When Being the Fifth Wheel Pays Off:
Wisdom of the Crowds with Costly Information

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written by

Matteo Michelini
(born April 30th, 1996 in Firenze, Italy)

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Date of the public defense:	**Members of the Thesis Committee:**
July 12th, 2021	Prof. dr. Ulle Endriss
	Dr. Davide Grossi
	Dr. Adrian Haret
	Dr. Arianna Novaro
	Prof dr. Yde Venema (*chair*)

Institute for Logic, Language and Computation

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Abstract

The Condorcet Jury Theorem (CJT) is considered one of the cornerstones of the *wisdom of the crowd*, i.e. the idea that large groups of people are better at tracking the truth than small ones. However, such a result is often criticized on the ground that its assumptions are unrealistic. In this thesis, we provide a new perspective on the truth-tracking potential of a group by introducing in the CJT setting a more realistic point of view. In particular, we consider a group whose members are interested in finding the right answer to a question, but where being *competent* is costly. Consequently, agents can either acquire information by spending effort and express a vote, or abstain. We model this scenario game-theoretically and characterize its equilibria. In addition, we also look at the social welfare of the game thus defined. We use our formal results to show that some of the claims of the CJT do not hold under our proposed assumptions. For example, we prove that larger groups do not necessarily produce better outcomes if the group size surpasses a certain threshold.

Next, we extend this basic setting in three different directions. Firstly, we study the possibility for agents of having different stakes in the matter, i.e. to be less or more interested. We prove that a group can be as accurate as possible only if the agents with the higher stakes vote, and similarly we show which groups are more efficient in terms of social welfare. Secondly, we describe the game in the case where agents may reach different levels of competence. In this scenario, we study the pure equilibria, and we also introduce other possible ways of characterizing this scenario further. We also discuss the notion of weight of a voter and the possibility of delegating.

Lastly, we consider the possibility for the agents of being connected through a network. We characterize equilibria on distinct classes of networks and we comment these results. We integrate our analytical results with a computer simulation of best response dynamics in those games.

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Contents

1 Introduction

1.1 Background and Motivation

The Right Decision. Making decisions can be as easy as choosing pasta rather than rice for your lunch or as hard as deciding whether to marry or not. However, sometimes there exists a good option and one (or more than one) bad option(s). In the same way as there is a correct answer to certain questions, there are situations in which a decision is correct and another is wrong, such that the case of a jury that is called to judge a defendant. It should always be possible, at least in principle, to tell if the defendant is guilty or not, if we were exactly aware of all the facts that happened. This is the reason why the jury is usually provided with as much evidence as possible and is then asked to take a decision: each piece of evidence is supposed to bring the jurors closer to the truth. There are may situations in which we, as a society, usually believe in the existence of a certain truth.[1] Although we are aware of the difficulties that we may encounter to find it, we also hope that, by asking the 'right people', we can get closer to the truth. This is, for example, the mission of the so-called *Good Judgement Project*, which aggregates the opinions of many people and tries to make reliable predictions on geopolitical phenomena.[2]

Experts or Crowds? However, there has been much debate in science and philosophy about how to form (and who to include in) the entities (institutions, groups, individuals) dedicated to this task, i.e. to find the correct answer. For example, Plato argued in favour of leaving this responsibility to *philosophers* (Sharples, 1994). His main point was that since those are the people who study the truth itself, there would be no point in asking anybody else. Still today, we acknowledge the existence of many groups of *experts*, whose opinions are more relevant when specific decisions are taken: pools of virologists, epidemiologists and doctors are formed to settle the policies against COVID-19, groups of economists

[1] Of course, this is not always the case. Often decisions are also about *values*, and in that case, we cannot say anything about 'correctness' of a decision.

[2] See the website `https://goodjudgment.com/` and some articles (Schoemaker and Tetlock, 2016; Tetlock, 2009; Ungar et al., 2012).

and brokers are chosen to predict market trends. On the other hand, many others (especially in most recent times) argued in favour of allowing bigger groups of people to take decisions. This is the idea of the *wisdom of the crowds*, i.e. the belief that the more people we involve in the decision the more likely they are to find the right answer. One of the most iconic experiments that has been carried out in this sense is the *weighting of the ox*. In such an experiment, Galton (1907) reported the final outcome of a weight judging competition where every participant was supposed to guess the weight of a fat ox in order to win a prize. Galton's collected data shows that the crowd was, on average, within 1 lb of the true weight. After his experiment, many other researchers assessed the efficiency of large crowds in many fields (Sunstein, 2006; Surowiecki, 2005). De Condorcet (1785) was probably one of the first philosophers who tried to support the idea of the wisdom of the crowds with a formal and theoretical argument.[3] He believed that politics was all about finding the correct answer to problems in the same way Plato did. Yet, unlike him, he wanted to show that for this reason, it would have been better to outsource these decisions to as many people as possible.

The Condorcet Jury Theorem (CJT). Condorcet proposed a simple formal model that was meant to compare the individual performance with the collective one (Grofman, Owen, and Scott L. Feld, 1983; Nitzan and Paroush, 2017). His model consists of a group that is asked to find the right answer in a binary choice. It assumes that each agent votes and aggregates the answers through majority rule. Then, given that each agent has a probability of finding the right answer, the model computes the overall probability of the group. Condorcet's model makes three important assumptions. First of all, it assumes that the agents are competent, i.e. each agent is more likely to find the right answer than the toss of a fair coin. Secondly, it considers homogeneous agents, i.e. each agent has the same probability of all others to find the correct answer. And, lastly, it assumes that the agents choose independently one from each other. Thus, the following claims can be shown through a formal proof.

(GBI) Groups are Better than Individuals: the group, as defined by the majority opinion, is at least as accurate as each of the individual members.

(LIB) Larger is Better: accuracy of the group improves as its size grows.

(ASY) The accuracy of the group approaches 1 asymptotically.

Although the assumptions are quite strong and it may be debatable if such a theorem really tells us something about reality, Condorcet's work has had a huge

[3] See either Nitzan and Paroush (1982) or Grofman, Owen, and Scott L. Feld (1983) for a modern formulation of Condorcet's results.

impact on the literature about *truth-tracking*. In the formal literature it contributed to the creation of a framework in which trying to answer questions concerning expertise and group accuracy. Besides this, it also became one of cornerstone of many contemporary philosophical political theories. For example, CJT directly fuelled the idea of an epistemic justification of democracy.[4]

Competence Requires Effort Many researchers questioned the adequacy of the three assumptions that Condorcet made.[5] Here we want to focus on the assumption about the *competence* of the agents. In particular, we question the idea that the competence comes for free. In our model, competence is an endogenous variable, obtained as the result of exerting effort. Indeed, it is demanding to find good pieces of information and even if the agents were provided with reliable pieces of evidence, the full understanding of them (and the translation into beliefs) still costs them some effort. We experience such effort every day, for example, when we try to make sense of the daily news, when we study or when we learn about pros and cons of a new diet. It may be true that if individuals were able to gather and learn information about the matter at stakes freely they would be also able to select the right option with no doubt, but what happens if this is not the case? Is still a *crowd* better than a bunch of experts? What if agents decide that learning is not worth it? Or what if we witness the phenomenon of free-riding (Albanese and Van Fleet, 1985)? Some members of the group may decide that the effort of the others is already enough and that there is no reason for them to also 'become competent'. Such questions become even more pressing when direct democracy is compared to other systems, which have among their advantages the possibility of allowing citizens to spend less effort in a decision. Indeed, one of the many strengths of representative systems is exactly this one: saving the majority of citizens from acquiring competence about things (Blum and Zuber, 2016).

Buying a Fridge (in a Shared Flat). Consider a group of five flatmates that has to decide what fridge to buy (as they broke their old one at a party). They decided to choose between two models of two different brands, and to do so through a majority vote. They may be aware of the fact that if they all put some effort in studying the technical specifications of fridges and vote independently they will be more likely to choose the most suited model, and still not do so. Instead, they may publicly discuss what they want to do and it may happen that just one of them, Adam, chooses to put effort in the decision and 'expresses a vote', whereas

[4]The epistemic justification consists of showing that the epistemic outcome of democracy is better than the outcome of any other political system (Estlund, 2008). The CJT has always been one of the most influential result backing up such an idea (Cohen, 1986; Goodin and Spiekermann, 2018).

[5]See the Section 1.3 for a good overview.

the others ‘abstain’. They may not care much about fridges, or they may just be super busy with other things. It is natural to model this scenario in terms of costs and benefits. The participation of all the five flatmates in the decision would have yielded a more accurate decision, but at the cost of a higher effort for each of them. Why Adam, though? Adam could be extremely skilled at evaluating house appliances. Or, he may have had a special interest in buying a good fridge, as he always buys food for the next three weeks. We reckon that the degree of interest that each flatmate has in the accuracy of the decision is indeed fundamental. And, similarly, the level of competence that each of the flatmates can reach plays an important role. Eventually, consider a situation in which the flatmates discuss the choice between ‘learning and voting’ and ‘not learning and then not voting’ in small groups. Adam and Bob may not be aware of the decisions of Charlie, Duncan and Elodie and viceversa. So, Elodie may decide she will learn information about fridges and vote and Adam may do the same, although if Elodie had had information about Adam’s choice she would have not done so. A final question questions arises naturally: how does information about other agents shapes the agent’s behaviour?

CJT May Not Work in Shared Flats if Information is Too Costly. In general, although the Condorcet Jury Theorem would have predicted that a group consisting of five members would have actually improved the quality of the decision compared to a household composed by two members, this was not the case for the fridge. Even if they were five they still reached the same accuracy that a single person would have had. Indeed, not all the people contributed to the decision. The aim of this thesis is exactly to investigate the behaviour of agents in situations where the group has to take a decision and information is costly. Since we believe that agents with different stakes in the matter may behave differently we also take into account this possibility. Someone may care more than someone else. In a similar fashion, we allow for heterogeneous competences among the agents, as someone may be more skilled in finding the truth. Last but not least, we introduce in our setting the possibility for agents to not be aware of all the other agents’ choices. Indeed, as we saw in the example, it can easily be that some decisions are taken without knowing about other people’s actions.

1.2 Our Contributions

We explore the setting of the Condorcet Jury Theorem assuming that every agent can acquire competence only by spending some effort. Such a competence corresponds to a probability of finding the right answer higher than the toss of a fair coin. And, the effort stands for the combination of the cost of *finding information* with the cost of *understanding information*. To model this scenario we introduce a game theoretical environment in which each agent can either decide to learn by spending effort or not. Each agent spends the same effort to reach a certain accuracy and each agent is interested in finding the truth. Consistently, if the agents learn information they also vote, whereas if they do not, they abstain.[1] The votes are aggregated through majority rule. Every agent makes her decision on the basis of the behaviour of the other agents. We mainly aim at understanding what are the pure Nash equilibria in such games and consequently, what is the accuracy that we can expect every group to produce (Leyton-Brown and Shoham, 2008; Nash, 1950). Indeed, such concepts allow predicting the players' choices, given that the players are interdependent. In addition, we also study the social welfare of these equilibria. We mainly study this game in four different settings.

1. Firstly, we consider the basic case in which all the agents may reach the same competence by spending the same effort. All the agents have full information about the behaviour of the other agents and they are also equally interested in finding the truth. With this basic setting we prove quite strong and fundamental results and, at the same time, we prepare the ground for the following variations.

2. Secondly, we consider groups where agents have different stakes in the matter. This is the case, when, for example, two of the five flatmates of the example above are more interested than the other three in finding a good fridge. The other features remain as in the basic setting.

3. Thirdly, we study situations in which agents are able to reach different levels of competences by spending the same effort. As we mentioned, Adam may be extremely quick and efficient in learning about fridges (it takes him the same effort to acquire way more confidence in the right decision, if compared with the other flatmates).

4. Eventually, we consider the possibility for groups of not being completely connected, i.e. we add an underlying network that describes which other agents a specific agent can see. Consequently, not everybody may have full information about everybody else. Indeed, Elodie may not be aware of the decision of Adam.

[1] Indeed, the effort could contain the cost of the act of voting itself, as it is paid by the agents if they decide to learn and vote. Indeed, there are often requirements that must be fulfilled to vote. These may include having an ID, going to certain places, etc. And, sometimes, voting even has an actual cost in terms of money. For example, this is the case of the so-called *open primaries* in Italy (`https://en.wikipedia.org/wiki/2019_Democratic_Party_(Italy)_leadership_election#Nicola_Zingaretti`).

In the basic setting of the CJT with just the addition of the cost of voting, we characterize both the pure Nash equilibria and the strong pure Nash equilibria. We show that there exist many pure Nash equilibria, and that in general increasing the size of the group beyond a certain threshold does not result in more agent participating in the process, and thus the accuracy of the group does not increase. We prove that this threshold depends on the effort. In addition, we describe the equilibria in terms of social welfare, and we show that the larger the size of the group the higher is the difference between the welfare of the equilibria and the maximum welfare that can be reached by the group trough a profile. We show that it is true that in an equilibrium the result `(GBI)` of CJT is respected, whereas `(LIB)` and `(ASY)` do not hold anymore.

We do the same for groups where agents have different stakes in the matter. We show how the maximal accuracy at an equilibrium can be computed and how this is influenced by the different stakes. Again, we show how the social welfare changes with respect to different stakes. Indeed, we prove that for the group to reach its highest accuracy at an equilibrium the people with the highest stakes should vote, and similarly, in order to reach the equilibrium with the highest social welfare.

Then, we take into account groups where agents can reach different levels of competences by spending the same amount of effort. Again, we characterize the equilibria and we also discuss briefly the social welfare and some general features of the model. We also discuss what could be the advantages of a delegative system in such a framework.

Eventually, we introduce an underlying network connecting the agents to assess what happens if each agent is just able to see the actions of a part of the other members. We characterize equilibria on certain types of networks and then with the help of computer simulations we investigate the general effects of a network on the accuracy of the groups. We make use of the concept of Best-Response-Dynamics. We show that the lower the number of the links the higher is the accuracy of the group. In addition, we show that also on networks, if there are agents with different stakes, those with the highest ones are more likely to vote. Eventually, we look at the possibility of finding stability if we also allow coalitions to move together (as in the case of Strong Nash Equilibrium).

Notably, we believe that our work lays the ground for a more detailed perspective on these types of games on networks, and consequently, for a full generalization and characterization of equilibria.

1.3 Related Literature

Our work has its roots in the vast body of literature that extends and refines the results of the Condorcet Jury Theorem. Here, we present an overview of the most fundamental works on the topic.[1] Many of these works can be considered as attempts of relaxing the strict assumptions of CJT and see what claims can still be proven (Dietrich, 2008). After this, we focus more carefully on a series of works with an approach closer to ours.

CJT in the Literature. A first group of researchers, following Condorcet's conjectures, studied the effects of heterogeneous competences on Condorcet's results. Scott L Feld and Grofman (1984) discuss how the addition of new members impacts the first two claims of CJT and Owen, Grofman, and Scott L. Feld (1989) characterize the distribution of competences that maximizes (or minimizes) the accuracy of a group majority. On the other hand, Paroush (1997) and Berend and Paroush (1998) use their results to look at under which conditions the asymptotic claim of CJT does not hold anymore. In a similar fashion, Kanazawa (1998) and Fey (2003) generalize the impact of heterogeneous competences on that claim. In addition, a similar line of investigation aims at figure out which is the best rule to aggregate votes when competences are heterogeneous. Nitzan and Paroush (1982) and Shapley and Grofman (1984) follow this approach, and they both show why a weighted majority rule is to be preferred to a simple majority rule in such a case.

A second direction of research consists in discussing how correlation among voters would impact CJT. Boland (1989), Ladha (1992) and Estlund (1994) study how correlation can be related to opinion leaders, exchange of information or communication. Likewise, Dietrich and List (2004) use bayesian nets to show how misleading evidence can influence the entire group.

A third group of researchers, like Austen-Smith and Banks (1996), T. J. Feddersen and Pesendorfer (1996) and McLennan (1998), aims at understanding how strategic voting can undermine the results. It is usually assumed in such a framework the possibility of people to prefer either one of the two options. Instead, Ben-Yashar and Milchtaich (2007) take into account people with the same common interest but with different priors. Eventually List and Goodin (2001) generalize the result with more than one option.

CJT and Costly Information. Besides these more general approaches there exists a well affirmed branch in the economic literature that takes into account a setting similar to ours. The basic idea is to look, through game theoretical notions, at when voters in a group acquire information and how much information they

[1] See also (Nitzan and Paroush, 2017) for a good summary.

acquire. We can divide this approach in two streams. Both these two directions assume that the information gained depends on the amount of effort made, and so agents can decide on the quantity of effort to make, i.e. competences are not fixed. Such an assumption comes mainly from Radner and Stiglitz (1984)'s results and similar works. Indeed, this is a first difference from our approach, as we chose to fix the amount of effort spendable by each agent, in order to look at other features of our system. The first stream is concerned with the design of a small committee. Mukhopadhaya (2003) and Persico (2004) look at how to choose the number of agents and how to select a voting rule in order to maximize the social welfare and the accuracy of the group. Indeed, both these works are explicitly directed at the problem of American juries. Similarly, Koriyama and Szentes (2009) define the optimal threshold for the number of jurors and study what happens if the committee surpasses that threshold. Eventually, Gershkov and Szentes (2009) show that the best way to produce optimal results is to ask the voters sequentially.

The second stream addresses the problem of information acquisition in large elections. It moves from the so-called *voting paradox* (Aidt, 2000), i.e. the fact that if voters were to behave rationally no information acquisition would happen in big elections as the actual benefit voters could have would be too little to justify an effort in terms of learning. Ghosal, Lockwood, et al. (2003) discuss the effect on voters of voluntary and compulsory participation. Martinelli (2006) shows that if the marginal cost for the acquiring of information is near zero there is still the possibility for rational voters to be well-informed. T. J. Feddersen (2004), T. Feddersen, Sandroni, et al. (2006) and T. Feddersen and Sandroni (2006) justify the actual behaviour of voters by introducing the notion of *ethical voters*, i.e. a category of voters that act for the 'good of the community'. They acquire information and vote 'consciously' in order to compensate the possible votes of ignorant agents but without having any information about them. Eventually, Tyson (2016) looks at how different policies can affect the motivation of the voters of voting. On a the same topic, there is also quite a good number of works from a more experimental perspective. We limit ourselves to mention some of the most well-known: Blais and Young (1999), Grosser and Seebauer (2016), and Levine and Palfrey (2007).

Fixed Effort and Networks. Our work differs from these mainly for two reasons. Firstly, we do not look at different amounts of effort, the cost that our agents can pay is not gradual: they either make an effort e or they do not. However, we allow for different levels of competences and for different stakes in the matter. On the other hand, we take into account networks and graphs, which is indeed something that the previous literature has not done. Nonetheless, in our basic setting (Chapter 2) the general takeaways are mostly in line with the results from this Literature. For these reasons our approach is more similar to Bloembergen, Grossi, and Lackner (2019)'s one. Such work, indeed, makes exactly the same use we did of the notion of effort and studies a similar voting game on networks. Although the focus is on the concept of *opinion representation*, i.e. the need of voters of seeing their opinion expressed by someone, the structure is very similar: voters either vote or delegate depending on how the agents around them behave. Akin to our work, they also study through a computer simulation the process of the iterated best response dynamics. Indeed, taking into account the possibility of our agents of being placed on a networks moves our work closer to a more computational perspective on voting theory.

