decisions to light compromises between relatives and friends. Throughout such contexts, some of the individuals whose opinions are to be aggregated into a collective decision might—being given the chance—choose to abstain from the procedure and not report a clear-cut personal judgment. For example, members of parliaments may abstain when they do not feel adequately informed about an issue at hand or when there exists some conflict of interest. More generally, individuals may not care about all the issues with which they

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are presented, or they may find the process of coming up with a concrete opinion costly, for any personal reason.

Abstentions are an essential part of collective decision making and are widely studied by practical and theoretical political scientists in election contexts (e.g., Pattie & Johnston, 2001; Perea, 2002; Plane & Gershtenson, 2004; Adams et al., 2006; Laruelle & Valenciano, 2011). Still, there is room for further analysis within formal models pertaining particularly to social choice theory. In this paper I delve deeper into judgment aggregation, a formal framework for collective decision making about binary (*yes*/*no*) issues linked to each other through logic (List, 2012; Grossi & Pigozzi, 2014; Endriss, 2016). Problems of judgment aggregation are central in various disciplines, like philosophy, law, economics, and artificial intelligence among others. In order to better illustrate the need for directing attention to abstentions (to which I also refer as *incomplete judgments*) in judgment aggregation, let us consider an example.

Example 1. Consider the board of a college in a small town, having to decide whether to offer fresh fruit on campus during the coming academic year. In the town there are only two farms (farm A and farm B) that can supply the college with fresh fruit. The board members are thus asked to express their judgments on three issues: whether a contract with farm A should be established, whether a contract with farm B should be established, and whether fresh fruit should be offered on campus. It happens that an 80% supermajority of the board members do not have any particular opinion about fruit, or about farm A, but they really dislike farm B (which is rumoured to follow unethical animal treatments). The rest of the board members report a clear judgment in favour of farm B and against farm A, while also supporting the offer of fruit on campus.

	FARM A?	FARM B?	FRUIT ON CAMPUS?
80%	–	No	–
20 %	No	Yes	Yes

How should the board decide? On the one hand, all members that expressed *some* opinion about farm A were negative regarding a contract with it, and all those that did not abstain with respect to the fruit issue were positive towards it. It would then be natural for the board to respect these unanimous opinions. On the other hand, a straightforward majority of the members were opposed to a contract with farm B, leading to an impossible situation for the board: it would need to be able to offer fresh fruit on campus without establishing a contract with any of the two providers. △

Binary decisions on logically interconnected issues feature in several application domains that differ from the one of Example 1 and where abstentions are frequent, such as political referendums, juridical cases, and companies' policy making. In general, a threshold associated with a specific proposition (e.g., establishing a new contract) is the minimum number of individuals that need to agree with that proposition in order for it to be collectively accepted. The starting point of the analysis that follows is that individuals who abstain on a given issue do not hold any positive or negative opinion about that issue—the two options (*yes* and *no*) are incomparable to them. For instance, we assume that the individuals who did not express a judgment about farm A in Example 1 merely did not have an opinion on the issue.

Contributions. The purpose of this paper is to formally introduce, in the framework of judgment aggregation, the class of quota rules based on thresholds for propositions that incorporate possibly incomplete individual judgments. Quota rules provide a simple—easy to compute and easy to explain—method for aggregating judgments that is commonly used in practice.[1] Although quota rules are very clearly defined in the complete case, original complications arise when incompleteness comes to play. Specifically, the way to count individuals who abstain is not straightforward: when building an aggregation rule, we may want to only rely on those individuals that actually report a positive or a negative judgment, or we may also want to take into consideration those that abstain. What option is better of course depends on the context. I present four alternative formulations of quota rules that capture different applications and study how they relate to each other.[2] I find that one of these formulations is the most general of all, hinging on thresholds that vary in the number of abstentions.

Moreover, I conduct a principled analysis of the new rules by defining a number of axioms that are pertinent to the aggregation of incomplete judgments and that extend classical axioms of the standard model for complete judgments. For all rules, I prove axiomatic characterisation theorems in line with known results for the complete case (Dietrich & List, 2007a). Topics I address concern, among others, the obstacles that may emerge when a given quota rule produces inconsistent outcomes. Indeed, the well-known trade-off in judgment aggregation between independent (issue-by-issue) aggregation and logical consistency evidently manifests itself in quota rules. Then, acknowledging that individuals may attempt to manipulate the collective decision by being untruthful in various occasions, I investigate whether there are rules immune to this kind of behaviour; I also characterise these rules axiomatically.

Related work. The idea of including incompleteness in studies of judgment aggregation is not new. Prior work in the area has already considered incomplete judgments, both at the collective and at the individual level. In the former direction, the assumption that an aggregation rule has to produce a collective

[1] For instance, the 1993 referendum about monarchy/parliamentarism in Brazil (`https://web.archive.org/web/20200501204603/https://en.wikipedia.org/wiki/1993_Brazilian_constitutional_referendum`) and the 1946 post-war referendum in Italy (`https://web.archive.org/save/https://en.wikipedia.org/wiki/1946_Italian_general_election`) both concerned correlated issues and employed quota rules. I am grateful to an anonymous reviewer for pointing this out.

[2] Note, though, that in this paper I do not distinguish between individuals who do not go to vote and those who submit a blank/invalid vote, although many aggregation mechanisms in practice do.

decision regarding all issues at stake has been relaxed by Gärdenfors (2006), Dietrich & List (2008), and Dokow & Holzman (2010) in the hope of circumventing typical impossibility results of the field stating that there does not exist any reasonable rule satisfying simultaneously a handful of desirable axioms. Allowing individuals to possibly abstain on some of the issues has been explored by the same authors as well, again in the light of some positive news regarding possibility results (with no great success). Dietrich & List (2010) also study a special quota rule, namely the majority rule, for the general case of possibly incomplete individual judgments—the goals of this paper differ from theirs, since Dietrich & List explore what domains of decision making are suitable for consistent majority aggregation, while I do not impose any constraints on the domain and study larger classes of quota rules.

Additional papers have specifically focused on the design of new aggregation rules tailored to incomplete inputs, which usually violate the basic independence axiom that quota rules satisfy. This paper is more similar in flavour to those. For instance, Slavkovik & Jamroga (2011) treat an abstention as a vote bearing a third value (in parallel to positive and negative values) and define a class of distance-based rules in this setting, while Terzopoulou et al. (2018) notice that incomplete judgments may have different sizes when regarded as sets, and introduce a class of rules based on weights that depend on these sizes. Jiang et al. (2018) construct a non-anonymous aggregation rule that hinges on a hierarchy over the individuals and show that it satisfies a number of desirable axioms for the incomplete setting.

Leaving aside the logical interactions between the issues at stake, there is a large pool of literature involved with referendums, and predominantly with votes on single binary choices. This stream of work was pioneered by May (1952), who proved that the majority rule is the only rule simultaneously satisfying the axioms of anonymity, neutrality, and monotonicity (for groups with an odd number of individuals). Given more than one issue, anonymity and monotonicity are also central in the axiomatisations of quota rules, while neutrality would force the thresholds of acceptance for the different propositions to coincide. Note that this first work on the topic by May hinged on complete individual opinions, but more recently scholars have relaxed this assumption.

Characteristically, Côrte-Real & Pereira (2004) analyse systems of referendums used by countries in the European Union by employing an axiomatic methodology. Côrte-Real & Pereira are specifically interested in a version of the non-show paradox (that is, a situation where an individual can improve the collective decision for herself by abstaining), which will also play a role in Section 6 of this paper. Many of their results can be translated in our model for the special case of a single issue. Still in the context of single-issue voting with approval and participation quota, Maniquet & Morelli (2015) are concerned with strategic individuals that may untruthfully abstain in order to obtain a more desirable outcome for themselves, and note that those rules that satisfy the property of monotonicity prevent such behaviour. Although this observation will also be important for us in Section 6, Maniquet & Morelli make further assumptions about the probabilistic information of the individuals concerning the number of their peers that abstain, on which their results heavily rely. On the contrary, I follow the most classical—qualitative rather than quantitative—path and study cases where individuals will *never* have the opportunity to benefit by being untruthful, no matter how the rest of the group behaves.

Other axiomatic works related to this paper have been conducted by Llamazares (2006) and by Houy (2007). Both authors examine voting rules on a single binary choice, defined with respect to the difference in number between those that support the given choice and those that oppose to it—they call this kind of aggregation methods *majority of difference*.[3] In our framework, such rules are called *marginal quota rules* in order to stay as close as possible to the—established in judgment aggregation—term *quota rules*. Considering the special case of judgment aggregation on a single proposition, the results presented in Section 4 subsume the characterisations obtained by Llamazares and Houy. Then, variants of the independence axiom in judgment aggregation essentially guarantee that in the most general case with several interconnected issues, quota rules apply in an issue-by-issue basis. Overall, this paper builds important links between two previously separate domains of research: first, single-issue voting rules with majority quota, and second, quota rules in judgment aggregation, for multiple interrelated issues.

Paper overview. The remainder of this paper is organised as follows. In Section 2 I introduce the basic judgment aggregation model together with notation and terminology. In Section 3 I formalise the new classes of quota rules for incomplete inputs and examine how they relate to each other logically. I axiomatise these rules in Section 4, and investigate their properties of collective rationality (i.e., whether they ensure complete and consistent outcomes) in Section 5. In Section 6 I concentrate on manipulative individuals that may misrepresent their judgments in order to obtain a better outcome for themselves and characterise the rules that are immune to this kind of behaviour. In Section 7 I conclude. The proofs of all formal results have been relegated to the Appendix.

2. The Model

The basic model I review follows standard models of the judgment aggregation literature (List & Pettit, 2002; List & Puppe, 2009; Grossi & Pigozzi, 2014; Endriss, 2016) and allows for incomplete individual judgments in a similar way to Terzopoulou et al. (2018).

We have a group of individuals $N = \{1, \dots, n\}$, with $n \geqslant 2$, that are asked to judge a set of binary (*yes*/*no*) issues. The agenda Φ contains a finite number of propositions (possibly complex formulas in propositional logic) of the form φ and $\neg\varphi$, denoting a positive and a negative judgment on the issue $\widetilde{\varphi}$, respectively.[4]

[3] Such binary voting rules were already mentioned by Fishburn (1973).

[4] Slightly abusing notation I assume that double negations cancel each other, i.e., that $\neg\neg\varphi = \varphi$.

Each individual i holds a judgment set $J_i \subseteq \Phi$ that formally captures her opinion on the issues at stake.[5] For instance, $\varphi \in J_i$ denotes that individual i has a positive (*yes*) judgment on $\widetilde{\varphi}$. Individual judgments are *logically consistent*, but not necessarily *complete*. This means that there may exist some issue $\widetilde{\varphi}$ such that $\varphi \notin J_i$ and $\neg\varphi \notin J_i$: in this case, we say that i *abstains* on $\widetilde{\varphi}$. Also, we say that individual i *supports/accepts* proposition φ if $\varphi \in J_i$, and she *rejects* proposition φ if she does not accept it, that is, if $\varphi \notin J_i$.

Given the individual judgments of all members of the group, we have a *profile* $\boldsymbol{J} = (J_1, \ldots J_n)$. We also write $(\boldsymbol{J}_{-i}, J_i')$ for the profile where individual i reports the judgment J_i', and all other individuals report the same judgments as in profile $\boldsymbol{J}$. We denote by $N_\varphi^{\boldsymbol{J}} = \{i \in N \mid \varphi \in J_i\}$ the set of individuals who accept φ in the profile $\boldsymbol{J}$, and we write $n_\varphi^{\boldsymbol{J}} = |N_\varphi^{\boldsymbol{J}}|$. Analogously, $N_{\widetilde{\varphi}}^{\boldsymbol{J}}$ is the set of individuals who submit a judgment (positive or negative) on the issue $\widetilde{\varphi}$ in the profile $\boldsymbol{J}$ (that is, $N_{\widetilde{\varphi}}^{\boldsymbol{J}} = N_\varphi^{\boldsymbol{J}} \cup N_{\neg\varphi}^{\boldsymbol{J}}$), and $n_{\widetilde{\varphi}}^{\boldsymbol{J}} = |N_{\widetilde{\varphi}}^{\boldsymbol{J}}|$.

Then, in order to obtain a collective decision, a (resolute) *aggregation rule for incomplete inputs* F maps every profile of individual judgments $\boldsymbol{J}$ to a non-empty subset of the agenda: $\emptyset \subset F(\boldsymbol{J}) \subseteq \Phi$. The set $F(\boldsymbol{J})$ may be incomplete, and it may also be logically inconsistent.

For instance, recall Example 1. There, we have three issues that the individuals judge: establishing a contract with farm A ($\widetilde{\varphi}$), establishing a contract with farm B ($\widetilde{\psi}$), and providing fresh fruit on campus ($\widetilde{\chi}$). That is, $\Phi = \{\varphi, \neg\varphi, \psi, \neg\psi, \chi, \neg\chi\}$. Moreover, the set $\{\neg\varphi, \neg\psi, \chi\}$ is logically inconsistent because fresh fruit cannot be offered if no supplier is selected.[6] Suppose that 100 people were eligible to express their opinions as board members. Then, we would have that $n_\varphi^{\boldsymbol{J}} = n_{\neg\chi}^{\boldsymbol{J}} = 0$, $n_{\neg\varphi}^{\boldsymbol{J}} = n_\chi^{\boldsymbol{J}} = 20$, $n_{\widetilde{\varphi}}^{\boldsymbol{J}} = n_{\widetilde{\chi}}^{\boldsymbol{J}} = 20$, $n_\psi^{\boldsymbol{J}} = 20$, $n_{\neg\psi}^{\boldsymbol{J}} = 80$, and $n_{\widetilde{\psi}}^{\boldsymbol{J}} = 100$.

