Proofs and Strategies
A Characterization of Classical and Intuitionistic Logic using Games with Explicit Strategies

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

We define two-player perfect information games characterizing classical and intuitionistic first-order validity. In short we enrich the language of first-order logic with two force markers denoting assertion and challenge. A two-player game is then a tree of states representing each players assertions and challenges and whose turn it is to move. A winning strategy for a player is a subtree of a game fulfilling some conditions. In particular we examine one of the players (the proponents) winning strategies for which we define several operations such as parallel, contraction, application, and composition. Using these operations we then establish a correspondence of strategies with derivations in the sequent calculus, giving us soundness and completeness for classical and intuitionistic logic. Additionally a close correspondence between composition and the cut-rule provides us with a method for cut-elimination. The constructive treatment of strategies gives them a computational interpretation which is of general interest for denotational semantics. The techniques developed may also be of use for many similar game-semantics.

Contents

1 Introduction

1.1 Motivation

The use of game semantics to characterize logical validity has two primary motivations, one is *foundational* and the other is *computational*. The foundational motivation is that game semantics gives an alternative explanation of logical validity not in terms of derivations or satisfiability but in terms of winnability of two-person perfect information games. This idea goes back to the *dialogue games* of Lorenzen and the Erlangen school [Lorenzen, 1958] which were created to model a *dialogue* or *argument* between a prover and doubter. The rules of the games would then fix the rules governing the logical connectives giving them a justification or meaning explanation. However, while the rules of these dialogue games are simple and intuitive, the actual games that result from the rules are quite complex infinite objects making it hard to establish a formal correspondence between a logic and a particular set of rules. Thus, it took Lorenzen and his followers quite some time before finding a set of rules that characterized *intuitionistic logic* and while the similarity between dialogue games and proof systems was noted early on, the first correct soundness and completeness proof of the dialogue games were produced by [Felscher, 1985].

The computational motivation of game semantics stems from the natural view of a two-player game as an interaction between a program and its environment. Thus game semantics gives an alternative denotational semantics of programming languages that relies on the notion of *strategy* rather than terms or proofs. The dialogue games have not been used in this sense and have historically been treated rather informally. There has been no notion of operations on strategies. In particular there has been no notion of a *composition* of strategies, which is all important from a computational perspective. It has been pointed our that the "technical record of the school seems rather bleak" [Girard, 1999]. Historically the first to define composition of strategies formally was [Joyal, 1977] following the combinatorial game theory of [Conway, 1976]. Recent game semantics in this vein have gone in different logical directions (e.g Linear logic [Blass, 1992, Abramsky et al., 1997], Scott's type language PCF [Hyland and Ong, 2000] and dependent type-theory [Yamada, 2016]). While these modern semantics are computational by design they have lost their intuitive nature. In particular there are no specific semantics for classical and intuitionistic logics.

Our goals here are to give a formalism that connects the philosophical foundations of the dialogue games of Lorenzen with the computational game theory of Conway. The purpose of this work is thus threefold:

1. To define a simple and intuitive game semantics for both classical and intuitionistic first-order logic where both games and strategies are combinatorial objects.

2. To develop a formal theory of these games and strategies and define operations on them. In particular we define *composition* of strategies.

3. To establish a formal correspondence between winning strategies for the games and derivations in sequent calculus. This will allow us to look at structural properties of the proof systems through the lens of games. In particular cut-elimination theorems will follow directly from the existence of composition of strategies.

1.2 Overview of the Thesis

The remainder of this thesis is divided into five sections:

Section 2. Preliminaries. This section defines preliminary notions that will be used throughout the thesis. The first part of the section defines a first order language, sequent calculi for classical and intuitionistic logic, and presents some general facts and notions about trees. The content in the first part are all standard. The second part of the section defines general games and strategies. The notions in this section are non-standard and specific to this thesis.

Section 3. Classical Games. This section firstly defines games for classical logic, followed by definitions of *basic games* and operations on strategies. Finally using the defined operations a correspondence is established between proponent winning strategies for basic games and derivations in the sequent calculus, proving soundness and completeness and also cut-elimination. A non-standard proof system which is a refinement of the standard proof-calculi and corresponds closely to the winning strategies is also introduced.

Section 4. Intuitionistic Games. What the previous sections does for classical logic, this section does for intuitionistic logic. It is to be noted that the constructions for intuitionistic strategies are similar to the ones in the classical case. Interesting differences arise in particular in the definition of the *parallel*-strategy.

Section 5. Comparison to Other Works. This section compares the games defined in this thesis with games in the tradition of Lorenzen and compares the operations defined here with similar operations in combinatorial game theory and linear logic. The section has two purposes: To explain to a reader familiar with the above systems the connection with the present work. Secondly, to give a reader unfamiliar with the above notions a reference point for further investigation.

Section 6. Conclusion and Further Work. The thesis ends with a summary of the results and a short discussion of several possible directions in which this type of game semantics may be extended.

2 Preliminaries

2.1 The language of first-order logic

In order to define proof systems and games for first-order logic we first specify a language to work with. The language $\mathcal{L}$ will be a simple first-order language not containing any function symbols.

Definition 2.1 (The language $\mathcal{L}$ of first-order logic)**.** We define the language $\mathcal{L}$ of first-order logic by fixing a set of *constants* and a set of *relation* symbols, each with an arity $n \in \mathbb{N}$, and a countable infinite set of *variables*.

- We define the *terms* t of $\mathcal{L}$ by the following grammar:

  $$t ::= x \mid c,$$

  where x is an arbitrary variable and c an arbitrary constant.

- We define an *atomic formula* A of $\mathcal{L}$ to be of the form $A = R(t_1, \ldots, t_n)$ or $A = \bot$, where $t_1, \ldots, t_n$ are terms and R is a n-ary relation symbol.

- We define the *formulas* φ of $\mathcal{L}$ by the following grammar:

  $$\varphi ::= \bot \mid A \mid \varphi \to \varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \forall x\varphi \mid \exists x\varphi,$$

  where A is an atomic formula and x is an arbitrary variable.

Definition 2.2 (Free Variables)**.** Given a set or multiset Γ of formulas in $\mathcal{L}$ we define its set of free variables $FV(\Gamma)$ inductively as follows:

$$\begin{aligned}
FV(y) &= \{y\} \\
FV(\bot) &= FV(c) = \emptyset \\
FV(R(t_1, \ldots, t_n)) &= FV(t_1) \cup \cdots \cup FV(t_n) \\
FV(\varphi * \psi) &= FV(\varphi) \cup FV(\psi) && \text{where } * \in \{\to, \wedge, \vee\} \\
FV(*y\varphi) &= FV(\varphi) \setminus \{y\} && \text{where } * \in \{\forall, \exists\} \\
FV(\{\varphi_1, \ldots, \varphi_n\}) &= FV(\varphi_1) \cup \cdots \cup FV(\varphi_n)
\end{aligned}$$

Where y is a variable, R an n-ary relation symbol and $t_1, \ldots, t_n$ are terms. We usually write $\varphi(x_1, \ldots, x_n)$ to indicate that $x_1, \ldots, x_n$ are free variables in φ.

Definition 2.3 (Substitution)**.** We write $\varphi[t/x]$ for the formula obtained by substituting any free occurrence of the variable x with the term t in φ. We define $\Gamma[t/x]$ by induction on the expressions of $\mathcal{L}$ as follows:

$$
\begin{aligned}
y[t/x] &= \begin{cases} y \text{ if } x \neq y \\ t \text{ else} \end{cases} \\
c[t/x] &= c \\
\bot[t/x] &= \bot \\
P(t_1, \ldots, t_n)[t/x] &= P(t_1[t/x], \ldots, t_n[t/x]) \\
(\varphi * \psi)[t/x] &= \varphi[t/x] * \psi[t/x] && \text{where } * \in \{\rightarrow, \wedge, \vee\} \\
(*y\varphi)[t/x] &= \begin{cases} *y\varphi[t/x] \text{ if } x \neq y \\ *y\varphi \text{ else} \end{cases} && \text{where } * \in \{\forall, \exists\} \\
\{\varphi_1, \ldots, \varphi_n\}[t/x] &= \{\varphi_1[t/x], \ldots, \varphi_n[t/x]\}
\end{aligned}
$$

Given that we want to avoid substitutions that change the meaning of a formula, for example $\exists x(x \neq y)[x/y] = \exists x(x \neq x)$, we also introduce the notion of a term t being *free for* x in a formula.

Definition 2.4. A term t is *free for* x in φ if

- φ is atomic.
- $\varphi = \alpha * \beta$ and t is free for x in α and β, where $* \in \{\rightarrow, \wedge, \vee\}$.
- $\varphi = *y\psi$ and if x is free in ψ, then y is not free in t and t is free for x in ψ, where $* \in \{\forall, \exists\}$.

Given a set or multiset of formulas Γ we say that t is free for x in Γ, if t is free for x in φ for any $\varphi \in \Gamma$.

2.2 Sequent Calculus

We define Gentzen-style sequential calculi for first-order classical and intuitionistic logics. Gentzen [Gentzen, 1935] originally introduced the LK and LJ proof systems for classical and intuitionistic logic. We will define systems $G3C$ and $G3I$ that are just as the standard $G3c$ and $G3i$ systems developed by Kleene, Troelstra and others, except that weakening and contraction are made explicit rules. The $G3c$ and $G3i$ systems have the property that the structural rules of weakening, contraction, and cut, are admissible. For proofs of any of the facts in this section we refer to [Negri and von Plato, 2001] and [Troelstra and Schwichtenberg, 2000].

Definition 2.5 (A sequent)**.** If Γ and Δ are finite multisets of formulas, then $\Gamma \Rightarrow \Delta$ is a sequent.

We let $\Gamma, \varphi \Rightarrow \Delta, \psi$ denote the sequent $\Gamma \cup \{\varphi\} \Rightarrow \Delta \cup \{\psi\}$, where $\Gamma \cup \{\varphi\}$ denotes the multiset that is just like Γ except with an additional occurrence of φ. The intended interpretation of the sequent $\Gamma \Rightarrow \Delta$ is that $\bigwedge \Gamma$ implies $\bigvee \Delta$.

We say that a sequent is *derivable* if it can be derived using a set of rules.

Definition 2.6. We write $\Gamma \vdash_c \Delta$ if $\Gamma \Rightarrow \Delta$ is derivable in $G3C$. We write $\Gamma \vdash_i \varphi$ if $\Gamma \Rightarrow \varphi$ is derivable in $G3I$. We define $G3C$ and $G3I$ as follows:

Table 1: The Sequent Calculus G3C

Ax	$\dfrac{}{\Gamma, A \Rightarrow \Delta, A}\ Ref$	$\dfrac{}{\Gamma, \bot \Rightarrow \Delta}\ Falsum$
$\wedge$	$\dfrac{\Gamma, \alpha, \beta \Rightarrow \Delta}{\Gamma, \alpha \wedge \beta \Rightarrow \Delta}\ \wedge_L$	$\dfrac{\Gamma \Rightarrow \Delta, \alpha \qquad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}\ \wedge_R$
$\vee$	$\dfrac{\Gamma, \alpha \Rightarrow \Delta \qquad \Gamma, \beta \Rightarrow \Delta}{\Gamma, \alpha \vee \beta \Rightarrow \Delta}\ \vee_L$	$\dfrac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}\ \vee_R$
$\rightarrow$	$\dfrac{\Gamma \Rightarrow \Delta, \alpha \qquad \Gamma, \beta \Rightarrow \Delta}{\Gamma, \alpha \rightarrow \beta \Rightarrow \Delta}\ \rightarrow_L$	$\dfrac{\Gamma, \alpha \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \rightarrow \beta}\ \rightarrow_R$
$\forall$	$\dfrac{\Gamma, \forall x\varphi(x), \varphi(t) \Rightarrow \Delta}{\Gamma, \forall x\varphi(x) \Rightarrow \Delta}\ \forall_L$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad x \notin FV(\Gamma \cup \Delta)}{\Gamma \Rightarrow \Delta, \forall x\varphi(x)}\ \forall_R$
$\exists$	$\dfrac{\Gamma, \varphi \Rightarrow \Delta \qquad x \notin FV(\Gamma \cup \Delta)}{\Gamma, \exists x\varphi(x) \Rightarrow \Delta}\ \exists_L$	$\dfrac{\Gamma \Rightarrow \Delta, \exists x\varphi(x), \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x\varphi(x)}\ \exists_R$
Weakening	$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ W_L$	$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \psi}\ W_R$
Contraction	$\dfrac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ C_L$	$\dfrac{\Gamma \Rightarrow \Delta, \psi, \psi}{\Gamma \Rightarrow \Delta, \psi}\ C_R$

Table 2: The Sequent Calculus G3I

Ax	$\dfrac{}{\Gamma, A \Rightarrow A}\ Ref$	$\dfrac{}{\Gamma, \bot \Rightarrow \psi}\ Falsum$
$\wedge$	$\dfrac{\Gamma, \alpha, \beta \Rightarrow \psi}{\Gamma, \alpha \wedge \beta \Rightarrow \psi}\ \wedge_L$	$\dfrac{\Gamma \Rightarrow \alpha \qquad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ \wedge_R$
$\vee$	$\dfrac{\Gamma, \alpha \Rightarrow \psi \qquad \Gamma, \beta \Rightarrow \psi}{\Gamma, \alpha \vee \beta \Rightarrow \psi}\ \vee_L$	$\dfrac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ \vee_R \qquad \dfrac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}$
$\rightarrow$	$\dfrac{\Gamma, \alpha \rightarrow \beta \Rightarrow \alpha \qquad \Gamma, \beta \Rightarrow \psi}{\Gamma, \alpha \rightarrow \beta \Rightarrow \psi}\ \rightarrow_L$	$\dfrac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta}\ \rightarrow_R$
$\forall$	$\dfrac{\Gamma, \forall x\varphi(x), \varphi(t) \Rightarrow \psi}{\Gamma, \forall x\varphi(x) \Rightarrow \psi}\ \forall_L$	$\dfrac{\Gamma \Rightarrow \varphi \qquad x \notin FV(\Gamma \cup \{\psi\})}{\Gamma \Rightarrow \forall x\varphi(x)}\ \forall_R$
$\exists$	$\dfrac{\Gamma, \varphi \Rightarrow \psi \qquad x \notin FV(\Gamma \cup \{\psi\})}{\Gamma, \exists x\varphi(x) \Rightarrow \psi}\ \exists_L$	$\dfrac{\Gamma \Rightarrow \varphi(t)}{\Gamma \Rightarrow \exists x\varphi(x)}\ \exists_R$
Weakening	$\dfrac{\Gamma \Rightarrow \psi}{\Gamma, \varphi \Rightarrow \psi}\ W_L$	
Contraction	$\dfrac{\Gamma, \varphi, \varphi \Rightarrow \psi}{\Gamma, \varphi \Rightarrow \psi}\ C_L$	

2.2.1 Properties of G3C and G3I

Admissible rules. A rule is *admissible* if its conclusion holds whenever its premisses holds.

Theorem 2.1. The following rules are admissible in $G3C$.

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \varphi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \qquad (Cut)$$

Theorem 2.2. The following rules are admissible in $G3I$

$$\frac{\Gamma \Rightarrow \varphi \qquad \varphi, \Gamma' \Rightarrow \psi}{\Gamma, \Gamma' \Rightarrow \psi} \qquad (Cut)$$

Invertible rules. A rule

$$\frac{X_1 \qquad \cdots \qquad X_n}{Y}$$

is *invertible* if $X_1, \ldots, X_n$ is derivable whenever Y is derivable.

Theorem 2.3. All the rules of $G3C$ except *weakening* are invertible.

Theorem 2.4. The rules of $\wedge_L$,$\wedge_R$,$\vee_L$,$\rightarrow_R$,$\exists_L$ $\forall_R$, and *contraction* are invertible in $G3I$.

Substitution.

Theorem 2.5 (Substitution lemma)**.**

- If $\Gamma \vdash_c \Delta$ and t is free for x in $\Gamma \cup \Delta$, then $\Gamma[t/x] \vdash_c \Delta[t/x]$.
- If $\Gamma \vdash_i \varphi$ and t is free for x in $\Gamma \cup \{\varphi\}$, then $\Gamma[t/x] \vdash_c \varphi[t/x]$.

2.3 Trees

The study of games is strongly linked to the study of trees since a game can be seen as a tree of positions or states. Thus, in this section we define some basic notions and properties of trees. The notion of tree that is used here is a basic notion of descriptive set theory. For proofs of any of the facts in this section we refer to [Srivastava, 1998].

Definition 2.7. Given a set X, a *sequence* $(x_1, x_2, x_3, \ldots)$ is a possible infinite tuple of objects from X.

We will use roman letters $s, t, u, \ldots$ to denote sequences and $a, b, c, \ldots$ to denote objects in the sequences.

Definition 2.8. Given two sequences $s = (x_1, \ldots, x_n)$ and $t = (y_1, \ldots)$ we denote their concatenation

$$st = (x_1, \ldots, x_n, y_1, \ldots)$$

We will most often write sa, instead of $s(a)$. We let $|s|$ denote the length of as sequence. We say that s is a prefix of u if $u = st$ for some sequence t.

Definition 2.9. A *tree* T on a set X is a prefix closed, (i.e., if $st \in T$ then $s \in T$) collection of finite sequences of elements of X. A *subtree* $S \subseteq T$ is a subset of T which is also a tree. We let $\mathbb{T}$ denote the class of all trees. We use the following terminology when discussing trees:

- A sequence $s \in T$ is a *leaf* if there is no $a \in X$ such that $sa \in T$.
- A tree T is *finitely splitting* if $\{sa \mid sa \in T, a \in X\}$ is finite for every $s \in T$.
- A *branch* of a tree T is a sequence s such that for all prefixes t of s, $t \in T$.
- A tree T is *well-founded* if it has no infinite branch.

Theorem 2.6 (Kőnig's lemma)**.** Let T be a finitely splitting infinite tree on X, then T has an infinite branch.

2.3.1 Induction on Trees

The methods of transfinite induction can be extended to induction on well-founded trees introducing a *rank*-function on trees that assigns an ordinal number to every well-founded tree.

Definition 2.10. The rank of a well-founded tree T is defined inductively as follows, where $u \in T$ is a leaf and $s \in T$ is a non-leaf:

$$\begin{aligned} rank(u) &= 0 \\ rank(s) &= \sup\{rank(t) + 1 \mid t \subset s, t \in T\} \\ rank(T) &= \sup\{rank(s) + 1 \mid s \in T\} \end{aligned}$$

Thus we define an order on well-founded trees as follows:

$$T <_T T' \iff rank(T) < rank(T')$$

The order is well-founded since the ordinals are well-founded. We extend this to ordering to finite multisets of well-founded trees as follows.

Definition 2.11.

$$\{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m\} \ll \{\alpha_1, \dots, \alpha_n, \alpha_{n+1}\}$$

where $\beta_1, \dots, \beta_m <_T \alpha_{n+1}$, for $n, m \in \mathbb{N}$ and $m \neq 0$.

Theorem 2.7. The relation $\ll$ is well-founded.

Proof. Suppose there is a infinite descending chain of finite multisets of trees:

$$M \gg M_1 \gg M_2 \gg M_3 \gg \dots,$$

from this chain we construct a tree S as follows:

$$\begin{aligned} S_0 &= \{(x) \mid x \in M\} \cup \{()\} \\ S_{n+1} &= S_n \cup \{s\alpha\beta \mid s\alpha \in S_n \text{ and } \beta \in M_{n+1} \text{ and } \beta <_T \alpha\} \\ S &= \bigcup_{n \in \mathbb{N}} S_n \end{aligned}$$

We have:

- The tree S is finitely branching, since M_i is finite for all $i \in \mathbb{N}$.
- The tree S is infinite since at least one element is added for each M_i by definition.

Thus by Kőnig's lemma there is an infinite branch

$$\alpha >_T \alpha_1 >_T \alpha_2 >_T \alpha_3 >_T \dots$$

of well founded trees, but this is impossible since $<_T$ is well-founded. $\square$

The relation is straightforwardly extended to finite ordered sequences of trees.

2.4 Games

Here we define the general concept of a *game*. While there are similar conceptions in the literature, this specific conception is original to this thesis. A *game* is defined given a set of *states*. A state a tuple $(G, \circ)$ consisting of a *position* G and a player $\circ$. Intuitively we interpret the state $(G, \circ)$ to say that the game is at position G and it's player $\circ$'s turn to move. We let the set of states be

$$\mathbf{States} = \mathbf{Pos} \times \mathbf{Players}.$$

We use capital letters

$$G, H, J, \dots,$$

to represent positions. We use $\circ$ and $\bullet$ to represent players where it's always the case that $\circ \neq \bullet$. In this thesis we will only consider two-player games, so we fix a set of players.

Definition 2.12 (Players)**.**

$$\mathbf{Players} = \{\mathcal{O}, \mathcal{P}\}.$$

We call the player $\mathcal{O}$ the *opponent* and the player $\mathcal{P}$ the *proponent*. While we have fixed the set of players most of the definitions in this section work for arbitrary sets of players. Before we define the games we define a *ruleset*, which gives the rules of the games.

Definition 2.13 (Rulesets)**.** A ruleset is a tuple $(\mathbf{States}, Act, M, Term)$, where

- $\mathbf{States} = \mathbf{Pos} \times \mathbf{Players}$ is a set of states.
- Act is a set of actions.
- $M \subseteq \mathbf{States} \times Act \times \mathbf{States}$ is a transition relation representing the legal moves which we require be *functional* in the following sense: If $(S, a, T) \in M$ and $(S, a, U) \in M$, then $T = U$.
- $Term \subseteq \mathbf{Pos} \times Players$ is a set of terminal states partitioned into terminal states $Term_\circ$ for each player $\circ \in \mathbf{Players}$.

We write

$$S \xrightarrow{a} T$$

for a *move* $(S, a, T) \in M$. If $S = (G\,;\circ)$ we say that it's a $\circ$-player move. Since M is functional, given a state $(G\,;\circ)$ and a move $(G\,;\circ) \xrightarrow{a} (H\,;\circ')$ we define

$$a(G\,;\circ) = (H\,;\circ')$$
$$aG = H.$$

Intuitively a *game* is a tree, where the vertices represents states of the game and the edges represents moves. A game ends when a *terminal state* is reached.

Definition 2.14 (Games)**.** Given a ruleset $(\mathbf{States}, Act, M, Term)$ and an initial state S, a game $\mathbb{G}$ is a tree of states such that:

- $S \in \mathbb{G}$.
- If $sT \in \mathbb{G}$, $T \notin Term$ and $T \xrightarrow{a} U$ for some $a \in Act$, then $sTU \in \mathbb{G}$.

Thus given a ruleset, any state S will determine a game. We therefore often call a state S a game.

2.5 Strategies

Intuitively a strategy is a method for a player of choosing a move in a game given their available information of the game. Following [Hyland, 1997] there are three major options when deciding what information the players are allowed to take into account when deciding their next move:

- A strategy may only take into account the last move.
- A strategy may take into account the whole history of the game.
- A strategy may take into account the current position.

The last option is usually called a *positional* or *memoryless* strategy. These are the types of strategies that will be considered in this thesis. There are now two isomorphic ways of looking at a strategy for a game $\mathbb{G}$ and a player $\circ$:

1. A strategy is a subtree
$$\sigma \subseteq \mathbb{G},$$
fulfilling some conditions.

2. A strategy is a function that takes any state $(H\,;\circ)$ and returns an action:
$$\sigma : (H\,;\circ) \mapsto a \in \mathit{Act}.$$

We will call the definitions of strategies corresponding to these views *extensional* and *intensional* respectively. The extensional strategies will be game specific while the intensional will not. We thus define

Definition 2.15 (Extensional Strategies)**.** An *extensional strategy* for a game $\mathbb{G}$ with initial state S, and a player $\circ$ is the smallest subtree $\sigma \subseteq \mathbb{G}$ such that:

- (Initial state). $S \in \sigma$.
- (Closure under opposing player moves). If $s(G\,;\bullet) \in \sigma$ and $s(G\,;\bullet)T \in \mathbb{G}$, then $s(G\,;\bullet)T \in \sigma$.
- (Determinism for player moves). If $s(G\,;\circ) \in \sigma$ and $s(G\,;\circ)T \in \mathbb{G}$, then there is exactly one state T' such that $s(G\,;\circ)T' \in \sigma$.
- (Memorylessness) If $s(G\,;\circ)T \in \sigma$ and $t(G\,;\circ)U \in \sigma$, then $T = U$.

We let $\mathit{Str}_\circ(S)$ denote the set of all $\circ$-player strategies on the game S.

2.5.1 Winning Strategies

The above definitions have been general in the sense that the set **Players** could have been arbitrary. When we now define the winning strategies we specifically consider the set of players $\mathbf{Players} = \{\mathcal{O}, \mathcal{P}\}$. To win the proponent must reach a terminal stat $T \in \mathbf{Term}_\mathcal{P}$, the opponent wins if she can prevent the proponent from reaching such a state or reaching a terminal state $T \in \mathbf{Term}_\mathcal{O}$.

Definition 2.16 (Winning Strategies)**.** We define winning strategies for both players.

1. A strategy $\sigma \in \mathit{Str}_\mathcal{P}(S)$ is winning for the proponent if:
   - There is no infinite branch of σ and for all leafs $S \in \sigma$ we have that $S \in \mathbf{Term}_\mathcal{P}$.
2. A strategy $\sigma \in \mathit{Str}_\mathcal{O}(S)$ is winning for the opponent if:
   - There is an infinite branch of σ or a leaf $S \in \sigma$ such that $S \notin \mathbf{Term}_\mathcal{P}$.