Voting Theory and Networks. In the latest years, the field of *Computational Social Choice* (Brandt et al., 2016) has become more and more interested in the effects of networks in social choice mechanisms, mainly because of the many empirical results about the impact of social networks (Abrams, Iversen, and Soskice, 2011; Kearns et al., 2009; Sokhey and McClurg, 2012; Yildiz et al., 2010). A good overview of the main directions is provided by Grandi (2017). A first line of research is, indeed, quite close to ours: it looks at how voters are influenced by each other when provided with a noisy signal and asked to vote. Conitzer (2013), A. D. Procaccia, Shah, and Sodomka (2015), Tsang, Doucette, and Hosseini (2015), Tsang (2018) and Doucette et al. (2019) all address this problem. Nonetheless, they all differ from our work as they do not look at game-theoretical interactions and at the possibility of not voting. A second line of research is that of iterative voting (Tsang and Larson, 2016) and another one that of coalitional games (Elkind, 2014). Eventually, lots of work has being done in delegative systems, where the underlying network is necessary to understand how the delegation process works. Caragiannis and Micha (2019) and A. D. Procaccia, Shah, and Sodomka (2015) look at the impacts of network and delegation in the system of *liquid democracy*, i.e. a voting system in which each agent can delegate to any other agent. They prove that liquid democracy may have a negative impact on the accuracy of the group. In this sense, our brief section about delegation in Chapter 3 can also be considered a good answer to their perspectives, as it briefly shows that if we take into account the effort of voting, i.e. the effort of learning, delegation may help increase the social welfare of a group in a situation of equilibrium. Escoffier, Gilbert, and Pass-Lanneau (2019) and Christoff and Grossi (2017) also take into account the results on a network for liquid democracy.

1.4 Overview of the Thesis

We will now give a quick outline of the thesis chapters.

Chapter 2 introduces the reader to needed notions in our voting scenario and defines important game theoretic notions. First of all, it states formally the Condorcet Jury Theorem. Then, it describes the use of the effort, the utility of the agents, their actions and the way in which social welfare and group accuracy are computed. We formally characterize pure Nash equilibria and discuss the social welfare. All the notions and the results we obtain here will then be used again in the following chapters.

Chapter 3, 4 and 5 look at the three variations to our basic setting. In Chapter 3, we introduce different stakes for each agent. In Chapter 4, we consider agents that can reach different levels of competence, still by making the same effort. Lastly, in Chapter 5, we formalize the introduction of a network. We then describe some of the networks we work with and characterize pure Nash equilibria on them. In the second part of Chapter 5, we describe our computer simulation and we show some of the results that can be obtained through that. Remarkably, in all these three chapters we build from the results of Chapter 2.

In the Conclusions we summarize our findings and we propose new directions to follow.

2 Group Accuracy when Competence is Costly

This chapter presents a basic scenario for group decisions when there is a right answer, the agents may acquire *competence* in order to vote, i.e. a certain probability of getting the right answer, and this competence is costly. We model this scenario as a game where players' payoffs depend on how good the group is at identifying the right answer. In Section 2.1 we first present the Condorcet Jury Theorem formally, then we extend that model to include the cost of acquiring competence and explain how such a model can be considered a game. In Section 2.2, we prove some results about pure Nash equilibria in this game. In Section 2.3, the concept of social welfare is taken into account. Eventually, in Section 2.4 we discuss the results and the model itself.

2.1 Preliminaries

In this section, we model the Condorcet Jury Theorem and expand that framework to include the new notions we want to focus on, i.e. those related to the cost of competence.

2.1.1 The Condorcet Jury Theorem

The Condorcet Jury Theorem is the starting point of our work (Grofman, Owen, and Scott L. Feld, 1983; Nitzan and Paroush, 2017). In the introduction, we briefly mentioned the general ideas of the theorem, here we state it explicitly.

Consider a set of alternatives $Z = \{z_1, z_2\}$ where one $z^* \in Z$ is the true alternative and a set of a voters $N = \{1, ..., n\}$ where n is odd. Each voter submits a vote $v_i \in Z$ and the votes, gathered in $\boldsymbol{v} = (v_1, ..., v_n)$ are aggregated through the majority rule $maj(\boldsymbol{v})$:

$$maj(\boldsymbol{v}) = z \text{ s.t. } z \in Z \text{ and } |\{i \in N | v_i = z\}| \geq \frac{|N| + 1}{2}. \tag{2.1}$$

There is a probability $P(v_i = z^*) = p_i$ that i gets it right, which is called i's competence. We model this with a random variable X_i such that:

$$X_i = \begin{cases} R & \text{if agent i gets it right, i.e. } v_i = z^* \\ W & \text{otherwise.} \end{cases} \tag{2.2}$$

Now, we shall make the following assumptions.

`(COM)` Competence: $p_i > \frac{1}{2}$.

`(HOM)` Homogeneity: $p_i = p_j = p$.

`(IND)` Independence: $P(X_i = u, X_j = v) = P(X_i = u)P(X_j = v)$ for any two agents $i, j \in N$ and $u, v \in \{R, W\}$.

We call $M(x, p)$ the function that given a competence level p for all the voters and a number of voters x returns the probability of the group of getting the true alternative w^*. Given our assumptions, $M(x, p)$ is so defined:

$$M(x, p) = \sum_{i \geq \frac{x+1}{2}}^{n} \binom{n}{i} \cdot p^i \cdot (1 - p)^{n-i}. \tag{2.3}$$

Thus, the CJT consists in proving that (Grofman, Owen, and Scott L. Feld, 1983):

`(GBI)` Groups are Better than Individuals. If $n > 1, M(n, p) > p$.

`(LIB)` Larger is Better. If $m > n$, $M(m, p) > M(n, p)$.

`(ASY)` Asymptotical Behaviour. If $n \to \infty$ then $M(n, p) \to 1$.

Notably, these are the results that we explained in natural language in the introduction. Indeed, the first result asserts that any group is always better than any single individual. The second one states that the larger the group the higher the accuracy. Eventually, the third one says that if the group size becomes infinitely large the overall accuracy of the group tends to 1.

2.1.2 The Cost of Competence - Extension of the Model

We extend the scenario described earlier so to include the cost, in terms of effort, that each agent pays in order to be *competent*, i.e. the cost of learning. Notably, we drop the notation that we do not consider necessary to not distract the reader. We consider a decision structure $D = \langle N, p, e \rangle$ where a group of agents $N = \{1, ..., n\}$ with $n \geq 2$ is faced with a binary decision. The decision is taken through a vote. Since there exists a right answer (between the two possible options), every member of the group gets a reward if the group manages to identify that answer. This time, each agent $i \in N$ can either vote or abstain, i.e. $A_i = \{0, 1\}$ is the set of actions available to agent i (where $a_i = 0$ means that agent i decides to abstain and $a_i = 1$ means that agent i decides to vote). When an agent votes, she votes for the right option with a probability $p \in (\frac{1}{2}, 1)$. We call p the competence level. However, voting with such a competence is costly. Indeed, if an agent chooses the action 1, i.e. she votes, she pays a cost of $e \in (0, \frac{1}{2})$. This effort represents the cost needed to learn information about the matter at stakes (and vote with an accuracy higher than $\frac{1}{2}$). The final decision is made by majority rule over the agents that vote. If there is a tie the choice is made by tossing a fair coin. If nobody votes we also toss a fair coin.

Let $\{0, 1\}^n$ be the set of all possible action profiles. We consider only pure strategies. In particular, let $\boldsymbol{a}$ be a profile and denote with $|\boldsymbol{a}|$ the number of people voting in a certain profile, i.e. $|\boldsymbol{a}| = \#\{i \in N | a_i = 1\}$.[1] This notation also allows us to speak, more or less precisely, about the *size* of a profile, or if that profile is an equilibrium, of the size of an equilibrium. Notably, we talk about 'voters' to refer to the agents that decided to vote in a certain profile $\boldsymbol{a}$. Instead, we talk about 'agents' when we include all the group members.

Note that if $|\boldsymbol{a}| = |\boldsymbol{a}'|$ for two profiles $\boldsymbol{a}$ and $\boldsymbol{a}'$ then the two profiles are equivalent, in the sense that they actually produce the same outcome under every aspect, as the voters do not differ in anything. So, from now on we will characterize a set of profiles just by talking about their number of voters, i.e. if a profile $\boldsymbol{a}$ with $|\boldsymbol{a}| = j$ has a property q then each profile $\boldsymbol{a}'$ with $|\boldsymbol{a}'| = j$ has that property q. In addition, we denote with $\boldsymbol{a}' = (a_i^*, \boldsymbol{a}_{-i})$ the profile that is exactly like $\boldsymbol{a}$, except that agent i takes action a_i' instead of action a_i.

Again, we assume that every voter casts her vote independently from the others, i.e. her probability of choosing an answer is independent from the probability of any other agent's choice. For integers x and k with $x \leq k$, we denote by $m(p, x, k)$ the probability that exactly k of x agents are correct, when all x agents

[1] The inspiration for this notation comes from the cardinality of a set. But the same method is also used to refer to the size of a bit-vector, e.g., $|1011| = 3$, with action profiles being kind of like bit-vectors.

vote and get the right answer with probability p. If agents vote independently, as we assume here, we have that:

$$m(p, x, k) = \binom{x}{k} p^k (1-p)^{x-k}. \tag{2.4}$$

We then define $M(x, p)$, i.e. the overall probability of x voters with accuracy p to select the option through *majority rule*:

$$M(x, p) = \begin{cases} \sum_{k=\frac{x+1}{2}}^{x} m(p, x, k) + \frac{1}{2} m(p, x, \frac{x}{2}) & \text{if } x \geq 0 \text{ and is even,} \\ \sum_{k=\frac{x+1}{2}}^{x} m(p, x, k) & \text{if } x \geq 0 \text{ and is odd,} \\ 0 & \text{if } x < 0. \end{cases} \tag{2.5}$$

Indeed, the overall accuracy consists of the sum of the probability of every combination of votes where the right option wins. And, in the case of an even number of votes, if there is a tie, the corresponding probability is multiplied by $1/2$ to represent the tossing of the fair coin. The last condition is for purely technical reasons, and will become apparent in Theorem 2.1. The utility for each agent is defined as follows:[2]

$$u_i(\boldsymbol{a}) = M(|\boldsymbol{a}|, p) - a_i e. \tag{2.6}$$

Indeed, every agent gets an utility that consists of the accuracy of the group $M(|\boldsymbol{a}|, p)$ minus the effort required to perform the taken action. Note that if the agent votes her effort will be equal to e, whereas if she abstains the effort will be equal to 0.

2.2 Basic Results and Pure Nash Equilibria

In this section we first have a look at some key features of our model. Then, we introduce the concept of equilibrium and we characterize the Nash equilibria in our model. We restrict ourselves to the concept of pure equilibrium.

[2]Note that this is not an actual utility but more an expected utility, as the actual utility would have been:

$$u_i(\boldsymbol{a}) = \begin{cases} 1 - a_i e & \text{if the group makes the right choice} \\ -a_i e & \text{if the group makes the wrong choice.} \end{cases}$$

2.2.1 Basic Results

The next proposition is just a direct consequence of how $M(x,p)$ is structured and is useful to get a feeling of the mathematical machinery behind our model. We omit the proofs for point 2 and 3 of the proposition as they can be found in other works regarding the Condorcet Jury Theorem, for example that carried out by grofman1983thirteen.

Proposition 2.1. Let x, y be two odd integers s.t. $x > y > 0$. Then,

1. $M(x,p) = M(x+1,p)$,
2. $M(x,p) > M(y,p) > M(0,p)$ and
3. $M(1,p) - M(0,p) > M(y+2,p) - M(y,p) > M(x+2,p) - M(x,p)$.

Proof. As we mentioned, we only prove Statement 1. The other statements can be found in (Grofman, Owen, and Scott L. Feld, 1983).

We introduce some notation. Let $x = 2m-1$ and $x+1 = 2m$ for some $m \in \mathbb{N}^+$. Call $P(z, l = y)$ the probability of having an l number of votes in favour of the right answer equal to y out of z. And let $P(z, l \geq y)$ the probability of the event of getting a number l of votes in favour of the right answer out of z higher or equal than y, i.e. $P(z, l \geq y) = \sum_{i=y}^{z} P(z, l = y)$. Consequently, $P(z, l = k) = m(p, z, k) = \binom{z}{k} p^k (1-p)^{z-k}$. And $P(z, l \geq \frac{z+1}{2}) = M(z,p)$ if z is odd.

Now, we shall consider our specific case. Fix the probability p. In particular, we have that $M(2m)$ can be computed from the probability $M(2m-1)$ in the following way.

$$\begin{aligned} M(2m) &= p \cdot P(2m-1, l \geq m) + (1-p) \cdot P(2m-1, l > m) + \\ &\quad + \frac{1}{2} p \cdot P(2m-1, l = m-1) + \frac{1}{2}(1-p) \cdot P(2m-1, l = m) \end{aligned}$$

Now, we have that $p \cdot P(2m-1, l \geq m) = p \cdot P(2m-1, l > m) + p \cdot P(2m-1, l = m)$. In addition, $p \cdot P(2m-1, l > m) + (1-p) \cdot P(2m-1, l > m) = P(2m-1, l > m)$. Now, consider the following equality:

$$\begin{aligned} &\frac{1}{2} p P(2m-1, l = m-1) + \frac{1}{2}(1-p) P(2m-1, l = m) = \\ &\quad = \frac{1}{2}[p \binom{2m-1}{m-1} p^{(m-1)} (1-p)^m] + \frac{1}{2}[(1-p) \binom{2m-1}{m} p^m (1-p)^{(m-1)}] = \\ &\quad = \frac{1}{2} \binom{2m-1}{m} [2p^m (1-p)^m] = \\ &\quad = (1-p) P(2m-1, l = m). \end{aligned}$$

Indeed, $\binom{2m-1}{m-1} = \binom{2m-1}{m}$. Thus, we can plug in our results in the initial equality.

$$\begin{aligned} M(2m) &= p \cdot P(2m-1, l=m) + P(2m-1, l>m) + (1-p)P(2m-1, l=m) = \\ &= P(2m-1, l=m) + P(2m-1, l>m) = \\ &= P(2m-1, l \geq m) \end{aligned}$$

Indeed, $P(2m-1, l \geq m) = M(2m-1)$. Thus, $M(2m) = M(2m-1)$.

$\square$

Intuitively, the group accuracy increases monotonically as the size of the group grows with a non zero jump when the size goes from x to $x+2$, and the increase from x to $x+2$ also decreases monotonically. Indeed, the increase is the largest possible when going from 0 voters to 1 voter, and then it decreases (Statement 2). Notably, $M(0,p) = 0$ and $M(1,p) = p$. In addition, the accuracy of an even group of voters is the same as the accuracy of a group with size smaller than one. This is a key of many of the next results.

Such a first result allows us to form an idea of how the accuracy of groups changes.

Example 2.1. *Consider a group $N = \{1,2,3,...,50\}$. Depending on the level of competence (or just 'competence') p of the voters the function $M(x,p)$ with $0 \leq x \leq n$ assumes different values. Although $M(x,p)$ increases with the number of voters (or stays the same if we are going from an odd to an even), it does so with different slopes depending on the accuracy of the members. See Figure 5.11 for an illustration.*

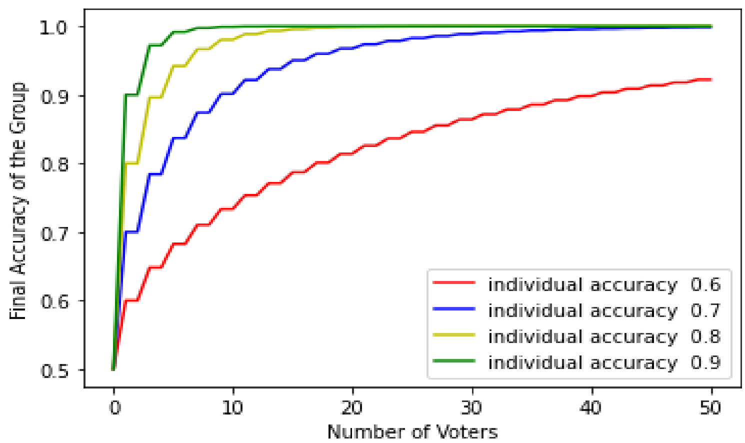

Figure 2.1: The accuracy of a group $N = \{1,2,3,...,50\}$ for four different levels of competences: $p \in \{0.6, 0.7, 0.8, 0.9\}$.

This is exactly what the Condorcet Jury Theorem tells us: if $p > \frac{1}{2}$ (as in all our cases) the more people in the group the more accurate the group itself. However, what happens in terms of behaviour if competence is costly and agents have the option to abstain from voting? If learning information about a certain matter is costly, then it may also be that not all the members of a group have an incentive to vote. And consequently, we may ask: is it still true that adding more people to a group will benefit the final decision? In order to answer this question, we start by looking at the concept of pure Nash equilibrium in our setting (Leyton-Brown and Shoham, 2008; Nash, 1950).

2.2.2 Pure Nash Equilibria

The concept of Nash equilibrium has been proposed in order to find the solution of a non-cooperative game (Nash, 1950). It describes a situation in which no agent would play differently. For many reason, it is often regarded as a good prediction of how the players will behave. Here, we use this notion to analyze our game. Being able to find the Nash equilibrium gives us insights about the most likely outcomes of the game.

First of all, we characterize generally the pure Nash equilibria of the game underlying a precise decision structure. Then, we look at how the equilibria can change with respect to the effort. Indeed, this is the main focus of our work. Eventually, we look for the pure Nash equilibrium with the highest accuracy. Such a solution represents the highest accuracy that can be reached by a certain decision structure. And, it allows us to draw a comparison with the results about the overall accuracy of the canonical situation of CJT, where there is no effort.

We start by defining formally a pure Nash equilibrium.

Definition 2.1. Let $\boldsymbol{a}$ be a profile. We say that the profile $\boldsymbol{a}$ is a Pure Nash Equilibrium (PNE) if no player i wants to unilaterally deviate from her assigned action a_i: $u_i(\boldsymbol{a}) \geq u_i(a'_i, \boldsymbol{a}_{-i})$ for any a'_i.

We can now prove the following result.

Theorem 2.1. Let $D = \langle N, e, p \rangle$ be a decision structure and $\boldsymbol{a} \in \{0,1\}^n$ be an action profile with $|\boldsymbol{a}| = k$. The profile $\boldsymbol{a}$ is a PNE iff $M(k+1,p) - M(k,p) \leq e \leq M(k,p) - M(k-1,p)$.

Proof. A profile $\boldsymbol{a}$ is a PNE iff for no agent i $u_i(\boldsymbol{a}) \geq u_i(a'_i, \boldsymbol{a}_{-i})$ by Definition 2.1. We proceed by first assuming that i is s.t. $a_i = 1$ and then assuming i s.t. $a_i = 0$. We show that i does not deivate in neither of the cases iff $M(k+1,p) - M(k,p) \leq e \leq M(k,p) - M(k-1,p)$.