3. Quota Rules for Incomplete Judgments

In this section I introduce four directions for formally generalising quota rules from the complete setting to the incomplete one. Then, I explore the logical relations between these new classes of rules.

In practice, incomplete judgments are (or should be) treated in several different ways depending on the context—i.e., on the specific situation at hand and the institution where the decision making takes place.

3.1. Formalisation

The threshold of acceptance for a proposition φ (when some of the individuals may abstain on the issue $\widetilde{\varphi}$) is defined based on (i) the *absolute number* of the supporters of φ, or (ii) the *margin* of those who support φ over those who

[5] I may sometimes also refer to a judgment set simply as judgment.

[6] We could also capture the logical interconnection between the three issues within the agenda by artificially formalising the third proposition as $\chi' := (\varphi \vee \psi) \wedge \chi$.

support $\neg\varphi$. In addition, I consider two versions of each one of the aforementioned cases, regarding whether or not the relevant threshold depends on the total number of abstentions: if it does, the threshold is called *variable*, otherwise it is called *invariable*.

We thus have four classes of quota rules. For simplicity of notation, I will write (a_φ) instead of $(a_\varphi)_{\varphi\in\Phi}$ and (a^k_φ) instead of $(a^k_\varphi)_{\varphi\in\Phi,k\in\{0,\dots,n\}}$ (and similarly for the marginal thresholds m).

The first class contains the rules according to which an absolute threshold has to be reached for a decision to be made, independently of the possible abstentions.

Definition 1. *Consider a class of thresholds (a_φ) with $a_\varphi \in \{0,\dots,n+1\}$ for all $\varphi\in\Phi$. The **invariable absolute quota rule** $F_{(a_\varphi)}$ is such that:*

$$F_{(a_\varphi)}(\boldsymbol{J}) = \{\varphi\in\Phi \mid n^{\boldsymbol{J}}_\varphi \geqslant a_\varphi\}$$

For example, according to article 27 of the UN Charter, decisions of the United Nations Security Council on procedural matters (φ) are confirmed if and only if there is an affirmative vote of at least nine (out of the fifteen) members of the council. So, an invariable absolute threshold $a_\varphi = 9$ is employed in practice.

The second class contains rules imposing a minimum margin between positive and negative votes, still independently of the possible abstentions.

Definition 2. *Consider a class of thresholds (m_φ) with $m_\varphi \in \{-n,\dots,n+1\}$ for all $\varphi\in\Phi$. The **invariable marginal quota rule** $F_{(m_\varphi)}$ is such that:*

$$F_{(m_\varphi)}(\boldsymbol{J}) = \{\varphi\in\Phi \mid n^{\boldsymbol{J}}_\varphi - n^{\boldsymbol{J}}_{\neg\varphi} \geqslant m_\varphi\}$$

In the Brexit referendum of 2016, the difference between the *Leave* and the *Remain* votes was not even 1.3 million, while the population of the United Kingdom is around 66 millions. An important decision with big consequences was decided by a very small margin (less than 2% of the British people). For such a crucial change to be implemented, many would find it desirable to have a rule requiring a minimum margin between positive and negative votes. Hence, although this was not the case in reality, an invariable marginal quota rule with threshold $m_\varphi = c$ for some large constant c would probably be appropriate.

Next, we have the class of rules that impose a threshold of acceptance for a proposition depending on the number of the reported judgments. In a variable threshold a^k_φ or m^k_φ, the parameter k codifies the number of reported judgments about the issue $\widetilde{\varphi}$ on a given profile.

Definition 3. *Consider a class of thresholds (a^k_φ) with $a^k_\varphi \in \{0,\dots,n+1\}$ for all $\varphi\in\Phi$, $k\leqslant n$. The **variable absolute quota rule** $F_{(a^k_\varphi)}$ is such that:*

$$F_{(a^k_\varphi)}(\boldsymbol{J}) = \{\varphi\in\Phi \mid n^{\boldsymbol{J}}_\varphi \geqslant a^{n^{\boldsymbol{J}}_{\widetilde{\varphi}}}_\varphi\}$$

The most popular way of conducting a referendum (for instance in Switzerland) is by materialising a proposal if and only if it is accepted by the majority of the submitted votes.[7] Such referendums use in practice the simple majority rule, with a variable absolute threshold $a^k_\varphi = \lceil \frac{k}{2} \rceil$, for all k.

For the last class of rules, a proposition φ is accepted if the difference between those that agree and those that disagree with it exceed a threshold that depends on the number of abstentions.

Definition 4. *Consider a class of thresholds (m^k_φ) with $m^k_\varphi \in \{-n, \ldots, n+1\}$ for all $\varphi \in \Phi$, $k \leqslant n$. The **variable marginal quota rule** $F_{(m^k_\varphi)}$ is such that:*

$$F_{(m^k_\varphi)}(\boldsymbol{J}) = \{\varphi \in \Phi \mid n^{\boldsymbol{J}}_\varphi - n^{\boldsymbol{J}}_{\neg\varphi} \geqslant m_\varphi^{n^{\boldsymbol{J}}_{\widetilde{\varphi}}}\}$$

In the Council of the European Union, an abstention on a matter decided by unanimity (e.g., a matter on taxation, family law, or citizenship) has the effect of a *yes* vote.[8] That is, a decision comes into force if and only if the votes in favour of it constitute all the non-abstaining votes—concretely, we have a variable marginal threshold $m^k_\varphi = k$, for all k.

Other commonly used rules can also be described within the model of quota rules presented above. For example:

- The absolute majority uses an invariable absolute threshold $a_\varphi = \lceil \frac{n}{2} \rceil$.
- A procedure with a certain quorum r (where a decision goes through if and only if at least r individuals do not abstain and from those a majority supports the decision under discussion) is equivalent to the implementation of a variable absolute threshold such that $a^k_\varphi = n + 1$ if $k < r$ and $a^k_\varphi = \lceil \frac{k}{2} \rceil$ if $k \geqslant r$.

For the variable absolute (or marginal) quota rules and for any specific threshold a^k_φ (or m^k_φ), if the threshold is at least $k + 1$, then proposition φ will never be accepted when k judgments about the issue $\widetilde{\varphi}$ are reported. This means that in practice, any rule with threshold larger than k will behave the same. Analogously, for variable marginal quota rules, all thresholds smaller than $-k$ will effectively be the same, meaning that the relevant proposition will always be accepted when k judgments on $\widetilde{\varphi}$ are reported.[9]

Moreover, the above definitions include a class of (invariable) rules called *trivial*: Trivial rules are such that, for all propositions $\varphi \in \Phi$, one of the following holds for all numbers $k = n^{\boldsymbol{J}}_{\widetilde{\varphi}}$ of reported judgments: (i) φ is always accepted (that is, the relevant absolute (marginal) threshold is equal to 0 $(-n)$), (ii) φ is

[7] `https://web.archive.org/web/20170110092314/https://www.ch.ch/en/referendum`

[8] `https://web.archive.org/web/20200429153617/https://www.consilium.europa.eu/en/council-eu/voting-system/unanimity/`

[9] The definitions I use may lack in elegance, including multiple thresholds with the same function, but are indispensable for establishing that the variable quota rules constitute the most general class of rules.

always rejected (that is, the relevant (absolute or marginal) threshold is equal to $n+1$), (*iii*) φ is accepted if and only if there is a unanimous support in favour of it (that is, the relevant (absolute or marginal) threshold is equal to n).

Note also that in this paper I only consider quota rules where the same type of threshold is associated with every proposition—in particular, we cannot have an invariable absolute threshold for some proposition φ and an invariable marginal threshold for a different proposition ψ within the same rule.

All types of rules I have described can be generalised to an even more general class of rules $F_{\alpha(\varphi),\beta(\varphi)}$, which depend on parameters $\alpha(\varphi), \beta(\varphi) \geqslant 0$, and are expressed as follows: $F_{\alpha(\varphi),\beta(\varphi)}(\boldsymbol{J}) = \{\varphi \in \Phi \mid n_{\varphi}^{\boldsymbol{J}} - \alpha(\varphi) \cdot n_{\neg\varphi}^{\boldsymbol{J}} \geqslant \beta(\varphi)\}$.[10] Although a full axiomatisation of this large class is certainly of interest, it does go beyond the main goal of this paper, which is to study and compare (based on the axiomatic method) the four different classes presented above.

3.2. Relations between different quota rules

Figure 1 depicts the space of quota rules for incomplete judgments. More precisely, interesting questions that Figure 1 answers look as follows: Do all provided definitions of Section 3.1 generate disjoint rules? Is one particular way of describing quota rules in judgment aggregation the most general one? If some definitions induce equivalent rules, how do these joint classes look like?

We next prove all relevant relations between our classes of rules. To start, variable quota rules clearly are more general than invariable quota rules, since variable thresholds can simply be constant in order to capture invariable ones.

But what about the relations between absolute and marginal thresholds? As we will see, the answers here differ completely across the classes of variable and invariable quota rules: invariable ones, divided into absolute and marginal, are proven to define two disjoint classes of rules (except for trivial cases), while variable ones, absolute and marginal, define exactly the same class of rules.

Proposition 1. *There is no non-trivial invariable absolute quota rule that coincides with an invariable marginal quota rule.*

Proposition 2. *Every variable absolute quota rule coincides with a variable marginal quota rule, and every variable marginal quota rule coincides with a variable absolute quota rule.*

Thus, for the rest of my analysis it suffices to study variable absolute quota rules (the obtained results will also hold for variable marginal quota rules).

4. Axiomatic Characterisations

In this section I discuss properties of aggregation rules (also known as *axioms*) that are tailored to the aggregation of incomplete individual judgments. Then, I characterise all classes of quota rules introduced in Section 3.

[10] I am thankful to an anonymous reviewer for this suggestion.

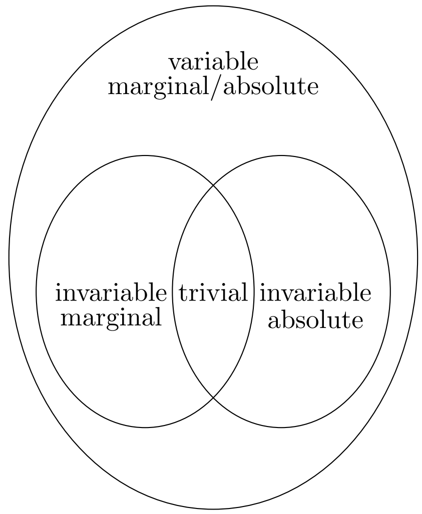

Figure 1: The space of quota rules for incomplete judgments.

4.1. Axioms

To start, I present some classical axioms—initially defined for complete individual judgments—that can be naturally defined for the incomplete case: *anonymity*, *completeness*, *complement-freeness*, and *consistency*. Except for anonymity (stating that all individuals should be treated equally during aggregation), the other axioms capture different notions of *collective rationality*.[11]

- The aggregation rule F satisfies **anonymity** (A) if for all permutations $\pi : N \to N$ and profiles $\boldsymbol{J} = (J_1, \ldots, J_n)$, it holds that $F(\boldsymbol{J}) = F(\pi(\boldsymbol{J}))$, where $\pi(\boldsymbol{J}) = (J_{\pi(1)}, \ldots, J_{\pi(n)})$.
- The aggregation rule F satisfies **completeness** (C) if for all profiles $\boldsymbol{J}$ and propositions $\varphi \in \Phi$, it holds that $\varphi \in F(\boldsymbol{J})$ or $\neg\varphi \in F(\boldsymbol{J})$.
- The aggregation rule F satisfies **complement-freeness** (CF) if for all profiles $\boldsymbol{J}$ and propositions $\varphi \in \Phi$, it is never the case that $\varphi \in F(\boldsymbol{J})$ and $\neg\varphi \in F(\boldsymbol{J})$.
- The aggregation rule F satisfies **consistency** (CN) if for all profiles $\boldsymbol{J}$ it is the case that $F(\boldsymbol{J})$ is a logically consistent set.

I now explore how other desirable properties of aggregation rules may be defined, extending their axiomatic counterparts for the complete case by specifically tak-

[11] I use the term "collective rationality" referring to properties that are desirable for the outcome of an aggregation process. Note that these properties do not necessarily need to coincide with the relevant ones for individual judgments. In particular, we may be interested in outcomes that are *complete*, although individual inputs may still be incomplete.

ing into account the potential incompleteness of the judgments. Interesting connections are also established between judgment aggregation with incompleteness and single-issue voting with abstentions.

I begin with the property of *monotonicity*, which, broadly speaking, states that extra support on a given proposition φ should never harm the collective acceptance of that proposition. Two clarifications are in order here. First, what exactly does the term "extra support" mean? Second, when is the collective acceptance of a proposition considered "harmed"? Relevant to the first question is the observation that an individual can accept a proposition φ that she was previously rejecting in two scenarios: (i) by including φ in her judgment that was before abstaining on the issue $\widetilde{\varphi}$ and (ii) by including φ in her judgment that was before containing the proposition $\neg\varphi$. The former action could be seen as less radical than the latter. Moreover, an individual can possibly promote proposition φ even more indirectly, by abstaining on $\widetilde{\varphi}$ instead of accepting $\neg\varphi$. Similarly, and regarding the second question above, a proposition φ can disappear from the collective outcome because $\neg\varphi$ is collectively accepted instead, or also independently of that (since collective judgment sets are not necessarily consistent, it may be the case that both φ and $\neg\varphi$ are included in the outcome). The following definitions will help us make these ideas concrete.