We let $\mathit{Str}^w_\circ(S)$ be the set of winning strategies of player $\circ$ on the game S. Following [Joyal, 1997] we define the intensional winning strategies for the proponent for a game using transfinite induction. Let
$$\mathbf{Move}(S) = \{a \mid a \in \mathit{Act}\, \exists T \in \mathbf{States} : S \xrightarrow{a} T \text{ and } S \notin \mathit{Term}\}.$$

Then the set of proponent winning strategies $\mathit{Str}^w_\mathcal{P}(S)$ is defined as follows:

Definition 2.17 (Intensional Winning Strategies)**.**

- $\dfrac{(G\,;\circ) \in Term_{\mathcal{P}}}{e \in Str^{w}_{\mathcal{P}}(G\,;\circ)}$
- $\dfrac{(G\,;{\scriptstyle\mathcal{P}}) \notin Term \qquad (a,\sigma) \in \mathbf{Move}(G\,;{\scriptstyle\mathcal{P}}) \times Str^{w}_{\mathcal{P}}(a(G\,;{\scriptstyle\mathcal{P}}))}{(a,\sigma) \in Str^{w}_{\mathcal{P}}(G\,;{\scriptstyle\mathcal{P}})}$
- $\dfrac{(G\,;{\scriptstyle\mathcal{O}}) \notin Term \qquad f : (a : \mathbf{Move}(G\,;{\scriptstyle\mathcal{O}})) \to Str^{w}_{\mathcal{P}}(a(G\,;{\scriptstyle\mathcal{O}}))}{f \in Str^{w}_{\mathcal{P}}(G\,;{\scriptstyle\mathcal{O}})}$

We call e the *empty strategy*. Thus for any game S any strategy $\sigma \in Str^{w}_{\mathcal{P}}(S)$ is a well-founded tree. This allows us to use induction on proponent winning strategies.

We will sometimes switch between the extensional and intensional views on strategies, thus most often we will write

$$(a, \sigma') \in Str_{\mathcal{P}}(G\,;{\scriptstyle\mathcal{P}}),$$

and

$$\lambda x.\sigma(x) \in Str_{\mathcal{P}}(G\,;{\scriptstyle\mathcal{O}}),$$

but also $s \in \sigma$ for $\sigma \in Str_{\mathcal{P}}(G\,;\circ)$.

2.5.2 Determinacy

We call a game *determined* if either of the player has a winning strategy. Given our definition of a winning strategy it follows immediately that all games are determined.

Theorem 2.8 (Determinacy)**.** For any state S we have:

$$Str^{w}_{\mathcal{P}}(S) \neq \emptyset \vee Str^{w}_{\mathcal{O}}(S) \neq \emptyset$$

3 Classical Games

We define games for classical validity along the lines of Lorenzen's dialogue games [Lorenzen, 1958]: A game is seen as modeling a formal debate or dialogue between the proponent who seeks to prove and the opponent who seeks to spoil the proponents attempts at proving. The two basic actions in a game are to assert a formula and to challenge an assertion. We thus write:

- $!_\circ\varphi$, meaning "player $\circ$ asserts φ".
- $?_\circ\varphi$, meaning "player $\circ$ is challenged why φ".

Intuitively the proponent wins if a state is reached where either an agreement is reached with the opponent, or if the opponent asserts falsum. The dialogue proceeds in alternating turns.

Example 3.1. From $P, P \to Q, Q \to R$ we derive R informally:

The game begins by the opponent asserting the premisses and the proponent taking up the challenge to defend the conclusion.

1. $!_\mathcal{O}P$, $!_\mathcal{O}P \to Q$, $!_\mathcal{O}Q \to R$, $?_\mathcal{P}R$

The proponent has the first move and begins by attacking the assertion $!_\mathcal{O}P \to Q$. To attack an assertion the attacker must assert the antecedent, the defender is then challenged for the succedent. The proponent may assert atomic propositions the opponent already agreed to without being challenged:

2. Proponent: You asserted $!_\mathcal{O}P$, so I assert $!_\mathcal{P}P$ and challenge $?_\mathcal{O}Q$.
3. Opponent : Ok, then to defend $?_\mathcal{O}Q$ I assert $!_\mathcal{O}Q$.

The proponent may then attack $!_\mathcal{O}Q \to R$, asserting $!_\mathcal{P}Q$ and challenging $?_\mathcal{O}R$.

4. Proponent: You asserted $!_\mathcal{O}Q$, so I assert $!_\mathcal{P}Q$ and challenge $?_\mathcal{O}R$.
5. Opponent : Ok, then to defend $?_\mathcal{O}R$ I assert $!_\mathcal{O}R$.

Finally then the proponent has a defence for asserting $!_\mathcal{P}R$, and thus can meet the original challenge $?_\mathcal{P}R$ and win the game.

6. Proponent: You asserted $!_\mathcal{O}R$ so I defend $?_\mathcal{P}R$ by asserting $!_\mathcal{P}R$.

Definition 3.1 (Game Language)**.** Adding the two *force markers* ? and ! to the first-order language $\mathcal{L}$ gives us the game language:

$$\mathcal{L}^{Game} = \{\, !_\circ\varphi \mid \varphi \in \mathcal{L}, \circ \in Players\} \cup \{\, ?_\circ\varphi \mid \varphi \in \mathcal{L}, \circ \in Players\}$$

We divide all assertions into *positive* and *negative*.

Definition 3.2 (Positive and Negative Assertions)**.**

Negative assertion	Positive assertion
$!_\circ\varphi \wedge \psi$	$!_\circ\varphi \vee \psi$
$!_\circ\varphi \to \psi$	–
$!_\circ\forall x\varphi(x)$	$!_\circ\exists x\varphi(x)$
$!_\mathcal{O} A$	$!_\mathcal{P} A$

For positive assertions the duty of the asserter is positive: To provide an example. Consequently a positive assertion may be unconditionally challenged, the asserter may then choose how to defend it. Conversely, the duty of the asserter of a negative assertion is to defend against counterexamples, thus an attacker may choose how to attack a negative assertion, the defender must then unconditionally defend it.

3.1 Classical Positions

A *position* in a classical game is then a finite multi-set of $\mathcal{L}^{Game}$ formulas representing each players assertions and challenges at a point in a dialogue.

Definition 3.3 (Set of Positions)**.**

$$\mathbf{Pos} = \{G \subseteq \mathcal{L}^{Game} \mid G \text{ is finite}\}$$

We define some simple operations on positions which will be used throughout the thesis when discussing positions.

Definition 3.4 (Operations on Positions)**.**

- Given a position G we define the *dual* position

$$G^d$$

Where,

$$\begin{aligned} \{\varphi_1, \ldots, \varphi_n\}^d &= \{\varphi_1^d, \ldots, \varphi_n^d\} \\ (\,?_\circ\varphi)^d &= ?_\bullet\varphi \\ (\,!_\circ\varphi)^d &= !_\bullet\varphi \end{aligned}$$

We immediately get that $(G^d)^d = G$. Intuitively, in the dual position players switch assertions and challenges with each other.

- Following Joyal we also define an *implication*:

$$G \multimap H := G^d, H$$

We list some useful identities.

$$\begin{aligned} G, (H, J) &= (G, H), J \\ G, H &= H, G \\ G, \emptyset &= G = \emptyset, G \\ G \multimap (H \multimap J) &= (G, H) \multimap J \end{aligned}$$

3.2 Classical Rules

Definition 3.5 (Classical Ruleset)**.** The classical games are defined given the classical ruleset (**States**, $Act, M, Term$), where

- **States** $=$ **Pos** $\times \{\mathcal{O}, \mathcal{P}\}$.
- The set of actions Act is defined using the following grammar:

$$a ::= D(\varphi) \mid D_i(\varphi) \mid D_t(\varphi) \mid A_t(\varphi) \mid A_i(\varphi) \mid A(\varphi).$$

  Where $i \in \{0, 1\}$, t is an arbitrary term and φ is an arbitrary formula in $\mathcal{L}$.

- The transition relations $M \subseteq$ **States** $\times\, Act \,\times$ **States** is defined given the following table:

$$\frac{(G,\ ?_{\circ}\varphi\,;\circ) \qquad !_{\circ}\varphi \text{ is negative}}{(G,\ !_{\circ}\varphi\,;\bullet)}\ D(\varphi) \qquad\qquad \frac{(G,\ !_{\bullet}\varphi\,;\circ) \qquad !_{\bullet}\varphi \text{ is positive}}{(G,\ ?_{\bullet}\varphi\,;\bullet)}\ A(\varphi)$$

$$(*)\ \frac{(G,\ !_{\bullet}\varphi_0 \wedge \varphi_1\,;\circ)}{(G,\ ?_{\bullet}\varphi_i\,;\bullet)}\ A_i(\varphi_0 \wedge \varphi_1) \qquad\qquad (*)\ \frac{(G,\ ?_{\circ}\varphi_0 \vee \varphi_1\,;\circ)}{(G,\ !_{\circ}\varphi_i\,;\bullet)}\ D_i(\varphi_0 \vee \varphi_1)$$

$$(*)\ \frac{(G,\ !_{\bullet}\forall x\varphi(x)\,;\circ)}{(G,\ ?_{\bullet}\varphi(t)\,;\bullet)}\ A_t(\forall x\varphi(x)) \qquad\qquad (*)\ \frac{(G,\ ?_{\circ}\exists x\varphi(x)\,;\circ)}{(G,\ !_{\circ}\varphi(t)\,;\bullet)}\ D_t(\exists x\varphi(x))$$

$$(*)\ \frac{(G,\ !_{\bullet}\varphi \to \psi\,;\circ)}{(G,\ !_{\circ}\varphi,\ ?_{\bullet}\psi\,;\bullet)}\ A(\varphi \to \psi) \qquad\qquad \frac{(G,\ ?_{\mathcal{P}}A\,;\mathcal{P})}{(G,\ !_{\mathcal{P}}A\,;\mathcal{O})}\ D(A)$$

  Side-condition:

  - Moves marked with $(*)$ are such that when the proponent is the active player the active formula is repeated and not cancelled, that is to say, the proponent may re-attack and re-defend these formulas.

  Note that the proponent cannot attack the opponents atomic assertions.

- The set of terminal states $Term = Term_{\mathcal{O}} \cup Term_{\mathcal{P}}$ is inductively defined as follows, where G is an arbitrary position.
  - $(G,\ ?_{\mathcal{P}}A\,;\mathcal{P}) \in Term_{\mathcal{P}}$, where $!_{\mathcal{O}}A \in G$.
  - $(G,\ !_{\mathcal{P}}A\,;\mathcal{O}) \in Term_{\mathcal{O}}$, where $!_{\mathcal{O}}A \notin G$.
  - $(G,\ !_{\bullet}\bot\,;\circ) \in Term_{\circ}$.
  - $(\emptyset\,;\circ) \in Term_{\bullet}$.

The rules can be given the following motivation or meaning explanation:

(a) To attack a conjunction is to challenge one of the conjuncts.

(b) To defend a disjunction is to assert one of the disjuncts.

(c) To attack an universal statement is to challenge an instance of it.

(d) To defend an existential statement is to assert an instance of it.

(e) To attack an implication is to assert the antecedent and challenge the consequent.

(f) Atomic formulas may be defended or attacked, however:

1. If the opponent challenges a proposition A which she has already asserted the opponent loses. This is known in the literature as an *ipse dixisti* or "you said so yourself"-condition: In a dialogue you shouldn't be able to question propositions you already agreed to.
2. If the proponent asserts a proposition A which the opponent has not asserted the proponent loses. The idea is that the proponent cannot defend an atomic proposition unless the opponent already has agreed to it.
3. Any player asserting falsum loses. This is taken to be the primitive meaning of falsum.
4. The last winning condition is a purely formal condition, meant to enforce who wins in the empty position.

This gives us a motivation for the "particular" parts of the rules, but what about the "structural" part of the rules that says that the proponent may re-attack and re-defend formulas while the opponent may not? There are two possible ways of defending the fairness of the structural part of the rules.

The first possible defense is pre-theoretical and rests on the view that the games seek to model a debate or dialogue between the proponent and the opponent. In such a dialogue it would be seen as unfair if the doubter could win by simply repeating previous assertions or challenges.

The second possible defense would be to actually allow the opponent to re-attack and re-defend formulas resulting in some new class of games and then showing that these games are equivalent to the games defined in this thesis with regards to winnability for the proponent.

We will not delve deeper into defenses of the structural parts of the rules in this thesis, however we will return to this last idea in the conclusion.

Example 3.2. Let $G = !_{\mathcal{O}}P,\ !_{\mathcal{O}}P \to Q,\ !_{\mathcal{O}}Q \to R$, we exhibit a winning strategy

$$\sigma \in Str^{w}_{\mathcal{P}}(G,\ ?_{\mathcal{P}}R\,;{\scriptstyle\mathcal{P}}):$$

$$\sigma = \begin{array}{c} (G,\ ?_{\mathcal{P}}R\,;{\scriptstyle\mathcal{P}}) \\ \hline (G,\ !_{\mathcal{P}}P,\ ?_{\mathcal{O}}Q,\ ?_{\mathcal{P}}R\,;{\scriptstyle\mathcal{O}}) \\ \hline (G,\ !_{\mathcal{P}}P,\ !_{\mathcal{O}}Q,\ ?_{\mathcal{P}}R\,;{\scriptstyle\mathcal{P}}) \\ \hline (G,\ !_{\mathcal{P}}P,\ !_{\mathcal{O}}Q,\ !_{\mathcal{P}}Q,\ ?_{\mathcal{O}}R,\ ?_{\mathcal{P}}R\,;{\scriptstyle\mathcal{O}}) \\ \hline (G,\ !_{\mathcal{P}}P,\ !_{\mathcal{O}}Q,\ !_{\mathcal{P}}Q,\ !_{\mathcal{O}}R,\ ?_{\mathcal{P}}R\,;{\scriptstyle\mathcal{P}}) \end{array}$$

Thus, the strategies are similar to upside down sequent calculus derivations with the major exceptions:

- The strategies do not split into different branches when an implication is attacked as in the rule:

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta} \to_L.$$

- Since the proponent loses if he asserts an atomic proposition that hasn't already been asserted by the opponent some instances of the sequent calculus rules are not "valid", for example:

$$\frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, A \to \psi \Rightarrow \Delta} \to_L.$$

3.3 Basic Positions

While this gives us a definition of the classical games on arbitrary states, we are in particular interested in games of the form $(\,!_{\mathcal{O}}\Gamma,\ ?_{\mathcal{P}}\Delta\,;{\scriptstyle\mathcal{P}})$, where

Notation 3.3.

$$?_\circ\{\varphi_1,\ldots,\varphi_n\} = \{\,?_\circ\varphi_1,\ldots,\ ?_\circ\varphi_n\}$$
$$!_\circ\{\varphi_1,\ldots,\varphi_n\} = \{\,!_\circ\varphi_1,\ldots,\ !_\circ\varphi_n\}$$

Since we will show that proponent winning strategies in these games correspond to sequent calculus derivations:

$$Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ ?_{\mathcal{P}}\Delta\,;{\scriptstyle\mathcal{P}}) \neq \emptyset \iff \Gamma \vdash_c \Delta.$$

We call a position of the above form a *basic classical position*. The intended interpretation of the basic positions are: The opponent asserts all formulas in Γ and the proponent takes on the duty to meet a challenge in Δ. We thus introduce some definitions and notations to reason about basic positions.

Definition 3.6 (Basic Classical Position)**.** A basic classical position is a position of the form $(\,!_{\mathcal{O}}\Gamma,\ ?_{\mathcal{P}}\Delta)$, for some finite multisets of formulas Γ and Δ.

In the rest of this section we write "basic positions" instead of "basic classical positions". We call a position *pre-basic* if any move the opponent makes on the position results in a basic game.

Definition 3.7 (Pre-basic Position)**.** A position G is *pre-basic* if $\mathbf{Move}(G\,;{\scriptstyle\mathcal{O}}) \neq \emptyset$ and for any $a \in \mathbf{Move}(G\,;{\scriptstyle\mathcal{O}})$: aG is basic.

We say that a state $(G\,;\circ)$ is basic (pre-basic) if G is basic (pre-basic). We use greek capital letters

$$\Phi, \Psi, \ldots$$

to represent single formula positions. We note some important properties of basic and pre-basic positions:

- The opponent has no legal moves in a basic position.
- The opponent has one legal move in pre-basic positions.
- All positions are of the form $G \multimap H$ where G and H are basic.
- All pre-basic positions are of the form $\Phi \multimap G$, where Φ and G are basic.
- Given a basic game $(G\,;\mathcal{P})$ and a proponent move $(G\,;\mathcal{P}) \xrightarrow{a} (aG\,;\mathcal{O})$ such that $a \neq A(\varphi \to \psi)$, we have that $aG = \Phi \multimap H$ for some pre-basic $\Phi \multimap H$.
- Given a basic game $(G\,;\mathcal{P})$ and a proponent move $(G\,;\mathcal{P}) \xrightarrow{a} (aG\,;\mathcal{O})$ such that $a = A(\varphi \to \psi)$, we have that $aG = \Phi, \Phi' \multimap G$ for some basic Φ and Φ.

Thus, if the proponent never uses the action $A(\varphi \to \psi)$ all reachable states $(H\,;\mathcal{P})$ in a basic game $(G\,;\mathcal{P})$ are basic and all reachable states $(H\,;\mathcal{O})$ are pre-basic. Unfortunately the action $A(\varphi \to \psi)$ complicates the situation by making some reachable states neither basic nor pre-basic. This is problematic since suppose we want to define a function

$$f : Str^w_{\mathcal{P}}(G\,;\mathcal{P}) \times Str^w_{\mathcal{P}}(H\,;\mathcal{P}) \to Str^w_{\mathcal{P}}(G, H\,;\mathcal{P})$$

by saying that $f(\sigma, \tau)$ is the strategy given by alternating between moves in σ and τ starting with a move from σ. If there is no action $A(\varphi \to \psi)$ in either σ nor τ this works perfectly well, a play on $(G, H\,;\mathcal{P})$ may look as follows:

$$(G, H\,;\mathcal{P}) \xrightarrow{a} (aG, H\,;\mathcal{O}) \xrightarrow{b} (baG, H\,;\mathcal{P}) \xrightarrow{c} (baG, cH\,;\mathcal{O}) \xrightarrow{d} (baG, dcH\,;\mathcal{P}) \xrightarrow{e} \cdots$$

However suppose the action $a = A(\varphi \to \psi)$, then the position baG is not basic and in-fact contains one extra legal move for the opponent, thus the play may proceed as follows:

$$(G, H\,;\mathcal{P}) \xrightarrow{a} (aG, H\,;\mathcal{O}) \xrightarrow{b} (baG, H\,;\mathcal{P}) \xrightarrow{c} (baG, cH\,;\mathcal{O}) \xrightarrow{d} (dbaG, cH\,;\mathcal{P}),$$

where the strategy σ is not defined for the state $(dbaG\,;\mathcal{P})$, and thus f is no longer well-defined. We will circumvent this problem by defining functions

- $l : Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{O}}\psi,\, !_{\mathcal{P}}\varphi\,;\mathcal{O}) \to Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{O}}\psi\,;\mathcal{O})$,
- $r : Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{O}}\psi,\, !_{\mathcal{P}}\varphi\,;\mathcal{O}) \to Str^w_{\mathcal{P}}(G,\, !_{\mathcal{P}}\varphi\,;\mathcal{O})$,

effectively splitting any game where $A(\varphi \to \psi)$ has been played into several subgames. The proponent may then play these games in parallel creating a strategy which we can contract back into a strategy for the original game using the functions

- $\| : Str^w_{\mathcal{P}}(G\,;\mathcal{O}) \times Str^w_{\mathcal{P}}(H\,;\mathcal{O}) \to Str^w_{\mathcal{P}}(G, H\,;\mathcal{O})$.
- $Con : Str^w_{\mathcal{P}}(G, G\,;\mathcal{P}) \to Str^w_{\mathcal{P}}(G\,;\mathcal{P})$.

Showing that these functions exists will be the major task in the following sections.

3.3.1 No Stalemate

Given our definition of a winning strategy, the opponent may in principle win a game $(G\,;\mathcal{P})$, where G is basic by reaching a state $(H\,;\mathcal{O})$ for which there is no further move. The game has reached a so called *stalemate*. For us to be able to define the left and right strategies, we do not want this to be the case, so we begin by showing.

Theorem 3.4 (No Stalemate)**.** If G is basic and $(H, \mathcal{O})$ is a reachable non-terminal state in a classical game $(G\,;\mathcal{P})$, then there exists a move

$$(H\,;\mathcal{O}) \xrightarrow{a} (aH\,;\mathcal{P}),$$

for some action $a \in \mathit{Actions}$.

Proof. Consider the move $(J\,;\mathcal{P}) \xrightarrow{a} (H\,;\mathcal{O})$ on the position preceding H, by cases

- The action a is an attack, then the opponent has a defense move in $(H, \mathcal{O})$.
- The action a is a defense of a formula φ, by cases
  - The assertion resulting from the defense is non-atomic, then there is an attack move for the opponent in $(H\,;\mathcal{O})$.
  - The assertion resulting from the defense $!_{\mathcal{P}}\bot$, but this is impossible since then $!_{\mathcal{P}}\bot \in H$ and thus $(H, \mathcal{O})$ is terminal.
  - The assertion resulting from the defense is $!_{\mathcal{P}}P$, then since $(H, \mathcal{O})$ is non-terminal $!_{\mathcal{O}}P \in H$ and thus also $!_{\mathcal{O}}P \in J$ thus $(J\,;\mathcal{P})$ is terminal. But this is impossible since then $(H\,;\mathcal{O})$ is not a reachable state. $\square$

Corollary 3.4.1. If $\sigma \in \mathit{Str}_{\mathcal{P}}(G\,;\mathcal{O})$ is a non-winning strategy where G is pre-basic, then there is a branch b of σ that is either infinite or ends in a terminal state $(H\,;\mathcal{O})$ or a stalemate state $(H\,;\mathcal{P})$.

Proof. By definition is σ is non-winning then there is a branch b of σ that is either infinite or ends in a terminal state $(H\,;\mathcal{O})$ or a stalemate state $(H\,;\mathcal{P})$ or in a stalemate state $(H\,;\mathcal{O})$. To exclude this last possibility it suffices to note that since $(G\,;\mathcal{O})$ is pre-basic it's not a stalemate state, and for all reachable states $(H\,;\mathcal{O})$ the opponent have a legal move by the above theorem. $\square$

3.4 Operations on Strategies

3.4.1 Weakening

Since the opponent has no legal moves in a basic position adding a basic position to an already winning position changes nothing with regards to winnability for the proponent, this is called *weakening*.

Lemma 3.5 (Weakening)**.** Let G be any position and H be a basic position, then there is a function

1. $Wk : Str^w_{\mathcal{P}}(G\,;{\scriptstyle\mathcal{P}}) \to Str^w_{\mathcal{P}}(G, H\,;{\scriptstyle\mathcal{P}})$
2. $Wk : Str^w_{\mathcal{P}}(G\,;{\scriptstyle\mathcal{O}}) \to Str^w_{\mathcal{P}}(G, H\,;{\scriptstyle\mathcal{O}})$

where

1. $Wk(\sigma) = \begin{cases} e \text{ if } (G, H\,;{\scriptstyle\mathcal{P}}) \in Term_{\mathcal{P}} \\ \sigma \text{ else.} \end{cases}$
2. $Wk(\sigma) = \sigma$.

3.4.2 The Copy-cat Strategy

The most important strategy in combinatorial games is the so called *copy-cat* or *identity* strategy where a player simply repeats the opposing players moves. We show that for the classical games and any given position G the proponent has a winning copy-cat strategy in the game $(G \multimap G\,;{\scriptstyle\mathcal{O}})$ where the proponent just repeats the opponents actions.

Theorem 3.6 (The Copy-cat strategy)**.** If G is any position there is a function

$$Id \in Str^w_{\mathcal{P}}(G \multimap G\,;{\scriptstyle\mathcal{O}}),$$

where

$$Id\,a = \begin{cases} (a, Id) \ \textit{if} \ (a(G \multimap G)\,;{\scriptstyle\mathcal{P}}) \notin Term \\ e \ \textit{else.} \end{cases}$$

For arbitrary $a \in \mathbf{Move}(G \multimap G\,;{\scriptstyle\mathcal{O}})$.

Proof. If $G = \emptyset$, then by definition $e \in Str^w_{\mathcal{P}}(\emptyset \multimap \emptyset\,;{\scriptstyle\mathcal{O}})$. If $G \neq \emptyset$, then eventually a state $(H,\, !_{\mathcal{O}}\bot\,;{\scriptstyle\mathcal{P}})$ or $(H,\, !_{\mathcal{O}}A,\, ?_{\mathcal{P}}A\,;{\scriptstyle\mathcal{P}})$ is reached, both of which are winning for the opponent. $\square$

Using the copy-cat strategy we can define a *modus ponens* strategy as follows:

Theorem 3.7 (Modus Ponens)**.** If G is a basic position then there is a strategy

$$Mp \in Str^w_{\mathcal{P}}(G,\, !_{\mathcal{O}}\varphi \to \psi,\, !_{\mathcal{O}}\varphi,\, ?_{\mathcal{P}}\psi\,;{\scriptstyle\mathcal{O}})$$

Proof. We construct it as follows:

$$\begin{aligned} Id &\in Str^w_{\mathcal{P}}(\,!_{\mathcal{P}}\varphi,\, ?_{\mathcal{O}}\psi,\, !_{\mathcal{O}}\varphi,\, ?_{\mathcal{P}}\psi\,;{\scriptstyle\mathcal{O}}) \\ Wk(Id) &\in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\varphi \to \psi,\, !_{\mathcal{P}}\varphi,\, ?_{\mathcal{O}}\psi,\, !_{\mathcal{O}}\varphi,\, ?_{\mathcal{P}}\psi\,;{\scriptstyle\mathcal{O}}) \\ (A(\varphi \to \psi), Wk(Id)) &\in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\varphi \to \psi,\, !_{\mathcal{O}}\varphi,\, ?_{\mathcal{P}}\psi\,;{\scriptstyle\mathcal{P}}) \\ Wk(A(\varphi \to \psi), Wk(Id)) &\in Str^w_{\mathcal{P}}(G,\, !_{\mathcal{O}}\varphi \to \psi,\, !_{\mathcal{O}}\varphi,\, ?_{\mathcal{P}}\psi\,;{\scriptstyle\mathcal{P}}) \end{aligned}$$ $\square$

3.4.3 Left and Right Strategies

Using No Stalemate we are now able to show that we can decompose a game $(G,\, ?_{\mathcal{O}}\psi,\, !_{\mathcal{O}}\varphi\,;\mathcal{O})$ into two components using the left and right strategies.