Assume i s.t. $a_i = 1$. Thus, $u_i(0, \boldsymbol{a}_{-i}) = M(|\boldsymbol{a}| - 1, p)$ and $u_i(\boldsymbol{a}) = M(|\boldsymbol{a}|, p) - e$. Thus, agent i does not deviate iff $M(|\boldsymbol{a}|, p) - e \geq M(|\boldsymbol{a}| - 1, p)$, i.e. iff $M(|\boldsymbol{a}|, p) - M(|\boldsymbol{a}| - 1, p) \geq e$.

Now, assume i s.t. $a_i = 0$. Thus, $u_i(1, \boldsymbol{a}_{-i}) = M(|\boldsymbol{a}| + 1, p) - e$ and $u_i(\boldsymbol{a}) = M(|\boldsymbol{a}|, p)$. Thus, agent i does not deviate iff $M(|\boldsymbol{a}|, p) \geq M(|\boldsymbol{a}| + 1, p) - e$, i.e. iff $M(|\boldsymbol{a}| + 1, p) - M(|\boldsymbol{a}|, p) \leq e$.

Consequently, no agent i would deviate iff $M_p(|\boldsymbol{a}| + 1) - M_p(|\boldsymbol{a}|) \leq e \leq M_p(|\boldsymbol{a}|) - M_p(|\boldsymbol{a}| - 1)$. And this corresponds to saying that $\boldsymbol{a}$ is a PNE iff $M_p(|\boldsymbol{a}| + 1) - M_p(|\boldsymbol{a}|) \leq e \leq M_p(|\boldsymbol{a}|) - M_p(|\boldsymbol{a}| - 1)$. □

This is already a full characterization of PNEs on our game. Notably, this theorem makes use of the fact that $M(x, p) = 0$ if $x < 0$. Indeed, we needed such an assumption to be able to generalize the reasoning to all the profiles. Of course, it is completely harmless. It comes into play when we analyze the profile $\boldsymbol{a}$ of size 0, as in that case, we have to compute $M(-1)$. By equating $M(-1)$ to 0 we are sure that $M(0) - M(-1) \geq 0$. Indeed, no agent could deviate to go to a profile where $|\boldsymbol{a}'| = -1$, and so such a profile should not be considered.

In order to better understand such result (and why our assumption was harmless) we shall have a look at the following corollary. This helps us to complete our first task of this section: understanding how PNEs change with respect to the effort.

Corollary 2.1. Let $D = \langle N, e, p \rangle$ be a decision structure.

1. If $e > p - \frac{1}{2}$ a profile $\boldsymbol{a}$ is a PNE iff $|\boldsymbol{a}| = 0$.
2. If $e = p - \frac{1}{2}$ a profile $\boldsymbol{a}$ is a PNE iff either $|\boldsymbol{a}| = 0$ or $|\boldsymbol{a}| = 1$.
3. If $e < p - \frac{1}{2}$ a profile $\boldsymbol{a}$ is a PNE iff $|\boldsymbol{a}|$ is odd and $M(|\boldsymbol{a}|, p) \geq M(|\boldsymbol{a}| - 1, p) + e$.

Proof. This is just a result of Theorem 2.1. The first and the second case are quite straightforward. In the first case, we see that $M(1, p) - M(0, p) \leq e \leq M(0, p) - M(-1, p)$. Notably, we make use of our assumption here, in the way we just mentioned above. In the second case, we have the same for 0 and 1. The third case is the less trivial. If $|\boldsymbol{a}|$ is odd, we have that $\boldsymbol{a}$ is a PNE iff $M(k+1, p) - M(k, p) \leq e$ is true by Proposition 2.1. The rest follows. □

This result is extremely interesting. Indeed, it provides us with a full understanding of the mechanism of equilibria. First of all, it proves that any decision structure $D = \langle N, e, p \rangle$ produces a game with at least one PNE. Indeed, for any value of e there exists a configuration that is stable. Secondly, it shows that if $e \leq p - \frac{1}{2}$ there is more than one size for which a profile can be a PNE. For this reason, as we mentioned in the beginning, we may be interested in understanding which of all these equilibria is the one with the highest overall accuracy. From now on, we say that a profile $\boldsymbol{a}$ is an mPNE (a *maximal Pure Nash Equilibrium*) if and only if for all $\boldsymbol{a}'$ such that $\boldsymbol{a'}$ is a PNE then $M(|\boldsymbol{a}|, p) \geq M(|\boldsymbol{a}'|, p)$. Note that in the present setting, there can be many profiles that are mPNEs, however all of them must have the same amount of voters. We can elaborate on this a bit further by making use of the following definitions.

Definition 2.2. Given p and e, let $n_{thr} = max\{m \in \mathbb{N} | M(m,p) - M(m-1,p) \geq e$ and m is odd$\}$. Given a decision structure $D = \langle N, e, p \rangle$, let $n_{eq} = max\{m \in N | M(m,p) - M(m-1,p) \geq e$ and m is odd$\}$.

Indeed, n_{thr} corresponds to the highest number of people voting in an equilibrium if we assume that $n \to \infty$. Instead, n_{eq} is the actual highest number of people voting in an equilibrium given that n has a finite value. Clearly, we have that $n_{eq} \leq n_{thr}$. However, every time $n \geq n_{thr}$, $n_{eq} = n_{thr}$. The following observation is just a result of Proposition 2.1 and of the fact that $M(n_{eq}, p) \geq M(|\boldsymbol{a}|, p)$ for any PNE $\boldsymbol{a}$. These two elements allow us to characterize further the size of an equilibrium.

Observation 2.1. *Let $D = \langle N, e, p \rangle$ be a decision structure. If $\boldsymbol{a}$ is a PNE then $M(|\boldsymbol{a}|, p) \leq M(n_{eq}, p)$.*

Consequently, a profile $\boldsymbol{a}$ is an mPNE if and only if $|\boldsymbol{a}| = n_{eq}$. Indeed, by Proposition 2.1, the larger the size the higher the accuracy of a group of voters. With this result, we could conclude this section. Indeed, we managed to characterize the equilibria, to study how the effort e influences them and to find the one with the highest accuracy, i.e. the one with more voters. However, before going on we shall have a quick look at the concept of *marginal utility*. Although such a notion is not explicitly used in our results, it may give the reader some more insights about the agents' behaviour. Indeed, it is also quite present in the economic literature about CJT and costly information (T. J. Feddersen and Pesendorfer, 1996; Martinelli, 2006; Persico, 2004). The marginal utility for agent i of action a_i wrt a_i' in a profile $\boldsymbol{a}$ is:

$$\Delta u_i(\boldsymbol{a}) = u_i(\boldsymbol{a}) - u_i(a_i', \boldsymbol{a}_{-i}) \tag{2.7}$$

where $a_i' \neq a_i$.

The marginal utility is what guides agents' choices. If the marginal utility is positive the agents will remain on the action they chose. In our case, the marginal utility for a voter (with $a_i = 1$ and $a_i' = 0$) is the following:

$$\Delta u_i(\boldsymbol{a}) = M(|\boldsymbol{a}|, p) - M(|\boldsymbol{a}| - 1, p) - e. \tag{2.8}$$

Theorem 2.1 relies on the concept of marginal utility. When does a voter want to change her action? When $M(p, |\boldsymbol{a}|) - M(|\boldsymbol{a}| - 1, p)$ is lower than e. The value $M(p, |\boldsymbol{a}|) - M(|\boldsymbol{a}| - 1, p)$ represents the *contribution* of a single agent to the accuracy of the group.[1] Such a value decreases as $|\boldsymbol{a}|$ grows, as shown in Figure 2.2.

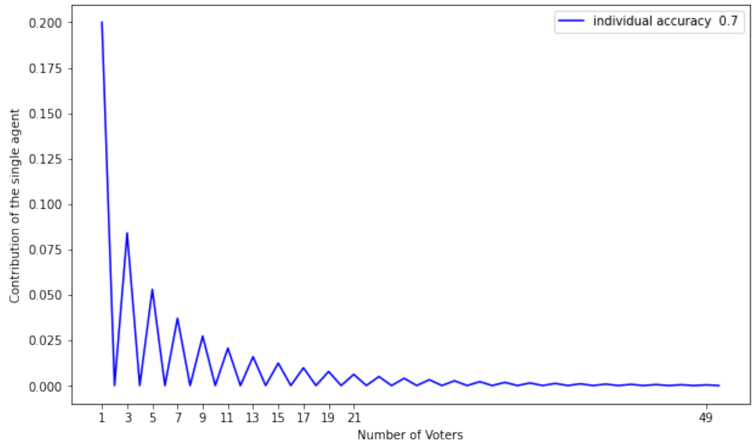

Figure 2.2: The contribution of a single agent in the group of voters decreases with the number of voters.

The individual contribution is 0 when the total number of people voting is even. Indeed, there is no point for one agent to vote if the size of the group is even (see, again Proposition 2.1). Yet, there is a positive contribution in all the other cases. Both formally and theoretically speaking, this contribution corresponds to what is usually referred to as *pivotality* of a voter in the literature (T. J. Feddersen and Pesendorfer, 1996; Martinelli, 2006; Persico, 2004). However, when the number of people gets quite high, this positive contribution gets very small. So, let $e = 0.025$: the gain from voting is not significant enough if there are already more than 8 other people voting. Consequently, all the groups with more than 7 people cannot be equilibria. The basic idea is that given a certain group of agents with n and p the higher is e, the lower n_{thr} becomes. We shall look at Figure 2.3 to get even a better grasp. Notably, the features of the equilibria depend only in minimal part on the number of agents in the group. The threshold n_{thr} depends only on e and p. So, two groups with different numbers of people n' and n (with same p and e) will have the same equilibria if n' and n are both greater or equal to

[1] We make extensive use of a similar notion later (namely in Chapter 3). However, for now we leave this name just mentioned.

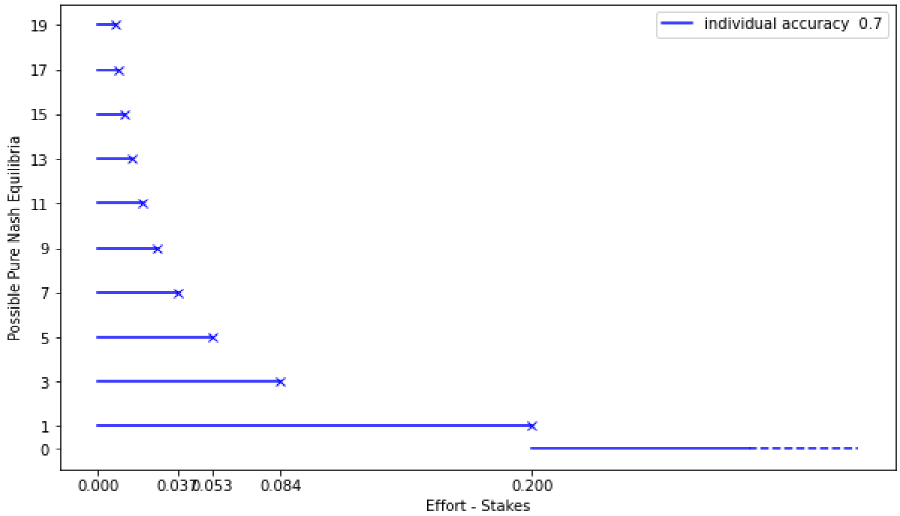

Figure 2.3: The possible sizes of equilibria that a group (formed by agents with level of competence $p = 0.7$) can reach depend on the effort. All the lines correspond to the equilibria of the depicted size.

n_{thr}. This is a major difference from the CJT. Increasing n in a decision structure not always guarantees an increase in the overall accuracy. A clear consequence is that if two groups have different levels of competence their overall accuracy can be different at their mPNEs. We shall have a look at Figure 2.4. Competence is highly relevant. And having lots of people with low competence does not always produce the same accuracy as few people with high competence, which seems to undermine another CJT result.[2] Large numbers do not compensate for the little competence of the individuals. However, notably, it is true that in case of lower competences the increase in terms of accuracy from the worst equilibrium to the best one is higher.

2.2.3 Strong Pure Nash Equilibria

As the agents are all members of the same group it seems quite interesting to also look at the equilibria when coalitions are involved. Indeed, since every agent is aware of other agents' actions, they may decide to change their actions together. These kind of equilibria are usually called *strong equilibria* (Aumann, 2016; Bernheim, Peleg, and Whinston, 1987). In this section, we characterize pure strong equilibria, and in a similar fashion as before, we look at how the effort influences such solutions. In particular, we also prove that there are cases in which an SPNE

[2] Look at the final section of this chapter for a full discussion of this hint.

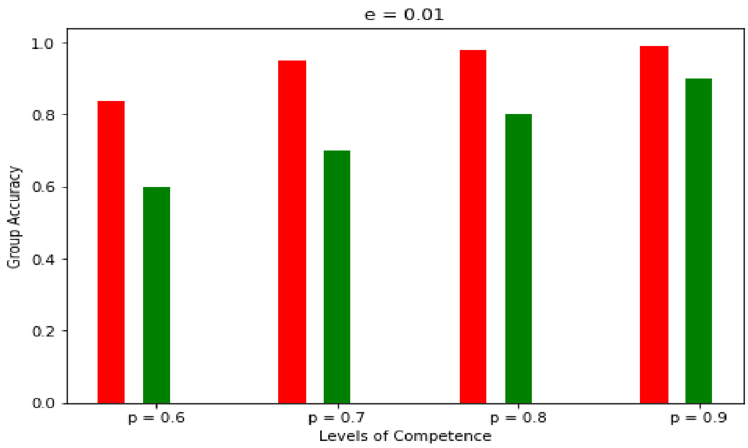

Figure 2.4: The red bars represent accuracy for the groups at the their mPNE, the green bars the accuracy at the equilibria with the lowest accuracy. The four different groups present the same number of voters, the same effort but different levels of competence $p \in \{0.6, 0.7, 0.8, 0.9\}$.

does not exist. This is highly relevant, as it shows that if agents form coalitions and take decisions together they may not be able to reach the stability.

We introduce some new definitions. We call a subset $K = \{k_1, \ldots, k_m\}$ of $N = \{1, ..., n\}$ a coalition with size $m \leq n$, and for each coalition we abbreviate the sequence $(a_{k_1}, \ldots, a_{k_m})$ of actions to $\boldsymbol{a}_K$ and $A_{k_1} \times \cdots \times A_{k_m}$ to A_K. Thus, we denote with $\boldsymbol{a}' = (\boldsymbol{a}_K, \boldsymbol{a}_{N \setminus K})$ the profile that is exactly like $\boldsymbol{a}$ except that each agent $i \in K$ takes the action she is assigned in $\boldsymbol{a}_K$.

Definition 2.3. We call a profile $\boldsymbol{a}$ a Strong Pure Nash Equilibrium (SPNE) iff for each coalition $K \subseteq N$ there does not exist $\boldsymbol{a}'_K \in A_K$ such that $u_i(\boldsymbol{a}'_K, \boldsymbol{a}_{N \setminus K}) > u_i(\boldsymbol{a})$ for all $i \in K$.

In particular, we say that a coalition K 'wants to deviate' or 'deviates' from a profile $\boldsymbol{a}$ iff $u_i(\boldsymbol{a}'_K, \boldsymbol{a}_{N \setminus K}) > u_i(\boldsymbol{a})$ for all $i \in K$. From now on, we take into account only cases s.t. $\boldsymbol{a}'_K$ is completely different from $\boldsymbol{a}_K$, i.e. for all $i \in K$, $a_i \neq a'_i$. Indeed, if there exists a coalition K that wants to deviate with a new profile $\boldsymbol{a}'_K$ that is partly similar to $\boldsymbol{a}_K$ we can just look at the subset of K that has a completely different one. Notably, as a coalition can be composed also by just one agent, if a profile $\boldsymbol{a}$ is an SPNE, it is also a PNE. The idea is that a coalition may move together to benefit all the agents. However, in order to find our Strong Equilibria, we do not really need to check that all the possible coalitions do not deviate.

Lemma 2.1. Let $D = \langle N, e, p \rangle$ be a decision structure and let $\boldsymbol{a}$ be an action profile. Let $K \subseteq N$ be a coalition s.t. there exists $i, j \in K$ with $a_i \neq a_j$. If K wants to deviate from $\boldsymbol{a}$, then there exists a coalition $K' \subseteq N$ s.t. for all $i, j \in K'$ $a_i = a_j$ that wants to deviate as well.

Proof. Call $|\boldsymbol{a}| - |(\boldsymbol{a}'_K, \boldsymbol{a}_{N\setminus K})| = l$. If $l = 0$ then $u_i(\boldsymbol{a}) = u_i(\boldsymbol{a}'_K, a_{N\setminus K})$ and so K would not deviate. If $l > 0$ there must be in K at least l elements that chose action 1 in $\boldsymbol{a}$ and action 0 in $\boldsymbol{a}'_K$. Build $K' \subseteq \{i \in K | a_i = 1\}$ s.t. $|K'| = l$. Thus, $|\boldsymbol{a}'_K, \boldsymbol{a}_{N\setminus K}| = |\boldsymbol{a}'_{K'}, \boldsymbol{a}_{N\setminus K'}|$. Consequently, $u_i(\boldsymbol{a}'_K, \boldsymbol{a}_{N\setminus K}) = u_i(\boldsymbol{a}'_{K'}, \boldsymbol{a}_{N\setminus K'})$. Thus, if K wants to deviate also K' wants to. If $l < 0$ let there must be in K at least l elements that chose action 0 in $\boldsymbol{a}$ and action 1 in $\boldsymbol{a}'_K$. Build $K' \subseteq \{i \in K | a_i = 0\}$ s.t. $|K'| = l$. Thus, $|\boldsymbol{a}'_K, \boldsymbol{a}_{N\setminus K}| = |\boldsymbol{a}'_{K'}, \boldsymbol{a}_{N\setminus K'}|$. Consequently, $u_i(\boldsymbol{a}'_K, \boldsymbol{a}_{N\setminus K}) = u_i(\boldsymbol{a}'_{K'}, \boldsymbol{a}_{N\setminus K'})$. Thus, if K wants to deviate also K' wants to. $\square$

Lemma 2.2. Let $D = \langle N, e, p \rangle$ be a decision structure and let $\boldsymbol{a}$ be an action profile. Consider $K \subseteq \{i \in N | a_i = 1\}$. If K wants to deviate from $\boldsymbol{a}$, then a coalition $K' \subseteq K$ with $|K'| = 1$ wants to deviate too.

Proof. Consider $i \in K$. Thus, $u_i(\boldsymbol{a}) = M(|\boldsymbol{a}|, p) - e$ and $u_i(\boldsymbol{a}'_K, \boldsymbol{a}_{N\setminus K}) = M(|\boldsymbol{a}| - |K|, p)$. By Proposition 2.1, $M(|\boldsymbol{a}| - |K|, p) \leq M(|\boldsymbol{a}| - 1, p)$. Thus, if K deviates, $M(|\boldsymbol{a}| - |K|, p) > M(|\boldsymbol{a}|, p) - e$, which implies that for the coalition $K' = \{i\}$ with $i \in K$, we also have $M(|\boldsymbol{a}| - |K'|, p) > M(|\boldsymbol{a}|, p) - e$. $\square$

Now, with the help of these two lemmas we prove our main result about SPNE.