Definition 5. *Given two judgment sets J and J' and a proposition φ, the support of φ **weakly increases** from J to J' (and weakly decreases from J' to J), denoted by $J \trianglelefteq_{\varphi} J'$, if $J \cap \{\varphi\} \subseteq J' \cap \{\varphi\}$. The relation $\triangleleft$ is defined as the strict part of $\trianglelefteq$. Moreover, we write $J =_{\varphi} J'$ if and only if $J \trianglelefteq_{\varphi} J'$ and $J' \trianglelefteq_{\varphi} J$.*

Definition 5 implies that $J' \trianglelefteq_{\varphi} J$ if and only if it is not the case that $J \triangleleft_{\varphi} J'$, for any two judgment sets J and J'.

Numerous versions of the monotonicity property can be defined, depending on the precise way we interpret the notion of support of a proposition. Here I discuss two such versions, a strong one and a weak one.[12]

- The aggregation rule F satisfies **monotonicity** (M) if for all propositions $\varphi \in \Phi$ and profiles $\boldsymbol{J} = (J_1, \ldots, J_i, \ldots, J_n)$, $\boldsymbol{J'} = (J_1, \ldots, J'_i, \ldots, J_n)$ with $J_i \triangleleft_{\varphi} J'_i$, it holds that $F(\boldsymbol{J}) \trianglelefteq_{\varphi} F(\boldsymbol{J'})$ and $F(\boldsymbol{J'}) \trianglelefteq_{\neg\varphi} F(\boldsymbol{J})$.

- The aggregation rule F satisfies **weak monotonicity** (WM) if for all propositions $\varphi \in \Phi$ and profiles $\boldsymbol{J} = (J_1, \ldots, J_i, \ldots, J_n)$, $\boldsymbol{J'} = (J_1, \ldots, J'_i, \ldots, J_n)$ with $J_i =_{\neg\varphi} \{\neg\varphi\}$ and $J'_i =_{\varphi} \{\varphi\}$, it holds that $F(\boldsymbol{J}) \trianglelefteq_{\varphi} F(\boldsymbol{J'})$.

When the input profiles are complete, monotonicity and weak monotonicity reduce to the same axiom, viz., the standard monotonicity axiom in judgment aggregation (Dietrich & List, 2007a).

I continue with *independence*, an axiom requiring that the positive judgments on $\widetilde{\varphi}$ and only those should play a role in the collective decision about φ. The

[12] Restricting attention to the special case of a single-issue agenda, weak monotonicity corresponds to the axiom that Houy (2007) calls "weak monotonicity 2".

importance of independence becomes evident for agendas with more than one issue, and will play a determinant role in our analysis of quota rules—indeed quota rules are among the few natural judgment aggregation rules that satisfy independence in the complete case.

This basic idea has two implicit parts: (*i*) the judgments on $\neg\varphi$ should not affect the collective outcome about φ, and (*ii*) the judgments on different issues $\widetilde{\psi}$ should not affect the collective outcome about φ. When individual judgments are complete, case (*ii*) is the only interesting one. But if we allow for incompleteness, this is not true anymore. I thus define, additionally to the most general independence property, a weaker version.[13]

- The aggregation rule F satisfies **independence** (I) if for all propositions $\varphi \in \Phi$ and profiles $\boldsymbol{J}, \boldsymbol{J'}$, whenever $N_\varphi^{\boldsymbol{J}} = N_\varphi^{\boldsymbol{J'}}$, it holds that $F(\boldsymbol{J}) =_\varphi F(\boldsymbol{J'})$.

- The aggregation rule F satisfies **weak independence** (WI) if for all propositions $\varphi \in \Phi$ and profiles $\boldsymbol{J}, \boldsymbol{J'}$, whenever $N_\varphi^{\boldsymbol{J}} = N_\varphi^{\boldsymbol{J'}}$ and $N_{\neg\varphi}^{\boldsymbol{J}} = N_{\neg\varphi}^{\boldsymbol{J'}}$, it holds that $F(\boldsymbol{J}) =_\varphi F(\boldsymbol{J'})$.

Next, the axiom of *cancellation* is particularly relevant when modelling incomplete individual judgments, and suggests that adding to a profile the same number of individuals supporting a proposition φ and its negation $\neg\varphi$ should not change the collective outcome on that proposition. Llamazares (2006) also defines cancellation within binary voting with one issue. However, the setting of this paper is more engaged, since the multiplicity of issues brings to light original interactions between the axioms of cancellation and independence.[14]

- The aggregation rule F satisfies **cancellation** (C) if for all propositions $\varphi \in \Phi$ and profiles $\boldsymbol{J} = (J_1, \ldots, J_i, J_j, \ldots, J_n)$, $\boldsymbol{J'} = (J_1, \ldots, J'_i, J'_j, \ldots, J_n)$ with $\varphi, \neg\varphi \notin J_i \cup J_j$ and $\varphi \in J'_i$, $\neg\varphi \in J'_j$, it holds that $\varphi \in F(\boldsymbol{J})$ if and only if $\varphi \in F(\boldsymbol{J'})$.

I shall stress here that both monotonicity and cancellation bear a flavour of weak independence: they suggest that we can restrict attention to a relevant proposition φ and inspect two profiles where some of the submitted judgments on different propositions ψ may change. This is a debatable feature of these definitions. Nonetheless, I insist on using them in order to facilitate the comparison of this work with two central results of the judgment aggregation literature where monotonicity is defined in an analogous manner: the axiomatisation result of quota rules for complete inputs (Dietrich & List, 2007a), and the characterisation of all rules immune to manipulation (Dietrich & List, 2007b). Besides

[13]Weak independence has previously been defined by Gärdenfors (2006) and Dietrich & List (2008) too.

[14]For details consult Table 1, showing that independence and cancellation are not simultaneously satisfied by any of the quota rules considered here.

keeping the connections with previous work, note that the definitions I use do not have substantial technical impact on the results of this paper, since all quota rules are already weakly independent.

Finally, quota rules in general do not satisfy another popular axiom in judgment aggregation, namely *neutrality.* Neutrality demands that all propositions be treated equally by an aggregation rule, while this is not the case for quota rules that assign different thresholds to different propositions. Similarly, quota rules violate *unbiasedness* (or sometimes called acceptance-rejection neutrality), which imposes the equal treatment between a proposition and its negation.

4.2. Characterisations

Given the axioms presented in Section 4.1, I characterise all different quota rules for incomplete individual judgments. The relevant proofs are inspired by the proof of Dietrich & List (2007a) for the characterisation of quota rules in the complete framework, which uses the axioms of anonymity, monotonicity, and independence.

Theorem 1. *An aggregation rule for incomplete inputs is an invariable absolute quota rule if and only if it satisfies simultaneously the axioms of anonymity, monotonicity, and independence.*

Theorem 2. *An aggregation rule for incomplete inputs is an invariable marginal quota rule if and only if it satisfies simultaneously the axioms of anonymity, monotonicity, weak independence, and cancellation.*

Theorem 3. *An aggregation rule for incomplete inputs is a variable quota rule if and only if it satisfies simultaneously the axioms of anonymity, weak monotonicity, and weak independence.*

Freixas & Zwicker (2009) have obtained an analogous result to Theorem 1, using the properites of anonymity and monotonicity to characterise a class of quota rules that always return a yes/no answer in single-issue voting. Theorems 2 and 3 generalise upon the results of Llamazares (2006) and Houy (2007), respectively, which—restricted to the case of single-issue voting—make use of the same axioms as the ones appearing in the above results, except for independence.[15] Essentially, the original aspect of our characterisations lies in the use of the independence axiom (or versions thereof), which is vacuous in the scope of an agenda with a single issue but prominent in judgment aggregation.

Table 1 demonstrates succinctly which axioms are satisfied by what rules. In the characterisation of invariable absolute quota rules we can replace monotonicity with weak monotonicity, because these two versions of monotonicity coincide under independence. On the contrary, weak monotonicity (together

[15]Specifically, Llamazares and Houy add a *Pareto* condition in order to characterise each particular quota rule within the larger classes. Also, Houy incorporates an additional monotonicity axiom that characterises the rules based on quorums among all variable quota rules.

Quota rules \ Axioms	A	M	WM	I	WI	C
invariable absolute	✓	✓	✓	✓	✓	×
invariable marginal	✓	✓	✓	×	✓	✓
variable	✓	×	✓	×	✓	×

Table 1: Axioms satisfied by quota rules. The coloured cells in each row illustrate the axioms that characterise the relevant class of rules.

with anonymity, weak independence, and cancellation) is not enough to characterise invariable marginal quota rules. Importantly, weak independence cannot replace independence in the characterisation of invariable absolute quota rules.

4.3. Axioms within classes of rules

When we say that an axiom is violated within a particular class of quota rules, what we really mean is that there is *some* rule in this class that provides a counterexample. But are there specific assumptions on the relevant thresholds for which axioms are satisfied within a class of rules in which they are generally violated? I answer this question next.

Proposition 3 states that no invariable absolute quota rule can satisfy cancellation, unless it is trivial. Put differently, cancellation is the property that differentiates between invariable marginal and invariable absolute quota rules. Analogously (but on the other direction), independence differentiates between invariable absolute and invariable marginal quota rules.

Proposition 3. *An invariable absolute (marginal) quota rule satisfies cancellation (independence) if and only if it is trivial.*

Although the conditions stated below (in Lemma 1) are very technical, they help us better understand not only the formal properties of the different quota rules, but also the relations between these rules. Specifically, from (a) and (b) we can deduce that a variable absolute quota rule that satisfies independence will also satisfy monotonicity, and will thus be an invariable absolute quota rule, by Theorem 1. So, we obtain Proposition 4.[16]

Lemma 1. *A variable absolute quota rule $F_{(a_\varphi^k)}$ satisfies*

(a) *monotonicity if and only if for all $\varphi \in \Phi$ and $k \in \{0, \ldots, n\}$,*

$$\begin{array}{llll} (1) & a_\varphi^{k+1} \leqslant a_\varphi^k + 1 & \textit{when} & a_\varphi^k \leqslant k \leqslant n-1 \quad \textit{and} \\ (2) & a_\varphi^k \leqslant a_\varphi^{k+1} & \textit{when} & a_\varphi^{k+1} \leqslant k; \end{array}$$

[16]In general, it is not true that independence implies monotonicity—interestingly, this is true within the class of quota rules.

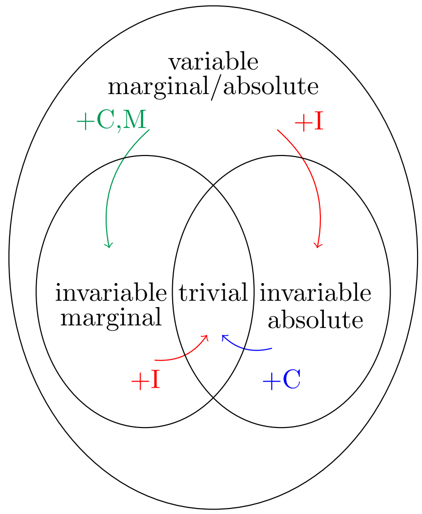

Figure 2: The space of quota rules for incomplete judgments and their properties.

(b) *independence if and only if for all $\varphi \in \Phi$ and $k, \ell \in \{0, \ldots, n\}$ with $k < \ell$,*

$$\begin{array}{llll} (1) & a^\ell_\varphi = a^k_\varphi & \textit{when} & a^k_\varphi \leqslant k \leqslant n-1 \qquad \textit{and} \\ (2) & k < a^\ell_\varphi & \textit{when} & a^k_\varphi \geqslant k+1 \ \textit{or}\ a^k_\varphi = n; \end{array}$$

(c) *cancellation if and only if for all $\varphi \in \Phi$ and $k, \ell \in \{0, \ldots, n-1\}$,*

$$\begin{array}{llll} (1) & a^{k+2}_\varphi = a^k_\varphi + 2 & \textit{when} & a^k_\varphi \leqslant k \leqslant n-2 \qquad \textit{and} \\ (2) & k+2 < a^{k+2}_\varphi & \textit{when} & a^k_\varphi \geqslant k+1 \ \textit{or}\ a^k_\varphi \geqslant n-1. \end{array}$$

Proposition 4. *A variable absolute (or marginal) quota rule satisfies independence if and only if it is an invariable absolute (or marginal) quota rule.*

Figure 2 graphically presents the observations of this section, illustrating how we can "move" through classes of rules via our axioms.

5. Collective Rationality

As we already know from the case of complete input profiles, notions of collective rationality such as completeness, complement-freeness, and consistency are not necessarily satisfied by quota rules.

The prominent example in judgment aggregation is posed by the majority rule, which collectively accepts a proposition when more than half of the individuals agree with this decision. The majority rule is straightforwardly complete for an odd number of individuals and complete inputs. However, it generates logical inconsistencies even in slightly complex agendas, like the one instantiated

	φ	ψ	$\varphi \wedge \psi$
Individual 1:	Yes	No	No
Individual 2:	No	Yes	No
Individual 3:	Yes	Yes	Yes
Majority:	Yes	Yes	No

Table 2: The discursive dilemma for complete individual judgments.

	φ	ψ	$\varphi \wedge \psi$
Individual 1:	Yes	–	–
Individual 2:	No	–	No
Individual 3:	–	Yes	–
Individual 4:	Yes	–	–
Simple majority:	Yes	Yes	No

Table 3: The discursive dilemma for incomplete individual judgments.

in the famous *discursive dilemma* (Pettit, 2001), including two propositions φ, ψ and their conjunction $\varphi \wedge \psi$. Table 2 illustrates such an inconsistent scenario.