Lemma 3.8. Let G be a basic position, then there are functions

1. $l : Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{O}}\psi,\, !_{\mathcal{P}}\varphi\,;\mathcal{O}) \to Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{O}}\psi\,;\mathcal{O})$
2. $r : Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{O}}\psi,\, !_{\mathcal{P}}\varphi\,;\mathcal{O}) \to Str^w_{\mathcal{P}}(G,\, !_{\mathcal{P}}\varphi\,;\mathcal{O})$

such that for all $\sigma \in Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{O}}\psi,\, !_{\mathcal{P}}\varphi\,;\mathcal{O})$, it holds that $l(\sigma) \leq_T \sigma$ and $r(\sigma) \leq_T \sigma$.

Proof. Assume $\sigma \in Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{O}}\psi,\, !_{\mathcal{P}}\varphi\,;\mathcal{O})$. Let $X = \bigcup Str_{\mathcal{P}}(G,\, ?_{\mathcal{O}}\psi\,;\mathcal{O})$, then let $l(\sigma) = \sigma \cap X$ be the restriction of the strategy σ to the game $(G,\, ?_{\mathcal{O}}\psi\,;\mathcal{O})$. Then we have that $l(\sigma)$ is a strategy $l(\sigma) \in Str_{\mathcal{P}}(G,\, ?_{\mathcal{O}}\psi\,;\mathcal{O})$, since it's just a restriction of σ to the plays where the opponent never attacks $!_{\mathcal{P}}\varphi$. Also it's a winning strategy:

$$l(\sigma) \in Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{O}}\psi\,;\mathcal{O}).$$

Since assume that $l(\sigma)$ is not winning, then since $(G,\, ?_{\mathcal{O}}\psi\,;\mathcal{O})$ is pre-basic by a corollary of No Stalemate if $l(\sigma)$ is not winning there is an branch b of $l(\sigma)$ that is infinite or ends in a terminal state $(H\,;\mathcal{O})$ or a stalemate state $(H\,;\mathcal{P})$. Now since $l(\sigma) \subseteq \sigma$ this branch is also in σ, thus σ would not be a winning strategy which is a contradiction. Also, since $l(\sigma) \subseteq \sigma$ we have that $l(\sigma) \leq_T \sigma$. Similarly, taking $r(\sigma) = \sigma \cap \bigcup Str_{\mathcal{P}}(G,\, !_{\mathcal{P}}\varphi\,;\mathcal{O})$, we get a

$$r(\sigma) \in Str^w_{\mathcal{P}}(G,\, !_{\mathcal{P}}\varphi\,;\mathcal{O}),$$

such that $r(\sigma) \leq_T \sigma$. □

3.4.4 Move-order Invariance

Suppose the proponent has a winning strategy $(a, \sigma) \in Str^w_{\mathcal{P}}(G\,;\mathcal{P})$ where the first action is a. If there is another action $b \in \mathbf{Move}(G\,;\mathcal{P})$ distinct from a we can ask if there is another winning strategy $(b, \sigma') \in Str^w_{\mathcal{P}}(G\,;\mathcal{P})$ starting with the action b instead. If there's always such a strategy for a given action b we say that b is *move-order invariant*. We will show that the actions $A(\varphi)$ where φ is positive, and $D(\varphi)$ where φ is negative are all move-order invariant.

Theorem 3.9. Let $(G,\, !_{\mathcal{O}}\varphi\,;\mathcal{P})$ be a non terminal state and φ positive and ψ negative composite formulas, then there is a function:

1. $Perm_{\varphi} : Str^w_{\mathcal{P}}(G,\, !_{\mathcal{O}}\varphi\,;\mathcal{P}) \to Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{O}}\varphi\,;\mathcal{O})$
2. $Perm_{\psi} : Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{P}}\psi\,;\mathcal{P}) \to Str^w_{\mathcal{P}}(G,\, !_{\mathcal{P}}\psi\,;\mathcal{O})$

Proof.

1. Suppose $(a, \sigma) \in Str^w_{\mathcal{P}}(G,\, !_{\mathcal{O}}\varphi\,;{\scriptstyle \mathcal{P}})$. We construct a strategy
$$Perm_\varphi((a,\sigma)) \in Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{O}}\varphi\,;{\scriptstyle \mathcal{O}}).$$
If $a = A(\varphi)$, let $Perm_\varphi((a,\sigma)) = \sigma$. Otherwise we define
$$Perm_\varphi((a,\sigma))(x) \in Str^w_{\mathcal{P}}(x(G,\, ?_{\mathcal{O}}\varphi\,;{\scriptstyle \mathcal{O}}))$$
for arbitrary $x \in \mathbf{Move}(G,\, ?_{\mathcal{O}}\varphi\,;{\scriptstyle \mathcal{O}})$. By cases:
   - Let the move be b on the G component of the game, then we construct a strategy in $Str^w_{\mathcal{P}}(bG,\, ?_{\mathcal{O}}\varphi\,;{\scriptstyle \mathcal{P}})$ as follows:
$$\begin{aligned}(a,\sigma) &\in Str^w_{\mathcal{P}}(G,\, !_{\mathcal{O}}\varphi\,;{\scriptstyle \mathcal{P}}) & \\ \sigma &\in Str^w_{\mathcal{P}}(aG,\, !_{\mathcal{O}}\varphi\,;{\scriptstyle \mathcal{O}}) & \\ \sigma(b) &\in Str^w_{\mathcal{P}}(baG,\, !_{\mathcal{O}}\varphi\,;{\scriptstyle \mathcal{P}}) & \text{in particular} \\ Perm_\varphi(\sigma(b)) &\in Str^w_{\mathcal{P}}(baG,\, ?_{\mathcal{O}}\varphi\,;{\scriptstyle \mathcal{O}}) & \text{by induction} \\ (a, Perm_\varphi(\sigma(b))) &\in Str^w_{\mathcal{P}}(bG,\, ?_{\mathcal{O}}\varphi\,;{\scriptstyle \mathcal{P}}) &\end{aligned}$$
   - Let the move be c on the $?_{\mathcal{O}}\varphi$ component of the game, then we construct a strategy in $Str^w_{\mathcal{P}}(G, c\,?_{\mathcal{O}}\varphi\,;{\scriptstyle \mathcal{P}})$ as follows:
$$\begin{aligned}\sigma(b) &\in Str^w_{\mathcal{P}}(baG,\, !_{\mathcal{O}}\varphi\,;{\scriptstyle \mathcal{P}}) & \text{where } b \text{ is arbitrary} \\ Perm_\varphi(\sigma(b)) &\in Str^w_{\mathcal{P}}(baG,\, ?_{\mathcal{O}}\varphi\,;{\scriptstyle \mathcal{O}}) & \text{by induction} \\ Perm_\varphi(\sigma(b))(c) &\in Str^w_{\mathcal{P}}(baG, c\,?_{\mathcal{O}}\varphi\,;{\scriptstyle \mathcal{P}}) & \text{in particular} \\ \lambda x.Perm_\varphi(\sigma(x))(c) &\in Str^w_{\mathcal{P}}(aG, c\,?_{\mathcal{O}}\varphi\,;{\scriptstyle \mathcal{O}}) & \\ (a, \lambda x.Perm_\varphi(\sigma(x)))(c)) &\in Str^w_{\mathcal{P}}(G, c\,?_{\mathcal{O}}\varphi\,;{\scriptstyle \mathcal{P}}) &\end{aligned}$$

2. Suppose $(a, \sigma) \in Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{P}}\psi\,;{\scriptstyle \mathcal{P}})$. We construct a strategy
$$Perm_\psi((a,\sigma)) \in Str^w_{\mathcal{P}}(G,\, !_{\mathcal{P}}\psi\,;{\scriptstyle \mathcal{O}}).$$
If $a = D(\varphi)$, let $Perm_\varphi((a,\sigma)) = \sigma$. Otherwise we define
$$Perm_\varphi((a,\sigma))(x) \in Str^w_{\mathcal{P}}(x(G,\, !_{\mathcal{P}}\psi\,;{\scriptstyle \mathcal{O}}))$$
for arbitrary $x \in \mathbf{Move}(G,\, !_{\mathcal{P}}\psi\,;{\scriptstyle \mathcal{O}})$. By cases:
   - Let the move be b on the G component of the game, then we construct a strategy in $Str^w_{\mathcal{P}}(bG,\, !_{\mathcal{P}}\psi\,;{\scriptstyle \mathcal{P}})$ as follows:
$$\begin{aligned}(a,\sigma) &\in Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{P}}\psi\,;{\scriptstyle \mathcal{P}}) & \\ \sigma &\in Str^w_{\mathcal{P}}(aG,\, ?_{\mathcal{P}}\psi\,;{\scriptstyle \mathcal{O}}) & \\ \sigma(b) &\in Str^w_{\mathcal{P}}(baG,\, ?_{\mathcal{P}}\psi\,;{\scriptstyle \mathcal{P}}) & \text{in particular} \\ Perm_\psi(\sigma(b)) &\in Str^w_{\mathcal{P}}(baG,\, !_{\mathcal{P}}\psi\,;{\scriptstyle \mathcal{O}}) & \text{by induction} \\ (a, Perm_\psi(\sigma(b))) &\in Str^w_{\mathcal{P}}(bG,\, !_{\mathcal{P}}\psi\,;{\scriptstyle \mathcal{P}}) &\end{aligned}$$

- Let the move be c on the $?_{\mathcal{O}}\varphi$ component of the game, then we construct a strategy in $Str^w_{\mathcal{P}}(G, c\,!_{\mathcal{P}}\psi\,;_{\mathcal{P}})$ as follows:

$$\begin{aligned} \sigma(b) &\in Str^w_{\mathcal{P}}(baG,\ ?_{\mathcal{P}}\psi\,;_{\mathcal{P}}) && \text{where } b \text{ is arbitrary} \\ Perm_{\psi}(\sigma(b)) &\in Str^w_{\mathcal{P}}(baG,\ !_{\mathcal{P}}\psi\,;_{\mathcal{O}}) && \text{by induction} \\ Perm_{\psi}(\sigma(b))(c) &\in Str^w_{\mathcal{P}}(baG, c\,!_{\mathcal{P}}\psi\,;_{\mathcal{P}}) && \text{in particular} \\ \lambda x.Perm_{\psi}(\sigma(x))(c) &\in Str^w_{\mathcal{P}}(aG, c\,!_{\mathcal{P}}\psi\,;_{\mathcal{O}}) \\ (a, \lambda x.Perm_{\psi}(\sigma(x))(c)) &\in Str^w_{\mathcal{P}}(G, c\,!_{\mathcal{P}}\psi\,;_{\mathcal{P}}) \end{aligned}$$

□

3.4.5 Contraction

Given a strategy $\sigma \in Str^w_{\mathcal{P}}(G, H, H\,;_{\mathcal{P}})$ where G and H are basic we would like to define a contraction $Con(\sigma) \in Str^w_{\mathcal{P}}(G, H\,;_{\mathcal{P}})$. Intuitively there should be such a strategy since the opponent should not gain anything from asserting a formula twice, and the proponent should not gain anything by being challenged twice about the same formula.

Theorem 3.10 (Contraction)**.** Let G be a basic position, then there is a function:

$$Con : Str^w_{\mathcal{P}}(G, H, H\,;_{\mathcal{P}}) \to Str^w_{\mathcal{P}}(G, H\,;_{\mathcal{P}})$$

Proof. We show this by showing there are functions

1. $Con : Str^w_{\mathcal{P}}(G,\ !_{\mathcal{O}}\varphi,\ !_{\mathcal{O}}\varphi\,;_{\mathcal{P}}) \to Str^w_{\mathcal{P}}(G,\ !_{\mathcal{O}}\varphi\,;_{\mathcal{P}})$
2. $Con : Str^w_{\mathcal{P}}(G,\ ?_{\mathcal{P}}\varphi,\ ?_{\mathcal{P}}\varphi\,;_{\mathcal{P}}) \to Str^w_{\mathcal{P}}(G,\ ?_{\mathcal{P}}\varphi\,;_{\mathcal{P}})$

1. Suppose $\sigma \in Str^w_{\mathcal{P}}(G,\ !_{\mathcal{O}}\varphi,\ !_{\mathcal{O}}\varphi\,;_{\mathcal{P}})$, by cases:
   - If $\varphi = A$ is an atom it is never attacked, thus let $Con(\sigma) = \sigma$.
   - If φ is positive, we construct a strategy $Con(\sigma) \in Str^w_{\mathcal{P}}(G,\ !_{\mathcal{O}}\varphi)$ as follows:

$$\begin{aligned} \sigma &\in Str^w_{\mathcal{P}}(G,\ !_{\mathcal{O}}\varphi,\ !_{\mathcal{O}}\varphi\,;_{\mathcal{P}}) \\ Perm_{\varphi}(\sigma) &\in Str^w_{\mathcal{P}}(G,\ ?_{\mathcal{O}}\varphi,\ !_{\mathcal{O}}\varphi\,;_{\mathcal{O}}) \\ Perm_{\varphi}(\sigma)(b) &\in Str^w_{\mathcal{P}}(G, b\,?_{\mathcal{O}}\varphi,\ !_{\mathcal{O}}\varphi\,;_{\mathcal{P}}) \\ Perm_{\varphi}(Perm_{\varphi}(\sigma)(b)) &\in Str^w_{\mathcal{P}}(G, b\,?_{\mathcal{O}}\varphi,\ ?_{\mathcal{O}}\varphi\,;_{\mathcal{O}}) \\ Perm_{\varphi}(Perm_{\varphi}(\sigma)(b))(b) &\in Str^w_{\mathcal{P}}(G, b\,?_{\mathcal{O}}\varphi, b\,?_{\mathcal{O}}\varphi\,;_{\mathcal{P}}) \\ Con(Perm_{\varphi}(Perm_{\varphi}(\sigma)(b))(b)) &\in Str^w_{\mathcal{P}}(G, b\,?_{\mathcal{O}}\varphi\,;_{\mathcal{P}}) \\ \lambda x.Con(Perm_{\varphi}(Perm_{\varphi}(\sigma)(x))(x)) &\in Str^w_{\mathcal{P}}(G,\ ?_{\mathcal{O}}\varphi\,;_{\mathcal{O}}) \\ (A(\varphi), \lambda x.Con(Perm_{\varphi}(Perm_{\varphi}(\sigma)(x))(x))) &\in Str^w_{\mathcal{P}}(G,\ !_{\mathcal{O}}\varphi\,;_{\mathcal{P}}) \end{aligned}$$

     Where b is an arbitrary defense of $?_{\mathcal{O}}\varphi$.
   - If φ is negative the proponent may re-attack φ, thus let $Con(\sigma) = \sigma$.

2. Suppose $\sigma \in Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{P}}\varphi,\, ?_{\mathcal{P}}\varphi\,;{\scriptstyle\mathcal{P}})$, by cases:

   - If $\varphi = A$ is an atom it is never defended, thus let $Con(\sigma) = \sigma$.
   - If φ is negative, we construct a strategy $Con(\sigma) \in Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{P}}\varphi)$ as follows:

$$
\begin{aligned}
\sigma &\in Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{P}}\varphi,\, ?_{\mathcal{P}}\varphi\,;{\scriptstyle\mathcal{P}})\\
Perm_{\varphi}(\sigma) &\in Str^w_{\mathcal{P}}(G,\, !_{\mathcal{P}}\varphi,\, ?_{\mathcal{P}}\varphi\,;{\scriptstyle\mathcal{O}})\\
Perm_{\varphi}(\sigma)(b) &\in Str^w_{\mathcal{P}}(G, b\,!_{\mathcal{P}}\varphi,\, ?_{\mathcal{P}}\varphi\,;{\scriptstyle\mathcal{P}})\\
Perm_{\varphi}(Perm_{\varphi}(\sigma)(b)) &\in Str^w_{\mathcal{P}}(G, b\,!_{\mathcal{P}}\varphi,\, !_{\mathcal{P}}\varphi\,;{\scriptstyle\mathcal{O}})\\
Perm_{\varphi}(Perm_{\varphi}(\sigma)(b))(b) &\in Str^w_{\mathcal{P}}(G, b\,!_{\mathcal{P}}\varphi, b\,!_{\mathcal{P}}\varphi\,;{\scriptstyle\mathcal{P}})\\
Con(Perm_{\varphi}(Perm_{\varphi}(\sigma)(b))(b)) &\in Str^w_{\mathcal{P}}(G, b\,!_{\mathcal{P}}\varphi\,;{\scriptstyle\mathcal{P}})\\
\lambda x.Con(Perm_{\varphi}(Perm_{\varphi}(\sigma)(x))(x)) &\in Str^w_{\mathcal{P}}(G,\, !_{\mathcal{P}}\varphi\,;{\scriptstyle\mathcal{O}})\\
(D(\varphi), \lambda x.Con(Perm_{\varphi}(Perm_{\varphi}(\sigma)(x))(x))) &\in Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{P}}\varphi\,;{\scriptstyle\mathcal{P}})
\end{aligned}
$$

     Where b is an arbitrary attack on $!_{\mathcal{P}}\varphi$.
   - φ is positive, but then the proponent may re-defend φ, thus let $Con(\sigma) = \sigma$. $\square$

3.4.6 The Parallel Strategy

Given two strategies $\sigma \in Str^w_{\mathcal{P}}(G\,;{\scriptstyle\mathcal{O}})$ and $\tau \in Str^w_{\mathcal{P}}(H\,;{\scriptstyle\mathcal{O}})$ we seek to define the parallel strategy $\sigma \parallel \tau \in Str^w_{\mathcal{P}}(G, H\,;{\scriptstyle\mathcal{O}})$. Intuitively this is the strategy where the proponent responds to any opponent move in the G or H component by a corresponding move from either σ or τ. This strategy is winning since eventually the proponent will reach a terminal state in either component, thus winning the composite game.

Theorem 3.11 (Parallel Strategy)**.** There is a function

$$\parallel : Str^w_{\mathcal{P}}(G\,;{\scriptstyle\mathcal{O}}) \times Str^w_{\mathcal{P}}(H\,;{\scriptstyle\mathcal{O}}) \to Str^w_{\mathcal{P}}(G, H\,;{\scriptstyle\mathcal{O}})$$

Proof. Consider an arbitrary action $a \in \mathbf{Move}(G, H\,;{\scriptstyle\mathcal{O}})$, without loss of generality let it be on the H component of the game, we define

$$(\sigma \parallel \tau)(a) \in Str^w_{\mathcal{P}}(G, aH\,;{\scriptstyle\mathcal{P}}),$$

by induction on $\tau(a) \in Str^w_{\mathcal{P}}(aH\,;{\scriptstyle\mathcal{P}})$. Take as inductive hypothesis that the parallel strategy $\sigma \parallel \tau'$ is defined for $\tau' <_T \tau$. By cases:

- The strategy is $\tau(a) = e$, thus $(aH\,;{\scriptstyle\mathcal{P}})$ is a terminal state, then also $(G, aH\,;{\scriptstyle\mathcal{P}})$ is terminal so let $(\sigma \parallel \tau)(a) = e$.
- The strategy is $\tau(a) = (b, \tau')$ and $\tau' \in Str^w_{\mathcal{P}}(baH\,;{\scriptstyle\mathcal{O}})$, we construct a strategy $(\sigma \parallel \tau)(a) \in Str^w_{\mathcal{P}}(G, aH\,;{\scriptstyle\mathcal{P}})$ as follows:

$$\dfrac{\dfrac{\sigma \in Str^w_{\mathcal{P}}(G\,;{\scriptstyle\mathcal{O}}) \qquad \tau' \in Str^w_{\mathcal{P}}(baH\,;{\scriptstyle\mathcal{O}})}{\sigma \parallel \tau' \in Str^w_{\mathcal{P}}(G, baH\,;{\scriptstyle\mathcal{O}})}}{(b, (\sigma \parallel \tau')) \in Str^w_{\mathcal{P}}(G, aH\,;{\scriptstyle\mathcal{P}})}$$

$\square$

3.4.7 Application and Composition

The standard technique to define a combination of strategies $\sigma \in Str^w_{\mathcal{P}}(G, H\,;{\scriptscriptstyle\mathcal{P}})$ and $\tau \in Str^w_{\mathcal{P}}(H \multimap J\,;{\scriptscriptstyle\mathcal{O}})$ in combinatorial game theory is by considering a so called swivel chair strategy [Siegel, 2013]: To show the proponent also has a winning strategy $Ap(\sigma, \tau) \in Str^w_{\mathcal{P}}(G, J\,;{\scriptscriptstyle\mathcal{P}})$, we set up this game and right below it a copy of the game H, H^d, we can imagine a play on the two component games proceeding as

$$\begin{array}{c}G\,,\ J\\ H\,,H^d\end{array} \xrightarrow{\mathcal{P}} \begin{array}{c}aG\,,\ J\\ H\ \,,H^d\end{array} \xrightarrow{\mathcal{O}} \begin{array}{c}baG\,,\ J\\ H\ \ ,H^d\end{array} \xrightarrow{\mathcal{P}} \cdots\,,$$

or

$$\begin{array}{c}G\,,\ J\\ H\,,H^d\end{array} \xrightarrow{\mathcal{P}} \begin{array}{c}G\ \,,\ J\\ aH\,,H^d\end{array} \xrightarrow{\mathcal{O}} \begin{array}{c}G\ \ ,\ \ J\\ aH\,,aH^d\end{array} \xrightarrow{\mathcal{P}} \cdots\,,$$

We then construct the strategy $Ap(\sigma, \tau) \in Str^w_{\mathcal{P}}(G, J\,;{\scriptscriptstyle\mathcal{P}})$ by forgetting the moves on the H components. However, this construction will not work straightforwardly for our games since we'll run into problems where responses may not always occur in the same component of the larger game, since we allow for backtracking for the opponent. We therefore proceed by defining two functions by induction to create a strategy similar to the swivel-chair strategy.

Theorem 3.12 (Application and Composition)**.** Let G, J, I be basic games and Φ, Ψ be basic single formula games, then there are functions:

1. $Ap : Str^w_{\mathcal{P}}(G, \Psi\,;{\scriptscriptstyle\mathcal{P}}) \times Str^w_{\mathcal{P}}(\Psi \multimap J\,;{\scriptscriptstyle\mathcal{O}}) \to Str^w_{\mathcal{P}}(G, J\,;{\scriptscriptstyle\mathcal{P}})$
2. $(-;-) : Str^w_{\mathcal{P}}(\Phi \multimap I, \Psi\,;{\scriptscriptstyle\mathcal{O}}) \times Str^w_{\mathcal{P}}(\Psi \multimap J\,;{\scriptscriptstyle\mathcal{O}}) \to Str^w_{\mathcal{P}}(\Phi \multimap I, J\,;{\scriptscriptstyle\mathcal{O}})$

Proof.