Theorem 2.2. Let $D = \langle N, e, p \rangle$ be a decision structure and $\boldsymbol{a} \in \{0, 1\}^n$ be an action profile with $|\boldsymbol{a}| = k$. The profile $\boldsymbol{a}$ is an SPNE iff $M(n, p) - M(k, p) \leq e \leq M(k, p) - M(k - 1, p)$.

Proof. Consider a profile $\boldsymbol{a}$. Such a profile is an SPNE iff no coalition wants to deviate. Let K be a coalition s.t. $K \subseteq \{i \in N | a_i = 0\}$ and $\boldsymbol{a}'_K = (1, 1, ..., 1)$. Consequently, $u_i(\boldsymbol{a}'_K, \boldsymbol{a}_{N\setminus K}) = M(|\boldsymbol{a}| + |K|, p) - e \leq M(n, p) - e$. Thus, K would not deviate iff $M(n, p) - M(|\boldsymbol{a}|, p) \leq e$. Now, let K be a coalition s.t. $K \subseteq \{i \in N | a_i = 1\}$ and $\boldsymbol{a}'_K = (0, 0, ..., 0)$. By Lemma 2.2, we have that it does not deviate if a subset of size 1 does not. Thus, K would not deviate iff $M(|\boldsymbol{a}|, p) - M(|\boldsymbol{a}| - 1, p) \geq e$. By Lemma 2.1, if these two types of coalitions do not deviate no coalition does so. Thus, $\boldsymbol{a}$ is SPNE iff $M(n, p) - M(k, p) \leq e \leq M(k, p) - M(k - 1, p)$. $\square$

The idea is that we need to be sure that just two types of coalitions do not deviate: the biggest coalition possible of agents switching from 0 to 1 and the one-agent coalition switching from 1 to 0. If those two do not deviate then no other else deviates. This is a result of the composition of Lemma 2.1 with Lemma 2.2 and another observation contained in the proof for Theorem 2.2. Lemma 2.1 tells us that no 'mixed' coalitions should be take into account, as if one of that type deviates than also a 'non-mixed' one would do the same. So, we are left with coalitions of non-voters and coalitions of voters. Lemma 2.1 states that if a coalition of m voters deviates from $\boldsymbol{a}$ then also a coalition of 1 voter deviates. And, similarly, we proved in the proof for Theorem 2.2 that if a coalition of m non-voters deviates from $\boldsymbol{a}$ also the coalition of non-voters of size $n - |\boldsymbol{a}|$ deviates. Thus, no coalition wants to deviate if the biggest coalition possible of agents switching from 0 to 1 and the one-agent coalition switching from 1 to 0 do not want to deviate. Notably, to block the deviation of the second type of coalition we mentioned is enough to impose the condition we also used for PNEs. Indeed, the characterization of an SPNE profile can be easily connected to that of mPNE.

Corollary 2.2. Let $D = \langle N, e, p \rangle$ be a decision structure. If a profile $\boldsymbol{a}$ is an SPNE then it is also an mPNE.

Proof. Assume $\boldsymbol{a}$ is an SPNE s.t. $|\boldsymbol{a}| = y$. Then assume that $\boldsymbol{a}'$ with $|\boldsymbol{a}'| = x$ is a PNE. Now, assume that $\boldsymbol{a}'$ is an mPNE. Thus, $M(x) \geq M(y)$. In particular, $M(x) \geq M(x-1) + e$ by Theorem 2.1. If $y \neq x$ then $M(x) > M(y) + e$ which is contradiction with $M(y) \geq M(n) - e$. Thus, $x = y$. $\square$

The idea is that if there was an equilibria with size n^+ and with higher accuracy than the SPNE (with size n^-), the coalition of non-voters in SPNE that would have been required to reach n^+ would have switched to 1, as we would have had $M(n^+) - M(n^-) \geq e$ and n^- would have not been an SPNE.

However, it is not always true that a profile $\boldsymbol{a}$ that is an mPNE, it is also an SPNE. Indeed, it may happen that a certain decision structure does not have an SPNE. The existence of an SPNE depends again on e, p and n. As we saw, to have an SPNE there must exists x such that $x \leq n$ and $M(n,p) - M(x,p) \leq e \leq M(x,p) - M(x-1,p)$. Notably, in the case for $x = 0$ the necessary and sufficient condition becomes that $e \geq M(n,p) - \frac{1}{2}$.

We shall look at the following examples.

Example 2.2. *Let* $e = 0.01$, $p = 0.6$ *and* $n = 100$. *Notably, there exists no SPNE. Indeed, an mPNE* $\boldsymbol{a}$ *is s.t.* $|\boldsymbol{a}| = 23$ *and* $M(23, 0.6) = 0.84$. *However, if every non-voter decides to vote the accuracy is* $M(n, 0.6) = M(100, 0.6) = 0.98$. *Since it does not hold that* $M(100, 0.6) - M(23, 0.6) \leq 0.01$ *we have that there exists no SPNE.*

Example 2.3. *Let* $e = 0.01$, $p = 0.9$ *and* $n = 100$. *Thus, every profile where a* 5 *people vote is an SPNE. Indeed, those profiles produce a group accuracy of* 0.992 *and* $M(100, 0.9) = 1$.

It is worth noting that if $M(n,p) \rightarrow 1$, but the function $M(x,p)$ increases slowly, the decision structure is less likely to have an SPNE. Indeed, the value for $M(x,p) - M(x-1,p)$ is quite low for each x and at the same time $M(n,p) - M(x,p)$ can be quite high.

The idea is that the lower the individual competence of each agent the higher the impact of allowing for coalitions. Indeed, if each agent has already a competence of 0.9, even if she cooperates with others the potential increase in the actual overall accuracy is not extremely higher than if she just votes alone. Instead, if an agent has a competence of 0.6 moving in coalitions makes a huge difference for her. If nobody is voting and she votes, the increase in overall accuracy is 0.1. If a group of fifty people switches together from action 0 to action 1, the increase will be around three times (or more). For this reason, the lower the competences of the agents the less likely is for a decision structure to have a strong equilibrium.

2.3 Social Welfare with Pure Nash Equilibria

So far we just looked at the individual utilities of our agents. However, it is also extremely interesting to take into account the collective perspective. An equilibrium is a situation which is considered *ideal* by every single agent (in the sense that nobody has a better move). Is it also ideal at a collective level? The concept of *social welfare* helps us to capture the idea of optimality for the collective perspective (Leyton-Brown and Shoham, 2008). Although this is not the only notion that can be used to represent collective satisfaction, it is definitely among the most common ones. If a certain profile maximizes the social welfare it represents the best situation for the group. However, a pure Nash equilibrium may also not maximize the social welfare, and may even be quite distant from the maximal value for it. In this section, we investigate our game from the perspective of the social welfare. In particular, we first define the social welfare and characterize the profiles that produce the maximal social welfare. Then, we describe in which decision structure an equilibrium maximizes the social welfare. Eventually, we introduce the concept of *price of anarchy*. This is a measure of how much the individual behaviour damages the group (in terms of social welfare). Such a concept allows us to draw a final perspective on the game.

The social welfare of the group with respect to a profile $\boldsymbol{a}$ is defined as:

$$sw(\boldsymbol{a}) = \sum_{i=0}^{n} u_i(\boldsymbol{a}) = n \cdot M(|\boldsymbol{a}|, p) - |\boldsymbol{a}| \cdot e. \tag{2.9}$$

If we put aside n and e, $sw(\boldsymbol{a})$ depends only on the number of the people voting. For this reason, overloading notation sw we also talk about $sw(x)$ for a number

$x \in N$ such that:

$$sw(x) = \begin{cases} n \cdot M(x,p) - x \cdot e & \text{if } 0 \leq x \leq n \\ 0 & \text{otherwise.} \end{cases} \tag{2.10}$$

Notably, the value for $x < 0$ is going to be used for technical reasons, in a similar fashion of what we saw for $M(x,p)$ in the previous section. Through the definition of social welfare we are able to judge how 'good' are certain profiles. We start by looking at how the function for social welfare behaves.

Proposition 2.2. Let $D = \langle N, e, p \rangle$ be a decision structure. Let $0 \leq x < y \leq n$ such that $x, y \in \mathbb{N}$:

1. $sw(x+1) < sw(x)$ if x is odd,
2. $sw(x+2) - sw(x) \geq sw(y+2) - sw(y)$,
3. if either $x = 0$ or x is odd, $sw(x) \geq sw(x+1)$ and $sw(x) \geq sw(x+2)$ then $sw(x) \geq sw(y)$, and
4. if y is odd, $sw(y+2) \geq sw(y)$ then $sw(y+2) \geq sw(x)$.

Proof. We prove point by point.

1. If x is odd $M(x,p) = M(x+1,p)$ by Proposition 2.1. Thus, $sw(x) - sw(x+1) = -e$.
2. The marginal increase $(sw(x+2) - sw(x))$ it is formed by two parts: $n \cdot (M(x,p) - M(x+2,p)$ and $2e$. By Proposition 2.1, we get that if $x < y$ then $M(x+2,p) - M(x,p) > M(y+2,p) - M(y,p)$. Thus, since $(sw(x+2) - sw(x)) - (sw(y+2) - sw(y)) = M(x,p) - M(x+2,p) - (M(y,p) - M(y+2,p))$, $sw(x+2) - sw(x) \geq sw(y+2) - sw(y)$.
3. We have two cases: either x is odd, or x is equal to 0. Assume x is odd. Consider z s.t. $z > x$ and z is odd. We have that $sw(z) = sw(x,p) + (sw(x+2,p) - sw(x,p)) + ... + (sw(i,p) - sw(i-2,p)) + ... + (sw(z,p) - sw(z-2,p)) = sw(x,p) + \sum_{i=x}^{z}(sw(i,p) - M(i-1,p))$. Since $sw(x+2,p) - sw(x,p) \leq 0$, $(sw(i,p) - M(i-2,p)) \leq 0$ for all i, by Statement 2. Consequently, $\sum_{i=x}^{z}(sw(i,p) - M(i-1,p)) \leq 0$ and $sw(x) \geq sw(z)$. For any $y \geq x$, either y is odd (and then we just proved our result) or y is even and then there exists z odd s.t. $sw(z) > sw(y)$ and $z > y$. Assume x is 0. Thus, $sw(0) > sw(1)$ and consequently, $sw(1) > sw(3)$ by Proposition 2.1. We apply to 1 the same reasoning we applied in the first part of this point and we have that $sw(1) \geq sw(y)$ for any y. Consequently, $sw(0) \geq sw(y)$ for any y.

4. The reasoning is similar to Statement 3. So, for any odd number z s.t. $z < y$ and z is odd we have that $sw(y+2) = sw(z,p) + (sw(z+2,p) - sw(z,p)) + ... + (sw(i+2,p) - sw(i,p)) + ... + (sw(y+2,p) - sw(y,p)) = sw(z,p) + \sum_{i=z}^{y+2}(sw(i+2,p) - M(i,p))$. Since $sw(y+2,p) - sw(y,p) \geq 0$ then $(sw(i+2,p) - M(i,p)) \geq sw(y+2) - sw(y) \geq 0$ for all i, by Statement 2. Consequently, $\sum_{i=z}^{y+2}(sw(i+2,p) - M(i,p)) \geq 0$ and $sw(y+2) \geq sw(z)$. For any $x < y+2$, either x is odd (and then we just proved our result) or x is even and then there exists z odd s.t. $sw(z) > sw(y)$ and $z < y$.

□

Interestingly, this corresponds to saying that if we consider the set of odd numbers plus 0 as domain for the function sw, we have that a local maximum is also a global maximum. So, if x is such that $sw(x-2) \leq sw(x) \geq sw(x+2)$ then x is a global maximum. We call an mSW a number x iff $sw(\boldsymbol{a}) \geq sw(\boldsymbol{a}')$ for all $\boldsymbol{a}'$ such that $|\boldsymbol{a}| = x$, i.e. if and only if x maximizes the social welfare.

Theorem 2.3. Let $D = \langle N, e, p \rangle$ be a decision structure. A profile $\boldsymbol{a}$ with $|\boldsymbol{a}| = k$ maximizes the social welfare (is a mSW) iff $sw(k) \geq sw(k+2)$, $sw(k) \geq sw(k+1)$ and $sw(k) \geq sw(k-2)$.

Proof. Let x s.t. $sw(x) \geq sw(x+2)$, $sw(x) \geq sw(x+1)$ and $sw(x) \geq sw(x-2)$. By Proposition 2.2, $sw(x) \geq sw(y)$ for all $y > x$. Similarly, $sw(x) \geq sw(y)$ for all $y < x$. □

This theorem gives us already some information about the maximum for the social welfare. Notably, we made use here of the assumption that $sw(x) = 0$ if $x < 0$ or $x > n$. Indeed, since a profile with size lower than 0 or greater than n cannot exist, we want to be sure that every profile has a higher social welfare than that. If $e \geq n(p - \frac{1}{2})$ then 0 maximizes sw. Indeed, from that follows $sw(0) > sw(1)$. Since 1 is odd, then also $sw(0) > sw(2)$. And, trivially, $sw(0) > sw(-1)$. Now, let n_{odd} the highest odd number, i.e. $n_{odd} = max\{x \leq n | x \text{ is odd } \}$. If $e \leq \frac{n(M(y,p) - M(y-2,p))}{2}$, then n_{odd} maximizes the social welfare. Eventually, we can prove the following.

Corollary 2.3. If $0 < x < n-1$ then x is mSW iff it is odd and such that

$$\frac{n(M(x+2,p) - M(x,p))}{2} \leq e \leq \frac{n(M(x,p) - M(x-2,p))}{2}$$

Proof. Let $x \in N$ such that $M(x+2,p) - M(x,p) \leq \frac{2e}{n} \leq M(x,p) - M(x-2,p)$. Since $\frac{2e}{n} \leq M(x,p) - M(x-2,p)$, $sw(x) \geq sw(x-2)$. Thus, by Proposition 3 we also know that since $sw(x)$ is higher than $sw(z)$ for all the $z < x$. Similarly, by Proposition 2.2, since $M(x+2,p) - M(x,p) \leq \frac{2e}{n}$ we know that $sw(x)$ is higher than $sw(z)$ for all the $z > x$. Thus, x maximizes the social welfare, i.e. x is mSW. $\square$

Notably, there can be more than one number that maximizes the social welfare. To have a clearer idea we can look at example and the corresponding Figure 2.5.

Example 2.4. *We can look at the same four groups we introduced at the beginning. Each group has 50 people, whose accuracy is, in turn, 0.6, 0.7, 0.8, 0.9. The following graphs show the social welfare with different efforts.*

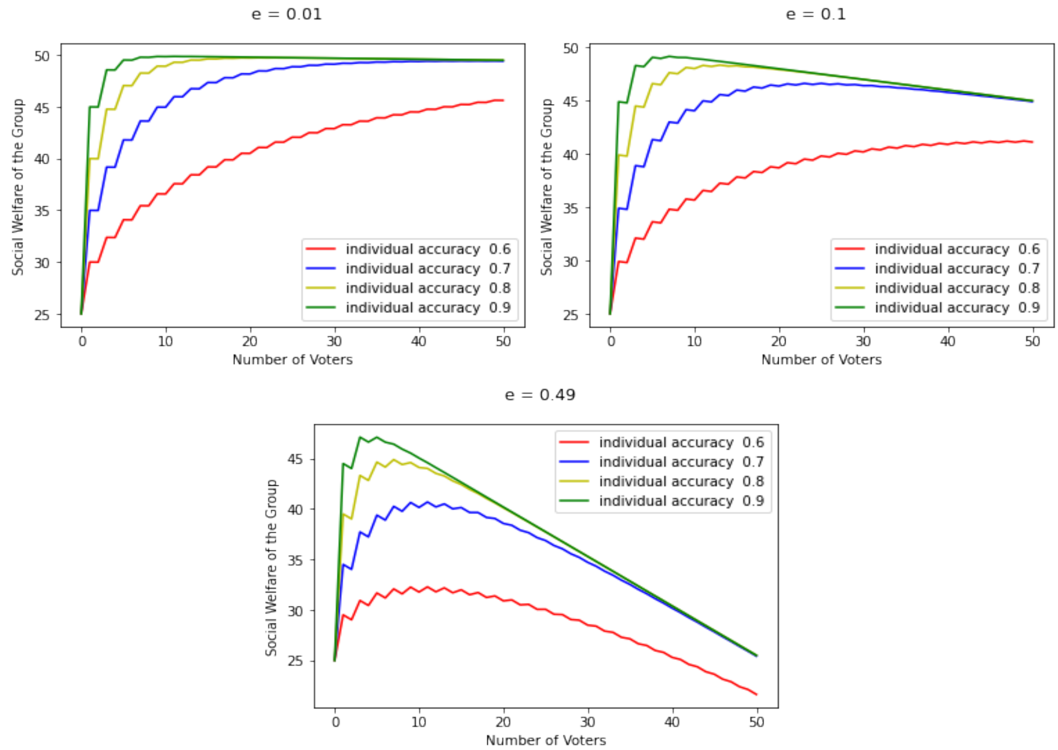

Figure 2.5: The function for the social welfare with $n = 50$ and three different values of effort: $e \in \{0.01, 0.1, 0.49\}$

As it can be seen in Figure 2.5, the higher is the effort, the lower is the number of voters at which the function reaches its maximum. The same thing can be said for the accuracy: the higher the accuracy the earlier is reached the maximum (this is extremely explicit in $e = 0.01$*).*

So far, we managed to characterize and study the maxima for sw. However, as we pointed out at the beginning of this section, our main aim is to evaluate the *quality* of the equilibria in terms of social welfare. This will give us an idea of the efficiency of different decision structures. In order to do so, we will follow two lines:

- we will show under which conditions a PNE maximizes the social welfare and
- we will look at the price of anarchy of the game.

Interestingly enough, we already have some information about the first point. Indeed, if $x = 0$ is a maximum for the social welfare it is also a PNE, namely an mPNE. This follows immediately from the fact that if $e \geq n(p - \frac{1}{2})$, we also have that $e > p - \frac{1}{2}$ (recall that $n \geq 2$). More in general we show the following.

Corollary 2.4. Let $D = \langle N, e, p\rangle$ be a decision structure and let x be mSW. Thus, $x \geq n_{eq}$.

Proof. Consider n_{eq}. There are three cases. Case 1: $e > p - \frac{1}{2}$. Thus, $n_{eq} = 0$. So, every x is s.t. $x \geq 0$. Case 2: $e = p - \frac{1}{2}$. Thus, since we assume $n \geq 2$ it must be that $e \geq n(p - \frac{1}{2})$. Thus, 0 is not a mSW. Consequently, for all x that can be mSW $x \geq 1$. Case 3: $e < p - \frac{1}{2}$. Thus, for n_{eq} holds $M(n_{eq}, p) - M(n_{eq} - 1) \geq e$ and therefore, $M(n_{eq}, p) - M(n_{eq} - 2, p) \geq e$. So, if $n \geq 2$ (as we assume it is), $M(n_{eq}, p) - M(n_{eq} - 2, p) \geq {}^{2e}/{}_{n}$. And, $sw(n_{eq}) - sw(n_{eq} - 2) \geq 0$. By Proposition 2.2, $sw(n_{eq}) > sw(y)$ for all $y < n_{eq}$. So, $x \geq n_{eq}$. □

Such a result provides us with two insights. Firstly, the point of mSW is always greater than the number of voters at any equilibrium. Secondly, it follows that if there is more than one value k for which a profile $\boldsymbol{a}$ is PNE if $|\boldsymbol{a}| = k$ then only the highest one can maximize social welfare. This allows us to draw a conclusion about another 'special case'. If $n_{eq} = n_{odd}$, then n_{eq} also maximizes the social welfare. Eventually we can introduce, composing the results from Theorem 2.1 and Theorem 2.3, a general observation.