Thus, a natural question arises: Can problematic cases such as the discursive dilemma be avoided by allowing individuals to abstain from expressing their judgment on some of the issues at stake? Although this direction may look promising at first, we will soon see that individual incompleteness is in fact detrimental to logical consistency, as soon as we require complete outcomes (indeed, if completeness of the collective decision is dropped, then inconsistencies can be trivially resolved).

Consider the simple majority rule, which accepts a proposition φ if more individuals report φ than $\neg\varphi$. This rule is guaranteed to induce complete outcomes only in the special cases where an odd number of individuals submit a judgment on $\widetilde{\varphi}$. Still, the discursive dilemma persists (consult Table 3).

Even worse, inconsistencies appear under incompleteness also for very simple agendas where only two propositions are incompatible together: consider for instance an agenda $\Phi = \{\varphi, \psi, \neg\varphi, \neg\psi\}$, with $\{\varphi, \psi\}$ being inconsistent, and see Table 4 for an inconsistent collective decision.

My next goal is to study the specific restrictions that can be imposed on the thresholds within the various classes of quota rules in order to guarantee that the relevant rules will not fail our desirable notions of collective rationality, for arbitrary agendas. The results that follow are in the spirit of existing results by Dietrich & List (2007a) concerning quota rules in the complete case, but shed light to further similarities and differences between the various classes of rules under the assumption of incompleteness.

Recognising that in applications it is of prime importance for a quota rule

	φ	ψ
Individual 1:	Yes	–
Individual 2:	No	Yes
Individual 3:	Yes	–
Simple majority:	Yes	Yes

Table 4: Inconsistent outcome $\{\varphi, \psi\}$ for incomplete individual judgments on a simple agenda.

to be collectively rational, I also discuss how easy it is to verify whether a class of thresholds complies with the relevant restrictions, when the agenda Φ is considered part of the input.

Let's start with the notion of completeness, which is essential for practical purposes where we need either a positive or a negative decision to be made by the group for every issue $\widetilde{\varphi}$ at stake.

Proposition 5. *The following hold.*

(a) *An invariable absolute quota rule with class of thresholds (a_φ) satisfies completeness if and only if for all propositions $\varphi \in \Phi$, it holds that*

$$a_\varphi = 0 \text{ or } a_{\neg\varphi} = 0.$$

(b) *An invariable marginal quota rule with class of thresholds (m_φ) satisfies completeness if and only if for all propositions $\varphi \in \Phi$,*

$$m_\varphi \leqslant \ell \text{ and } m_{\neg\varphi} \leqslant -\ell + 1 \text{ for some } \ell \in \{-n, \dots, n+1\}.$$

(c) *A variable absolute quota rule with class of thresholds (a_φ^k) satisfies completeness if and only if for all propositions $\varphi \in \Phi$ and $k \in \{0, \dots, n\}$, it holds that*

$$b_\varphi^k + b_{\neg\varphi}^k \leqslant k + 1,$$

where $b_\varphi^k = \min\{a_\varphi^k, k+1\}$ for all $\varphi \in \Phi$.

It is computationally easy (polynomial in the number of individuals and the size of the agenda) to check whether the conditions of Proposition 5 hold for a given quota rule. Also, a quite surprising fact is now brought to light: an invariable absolute quota rule satisfies completeness only if it is trivial; said differently, it is necessary that every issue is always mapped to a positive or a negative collective decision, independently of the judgments of the individuals. This observation provides an argument against many—otherwise reasonable—invariable absolute quota rules, like the absolute majority rule.

I continue with complement-freeness, which intuitively suggests that a group should not simultaneously approve and disapprove any given issue.

Proposition 6. *The following hold.*

(*a*) *An invariable absolute quota rule with class of thresholds* (a_φ) *satisfies complement-freeness if and only if for all propositions* $\varphi \in \Phi$*, it holds that*

$$a_\varphi + a_{\neg\varphi} \geqslant n + 1.$$

(*b*) *An invariable marginal quota rule with class of thresholds* (m_φ) *satisfies complement-freeness if and only if for all propositions* $\varphi \in \Phi$*, it holds that*

$$(i)\ n+1 \in \{m_\varphi, m_{\neg\varphi}\} \text{ or } (ii)\ m_\varphi, m_{\neg\varphi} \neq 0 \text{ and } m_\varphi + m_{\neg\varphi} > 0.$$

(*c*) *A variable absolute quota rule with class of thresholds* (a^k_φ) *satisfies the property of complement-freeness if and only if for all propositions* $\varphi \in \Phi$ *and* $k \in \{0, \ldots, n\}$*, it holds that*

$$a^k_\varphi + a^k_{\neg\varphi} \geqslant k + 1.$$

The conditions of Proposition 6 are also computationally tractable (i.e., polynomial in the number of individuals and the size of the agenda).

Finally I examine consistency, which is stronger than complement-freeness. We say that a subset of the agenda $Z \subseteq \Phi$ is *minimally inconsistent* (mi) if it is a logically inconsistent set that has only logically consistent strict subsets. For instance, in Table 4 above we have an agenda Φ with three minimally inconsistent subsets: $\{\varphi, \neg\varphi\}$, $\{\psi, \neg\psi\}$, and $\{\varphi, \psi\}$. Note, for instance, that $\{\varphi, \neg\varphi, \psi\}$ is also inconsistent, but not minimally, since it has strict subsets that are inconsistent as well.

Proposition 7. *The following hold.*

(*a*) *An invariable absolute quota rule with class of thresholds* (a_φ) *satisfies consistency if and only if for all mi sets* $Z \subseteq \Phi$*, it is the case that*

$$\sum_{\varphi \in Z} a_\varphi > n(|Z| - 1) \text{ or there exists } \varphi \in Z \text{ with } a_\varphi = n + 1.$$

(*b*) *An invariable marginal quota rule with class of thresholds* (m_φ) *satisfies consistency if and only if for all mi sets* $Z \subseteq \Phi$*, it is the case that*

$$\sum_{\varphi \in Z} m_\varphi > n(|Z| - 1) \text{ or there exists } \varphi \in Z \text{ with } m_\varphi = n + 1.$$

(*c*) *A variable absolute quota rule with class of thresholds* (a^k_φ) *satisfies consistency if and only if*

$$a^k_\varphi + a^k_{\neg\varphi} \geqslant k + 1 \text{ for all } k \in \{0, \ldots, n\} \text{ and } \varphi \in \Phi$$

and for all mi sets $Z \subseteq \Phi$ *such that* $Z \neq \{\varphi, \neg\varphi\}$ *for all* $\varphi \in \Phi$*,*

$$\text{either there exists } \varphi \in Z \text{ with } a^k_\varphi \geqslant k + 1 \text{ for all } k \in \{0, \ldots, n\}$$

$$\text{or } \sum_{\varphi \in Z} \min_{k : a^k_\varphi \leqslant k} a^k_\varphi > n(|Z| - 1).$$

Proposition 7 is powerful, since it permits several interesting observations to be made. The first of those concerns again the common rule of simple majority, written as an invariable marginal quota rule with quota $m_\varphi = 1$, for all $\varphi \in \Phi$. In the complete framework, the domains where the standard majority rule is consistent have been characterised by Nehring & Puppe (2007)—we know that consistency is guaranteed under an agenda Φ if and only if all mi sets of Φ are of size at most two (we then say that Φ has the *median property*).[17] Now, using Proposition 7(b), a strong negative result for the incomplete case arises.

Proposition 8. *For any agenda Φ, the simple majority rule for incomplete individual judgments is inconsistent.*

Can we possibly bring out better news, by examining whether other invariable quota rules may simultaneously be complete and consistent? Unfortunately, the answer is negative (recalling that all invariable absolute quota rules are also incomplete, unless they are trivial).

Theorem 4. *For any agenda Φ, there exists no complete and consistent, non-trivial invariable marginal quota rule for incomplete individual judgments.*

Intuitively, in the risk of facing an inconsistent outcome under *some* number of abstentions, invariable quota rules that are also complete must be trivial. Thus, variable quota rules present an immediate advantage in contexts where completeness and consistency are desired. Consider for instance the simple agenda of Table 4, and assume that the group of individuals participating in the collective decision is of an odd number. Then, there exists a variable quota rule that is complete and consistent, and functions as follows: it always accepts φ and $\neg\psi$ as the collective decision, besides the case where all individuals report their judgments on both issues; in the latter case of unanimous participation, the rule follows the opinion of the majority on each issue.

To fill the remaining picture, note that checking whether a quota rule is consistent (or whether the conditions of Proposition 7 hold) is not an easy task in general, because all mi subsets of the agenda need to be considered. In fact, this task is Π_2^p-complete in the size of the agenda. Π_2^p is the class of problems for which we can give a polynomial algorithm that decides the correctness of a certificate for the violation of the relevant condition, assuming that this algorithm has access to a SAT (satisfiability) oracle. This fact relies on the result of Endriss et al. (2012), who showed that checking whether the majority rule is consistent in the complete framework is Π_2^p-complete. Since consistency can be hard to verify for complex agendas and is violated by many quota rules, one may want to compromise and simply require complement-freeness.

[17] Dietrich & List (2010) also study the domain restrictions that guarantee majority consistency for incomplete individual judgments, but the majority definition they use is different than ours: they say that a proposition φ belongs to the collective decision if and only if more individuals accept φ than abstain on $\widetilde{\varphi}$ or accept $\neg\varphi$.

To sum up, the scope of this section was twofold, serving both a practical and a theoretical purpose. On the one hand, I have presented concrete restrictions on the thresholds inducing rules that enjoy different properties of collective rationality. The obtained results also imply that verifying whether a quota rule is collectively rational under a given interpretation is never harder in the more general incomplete setting compared to the framework of complete inputs. Thus, loosely speaking, the framework's expressivity can be increased without giving up on its complexity. On the other hand, I have specifically found that admitting incomplete judgments in the input of the quota rules does not save us from the troubles of logical inconsistency; on the contrary, several typical rules like the simple majority rule perform very badly in that respect.

6. Manipulability

Collective decision making is commonly used to find a compromise (or consensus) between the individuals involved. But this may not be enough for individuals who can see opportunities arising to manipulate the group decision and obtain a better outcome for themselves. Such individuals may often misrepresent their truthful judgments. The problem of *manipulation* has been initially studied in the context of judgment aggregation by Dietrich & List (2007b) and has been receiving increasing attention until very recently (Baumeister et al., 2013; Terzopoulou & Endriss, 2019; Botan & Endriss, 2020). In this section I offer a broader account, considering individuals who have the freedom to report incomplete judgments, and may thus lie both by hiding their truthful judgment on an issue or by inventing a new untruthful judgment, as well as by flipping their judgment (e.g., from positive to negative) on an issue.[18]

It has in fact been observed experimentally that in referendums imposing participation quorums, individuals that expect to be in the minority often abstain (Aguiar-Conraria & Magalhães, 2010). This also relates to the well-known *no-show paradox* of voting theory (Fishburn & Brams, 1983), stressing there are situations where individuals may achieve a preferable outcome by not participating in the collective decision making process. Consider the following example.

Example 2. In continuation of Example 1, suppose that Isabelle (individual i), the president of the college, does not want to offer fresh fruit on campus. Suppose also that there exists a quorum of 21% that needs to be reached in order for the collective decision to have an effect: at least 21 out of the 100 members of the board must express *some* judgment on the fruit issue, and at least half of the reported judgments must be supportive, for the board to decide to offer the fruit. If 20 people wish to have the fruit and these are all the board members that actually have an opinion on the issue, then Isabelle has two options: to express her judgment and end up having to offer the fruit, or to abstain (even if she truthfully has an opinion) and to achieve her preferred outcome. $\triangle$

[18]Note that under the assumption of complete judgments, an individual can only lie by flipping her judgment on an issue.

Suppose more generally that individual i is truthfully in favour of some proposition $\varphi \in \Phi$. Then, depending on the context, individual i may try to force φ into the collective outcome, exclude $\neg\varphi$ from the collective outcome, or both.

Example 3. Imagine a local referendum within a municipality where, in case both φ and $\neg\varphi$ end up in the collective outcome, the major is going to make the final decision either by accepting φ or by accepting $\neg\varphi$, according to her own judgment. If individual i is optimistic and believes that the major will agree with her, then she will find it sufficient to have φ in the outcome and give the major the option to choose it; if i is pessimistic and believes that the major—having the option—will explicitly disagree with her, then she will try to exclude $\neg\varphi$ from the outcome. $\triangle$

Below I formalise the notion of manipulation in judgment aggregation with possibly incomplete inputs. I take into account all possible types of manipulation that may occur according to our discussion above. Specifically, I make two main assumptions: (i) that an individual with a truthful positive judgment on $\widetilde{\varphi}$ will not see the opportunity to manipulate against φ or favourably to $\neg\varphi$,[19] and (ii) that an individual who truthfully abstains on $\widetilde{\varphi}$ will not see the opportunity to manipulate regarding $\widetilde{\varphi}$.[20]

Definition 6. *An individual i with truthful judgment J_i is said to have an* ***opportunity to manipulate*** *the aggregation rule F on proposition φ if $\varphi \in J_i$ and there exist a profile $\boldsymbol{J} = (\boldsymbol{J}_{-i}, J_i)$ and an untruthful judgment J_i' such that*

$$F(\boldsymbol{J}_{-i}, J_i) \lhd_{\varphi} F(\boldsymbol{J}_{-i}, J_i') \quad \textit{or} \quad F(\boldsymbol{J}_{-i}, J_i') \lhd_{\neg\varphi} F(\boldsymbol{J}_{-i}, J_i).$$

The aggregation rule F is *manipulable* if there exists an individual i with truthful judgment J_i that has an opportunity to manipulate F on some proposition $\varphi \in \Phi$. Otherwise F is *immune to manipulation*.