1. Let $(\sigma, \tau) \in Str^w_{\mathcal{P}}(G, \Psi\,;{\scriptscriptstyle\mathcal{P}}) \times Str^w_{\mathcal{P}}(\Psi \multimap J\,;{\scriptscriptstyle\mathcal{O}})$. By induction on $\sigma \in Str^w_{\mathcal{P}}(G, \Psi\,;{\scriptscriptstyle\mathcal{P}})$, with subinduction on the formula Ψ, by cases:
   - The strategy is $\sigma = e$, thus $(G, \Psi\,;{\scriptscriptstyle\mathcal{P}})$ is a terminal state, by cases
     - The state $(G\,;{\scriptscriptstyle\mathcal{P}})$ is terminal, then also $(G, J\,;{\scriptscriptstyle\mathcal{P}})$ is terminal, so let
     $$Ap(\sigma, \tau) = e.$$
     - The state $(G, \Psi\,;{\scriptscriptstyle\mathcal{P}}) = (G',\ ?_{\mathcal{P}}A,\ !_{\mathcal{O}}A\,;{\scriptscriptstyle\mathcal{P}})$ where $\Psi = !_{\mathcal{O}}A$. Since $(\,!_{\mathcal{P}}A, J\,;{\scriptscriptstyle\mathcal{O}})$ is non-terminal also $!_{\mathcal{O}}A \in J$, but then also $(G',\ ?_{\mathcal{P}}A, J\,;{\scriptscriptstyle\mathcal{P}})$ is terminal so let
     $$Ap(\sigma, \tau) = e.$$
     - The state $(G, \Psi\,;{\scriptscriptstyle\mathcal{P}}) = (G'\,!_{\mathcal{O}}A,\ ?_{\mathcal{P}}A\,;{\scriptscriptstyle\mathcal{P}})$ where $\Psi = ?_{\mathcal{P}}A$, then $\tau \in Str^w_{\mathcal{P}}(\,?_{\mathcal{O}}A, J\,;{\scriptscriptstyle\mathcal{O}})$. Thus $\tau(D(A)) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}A, J\,;{\scriptscriptstyle\mathcal{P}})$, thus $Wk(\tau(D(A))) \in Str^w_{\mathcal{P}}(G',\ !_{\mathcal{O}}A, J\,;{\scriptscriptstyle\mathcal{P}})$, so let
     $$Ap(\sigma, \tau) = Wk(\tau(D(A))).$$

- The strategy is $\sigma = (a, \sigma')$ and the first move is on the G-component of the game, we get two cases:
  (a) The position $aG, \Psi = \Phi \multimap \Psi, H$ for some basic games Φ and H, i.e $a \neq A(\varphi_1 \to \varphi_2)$, thus
  $$\sigma' \in Str^w_{\mathcal{P}}(\Phi \multimap H, \Psi \,;\mathcal{O})$$
  we construct a strategy $Ap(\sigma, \tau) \in Str^w_{\mathcal{P}}(G, J \,;\mathcal{P})$ as follows:
  $$\dfrac{\dfrac{\sigma' \in Str^w_{\mathcal{P}}(\Phi \multimap H, \Psi \,;\mathcal{O}) \qquad \tau \in Str^w_{\mathcal{P}}(\Psi \multimap J \,;\mathcal{O})}{\sigma';\tau \in Str^w_{\mathcal{P}}(\Phi \multimap H, J \,;\mathcal{O})}}{(a, (\sigma';\tau)) \in Str^w_{\mathcal{P}}(G, J \,;\mathcal{P})}$$
  (b) The position $aG, \Psi = \Phi_1, \Phi_2 \multimap G, \Psi$ for some basic games Φ_1 and Φ_2, i.e $a = A(\varphi_1 \to \varphi_2)$, thus
  $$\sigma' \in Str^w_{\mathcal{P}}(\Phi_1, \Phi_2 \multimap G, \Psi \,;\mathcal{O})$$
  we construct a strategy $Ap(\sigma, \tau) \in Str^w_{\mathcal{P}}(G, J \,;\mathcal{P})$ by first constructing two strategies:
  $$\dfrac{\dfrac{\sigma' \in Str^w_{\mathcal{P}}(\Phi_1, \Phi_2 \multimap G, \Psi \,;\mathcal{O})}{l(\sigma') \in Str^w_{\mathcal{P}}(\Phi_1 \multimap G, \Psi \,;\mathcal{O})} \qquad \tau \in Str^w_{\mathcal{P}}(\Psi \multimap J \,;\mathcal{O})}{l(\sigma');\tau \in Str^w_{\mathcal{P}}(\Phi_1 \multimap G, J \,;\mathcal{O})}$$
  and
  $$\dfrac{\dfrac{\sigma' \in Str^w_{\mathcal{P}}(\Phi_1, \Phi_2 \multimap G, \Psi \,;\mathcal{O})}{r(\sigma') \in Str^w_{\mathcal{P}}(\Phi_2 \multimap G, \Psi \,;\mathcal{O})} \qquad \tau \in Str^w_{\mathcal{P}}(\Psi \multimap J \,;\mathcal{O})}{r(\sigma');\tau \in Str^w_{\mathcal{P}}(\Phi_2 \multimap G, J \,;\mathcal{O})}$$
  Then playing them in parallel and contracting:
  $$\dfrac{\dfrac{\dfrac{l(\sigma');\tau \in Str^w_{\mathcal{P}}(\Phi_1 \multimap G, J \,;\mathcal{O}) \qquad r(\sigma');\tau \in Str^w_{\mathcal{P}}(\Phi_2 \multimap G, J \,;\mathcal{O})}{(l(\sigma');\tau) \parallel (r(\sigma');\tau) \in Str^w_{\mathcal{P}}(\Phi_1, \Phi_2 \multimap G, J, G, J \,;\mathcal{O})}}{(a, (l(\sigma');\tau) \parallel (r(\sigma');\tau)) \in Str^w_{\mathcal{P}}(G, G, J, J \,;\mathcal{P})}}{Con((a, (l(\sigma');\tau) \parallel (r(\sigma');\tau))) \in Str^w_{\mathcal{P}}(G, J \,;\mathcal{P})}$$

- The strategy is $\sigma = (a, \sigma')$ and the first move is on the Ψ-component of the game, we get three cases:
  (a) The position $G, a\Psi = \Phi \multimap G$, i.e $a = D(\varphi)$ for negative φ or $a = A(\varphi)$ for positive φ. Thus,
  $$\sigma' \in Str^w_{\mathcal{P}}(\Phi \multimap G \,;\mathcal{O}).$$
  We construct a strategy in $Str^w_{\mathcal{P}}(G, J \,;\mathcal{P})$ as follows:
  $$\dfrac{\sigma' \in Str^w_{\mathcal{P}}(\Phi \multimap G \,;\mathcal{O}) \qquad \dfrac{\tau \in Str^w_{\mathcal{P}}(\Psi \multimap J \,;\mathcal{O})}{\tau(a) \in Str^w_{\mathcal{P}}(\Phi, J \,;\mathcal{P})}}{Ap(\tau(a), \sigma') \in Str^w_{\mathcal{P}}(G, J \,;\mathcal{P})}$$

(b) The position $G, a\Psi = \Phi \multimap G, \Psi$, i.e $a = D(\varphi)$ for positive φ or $a = A(\varphi)$ for negative $\varphi \neq \varphi_1 \to \varphi_2$. Thus,

$$\sigma' \in Str^w_{\mathcal{P}}(\Phi \multimap G, \Psi\,;\mathcal{O})$$

We construct a strategy in $Str^w_{\mathcal{P}}(G, J\,;\mathcal{P})$ as follows:

$$\dfrac{\dfrac{\dfrac{\sigma' \in Str^w_{\mathcal{P}}(\Phi \multimap G, \Psi\,;\mathcal{O}) \qquad \tau \in Str^w_{\mathcal{P}}(\Psi \multimap J\,;\mathcal{O})}{(\sigma';\tau) \in Str^w_{\mathcal{P}}(\Phi \multimap G, J\,;\mathcal{O})} \qquad \dfrac{\tau \in Str^w_{\mathcal{P}}(\Psi \multimap J\,;\mathcal{O})}{\tau(a) \in Str^w_{\mathcal{P}}(\Phi, J\,;\mathcal{P})}}{Ap(\tau(a), (\sigma';\tau)) \in Str^w_{\mathcal{P}}(G, J, G\,;\mathcal{P})}}{Con((Ap(\tau(a), (\sigma';\tau))) \in Str^w_{\mathcal{P}}(G, J\,;\mathcal{P})}$$

(c) The position $G, a\Psi = \Phi_1, \Phi_2 \multimap G, \Psi$ for some basic games Φ_1 and Φ_2, i.e $a = A(\varphi_1 \to \varphi_2)$. This is the same as a previous case so let

$$Ap(\sigma, \tau) = Con((a, (l(\sigma');\tau) \parallel (r(\sigma');\tau)))$$

2. Let $(\sigma, \tau) \in Str^w_{\mathcal{P}}(\Phi \multimap I, \Psi\,;\mathcal{O}) \times Str^w_{\mathcal{P}}(\Psi \multimap J\,;\mathcal{O})$. By induction on $\sigma \in Str^w_{\mathcal{P}}(\Phi \multimap I, \Psi\,;\mathcal{O})$. For arbitrary $a \in \mathbf{Move}(\Phi \multimap I, \Psi\,;\mathcal{O})$ we note that $a(\Phi \multimap I, \Psi) = (a\Phi, I, \Psi)$ is a basic position and there is a strategy:

$$\sigma(a) \in Str^w_{\mathcal{P}}(a\Phi, I, \Psi\,;\mathcal{P}),$$

we construct a strategy $\sigma;\tau \in Str^w_{\mathcal{P}}(\Phi \multimap I, J\,;\mathcal{O})$ as follows:

$$\dfrac{\dfrac{\sigma(a) \in Str^w_{\mathcal{P}}(a\Phi, I, \Psi\,;\mathcal{P}) \qquad \tau \in Str^w_{\mathcal{P}}(\Psi \multimap J\,;\mathcal{O})}{Ap(\sigma(a), \tau) \in Str^w_{\mathcal{P}}(a\Phi, I, J\,;\mathcal{P})}}{\lambda a.Ap(\sigma(a), \tau) \in Str^w_{\mathcal{P}}(\Phi \multimap I, J\,;\mathcal{O})}$$

□

3.4.8 Cut Elimination

Recall that the following cut rule is admissible in $G3C$:

$$\dfrac{\Gamma \Rightarrow \varphi, \Delta \qquad \varphi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}.$$

Similarly, for the strategies we have the following function:

Theorem 3.13. Let G and H and J be basic games, then there is a cut function:

$$Cut_\varphi : Str^w_{\mathcal{P}}(G, H, !_{\mathcal{O}}\varphi\,;\mathcal{P}) \times Str^w_{\mathcal{P}}(G, J, ?_{\mathcal{P}}\varphi\,;\mathcal{P}) \to Str^w_{\mathcal{P}}(G, H, J\,;\mathcal{P})$$

Proof. By cases:

- The assertion $!_{\mathcal{O}}\varphi$ is positive, then we construct a strategy in $Str^w_{\mathcal{P}}(G, H, J\,;\mathcal{P})$ as follows:

$$\dfrac{\dfrac{\dfrac{\sigma \in Str^w_{\mathcal{P}}(G, H, !_{\mathcal{O}}\varphi\,;\mathcal{P})}{Perm_\varphi(\sigma) \in Str^w_{\mathcal{P}}(G, H, ?_{\mathcal{O}}\varphi\,;\mathcal{O})} \qquad \tau \in Str^w_{\mathcal{P}}(G, J, ?_{\mathcal{P}}\varphi\,;\mathcal{P})}{Ap(Perm_\varphi(\sigma), \tau) \in Str^w_{\mathcal{P}}(G, G, H, J\,;\mathcal{P})}}{Con(Ap(Perm_\varphi(\sigma), \tau)) \in Str^w_{\mathcal{P}}(G, H, J\,;\mathcal{P})}$$

- The assertion $!_{\mathcal{O}}\varphi$ is negative, then we construct a strategy in $Str^w_{\mathcal{P}}(G, H, J\,;_{\mathcal{P}})$ as follows:

$$\dfrac{\dfrac{\dfrac{\sigma \in Str^w_{\mathcal{P}}(G, H,\, !_{\mathcal{O}}\varphi\,;_{\mathcal{P}})}{\lambda x.\sigma \in Str^w_{\mathcal{P}}(G, H,\, ?_{\mathcal{O}}\varphi\,;_{\mathcal{O}})} \qquad \tau \in Str^w_{\mathcal{P}}(G, J,\, ?_{\mathcal{P}}\varphi\,;_{\mathcal{P}})}{Ap(\lambda x.\sigma, \tau) \in Str^w_{\mathcal{P}}(G, G, H, J\,;_{\mathcal{P}})}}{Con(Ap(\lambda x.\sigma, \tau)) \in Str^w_{\mathcal{P}}(G, H, J\,;_{\mathcal{P}})}$$

□

3.5 Correspondence of Strategies and Proofs

We will now prove soundness and completeness for proponent winning strategies of basic games with regards to $G3C$, that is we will show

$$\Gamma \vdash_c \Delta \iff Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?_{\mathcal{P}}\Delta\,;_{\mathcal{P}}).$$

In fact we will actually show something stronger since we can derive functions f and g from the proof, such that:

$$f : Der(\Gamma \vdash_c \Delta) \to Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?_{\mathcal{P}}\Delta\,;_{\mathcal{P}})$$
$$g : Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?_{\mathcal{P}}\Delta\,;_{\mathcal{P}}) \to Der(\Gamma \vdash_c \Delta).$$

Where $Der(\Gamma \vdash_c \Delta)$ are the set of derivations of the sequent $\Gamma \Rightarrow \Delta$. Thus from a derivation we have a method for constructing a strategy, and from a strategy we have a method for constructing a derivation.

3.5.1 Soundness

Theorem 3.14 (Soundness)**.**

$$\text{If } Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?_{\mathcal{P}}\Delta\,;_{\mathcal{P}}) \neq \emptyset, \text{ then } \Gamma \vdash_c \Delta.$$

Proof. As inductive hypothesis we take the following:

1. If $Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?_{\mathcal{P}}\Delta\,;_{\mathcal{P}}) \neq \emptyset$, then $\Gamma \vdash_c \Delta$
2. If $Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?_{\mathcal{O}}\varphi,\, ?_{\mathcal{P}}\Delta\,;_{\mathcal{O}}) \neq \emptyset$, then $\Gamma, \varphi \vdash_c \Delta$
3. If $Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, !_{\mathcal{P}}\varphi,\, ?_{\mathcal{P}}\Delta\,;_{\mathcal{O}}) \neq \emptyset$, then $\Gamma \vdash_c \Delta, \varphi$

1. By induction on $\sigma \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?_{\mathcal{P}}\Delta\,;_{\mathcal{P}})$.
   - The strategy is $\sigma = e$, that is the state $(\,!_{\mathcal{O}}\Gamma,\, ?_{\mathcal{P}}\Delta\,;_{\mathcal{P}})$ is terminal, by cases:
     - $A \in \Gamma$ and $A \in \Delta$, then $\Gamma \vdash_c \Delta$.
     - $\bot \in \Gamma$, then $\Gamma \vdash_c \Delta$.
   - The strategy is $\sigma = (a, \sigma')$, the opening action a is a defense of φ, thus $\Delta = \Delta', \varphi$. By cases:

- The formula φ is negative, then $\sigma' \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, !_{\mathcal{P}}\varphi,\, ?_{\mathcal{P}}\Delta'\,;\mathcal{O})$, then by induction $\Gamma \vdash_c \Delta', \varphi$.
- The formula φ is positive, then by cases:
  * The formula is $\varphi = \alpha \vee \beta$, without loss of generality

    $$\sigma' \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, !_{\mathcal{P}}\alpha,\, ?_{\mathcal{P}}\Delta\,;\mathcal{O}),$$

    thus by induction $\Gamma \vdash_c \Delta, \alpha$ by weakening $\Gamma \vdash_c \Delta, \alpha, \beta$, thus $\Gamma \vdash_c \Delta, \alpha \vee \beta$, thus by contraction $\Gamma \vdash_c \Delta$.
  * The formula is $\varphi = \exists x \alpha(x)$, without loss of generality

    $$\sigma' \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, !_{\mathcal{P}}\alpha(t),\, ?_{\mathcal{P}}\Delta\,;\mathcal{O}),$$

    thus by induction $\Gamma \vdash_c \Delta, \alpha(t)$, thus $\Gamma \vdash_c \Delta, \exists x \alpha(x)$, thus by contraction $\Gamma \vdash_c \Delta$.
  * The formula is $\varphi = A$, but this is impossible since then already $!_{\mathcal{O}}A \in !_{\mathcal{O}}\Gamma$, thus the state $(\,!_{\mathcal{O}}\Gamma,\, ?_{\mathcal{P}}\Delta\,;\mathcal{P})$ would be terminal.

- The strategy is $\sigma = (a, \sigma')$, the opening action a is an attack of φ, thus $\Gamma = \Gamma', \varphi$. By cases:
  - The formula φ is positive, then $\sigma' \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma',\, ?_{\mathcal{O}}\varphi,\, ?_{\mathcal{P}}\Delta\,;\mathcal{O})$, thus by induction $\Gamma', \varphi \vdash_c \Delta$.
  - The formula φ is negative, then by cases:
    * The formula is $\varphi = A$, then $\sigma(a) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?_{\mathcal{P}}A,\, ?_{\mathcal{P}}\Delta\,;\mathcal{P})$, thus by IH $\Gamma \vdash_c \Delta, A$.
    * The formula is $\varphi = \alpha \wedge \beta$, without loss of generality

      $$\sigma' \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?_{\mathcal{O}}\alpha,\, ?_{\mathcal{P}}\Delta\,;\mathcal{O}),$$

      thus by induction we have that $\Gamma, \alpha \vdash_c \Delta$, by weakening $\Gamma, \alpha, \beta \vdash_c \Delta$, thus also $\Gamma, \alpha \wedge \beta \vdash_c \Delta$, thus by contraction $\Gamma \vdash_c \Delta$.
    * The formula is $\varphi = \alpha \to \beta$, then $\sigma' \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?_{\mathcal{O}}\beta,\, !_{\mathcal{P}}\alpha,\, ?_{\mathcal{P}}\Delta\,;\mathcal{O})$, thus:

      $$l(\sigma') \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?_{\mathcal{O}}\beta,\, ?_{\mathcal{P}}\Delta\,;\mathcal{O})$$
      $$r(\sigma') \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, !_{\mathcal{P}}\alpha,\, ?_{\mathcal{P}}\Delta\,;\mathcal{O})$$

      Thus by induction $\Gamma, \beta \vdash_c \Delta$ and $\Gamma \vdash_c \alpha, \Delta$, thus $\Gamma, \alpha \to \beta \vdash_c \Delta$, thus by contraction $\Gamma \vdash_c \Delta$.
    * The formula is $\varphi = \forall x \alpha(x)$, without loss of generality

      $$\sigma' \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?_{\mathcal{O}}\alpha(t),\, ?_{\mathcal{P}}\Delta\,;\mathcal{O}),$$

      then by induction $\Gamma, \alpha(t) \vdash_c \Delta$, then $\Gamma, \forall x \alpha(x) \vdash_c \Delta$, thus by contraction $\Gamma \vdash_c \Delta$.

2. By induction on $\sigma \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?_{\mathcal{O}}\varphi,\, ?_{\mathcal{P}}\Delta\,;{\scriptstyle\mathcal{O}})$, let a be the opponents opening action.

   - The formula φ is negative, then $\sigma(a) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, !_{\mathcal{O}}\varphi,\, ?_{\mathcal{P}}\Delta\,;{\scriptstyle\mathcal{P}})$, thus by induction $\Gamma, \varphi \vdash_c \Delta$.
   - The formula φ is positive, by cases:
     - The formula is $\varphi = \exists x\alpha(x)$, then

       $$\sigma(D_t(\varphi)) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, !_{\mathcal{O}}\alpha(t),\, ?_{\mathcal{P}}\Delta\,;{\scriptstyle\mathcal{P}}),$$

       for any term t, thus in particular by induction $\Gamma, \alpha(x) \vdash_c \Delta$ for some $x \notin FV(\Gamma \cup \Delta)$, thus $\Gamma, \exists x\alpha(x) \vdash_c \Delta$.
     - The formula is $\varphi = \alpha \vee \beta$, then the proponent has winning strategies

       $$\sigma(D_0(\varphi)) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, !_{\mathcal{O}}\alpha,\, ?_{\mathcal{P}}\Delta\,;{\scriptstyle\mathcal{P}})$$
       $$\sigma(D_1(\varphi)) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, !_{\mathcal{O}}\beta,\, ?_{\mathcal{P}}\Delta\,;{\scriptstyle\mathcal{P}})$$

       Thus by induction $\Gamma, \alpha \vdash_c \Delta$ and $\Gamma, \beta \vdash_c \Delta$, thus $\Gamma, \alpha \vee \beta \vdash_c \Delta$.

3. By induction on $\sigma \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, !_{\mathcal{P}}\varphi,\, ?_{\mathcal{P}}\Delta\,;{\scriptstyle\mathcal{O}})$, let a be the opponents opening action.

   - The formula φ is positive, then $\sigma(a) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?_{\mathcal{P}}\varphi,\, ?_{\mathcal{P}}\Delta\,;{\scriptstyle\mathcal{P}})$, thus by induction $\Gamma \vdash_c \Delta, \varphi$.
   - The formula φ is negative, then by cases:
     - The formula is $\varphi = \alpha \wedge \beta$, thus

       $$\sigma(A_0(\varphi)) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?_{\mathcal{P}}\alpha,\, ?_{\mathcal{P}}\Delta\,;{\scriptstyle\mathcal{P}})$$
       $$\sigma(A_1(\varphi)) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?_{\mathcal{P}}\beta,\, ?_{\mathcal{P}}\Delta\,;{\scriptstyle\mathcal{P}})$$

       Thus by induction $\Gamma \vdash_c \Delta, \alpha$ and $\Gamma \vdash_c \Delta, \beta$, thus $\Gamma \vdash_c \Delta, \alpha \wedge \beta$.
     - The formula is $\varphi = \alpha \to \beta$, then

       $$\sigma(A(\varphi)) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, !_{\mathcal{O}}\alpha,\, ?_{\mathcal{P}}\beta,\, ?_{\mathcal{P}}\Delta\,;{\scriptstyle\mathcal{P}}),$$

       thus by induction $\Gamma, \alpha \vdash_c \beta, \Delta$, thus $\Gamma \vdash_c \alpha \to \beta, \Delta$.
     - The formula is $\varphi = \forall x\alpha(x)$, then

       $$\sigma(A_t(\varphi)) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?_{\mathcal{P}}\alpha(t),\, ?_{\mathcal{P}}\Delta\,;{\scriptstyle\mathcal{P}}),$$

       for any term t, then in particular by induction $\Gamma \vdash_c \Delta, \alpha(x)$ for some $x \notin FV(\Gamma \cup \Delta)$, then $\Gamma \vdash_c \Delta, \forall x\alpha(x)$. $\square$

3.5.2 Completeness

Theorem 3.15 (Completeness)**.** Let $G = !_{\mathcal{O}}\Gamma,\ ?_{\mathcal{P}}\Delta$, then

$$\text{If } \Gamma \vdash_c \Delta, \text{ then } Str^w_{\mathcal{P}}(G\,;{\scriptstyle\mathcal{P}}) \neq \emptyset.$$

Proof. By induction on the height of the derivation of the sequent $\Gamma \Rightarrow \Delta$.

- (Basecase) $\Gamma \Rightarrow \Delta$ is an instance of an axiom, we have two cases:
  - We have $A \in \Gamma$ and $A \in \Delta$, then $(G\,;{\scriptstyle\mathcal{P}})$ is a terminal state, so
  $$Str^w_{\mathcal{P}}(G\,;{\scriptstyle\mathcal{P}}) \neq \emptyset.$$
  - We have $\bot \in \Gamma$, then $(G\,;{\scriptstyle\mathcal{P}})$ is a terminal state, so
  $$Str^w_{\mathcal{P}}(G\,;{\scriptstyle\mathcal{P}}) \neq \emptyset.$$
- The last rule used in the derivation is $\wedge_R$ thus $G = H,\ ?_{\mathcal{P}}\varphi \wedge \psi$, then by induction we have:
  $$\begin{gathered}\sigma \in Str^w_{\mathcal{P}}(H,\ ?_{\mathcal{P}}\varphi\,;{\scriptstyle\mathcal{P}})\\ \tau \in Str^w_{\mathcal{P}}(H,\ ?_{\mathcal{P}}\psi\,;{\scriptstyle\mathcal{P}}).\end{gathered}$$
  Let
  $$\begin{gathered}\rho(A_0(\varphi \wedge \psi)) = \sigma\\ \rho(A_1(\varphi \wedge \psi)) = \tau.\end{gathered}$$
  we construct a strategy in $Str^w_{\mathcal{P}}(G\,;{\scriptstyle\mathcal{P}})$ as follows:
  $$\frac{\rho \in Str^w_{\mathcal{P}}(H,\ !_{\mathcal{P}}\varphi \wedge \psi\,;{\scriptstyle\mathcal{O}})}{(D(\varphi \wedge \psi), \rho) \in Str^w_{\mathcal{P}}(G\,;{\scriptstyle\mathcal{P}})}\ .$$
- The last rule used in the derivation is $\wedge_L$, thus $G = H,\ !_{\mathcal{O}}\varphi \wedge \psi$, then by induction and weakening we have:
  $$\frac{\sigma \in Str^w_{\mathcal{P}}(H,\ !_{\mathcal{O}}\varphi,\ !_{\mathcal{O}}\psi\,;{\scriptstyle\mathcal{P}})}{Wk(\sigma) \in Str^w_{\mathcal{P}}(G,\ !_{\mathcal{O}}\varphi,\ !_{\mathcal{O}}\psi\,;{\scriptstyle\mathcal{P}})}$$
  We construct a strategy in $Str^w_{\mathcal{P}}(G\,;{\scriptstyle\mathcal{P}})$ as follows:

$$\dfrac{\dfrac{Wk(\sigma) \in Str^w_{\mathcal{P}}(G,\ !_{\mathcal{O}}\varphi,\ !_{\mathcal{O}}\psi\,;{\scriptstyle\mathcal{P}}) \qquad \dfrac{Id \in Str^w_{\mathcal{P}}(G,\ ?_{\mathcal{O}}\varphi,\ ?_{\mathcal{P}}\varphi\,;{\scriptstyle\mathcal{O}})}{(a, Id) \in Str^w_{\mathcal{P}}(G,\ ?_{\mathcal{P}}\varphi\,;{\scriptstyle\mathcal{P}})}}{Cut_\varphi(Wk(\sigma), (a, Id)) \in Str^w_{\mathcal{P}}(G,\ !_{\mathcal{O}}\psi\,;{\scriptstyle\mathcal{P}})} \qquad \dfrac{Id \in Str^w_{\mathcal{P}}(G,\ ?_{\mathcal{O}}\psi,\ ?_{\mathcal{P}}\psi\,;{\scriptstyle\mathcal{O}})}{(b, Id) \in Str^w_{\mathcal{P}}(G,\ ?_{\mathcal{P}}\psi\,;{\scriptstyle\mathcal{P}})}}{Cut_\psi((b, Id), Cut_\varphi(Wk(\sigma), (a, Id))) \in Str^w_{\mathcal{P}}(G\,;{\scriptstyle\mathcal{P}})}$$

  Where $a = A_0(\varphi \vee \psi)$ and $b = A_1(\varphi \vee \psi)$.