Observation 2.2. *Let $D = \langle N, e, p\rangle$ be a decision structure. A profile $\boldsymbol{a}$ with $0 < |\boldsymbol{a}| < n - 1$ is both a PNE and an mSW iff*

$$M(|\boldsymbol{a}|, p) - M(|\boldsymbol{a}| - 1, p) \geq e \geq \frac{n[M(|\boldsymbol{a}| + 2, p) - M(|\boldsymbol{a}|, p)]}{2}.$$

Indeed, since $n \geq 2$ we have that $\frac{n[M(|\boldsymbol{a}|+2,p)-M(|\boldsymbol{a}|,p)]}{2} \geq M(|\boldsymbol{a}| + 1, p) - M(|\boldsymbol{a}|, p)$ and that $\frac{n[M(|\boldsymbol{a}|,p)-M(|\boldsymbol{a}|-2,p)]}{2} \geq M(|\boldsymbol{a}|, p) - M(|\boldsymbol{a}| - 1, p)$. Remarkably, the conditions for this observation are not that different from those of Strong Equilibria. Indeed, the mechanism is quite similar.

We managed to describe under which conditions a PNE is also an mSW. We conclude this part (before looking at the price of anarchy) by proving something similar to a 'limitation' result.

Corollary 2.5. Let $D = \langle N, e, p \rangle$ a decision structure. If $n > \frac{1}{-p^2+p}$, and n_{eq} maximizes the social welfare then $n_{eq} = 0$ or $n_{eq} = n_{odd}$.

Proof. Assume $n > \frac{1}{-p^2+p}$. By contraposition, assume $0 < n_{eq} < n-1$ s.t. n_{eq} is an mSW. By Proposition 2.1, $M(n_{eq}, p) - M(n_{eq}-1, p) \geq e \geq \frac{n(M(n_{eq}+2,p) - M(n_{eq},p)]}{2}$. This implies $M(n_{eq}, p) - M(n_{eq}-1, p) \geq \frac{n(M(n_{eq}+2,p) - M(n_{eq},p)]}{2}$. Consequently, $n \leq \frac{M(n_{eq},p) - M(n_{eq}-1,p)}{M(n_{eq}+2,p) - M(n_{eq},p)}$. In addition, $\frac{M(1,p)-\frac{1}{2}}{M(3,p)-M(1,p)} \geq \frac{M(n_{eq},p) - M(n_{eq}-1,p)}{M(n_{eq}+2,p) - M(n_{eq},p)}$ by Proposition 2.1. Indeed, the ratio between $M(n_{eq}, p) - M(n_{eq}-1, p)$ and $M(n_{eq}+2, p) - M(n_{eq}, p)$ is as big as possible in the lowest numbers. Thus, $n \leq \frac{M(1,p)-\frac{1}{2}}{M(3,p)-M(1,p)}$. Furthermore, $2(M(1,p) - \frac{1}{2}) = 2p-1$ and $(M(3,p) - M(1,p) = (3p^2 + p^3 - p)$. Thus, $M(n_{eq}, p) - \frac{1}{2} \geq \frac{n(M(3,p)-M(1,p))}{2}$ iff $2p-1 \geq n(3p^2+p^3-p)$ iff $1 \geq n(p^2+p)$, i.e. $n \leq \frac{1}{-p^2+p}$ □

This result shows that if n is higher than a certain threshold, only a profile of size n_{odd} or 0 can be both a PNE and an mSW. Relevantly, the threshold for n is surprisingly low.

In Figure 2.6 we give a visual illustration of this result. If $p = 0.6$ then as soon as $n \geq 5$, the only possible values that can be both PNE and mSW are 0 and n_{odd}.

The main difference between the social welfare perspective and the 'individual one' is that in the collective perspective the effort of learning is valued more. For this reason, as we saw in Corollary 2.4, the maximization of the social welfare requires always an equal or greater number of voters than any equilibrium. The gap between the 'value' of the effort for the single agent and the 'value' of the effort for the community increases with the size of the community, as the value for the community increases and that for the single agent remains fixed. Indeed, as showed in Corollary 2.5, if n passes a certain threshold, then no equilibrium can maximize sw, unless we are in a 'trivial case' like 0 or n_{odd}.

Eventually, we look at the value for the price of anarchy (*PoA*). The price of anarchy is usually a measure that gives an idea of how the efficiency of a system degrades due to selfish behavior of its agents in the worst case. It represents an opposite perspective with respect to what we did so far. So far, we looked at when an mPNE could have maximized the social welfare. Now, we looked at the value of social welfare for the equilibrium with the lowest one.

Definition 2.4. Let $D = \langle N, e, p \rangle$ be a decision structure. Let $\boldsymbol{a}$ be a PNE s.t. if $\boldsymbol{a}'$ is a PNE then $sw(\boldsymbol{a}) \leq sw(\boldsymbol{a}')$. And let $0 \leq x \leq n$ be an mSW. Thus, *PoA* for D is so defined:

$$\textit{PoA} = \frac{sw(|\boldsymbol{a}|)}{sw(x)}. \tag{2.11}$$

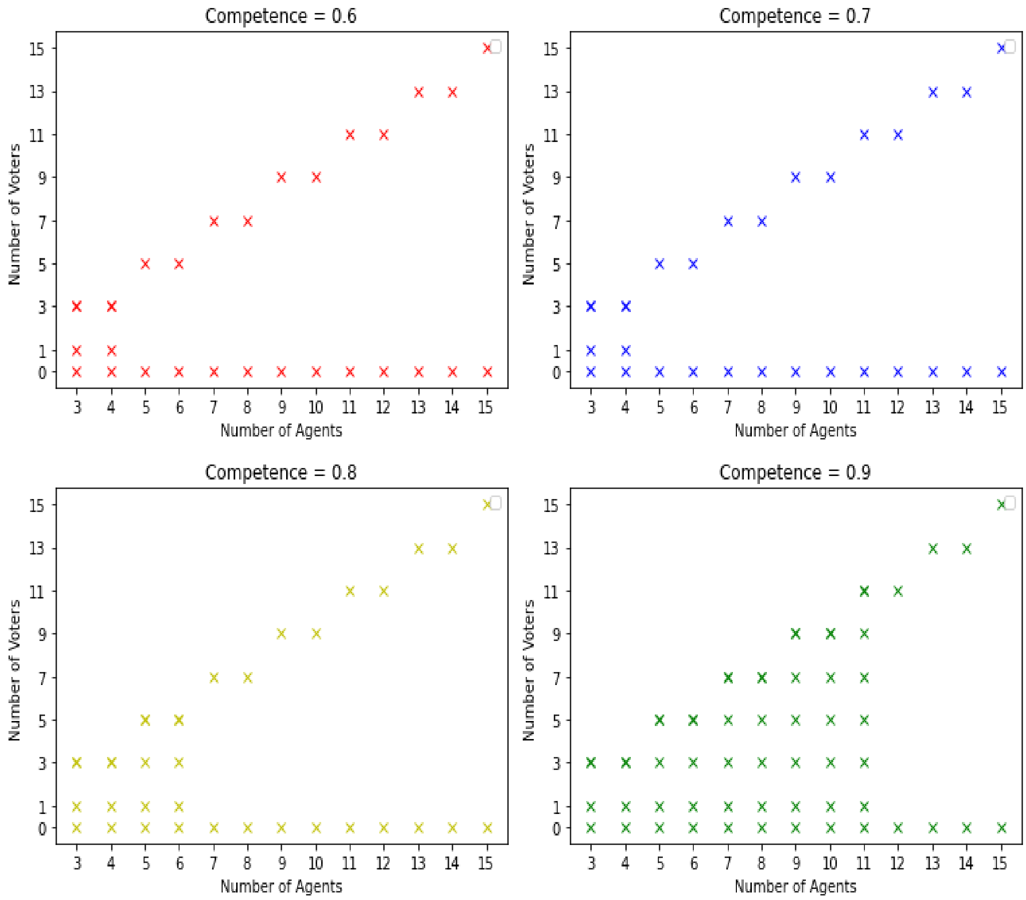

Figure 2.6: For each level of competence $p \in \{0.6, 0.7, 0.8, 0.9\}$ we plot on the y-axis what can be the size of an equilibrium that maximizes the social welfare for groups of different sizes (on the x-axis).

As it stands, the price of anarchy measures the quality of the *worst* equilibrium. This means that if *PoA* is high the individual behaviour impacts harmfully on the social welfare of the group, as the social welfare obtained at the equilibrium is quite low compared to the potential one. If $PoA = 1$, the interest of the individuals exactly coincides with the interest of the group.

Proposition 2.3. Let $D = \langle N, e, p \rangle$ be a decision structure and x be an mSW.

- If $e < p - \frac{1}{2}$:

$$PoA = \frac{M(x,p) - \frac{xe}{n}}{p - \frac{e}{n}}.$$

- If $e \geq p - \frac{1}{2}$ there are two cases:
  - if $e > n(p - \frac{1}{2})$ then *PoA* $= 1$ and
  - if $e \leq n(p - \frac{1}{2})$ then *PoA* $= 2(M(x,p) - \frac{xe}{n})$.

Proof. Assume $e < p - \frac{1}{2}$. Thus, by Corollary 2.1 and by Proposition 2.2 a profile $\boldsymbol{a}$ with $|\boldsymbol{a}| = 1$ is the PNE with the lowest social welfare. So, *PoA* $= \frac{M(x,p) - \frac{xe}{n}}{p - \frac{e}{n}}$.

If $e \geq p - \frac{1}{2}$ the profile $\boldsymbol{a}$ with $|\boldsymbol{a}| = 0$ is the one with the lowest social welfare. Indeed, by Proposition 2.2 and by Corollary 2.4, we know that sw has a global maximum and that $sw(1) > sw(0)$. Now, we look at the two statements. If $e > n(p - \frac{1}{2})$ then 0 maximizes the social welfare and consequently *PoA* $= 1$. If $e \leq n(p - \frac{1}{2})$ then *PoA* $= 2(M(x,p) - \frac{xe}{n})$, as $sw(0) = \frac{n}{2}$. □

Interestingly enough, the price of anarchy is equal to 1 only if $e > n(p - \frac{1}{2})$. This comes at the cost of not producing any increase of accuracy whatsoever, and it should not be a desirable outcome. Remarkably, this is also quite unlikely.[1] In all the other situations the price of anarchy is always greater than 1. We may wonder how it varies depending on n. The answer is in the last observation of this chapter and in Figure 2.7.

Observation 2.3. *For any decision structure $D = \langle N, e, p \rangle$, if $e > n(p - \frac{1}{2})$ the maximum value that PoA can take is, when $n \to \infty$: PoA $\to \frac{1}{p}$. The minimum value that it can take is when $n \to 2$: PoA $\to 1$.*

To have a better understanding of how price of anarchy changes with respect to the number of agents in a group we can have a look at Figure 2.7. In line with Observation 2.3, the increase in the size of the group increases also the price of anarchy. In addition, the price of anarchy grows relevantly more if the individual competence is lower. Indeed, as we mentioned, if $p = 0.9$, the price of anarchy reaches $\frac{1}{0.9}$ at its maximum, whereas if $p = 0.6$, it reaches $\frac{1}{6}$.

2.4 Summary, Discussion and Takeaways

The results we obtained in this chapter are quite fundamental for the entire work. They not only prepare the field for the results to come but also already address quite satisfyingly some of the questions we asked in the introduction.

[1] For example, if $p \geq 0.75$ this can never be the case. Indeed, $e < \frac{n}{4}$.

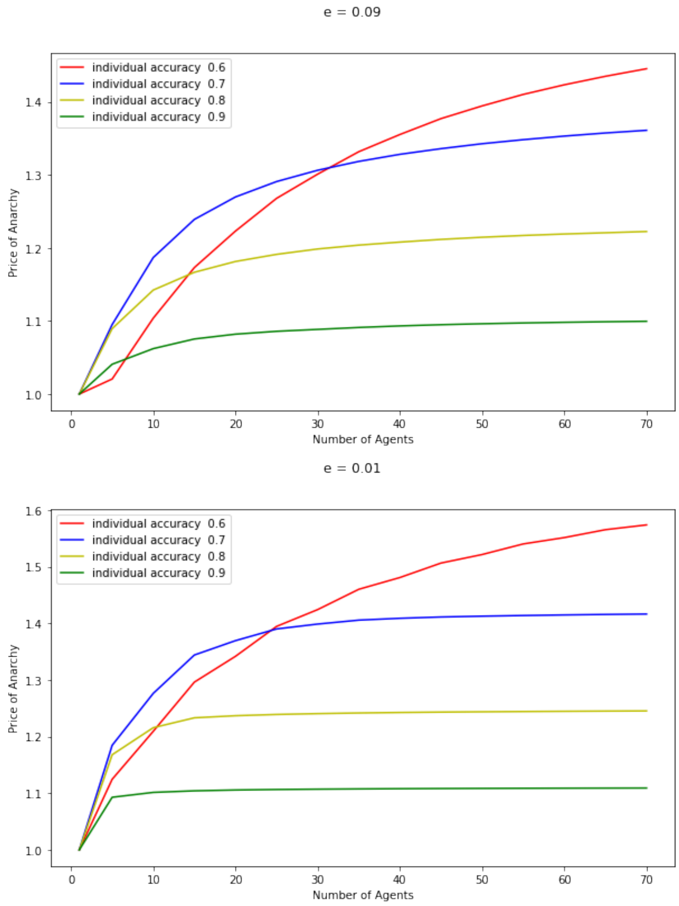

Figure 2.7: On the x-axis the number n and on the y-axis the value for *PoA*, given that $e < p - \frac{1}{2}$.

Revisiting CJT results. We shall compare this perspective to the one of the CJT. First, we must observe that statement (LIB) of the Condorcet Jury Theorem (see Section 1.1) does not hold anymore. There exists a threshold n_{thr} such that adding more agents to the group does not modify the possible equilibria. And consequently, it does not increase the group accuracy. Indeed, the agents that may be added would just free-ride. A big group is not always better than a small one.

For a similar reason, statement (ASY) does not hold either. The only case in which the overall accuracy tends to 1 with $n \to \infty$ is when $e \to 0$. For this reason, the competence level of the agents is also quite relevant for the overall accuracy of the group. If CJT were to be applied in a real scenario, we could have been sure that if people were all sufficiently competent (but not experts), their large number would have still produced the right answer with quite a high probability. Here, instead, if all the people in a group are very little competent (e.g. $p = 0.55$), the group may never reach a 'high' probability (such as $p = 0.9$), no matter how many people the group contains, as n_{thr} may just be too low.

However, it is true that somehow, being in a group does not harm the individual. Indeed, if the individual considered voting already worth the effort if she was alone, in the group she would also get an accuracy at least equal or higher than being by herself. Interestingly, these results are in line with many of the other works that approached a similar setting (T. J. Feddersen, 2004; Martinelli, 2006; Persico, 2004). Of course, as our model frames the problem slightly differently this is a proof of the *robustness* of these results.

Social Welfare and Strong Equilibria. The number of people in the group can affect the group in terms of stability. Indeed, the higher n the less likely is an SPNE to exist. On the other hand, results on social welfare highlight another type of perspective which is extremely fundamental. They give us an idea of how much individual behaviour is *harmful* for the system. We showed that the cases in which an equilibrium can also produce a maximization of social welfare are few. In the majority of cases, it is reasonable to assume that the maximization of social welfare cannot be reached at an equilibrium. This is a clear example of how phenomena of free-riding often coincide with poor efficiency. Our observations about the price of anarchy tell us something about this matter too. Notably, if $n \to \infty$ the price of anarchy is maximum. Indeed, in that case the potential of the group would be quite high, but the actual welfare at equilibria it produces is relevantly lower.

Model features. Now, we shall briefly discuss the features of our model. First of all, we must have a look at the idea of effort e. As we mentioned in Chapter 1, we introduced the effort to represent the cost of learning, i.e. of *acquiring competence*. It may either stand for the cost of *finding information* or for the cost of *understanding information* or for the combination of them. We explicitly made the decision of not connoting it any further, as we wanted to leave many possible interpretations open. Indeed, it could also represent the cost of the act of voting itself, as we explained in Chapter 1, as it is paid by the agents if and only if they decide to vote (action 1). The choice of assuming the effort to be a constant and to be fixed for everybody allowed us to leave every possibility open. Instead, in other works, for example that of Persico (2004), the effort is modeled to better represent the cost of learning information and is described through a more complex function. Keeping the effort fixed also enabled our analysis to focus more effectively on other elements of the framework. Secondly, the reader may wonder why we took into account only pure Nash equilibria. The answer is indeed very simple. Although we believe that looking into mixed equilibria could represent an interesting expansion of the present work, we also think that looking at pure ones was more than enough to get an idea of the agents' behaviour in such a setting, which was indeed our aim from the beginning.

Voting on many issues. Before concluding this chapter, we shall draw an interesting comparison. Consider taking a certain group and asking it to vote on different issues at different times. At every round, each agent should decide what to do without knowing about what the other agents will do. And she can adjust her strategy based on the outcomes of the previous votes. Interestingly, such a setting would resemble closely the one described in the El Farol game (Arthur, 1994; Whitehead et al., 2008). In both the games, the group of player must divide efficiently in two groups and if it fails to do so, the agents may not be satisfied. On the other hand, there are also some differences. In the original game of El Farol, if the number of people at the bar is too large they all get a negative payoff, whereas the people at home do not really get any negative or positive utility. Instead, here, the people that do not vote have a reward that still depends on the number of people voting. There seems to be a higher level of correlation between the two groups in terms of utilities. We believed that investigating such a link further may produce extremely interesting results.

3 Those who Care and Those who Do Not

In Chapter 2 we considered homogeneous groups, i.e. groups where all agents have the same features. Yet, what if the group is heterogeneous? In this chapter, we take into account a specific form of heterogeneity, namely heterogeneity in *stakes*: every agent may have different stakes in the matter. The concept of stakes captures the idea that certain decisions have higher impact on some people and lower on others. This depends on the fact that not everybody has the same interests, values, and living situation as everybody else. Although it may seem hard to exactly quantify these stakes, it is definitely possible to give a rough estimate (sometimes also just based on few factors) of their value, as it is undeniable that they play an important role in the motivation that moves a person to get involved in a decision. Indeed, in a realistic scenario, not all the people of a group care equally for the accuracy of the vote. Consider a group of people that has to decide whether to build a garden or not in a certain area. Every member of this group may have about the same level of expertise (the typical situation would be where nobody has any special knowledge of gardens), and so the same level of competence, but some of them may be more interested in the matter than others as they live closer to that area.

In Section 3.1 we give a formalization of this situation. In Section 3.2 we take into account pure Nash equilibria and strong pure Nash equilibria in this new framework. Notably we make use of many of the results we obtained in Chapter 2. In Section 3.3 we consider the social welfare of these equilibria. Eventually, we briefly discuss our results.