The notion of manipulation used in this paper—in line with the original one by Dietrich & List (2007b)—does not involve any definition of individual *preferences*: a manipulation act by an individual may be triggered because of several behavioural reasons. Examples 2 and 3 illustrate concrete contexts where the *opportunities* of individuals to manipulate indeed translate into *incentives* to manipulate. The detailed investigation of the relation between preference-free and preference-based notions of manipulation in judgment aggregation with

[19] This assumption may be violated, in case an individual has a reason to want her truthful judgment collectively refuted. In the context of this paper, I exclude such special circumstances and assume that the truthful opinions of the individuals about the issues at stake—in case they exist—overrule other possible motives.

[20] This assumption may also be violated for individuals that should truthfully abstain on an issue $\widetilde{\varphi}$ but may still try to steer the collective outcome towards the positive or the negative side as far as $\widetilde{\varphi}$ is concerned. When accounting for this kind of manipulation, monotonicity ceases to be a sufficient property for a quota rule's immunity to manipulation. I discuss this in more detail towards the end of this section.

incompleteness is a topic for another paper.[21]

Intuitively, the manipulation illustrated in Example 2 was possible due to a failure of monotonicity: additional support on proposition φ was able to turn the collective outcome against φ. Next, I prove more generally that monotonicity together with weak independence are sufficient conditions for any aggregation rule to be immune to manipulation (Lemma 2). This means in particular that all monotonic quota rules for incomplete judgments are immune to manipulation.

Lemma 2. *Any aggregation rule for incomplete inputs that is monotonic and weakly independent is also immune to manipulation.*

I further show that monotonicity is a necessary condition to be satisfied by any non-manipulable aggregation rule for incomplete inputs (Lemma 3). Hence, we can identify exactly those quota rules for incomplete inputs that prevent manipulation (Proposition 9).

Lemma 3. *Any aggregation rule for incomplete inputs that is immune to manipulation must be monotonic.*

Proposition 9. *A quota rule for incomplete inputs is immune to manipulation if and only if it is monotonic, that is, if and only if it is an invariable (absolute or marginal) quota rule.*

Note that the violation of weak independence (as opposed to that of monotonicity) does not directly imply the manipulability of an aggregation rule. To see this, suppose that an individual truthfully abstaining on $\widetilde{\varphi}$ unilaterally alters her judgment regarding a different issue $\widetilde{\psi}$, causing a change in the collective outcome on $\widetilde{\varphi}$. Although this is a violation of weak independence, the relevant definitions do not prescribe that it constitutes an opportunity for manipulation on φ. If, nonetheless, we modify our definitions to incorporate the possible opportunities that individuals who abstain on an issue may find to manipulate, then monotonicity will not anymore provide a safety net against the manipulability of a rule: given a monotonic rule, an individual may strategically add a proposition φ in her judgment set in order to promote φ's acceptance in the outcome (or, arguably more plausibly, in order to harm the collective acceptance of $\neg\varphi$), even if the individual should truthfully abstain on $\widetilde{\varphi}$. This observation contrasts the well-known result of Dietrich & List (2007b), who characterised all non-manipulable aggregation rules for the special case of complete inputs in terms of the two axioms of independence and monotonicity.

I close this section by providing a characterisation of all non-manipulable aggregation rules for incomplete inputs, which generalises the aforementioned

[21] Note that much of the recent work on strategic behaviour in judgment aggregation with complete inputs relies on preferences, like those defined based on the *Hamming distance*. The classical characterisation of Dietrich & List (2007b) does not hold for such classes of preferences, which pose additional restrictions. See Baumeister et al. (2017) for a survey on the topic.

result by Dietrich & List. To that end, I define an even weaker independence axiom that gives the freedom to individuals that abstain on $\widetilde{\varphi}$ to still influence, indirectly, the collective decision on $\widetilde{\varphi}$. More precisely, *very weak independence* captures the idea that the collective outcome on a proposition φ should only depend on the judgments regarding $\widetilde{\varphi}$ of the individuals that have an opinion directly on $\widetilde{\varphi}$ or possibly on the judgments regarding different issues $\widetilde{\psi}$ of the individuals that abstain on $\widetilde{\varphi}$.

- The aggregation rule F satisfies **very weak independence** (VWI) if for all propositions $\varphi \in \Phi$ and profiles $\boldsymbol{J} = (J_1, \ldots, J_n)$, $\boldsymbol{J'} = (J'_1, \ldots, J'_n)$, whenever $N^{\boldsymbol{J}}_{\varphi} = N^{\boldsymbol{J'}}_{\varphi}$, $N^{\boldsymbol{J}}_{\neg\varphi} = N^{\boldsymbol{J'}}_{\neg\varphi}$, and $J_i = J'_i$ for all $i \notin N^{\boldsymbol{J}}_{\widetilde{\varphi}}$, it holds that $F(\boldsymbol{J}) =_{\varphi} F(\boldsymbol{J'})$.

Theorem 5. *An aggregation rule for incomplete inputs is immune to manipulation if and only if it is monotonic and very weakly independent.*

7. Conclusion

To conclude, I have defined three distinct classes of quota rules in the framework of judgment aggregation that naturally account for possibly incomplete inputs. These new rules can capture plenty of scenarios of practical interest where collective decision making takes place and individuals have the option to abstain. I have conducted a principled analysis of the new rules, by characterising them axiomatically and exploring further properties they may satisfy regarding notions of collective rationality and problems of manipulability. The results presented in this paper bridge and extend known results that have so far been restricted to the special case of complete inputs or single-issue voting. In addition, a new contrast point is discovered between the two main types of thresholds in quota rules for incomplete judgments, viz. variable and invariable ones (that do and do not depend on the number of individuals that abstain, respectively): the former are superior in contexts that prioritise collective rationality (i.e., complete and consistent outcomes), while the latter have more leverage regarding immunity to manipulation.

This work opens up several avenues for future research. Although incomplete judgments are very reasonable in many contexts, most of the rules studied in the judgment aggregation literature to date rely on complete inputs; besides quota rules with which we are concerned in this paper, it would be intriguing to examine how other standard rules behave in the general case involving abstentions. Moreover, further questions arise specifically with respect to manipulability and strategic behaviour within judgment aggregation with abstentions. I have assumed that individuals who abstain do not find any opportunity to manipulate towards the positive or the negative side of the issue at stake. But as I briefly mentioned in the introduction, an individual may also submit an incomplete judgment because of some conflict of interest, in which case she actually has a settled opinion that she simply cannot report. Such an individual may also try to manipulate the outcome, possibly by lying about the relevant conflict. It would be interesting to investigate what kinds of rules can prevent manipulation in these cases, as well as in other settings with different individual incentives.

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Appendix: Proofs

Proof of Proposition 1. Take an arbitrary non-trivial invariable absolute quota rule $F_{(a_\varphi)}$. For some $\varphi \in \Phi$, it is the case that $a_\varphi \neq 0$, $a_\varphi \neq n+1$, and $a_\varphi \neq n$ (thus, $1 \leqslant a_\varphi \leqslant n-1$). Suppose—aiming for a contradiction—that $F_{(a_\varphi)}$ coincides with some invariable marginal quota rule $F_{(m_\varphi)}$. Consider a profile $\boldsymbol{J}$ where exactly a_φ individuals accept φ and 1 individual accepts $\neg\varphi$ (i.e., $n^{\boldsymbol{J}}_\varphi = a_\varphi$ and $n^{\boldsymbol{J}}_{\neg\varphi} = 1$)—this is possible because $a_\varphi + 1 \leqslant n$. Then, it holds that $\varphi \in F_{(a_\varphi)}(\boldsymbol{J})$, so it must also be the case that $\varphi \in F_{(m_\varphi)}(\boldsymbol{J})$. But for $\varphi \in F_{(m_\varphi)}(\boldsymbol{J})$ to hold, we must have that $m_\varphi \leqslant n^{\boldsymbol{J}}_\varphi - n^{\boldsymbol{J}}_{\neg\varphi} = a_\varphi - 1$.

Now, since $a_\varphi - 1 \geqslant 0$, we can consider a different profile $\boldsymbol{J'}$ where exactly $a_\varphi - 1$ individuals accept φ and no individual accepts $\neg\varphi$ (i.e., $n^{\boldsymbol{J'}}_\varphi = a_\varphi - 1$ and $n^{\boldsymbol{J'}}_{\neg\varphi} = 0$). Since $m_\varphi \leqslant n^{\boldsymbol{J'}}_\varphi - n^{\boldsymbol{J'}}_{\neg\varphi} = a_\varphi - 1$, we have that $\varphi \in F_{(m_\varphi)}(\boldsymbol{J'})$. Thus, we must also have that $\varphi \in F_{(a_\varphi)}(\boldsymbol{J'})$. But $n^{\boldsymbol{J'}}_\varphi = a_\varphi - 1 < a_\varphi$, which means that $\varphi \notin F_{(a_\varphi)}(\boldsymbol{J'})$, and we reached a contradiction. $\square$

Proof of Proposition 2. Consider a variable absolute quota rule $F_{(a^k_\varphi)}$. We need to construct a suitable variable marginal quota rule $F_{(m^k_\varphi)}$ and show that $F_{(a^k_\varphi)}(\boldsymbol{J}) = F_{(m^k_\varphi)}(\boldsymbol{J})$ for all profiles $\boldsymbol{J}$. For all propositions φ and numbers k, we define

$$m^k_\varphi = \begin{cases} k+1 & \text{if } a^k_\varphi \geqslant k+1 \\ 2a^k_\varphi - k & \text{otherwise.} \end{cases}$$

Take an arbitrary profile $\boldsymbol{J}$ and a proposition $\varphi \in \Phi$ with $\varphi \in F_{(a^k_\varphi)}(\boldsymbol{J})$. By definition of the variable absolute quota rule, $\varphi \in F_{(a^k_\varphi)}(\boldsymbol{J})$ means that $n^{\boldsymbol{J}}_\varphi \geqslant a^{n^{\boldsymbol{J}}_{\tilde\varphi}}_\varphi$. But we also have that $n^{\boldsymbol{J}}_\varphi + n^{\boldsymbol{J}}_{\neg\varphi} = n^{\boldsymbol{J}}_{\tilde\varphi}$. Thus, $n^{\boldsymbol{J}}_\varphi \geqslant a^{n^{\boldsymbol{J}}_{\tilde\varphi}}_\varphi$ implies that $n^{\boldsymbol{J}}_{\neg\varphi} \leqslant n^{\boldsymbol{J}}_{\tilde\varphi} - a^{n^{\boldsymbol{J}}_{\tilde\varphi}}_\varphi$, from which we can derive that $n^{\boldsymbol{J}}_\varphi - n^{\boldsymbol{J}}_{\neg\varphi} \geqslant 2a^{n^{\boldsymbol{J}}_{\tilde\varphi}}_\varphi - n^{\boldsymbol{J}}_{\tilde\varphi} = m^{n^{\boldsymbol{J}}_{\tilde\varphi}}_\varphi$ (since φ is accepted by $F_{(a^k_\varphi)}$ in the profile $\boldsymbol{J}$, we know that $a^k_\varphi, m^k_\varphi \leqslant k$). Hence, by definition of the variable marginal quota rule we conclude that $\varphi \in F_{(m^k_\varphi)}(\boldsymbol{J})$ and so that $F_{(a^k_\varphi)}(\boldsymbol{J}) \subseteq F_{(m^k_\varphi)}(\boldsymbol{J})$.

Now, take a proposition $\varphi \in \Phi$ with $\varphi \in F_{(m^k_\varphi)}(\boldsymbol{J})$. By definition of the variable marginal quota rule, $\varphi \in F_{(m^k_\varphi)}(\boldsymbol{J})$ means that $n^{\boldsymbol{J}}_\varphi - n^{\boldsymbol{J}}_{\neg\varphi} \geqslant m^{n^{\boldsymbol{J}}_{\tilde{\varphi}}}_\varphi$. Since we also know that $n^{\boldsymbol{J}}_\varphi + n^{\boldsymbol{J}}_{\neg\varphi} = n^{\boldsymbol{J}}_{\tilde{\varphi}}$, we obtain that $2n^{\boldsymbol{J}}_\varphi \geqslant m^{n^{\boldsymbol{J}}_{\tilde{\varphi}}}_\varphi + n^{\boldsymbol{J}}_{\tilde{\varphi}} = 2a^{n^{\boldsymbol{J}}_{\tilde{\varphi}}}_\varphi - n^{\boldsymbol{J}}_{\tilde{\varphi}} + n^{\boldsymbol{J}}_{\tilde{\varphi}}$ (again, since φ is accepted by $F_{(m^k_\varphi)}$ in the profile $\boldsymbol{J}$, we know that $m^k_\varphi, a^k_\varphi \leqslant k$). Equivalently, $n^{\boldsymbol{J}}_\varphi \geqslant a^{n^{\boldsymbol{J}}_{\tilde{\varphi}}}_\varphi$, which implies that $\varphi \in F_{(a^k_\varphi)}(\boldsymbol{J})$. Thus, we also have that $F_{(m^k_\varphi)}(\boldsymbol{J}) \subseteq F_{(a^k_\varphi)}(\boldsymbol{J})$, concluding that $F_{(a^k_\varphi)}(\boldsymbol{J}) = F_{(m^k_\varphi)}(\boldsymbol{J})$.