- The last rule used in the derivation is $\vee_R$, thus $G = H,\ ?_{\mathcal{P}}\varphi \vee \psi$, then by induction and weakening we have:

$$\frac{\sigma \in Str^w_{\mathcal{P}}(H,\ ?_{\mathcal{P}}\varphi,\ ?_{\mathcal{P}}\psi\,;_{\mathcal{P}})}{Wk(\sigma) \in Str^w_{\mathcal{P}}(G,\ ?_{\mathcal{P}}\varphi,\ ?_{\mathcal{P}}\psi\,;_{\mathcal{P}})}$$

  We construct a strategy in $Str^w_{\mathcal{P}}(G\,;_{\mathcal{P}})$ as follows:

$$\frac{\dfrac{Wk(\sigma) \in Str^w_{\mathcal{P}}(G,\ ?_{\mathcal{P}}\varphi,\ ?_{\mathcal{P}}\psi\,;_{\mathcal{P}}) \qquad \dfrac{Id \in Str^w_{\mathcal{P}}(G,\ !_{\mathcal{O}}\varphi,\ !_{\mathcal{P}}\varphi\,;_{\mathcal{O}})}{(a, Id) \in Str^w_{\mathcal{P}}(G,\ !_{\mathcal{O}}\varphi\,;_{\mathcal{P}})}}{Cut_{\varphi}(Wk(\sigma), (a, Id)) \in Str^w_{\mathcal{P}}(G,\ ?_{\mathcal{P}}\psi\,;_{\mathcal{P}})} \qquad \dfrac{Id \in Str^w_{\mathcal{P}}(G,\ !_{\mathcal{O}}\psi,\ !_{\mathcal{P}}\psi\,;_{\mathcal{O}})}{(b, Id) \in Str^w_{\mathcal{P}}(G,\ !_{\mathcal{O}}\psi\,;_{\mathcal{P}})}}{Cut_{\psi}(Cut_{\varphi}(Wk(\sigma), (a, Id)), (b, Id)) \in Str^w_{\mathcal{P}}(G\,;_{\mathcal{P}})}$$

  Where $a = D_0(\varphi \vee \psi)$ and $b = D_1(\varphi \vee \psi)$.

- The last rule used in the derivation is $\vee_L$, thus $G = H,\ !_{\mathcal{O}}\varphi \vee \psi$, then by induction there are strategies:

$$\sigma \in Str^w_{\mathcal{P}}(H,\ !_{\mathcal{O}}\varphi\,;_{\mathcal{P}})$$
$$\tau \in Str^w_{\mathcal{P}}(H,\ !_{\mathcal{O}}\psi\,;_{\mathcal{P}}).$$

  Let

$$\rho(D_0(\varphi \vee \psi)) = \sigma$$
$$\rho(D_1(\varphi \vee \psi)) = \tau.$$

  We construct a strategy in $Str^w_{\mathcal{P}}(G\,;_{\mathcal{P}})$ as follows:

$$\frac{\rho \in Str^w_{\mathcal{P}}(H,\ ?_{\mathcal{O}}\varphi \vee \psi\,;_{\mathcal{O}})}{(D(\varphi \vee \psi), \rho) \in Str^w_{\mathcal{P}}(G\,;_{\mathcal{P}})}\ .$$

- The last rule used in the derivation is $\to_R$, thus $G = H,\ ?_{\mathcal{P}}\varphi \to \psi$, then by induction there is a strategy

$$\sigma \in Str^w_{\mathcal{P}}(H,\ !_{\mathcal{O}}\varphi,\ ?_{\mathcal{P}}\psi\,;_{\mathcal{P}}).$$

  We construct a strategy in $Str^w_{\mathcal{P}}(G\,;_{\mathcal{P}})$ as follows:

$$\frac{\dfrac{\sigma \in Str^w_{\mathcal{P}}(H,\ !_{\mathcal{O}}\varphi,\ ?_{\mathcal{P}}\psi\,;_{\mathcal{P}})}{\lambda x.\sigma \in Str^w_{\mathcal{P}}(H,\ !_{\mathcal{P}}\varphi \to \psi)}}{(D(\varphi \to \psi), \lambda x.\sigma) \in Str^w_{\mathcal{P}}(G\,;_{\mathcal{P}})}\ .$$

- The last rule used in the derivation is $\to_L$, thus $G = H,\ !_{\mathcal{O}}\varphi \to \psi$, then by induction there are strategies

$$\sigma \in Str^w_{\mathcal{P}}(G,\ ?_{\mathcal{P}}\varphi\,;_{\mathcal{P}})$$
$$\tau \in Str^w_{\mathcal{P}}(G,\ !_{\mathcal{O}}\psi\,;_{\mathcal{P}}).$$

We construct a strategy in $Str^w_{\mathcal{P}}(G\,;{\mathcal{P}})$ as follows:

$$\dfrac{\dfrac{\sigma \in Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{P}}\varphi\,;{\mathcal{P}}) \qquad Mp \in Str^w_{\mathcal{P}}(G,\, !_{\mathcal{O}}\varphi,\, ?_{\mathcal{P}}\psi\,;{\mathcal{P}})}{Cut_\varphi(Mp,\sigma) \in Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{P}}\psi\,;{\mathcal{P}})} \qquad \tau \in Str^w_{\mathcal{P}}(G,\, !_{\mathcal{O}}\psi\,;{\mathcal{P}})}{Cut_\psi(\tau, Cut_\varphi(Mp,\sigma)) \in Str^w_{\mathcal{P}}(G\,;{\mathcal{P}})}$$

- The last rule used in the derivation is $\exists_R$, thus $G = H,\, ?_{\mathcal{P}}\exists x\varphi(x)$, then by induction there is a strategy

$$\sigma \in Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{P}}\varphi(t)\,;{\mathcal{P}}).$$

  We construct a strategy in $Str^w_{\mathcal{P}}(G\,;{\mathcal{P}})$ as follows:

$$\dfrac{\sigma \in Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{P}}\varphi(t)\,;{\mathcal{P}}) \qquad \dfrac{Id \in Str^w_{\mathcal{P}}(G,\, !_{\mathcal{P}}\varphi(t),\, !_{\mathcal{O}}\varphi(t)\,;{\mathcal{O}})}{(a, Id) \in Str^w_{\mathcal{P}}(G,\, !_{\mathcal{O}}\varphi(t)\,;{\mathcal{P}})}}{Cut_\varphi((a, Id), \sigma) \in Str^w_{\mathcal{P}}(G\,;{\mathcal{P}})}$$

  Where $a = D_t(\exists x\varphi(x)$, the case for $\forall_L$ is similar.

- The last rule used in the derivation is $\forall_R$, thus $G = H,\, ?_{\mathcal{P}}\forall x\varphi(x)$, by theorem 2.5, we have that $\Gamma \vdash_c \varphi(t), \Delta$ for any term t, thus by induction for any term t there is a strategy:

$$\sigma_t \in Str^w_{\mathcal{P}}(H,\, ?_{\mathcal{P}}\varphi(t)\,;{\mathcal{P}}).$$

  Let $\rho(A_t(\forall x\varphi(x))) = \sigma_t$. We construct a strategy in $Str^w_{\mathcal{P}}(G\,;{\mathcal{P}})$ as follows:

$$\dfrac{\lambda x.\rho(x) \in Str^w_{\mathcal{P}}(H,\, !_{\mathcal{P}}\forall x\varphi(x)\,;{\mathcal{O}})}{(D(\forall x\varphi(x)), (\lambda x.\rho(x))) \in Str^w_{\mathcal{P}}(G\,;{\mathcal{P}})}$$

  The case for $\exists_L$ is similar. $\square$

3.5.3 Adequacy for the System G3C*

The soundness and completeness proofs identify winning strategies with proofs in $G3C$. In fact we can do a little better than that since if we inspect the rules of the game we see that in a winning strategy:

- The assertion $!_{\mathcal{O}}A \to \psi$ is never attacked, unless the opponent has already stated $!_{\mathcal{O}}A$.
- The left side of challenge $?_{\mathcal{P}}A \vee \psi$ is never defended unless the opponent has already stated $!_{\mathcal{O}}A$.
- The right side of challenge $?_{\mathcal{P}}\psi \vee A$ is never defended unless the opponent has already stated $!_{\mathcal{O}}A$.
- The challenge $?_{\mathcal{P}}\exists xA(x)$ is never defended unless the opponent has already stated $!_{\mathcal{O}}A(t)$ for some term t.

Thus, we can actually define a proof system $G3C^*$, to which the winning strategies corresponds more closely.

Definition 3.8. Let $G3C^*$ be just as $G3C$ except that we replace the rule $\to_L$ with two rules, one atomic and one non-atomic:

$$\frac{\Gamma, A, \psi \Rightarrow \Delta}{\Gamma, A, A \to \psi \Rightarrow \Delta} \to_{At} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma, \psi \Rightarrow \Delta \qquad \varphi \text{ non-atomic}}{\Gamma, \varphi \to \psi \Rightarrow \Delta} \to_L$$

We could also similarly replace $\vee_R$ and $\exists_R$, however this is not as interesting of a change as the above. If $\Gamma \Rightarrow \Delta$ is derivable in $G3C^*$ we write $\Gamma \vdash_c^* \Delta$. In particular the rule $\to_{At}$ is interesting since it shrinks the search space for proofs by putting a restriction on how derivations may be produced.

Theorem 3.16.

$$\Gamma \vdash_c^* \Delta \iff \Gamma \vdash_c \Delta$$

Proof. For the left to right direction we note that all rules of $G3C^*$ are admissible in $G3C$, in particular the rule $\to_{At}$ corresponds to:

$$\frac{\Gamma, A \Rightarrow A, \Delta \qquad \Gamma, A, \psi \Rightarrow \Delta}{\Gamma, A, A \to \psi \Rightarrow \Delta} \quad .$$

For right to left, if $\Gamma \vdash_c \Delta$, then by completeness $Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ ?_{\mathcal{P}}\Delta\,;{\scriptstyle \mathcal{P}}) \neq \emptyset$ it is then immediate that $\Gamma \vdash_c^* \Delta$. □

3.5.4 Cut Elimination for G3C and G3C*

Given that we have the cut strategy we have the following:

$$\frac{\sigma \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ ?_{\mathcal{P}}\Delta,\ !_{\mathcal{O}}\varphi\,;{\scriptstyle \mathcal{P}}) \qquad \tau \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma',\ ?_{\mathcal{P}}\Delta',\ ?_{\mathcal{P}}\varphi\,;{\scriptstyle \mathcal{P}})}{Cut_\varphi(\sigma,\tau) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ ?_{\mathcal{P}}\Delta,\ !_{\mathcal{O}}\Gamma',\ ?_{\mathcal{P}}\Delta'\,;{\scriptstyle \mathcal{P}})} \quad .$$

We immediately get that the cut rule

$$(Cut) \qquad \frac{\Gamma \Rightarrow \varphi \qquad \varphi, \Gamma' \Rightarrow \psi}{\Gamma, \Gamma' \Rightarrow \psi}$$

is admissible in $G3C$ and $G3C^*$. Furthermore, if we allow $G3C$ or $G3C^*$ derivations containing applications of the cut-rule, we can add the following case to the completeness proof:

- The last rule in the derivation of $\Gamma \vdash_c \Delta$ was cut with cut-formula φ. By induction we have strategies:

  $$\sigma \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma_0,\ !_{\mathcal{O}}\varphi,\ ?_{\mathcal{P}}\Delta_0\,;{\scriptstyle \mathcal{P}})$$
  $$\tau \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma_1,\ ?_{\mathcal{P}}\varphi,\ ?_{\mathcal{P}}\Delta_1\,;{\scriptstyle \mathcal{P}}).$$

  Where $\Gamma = \Gamma_0, \Gamma_1$ and $\Delta = \Delta_0, \Delta_1$. Thus, we have a strategy

  $$Cut_\varphi(\sigma,\tau) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ ?_{\mathcal{P}}\Delta\,;{\scriptstyle \mathcal{P}}).$$

Using the soundness part of the proof this strategy can then be transferred back to a $G3C$ or $G3C^*$ derivation not containing cut. Thus effectively eliminating the cut from the proof.

4 Intuitionistic Games

To turn the games intuitionistic we introduce three changes:

- No backtracking on disjunction or the existential quantifier.
- Last question answered first.
- All challenges must be met for the proponent to win. (Unless the opponent asserts falsum).

To enforce that the last challenge is answered first we extend the game language by indexing all challenges by a natural number.

Definition 4.1 (Game Language)**.**

$$\mathcal{L}^{Game} = \{\, !_{\circ}\varphi \mid \varphi \in \mathcal{L}, \circ \in Players\} \cup \{\, ?^{n}_{\circ}\varphi \mid \varphi \in \mathcal{L}, \circ \in Players, n \in \mathbb{N}\}$$

The sole purpose of the number is the demarcate in which order the challenges where introduces.

4.1 Intuitionistic Positions

A *position* in a intuitionistic game is then a finite multi-set of $\mathcal{L}^{Game}$.

Definition 4.2 (Set of Positions)**.**

$$\mathbf{Pos} = \{G \subseteq \mathcal{L}^{Game} \mid G \text{ is finite}\}$$

Consequently we define the operations on intuitionistic positions similarly to the classical case.

4.2 Intuitionistic Rules

Definition 4.3 (Intuitionistic Ruleset)**.** The intuitionistic games are defined given the intuitionistic ruleset $(\mathbf{States}, Act, M, Term)$, where

- $\mathbf{States} = \mathbf{Pos} \times \{\mathcal{O}, \mathcal{P}\}$.
- The set of actions Act is defined using the following grammar:

  $$a ::= D(\varphi) \mid D_i(\varphi) \mid D_t(\varphi) \mid A_t(\varphi) \mid A_i(\varphi) \mid A(\varphi).$$

  Where $i \in \{0, 1\}$, t is an arbitrary term and φ is an arbitrary formula in $\mathcal{L}$.
- The transition relations $M \subseteq \mathbf{States} \times Act \times \mathbf{States}$ is defined given the following table:

$$\frac{(G,\, ?^n_\circ \varphi\,;\circ) \qquad !_\circ\varphi \text{ is negative}}{(G,\, !_\circ\varphi\,;\bullet)}\, D(\varphi) \qquad \frac{(G,\, !_\bullet \varphi\,;\circ) \qquad !_\bullet\varphi \text{ is positive}}{(G,\, ?^{n+1}_\bullet\varphi\,;\bullet)}\, A(\varphi)$$

$$(*) \quad \frac{(G,\, !_\bullet \varphi_0 \wedge \varphi_1\,;\circ)}{(G,\, ?_\bullet \varphi_i\,;\bullet)}\, A_i(\varphi_0 \wedge \varphi_1) \qquad \frac{(G,\, ?^n_\circ \varphi_0 \vee \varphi_1\,;\circ)}{(G,\, !_\circ \varphi_i\,;\bullet)}\, D_i(\varphi_0 \vee \varphi_1)$$

$$(*) \quad \frac{(G,\, !_\bullet \forall x\varphi(x)\,;\circ)}{(G,\, ?_\bullet \varphi(t)\,;\bullet)}\, A_t(\forall x\varphi(x)) \qquad \frac{(G,\, ?^n_\circ \exists x\varphi(x)\,;\circ)}{(G,\, !_\circ \varphi(t)\,;\bullet)}\, D_t(\exists x\varphi(x))$$

$$(*) \quad \frac{(G,\, !_\bullet \varphi \to \psi\,;\circ)}{(G,\, !_\circ\varphi,\, ?^{n+1}_\bullet \psi\,;\bullet)}\, A(\varphi \to \psi) \qquad (*) \quad \frac{(G,\, !_\mathcal{O} A,\, ?^n_\mathcal{P} A\,;\mathcal{P})}{(G,\, !_\mathcal{O} A\,;\mathcal{O})}\, D(A)$$

Side-conditions:

- Moves marked with $(*)$ are such that when the proponent is the active player the active formula is repeated and not cancelled.
- $?^n_\circ\varphi$ is the latest challenge to player $\circ$, that is $n \geq i$ for all $?^i_\circ\psi \in G$.

- The set of terminal states $Term = Term_\mathcal{O} \cup Term_\mathcal{P}$ is inductively defined as follows, where G is an arbitrary position.
  - $(G\,;\mathcal{P}) \in Term_\mathcal{P}$, where
    - **Challenges**$(G\,;\mathcal{P}) \neq 0$.
    - All challenges $?^n_\mathcal{P}\varphi \in G$ are atomic.
    - For all $?^n_\mathcal{P}\varphi \in G$ there is a $!_\mathcal{O}\varphi \in G$.
  - $(G,\, !_\mathcal{P} A\,;\mathcal{O}) \in Term_\mathcal{O}$, where $!_\mathcal{O} A \notin G$.
  - $(G,\, !_\bullet \bot\,;\circ) \in Term_\circ$.
  - $(\emptyset\,;\circ) \in Term_\bullet$.

4.3 Basic Positions

As with the classical games we are in interested in games of a particular form, in this case $(\,!_\mathcal{O}\Gamma,\, ?^n_\mathcal{P}\psi\,;\mathcal{P})$, since proponent winning strategies in these games correspond to derivations in the sequent calculus:

$$Str^w_\mathcal{P}(\,!_\mathcal{O}\Gamma,\, ?^n_\mathcal{P}\psi\,;\mathcal{P}) \neq \emptyset \iff \Gamma \vdash_i \psi.$$

We call positions such as the above *basic intuitionistic states.*

Definition 4.4. A basic intuitionistic position is a position of the form $(\,!_\mathcal{O}\Gamma,\, ?^n_\mathcal{P}\psi)$, for some finite multisets of formulas Γ and a formula ψ and a number $n \in \mathbb{N}$.

In this section we will use “basic position” to refer to a “basic intuitionistic position”. We define the positions G^d and $G \multimap H$ as in the classical case. A pre-basic position is then defined in the same way as in the classical case. The following properties now holds for basic and pre-basic positions:

- The opponent has no legal moves in a basic position.
- The opponent has one legal move in pre-basic positions.
- The pre-basic positions have two forms:
  - $?^n_{\mathcal{P}}\varphi \multimap G$, where G is basic.
  - $!_{\mathcal{O}}\varphi \multimap !_{\mathcal{O}}\Gamma$.

  That is $\Phi \multimap G$ is no longer necessary a pre-basic position if G is basic and Φ is pre-basic, consider for example $(\,!_{\mathcal{P}}\varphi \vee \psi,\ ?^n_{\mathcal{P}}\alpha\,;\mathcal{O})$ with the successor state $(\,?^m_{\mathcal{P}}\varphi \vee \psi,\ ?^n_{\mathcal{P}}\alpha\,;\mathcal{P})$.
- Given a basic game G and a proponent move $(G\,;\mathcal{O}) \xrightarrow{a} (aG\,;\mathcal{P})$ such that $a \neq A(\varphi \to \psi)$, we have that $aG = \Phi \multimap H$ for some pre-basic $\Phi \multimap H$.
- Given a basic game G and a proponent move $(G\,;\mathcal{O}) \xrightarrow{a} (aG\,;\mathcal{P})$ such that $a = A(\varphi \to \psi)$, we have that $aG = \Phi, \Phi' \multimap G$ for some basic Φ and Φ.

Thus, in contrast to the classical positions: Even if Φ and G are basic $\Phi \multimap G$ is not necessarily pre-basic. We must therefore take better care in considering pre-basic positions.

4.3.1 No Stalemate

Given our definition of a winning strategy, the opponent may win a game $(G\,;\mathcal{P})$, where G is basic by reaching a state $(H\,;\mathcal{O})$ for which there is no further move. As in the classical case we do not want this to be able to happen, so we begin by showing the no stalemate lemma as in the classical case. This is somewhat harder in the intuitionistic case because in a game the proponent may defend an atomic formula P, leading to a state $(G,\ !_{\mathcal{P}}P\,;\mathcal{O})$ where it's not obvious that the opponent has an additional move. This was prevented in the classical case, since the preceding state $(G,\ ?^n_{\mathcal{P}}P\,;\mathcal{P})$ was already terminal. Now in a intuitionistic game this is no longer the case if there is some remaining challenge $?^m_{\mathcal{P}}\varphi \in G$. Now we claim that in fact:

$$\mathbf{Challenges}(G\,;\mathcal{O}) = \mathbf{Challenges}(G\,;\mathcal{P}),$$

where $\mathbf{Challenges}(G, \circ)$ is the number of $\circ$-player challenges in G. Thus the opponent must have a defensive move in $(G\,;\mathcal{O})$. Thus we begin by proving the following lemma:

Lemma 4.1. If $(G, \circ)$ is a non-terminal state in a basic game $(\,!_{\mathcal{O}}\Gamma,\ ?^n_{\mathcal{P}}\psi\,;\mathcal{P})$ reachable in k steps, then

$$\text{If } k \text{ is even, then } \mathbf{Challenges}(G\,;\mathcal{O}) + 1 = \mathbf{Challenges}(G\,;\mathcal{P})$$
$$\text{If } k \text{ is odd, then } \mathbf{Challenges}(G\,;\mathcal{O}) = \mathbf{Challenges}(G\,;\mathcal{P}).$$

Proof. By induction on $k \in \mathbb{N}$.

- (Basecase) $k = 0$. For the initial state $(\,?_{\mathcal{O}}\Gamma,\ ?^n_{\mathcal{P}}\psi\,;\mathcal{P})$ we have immediately

$$\mathbf{Challenges}(G\,;\mathcal{O}) + 1 = 1 = \mathbf{Challenges}(G\,;\mathcal{P})$$

- (Inductive case) $k > 0$ is even. Suppose the preceding state is $(H, \mathcal{O})$, by induction $\mathbf{Challenges}(H\,;\mathcal{O}) = \mathbf{Challenges}(H\,;\mathcal{P})$, we show that

$$\mathbf{Challenges}(G\,;\mathcal{O}) + 1 = \mathbf{Challenges}(G\,;\mathcal{P}),$$

  by cases consider the move $(H\,;\mathcal{O}) \xrightarrow{a} (G\,;\mathcal{P})$:

  - a is a attack move, thus
    1. $\mathbf{Challenges}(H\,;\mathcal{P}) + 1 = \mathbf{Challenges}(G\,;\mathcal{P})$.
    2. $\mathbf{Challenges}(H\,;\mathcal{O}) = \mathbf{Challenges}(G\,;\mathcal{O})$.

    thus

$$\begin{aligned}\mathbf{Challenges}(G\,;\mathcal{O}) + 1 &= \mathbf{Challenges}(H\,;\mathcal{O}) + 1 && \text{by (2)}\\ &= \mathbf{Challenges}(H\,;\mathcal{P}) + 1 && \text{by (induction)}\\ &= \mathbf{Challenges}(G\,;\mathcal{P}) && \text{by (1).}\end{aligned}$$

  - If a is a defense move, thus
    1. $\mathbf{Challenges}(H\,;\mathcal{P}) = \mathbf{Challenges}(G\,;\mathcal{P})$.
    2. $\mathbf{Challenges}(H\,;\mathcal{O}) = \mathbf{Challenges}(G\,;\mathcal{O}) + 1$.

    thus

$$\begin{aligned}\mathbf{Challenges}(G\,;\mathcal{O}) + 1 &= \mathbf{Challenges}(H\,;\mathcal{O}) && \text{by (1)}\\ &= \mathbf{Challenges}(H\,;\mathcal{P}) && \text{by (induction)}\\ &= \mathbf{Challenges}(G\,;\mathcal{P}) && \text{by (2).}\end{aligned}$$

- (Inductive case) $k > 0$ is odd. Suppose the preceding state is $(H, \mathcal{P})$, by induction $\mathbf{Challenges}(H\,;\mathcal{O}) + 1 = \mathbf{Challenges}(H\,;\mathcal{P})$, we show that

$$\mathbf{Challenges}(G\,;\mathcal{O}) = \mathbf{Challenges}(G\,;\mathcal{P}),$$

  by cases consider the move $(H\,;\mathcal{P}) \xrightarrow{a} (G\,;\mathcal{O})$:

  - If a is an attack move then,
    1. $\mathbf{Challenges}(H\,;\mathcal{O}) + 1 = \mathbf{Challenges}(G\,;\mathcal{O})$.
    2. $\mathbf{Challenges}(H\,;\mathcal{P}) = \mathbf{Challenges}(G\,;\mathcal{P})$.

    thus

$$\begin{aligned}\mathbf{Challenges}(G\,;\mathcal{O}) &= \mathbf{Challenges}(H\,;\mathcal{O}) + 1 && \text{by (1)}\\ &= \mathbf{Challenges}(H\,;\mathcal{P}) && \text{by (induction)}\\ &= \mathbf{Challenges}(G\,;\mathcal{P}) && \text{by (2).}\end{aligned}$$

– If a is a defense move then,

1. $\mathbf{Challenges}(H\,;\mathcal{P}) = \mathbf{Challenges}(G\,;\mathcal{P}) + 1$.
2. $\mathbf{Challenges}(H\,;\mathcal{O}) = \mathbf{Challenges}(G\,;\mathcal{O})$.

thus

$$\begin{aligned}\mathbf{Challenges}(G\,;\mathcal{O}) &= \mathbf{Challenges}(H\,;\mathcal{O}) && \text{by (1)}\\ &= \mathbf{Challenges}(H\,;\mathcal{P}) - 1 && \text{by (induction)}\\ &= \mathbf{Challenges}(G\,;\mathcal{P}) && \text{by (2).}\end{aligned}$$

□

Theorem 4.2 (No Stalemate)**.** If $(H, \mathcal{O})$ is a reachable non-terminal state in a basic game $(G\,;\mathcal{P})$, then there exists a move

$$(H\,;\mathcal{O}) \xrightarrow{a} (aH\,;\mathcal{P}),$$

for some action $a \in Act$.

Proof. Consider the move $(J\,;\mathcal{P}) \xrightarrow{a} (H\,;\mathcal{O})$ on the state preceding $(H, \mathcal{O})$, by cases

- The action a is an attack, then the opponent has a defense move in $(H, \mathcal{O})$.
- The action a is a defense of a formula φ, by cases
  - The assertion resulting from the defense is non-atomic, then there is an attack move for the opponent in $(H, \mathcal{O})$.
  - The assertion resulting from the defense is $!_{\mathcal{P}}\bot$, but this is impossible since $(H\,;\mathcal{O})$ is non-terminal.
  - The assertion resulting from the defense is $!_{\mathcal{P}}P$, now since $(J\,;\mathcal{P})$ is non-terminal we have $\mathbf{Challenges}(J\,;\mathcal{P}) > 1$, thus we also have that $\mathbf{Challenges}(H\,;\mathcal{P}) = \mathbf{Challenges}(H\,;\mathcal{O}) \geq 1$, thus the opponent has a defense move in $(H, \mathcal{O})$. □

Corollary 4.2.1. If $\sigma \in Str_{\mathcal{P}}(G\,;\mathcal{O})$ is a non-winning strategy where G is pre-basic, then there is a branch b of σ that is either infinite or ends in a terminal state $(H\,;\mathcal{O})$ or a stalemate state $(H\,;\mathcal{P})$.