3.1 Preliminaries

We start by generalising some of the notions we already have and add some new definitions, that will turn out to be useful in this new setting.

First of all, we shall start again from a group $N = \{1, ..., n\}$ of agents. Then, we introduce the variable $b_i \in \mathbb{R}^+$ to represent the stakes of agent i, such that if agent i cares more for a certain decision than agent j, we have that $b_i > b_j$. We use the vector $\boldsymbol{b} = (b_1, ..., b_n)$ to collect the stakes of the agents in a group. Each agent $i \in N$ may choose between 'vote' and 'abstain' as before, i.e. $a_i = \{0, 1\}$, and again, if she chooses action 1 she incurs in a cost e and she votes with accuracy p. The action profiles are defined as in Chapter 2. We generalise the notion of decision structure which now becomes $D = \langle N, p, e, \boldsymbol{b} \rangle$. In order to be able to better work with vectors of stakes we also make use of the notation $b^1_{min}(\boldsymbol{a})$ to indicate the lowest value of the stakes among the voters in $\boldsymbol{a}$, i.e.

$$b^1_{min}(\boldsymbol{a}) = \begin{cases} min\{b_i | a_i = 1\} & \text{if } |\boldsymbol{a}| > 0 \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1}$$

And, in a similar fashion, we denote with $b^0_{max}(\boldsymbol{a})$ the highest value of the stakes among the stakes of the non-voters in $\boldsymbol{a}$, i.e.

$$b^0_{max}(\boldsymbol{a}) = \begin{cases} max\{b_i | a_i = 0\} & \text{if} |\boldsymbol{a}| < n \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Notably, the value for $b^1_{min}(\boldsymbol{a})$ when $|\boldsymbol{a}| = 0$ and the value for $b^0_{max}(\boldsymbol{a})$ when $|\boldsymbol{a}| = n$ were attributed for technical reasons, which will become apparent in Theorem 3.1. Indeed, the concept itself of $b^1_{min}(\boldsymbol{a})$ does not make sense if $|\boldsymbol{a}| = 0$, and similarly for $b^0_{max}(\boldsymbol{a})$ if $|\boldsymbol{a}| = n$. Eventually, we rewrite the formula for the utility:

$$u_i(\boldsymbol{a}) = b_i \cdot M(|\boldsymbol{a}|, p) - a_i e. \tag{3.3}$$

Indeed, the reward coming from the group being accurate scales with how much the agent cares for the matter, i.e. it is multiplied by b_i. So, for example, in the scenario we proposed above if an agent i lives closer to the park than an agent j, $b_i > b_j$ and consequently, the accuracy of the choice benefits i more than j. Notably, this formula generalizes Formula 2.6 (where b_i can be assumed to be equal to 1).

3.2 Basic Results and Pure Nash Equilibria

As we already mentioned, equilibria are the tools we use to predict the agents' behaviour. We dedicate this section to describe them and explain the rationale behind them. First, we give a full characterization of them; then, we look at the equilibria where the group has the highest overall accuracy and eventually we also take into account strong equilibria. In order to do so, we employ many results from Chapter 2 and, in general, we compare the results obtained there with what we prove here.

3.2.1 Pure Nash Equilibria

First of all, it is important to note that as far as the function $M(x, p)$ is concerned nothing changes from the previous chapter. Indeed, Proposition 2.1 is still valid and the way the curve for $M(x, p)$ behaves for a decision structure $D = \langle N, p, e \rangle$ is exactly the same as for a decision structure $D' = \langle N, p, e, \boldsymbol{b} \rangle$. Indeed, what is fundamentally different is how people decide to vote or not. In order to account for the differences with Chapter 2, but basically following the same structure, we start by characterizing the equilibria.

Theorem 3.1. Let $D = \langle N, e, p, \boldsymbol{b} \rangle$ be a decision structure and $\boldsymbol{a} \in \{0, 1\}^n$ be an action profile with $|\boldsymbol{a}| = k$. The profile $\boldsymbol{a}$ is a PNE iff $b^0_{max}(\boldsymbol{a})[M(k+1, p) - M(k, p)] \leq e \leq b^1_{min}(\boldsymbol{a})[M(k, p) - M(k-1, p)]$.

Proof. A profile $\boldsymbol{a}$ is a PNE iff for no agent i $u_i(\boldsymbol{a}) \geq u_i(a'_i, \boldsymbol{a}_{-i})$ by Definition 2.1. Assume i s.t. $a_i = 1$. Thus, $u_i(0, \boldsymbol{a}_{-i}) = b_i M(|\boldsymbol{a}| - 1, p)$ and $u_i(\boldsymbol{a}) = b_i M(|\boldsymbol{a}|, p) - e$. Thus, agent i would not deviate iff $M(|\boldsymbol{a}|, p) - e \geq M(|\boldsymbol{a}| - 1, p)$, i.e. iff $b_i[M(|\boldsymbol{a}|, p) - M(|\boldsymbol{a}| - 1, p)] \geq e$. Since $b_i \geq b^1_{min}(\boldsymbol{a})$ by definition (and it can be that $b_i = b^1_{min}(\boldsymbol{a})$), agent i would not deviate iff $b^1_{min}(\boldsymbol{a})[M(|\boldsymbol{a}|, p) - M(|\boldsymbol{a}| - 1, p)] \geq e$.

Now, assume i s.t. $a_i = 0$. Thus, $u_i(1, \boldsymbol{a}_{-i}) = b_i M(|\boldsymbol{a}|+1, p) - e$ and $u_i(\boldsymbol{a}) = b_i M(|\boldsymbol{a}|, p)$. Thus, agent i would not deviate iff $b_i M(|\boldsymbol{a}|, p) \geq b_i M(|\boldsymbol{a}|+1, p) - e$, i.e. iff $b_i[M(|\boldsymbol{a}|+1, p) - M(|\boldsymbol{a}|, p)] \leq e$. Since $b_i \leq b^0_{max}(\boldsymbol{a})$ by definition (and it can be that $b_i = b^0_{max}(\boldsymbol{a})$), agent i would not deviate iff $b^0_{max}(\boldsymbol{a})[M(|\boldsymbol{a}| + 1, p) - M(|\boldsymbol{a}|, p)] \leq e$.

Consequently, no agent i would deviate iff $b^0_{max}(\boldsymbol{a})[M_p(|\boldsymbol{a}|+1) - M_p(|\boldsymbol{a}|)] \leq e \leq b^1_{min}(\boldsymbol{a})[M_p(|\boldsymbol{a}|) - M_p(|\boldsymbol{a}| - 1)]$. And this corresponds to saying that $\boldsymbol{a}$ is a PNE iff $b^0_{max}(\boldsymbol{a})[M_p(|\boldsymbol{a}| + 1) - M_p(|\boldsymbol{a}|)] \leq e \leq b^1_{min}(\boldsymbol{a})[M_p(|\boldsymbol{a}|) - M_p(|\boldsymbol{a}| - 1)]$. □

This is a full characterization of PNEs in the game associated with our decision structure. Notably, this theorem is very similar to Theorem 2.1. The only relevant difference is indeed the use of the notion of stakes. The individual value that each agent assigns to the possibility of changing her action is now dependent on the stakes. Remarkably, to be sure that no one wants to deviate we have to be sure that the voter with the lowest stakes does not want to abstain and the non-voter with the highest stakes does not want to vote.

The technical values we assigned to $b^1_{min}(\boldsymbol{a})$ and $b^0_{max}(\boldsymbol{a})$ are useful in this theorem as they allow us to give a full characterization. Indeed, when looking at a profile $\boldsymbol{a}$ such that $|\boldsymbol{a}| = 0$, we want to be sure that it does not make sense to switch to a profile $\boldsymbol{a}'$ with $|\boldsymbol{a}'| = -1$. And, for this reason, we assign to the hypothetical voter (which clearly does not exist) in $\boldsymbol{a}$ the value of ∞ so that she does not want to go away (and so a profile with size -1 cannot even be taken into consideration). A similar reasoning is also done when a profile $\boldsymbol{a}$ is such that $|\boldsymbol{a}| = n$. We assign to the hypothetical non-voter in $\boldsymbol{a}$ (which, again, does not exist) stakes equal to 0 so that we are sure it does not make sense to go to a profile with size $n + 1$.

Following the lines of Corollary 2.1, we can have a look further at how the effort influences the game by changing the possible equilibria. Again, we can distinguish three cases. Relevantly, they all depend on the value for the stakes of the most interested person in the group. We denote with b^+ this concept, i.e. $b^+ = max\{b_i | i \in N\}$.

Corollary 3.1. Let $D = \langle N, e, p, \boldsymbol{b}\rangle$ and let $\boldsymbol{a}$ be an action profile.

1. If $e > b^+(p - \frac{1}{2})$ a profile $\boldsymbol{a}$ is a PNE iff $|\boldsymbol{a}| = 0$.
2. If $e = b^+(p - \frac{1}{2})$ a profile $\boldsymbol{a}$ is a PNE iff either $|\boldsymbol{a}| = 0$ or $|\boldsymbol{a}| = 1$ and $a_i = 1$ for i s.t. $b_i = k$.
3. If $e < b^+(p - \frac{1}{2})$ a profile $\boldsymbol{a}$ is a PNE iff $|\boldsymbol{a}|$ is odd and $M(|\boldsymbol{a}|, p) \geq M(|\boldsymbol{a}| - 1, p) + \frac{e}{b^1_{min}(\boldsymbol{a}))}$.

Proof. This is just a result of Theorem 3.1. The first and the second case are quite straightforward. In the first case, we see that $M(1, p) - M(0, p) \leq e \leq M(0, p) - M(-1, p)$. Notably, $e \leq b^1_{min}(\boldsymbol{a})M(0, p) - M(-1, p)$ is trivially true. In the second case, we have the same for 0 and 1. The third case is the less trivial. If $|\boldsymbol{a}|$ is odd, we have that $\boldsymbol{a}$ is a PNE iff $M(k + 1, p) - M(k, p) \leq e$ is true by Proposition 2.1. In this case, we employed the fact that $b^0_{max}((1, 1, 1, ..., 1))$ is zero, as we cannot have more than n voters. The rest follows. $\square$

Also here, if e is not too big, we have that the only profiles that can be PNEs are those with an odd number of voters. The mechanism is exactly the same as in Chapter 2. If $|\boldsymbol{a}|$ is odd $b^0_{max}(\boldsymbol{a})[M(k+1, p) - M(k, p)] \leq e$ always holds (except for 0). Instead, if $|\boldsymbol{a}|$ is even $e \leq b^1_{min}(\boldsymbol{a})[M(k, p) - M(k-1, p)]$ never holds (except for 0). Recall that this is a direct consequence of Proposition 2.1. However, something changed compared to Corollary 2.1. In that case, who were the voters did not matter: only the size of the profile was relevant. In this chapter things are different under this aspect. Indeed, depending on who votes and who does not, the values for $b^1_{min}(\boldsymbol{a})$ and $b^0_{max}(\boldsymbol{a})$ change. However, we have to be careful in understanding where the change lies. Each agent still brings to the overall accuracy the same *contribution*. So, the change is not in the increase of the overall accuracy. Indeed, the change lies in how much agents value this contribution. Some value it more, some less. Such a detail becomes extremely important given that we are interested to characterize the profiles that yield the equilibria with the highest accuracy (mPNE).

Now, in order to be able to characterize the mPNE, we need to introduce some useful notions. Since they make use of the results we obtained in Chapter 2, we comment and review them here. Notably, in Chapter 2 we did not mention the stakes at all, but all the results that we got there can be easily considered to hold for a decision structure $D = \langle N, e, p, \boldsymbol{b} \rangle$ where $b_i = b$, i.e. an homogeneous decision structure. Indeed, the utility we considered in Chapter 2 (see Equation 2.6) was defined in terms of effort: $u_i(\boldsymbol{a}) = M(|\boldsymbol{a}|, p) - a_i e$. However, we can easily think the value for the effort e as a value for the term $\frac{e}{b}$. For example, Corollary 2.1 could say something like "if $\frac{e}{b} < p - \frac{1}{2}$...". So, the utility we use in Chapter 1 can be rewritten:

$$u_i(\boldsymbol{a}) = M(|\boldsymbol{a}|, p) - a_i \frac{e}{b}. \tag{3.4}$$

And, interestingly, for our purpose, this corresponds to

$$u_i(\boldsymbol{a}) = b \cdot M(|\boldsymbol{a}|, p) - a_i e, \tag{3.5}$$

as Equation 3.4 is the result of the normalization over b of Equation 3.5, i.e. they are equivalent.[1] Consequently, in order to know what happens in a homogeneous decision structure with $b_i = b$ is enough to look at the results in Chapter 2 and substituting the value for $\frac{e}{b}$ in the place of e. For this reason, from now on we refer to an homogeneous decision structure either with $D = \langle N, e, p \rangle$ or with $D = \langle N, p, e, b \rangle$. Relevantly, a decision structure $D = \langle N, e, p \rangle$ is equivalent to $D' = \langle N, p, e', b' \rangle$ iff $e = \frac{e'}{b'}$. With this in mind, we shall look at the following definition.

Definition 3.1. Let $D = \langle N, e, p, \boldsymbol{b} \rangle$ be a decision structure. We define the following notation.

- $g : \mathbb{R} \mapsto \mathcal{P}(N)$ is a function s.t. $g(b_j) = \{i \in N | b_i \geq b_j\}$, i.e. $g(b_j)$ is the set of agents with stakes higher or equal than b_i.

[1] The utility matters only for the equation $u_i(\boldsymbol{a}) - u_i(\boldsymbol{a}')$ and in this case the multiplication for b does not matter.

- $n_{hom} : \{b_1, ..., b_n\} \mapsto \mathbb{N}$ is a function that takes a value b_i and returns the value n_{eq} for the homogeneous decision structure $D' = \langle\{1, ..., |g(b_i)|\}, p, e, b_i\rangle$. Indeed, recall that n_{eq} is the highest size that could be reached by a PNE in D'.

- Let $N_H = \{x \in N|$ there exists b_i *s.t.* $n_{hom}(b_i) = x\}$ and let $n_h = maxN_H$ and b_h the value that corresponds to the b_i s.t. $n_{hom}(b_i) = n_h$. The set N_H collects all the values for n_{hom} and n_h is the maximum among them.

For each $x \in N_H$ there can more than one value among the b_i. We assign b_h to the lowest among them without loss of generality.

Theorem 3.2. Let $D = \langle N, e, p, \boldsymbol{b}\rangle$ be a decision structure. A profile $\boldsymbol{a}$ is an mPNE iff $|\boldsymbol{a}| = n_h$ and if $a_i = 1$ then $i \in g(b_h)$.

Proof. Consider the profile $\boldsymbol{a}$ such that $|\boldsymbol{a}| = n_h$ and if $a_i = 1$ then $i \in g(b_h)$. We first prove that $\boldsymbol{a}$ is an equilibrium. All the voters have equal or greater stakes than b_h. In addition, since $n_{hom}(b_h) = n_h$, a profile $\boldsymbol{a}$ would be an equilibrium in a decision structure $D' = \langle N, p, e, b_h\rangle$. Thus, $M(n_h, p) - M(n_h - 1, p) \geq \frac{e}{b_h} \geq M(n_h+1, p) - M(n_h, p)$ by Theorem 2.1. Since $b^1_{min}(\boldsymbol{a}) = b_h$ and $b^0_{max}(\boldsymbol{a}) \leq b_h$, we have that $b^1_{min}(\boldsymbol{a})[M(n_h, p) - M(n_h - 1, p)] \geq e \geq b^0_{max}(\boldsymbol{a})[M(n_h + 1, p) - M(n_h, p)]$. Consequently, $\boldsymbol{a}$ is a PNE by Theorem 3.1.

Now, we prove that it is an mPNE. Towards a contradiction, assume there exists a profile $\boldsymbol{a}'$ that is an equilibrium with a higher accuracy, which would imply an amount of voters n_l s.t. $n_l > n_h$, by how $M(n, p)$ is built. This means that $n_l \notin N_H$. Thus, there exists no b_i s.t. $n_{hom}(b_i) = n_l$. Thus, either n_l is even, or there exists no set $K \subseteq N$ s.t. $|K| = n_l$ and $n_l \leq n_{hom}(b_i)$ for all $i \in K$. In the first case (n_l is even) it cannot be an equilibrium as for each agent j s.t. $a_j = 1$, $u_j(0, \boldsymbol{a}'_{-j}) > u_i(\boldsymbol{a}')$. The other case implies that for some i s.t. $a'_i = 1$ we have that $n_l > n_{hom}(b_i)$. This implies that a profile $\boldsymbol{a}$ with $n_l = |\boldsymbol{a}|$ would not be an equilibrium in the correspondent homogeneous group. Thus, $M(n_l, p) - M(n_l - 1, p) < {}^{e}/_{b_i}$. Consequently, i would deviate. This is a contradiction. □

It may be helpful for our understanding to repeat here how the process we defined above works. We form different groups, each with a different lowest level of stakes, i.e. the different sets $g(b_j)$. For each of these groups we look at what is the biggest size of an equilibrium in that group. Since there are people with different stakes in each group, to have an equilibrium we must make sure that each agent i s.t. $a_i = 1$ has a sufficiently high b_i to vote. In order to do so, we check that the voter with the lowest stakes wants to vote, i.e. the agent j in the case for $g(b_j)$; in this way we are sure that all the agents in $g(b_j)$ would vote. Thus, we just compare all these size values and we take the highest.

We can have a look at an example.

Example 3.1. *Consider a decision structure* $D = \langle N, e, p, \mathbf{b} \rangle$ *with* $N = \{1, 2, ..., 15\}$*, the following vector of stakes* $\mathbf{b} = (600, 600, 400, 400, 400, 200, 200, 100, 50, 50, 50, 50, 20, 20, 20)$ *and* $p = 0.7$*. We assume that* $e = 0.49$*. Now, for each* b_i *we look at the amount of people that would vote in an homogeneous group with* $b = b_i$*,* $n = g(b_i)$ *(red line) and at the number of people at an mPNE for the decision structure* $D' = \langle N, p, e, b_i \rangle$*, i.e.* n_{eq} *(blue line) in Figure 3.1.*

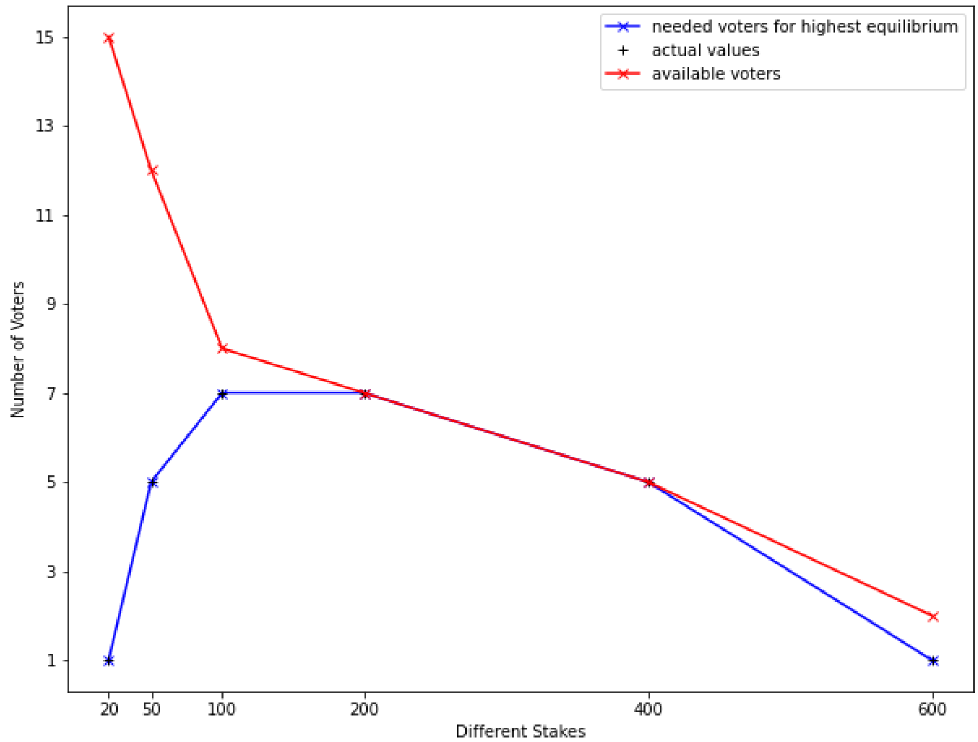

Figure 3.1: For any b_i we plot $g(b_i)$ (available voters) and $n_{hom}(b_i)$. This allows us to see how we can choose n_h.