Finally, the fact that every variable marginal quota rule coincides with some variable absolute quota rule can be proven analogously. $\square$

Proof of Theorem 1. This proof directly follows the proof of Theorem 1 by Dietrich & List (2007a), for quota rules in the complete setting. $\square$

Proof of Theorem 2. To check that invariable marginal quota rules satisfy all the axioms of anonymity, weak independence, monotonicity, and cancellation is easy. For the other direction, consider an arbitrary aggregation rule F satisfying all the axioms of the hypothesis, and an arbitrary proposition $\varphi \in \Phi$. If φ never belongs to the collective outcome, no matter what the input profile is, then we can take $m_\varphi = n+1$. Otherwise, there exists a profile $\boldsymbol{J}$ such that $\varphi \in F(\boldsymbol{J})$. So, we can consider a specific such profile $\boldsymbol{J} \in \text{argmin}_{\boldsymbol{J}:\varphi \in F(\boldsymbol{J})} n^{\boldsymbol{J}}_\varphi - n^{\boldsymbol{J}}_{\neg\varphi}$ and define the number m_φ as:

$$m_\varphi = \min_{\boldsymbol{J}:\varphi \in F(\boldsymbol{J})} n^{\boldsymbol{J}}_\varphi - n^{\boldsymbol{J}}_{\neg\varphi}$$

From anonymity and weak independence, it follows that φ will belong to the collective outcome for every profile where propositions φ and $\neg\varphi$ have the same number of supporters as in $\boldsymbol{J}$. Then, monotonicity implies that φ will belong to the collective outcome also for every profile in which the support of φ increases or the support of $\neg\varphi$ decreases with respect to the relevant support in $\boldsymbol{J}$. Moreover, the axiom of cancellation suggests that by increasing (or decreasing) the support of φ and $\neg\varphi$ to the same degree, φ will still belong to the collective outcome. Formally, we have that $\varphi \in F(\boldsymbol{J'})$ for all profiles $\boldsymbol{J'}$ with $n^{\boldsymbol{J'}}_\varphi - n^{\boldsymbol{J'}}_{\neg\varphi} \geqslant n^{\boldsymbol{J}}_\varphi - n^{\boldsymbol{J}}_{\neg\varphi} = m_\varphi$. Finally, by definition of m_φ, we have that $\varphi \notin F(\boldsymbol{J'})$ for all profiles $\boldsymbol{J''}$ with $n^{\boldsymbol{J''}}_\varphi - n^{\boldsymbol{J''}}_{\neg\varphi} < n^{\boldsymbol{J}}_\varphi - n^{\boldsymbol{J}}_{\neg\varphi} = m_\varphi$. We conclude that F coincides on φ with the invariable marginal quota rule associating with φ the threshold m_φ. $\square$

Proof of Theorem 3. We will prove the statement for variable absolute quota rules. To check that variable absolute quota rules satisfy all the axioms of anonymity, weak independence, and weak monotonicity is easy. For the other direction, we will repeat the following argument for all numbers $k \in \{0, \dots, n\}$: If φ never belongs to the collective outcome for profiles $\boldsymbol{J}$ with $n^{\boldsymbol{J}}_{\tilde{\varphi}} = k$, then we can take $m^k_\varphi = k+1$. Otherwise, there exists a profile $\boldsymbol{J}$ where exactly k individuals report an opinion on φ such that $\varphi \in F(\boldsymbol{J})$. So, we can consider a specific such profile $\boldsymbol{J}$ with the smallest number of supporters of φ and define the number m^k_φ as:

$$m^k_\varphi = \min_{\substack{\boldsymbol{J}:\varphi\in F(\boldsymbol{J})\\ n^{\boldsymbol{J}}_{\widetilde{\varphi}}=k}} n^{\boldsymbol{J}}_\varphi$$

From anonymity and weak independence, we have that φ will belong to the collective outcome for every profile where proposition φ has the same number of supporters as in $\boldsymbol{J}$ and the number of individuals reporting a judgment on the issue $\widetilde{\varphi}$ remains the same. Then, weak monotonicity implies that φ will belong to the collective outcome also for every profile in which the support for φ increases while still the number of individuals reporting a judgment on the issue $\widetilde{\varphi}$ remains the same. Formally, $\varphi \in F(\boldsymbol{J'})$ for all profiles $\boldsymbol{J'}$ with $n^{\boldsymbol{J'}}_\varphi \geqslant n^{\boldsymbol{J}}_\varphi = m^k_\varphi$ and $n^{\boldsymbol{J}}_{\widetilde{\varphi}} = k$. Finally, by definition of m_φ, we have that $\varphi \notin F(\boldsymbol{J'})$ for all profiles $\boldsymbol{J'}$ with $n^{\boldsymbol{J'}}_\varphi < m^k_\varphi$ and $n^{\boldsymbol{J}}_{\widetilde{\varphi}} = k$. We conclude that F coincides on φ with the variable absolute quota rule associating with φ (in profiles with k individuals reporting a judgment on $\widetilde{\varphi}$) the threshold m^k_φ. □

Proof of Proposition 3. Clearly, any trivial quota rule satisfies cancellation. Consider a non-trivial invariable absolute quota rule $F_{(a_\varphi)}$. We know that there exists some $\varphi \in \Phi$ such that $1 \leqslant a_\varphi \leqslant n-1$. Consider the profiles $\boldsymbol{J}$ and $\boldsymbol{J'}$, where the number of individuals who accept propositions φ or $\neg\varphi$ are depicted as follows:

$$\begin{array}{rl}
\boldsymbol{J}: & \overbrace{\varphi\ldots\varphi\varphi}^{a_\varphi}\,\neg\varphi \quad \overbrace{-\ldots-}^{n-a_\varphi-1} \\
\boldsymbol{J'}: & \overbrace{\varphi\ldots\varphi}^{a_\varphi-1}-\,- \quad \overbrace{-\ldots-}^{n-a_\varphi-1}
\end{array}$$

Proposition φ is accepted in $\boldsymbol{J}$ and rejected in $\boldsymbol{J'}$, thus cancellation is violated.

Analogously, we can prove that only trivial (invariable marginal) quota rules satisfy independence. □

Proof of Lemma 1. The conditions follow from a careful analysis of the axioms.

(a). $F_{(a^k_\varphi)}$ is monotonic if and only if, whenever we add extra support to a proposition φ in a profile with k reported judgments on the issue $\widetilde{\varphi}$, (i) the new threshold on φ is reached in case the old threshold on φ was reached and (ii) the new threshold on $\neg\varphi$ is not reached if the old threshold on $\neg\varphi$ was not reached. To "add extra support" includes two cases here: first, the case where an individual accepts φ instead of $\neg\varphi$, and second, the case where an individual accepts φ instead of abstaining on $\widetilde{\varphi}$.

For the first case, where the number of reported judgments on $\widetilde{\varphi}$ remains the same, conditions (i) and (ii) are trivially satisfied.

For the second case, it is easy to see the following: If $a^{k+1}_\varphi \leqslant a^k_\varphi + 1$ whenever $a^k_\varphi \leqslant k \leqslant n-1$, then condition (i) will be satisfied, and if $a^k_\varphi \leqslant a^{k+1}_\varphi$ whenever $a^{k+1}_\varphi \leqslant k$ (specifically for proposition $\neg\varphi$), then condition (ii) will be satisfied.

On the other hand, if there exists $\psi \in \Phi$ such that $a_\psi^{k+1} > a_\psi^k + 1$ for $a_\psi^k \leqslant k \leqslant n-1$, we can construct a profile $\boldsymbol{J}$ where k individuals report a judgment on $\widetilde{\psi}$ and a_ψ^k of them accept ψ (meaning that ψ will be collectively accepted on $\boldsymbol{J}$), and a different profile $\boldsymbol{J'}$ where $k+1$ individuals report a judgment on $\widetilde{\psi}$ and a_ψ^k+1 of them accept ψ (meaning that ψ will be collectively rejected on $\boldsymbol{J'}$), violating condition (i). Similarly, if there exists $\psi \in \Phi$ such that $a_\psi^k > a_\psi^{k+1}$ for $a_\psi^{k+1} \leqslant k$, we can construct a profile $\boldsymbol{J}$ where k individuals report a judgment on $\widetilde{\psi}$ and a_ψ^{k+1} of them accept ψ (meaning that ψ will be collectively rejected on $\boldsymbol{J}$), and a different profile $\boldsymbol{J'}$ where $k+1$ individuals report a judgment on $\widetilde{\psi}$ and still a_ψ^{k+1} of them accept ψ (that is, proposition $\neg\psi$ obtained extra support), meaning that ψ will be collectively accepted on $\boldsymbol{J'}$ and violating condition (ii).

(b). For the"if" direction: Consider two arbitrary profiles $\boldsymbol{J}$ and $\boldsymbol{J'}$ such that $n_\varphi^{\boldsymbol{J}} = n_\varphi^{\boldsymbol{J'}}$ and suppose that they have a different number of individuals reporting a judgment on $\widetilde{\varphi}$ (otherwise independence holds trivially for the variable absolute quota rule $F_{a_\varphi^k}$). Take $n_{\widetilde{\varphi}}^{\boldsymbol{J}} = k$ and $n_{\widetilde{\varphi}}^{\boldsymbol{J'}} = \ell$, with $k < \ell$ (the case where $\ell > k$ is symmetric). We will show that the axiom of independence is satisfied with respect to $\boldsymbol{J}$ and $\boldsymbol{J'}$.

If $a_\varphi^k \leqslant k \leqslant n-1$, then by condition (1) we have that $a_\varphi^\ell = a_\varphi^k$. This means that φ will be treated the same in the profiles $\boldsymbol{J}$ and $\boldsymbol{J'}$ and independence will be satisfied.

If $a_\varphi^k \geqslant k+1$ or $a_\varphi^k = n$, then by condition (2) we have that $k < a_\varphi^\ell$. In this case, whenever φ is collectively accepted in $\boldsymbol{J}$, independence is vacuoulsy satisfied (because $a_\varphi^k \geqslant k+1$ directly implies the rejection of φ, while when $a_\varphi^k = n$ and φ is accepted, there is no extra support to be added on $\neg\varphi$ in order to obtain the profile $\boldsymbol{J'}$). Now, whenever φ is collectively rejected in $\boldsymbol{J}$, it will hold that $k < a_\varphi^k$, so $a_\varphi^k \geqslant k+1$. Then, $n_\varphi^{\boldsymbol{J'}} = n_\varphi^{\boldsymbol{J}} \leqslant k < a_\varphi^\ell$, therefore φ will be rejected in the profile $\boldsymbol{J'}$ as well, and thus independence is satisfied.

For the "only if" direction: We will work on the contrapositive. We will show that if condition (1) or condition (2) does not hold, then we can have two profiles $\boldsymbol{J}$ and $\boldsymbol{J'}$ on which the axiom of independence is violated.

If condition (1) does not hold, then there are $\varphi \in \Phi$ and $k, \ell \in \{0, \dots, n\}$, $k < \ell$, with $a_\varphi^k \leqslant k \leqslant n-1$ such that $a_\varphi^\ell \neq a_\varphi^k$. If $a_\varphi^\ell < a_\varphi^k$, consider the following profiles: $\boldsymbol{J}$, where a_φ^ℓ individuals accept φ and $k - a_\varphi^\ell$ individuals accept $\neg\varphi$ (thus $n_{\widetilde{\varphi}}^{\boldsymbol{J}} = k$) and $\boldsymbol{J'}$, where a_φ^ℓ individuals (the same as in $\boldsymbol{J}$) accept φ and $\ell - a_\varphi^\ell$ individuals accept $\neg\varphi$ (thus $n_{\widetilde{\varphi}}^{\boldsymbol{J'}} = \ell$). Then, φ will be rejected in $\boldsymbol{J}$ but accepted in $\boldsymbol{J'}$, violating independence. If $a_\varphi^k < a_\varphi^\ell$, the construction is symmetric.

If condition (2) does not hold, then there are $\varphi \in \Phi$ and $k, \ell \in \{0, \dots, n\}$, $k < \ell$, with $a_\varphi^k \geqslant k+1$ or $a_\varphi^k = n$ such that $a_\varphi^\ell \leqslant k$. We construct a profile $\boldsymbol{J}$ with exactly k individuals judging the issue $\widetilde{\varphi}$, and all of them accepting φ. Since $k < a_\varphi^k$, the proposition φ will be rejected in $\boldsymbol{J}$. Then, we construct another profile $\boldsymbol{J'}$ with exactly ℓ individuals judging the issue $\widetilde{\varphi}$, and exactly k

of them (the same as in $\boldsymbol{J}$) accepting φ. Since $k \geqslant a^{\ell}_{\varphi}$, proposition φ will be accepted in $\boldsymbol{J'}$. Thus, independence is violated.

(c). The proof is analogous to that of part (b). $\square$

Proof of Proposition 5. We show each case separately.

(a). Consider a proposition $\varphi \in \Phi$ and a profile $\boldsymbol{J}$ with $n^{\boldsymbol{J}}_{\varphi} = n^{\boldsymbol{J}}_{\neg\varphi} = 0$. For $F_{(a_\varphi)}(\boldsymbol{J})$ to be complete, it must be the case that $a_\varphi = 0$ or $a_{\neg\varphi} = 0$. And obviously, if $a_\varphi = 0$ or $a_{\neg\varphi} = 0$ for all $\varphi \in \Phi$, then $F_{(a_\varphi)}$ is complete.