Proof. The proof is the same as in the classical case. □

4.4 Operations on Strategies

Since the opponent has no legal moves in a position $!_{\mathcal{O}}\Gamma$ adding such a position to an already winning position changes nothing with regards to winnability for the proponent.

Lemma 4.3 (Weakening)**.** Let G be any position and $\Gamma \subset \mathcal{L}$ a finite multiset, then there is a function

1. $Wk : Str^w_{\mathcal{P}}(G\,;{\scriptstyle\mathcal{P}}) \to Str^w_{\mathcal{P}}(G,\,!_{\mathcal{O}}\Gamma\,;{\scriptstyle\mathcal{P}})$
2. $Wk : Str^w_{\mathcal{P}}(G\,;{\scriptstyle\mathcal{O}}) \to Str^w_{\mathcal{P}}(G,\,!_{\mathcal{O}}\Gamma\,;{\scriptstyle\mathcal{O}})$

where

1. $Wk(\sigma) = \begin{cases} e \text{ if } (G,\,!_{\mathcal{O}}\Gamma\,;{\scriptstyle\mathcal{P}}) \in Term_{\mathcal{P}} \\ \sigma \text{ else.} \end{cases}$
2. $Wk(\sigma) = \sigma$.

4.4.1 The Copy-cat Strategy

For any given position G the proponent has a winning strategy called the copy-cat or identity strategy in the game $(G \multimap G\,;{\scriptstyle\mathcal{O}})$ where the proponent repeats the opponents actions. To define this strategy we must again make a slight adjustment to the classical definition.

Definition 4.5 (The Copy-cat Strategy)**.** If G is any position there is a function

$$Id \in Str^w_{\mathcal{P}}(G \multimap G\,;{\scriptstyle\mathcal{O}}),$$

Proof. For any position G and finite $\Gamma \subseteq \mathcal{L}$, we show there is a function:

$$Id \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma, G, G^d\,;{\scriptstyle\mathcal{O}}).$$

Consider arbitrary $a \in \mathbf{Move}(\,!_{\mathcal{O}}\Gamma, G, G^d\,;{\scriptstyle\mathcal{O}})$. We define

$$Id(a) \in Str^w_{\mathcal{P}}(a(G, G^d,\,!_{\mathcal{O}}\Gamma)\,;{\scriptstyle\mathcal{P}})$$

by induction on $Str^w_{\mathcal{P}}(a(G, G^d,\,!_{\mathcal{O}}\Gamma)\,;{\scriptstyle\mathcal{P}})$.

- The state $a(G, G^d,\,!_{\mathcal{O}}\Gamma\,;{\scriptstyle\mathcal{O}})$ is terminal, then let $Id(a) = e$.
- The action $a \neq A(A)$ for some atomic proposition A, in which case:

  $$aa(G, G^d,\,!_{\mathcal{O}}\Gamma\,;{\scriptstyle\mathcal{O}}) = (H, H^d,\,!_{\mathcal{O}}\Gamma'\,;{\scriptstyle\mathcal{O}}),$$

  for some position H and $\Gamma' \subseteq \mathcal{L}$.

  Thus we construct $Id(a) \in Str^w_{\mathcal{P}}(a(G, G^d,\,!_{\mathcal{O}}\Gamma\,;{\scriptstyle\mathcal{O}}))$ as follows:

  $$Id \in Str^w_{\mathcal{P}}(aa(G, G^d,\,!_{\mathcal{O}}\Gamma\,;{\scriptstyle\mathcal{O}}))$$
  $$(a, Id) \in Str^w_{\mathcal{P}}(a(G, G^d,\,!_{\mathcal{O}}\Gamma\,;{\scriptstyle\mathcal{O}})).$$

- The action $a = A(A)$ for some atomic proposition A, thus

  $$a(G, G^d,\,!_{\mathcal{O}}\Gamma\,;{\scriptstyle\mathcal{O}}) = (H, H^d,\,?_{\mathcal{P}}A,\,!_{\mathcal{O}}A,\,!_{\mathcal{O}}\Gamma\,;{\scriptstyle\mathcal{P}}).$$

  Thus we construct $Id(a) \in Str^w_{\mathcal{P}}(a(G, G^d,\,!_{\mathcal{O}}\Gamma\,;{\scriptstyle\mathcal{O}}))$ as follows:

  $$Id \in Str^w_{\mathcal{P}}(H, H^d,\,!_{\mathcal{O}}\Gamma\,;{\scriptstyle\mathcal{O}})$$
  $$(D(A), Id) \in Str^w_{\mathcal{P}}(a(G, G^d,\,!_{\mathcal{O}}\Gamma\,;{\scriptstyle\mathcal{O}})).$$

∎

Using the copy-cat strategy we can define a *modus ponens* strategy as follows:

Theorem 4.4 (Modus Ponens)**.** If $\Gamma \subseteq \mathcal{L}$ is a finite multiset then there is a strategy
$$Mp \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, !_{\mathcal{O}}\varphi \to \psi,\, !_{\mathcal{O}}\varphi,\, ?_{\mathcal{P}}\psi\,;{\mathcal{P}}).$$

Proof. We construct it as follows:

$$\begin{aligned}
Id &\in Str^w_{\mathcal{P}}(\,!_{\mathcal{P}}\varphi,\, ?_{\mathcal{O}}\psi,\, !_{\mathcal{O}}\varphi,\, ?_{\mathcal{P}}\psi\,;{\mathcal{O}})\\
Wk(Id) &\in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\varphi \to \psi,\, !_{\mathcal{P}}\varphi,\, ?_{\mathcal{O}}\psi,\, !_{\mathcal{O}}\varphi,\, ?_{\mathcal{P}}\psi\,;{\mathcal{O}})\\
(A(\varphi \to \psi), Wk(Id)) &\in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\varphi \to \psi,\, !_{\mathcal{O}}\varphi,\, ?_{\mathcal{P}}\psi\,;{\mathcal{P}})\\
Wk(A(\varphi \to \psi), Wk(Id)) &\in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, !_{\mathcal{O}}\varphi \to \psi,\, !_{\mathcal{O}}\varphi,\, ?_{\mathcal{P}}\psi\,;{\mathcal{P}})
\end{aligned}$$ ∎

4.4.2 Left and Right Strategies

Using No Stalemate we are now able to show that we can decompose a game $(G,\, ?^n_{\mathcal{O}}\psi,\, !_{\mathcal{P}}\varphi,\, ?^m_{\mathcal{P}}\alpha\,;{\mathcal{O}})$ into two components using the left and right strategies.

Lemma 4.5. Let $(G,\, ?^m_{\mathcal{P}}\alpha)$ be a basic position, then there are functions

1. $l : Str^w_{\mathcal{P}}(G,\, ?^n_{\mathcal{O}}\psi,\, !_{\mathcal{P}}\varphi,\, ?^m_{\mathcal{P}}\alpha\,;{\mathcal{O}}) \to Str^w_{\mathcal{P}}(G,\, ?^n_{\mathcal{O}}\psi,\, ?^m_{\mathcal{P}}\alpha\,;{\mathcal{O}})$
2. $r : Str^w_{\mathcal{P}}(G,\, ?^n_{\mathcal{O}}\psi,\, !_{\mathcal{P}}\varphi,\, ?^m_{\mathcal{P}}\alpha\,;{\mathcal{O}}) \to Str^w_{\mathcal{P}}(G,\, !_{\mathcal{P}}\varphi\,;{\mathcal{O}})$

such that for all $\sigma \in Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{O}}\psi,\, !_{\mathcal{P}}\varphi\,;{\mathcal{O}})$, it holds that $l(\sigma) \leq_T \sigma$ and $r(\sigma) \leq_T \sigma$.

Proof.

1. Assume we have a winning strategy $\sigma \in Str^w_{\mathcal{P}}(G,\, ?^n_{\mathcal{O}}\psi,\, !_{\mathcal{P}}\varphi,\, ?^m_{\mathcal{P}}\alpha\,;{\mathcal{O}})$. Let $X = \bigcup Str^w_{\mathcal{P}}(G,\, ?^n_{\mathcal{O}}\psi,\, ?^m_{\mathcal{P}}\alpha\,;{\mathcal{O}})$ be all possible sequents of moves in the game $(G,\, ?^n_{\mathcal{O}}\psi,\, ?^m_{\mathcal{P}}\alpha\,;{\mathcal{O}})$, then let $l(\sigma) = \sigma \cap X$ be the restriction of the strategy σ to the game $(G,\, ?^n_{\mathcal{O}}\psi,\, ?^m_{\mathcal{P}}\alpha\,;{\mathcal{O}})$. Then we have that $l(\sigma)$ is a strategy $l(\sigma) \in Str_{\mathcal{P}}(G,\, ?^n_{\mathcal{O}}\psi,\, ?^m_{\mathcal{P}}\alpha\,;{\mathcal{O}})$, since it's just a restriction of σ to the plays where the opponent never attacks $!_{\mathcal{P}}\varphi$. Also it's a winning strategy:
$$l(\sigma) \in Str^w_{\mathcal{P}}(G,\, ?^n_{\mathcal{O}}\psi,\, ?^m_{\mathcal{P}}\alpha\,;{\mathcal{O}}).$$
   Since assume that $l(\sigma)$ is not winning, then since $(G,\, ?^n_{\mathcal{O}}\psi,\, ?^m_{\mathcal{P}}\alpha\,;{\mathcal{O}})$ is pre-basic by a corollary of No Stalemate if $l(\sigma)$ is not winning there is an branch b of $l(\sigma)$ that is infinite or ends in a terminal state $(H\,;{\mathcal{O}})$ or a stalemate state $(H\,;{\mathcal{P}})$. Now since $l(\sigma) \subseteq \sigma$ this branch is also in σ, thus σ would not be a winning strategy which is a contradiction. Also, since $l(\sigma) \subseteq \sigma$ we have that $l(\sigma) \leq \sigma$.

2. Assume we have a winning strategy $\sigma \in Str^w_{\mathcal{P}}(G,\, ?^n_{\mathcal{O}}\psi,\, !_{\mathcal{P}}\varphi,\, ?^m_{\mathcal{P}}\alpha\,;\mathcal{O})$. Let $X = \bigcup Str^w_{\mathcal{P}}(G,\, !_{\mathcal{P}}\varphi,\, ?^m_{\mathcal{P}}\alpha\,;\mathcal{O})$ be all possible sequents of moves in the game $(G,\, !_{\mathcal{P}}\varphi,\, ?^m_{\mathcal{P}}\alpha\,;\mathcal{O})$, then let $l(\sigma) = \sigma \cap X$ be the restriction of the strategy σ to the game $(G,\, !_{\mathcal{P}}\varphi,\, ?^m_{\mathcal{P}}\alpha\,;\mathcal{O})$. Then we have that $l(\sigma)$ is a strategy $l(\sigma) \in Str_{\mathcal{P}}(G,\, !_{\mathcal{P}}\varphi,\, ?^m_{\mathcal{P}}\alpha\,;\mathcal{O})$, since it's just a restriction of σ to the plays where the opponent never defends $?^n_{\mathcal{O}}\psi$. Now we also get that

$$l(\sigma) \in Str_{\mathcal{P}}(G,\, !_{\mathcal{P}}\varphi\,;\mathcal{O}).$$

Since $Str_{\mathcal{P}}(G,\, !_{\mathcal{P}}\varphi\,;\mathcal{O}) = Str_{\mathcal{P}}(G,\, !_{\mathcal{P}}\varphi,\, ?^m_{\mathcal{P}}\alpha\,;\mathcal{O})$, since the proponent can never defend $?^m_{\mathcal{P}}\alpha$. Since suppose the proponent reached a state $(H,\, ?^m_{\mathcal{P}}\alpha\,;\mathcal{P})$ where $?^m_{\mathcal{P}}\alpha$ could be defended, then by definition $\mathbf{Challenges}(H\,;\mathcal{P}) = 0$, but then since $(G,\, !_{\mathcal{P}}\varphi,\, ?^m_{\mathcal{P}}\alpha\,;\mathcal{O})$ is pre-basic, we would have by lemma 4.1 that $\mathbf{Challenges}(H\,;\mathcal{O}) < \mathbf{Challenges}(H\,;\mathcal{P}) = 0$ which is impossible. Thus $?^m_{\mathcal{P}}\alpha$ is never defended, and $l(\sigma) \in Str_{\mathcal{P}}(G,\, !_{\mathcal{P}}\varphi\,;\mathcal{O})$. By the same reasoning as in the previous case we then get that

$$r(\sigma) \in Str^w_{\mathcal{P}}(G,\, !_{\mathcal{P}}\varphi\,;\mathcal{O}),$$

and $r(\sigma) \leq \sigma$. □

4.4.3 Move-order Invariance

We will show that the actions $A(\varphi)$ where φ is positive are all move-order invariant.

Theorem 4.6. Let $(G,\, !_{\mathcal{O}}\varphi\,;\mathcal{P})$ be a non terminal state and φ positive, then there is a function:

$$Perm_\varphi : Str^w_{\mathcal{P}}(G,\, !_{\mathcal{O}}\varphi\,;\mathcal{P}) \to Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{O}}\varphi\,;\mathcal{O})$$

Proof. Suppose $(a,\sigma) \in Str^w_{\mathcal{P}}(G,\, !_{\mathcal{O}}\varphi\,;\mathcal{P})$. We construct a strategy

$$Perm_\varphi((a,\sigma)) \in Str^w_{\mathcal{P}}(G,\, ?_{\mathcal{O}}\varphi\,;\mathcal{O}).$$

If $a = A(\varphi)$, let $Perm_\varphi((a,\sigma)) = \sigma$. Otherwise we define

$$Perm_\varphi((a,\sigma))(x) \in Str^w_{\mathcal{P}}(x(G,\, ?_{\mathcal{O}}\varphi\,;\mathcal{O}))$$

for arbitrary $x \in \mathbf{Move}(G,\, ?_{\mathcal{O}}\varphi\,;\mathcal{O})$. By cases:

- Let the move be b on the G component of the game, then we construct a strategy in $Str^w_{\mathcal{P}}(bG,\, ?^n_{\mathcal{O}}\varphi\,;\mathcal{P})$ as follows:

$$\begin{aligned}
(a,\sigma) &\in Str^w_{\mathcal{P}}(G,\, !_{\mathcal{O}}\varphi\,;\mathcal{P}) & \\
\sigma &\in Str^w_{\mathcal{P}}(aG,\, !_{\mathcal{O}}\varphi\,;\mathcal{O}) & \\
\sigma(b) &\in Str^w_{\mathcal{P}}(baG,\, !_{\mathcal{O}}\varphi\,;\mathcal{P}) & \text{in particular} \\
Perm_\varphi(\sigma(b)) &\in Str^w_{\mathcal{P}}(baG,\, ?^n_{\mathcal{O}}\varphi\,;\mathcal{O}) & \text{by induction} \\
(a, Perm_\varphi(\sigma(b))) &\in Str^w_{\mathcal{P}}(bG,\, ?^n_{\mathcal{O}}\varphi\,;\mathcal{P}) &
\end{aligned}$$

- Let the move be c on the $?^n{}_{\mathcal{O}}\varphi$ component of the game, then we construct a strategy in $Str^w_{\mathcal{P}}(G, c\,?^n_{\mathcal{O}}\varphi\,;_{\mathcal{P}})$ as follows:

$$
\begin{array}{rll}
\sigma(b) \in Str^w_{\mathcal{P}}(baG,\ !_{\mathcal{O}}\varphi\,;_{\mathcal{P}}) & & \text{where } b \text{ is arbitrary} \\
Perm_\varphi(\sigma(b)) \in Str^w_{\mathcal{P}}(baG,\ ?^n_{\mathcal{O}}\varphi\,;_{\mathcal{O}}) & & \text{by induction} \\
Perm_\varphi(\sigma(b))(c) \in Str^w_{\mathcal{P}}(baG, c\,?^n_{\mathcal{O}}\varphi\,;_{\mathcal{P}}) & & \text{in particular} \\
\lambda x.Perm_\varphi(\sigma(x))(c) \in Str^w_{\mathcal{P}}(aG, c\,?^n_{\mathcal{O}}\varphi\,;_{\mathcal{O}}) & & \\
(a, \lambda x.Perm_\varphi(\sigma(x)))(c)) \in Str^w_{\mathcal{P}}(G, c\,?^n_{\mathcal{O}}\varphi\,;_{\mathcal{P}}) & &
\end{array}
$$

□

4.4.4 Contraction

We show contraction in the intuitionistic case only for opponent assertions.

Theorem 4.7 (Contraction)**.** Let G be a basic position, then there is a contraction:

1. $Con : Str^w_{\mathcal{P}}(G,\ !_{\mathcal{O}}\varphi,\ !_{\mathcal{O}}\varphi\,;_{\mathcal{P}}) \to Str^w_{\mathcal{P}}(G,\ !_{\mathcal{O}}\varphi\,;_{\mathcal{P}})$

Proof. By induction on φ.

1. By cases:
   - If $\varphi = A$ is an atom it is never attacked, thus let $Con(\sigma) = \sigma$.
   - If φ is positive, we construct a strategy $Con(\sigma) \in Str^w_{\mathcal{P}}(G,\ !_{\mathcal{O}}\varphi)$ as follows:

$$
\begin{array}{rl}
\sigma \in & Str^w_{\mathcal{P}}(G,\ !_{\mathcal{O}}\varphi,\ !_{\mathcal{O}}\varphi\,;_{\mathcal{P}}) \\
Perm_\varphi(\sigma) \in & Str^w_{\mathcal{P}}(G,\ ?_{\mathcal{O}}\varphi,\ !_{\mathcal{O}}\varphi\,;_{\mathcal{O}}) \\
Perm_\varphi(\sigma)(b) \in & Str^w_{\mathcal{P}}(G, b\,?^n_{\mathcal{O}}\varphi,\ !_{\mathcal{O}}\varphi\,;_{\mathcal{P}}) \\
Perm_\varphi(Perm_\varphi(\sigma)(b)) \in & Str^w_{\mathcal{P}}(G, b\,?^n_{\mathcal{O}}\varphi,\ ?^n_{\mathcal{O}}\varphi\,;_{\mathcal{O}}) \\
Perm_\varphi(Perm_\varphi(\sigma)(b))(b) \in & Str^w_{\mathcal{P}}(G, b\,?^n_{\mathcal{O}}\varphi, b\,?^n_{\mathcal{O}}\varphi\,;_{\mathcal{P}}) \\
Con(Perm_\varphi(Perm_\varphi(\sigma)(b))(b)) \in & Str^w_{\mathcal{P}}(G, b\,?^n_{\mathcal{O}}\varphi\,;_{\mathcal{P}}) \\
\lambda x.Con(Perm_\varphi(Perm_\varphi(\sigma)(x))(x)) \in & Str^w_{\mathcal{P}}(G,\ ?^n_{\mathcal{O}}\varphi\,;_{\mathcal{O}}) \\
(A(\varphi), \lambda x.Con(Perm_\varphi(Perm_\varphi(\sigma)(x))(x))) \in & Str^w_{\mathcal{P}}(G,\ !_{\mathcal{O}}\varphi\,;_{\mathcal{P}})
\end{array}
$$

     Where b is an arbitrary defense of $?^n_{\mathcal{O}}\varphi$.

□

4.4.5 The Parallel Strategy

For the intuitionistic games we would like to define the parallel strategy along the lines of the classical games, however for intuitionistic games a parallel strategy does not always exist for arbitrary games so instead we define it for arbitrary pre-basic games.

Theorem 4.8 (Intuitionistic Parallel Strategy)**.** Let $G_1, \ldots, G_n$ be pre-basic games, then there is a function:

1. $\|: Str^w_{\mathcal{P}}(G_1 ; \mathcal{O}) \times \cdots \times Str^w_{\mathcal{P}}(G_n ; \mathcal{O}) \to Str^w_{\mathcal{P}}(G_1, \ldots, G_n ; \mathcal{O})$

Proof. By induction on $n \in \mathbb{N}$ with subinduction on $\sigma_1, \ldots, \sigma_n \in Str^w_{\mathcal{P}}(G_1 ; \mathcal{O}) \times \cdots \times Str^w_{\mathcal{P}}(G_n ; \mathcal{O})$. Suppose for induction that the parallel strategy exists for strategies $\tau_1, \ldots, \tau_m$ where $m < n$. Consider an arbitrary move

$$a \in \mathbf{Move}(G_1, \ldots, G_n ; \mathcal{O}),$$

without loss of generality let it be on the G_j component of the game. Thus we have $\sigma_j \in Str^w_{\mathcal{P}}(G_j, \mathcal{O})$ and $\sigma_j(a) \in Str^w_{\mathcal{P}}(aG_j, \mathcal{P})$. Let $I = \{1, \ldots, n\} \setminus \{j\}$. We define

$$\| \, (\sigma_1, \ldots, \sigma_n)(a) \in Str^w_{\mathcal{P}}((\bigcup_{i \in I} G_i), aG_j ; \mathcal{P}),$$

by induction on $\sigma_j(a) \in Str^w_{\mathcal{P}}(aG_j ; \mathcal{P})$. By cases:

- The strategy $\sigma_j(a) = e$, that is $(aG_j ; \mathcal{P})$ is a terminal state, by cases:
  - If $!_{\mathcal{O}}\bot \in aG_j$ also $Str^w_{\mathcal{P}}((\bigcup_{i \in I} G_i), aG_j ; \mathcal{P})$ is a terminal state, so let
  $$\| \, (\sigma_1, \ldots, \sigma_n)(a) = e.$$
  - If $!_{\mathcal{O}}A \in aG_j$ and $?^n_{\mathcal{P}}A \in aG_j$, then defending $?^n_{\mathcal{P}}A$, the proponent can reach a state $(D(A)aG_j ; \mathcal{O})$ that has no further moves for proponent or opponent and which is not terminal. Let $\sigma_1, \ldots, \sigma_n - \sigma_j$ be just as $\sigma_1, \ldots, \sigma_n$ without σ_j, then by induction we have that
  $$\| \, (\sigma_1, \ldots, \sigma_n - \sigma_j) \in Str^w_{\mathcal{P}}(\bigcup_{i \in I} G_i ; \mathcal{O})$$
  thus by weakening
  $$Wk(\| \, (\sigma_1, \ldots, \sigma_n - \sigma_j)) \in Str^w_{\mathcal{P}}((\bigcup_{i \in I} G_i), D(A)aG_j ; \mathcal{O})$$
  thus
  $$(D(A), Wk(\| \, (\sigma_1, \ldots, \sigma_n - \sigma_j))) \in Str^w_{\mathcal{P}}(\bigcup_{i \in I} G_i, aG_j ; \mathcal{P}).$$
- The strategy is $\sigma(a) = (b, \sigma')$. Suppose for induction the parallel strategy is defined for $\tau_1, \ldots, \tau_k \ll \sigma_1, \ldots, \sigma_n$. Consider the action b by cases:
  - The action $b \neq A(\varphi \to \psi)$, thus baG_j is a pre-basic position, thus by induction since $(\sigma_1, \ldots, \sigma_n - \sigma_j, \sigma') \ll (\sigma_1, \ldots, \sigma_n)$:
  $$\| \, (\sigma_1, \ldots, \sigma_n - \sigma_j, \sigma') \in Str^w_{\mathcal{P}}((\bigcup_{i \in I} G_i), baG_j ; \mathcal{O})$$
  so
  $$(b, \| \, (\sigma_1, \ldots, \sigma_n - \sigma_j, \sigma')) \in Str^w_{\mathcal{P}}((\bigcup_{i \in I} G_i), aG_j ; \mathcal{P}).$$

– The action $b = A(\varphi \to \psi)$, then $baG_j = !_{\mathcal{P}}\varphi,\ ?^m_{\mathcal{O}}\psi,\ !_{\mathcal{O}}\Gamma,\ ?^n_{\mathcal{P}}\alpha$, where $aG_j = !_{\mathcal{O}}\Gamma,\ ?^n_{\mathcal{P}}\alpha$. Thus, there are strategies

$$l(\sigma') \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ ?^m_{\mathcal{O}}\psi,\ ?^n_{\mathcal{P}}\alpha\,;{\scriptstyle\mathcal{O}}) \text{ and } r(\sigma') \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ !_{\mathcal{P}}\varphi\,;{\scriptstyle\mathcal{O}})$$

Then by induction since $(\sigma_1,\ldots,\sigma_n - \sigma_j, l(\sigma'), r(\sigma')) \ll (\sigma_1,\ldots,\sigma_n)$:

$$\|\,(\sigma_1,\ldots,\sigma_n - \sigma_j, l(\sigma'), r(\sigma')) \in Str^w_{\mathcal{P}}((\bigcup_{i\in I} G_i),\ !_{\mathcal{O}}\Gamma,\ ?^m_{\mathcal{O}}\psi,\ ?^n_{\mathcal{P}}\alpha,\ !_{\mathcal{O}}\Gamma,\ !_{\mathcal{P}}\varphi\,;{\scriptstyle\mathcal{O}})$$

so

$$(b, \|\,(\sigma_1,\ldots,\sigma_n - \sigma_j, l(\sigma'), r(\sigma'))) \in Str^w_{\mathcal{P}}((\bigcup_{i\in I} G_i),\ !_{\mathcal{O}}\Gamma,\ !_{\mathcal{O}}\Gamma,\ ?^n_{\mathcal{P}}\alpha\,;{\scriptstyle\mathcal{P}})$$

then by contraction

$$Con(b, \|\,(\sigma_1,\ldots,\sigma_n - \sigma_j, l(\sigma'), r(\sigma'))) \in Str^w_{\mathcal{P}}((\bigcup_{i\in I} G_i), aG_j\,;{\scriptstyle\mathcal{P}}). \quad \square$$

4.4.6 Application and Composition

Application and composition of strategies are defined as in the classical case, except that we now require the states $(\Psi \multimap J)$ and $(\Phi \multimap I, \Psi\,;{\scriptstyle\mathcal{O}})$ to be pre-basic.