We highlighted the points that are in N_H *with a little black cross. As it can be seen the highest is* 7*, i.e.* $n_h = 7$*, with* $b_h = 50$*.*

This result is quite interesting as it exemplifies two main points. First of all, it shows that the highest accuracy in an equilibrium is reached only if the people with the highest stakes vote. Secondly, it makes clear that to have an idea of what are the features of the equilibria (and first of all the accuracy that each equilibrium can produce) the only thing that matters is the value b_h. Recall that b_h is the stakes value that would yield the highest n_{hom}, as explained in Definition 3.1. It does not matter how is $\boldsymbol{b}$ structured as soon as b_h remains the same. This implies, indeed, that the length of of $\boldsymbol{b}$ does not matter either, and that adding more agents does not change anything, if it does not change b_h. This may remind the reader of the discussion in Chapter 2 about n_{thr} and n_{eq} for the homogeneous case. If adding voters did not change n_{eq} (and this was the case when $n \geq n_{thr}$), then the size of an mPNE would have remained the same. Indeed, the value for n_h in this chapter is exactly corresponding to the value n_{eq} for Chapter 2: the size of mPNEs. However, we prefer to keep them separated with the notation, as n_h has been defined for heterogeneous structure and n_{eq} for homogeneous ones. We shall look at the following example.

Example 3.2. *Consider a decision structure $D = \langle N, e, p, \boldsymbol{b}\rangle$ with $\boldsymbol{b} = (2000, 2000, 1000, 1000, 300)$, $p = 0.7$ and $e = 0.49$. We have that $b_h = 600$ and consequently $\boldsymbol{a} = (1, 1, 1, 1, 1)$ is an equilibrium and produces the highest accuracy possible, i.e. 0.837. Now, consider another decision structure D' with the following features: $\boldsymbol{b}' = \{2000, 2000, 1000, 1000, 600, 40, 30, 24, 20\}$, $p = 0.7$ and $e' = 0.49$. We still get $b'_h = 600$ and consequently $n'_h = n_h$ remains the size of the equilibrium with the highest accuracy possible. And the same thing happens if we just change the values of the highest bs as soon as they remain higher than b_h, for example, in a vector $\boldsymbol{b} = \{2000, 1000, 600, 600, 600, 40, 30, 24, 20\}$.*

3.2.2 Strong Pure Nash Equilibria

We shall now have a look at strong equilibria. Looking at when coalitions deviate was quite interesting in the previous chapter and we want to compare the results. Notably Lemma 2.2 and Lemma 2.1 still hold, as they do not assume anything about the homogeneity of the stakes of the agents. Recall that Lemma 2.2 states that if a coalition of m non-voters deviates from $\boldsymbol{a}$ then also a coalition of one non-voter deviates. The reason for this is that $M(|\boldsymbol{a}| - 1, p) \geq M(|\boldsymbol{a}| - x, p)$ for any $x \geq 1$ and for any $\boldsymbol{a}$. And, so if a group of agents benefits from leaving the group of voters also a single one would do so. Instead, Lemma 2.1 states that if a 'mixed' coalition deviates then also a 'non-mixed' one deviates. With the help of these two lemmas we can prove the following theorem.

Theorem 3.3. Let $D = \langle N, e, p, \boldsymbol{b}\rangle$ be a decision structure with $|N| = n$ and $\boldsymbol{a} \in \{0, 1\}^n$ be an action profile with $|\boldsymbol{a}| = s$. The profile $\boldsymbol{a}$ is an SPNE iff for all i s.t. $a_i = 0$, $b_i[M(s + x, p) - M(s, p)] \leq e \leq b^1_{min}(\boldsymbol{a})[M(s, p) - M(s - 1, p)]$ where $x = |\{j \in g(b_i)|a_j = 0\}|$.

Proof. Consider a profile $\boldsymbol{a}$. Such a profile is an SPNE iff no coalition deviates. Let K be a coalition s.t. $K \subseteq \{i \in N|a_i = 0\}$ and $\boldsymbol{a}'_K = (1, 1, ..., 1)$. Consequently, $K \subseteq \{j \in g(b_i)|a_j = 0\}$ for some i. Consequently, $u_i(\boldsymbol{a}'_K, \boldsymbol{a}_{N\setminus K}) > u_i(\boldsymbol{a})$ iff $b_iM(|\boldsymbol{a}| + |K|, p) - e = b_iM(s + x, p) - e > b_iM(s, p)$. Thus, K would not deviate iff $b_i[M(s + x, p) - M(s, p)] \leq e$. Now, let K be a coalition s.t. $K \subseteq \{i \in N|a_i = 1\}$ and $\boldsymbol{a}'_K = (0)$. Since $u_i(\boldsymbol{a}) = b_iM(s, p) - e$ we have that $u_i(\boldsymbol{a}) \geq u_i(\boldsymbol{a}'_K, \boldsymbol{a}_{N\setminus K})$ iff $M(s, p) - M(s - 1, p) \geq \frac{e}{b_i}$. Since $b_i \geq b^1_{min}(\boldsymbol{a})$ then $u_i(\boldsymbol{a}) \geq u_i(\boldsymbol{a}'_K, \boldsymbol{a}_{N\setminus K})$ iff $M(s, p) - M(s - |K|, p) \geq \frac{e}{b^1_{min}(\boldsymbol{a})}$ for any possible K containing exactly one voter. Thus, K would not deviate iff $b^1_{min}(\boldsymbol{a})[M(|\boldsymbol{a}|, p) - M(|\boldsymbol{a}| - 1, p)] \geq e$. By Lemma 2.1 and Lemma 2.2, if these two types of coalitions do not deviate no coalition does so. Thus, $\boldsymbol{a}$ is SPNE iff $M(n, p) - M(s, p) \leq e \leq M(s, p) - M(s - 1, p)$. □

It is worth to understand what changes between this theorem and that about SPNE in the previous chapter (Theorem 2.2). In the previous chapter we could be sure that if there was at least a coalition that wanted to switch – going from not voting to voting –, then the biggest coalition possible (that containing all the non-voters) would have also wanted to switch. This is not the case here, as the different stakes make this process a bit more obscure. Indeed, in this case the biggest coalition possible may contain an agent with a very low b_i who would never want to vote. So, we have to make sure that all the possible coalitions that can be formed do not want to switch to action 1. And this is done, by checking all the possible different stakes b_i of people outside the group of voters. However, we are still sure, as we were for Theorem 2.2, that if a profile is a pure Nash equilibrium, then no coalition of non-voters may want to switch. Indeed, this is exactly the result of Lemma 2.2 and it produces the following corollary.

Corollary 3.2. Let $D = \langle N, e, p, \boldsymbol{b}\rangle$ be a decision structure. If $\boldsymbol{a}$ is an SPNE then $|\boldsymbol{a}| = n_h$.

Proof. The proof proceeds similarly as the proof for Corollary 2.2. Since every $\boldsymbol{a}$ with $|\boldsymbol{a}| = n_h$ is an mPNE, every other $\boldsymbol{a}'$ s.t. $\boldsymbol{a}'$ is a PNE cannot be an SPNE, because going from that to $\boldsymbol{a}$ is always advantageous for every coalition of non-voters in $\boldsymbol{a}'$ of size $n_h - |\boldsymbol{a}'|$. □

This corollary consists in recognizing that a profile $\boldsymbol{a}$ to be an SPNE must also be an mPNE. As in our basic setting, not all the groups have a strong equilibrium and such existence depends on all the values that occur in vector $\boldsymbol{b}$ (and namely on n) and not only on b_h. Indeed, it is easy to find two groups were the values for b_h are the same, but since $\boldsymbol{b}'$ and $\boldsymbol{b}$ are different in one group there exists an SPNE and in the other there is not.

Example 3.3. *Consider the same decision structure we saw before in Example 3.2 with the following* $\boldsymbol{b} = \{1000, 1000, 500, 500, 300\}$ *and* $p = 0.7$. *In this case,* $b_h =$

300 and consequently $\boldsymbol{a}$ with $|\boldsymbol{a}| = 5$ is an SPNE. Now, take into account another decision structure with the following features: $\boldsymbol{b} = \{1000, 1000, 500, 500, 300, 20, 20, 20, 20\}$ and $p = 0.7$. Although we have that $\boldsymbol{a} = (1, 1, 1, 1, 1, 0, 0, 0, 0)$ is still an mPNE, in this case it is not an SPNE. Indeed, $M(9, p) = 0.90$ and $M(5, p) = 0.837$. We have that $M(9, p) - M(5, p) > \frac{1}{b_i}$ for $b_i = 20$. So that cannot be an SPNE.

3.3 Social Welfare

As we did in the Chapter 2, we now take into account the welfare of our decision structure. Relevantly here, we decided to just characterize the equilibria in terms of social welfare, without looking at the concept of price of anarchy. We will discuss at the end of this section the reason why we did not consider this important. The formula for the social welfare is the same as before (and consequently Proposition 2.2 holds). Interestingly enough the value n_h is not relevant when we consider this perspective. Instead, what matters here is the average of the values of $\boldsymbol{b}$, which we will denote with $\bar{b}$, i.e.

$$\bar{b} = \sum_{i=1}^{n} b_i. \tag{3.6}$$

First of all, we shall prove the following result.

Theorem 3.4. In a decision structure $D = \langle N, e, p, \boldsymbol{b} \rangle$, a strategy $\boldsymbol{a}$ maximizes the social welfare iff $|\boldsymbol{a}|$ maximizes the social welfare of a homogeneous decision structure $D' = \langle N, p, e, \bar{b} \rangle$.

Proof. Consider the formula for the social welfare (Equation 2.9). We have that

$$sw(|\boldsymbol{a}|) = \sum_{i=0}^{n} u_i(\boldsymbol{a}) = M(|\boldsymbol{a}|, p) \sum_{i=0}^{n} b_i - |\boldsymbol{a}| \cdot e.$$

Consequently, $sw(\boldsymbol{a}) = M(|\boldsymbol{a}|, p)n\bar{b} - |\boldsymbol{a}| \cdot e$. Indeed, this is the exact formula for the setting we saw in Chapter 2. As a consequence, to find the number of voters that maximize the social welfare of$D$ we shall look the number of voters that maximizes the social welfare for D'. □

Notably, this assimilates our heterogeneous decision structure to an homogeneous one. How do the equilibria, and in particular, the most accurate ones, in terms of social welfare? A first insight is a direct consequence of Theorem 3.4.

Observation 3.1. *Let $D = \langle N, e, p, \boldsymbol{b} \rangle$ be a decision structure. If $n_h < n_{hom}(\bar{b})$ then n_h does not maximize the social welfare.*

Indeed, in a decision structure $D' = \langle N', p', e, b' \rangle$ with $p' = p$, $n' = n$ and $b' = \overline{b}$ we know that $n'_{eq} \leq x'$ with x' being mSW for D'. Since the highest number of people at the equilibrium for D is given by n_h, then $n_h < n'_{eq} \leq x'$.

This is why a group where b_h is substantially smaller than $\overline{b}$ is not efficient in terms of social welfare. Since $\overline{b}$ may vary without changing at all b_h it may be useful to visualize how does the maximum social welfare changes with $\overline{b}$. We shall have a look at Figure 3.2.

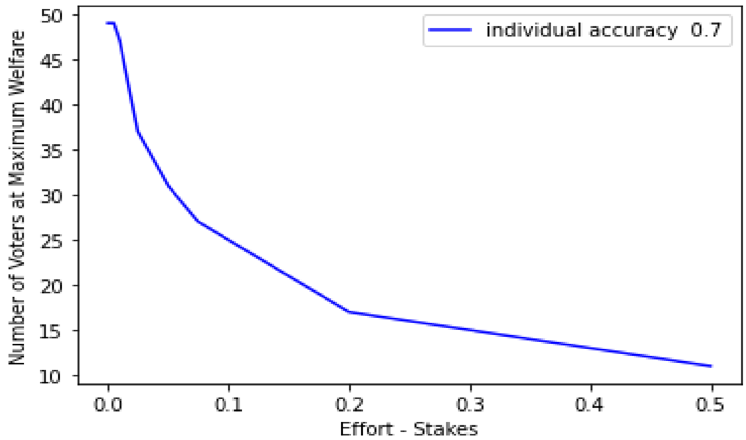

Figure 3.2: On the x-axis there is the value of $\frac{\overline{b}}{e}$ and on the y-axis the respective value of the number of voters at the maximum social welfare.

More in general, we can also prove the following result. Remember that we proved with Theorem 3.2 that n_h has the same role of what we denoted with n_{eq} in the previous chapter.

Corollary 3.3. For any possible decision structure $D = \langle N, e, p, \boldsymbol{b} \rangle$ such that x is mSW, $n_h \leq x$.

Proof. Assume that there exists a group for which this is false. Let $\boldsymbol{a}$ be a profile with $|\boldsymbol{a}| = n_h$. Let x be the number of voters that maximizes the social welfare s.t. $x < |\boldsymbol{a}|$ toward a contradiction. Thus, we have that for a homogeneous group $D = \langle N, p, e, b' \rangle$ with $b' = \overline{b}$, $nb'M(x,p) - ex$ is the maximum and consequently, $nb'M(x,p) - ex$ is the maximum also for the initial decision structure. However, by how we constructed n_h out of $\boldsymbol{a}$ (Theorem 3.2 and Definition 3.1), there exists a $b_i \in \boldsymbol{b}$ s.t. $b_i(M(|\boldsymbol{a}|,p) - M(|\boldsymbol{a}|-1,p)) \geq e$ and for all j s.t. $a_j = 1, b_j \geq b_i$. Thus, $M(|\boldsymbol{a}|,p) - M(|\boldsymbol{a}|-1,p) \geq \frac{e}{b_j}$. Consequently, $M(|\boldsymbol{a}|,p) - M(|\boldsymbol{a}|-1,p) > \frac{e \cdot y}{b_j \cdot |\boldsymbol{a}|}$ where $y = |\boldsymbol{a}| - x$, as $y \leq |\boldsymbol{a}|$. As a result, $b_i \cdot |\boldsymbol{a}| \cdot (M(|\boldsymbol{a}|,p) - M(|\boldsymbol{a}|-1,p)) >$

$e \cdot y$. In addition, $nb' \geq |\boldsymbol{a}|b_i$ as there exists at least $|\boldsymbol{a}| = n_h$ members of the group whose stakes are equal or higher than b_i and nb' corresponds to the sum of all the stakes. So,

$$nb' \cdot (M(|\boldsymbol{a}|, p) - M(|\boldsymbol{a}| - 1, p)) > e \cdot (|\boldsymbol{a}| - x)$$

. Yet, the consequence of this is the following:

$$nb' \cdot M(|\boldsymbol{a}|, p) - e|\boldsymbol{a}| - (nb'M(|\boldsymbol{a}| - 1, p) - ex) > 0$$

This is in contradiction with what we stated in the beginning as $\boldsymbol{a}$ has a higher social welfare than a profile with x voters. □

Such a result states that the number of voters that maximizes the social welfare is either greater or equal than the number of voters for profiles that are PNEs. Notably, from this result directly follows that for all $\boldsymbol{a}'$ s.t. $\boldsymbol{a}'$ is a PNE we have that $sw(n_h) \geq sw(\boldsymbol{a}')$. This means that the equilibria that present the highest accuracy among all the possible equilibria are also those that present the highest social welfare among them. Indeed, by Proposition 2.2 a local maximum is also a global one. This is extremely interesting. Indeed, it shows that the 'relationship' between individual behaviour and social welfare is exactly the one we described in Chapter 2. Again, the social welfare values more the effort made by the single agent than the agent herself.

We decided to not delve more in the matter of social welfare as the results are notihing different from what we saw in the basic setting. We do not look at when an equilibrium maximizes the social welfare as it is again just the composition of Theorem 3.4 and of Theorem 3.1. And similarly, we do not take into account the notion of price of anarchy, as the results that we obtained in Chapter 2 can be basically repeated here once we substituted the value for n with $\bar{b}n$. Indeed, the price of anarchy is concerned with the 'worst equilibria' and as we saw in the discussion of Theorem 3.1, these are basically the same as in Chapter 2.

However, before concluding the chapter we want to introduce a comparison between two decision structures that have the same values for $n \cdot \bar{b}$, same competence levels and effort but different values for n_h.

Observation 3.2. *Let $D = \langle N, e, p, \boldsymbol{b}\rangle$ and $D' = \langle N', p', e', \boldsymbol{b}\rangle$ be two decision structures with $n \cdot \bar{b} = n' \cdot \bar{b}'$, $p = p'$, $e = e'$ and $n_h > n'_h$. Thus, the structure with the highest n_h produces a higher (or equal) social welfare in its highest equilibria.*

The reasoning behind is straightforward. We know that $sw(n_h) \geq sw(n'_h)$ as $n \cdot \bar{b} = n' \cdot \bar{b}'$. Indeed, the function for the social welfare is single peaked and we know that $sw(x) = sw(x') \geq sw(n_h) > sw(n'_h)$ with x and x' being mSW respectively for D and D'. To get a clearer idea we shall have a look at an example.

Example 3.4. *Consider two decision structures D and D'. Let $N = \{1, 2, ..., 40\}$ with 15 people having $b = 100$ and other 25 people $b = 10$. Instead, $N' = \{1, 2, ..., 40\}$ with 30 people having $b' = 50$ and 10 people having $b' = 25$. Both the structures have the level of competence at 0.7 and they have the same stakes mean equal to 43.75. However, we have that $n_h = 15$ with final accuracy equal to 0.94 for D (and social welfare for it equal to 1647) and that $n'_h = 11$ with final accuracy for it equal to 0.92 and social welfare 1602 for D'.*

There is an interesting reflection to make. Between two structures with around the same total stakes involved in the question (indeed $\sum_{i=1}^{n} b_i = n\bar{b}$), the more 'efficient' (in terms of accuracy and social welfare) structure is not the one where each agent has the same stakes involved in the matter as everybody else. Indeed, it appears to be more efficient a group where there is a small amount (but not to small) of people with high stakes and the rest of the stakes are low (this seems to be the takeaway from Example 3.4 and Observation 3.2). This finding seems to be quite relevant to many realistic situations. It could happen that a certain neighbourhood has to express the opinion on two different matters. The first one may concern all the citizens (for example the construction of a park) and the second one concerns a smaller group (for example the construction of a boxing gym – we are assuming that the majority of the inhabitants are not boxers). It seems to be rational, based on the results we proved here, to expect a bigger participation of the population in the second case. Indeed, in that case all the boxers would take part in the decision (whereas just a small amount of citizen will actually join actively in the first case).