(b). First, suppose that for all propositions $\varphi \in \Phi$, there exists $\ell \in \{-n, \dots, n+1\}$ such that $m_\varphi \leqslant \ell$ and $m_{\neg\varphi} \leqslant -\ell + 1$. Consider an arbitrary proposition φ with corresponding ℓ. Then, for an arbitrary profile $\boldsymbol{J}$, if $n^{\boldsymbol{J}}_{\varphi} - n^{\boldsymbol{J}}_{\neg\varphi} \geqslant \ell \geqslant m_\varphi$, we will have that $\varphi \in F_{(m_\varphi)}(\boldsymbol{J})$. If $n^{\boldsymbol{J}}_{\varphi} - n^{\boldsymbol{J}}_{\neg\varphi} < \ell$, which means that $n^{\boldsymbol{J}}_{\neg\varphi} - n^{\boldsymbol{J}}_{\varphi} > -\ell$, or that $n^{\boldsymbol{J}}_{\neg\varphi} - n^{\boldsymbol{J}}_{\varphi} \geqslant -\ell + 1 \geqslant m_{\neg\varphi}$, we will have that $\neg\varphi \in F_{(m_\varphi)}(\boldsymbol{J})$. Thus, in any case, $F_{(m_\varphi)}(\boldsymbol{J})$ will be complete.

Second, suppose that there exists a proposition $\varphi \in \Phi$ such that for all $\ell \in \{-n, \dots, n+1\}$ it is the case that $m_\varphi > \ell$ or $m_{\neg\varphi} > -\ell+1$. We will construct a profile $\boldsymbol{J}$ such that $F_{(m_\varphi)}(\boldsymbol{J})$ will be incomplete. Take $\ell' \in \{-n, \dots, n\}$ to be the largest number for which $m_\varphi > \ell'$ (we know that this number exists, since it must at least be the case that $m_\varphi > -n$, because otherwise it should be $m_{\neg\varphi} > n + 1$, which is impossible). Then, we know that $m_\varphi \leqslant \ell' + 1$, which implies that $m_{\neg\varphi} > -\ell'$. Suppose without loss of generality that $\ell' \geqslant 0$ (the case for $\ell' < 0$ is symmetric). Consider the profile $\boldsymbol{J}$ where $n^{\boldsymbol{J}}_{\varphi} = \ell'$ and $n^{\boldsymbol{J}}_{\neg\varphi} = 0$. Then, $n^{\boldsymbol{J}}_{\varphi} - n^{\boldsymbol{J}}_{\neg\varphi} = \ell' < m_\varphi$, and also $n^{\boldsymbol{J}}_{\neg\varphi} - n^{\boldsymbol{J}}_{\varphi} = -\ell' < m_{\neg\varphi}$. Thus, $\varphi \notin F_{(m_\varphi)}(\boldsymbol{J})$ and $\neg\varphi \notin F_{(m_\varphi)}(\boldsymbol{J})$, so $F_{(m_\varphi)}(\boldsymbol{J})$ is incomplete.

(c). First, suppose that $b^k_\varphi + b^k_{\neg\varphi} \leqslant k + 1$ for all propositions $\varphi \in \Phi$ and $k \in \{0, \dots, n\}$. We will show that $F_{(a^k_\varphi)}$ is complete. Assume, working towards a contradiction, that $F_{(a^k_\varphi)}$ is not complete. Then, there exist a proposition $\varphi \in \Phi$ and a profile $\boldsymbol{J}$ with $n^{\boldsymbol{J}}_{\varphi} = k$ such that neither φ nor $\neg\varphi$ belong to the collective outcome. This means that

$$\begin{aligned} n^{\boldsymbol{J}}_{\varphi} &< a^k_\varphi \qquad \text{and} \\ n^{\boldsymbol{J}}_{\neg\varphi} &< a^k_{\neg\varphi}. \end{aligned}$$

Since $n^{\boldsymbol{J}}_{\varphi} \leqslant k$ and $n^{\boldsymbol{J}}_{\neg\varphi} \leqslant k$, it will also be the case that

$$\begin{aligned} n^{\boldsymbol{J}}_{\varphi} &< b^k_\varphi \qquad \text{and} \\ n^{\boldsymbol{J}}_{\neg\varphi} &< b^k_{\neg\varphi}. \end{aligned}$$

And since b^k_φ and $b^k_{\neg\varphi}$ are natural numbers,

$$\begin{aligned} n^{\boldsymbol{J}}_{\varphi} &\leqslant b^k_\varphi - 1 \qquad \text{and} \\ n^{\boldsymbol{J}}_{\neg\varphi} &\leqslant b^k_{\neg\varphi} - 1. \end{aligned}$$

We conclude that $k = n^{\boldsymbol{J}}_{\varphi} + n^{\boldsymbol{J}}_{\neg\varphi} \leqslant b^k_{\varphi} + b^k_{\neg\varphi} - 2 \leqslant k-1$, which is a contradiction.

For the other direction, suppose there exists a proposition φ and a number $k \in \{0, \ldots, n\}$ such that $b^k_{\varphi} + b^k_{\neg\varphi} > k+1$. We will assume that $F_{(m_\varphi)}(\boldsymbol{J})$ is complete for all profiles $\boldsymbol{J}$ and will reach a contradiction. Consider the profile $\boldsymbol{J}^0$ where $n^{\boldsymbol{J}}_{\varphi} = k$ and $n^{\boldsymbol{J}}_{\neg\varphi} = 0$. If $b^k_{\neg\varphi} \leqslant 0$, it follows from the hypothesis that $b^k_{\varphi} > k+1$, which violates the definition of b^k_{φ}. So, $a^k_{\varphi} \geqslant b^k_{\varphi} > 0$, and in order for $F_{(m_\varphi)}(\boldsymbol{J}^0)$ to be complete, we need $a^k_{\varphi} \leqslant k$, which implies $b^k_{\varphi} \leqslant k$, and thus $b^k_{\neg\varphi} > 1$. If $k = 0$, we already have a contradiction because a^k_{φ} implies that φ cannot be in the outcome, so we need $b^k_{\neg\varphi} \leqslant 1$ in order for $\neg\varphi$ to be in the outcome. If $k > 0$, consider the profile $\boldsymbol{J}^1$ where $n^{\boldsymbol{J}}_{\varphi} = k-1$ and $n^{\boldsymbol{J}}_{\neg\varphi} = 1$. From $a^k_{\neg\varphi} \geqslant b^k_{\neg\varphi} > 1$, it follows that $\neg\varphi \notin F_{(m_\varphi)}(\boldsymbol{J}^1)$. So, we must have $b^k_{\varphi} \leqslant k-1$, and thus $b^k_{\neg\varphi} > 2$. If $k = 1$, we have a contradiction as before, from the definition of $b^k_{\neg\varphi}$. Otherwise, we continue in the same way, considering profiles $\boldsymbol{J}^2, \ldots, \boldsymbol{J}^{k-1}$, and concluding that that we need $b^k_{\varphi} \leqslant 1$ and thus $b^k_{\neg\varphi} > k$, which means that $\neg\varphi$ can never belong to the collective outcome. Finally, let us consider the profile $\boldsymbol{J}^k$ where $n^{\boldsymbol{J}}_{\varphi} = 0$ and $n^{\boldsymbol{J}}_{\neg\varphi} = k$. It must hold that $\varphi \in F_{(a^k_\varphi)}(\boldsymbol{J})$, which implies that $a^k_{\varphi} \leqslant 0$ and thus $b^k_{\varphi} \leqslant 0$. But then, we must have $b^k_{\neg\varphi} > k+1$, which is impossible, and we reached a contradiction. □

Proof of Proposition 6. We show each case separately.

(a). Suppose that $F_{(a_\varphi)}(\boldsymbol{J})$ is not complement-free. This means that there exists a profile $\boldsymbol{J}$ and a proposition $\varphi \in \Phi$ such that $\varphi \in F_{(a_\varphi)}(\boldsymbol{J})$ and $\neg\varphi \in F_{(a_\varphi)}(\boldsymbol{J})$. Thus, $a_\varphi \leqslant n^{\boldsymbol{J}}_{\varphi}$ and $a_{\neg\varphi} \leqslant n^{\boldsymbol{J}}_{\neg\varphi}$, which means that $a_\varphi + a_{\neg\varphi} \leqslant n^{\boldsymbol{J}}_{\varphi} + n^{\boldsymbol{J}}_{\neg\varphi} \leqslant n < n+1$.

For the other direction, suppose that there exists a proposition $\varphi \in \Phi$ such that $a_\varphi + a_{\neg\varphi} < n+1$. Then, $a_\varphi + a_{\neg\varphi} \leqslant n$, so we can construct a profile $\boldsymbol{J}$ in which at least a_φ individuals accept φ and at least $a_{\neg\varphi}$ individuals accept $\neg\varphi$. Then, $\varphi \in F_{(a_\varphi)}(\boldsymbol{J})$ and $\neg\varphi \in F_{(a_\varphi)}(\boldsymbol{J})$, and $F_{(a_\varphi)}(\boldsymbol{J})$ is not complement-free.

(b). We start by assuming that $F_{(m_\varphi)}(\boldsymbol{J})$ is not complement-free for some profile $\boldsymbol{J}$. Thus, there exists a proposition φ such that $\varphi, \neg\varphi \in F_{(m_\varphi)}(\boldsymbol{J})$. So, $n^{\boldsymbol{J}}_{\varphi} - n^{\boldsymbol{J}}_{\neg\varphi} \leqslant m_\varphi$ and $n^{\boldsymbol{J}}_{\neg\varphi} - n^{\boldsymbol{J}}_{\varphi} \leqslant m_{\neg\varphi}$. Therefore, $0 \leqslant m_\varphi + m_{\neg\varphi}$ and condition (ii) is violated. In addition, condition (i) must be violated for all propositions φ as well, because if it was satisfied, then either φ or its complement would never belong to the collective outcome, meaning that $F_{(m_\varphi)}(\boldsymbol{J})$ would be complement-free.

Now assume that conditions (i) and (ii) are simultaneously violated. Suppose that (ii) is violated because $m_\varphi = 0$ (the argument is symmetric for $m_{\neg\varphi} = 0$). Then, φ belongs to the collective outcome for every input profile. But from the violation of (i), $m_{\neg\varphi} \neq n+1$, so there exists a profile for which $\neg\varphi$ belongs to the collective outcome as well, making the rule $F_{(m_\varphi)}$ non-complement-free. But (ii) may also be violated because $m_\varphi + m_{\neg\varphi} \leqslant 0$. Since (from the violation of (i)) $m_\varphi \leqslant n$, we can construct a profile $\boldsymbol{J}$ with exactly m_φ individuals accepting proposition φ and none accepting proposition $\neg\varphi$. Then, $n^{\boldsymbol{J}}_\varphi - n^{\boldsymbol{J}}_{\neg\varphi} = m_\varphi$, so $\varphi \in F_{(m_\varphi)}(\boldsymbol{J})$. But $n^{\boldsymbol{J}}_{\neg\varphi} - n^{\boldsymbol{J}}_\varphi = -m_\varphi \geqslant m_{\neg\varphi}$, so $\neg\varphi \in F_{(m_\varphi)}(\boldsymbol{J})$. We conclude that $F_{(m_\varphi)}(\boldsymbol{J})$ is not complement-free.

(c). Analogous to part (a). $\square$

Proof of Proposition 7. The proof technique for each case is similar, and is inspired by the relevant proof of Dietrich & List (2007a).

(a). Follows immediately from Dietrich & List (2007a).

(b). Similar to (and simpler than) part (c).

(c). For the one direction, assume that a variable absolute quota rule $F = F_{(a^k_\varphi)}$ is not consistent. We will prove that the conditions of the statement are violated.

We know that there must exist some profile $\boldsymbol{J}$ such that $F(\boldsymbol{J})$ is inconsistent. Take $Z \subseteq F(\boldsymbol{J})$ a mi set. If $\varphi, \neg\varphi \in Z$, then it must be the case that $a^k_\varphi + a^k_{\neg\varphi} \leqslant k$ (so that both φ and $\neg\varphi$ are collectively accepted), and we are done.

Thus, suppose that $\{\varphi, \neg\varphi\} \nsubseteq Z$. Then, for every proposition $\varphi \in Z$, since φ belongs to the collective outcome, there exists $k = n^{\boldsymbol{J}}_{\tilde{\varphi}}$ with $a^k_\varphi \leqslant n^k_\varphi \leqslant k$. We will next also prove that $\sum_{\varphi \in Z} \min_{k : a^k_\varphi \leqslant k} a^k_\varphi \leqslant n(|Z| - 1)$. Exactly $n - n^{\boldsymbol{J}}_\varphi$ individuals reject each proposition $\varphi \in Z$, hence the propositions $\varphi \in Z$ are rejected in profile $\boldsymbol{J}$ in total $\sum_{\varphi \in Z} n - n^{\boldsymbol{J}}_\varphi$ times. But since Z is inconsistent, each of the n individuals rejects at least one proposition in Z. Thus,

$$\sum_{\varphi \in Z} n - n^{\boldsymbol{J}}_\varphi \geqslant n.$$

Moreover, since $a_\varphi^{n^{\boldsymbol{J}}_{\tilde{\varphi}}} \leqslant n^{\boldsymbol{J}}_\varphi \leqslant n^{\boldsymbol{J}}_{\tilde{\varphi}}$, we have that $n^{\boldsymbol{J}}_\varphi \geqslant \min_{k : a^k_\varphi \leqslant k} a^k_\varphi$ for all $\varphi \in \Phi$. So,

$$\begin{aligned} \sum_{\varphi \in Z} n - \min_{k : a^k_\varphi \leqslant k} a^k_\varphi \geqslant n &\Leftrightarrow \\ n|Z| - \sum_{\varphi \in Z} \min_{k : a^k_\varphi \leqslant k} a^k_\varphi \geqslant n &\Leftrightarrow \\ \sum_{\varphi \in Z} \min_{k : a^k_\varphi \leqslant k} a^k_\varphi \leqslant n(|Z| - 1). \end{aligned}$$

And we are done.