Theorem 4.9 (Application and Composition)**.** Let (G, Ψ) be a basic position and let $(\Psi \multimap J)$ and $(\Phi \multimap I, \Psi\,;{\scriptstyle\mathcal{O}})$ be pre-basic positions, then there are functions:

1. $Ap : Str^w_{\mathcal{P}}(G, \Psi\,;{\scriptstyle\mathcal{P}}) \times Str^w_{\mathcal{P}}(\Psi \multimap J\,;{\scriptstyle\mathcal{O}}) \to Str^w_{\mathcal{P}}(G, J\,;{\scriptstyle\mathcal{P}})$
2. $(-;-) : Str^w_{\mathcal{P}}(\Phi \multimap I, \Psi\,;{\scriptstyle\mathcal{O}}) \times Str^w_{\mathcal{P}}(\Psi \multimap J\,;{\scriptstyle\mathcal{O}}) \to Str^w_{\mathcal{P}}(\Phi \multimap I, J\,;{\scriptstyle\mathcal{O}})$

Proof.

1. Let $(\sigma, \tau) \in Str^w_{\mathcal{P}}(G, \Psi\,;{\scriptstyle\mathcal{P}}) \times Str^w_{\mathcal{P}}(\Psi \multimap J\,;{\scriptstyle\mathcal{O}})$, by induction on $\sigma \in Str^w_{\mathcal{P}}(G, \Psi\,;{\scriptstyle\mathcal{P}})$, with subinduction on the formula Ψ:

   - The strategy $\sigma = e$, the state $(G, \Psi\,;{\scriptstyle\mathcal{P}})$ is terminal, by cases
     – The state $(G\,;{\scriptstyle\mathcal{P}})$ is terminal, by cases
       * If $G = G',\ !_{\mathcal{O}}\bot$, then $(G, J\,;{\scriptstyle\mathcal{P}})$ is terminal.
       * If $G = G',\ !_{\mathcal{O}}A,\ ?^n_{\mathcal{P}}A$, then $J = !_{\mathcal{O}}\Gamma$ for some $\Gamma \subseteq \mathcal{L}$, since $\Psi = !_{\mathcal{O}}\psi$. Then $(G, J\,;{\scriptstyle\mathcal{P}})$ is terminal.

       In both cases let
       $$Ap(\sigma,\tau) = e.$$
     – The state $(G, \Psi\,;{\scriptstyle\mathcal{P}}) = (G',\ ?^n_{\mathcal{P}}A,\ !_{\mathcal{O}}A)$ where $\Psi = !_{\mathcal{O}}A$. Since $(\,!_{\mathcal{P}}A, J\,;{\scriptstyle\mathcal{O}})$ is non-terminal also $!_{\mathcal{O}}A \in J$, but then also $(G',\ !_{\mathcal{P}}A, J\,;{\scriptstyle\mathcal{P}})$ is terminal so let
       $$Ap(\sigma,\tau) = e.$$

– The state $(G, \Psi\,; {\scriptscriptstyle \mathcal{P}}) = (G', \,!_{\mathcal{O}}A, \,?^{n}_{\mathcal{P}}A)$ where $\Psi = ?^{n}_{\mathcal{P}}A$, then $\tau \in Str^{w}_{\mathcal{P}}(\,?^{n}_{\mathcal{O}}A, J\,; {\scriptscriptstyle \mathcal{O}})$. Thus $\tau(D(A)) \in Str^{w}_{\mathcal{P}}(\,!_{\mathcal{O}}A, J\,; {\scriptscriptstyle \mathcal{P}})$, thus $Wk(\tau(D(A))) \in Str^{w}_{\mathcal{P}}(G', \,!_{\mathcal{O}}A, J\,; {\scriptscriptstyle \mathcal{P}})$, so let

$$Ap(\sigma, \tau) = \tau(D(A))$$

- The strategy is $\sigma = (a, \sigma')$ and the first move is on the G-component of the game, we get two cases:

(a) The position $aG, \Psi = \Phi \multimap H, \Psi$ for some basic Φ and H, , i.e $a \neq A(\varphi_1 \to \varphi_2)$, thus

$$\sigma' \in Str^{w}_{\mathcal{P}}(\Phi \multimap H, \Psi\,; {\scriptscriptstyle \mathcal{O}}),$$

we construct a strategy $Ap(\sigma, \tau) \in Str^{w}_{\mathcal{P}}(G, J\,; {\scriptscriptstyle \mathcal{P}})$ as follows:

$$\cfrac{\cfrac{\sigma' \in Str^{w}_{\mathcal{P}}(\Phi \multimap H, \Psi\,; {\scriptscriptstyle \mathcal{O}}) \qquad \tau \in Str^{w}_{\mathcal{P}}(\Psi \multimap J\,; {\scriptscriptstyle \mathcal{O}})}{\sigma'; \tau \in Str^{w}_{\mathcal{P}}(\Phi \multimap H, J\,; {\scriptscriptstyle \mathcal{O}})}}{(a, \sigma'; \tau) \in Str^{w}_{\mathcal{P}}(G, J\,; {\scriptscriptstyle \mathcal{P}})}$$

(b) The position $aG, \Psi = ?^{m}_{\mathcal{O}}\varphi_1, \,!_{\mathcal{P}}\varphi_2, G, \Psi,$, i.e $a = A(\varphi_1 \to \varphi_2)$, thus

$$\sigma' \in Str^{w}_{\mathcal{P}}(\,?^{m}_{\mathcal{O}}\varphi_1, \,!_{\mathcal{P}}\varphi_2, G, \Psi\,; {\scriptscriptstyle \mathcal{O}}),$$

we construct a strategy $Ap(\sigma, \tau) \in Str^{w}_{\mathcal{P}}(G, J\,; {\scriptscriptstyle \mathcal{P}})$ by cases:

– The position $G, \Psi = G, \,!_{\mathcal{O}}\psi$, let

$$K = \begin{cases} G' \text{ if } G = G', \,?^{n}_{\mathcal{P}}\alpha \\ G \text{ else.} \end{cases}$$

Then there are two strategies:

$$l(\sigma') \in Str^{w}_{\mathcal{P}}(G, \,?^{m}_{\mathcal{O}}\varphi_1, \Psi\,; {\scriptscriptstyle \mathcal{O}})$$
$$r(\sigma') \in Str^{w}_{\mathcal{P}}(K, \,!_{\mathcal{P}}\varphi_2, \Psi\,; {\scriptscriptstyle \mathcal{O}}),$$

thus we construct two strategies

$$\cfrac{l(\sigma') \in Str^{w}_{\mathcal{P}}(\,?^{m}_{\mathcal{O}}\varphi_1, G, \Psi\,; {\scriptscriptstyle \mathcal{O}}) \qquad \tau \in Str^{w}_{\mathcal{P}}(\Psi \multimap J\,; {\scriptscriptstyle \mathcal{O}})}{l(\sigma'); \tau \in Str^{w}_{\mathcal{P}}(\,?^{m}_{\mathcal{O}}\varphi_1, G, J\,; {\scriptscriptstyle \mathcal{O}})},$$

and

$$\cfrac{r(\sigma') \in Str^{w}_{\mathcal{P}}(, \,!_{\mathcal{P}}\varphi_2, K, \Psi\,; {\scriptscriptstyle \mathcal{O}}) \qquad \tau \in Str^{w}_{\mathcal{P}}(\Psi \multimap J\,; {\scriptscriptstyle \mathcal{O}})}{r(\sigma'); \tau \in Str^{w}_{\mathcal{P}}(\,!_{\mathcal{P}}\varphi_2, K, J\,; {\scriptscriptstyle \mathcal{O}})}$$

Playing them in parallel yields:

$$\cfrac{\cfrac{(l(\sigma'); \tau) \parallel (r(\sigma'); \tau) \in Str^{w}_{\mathcal{P}}(\,?^{m}_{\mathcal{O}}\varphi_1, \,!_{\mathcal{P}}\varphi_2, G, K, J, J\,; {\scriptscriptstyle \mathcal{O}})}{(a, (l(\sigma'); \tau) \parallel (r(\sigma'); \tau)) \in Str^{w}_{\mathcal{P}}(G, K, J, J\,; {\scriptscriptstyle \mathcal{P}})}}{Con(a, (l(\sigma'); \tau) \parallel (r(\sigma'); \tau)) \in Str^{w}_{\mathcal{P}}(G, J\,; {\scriptscriptstyle \mathcal{P}})}$$

– The position $G, \Psi = G,\ ?^n_{\mathcal{P}}\psi$, then there are two strategies:

$$l(\sigma') \in Str^w_{\mathcal{P}}(\ ?^m_{\mathcal{O}}\varphi_1, G, \Psi\,;_{\mathcal{O}})$$
$$r(\sigma') \in Str^w_{\mathcal{P}}(\ !_{\mathcal{P}}\varphi_2, G\,;_{\mathcal{O}}),$$

we construct a strategy in $Str^w_{\mathcal{P}}(G, J\,;_{\mathcal{P}})$ by first construction a strategy:

$$\frac{l(\sigma') \in Str^w_{\mathcal{P}}(\ ?^m_{\mathcal{O}}\varphi_1, G, \Psi\,;_{\mathcal{O}}) \qquad \tau \in Str^w_{\mathcal{P}}(\Psi \multimap J\,;_{\mathcal{O}})}{l(\sigma');\tau \in Str^w_{\mathcal{P}}(\ ?^m_{\mathcal{O}}\varphi_1, G, J\,;_{\mathcal{O}})}$$

Playing in parallel yields:

$$\cfrac{\cfrac{(r(\sigma');\tau) \parallel (l(\sigma');\tau) \in Str^w_{\mathcal{P}}(\ ?^m_{\mathcal{O}}\varphi_1,\ !_{\mathcal{P}}\varphi_2, G, G, J, J\,;_{\mathcal{O}})}{(a, (r(\sigma');\tau) \parallel (l(\sigma');\tau)) \in Str^w_{\mathcal{P}}(G, J, J\,;_{\mathcal{P}})}}{Con(a, (r(\sigma');\tau) \parallel (l(\sigma');\tau)) \in Str^w_{\mathcal{P}}(G, J\,;_{\mathcal{P}})}$$

- The strategy is $\sigma = (a, \sigma')$ and the first move is on the Ψ-component of the game, we get three cases:

(a) The position $G, a\Psi = \Phi \multimap G$, i.e $a = D(\varphi)$ for negative φ or $a = A(\varphi)$ for positive φ. Thus,

$$\sigma' \in Str^w_{\mathcal{P}}(\Phi \multimap G\,;_{\mathcal{O}}),$$

we construct a strategy $Ap(\sigma, \tau) Str^w_{\mathcal{P}}(G, J\,;_{\mathcal{P}})$ as follows:

$$\frac{\sigma' \in Str^w_{\mathcal{P}}\Phi \multimap G\,;_{\mathcal{O}}) \qquad \cfrac{\tau \in Str^w_{\mathcal{P}}(\Psi \multimap J\,;_{\mathcal{O}})}{\tau(a) \in Str^w_{\mathcal{P}}(\Phi, J\,;_{\mathcal{O}})}}{Ap(\sigma', \tau(a)) \in Str^w_{\mathcal{P}}(G, J\,;_{\mathcal{P}})}$$

(b) The position $G, a\Psi = \Phi \multimap G, \Psi$, i.e $a = D(\varphi)$ for positive φ or $a = A(\varphi)$ for negative $\varphi \neq \varphi_1 \to \varphi_2$. Thus,

$$\sigma' \in Str^w_{\mathcal{P}}(\Phi \multimap G, \Psi\,;_{\mathcal{O}}),$$

we construct a strategy $Ap(\sigma, \tau) \in Str^w_{\mathcal{P}}(G, J\,;_{\mathcal{P}})$ as follows:

$$\cfrac{\cfrac{\cfrac{\sigma' \in Str^w_{\mathcal{P}}(\Phi \multimap G, \Psi\,;_{\mathcal{O}}) \qquad \tau \in Str^w_{\mathcal{P}}(\Psi \multimap J\,;_{\mathcal{O}})}{\sigma';\tau \in Str^w_{\mathcal{P}}(\Phi \multimap G, J\,;_{\mathcal{O}})} \qquad \cfrac{\tau \in Str^w_{\mathcal{P}}(\Psi \multimap J\,;_{\mathcal{O}})}{\tau(a) \in Str^w_{\mathcal{P}}(\Phi, J\,;_{\mathcal{O}})}}{Ap((\sigma';\tau), \tau(a)) \in Str^w_{\mathcal{P}}(G, J, J\,;_{\mathcal{P}})}}{Con(Ap((\sigma';\tau), \tau(a)) \in Str^w_{\mathcal{P}}(G, J\,;_{\mathcal{P}}))}$$

(c) The position $G, a\Psi = \Phi_1, \Phi_2 \multimap G, \Psi$ for some basic games Φ_1 and Φ_2, i.e $a = A(\varphi_1 \to \varphi_2)$. This is the same as a previous case so let

$$Ap(\sigma, \tau) \in Str^w_{\mathcal{P}}(G, J\,;_{\mathcal{P}}).$$

2. Let $(\sigma,\tau) \in Str^w_{\mathcal{P}}(\Phi \multimap I, \Psi\,;_{\mathcal{O}}) \times Str^w_{\mathcal{P}}(\Psi \multimap J\,;_{\mathcal{O}})$, by induction on $\sigma \in Str^w_{\mathcal{P}}(\Phi \multimap I, \Psi\,;_{\mathcal{O}})$. For arbitrary $a \in \mathbf{Move}(\Phi \multimap I, \Psi\,;_{\mathcal{O}})$ we note that $a(\Phi \multimap I, \Psi) = (a\Phi, I, \Psi)$ is a basic position and there is a strategy:

$$\sigma(a) \in Str^w_{\mathcal{P}}(a\Phi, I, \Psi\,;_{\mathcal{P}}),$$

we construct a strategy $\sigma \parallel \tau \in Str^w_{\mathcal{P}}(\Phi \multimap I, J\,;_{\mathcal{O}})$ as follows:

$$\dfrac{\dfrac{\sigma(a) \in Str^w_{\mathcal{P}}(a\Phi, I, \Psi\,;_{\mathcal{P}}) \qquad \tau \in Str^w_{\mathcal{P}}(\Psi \multimap J\,;_{\mathcal{O}})}{Ap(\sigma(a),\tau) \in Str^w_{\mathcal{P}}(a\Phi, I, J\,;_{\mathcal{P}})}}{\lambda x.Ap(\sigma(x),\tau) \in Str^w_{\mathcal{P}}(\Phi \multimap I, J\,;_{\mathcal{O}})}$$

□

4.4.7 Cut Elimination

Recall that the following cut rule is admissible in $G3I$:

$$\dfrac{\Gamma \Rightarrow \varphi \qquad \varphi, \Gamma' \Rightarrow \psi}{\Gamma, \Gamma' \Rightarrow \psi}$$

Similarly, for the strategies we have the following function:

Theorem 4.10. Let $G = !_{\mathcal{O}}\Gamma$, $J = !_{\mathcal{O}}\Gamma'$ and H be basic, then there is a function:

$$Cut_\varphi : Str^w_{\mathcal{P}}(G, H, !_{\mathcal{O}}\varphi\,;_{\mathcal{P}}) \times Str^w_{\mathcal{P}}(G, J, ?^n_{\mathcal{P}}\varphi\,;_{\mathcal{P}}) \to Str^w_{\mathcal{P}}(G, H, J\,;_{\mathcal{P}})$$

Proof. By cases:

- The assertion $!_{\mathcal{O}}\varphi$ is positive, then we construct a strategy in $Str^w_{\mathcal{P}}(G, H, J\,;_{\mathcal{P}})$ as follows:

$$\dfrac{\dfrac{\dfrac{\sigma \in Str^w_{\mathcal{P}}(G, H, !_{\mathcal{O}}\varphi\,;_{\mathcal{P}})}{Perm_\varphi(\sigma) \in Str^w_{\mathcal{P}}(G, H, ?^n_{\mathcal{O}}\varphi\,;_{\mathcal{O}})} \qquad \tau \in Str^w_{\mathcal{P}}(G, J, ?^n_{\mathcal{P}}\varphi\,;_{\mathcal{P}})}{Ap(Perm_\varphi(\sigma),\tau) \in Str^w_{\mathcal{P}}(G, G, H, J\,;_{\mathcal{P}})}}{Con(Ap(Perm_\varphi(\sigma),\tau)) \in Str^w_{\mathcal{P}}(G, H, J\,;_{\mathcal{P}})}$$

- The assertion $!_{\mathcal{O}}\varphi$ is negative, then we construct a strategy in $Str^w_{\mathcal{P}}(G, H, J\,;_{\mathcal{P}})$ as follows:

$$\dfrac{\dfrac{\dfrac{\sigma \in Str^w_{\mathcal{P}}(G, H, !_{\mathcal{O}}\varphi\,;_{\mathcal{P}})}{\lambda a.\sigma \in Str^w_{\mathcal{P}}(G, H, ?^n_{\mathcal{O}}\varphi\,;_{\mathcal{O}})} \qquad \tau \in Str^w_{\mathcal{P}}(G, J, ?^n_{\mathcal{P}}\varphi\,;_{\mathcal{P}})}{Ap(\lambda a.\sigma,\tau) \in Str^w_{\mathcal{P}}(G, G, H, J\,;_{\mathcal{P}})}}{Con(Ap(\lambda a.\sigma,\tau) \in) Str^w_{\mathcal{P}}(G, H, J\,;_{\mathcal{P}})}$$

□

4.5 Correspondence of Strategies and Proofs

We will now prove soundness and completeness of strategies with regards to $G3I$, that is we will show for arbitrary $n \in \mathbb{N}$

$$\Gamma \vdash_i \psi \iff Str^w_{\mathcal{P}}(!_{\mathcal{O}}\Gamma, ?^n_{\mathcal{P}}\psi\,;_{\mathcal{P}}).$$

In fact as in the classical case we can derive functions f and g from the proofs such that

$$f : Der(\Gamma \vdash_i \psi) \to Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ ?^n_{\mathcal{P}}\psi\,;{\scriptstyle\mathcal{P}})$$
$$g : Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ ?^n_{\mathcal{P}}\psi\,;{\scriptstyle\mathcal{P}}) \to Der(\Gamma \vdash_i \psi),$$

where $Der(\Gamma \vdash_i \psi)$ are the set of derivations of the sequent $\Gamma \Rightarrow \psi$.

4.5.1 Soundness

Theorem 4.11 (Soundness)**.** For arbitrary $n \in \mathbb{N}$

$$\text{If } Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ ?^n_{\mathcal{P}}\psi\,;{\scriptstyle\mathcal{P}}) \neq \emptyset, \text{ then } \Gamma \vdash_i \psi$$

Proof. As inductive hypothesis we take the following:

1. If $Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ ?^n_{\mathcal{P}}\psi\,;{\scriptstyle\mathcal{P}}) \neq \emptyset,$ then $\Gamma \vdash_i \psi$
2. If $Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ ?_{\mathcal{O}}\varphi,\ ?^n_{\mathcal{P}}\psi\,;{\scriptstyle\mathcal{O}}) \neq \emptyset,$ then $\Gamma, \varphi \vdash_i \psi$
3. If $Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ !_{\mathcal{P}}\varphi\,;{\scriptstyle\mathcal{O}}) \neq \emptyset,$ then $\Gamma \vdash_i \varphi$

1. By induction on $\sigma \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ ?^n_{\mathcal{P}}\psi\,;{\scriptstyle\mathcal{P}})$.
   - The strategy is $\sigma = e$, that is the state $(\,!_{\mathcal{O}}\Gamma,\ ?^n_{\mathcal{P}}\psi\,;{\scriptstyle\mathcal{P}})$ is terminal, by cases:
     - $A \in \Gamma$ and $\psi = A$, then $\Gamma \vdash_i \psi$.
     - $\bot \in \Gamma$, then $\Gamma \vdash_i \psi$.
   - The strategy is $\sigma = (a, \sigma')$, the opening action a is a defense of ψ. By cases:
     - The formula ψ is negative, then $\sigma' \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ !_{\mathcal{P}}\psi\,;{\scriptstyle\mathcal{O}})$, then by induction $\Gamma \vdash_i \psi$.
     - The formula ψ is positive, then by cases:
       * The formula is $\psi = \alpha \vee \beta$, without loss of generality

         $$\sigma' \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ !_{\mathcal{P}}\alpha\,;{\scriptstyle\mathcal{O}}),$$

         thus by induction $\Gamma \vdash_i \alpha$, thus $\Gamma \vdash_i \alpha \vee \beta$.
       * The formula is $\psi = \exists x \alpha(x)$, without loss of generality

         $$\sigma' \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ !_{\mathcal{P}}\alpha(t)\,;{\scriptstyle\mathcal{O}}),$$

         thus by induction $\Gamma \vdash_i \alpha(t)$, thus $\Gamma \vdash_i \exists x \alpha(x)$.
       * The formula is $\varphi = A$, but this is impossible since then already $!_{\mathcal{O}}A \in !_{\mathcal{O}}\Gamma$, thus the state $(\,!_{\mathcal{O}}\Gamma,\ ?^n_{\mathcal{P}}\psi\,;{\scriptstyle\mathcal{P}})$ would be terminal.
   - The strategy is $\sigma = (a, \sigma')$, the opening action a is an attack on φ, thus $\Gamma = \Gamma', \varphi$. By cases:

- The formula φ is positive, then $\sigma' \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma',\ ?_{\mathcal{O}}\varphi,\ ?^n_{\mathcal{P}}\psi\,;\mathcal{O})$, thus by induction $\Gamma', \varphi \vdash_i \psi$.
- The formula φ is negative, then by cases:
  * The formula is $\varphi = \alpha \wedge \beta$, without loss of generality

$$\sigma' \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ ?_{\mathcal{O}}\alpha,\ ?^n_{\mathcal{P}}\psi\,;\mathcal{O}),$$

    thus by induction we have that $\Gamma, \alpha \vdash_i \psi$, by weakening $\Gamma, \alpha, \beta \vdash_i \psi$, thus also $\Gamma, \alpha \wedge \beta \vdash_i \psi$, thus by contraction $\Gamma \vdash_i \psi$.
  * The formula is $\varphi = \alpha \to \beta$, then $\sigma' \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ ?_{\mathcal{O}}\beta,\ !_{\mathcal{P}}\alpha,\ ?^n_{\mathcal{P}}\psi\,;\mathcal{O})$, thus:

$$l(\sigma') \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ ?_{\mathcal{O}}\beta,\ ?^n_{\mathcal{P}}\psi\,;\mathcal{O})$$
$$r(\sigma') \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ !_{\mathcal{P}}\alpha\,;\mathcal{O})$$

    Thus by induction $\Gamma, \beta \vdash_i \psi$ and $\Gamma \vdash_i \alpha$, thus $\Gamma, \alpha \to \beta \vdash_i \psi$, thus by contraction $\Gamma \vdash_i \psi$.
  * The formula is $\varphi = \forall x \alpha(x)$, without loss of generality

$$\sigma' \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ ?_{\mathcal{O}}\alpha(t),\ ?^n_{\mathcal{P}}\psi\,;\mathcal{O}),$$

    then by induction $\Gamma, \alpha(t) \vdash_i \psi$, then $\Gamma, \forall x \alpha(x) \vdash_i \psi$, thus by contraction $\Gamma \vdash_i \psi$.

2. By induction on $\sigma \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ ?_{\mathcal{O}}\varphi,\ ?^n_{\mathcal{P}}\psi\,;\mathcal{O})$, let a be the opponents opening action.

   - The formula φ is positive, then $\sigma(a) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ !_{\mathcal{O}}\varphi,\ ?^n_{\mathcal{P}}\psi\,;\mathcal{P})$, thus by induction $\Gamma, \varphi \vdash_i \psi$.
   - The formula φ is negative, by cases:
     - The formula is $\varphi = \exists x \alpha(x)$, then

$$\sigma(D_t(\varphi)) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ !_{\mathcal{O}}\alpha(t),\ ?^n_{\mathcal{P}}\psi\,;\mathcal{P}),$$

       for any term t, thus in particular by induction $\Gamma, \alpha(x) \vdash_i \psi$ for some $x \notin FV(\Gamma \cup \{\alpha\})$, thus $\Gamma, \exists x \alpha(x) \vdash_i \psi$.
     - The formula is $\varphi = \alpha \vee \beta$, then the proponent has winning strategies

$$\sigma(D_0(\varphi)) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ !_{\mathcal{O}}\alpha,\ ?^n_{\mathcal{P}}\psi\,;\mathcal{P})$$
$$\sigma(D_1(\varphi)) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ !_{\mathcal{O}}\beta,\ ?^n_{\mathcal{P}}\psi\,;\mathcal{P})$$

       Thus by induction $\Gamma, \alpha \vdash_i \psi$ and $\Gamma, \beta \vdash_i \psi$, thus $\Gamma, \alpha \vee \beta \vdash_i \psi$.

3. By induction on $\sigma \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\ !_{\mathcal{P}}\varphi\,;\mathcal{O})$, let a be the opponents opening action.

- The formula φ is positive, then $\sigma(a) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?^n_{\mathcal{P}}\varphi\,;{\scriptscriptstyle\mathcal{P}})$, thus by induction $\Gamma \vdash_i \varphi$.
- The formula φ is negative, then by cases:
  - The formula is $\varphi = A$, then $\sigma(a) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?^n_{\mathcal{P}}A\,;{\scriptscriptstyle\mathcal{P}})$, thus by induction $\Gamma \vdash_i A$.
  - The formula is $\varphi = \alpha \wedge \beta$, thus

$$\sigma(A_0(\varphi)) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?^n_{\mathcal{P}}\alpha\,;{\scriptscriptstyle\mathcal{P}})$$
$$\sigma(A_1(\varphi)) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?^n_{\mathcal{P}}\beta\,;{\scriptscriptstyle\mathcal{P}})$$

    Thus by induction $\Gamma \vdash_i \alpha$ and $\Gamma \vdash_i \beta$, thus $\Gamma \vdash_i \alpha \wedge \beta$.
  - The formula is $\varphi = \alpha \to \beta$, then

$$\sigma(A(\varphi)) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, !_{\mathcal{O}}\alpha,\, ?^n_{\mathcal{P}}\beta\,;{\scriptscriptstyle\mathcal{P}}),$$

    thus by induction $\Gamma, \alpha \vdash_i \beta$, thus $\Gamma \vdash_i \alpha \to \beta$.
  - The formula is $\varphi = \forall x\alpha(x)$, then

$$\sigma(A_t(\varphi)) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?^n_{\mathcal{P}}\alpha(t)\,;{\scriptscriptstyle\mathcal{P}}),$$

    for any term t, then in particular by induction $\Gamma \vdash_i \alpha(x)$ for some $x \notin FV(\Gamma \cup \{\alpha\})$, then $\Gamma \vdash_i \forall x\alpha(x)$. $\square$

4.5.2 Completeness

Theorem 4.12 (Completeness)**.** Let $G = !_{\mathcal{O}}\Gamma,\, ?^n_{\mathcal{P}}\alpha$, then

$$\text{If } \Gamma \vdash_i \alpha, \text{ then } Str^w_{\mathcal{P}}(G\,;{\scriptscriptstyle\mathcal{P}}) \neq \emptyset$$

Proof. By induction on the height of the derivation of the sequent $\Gamma \Rightarrow \psi$.

- (Basecase) $\Gamma \Rightarrow \psi$ is an instance of an axiom, we have two cases:
  - We have $A \in \Gamma$ and $\alpha = A$, then $(G\,;{\scriptscriptstyle\mathcal{P}})$ is a terminal state, so

$$Str^w_{\mathcal{P}}(G\,;{\scriptscriptstyle\mathcal{P}}) \neq \emptyset.$$

  - We have $\perp \in \Gamma$, then $(G\,;{\scriptscriptstyle\mathcal{P}})$ is a terminal state, so

$$Str^w_{\mathcal{P}}(G\,;{\scriptscriptstyle\mathcal{P}}) \neq \emptyset.$$

- The last rule used in the derivation is $\wedge_R$ thus $G = H,\, ?^n_{\mathcal{P}}\varphi \wedge \psi$, then by induction we have:

$$\sigma \in Str^w_{\mathcal{P}}(H,\, ?^n_{\mathcal{P}}\varphi\,;{\scriptscriptstyle\mathcal{P}})$$
$$\tau \in Str^w_{\mathcal{P}}(H,\, ?^n_{\mathcal{P}}\psi\,;{\scriptscriptstyle\mathcal{P}}).$$

Let

$$\rho(A_0(\varphi \wedge \psi)) = \sigma$$
$$\rho(A_1(\varphi \wedge \psi)) = \tau.$$

We construct a strategy in $Str^w_{\mathcal{P}}(G\,;{\scriptscriptstyle\mathcal{P}})$ as follows:

$$\frac{\rho \in Str^w_{\mathcal{P}}(H,\, !_{\mathcal{P}}\varphi \wedge \psi\,;{\scriptscriptstyle\mathcal{O}})}{(D(\varphi \wedge \psi), \rho) \in Str^w_{\mathcal{P}}(G\,;{\scriptscriptstyle\mathcal{P}})}\ .$$

- The last rule used in the derivation is $\wedge_L$, thus $G = H,\, !_{\mathcal{O}}\varphi \wedge \psi$, then by induction and weakening we have:

$$\frac{\sigma \in Str^w_{\mathcal{P}}(H,\, !_{\mathcal{O}}\varphi,\, !_{\mathcal{O}}\psi\,;{\scriptscriptstyle\mathcal{P}})}{Wk(\sigma) \in Str^w_{\mathcal{P}}(G,\, !_{\mathcal{O}}\varphi,\, !_{\mathcal{O}}\psi\,;{\scriptscriptstyle\mathcal{P}})}$$

We construct a strategy in $Str^w_{\mathcal{P}}(G\,;{\scriptscriptstyle\mathcal{P}})$ as follows:

$$\frac{\dfrac{Wk(\sigma) \in Str^w_{\mathcal{P}}(G,\, !_{\mathcal{O}}\varphi,\, !_{\mathcal{O}}\psi\,;{\scriptscriptstyle\mathcal{P}}) \qquad \dfrac{Id \in Str^w_{\mathcal{P}}(\, !_{\mathcal{O}}\varphi \wedge \psi,\, ?^m_{\mathcal{O}}\varphi,\, ?^n_{\mathcal{P}}\varphi\,;{\scriptscriptstyle\mathcal{O}})}{(a, Id) \in Str^w_{\mathcal{P}}(\, !_{\mathcal{O}}\varphi \wedge \psi,\, ?^n_{\mathcal{P}}\varphi\,;{\scriptscriptstyle\mathcal{P}})}}{Cut_{\varphi}(Wk(\sigma), (a, Id)) \in Str^w_{\mathcal{P}}(G,\, !_{\mathcal{O}}\psi\,;{\scriptscriptstyle\mathcal{P}})} \qquad \dfrac{Id \in Str^w_{\mathcal{P}}(\, !_{\mathcal{O}}\varphi \wedge \psi,\, ?^m_{\mathcal{O}}\psi,\, ?^n_{\mathcal{P}}\psi\,;{\scriptscriptstyle\mathcal{O}})}{(b, Id) \in Str^w_{\mathcal{P}}(\, !_{\mathcal{O}}\varphi \wedge \psi,\, ?^n_{\mathcal{P}}\psi\,;{\scriptscriptstyle\mathcal{P}})}}{Cut_{\psi}((b, Id), Cut_{\varphi}(Wk(\sigma), (a, Id))) \in Str^w_{\mathcal{P}}(G\,;{\scriptscriptstyle\mathcal{P}})}$$

Where $a = A_0(\varphi \vee \psi)$ and $b = A_1(\varphi \vee \psi)$.

- The last rule used in the derivation is $\vee_R$, thus $G = H,\, ?^n_{\mathcal{P}}\varphi \vee \psi$, then by induction we have:

$$Str^w_{\mathcal{P}}(H,\, ?^n_{\mathcal{P}}\varphi\,;{\scriptscriptstyle\mathcal{P}}) \neq \emptyset \text{ or } Str^w_{\mathcal{P}}(H,\, ?^n_{\mathcal{P}}\psi\,;{\scriptscriptstyle\mathcal{P}}) \neq \emptyset,$$

without loss of generality we construct a strategy in $Str^w_{\mathcal{P}}(G\,;{\scriptscriptstyle\mathcal{P}})$ as follows:

$$\frac{\sigma \in Str^w_{\mathcal{P}}(H,\, ?^n_{\mathcal{P}}\varphi\,;{\scriptscriptstyle\mathcal{P}}) \qquad \dfrac{Id \in Str^w_{\mathcal{P}}(\, !_{\mathcal{P}}\varphi,\, !_{\mathcal{O}}\varphi\,;{\scriptscriptstyle\mathcal{O}})}{(a, Id) \in Str^w_{\mathcal{P}}(\, ?^n_{\mathcal{P}}\varphi \vee \psi,\, !_{\mathcal{O}}\varphi\,;{\scriptscriptstyle\mathcal{P}})}}{Cut_{\varphi}(\sigma, (a, Id)) \in Str^w_{\mathcal{P}}(G\,;{\scriptscriptstyle\mathcal{P}})}$$

Where $a = D_0(\varphi \vee \psi)$.

- The last rule used in the derivation is $\vee_L$, thus $G = H,\, !_{\mathcal{O}}\varphi \vee \psi$, then by induction there are strategies:

$$\sigma \in Str^w_{\mathcal{P}}(H,\, !_{\mathcal{O}}\varphi\,;{\scriptscriptstyle\mathcal{P}})$$
$$\tau \in Str^w_{\mathcal{P}}(H,\, !_{\mathcal{O}}\psi\,;{\scriptscriptstyle\mathcal{P}}).$$

Let

$$\rho(D_0(\varphi \vee \psi)) = \sigma$$
$$\rho(D_1(\varphi \vee \psi)) = \tau.$$

We construct a strategy in $Str^w_{\mathcal{P}}(G\,;{\scriptscriptstyle\mathcal{P}})$ as follows:

$$\frac{\rho \in Str^w_{\mathcal{P}}(H,\, ?^n_{\mathcal{O}}\varphi \vee \psi\,;{\scriptscriptstyle\mathcal{O}})}{(D(\varphi \vee \psi), \rho) \in Str^w_{\mathcal{P}}(G\,;{\scriptscriptstyle\mathcal{P}})}$$

- The last rule used in the derivation is $\rightarrow_R$, thus $G = H,\, ?^n_{\mathcal{P}}\varphi \rightarrow \psi$, then by induction:

$$Str^w_{\mathcal{P}}(H,\, !_{\mathcal{O}}\varphi,\, ?^n_{\mathcal{P}}\psi\,;{\scriptscriptstyle\mathcal{P}}) \neq \emptyset.$$

We construct a strategy in $Str^w_{\mathcal{P}}(G\,;{\scriptscriptstyle\mathcal{P}})$ as follows:

$$\cfrac{\cfrac{\sigma \in Str^w_{\mathcal{P}}(H,\, !_{\mathcal{O}}\varphi,\, ?^n_{\mathcal{P}}\psi\,;{\scriptscriptstyle\mathcal{P}})}{\lambda x.\sigma \in Str^w_{\mathcal{P}}(H,\, !_{\mathcal{P}}\varphi \rightarrow \psi)}}{(D(\varphi \rightarrow \psi), \lambda x.\sigma) \in Str^w_{\mathcal{P}}(G\,;{\scriptscriptstyle\mathcal{P}})}\ .$$

- The last rule used in the derivation is $\rightarrow_L$, thus $G = H,\, !_{\mathcal{O}}\varphi \rightarrow \psi$, let $H = J,\, ?^n_{\mathcal{P}}\alpha$, then by induction and weakening we have

$$Str^w_{\mathcal{P}}(J,\, ?^n_{\mathcal{P}}\varphi\,;{\scriptscriptstyle\mathcal{P}}) \neq \emptyset$$
$$Str^w_{\mathcal{P}}(G,\, !_{\mathcal{O}}\psi\,;{\scriptscriptstyle\mathcal{P}}) \neq \emptyset,$$

we construct a strategy in $Str^w_{\mathcal{P}}(G\,;{\scriptscriptstyle\mathcal{P}})$ as follows:

$$\cfrac{\cfrac{\sigma \in Str^w_{\mathcal{P}}(J,\, ?^n_{\mathcal{P}}\varphi\,;{\scriptscriptstyle\mathcal{P}}) \qquad Mp \in Str^w_{\mathcal{P}}(J,\, !_{\mathcal{O}}\varphi,\, ?^n_{\mathcal{P}}\psi\,;{\scriptscriptstyle\mathcal{P}})}{Cut_\varphi(\sigma, Mp) \in Str^w_{\mathcal{P}}(J,\, ?^n_{\mathcal{P}}\psi\,;{\scriptscriptstyle\mathcal{P}})} \qquad \tau \in Str^w_{\mathcal{P}}(G,\, !_{\mathcal{O}}\psi\,;{\scriptscriptstyle\mathcal{P}})}{Cut_\psi(Cut_\varphi(\sigma, Mp), \tau) \in Str^w_{\mathcal{P}}(G\,;{\scriptscriptstyle\mathcal{P}})}\ .$$

- The last rule used in the derivation is $\exists_R$, thus $G = H,\, ?^n_{\mathcal{P}}\exists x\varphi(x)$, then by induction, for some term t:

$$Str^w_{\mathcal{P}}(H,\, ?^n_{\mathcal{P}}\varphi(t)\,;{\scriptscriptstyle\mathcal{P}}) \neq \emptyset$$

we construct a strategy in $Str^w_{\mathcal{P}}(G\,;{\scriptscriptstyle\mathcal{P}})$ as follows:

$$\cfrac{\sigma \in Str^w_{\mathcal{P}}(H,\, ?^n_{\mathcal{P}}\varphi(t)\,;{\scriptscriptstyle\mathcal{P}}) \qquad \cfrac{Id \in Str^w_{\mathcal{P}}(\,!_{\mathcal{P}}\varphi(t),\, !_{\mathcal{O}}\varphi(t)\,;{\scriptscriptstyle\mathcal{O}})}{(a, Id) \in Str^w_{\mathcal{P}}(\,?^n_{\mathcal{P}}\exists x\varphi(x),\, !_{\mathcal{O}}\varphi(t)\,;{\scriptscriptstyle\mathcal{P}})}}{Cut_\varphi(\sigma, (a, Id)) \in Str^w_{\mathcal{P}}(G\,;{\scriptscriptstyle\mathcal{P}})}$$

Where $a = D_t(\exists x\varphi(x)$, the case for $\forall_L$ is similar.

- The last rule used in the derivation is $\forall_R$, thus $G = H,\, ?^n_{\mathcal{P}}\forall x\varphi(x)$, by theorem 2.5, we have that $\Gamma \vdash_i \varphi(t), \psi$ for any term t, thus by induction for any term t there is a strategy:

$$\sigma_t \in Str^w_{\mathcal{P}}(H,\, ?^n_{\mathcal{P}}\varphi(t)\,;{\scriptscriptstyle\mathcal{P}}).$$

Let $\rho(A_t(\forall x\varphi(x))) = \sigma_t$. We construct a strategy in $Str^w_{\mathcal{P}}(G\,;{\scriptscriptstyle\mathcal{P}})$ as follows:

$$\frac{\lambda x.\rho(x) \in Str^w_{\mathcal{P}}(H,\, !_{\mathcal{P}}\forall x\varphi(x)\,;{\scriptscriptstyle\mathcal{O}})}{(D(\forall x\varphi(x)), (\lambda x.\rho(x))) \in Str^w_{\mathcal{P}}(G\,;{\scriptscriptstyle\mathcal{P}})}$$

The case for $\exists_L$ is similar. $\square$

4.5.3 Adequacy for the System G3I*

Just as in the classical case we can also identify winning strategies with proofs in a system $G3I^*$ by making the following observations:

- The assertion $!_{\mathcal{O}} A \to \psi$ is never attacked, unless the opponent has already stated $!_{\mathcal{O}} A$.
- The left side of challenge $?^n_{\mathcal{P}} A \vee \psi$ is never defended unless the opponent has already stated $!_{\mathcal{O}} A$.
- The right side of challenge $?^n_{\mathcal{P}} \psi \vee A$ is never defended unless the opponent has already stated $!_{\mathcal{O}} A$.
- The challenge $?^n_{\mathcal{P}} \exists x A(x)$ is never defended unless the opponent has already stated $!_{\mathcal{O}} A(t)$ for some term t.

Definition 4.6. Let $G3I^*$ be just as $G3I$ except that we replace the rule $\to_L$ with two rules, one atomic and one non-atomic:

$$\frac{\Gamma, A, \psi \Rightarrow \beta}{\Gamma, A, A \to \psi \Rightarrow \beta} \to_{At} \qquad \frac{\Gamma, \varphi \to \psi \Rightarrow \varphi \qquad \Gamma, \psi \Rightarrow \beta \qquad \varphi \text{ non-atomic}}{\Gamma, \varphi \to \psi \Rightarrow \beta} \to_L$$

If $\Gamma \Rightarrow \psi$ is derivable in $G3I^*$ we write $\Gamma \vdash_i^* \psi$. As in the classical case the rule $\to_{At}$ is interesting since it shrinks the search space for proofs since it puts a restriction on how some derivations may be produced.

Theorem 4.13.

$$\Gamma \vdash_i^* \psi \iff \Gamma \vdash_i \psi$$

Proof. For the left to right direction we note that all rules of $G3I^*$ are admissible in $G3I$, in particular the rule $\to_{At}$ corresponds to:

$$\frac{\Gamma, A \Rightarrow A \qquad \Gamma, A, \psi \Rightarrow \alpha}{\Gamma, A, A \to \psi \Rightarrow \alpha} .$$

For right to left, if $\Gamma \vdash_i \psi$, then by completeness $Str^w_{\mathcal{P}}(!_{\mathcal{O}}\Gamma,\, ?^n_{\mathcal{P}}\psi\,;_{\mathcal{P}}) \neq \emptyset$ it is then immediate that $\Gamma \vdash_i^* \psi$. □

4.5.4 Cut Elimination for G3I and G3I*

Given that we have the cut strategy we have the following:

$$\frac{\sigma \in Str^w_{\mathcal{P}}(!_{\mathcal{O}}\Gamma,\, ?^n_{\mathcal{P}}\psi,\, !_{\mathcal{O}}\varphi\,;_{\mathcal{P}}) \qquad \tau \in Str^w_{\mathcal{P}}(!_{\mathcal{O}}\Gamma',\, ?^n_{\mathcal{P}}\varphi\,;_{\mathcal{P}})}{Cut_\varphi(\sigma,\tau) \in Str^w_{\mathcal{P}}(!_{\mathcal{O}}\Gamma,\, ?^n_{\mathcal{P}}\psi,\, !_{\mathcal{O}}\Gamma'\,;_{\mathcal{P}})} .$$

We immediately get that the cut rule

$$(Cut) \qquad \frac{\Gamma \Rightarrow \varphi \qquad \varphi, \Gamma' \Rightarrow \psi}{\Gamma, \Gamma' \Rightarrow \psi}$$

is admissible in $G3I$ and $G3I^*$. Furthermore, if we allow $G3I$ or $G3I^*$ derivations containing applications of the cut-rule, we can add the following case to the completeness proof:

- The last rule in the derivation of $\Gamma \vdash_i \psi$ was cut with cut-formula φ. By induction we have strategies:

$$\sigma \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma_0,\, !_{\mathcal{O}}\varphi,\, ?^n_{\mathcal{P}}\psi\,;_{\mathcal{P}})$$
$$\tau \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma_1,\, ?^m_{\mathcal{P}}\varphi\,;_{\mathcal{P}}).$$

  Where $\Gamma = \Gamma_0, \Gamma_1$. Thus, we have a strategy

$$Cut_{\varphi}(\sigma, \tau) \in Str^w_{\mathcal{P}}(\,!_{\mathcal{O}}\Gamma,\, ?^n_{\mathcal{P}}\psi\,;_{\mathcal{P}}).$$

Using the soundness part of the proof this strategy can then be transferred back to a $G3I$ or $G3I^*$ derivation not containing cut. Thus effectively eliminating the cut from the proof.

5 Comparison to Other Works

The games presented in this thesis are strongly related to the dialogue games of [Lorenzen and Lorenz., 1978] which can be characterized by *particular rules* defining admissible attacks and defenses on particular formulas and *structural rules* defining the general play of the game. The standard particular rules for dialogue games where first defined in [Lorenz, 1961] and are laid out in the following table:

Claim	Attack	Defense
$\varphi \wedge \psi$	$?_l$	φ
	$?_r$	ψ
$\varphi \vee \psi$	$?$	φ
		ψ
$\varphi \to \psi$	φ	ψ
$\neg\varphi$	φ	
$\forall x \varphi(x)$	$?_n$	$\varphi(n)$
$\exists x \varphi(x)$	$?$	$\varphi(n)$

Thus, basically the same as the rules defined in this thesis. Following [Krabbe, 1985] and [Felscher, 1985] the structural rules for dialogue games may be divided into four types D, Di, E and Ei. The Di-dialogues have the same structural conditions for moves as in our intuitionistic games with the exceptions that:

- In our games there exist winning states for the opponent while in the Di-dialogues no such states exists.
- In our games negative formulas may only be defended once and positive formulas may only be attacked once by either player while in the Di-dialogues the proponent may attack and defend formulas indefinitely.
- In our games the proponent wins only if he can meet all atomic challenges while in the Di-dialogues the proponent wins if he meets any challenge. (This is similar to the winning conditions for classical games).

The D and E-dialogues are the same as the Di and Ei-dialogues except that they are played according to the *normal play convention*: If a player has no moves the opposing player wins. As far as I am aware all soundness and completeness proofs of the D and Di-dialogues are done by first reducing the winning strategies to strategies in E-dialogues, which are just as the D-dialogues with the additional rule:

- (E) The opponent can only respond to the proponents immediately preceding move.

Then a correspondence is established between proponent winning strategies in E-dialogues and derivations in some suitable proof system. This is done in [Krabbe, 1985] and [Felscher, 1985] for intuitionistic logic. For classical logic the games considered are typically of the E-type from the beginning. there are proofs for such dialogues in [Fermüller, 2003], [Sørensen and Urzyczyn, 2007], and a short proof in [Alama et al., 2011] which relies on techniques from the previous paper.

The reductions of D to E-dialogues considered in the literature are complicated and often rather informal steming from the fact that games and strategies are not treated combinatorially in any of the above works. Consequently, there are no notions of copy-cat strategies, parallel strategies, composition of strategies, and so on. Using the tools considered in this thesis it seems possible to establish the correspondence of D and E by using the notions of *left* and *right*-strategies, the *parallel*-strategy and *contraction*.

Historically the first to treat strategies as combinatorial objects was Joyal [Joyal, 1977] following the combinatorial game theory of [Conway, 1976]. There are two precursors to the operation of union of games and dual of games as considered in this thesis: the *disjunctive sum* and *negative* of the combinatorial games of Conway [Conway, 1976, Berlekamp et al., 1982], and the *parallel sum* and *dual* of linear logic games, [Abramsky et al., 1997, Blass, 1992, Hyland and Ong, 2000]. While the duality operations are virtually the same in the above works the sum operations are somewhat different. We give a non-formal description of these operations.

- The moves in the disjunctive sum $G + H$ are the moves in each component games put together.
- The moves in the parallel sum $G \oplus H$ are the moves in each component game put together; however, when the proponent moves in either of the two component games, the opponent must respond in the same component.

The operation of union of games considered in this thesis closely resembles the disjunctive sum in that the games are played *concurrently* rather than in parallel. This difference disappears if we consider games on formulas which do not contain implications, since then the opponent has no way of backtracking.

6 Conclusion and Further Work

In summary we have defined game semantics for both classical and intuitionistic first-order logic with explicit strategies. The main results has been to establish a correspondence of proponent winning strategies with derivations in the calculi $G3C$ and $G3I$. Furthermore, the correspondence has allowed us to prove adequacy for the restricted calcli $G3C^*$ and $G3I^*$ and in addition also to prove cut-elimination for all the above proof systems.

To arrive at these results properties of the games and winning strategies have been examined. In particular, we have explicitly defined several operations on strategies that may be of further interest for similar game semantics. In particular the operations of *application* and *composition* have been defined giving the strategies a computational interpretation.

6.1 Further Research

We conclude with a discussion of possible variations and extensions of the game semantics defined.

Defining games following the normal play convention. As we saw in section 5 this is the standard winning condition for several game semantics. Changing the rules in this way would likely simplify the operations on games while still enabling us to define games for both classical and intuitionistic logic.

Defining games for different logical systems. The most obvious candidates being different intermediary logics, linear logic, and type theory. For type theory in particular games can be constructed where the objective of the proponent is to provide terms corresponding to types. Thus, the winning move for the proponent would be assigning an already existing term of a type to the same type:

$$\frac{(G,\, ?_{\mathcal{P}} e : A\, ;\, \mathcal{P}) \qquad !_{\mathcal{O}} d : A \in G}{(G,\, !_{\mathcal{P}} e = d : A\, ;\, \mathcal{O})}$$

If we consider a language with implication as its single connective it suffices to have the rules:

$$\frac{(G,\, ?_{\circ} e : \varphi\, ;\, \circ)}{(G,\, !_{\circ} e : \varphi\, ;\, \bullet)} \qquad \frac{(G,\, !_{\bullet} f : \varphi \to \psi\, ;\, \circ)}{(G,\, !_{\circ} y : \varphi,\, ?_{\bullet}(f\, y) : \psi\, ;\, \bullet)}$$

The resulting games then give a formal basis for defining functions by pattern-matching.

Defining games allowing for an infinite number of attacks and defenses for both players. This would make the games more "fair" since both players have the same "strength". Also, intuitively in a type-theoretic or linear logic game we would want a strategy $\sigma \in Str^{w}_{\mathcal{P}}(\mathbb{N} \to \mathbb{N}\, ;\, \mathcal{O})$ to correspond to an arbitrary function $f : \mathbb{N} \to \mathbb{N}$, however if σ is required to be well-founded σ can at most correspond to an arbitrary computable function. To define such infinite games a criteria is needed to distinguish proponent and opponent winning infinite plays.

Examine opponent winning strategies. This could possibly reduce proving negative claims about the proof systems, such as underivability to proving positive claims about the existence of a opponent winning strategy. In fact any claim that there exists a function $f : Str^w_{\mathcal{P}}(G) \to Str^w_{\mathcal{P}}(H)$ can be translated by contraposition to the claim that there exist a function $g : Str^w_{\mathcal{O}}(H) \to Str^w_{\mathcal{O}}(G)$. Furthermore, examining the opponent winning strategies could help to design so called refutation systems since the following relation holds between opponent strategies and refutations

$$Str^w_{\mathcal{O}}(\,!_{\mathcal{O}}\Gamma,\, ?_{\mathcal{P}}\Delta) \iff \Gamma \not\vdash \Delta.$$

To examine the opponent winning strategies closer the following questions would need to be answered: How do we define opponent strategies? How do we show that an opponent winning strategy is winning? Are there any interesting operations at all for the opponent winning strategies?

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