For the other direction, it is easy to see that if $a^k_\varphi + a^k_{\neg\varphi} \geqslant k + 1$ for some $\varphi \in \Phi$, then the relevant quota rule is inconsistent. So suppose that there is some mi set $Z \neq \{\varphi, \neq \varphi\}$ for all $\varphi \in \Phi$, such that $\sum_{\varphi \in Z} \min_{k : a^k_\varphi \leqslant k} a^k_\varphi \leqslant n(|Z| - 1)$ (or equivalently $\sum_{\varphi \in Z} n - \min_{k : a^k_\varphi \leqslant k} a^k_\varphi \geqslant n$) and for all $\varphi \in \Phi$ there exists $k \in \{0, \ldots, n\}$ with $a^k_\varphi \leqslant k$ (so $\min_{k : a^k_\varphi \leqslant k} a^k_\varphi$ is well-defined). We will construct a profile $\boldsymbol{J} = (J_1, \ldots, J_n)$ such that $F_{(a^k_\varphi)}(\boldsymbol{J})$ will be inconsistent. Since Z

is minimally inconsistent, we know that for each $\varphi \in \Phi$, $Z \setminus \{\varphi\}$ is consistent. Thus, we can extend $Z \setminus \{\varphi\}$ to a complete and consistent judgment set, denoted $J_{\neg\varphi}$. Since $\sum_{\varphi \in Z} n - \min_{k: a^k_\varphi \leqslant k} a^k_\varphi \geqslant n$, we know that it is possible to assign to every individual i exactly one proposition $\varphi_i \in Z$ in a way that each $\varphi \in Z$ is assigned to at most $n - \min_{k: a^k_\varphi \leqslant k} a^k_\varphi$ individuals (the idea is that the individuals will reject their assigned propositions). Then, let us define each $J_i \subseteq J_{\neg\varphi_i}$ as a subset of $J_{\neg\varphi_i}$ so that:

$$n_{\widetilde{\varphi}}^{(J_1,\ldots,J_n)} \in \operatorname*{argmin}_{k: a^k_\varphi \leqslant k} a^k_\varphi$$
$$\text{and } n_{\varphi}^{(J_1,\ldots,J_n)} \geqslant \min_{k: a^k_\varphi \leqslant k} a^k_\varphi$$

This is possible because, for each $\varphi \in Z$, at most $n - \min_{k: a^k_\varphi \leqslant k} a^k_\varphi$ individuals can reject φ, hence at least $\min_{k: a^k_\varphi \leqslant k} a^k_\varphi$ individuals can accept φ (intuitively, we first delete as many acceptances of $\neg\varphi$ as necessary and then as many acceptances of φ as necessary, in order to get the required k). We conclude that $\varphi \in F(\boldsymbol{J})$ for all $\varphi \in Z$, and we thus have an inconsistent $F_{(a^k_\varphi)}(\boldsymbol{J})$. $\square$

Proof of Proposition 8. Consider an agenda Φ with some $\varphi, \neg\varphi \in \Phi$, and the simple majority rule such that $m_\varphi = 1$, for all $\varphi \in \Phi$. Take the mi subset $Z = \{\varphi, \neg\varphi\} \subseteq \Phi$. If the simple majority rule was consistent, then Proposition 7(b) would imply that

$$\begin{aligned} \sum_{\varphi \in Z} m_\varphi &> n(|Z| - 1) && \Leftrightarrow \\ |Z| &> n(|Z| - 1) && \Leftrightarrow \\ \frac{|Z|}{|Z| - 1} &> n \end{aligned}$$

But $2 \geqslant \frac{|Z|}{|Z|-1} > n$, which is a contradiction. $\square$

Proof of Theorem 4. Consider an agenda Φ with some $\varphi, \neg\varphi \in \Phi$, and a non-trivial invariable marginal quota rule. Take the mi subset $Z = \{\varphi, \neg\varphi\} \subseteq \Phi$. If the rule was consistent, then Proposition 7(b) and Proposition 5(b) would imply that

$$n = n(|Z| - 1) < m_\varphi + m_{\neg\varphi} \leqslant 1,$$

which is a contradiction. $\square$

Proof of Lemma 2. Aiming for a contradiction, consider an aggregation rule F and suppose that some individual i has an opportunity to manipulate F on some proposition φ with $\varphi \in J_i$. If F is weakly independent, i cannot unilaterally change the outcome on φ by keeping her judgment on $\widetilde{\varphi}$ the same. So, assume that i decides to report an untruthful judgment J'_i with $J_i \neq_\varphi J'_i$ instead. This means that $J'_i \lhd_\varphi J_i$. If F is monotonic, it will hold that $F(\boldsymbol{J}_{-i}, J'_i) \unlhd_\varphi F(\boldsymbol{J}_{-i}, J_i)$ and $F(\boldsymbol{J}_{-i}, J_i) \unlhd_{\neg\varphi} F(\boldsymbol{J}_{-i}, J'_i)$. It follows from Definition 6 that i will not have an opportunity to manipulate on φ, which contradicts our hypothesis. $\square$

Proof of Lemma 3. Consider an aggregation rule F that is not monotonic. Then, there exist a proposition $\varphi \in \Phi$ and two profiles $\boldsymbol{J} = (\boldsymbol{J}_{-i}, J_i), \boldsymbol{J'} = (\boldsymbol{J}_{-i}, J'_i)$ such that $J'_i \lhd_\varphi J_i$ and either $F(\boldsymbol{J}_{-i}, J_i) \lhd_\varphi F(\boldsymbol{J}_{-i}, J'_i)$ or $F(\boldsymbol{J}_{-i}, J'_i) \unlhd_{\neg\varphi} F(\boldsymbol{J}_{-i}, J_i)$. Clearly, an individual i with truthful judgment J_i will have an opportunity to manipulate F on proposition φ. $\square$

Proof of Theorem 5. The "if" direction is proven analogously to Lemma 2. For the "only if" direction, we already know from Lemma 3 that any non-monotonic aggregation rule for incomplete inputs will be susceptible to manipulation. It remains to show that any aggregation rule for incomplete inputs that is not very weakly independent will be susceptible to manipulation.

Consider a rule F that violates very weak independence. Then, there exist some proposition $\varphi \in \Phi$ and profiles $\boldsymbol{J}, \boldsymbol{J'}$ such that $N_\varphi^{\boldsymbol{J}} = N_\varphi^{\boldsymbol{J'}}$, $N_{\neg\varphi}^{\boldsymbol{J}} = N_{\neg\varphi}^{\boldsymbol{J'}}$, and $J_i = J'_i$ for all $i \notin N_{\widetilde{\varphi}}^{\boldsymbol{J}}$, but $F(\boldsymbol{J}) \neq_\varphi F(\boldsymbol{J'})$. Suppose that $F(\boldsymbol{J}) \lhd_\varphi F(\boldsymbol{J'})$ (the case where $F(\boldsymbol{J'}) \lhd_\varphi F(\boldsymbol{J})$ is analogous). For $j \in \{2, \ldots, n+1\}$, we define

$$\boldsymbol{J^j} = (J'_1, \ldots, J'_{j-1}, J_j, \ldots, J_n)$$

and

$$\boldsymbol{J^1} = \boldsymbol{J} = (J_1, \ldots, J_n).$$

Then, we can identify an individual i such that

$$i \in \operatorname*{argmin}_{j \in \{1,\ldots,n\}} \boldsymbol{J^j} \lhd_\varphi \boldsymbol{J^{j+1}}.$$

In the two profiles $\boldsymbol{J^i}$ and $\boldsymbol{J^{i+1}}$ exactly the same individuals support φ, exactly the same individuals support $\neg\varphi$, and the individuals that abstain on $\widetilde{\varphi}$ report exactly the same judgments. Since $\boldsymbol{J^i} \neq \boldsymbol{J^{i+1}}$, we know that $J_i \neq J'_i$, and so individual i does not abstain on $\widetilde{\varphi}$. Now if $\varphi \in J_i$, individual i has an opportunity to manipulate by reporting J'_i instead of J_i; if $\neg\varphi \in J_i$ (which also means that $\neg\varphi \in J'_i$), she has an opportunity to manipulate when truthfully holding the judgment J'_i by reporting J_i instead. $\square$

References

Adams, J., Dow, J., & Merrill, S. (2006). The political consequences of alienation-based and indifference-based voter abstention: Applications to presidential elections. *Political Behavior*, *28*, 65–86.

Aguiar-Conraria, L., & Magalhães, P. C. (2010). Referendum design, quorum rules and turnout. *Public Choice*, *144*, 63–81.

Baumeister, D., Erdélyi, G., Erdélyi, O. J., & Rothe, J. (2013). Computational aspects of manipulation and control in judgment aggregation. In *Proceedings of the 3rd International Conference on Algorithmic Decision Theory (ADT)* (pp. 71–85). Springer.

Baumeister, D., Rothe, J., & Selker, A.-K. (2017). *Strategic behavior in judgment aggregation*. Lulu.com.

Botan, S., & Endriss, U. (2020). Majority-strategyproofness in judgment aggregation. In *Proceedings of the 19th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*.

Côrte-Real, P. P., & Pereira, P. T. (2004). The voter who wasn't there: Referenda, representation and abstention. *Social Choice and Welfare*, *22*, 349–369.

Dietrich, F., & List, C. (2007a). Judgment aggregation by quota rules: Majority voting generalized. *Journal of Theoretical Politics*, *19*, 391–424.

Dietrich, F., & List, C. (2007b). Strategy-proof judgment aggregation. *Economics and Philosophy*, *23*, 269–300.

Dietrich, F., & List, C. (2008). Judgment aggregation without full rationality. *Social Choice and Welfare*, *31*, 15–39.

Dietrich, F., & List, C. (2010). Majority voting on restricted domains. *Journal of Economic Theory*, *145*, 512–543.

Dokow, E., & Holzman, R. (2010). Aggregation of binary evaluations with abstentions. *Journal of Economic Theory*, *145*, 544–561.

Endriss, U. (2016). Judgment aggregation. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, & A. D. Procaccia (Eds.), *Handbook of Computational Social Choice* (pp. 399–426). Cambridge University Press.

Endriss, U., Grandi, U., & Porello, D. (2012). Complexity of judgment aggregation. *Journal of Artificial Intelligence Research*, *45*, 481–514.

Fishburn, P. C. (1973). *The theory of social choice*. Princeton University Press.

Fishburn, P. C., & Brams, S. J. (1983). Paradoxes of preferential voting. *Mathematics Magazine*, *56*, 207–214.

Freixas, J., & Zwicker, W. S. (2009). Anonymous yes–no voting with abstention and multiple levels of approval. *Games and Economic Behavior*, *67*, 428–444.

Gärdenfors, P. (2006). A representation theorem for voting with logical consequences. *Economics and Philosophy*, *22*, 181–190.

Grossi, D., & Pigozzi, G. (2014). *Judgment Aggregation: A Primer*. Synthesis Lectures on Artificial Intelligence and Machine Learning. Morgan & Claypool Publishers.

Houy, N. (2007). Some further characterizations for the forgotten voting rules. *Mathematical Social Sciences*, *53*, 111–121.

Jiang, G., Zhang, D., & Perrussel, L. (2018). A hierarchical approach to judgment aggregation with abstentions. *Computational Intelligence*, *34*, 104–123.

Laruelle, A., & Valenciano, F. (2011). Majorities with a quorum. *Journal of Theoretical Politics*, *23*, 241–259.

List, C. (2012). The theory of judgment aggregation: An introductory review. *Synthese*, *187*, 179–207.

List, C., & Pettit, P. (2002). Aggregating sets of judgments: An impossibility result. *Economics and Philosophy*, *18*, 89–110.

List, C., & Puppe, C. (2009). Judgment aggregation: A survey. In P. Anand, P. Pattanaik, & C. Puppe (Eds.), *Handbook of Rational and Social Choice* (pp. 457–482). Oxford University Press.

Llamazares, B. (2006). The forgotten decision rules: Majority rules based on difference of votes. *Mathematical Social Sciences*, *51*, 311–326.

Maniquet, F., & Morelli, M. (2015). Approval quorums dominate participation quorums. *Social Choice and Welfare*, *45*, 1–27.

May, K. O. (1952). A set of independent necessary and sufficient conditions for simple majority decision. *Econometrica: Journal of the Econometric Society*, (pp. 680–684).

Nehring, K., & Puppe, C. (2007). The structure of strategy-proof social choice—part I: General characterization and possibility results on median spaces. *Journal of Economic Theory*, *135*, 269–305.

Pattie, C., & Johnston, R. (2001). A low turnout landslide: Abstention at the british general election of 1997. *Political Studies*, *49*, 286–305.

Perea, E. A. (2002). Individual characteristics, institutional incentives and electoral abstention in Western europe. *European Journal of Political Research*, *41*, 643–673.

Pettit, P. (2001). Deliberative democracy and the discursive dilemma. *Philosophical Issues*, *11*, 268–299.

Plane, D. L., & Gershtenson, J. (2004). Candidates' ideological locations, abstention, and turnout in US midterm senate elections. *Political Behavior*, *26*, 69–93.

Slavkovik, M., & Jamroga, W. (2011). Distance-based judgment aggregation of three-valued judgments with weights. In *Proceedings of the IJCAI Workshop on Social Choice and Artificial Intelligence* (pp. 81–87).

Terzopoulou, Z., & Endriss, U. (2019). Strategyproof judgment aggregation under partial information. *Social Choice and Welfare*, *53*, 415–442.

Terzopoulou, Z., Endriss, U., & de Haan, R. (2018). Aggregating incomplete judgments: Axiomatisations for scoring rules. In *Proceedings of the 7th International Workshop on Computational Social Choice (COMSOC)*.