3.4 Summary, Discussion and Takeaways

This chapter aimed at showing whether or not allowing for different stakes could have changed the picture we described in Chapter 2. Indeed, throughout all the chapter we put the focus on highlighting differences and similarities.

Comparing the results. In general, the results we obtained here are mainly in line with what we saw in Chapter 2. We proved that adding more agents does not increase the accuracy of the group unless it changes the value for b_h (see Definition 3.1 and Theorem 3.2). The reason for this is that the increase of n does not necessarily increase the number of the voters at an equilibrium. As we have already seen, this is in contrast with the results of CJT (and in line with those of Chapter 2). However, there is a difference that must be highlighted. In Chapter 2, each agent had the same interest in the matter, and so just by knowing p and e we could have determined the limit n_{thr} (see Definition 2.2). Consequently, if $n \geq n_{thr}$ adding any type of agents would have not changed the equilibria. In the case for different stakes it may be that adding an agent with the same stakes of those already in the group does not change n_h (the size of the biggest equilibrium), whereas adding one (or more) agent(s) with relevantly higher stakes may actually increase n_h and consequently the accuracy at the mPNE. This seems to capture the fact that crowds may be more efficient than small groups, but only if they are 'very interested' crowds.

As far as the social welfare is concerned, we also acknowledged a similar situation to the one we described in Chapter 2. The higher is the number of agents, the greater the difference between the maximum for the social welfare and the social welfare at the equilibria. Quite relevantly, the similarity of the results is well shown by the fact that almost all the main theorems in this chapter rely on results from the previous one.

Some novelties. This framework provides us with more insights about the phenomenon too. Two are the lessons that can be drawn. First, we showed that an mPNE can be reached only if people with stakes higher than a certain threshold b_h vote (Theorem 3.2). This means that any equilibrium where the most interested agents are not voting is not ideal. Secondly, this framework allows us to draw a comparison between two groups with very similar attributes in the 'community perspective' ($n\bar{b}$) and slightly different ones in the individual points of view (see Observation 3.2). This is quite a success, and it seems to say that groups more diverse (in terms of stakes) may produce more efficient equilibria than homogeneous ones under certain conditions. Indeed, a good direction of research may be a deeper investigation of these cases and a more precise formalization of which groups should be preferred to others, as more 'efficient'.

4 Experts and Amateurs

In this fourth chapter we introduce another modification to our model. So far we only took into account situations in which each agent had the same level of competence as all the others. What if agents by spending the same amount of effort reach different levels of competence? Who is going to vote then? We dedicate Section 4.1 to formalize this scenario and to understand how it works. In Section 4.2, we characterize the pure Nash equilibria, and we introduce a conjecture on these types of profiles that could yield interesting results. In Section 4.3, we briefly discuss the social welfare of the game. Eventually, in Section 4.4 we take into account the possibility of delegation without introducing any formal results, but just showing through examples how and why the mechanism of delegation could be taken into account.

4.1 Preliminaries

We start by redefining some of the notions we already have and by adding some new definitions, which turn out useful in this new setting.

First of all, we add to a decision structure D a vector $\boldsymbol{p}_N = (p_1, ..., p_n)$ that collects the different levels of competence of the agents. Indeed, we say that p_i is the *competence*, or the *level of competence* of agent i. We assume $b = 1$ to be the same for everybody, to not needlessly complicate our scenario.[1] Consequently, the decision structure we refer to in this chapter is the following: $D = \langle N, e, \boldsymbol{p}_N \rangle$. We use in general the notation $\boldsymbol{p}$ to denote a vector of competences. In order to be able to work and describe such vectors we introduce some new notions. We call $|\boldsymbol{p}|$ the length of $\boldsymbol{p}$ and $N(\boldsymbol{p})$ the set of the agents s.t. if $i \in N(\boldsymbol{p})$ then p_i occurs in $\boldsymbol{p}$. So, $N(\boldsymbol{p}_N) = N$. Now, we can redefine the formula for the accuracy of a group of voters whose levels of competence are described by $\boldsymbol{p}$. Indeed, the idea is still the same as the one discussed in Chapter 1, but Equation 2.5 needs some changes. In order to do so we call WC the set of winning coalitions for a vector of voters $\boldsymbol{p}$, i.e. $WC = \{S \subseteq N(\boldsymbol{p}) || S| \geq \frac{|\boldsymbol{p}|+1}{2}\}$. Similarly, we call TC the set of

[1] This is the assumption we already made in Chapter 2.

tie coalitions, i.e. $TC = \{S \subseteq N(\boldsymbol{p}) || S| = \frac{|\boldsymbol{p}|}{2}\}$. We take $M(\boldsymbol{p})$ to represent the probability that a majority among the voters with vector $\boldsymbol{p}$ gets the answer right and we let $M : (0,1)^{n(\boldsymbol{p})} \to \mathbb{R}$ be the following function:

$$M(\boldsymbol{p}) = \begin{cases} \sum_{S \in WC} \prod_{i \in S} p_i \prod_{i \notin S}(1 - p_i) & \text{if } |\boldsymbol{p}| \text{ is odd} \\ \sum_{S \in WC} \prod_{i \in S} p_i \prod_{i \notin S}(1 - p_i) + & \\ \quad + \frac{1}{2} \cdot \sum_{S \in TC} \prod_{i \in S} p_i \prod_{i \notin S}(1 - p_i) & \text{if } |\boldsymbol{p}| \geq 0 \text{ and is even} \\ \frac{1}{2} & \text{if } |\boldsymbol{p}| = 0. \end{cases} \tag{4.1}$$

Indeed, first, we look at the coalitions of voters with size bigger than half of the total of voters, i.e. the sets $S \subseteq N(\boldsymbol{p}), |S| \geq \frac{|\boldsymbol{p}|+1}{2}$. Then we compute the probability for each coalition to exactly describe the set of people voting for the right answer. In case the total $|\boldsymbol{p}|$ is even, we add to that the probability that a coalition with exactly half of the total voters gets the correct answer. Consequently, if $\boldsymbol{p}$ is an empty vector, the probability is equal to $\frac{1}{2}$. To compute $M(\boldsymbol{p})$ the order of the elements in the vector does not matter. Consequently, from now on every time we consider a vector $\boldsymbol{p}$ we assume that it is ordered in descending order.[2] Thus, $\boldsymbol{p} = \boldsymbol{p'}$ is true iff $\boldsymbol{p}$ and $\boldsymbol{p'}$ have the same elements, i.e. for all i, $p_i = p'_i$. In addition, we also use the notation $\boldsymbol{p} + p_i$ and $\boldsymbol{p} - p_i$ to add and eliminate elements from a vector. We define formally two operations.

$$\boldsymbol{p} + p_i = (p_1, ..., p_N) + p_i = (p_1, ..., p_i, ..., p_N) \tag{4.2}$$

s.t. $(p_1, ..., p_i, ..., p_N)$ is still in descending order. And, similarly,

$$\boldsymbol{p} - p_i = \begin{cases} (p_1, ...p_{i-1}, p_{i+1}, ..., p_N) & \text{if } p_i \text{ occurs in } \boldsymbol{p} \\ (0) & \text{otherwise.} \end{cases} \tag{4.3}$$

Notably, the second condition is for purely technical reasons, and will become apparent in Theorem 4.1. The space of the actions is the same as in Chapter 1, i.e. $a_i = \{1, 0\}$. Thus, we have that $\boldsymbol{a} \in \{0,1\}^n$. We call $\boldsymbol{p}(\boldsymbol{a})$ the vector of levels of competence of the people voting in a profile $\boldsymbol{a}$. So, $|\boldsymbol{p}(\boldsymbol{a})| = |\boldsymbol{a}|$. Notably, two profiles $\boldsymbol{a}$ and $\boldsymbol{a'}$ may turn out to have the same vector $\boldsymbol{p}(\boldsymbol{a}) = \boldsymbol{p'}(\boldsymbol{a'})$ even if they do not have the same voters. This is indeed reasonable as these two profiles do not differ in anything in terms of our analysis. In fact, as in the Chapter 1, if $\boldsymbol{p}(\boldsymbol{a}) = \boldsymbol{p}(\boldsymbol{a'})$ then if $\boldsymbol{a}$ has a property also $\boldsymbol{a'}$ has it. Clearly, this is due to the fact that the accuracy of a profile $\boldsymbol{a}$ just depends on the corresponding vector $\boldsymbol{p}(\boldsymbol{a})$. In

[2] So, basically also $\boldsymbol{p}_N$ is ordered, which implies that the agents are numbered such that $p_i \geq p_j$ if $i < j$.

a similar way to what we did in Chapter 3 we also denote with $p^1_{min}(\boldsymbol{a})$ the value of the competence of the least accurate agent among the voters, i.e.

$$p^1_{min}(\boldsymbol{a}) = \begin{cases} min\{p_i | a_i = 1\} & \text{if } |\boldsymbol{a}| > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

And, we denote with $p^0_{max}(\boldsymbol{a})$ the value of the competence of the most accurate agent among the non-voters, i.e.

$$p^0_{max}(\boldsymbol{a}) = \begin{cases} max\{p_i | a_i = 0\} & \text{if } |\boldsymbol{a}| < n \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

Given the above equation for M we can now also redefine the utility of an agent i:

$$u_i(\boldsymbol{a}) = M(\boldsymbol{p}(\boldsymbol{a})) - a_i e. \tag{4.6}$$

On this basis, we shall now proceed to analyze the new game.

4.2 Basic Results and Pure Nash Equilibria

In this section, we have a look at how the mechanism is structured and at how we can characterize pure Nash equilibria. Then, we introduce a conjecture about the kinds of profiles we are studying now and we discuss what kinds of directions could be taken if that is proven to be true.

4.2.1 Pure Nash Equilibria

First of all, we shall have a look at an example to better understand the new formula for the accuracy of the majority.

Example 4.1. *Let $\boldsymbol{p} = (p_1, p_2, p_3)$. Our formula says that if they all vote the final accuracy can be computed following these lines:*

$$M(\boldsymbol{p}) = p_1 p_2 p_3 + p_1 p_2 (1 - p_3) + p_1 (1 - p_2) p_3 + (1 - p_1) p_2 p_3.$$

Notably, since the p_is are different the function $M(\boldsymbol{p})$ does not present all the regularities of the function $M(n, p)$ of Chapter 2. Indeed, it can be that adding more voters decreases the general accuracy, and it can also be that going from an odd number of voters x to $x + 1$ voters increases the accuracy . We will discuss a method to regularize this function, by restricting its domain in Subsection 4.2.2. We say that two vectors $\boldsymbol{p}$ and $\boldsymbol{p}'$ are s.t. $\boldsymbol{p} \succ \boldsymbol{p}'$ if $|\boldsymbol{p}| = |\boldsymbol{p}'|$ and for all $0 \leq i \leq |\boldsymbol{p}|$ we have that $p_i \geq p'_i$. It is easy to see that the following result holds. Relevantly, in our proof we make use of Lemma 1 in (Paroush, 1997).

Proposition 4.1. If $\boldsymbol{p}, \boldsymbol{p'}$ are two vectors of competences $\boldsymbol{p} \succ \boldsymbol{p'}$ then $M(\boldsymbol{p}) \geq M(\boldsymbol{p'})$.

Proof. The proposition can be shown through little changes. Let $\boldsymbol{p} = (p_1, ..., p_m)$ be a vector. By Lemma 1 (Paroush, 1997), we know that if $p_m \geq p'_m$ then $M(\boldsymbol{p}) \geq M(\boldsymbol{p''})$ with $\boldsymbol{p}'' = (p_1, ..., p'_m)$. We can repeat the same reasoning until we arrive at building the vector $\boldsymbol{p'}$. Thus, $M(\boldsymbol{p}) \geq M(\boldsymbol{p'})$. □

The idea is very intuitive: if the voters are more competent, the overall accuracy will be higher. Now, as we did in the previous chapters we can characterize the equilibria in this framework.

Theorem 4.1. Let $D = \langle N, e, \boldsymbol{p}_N \rangle$ be a decision structure and $\boldsymbol{a} \in \{0,1\}^n$ be an action profile with $\boldsymbol{p}(\boldsymbol{a}) = \boldsymbol{p}$. The profile $\boldsymbol{a}$ is a PNE iff $M(\boldsymbol{p} + p^0_{max}(\boldsymbol{a})) - M(\boldsymbol{p}) \leq e \leq M(\boldsymbol{p}) - M(\boldsymbol{p} - p^1_{min}(\boldsymbol{a}))$.

Proof. A profile $\boldsymbol{a}$ is a PNE iff for any agent i $u_i(\boldsymbol{a}) \geq u_i(a'_i, \boldsymbol{a}_{-i})$ by Definition 2.1. Assume i s.t. $a_i = 1$. Thus, $u_i(0, \boldsymbol{a}_{-i}) = M(\boldsymbol{p} - p_i)$ and $u_i(\boldsymbol{a}) = M(\boldsymbol{p}) - e$. Thus, agent i does not deviate iff $M(\boldsymbol{p}) - e \geq M(\boldsymbol{p} - p_i)$, i.e. iff $M(\boldsymbol{p}) - M(\boldsymbol{p} - p_i) \geq e$. In addition, $M(\boldsymbol{p} - p_i) \geq M(\boldsymbol{p} - p^1_{min}(\boldsymbol{a}))$ for all i with $a_i = 1$, by Proposition 4.1 and construction of $p^1_{min}(\boldsymbol{a})$. Thus, for all i, s.t. $a_i = 1$ $u_i(\boldsymbol{a}) \geq u_i(a'_i, \boldsymbol{a}_{-i})$ iff $e \leq M(\boldsymbol{p}) - M(\boldsymbol{p} - p^1_{min}(\boldsymbol{a}))$.

Now, assume i s.t. $a_i = 0$. Thus, $u_i(1, \boldsymbol{a}_{-i}) = M(\boldsymbol{p} + p_i) - e$ and $u_i(\boldsymbol{a}) = M(\boldsymbol{p})$. Thus, agent i does not deviate iff $M(\boldsymbol{p}) \geq M(\boldsymbol{p} + p_i) - e$, i.e. iff $M(\boldsymbol{p} + p_i) - M(\boldsymbol{p}) \leq e$. In addition, $M(\boldsymbol{p} + p_i) \leq M(\boldsymbol{p} + p^0_{max}(\boldsymbol{a}))$ for all i with $a_i = 0$, by Proposition 4.1 and construction of $p^0_{max}(\boldsymbol{a})$. Thus, for all i, s.t. $a_i = 0$ $u_i(\boldsymbol{a}) \geq u_i(a'_i, \boldsymbol{a}_{-i})$ iff $M(\boldsymbol{p} + p^0_{max}(\boldsymbol{a}) - M(\boldsymbol{p}) \leq e$.

Consequently, $\boldsymbol{a}$ is a PNE iff $M(\boldsymbol{p} + p^0_{max}(\boldsymbol{a}) - M(\boldsymbol{p}) \leq e \leq M(\boldsymbol{p}) - M(\boldsymbol{p} - p^1_{min}(\boldsymbol{a}))$. □

Notably, the result is not so different from what we saw in the previous chapters. In a similar fashion as before, we used here the technical assumptions we made in the previous section, in order to have a general characterization. In fact, by assuming that $\boldsymbol{p} - p_i = (0)$ if p_i does not occur in $\boldsymbol{p}$ we are sure that the profile with size -1 is not taken into consideration as $M((0)) = 0$. The same happens for the vector $\boldsymbol{p}_N$. If we try to add an element to it, we are sure that we are adding an element with value 0, which makes sure that $M(\boldsymbol{p}_N + 0) - M(p_N) = 0 \leq 0$.[1]

However, here the *contribution* of each agent becomes more relevant than before. Indeed, working with different levels of competence implies that in a group

[1] This can be easily verifies by computing the value for $M(\boldsymbol{p}_N + 0) - M(p_N)$.

of voters the contribution of one of them may be different from the contribution of the others. We can call *weight*[2] of i the value $M(\boldsymbol{p}) - M(\boldsymbol{p} - p_i)$, i.e.

$$W_{\boldsymbol{p}}(i) = M(\boldsymbol{p}) - M(\boldsymbol{p} - p_i). \tag{4.7}$$

This corresponds to what is usually referred to in the literature as the possibility for i of being pivotal in the vote[3]. A voter can consider herself pivotal any time the decision is entirely depending on her. Consequently, the probability of being pivotal corresponds to the probability of the other voters of splitting equally between the two possible answers.[4] This corresponds exactly to $M(\boldsymbol{p}) - M(\boldsymbol{p} - p_i)$. In our case such a probability decreases with two factors: the increase in the level of competence of the rest of the voters' (the more competent they are the more likely that they will not be voting for the wrong option) and the increase in the number of people (the more people the less likely that they will exactly split in half). And, of course, the higher is the competence of agent i the higher is her contribution. Multiple levels of competence allow us to appreciate the importance of this concept. However, we did mention in Section 2.2 the concept of *contribution*, which is, indeed, similar. In characterizing the equilibria we make sure that the contribution of the least competent agent is high enough to justify her effort. Indeed, this assures us that it will be the same for all the other voters. And, the notation of $M(\boldsymbol{p} + p^0_{max}(\boldsymbol{a})) - M(\boldsymbol{p})$ can be easily thought as the potential weight of an agent in the case she joins the group of voters. Interestingly enough, Theorem 4.1 allows us to look at how effort influences the equilibria. Again, as we did in Chapter 2 and Chapter 3 we dedicate our next corollary to this.

Corollary 4.1. Let $D = \langle N, e, \boldsymbol{p}_N \rangle$ be a decision structure with $p^+ = max\{p_i | i \in N\}$.

1. If $e > p^+ - \frac{1}{2}$ a profile $\boldsymbol{a}$ is a PNE iff $|\boldsymbol{a}| = 0$.
2. If $e = p^+ - \frac{1}{2}$ a profile $\boldsymbol{a}$ is a PNE iff either $|\boldsymbol{a}| = 0$ or $|\boldsymbol{a}| = 1$ and $a_i = 1$ iff $p_i = p$.
3. If $e < p^+ - \frac{1}{2}$ a profile $\boldsymbol{a}$ is a PNE iff $M(\boldsymbol{p}) \geq M(\boldsymbol{p} - p^1_{min}(\boldsymbol{a})) + e$ and $M(\boldsymbol{p}) \geq M(\boldsymbol{p} + p^0_{max}(\boldsymbol{a}) - e$.

Proof. The corollary is entirely dependent on Theorem 4.1. In the case for $e > p^+ - \frac{1}{2}$, we can see that the conditions in Theorem 4.1 hold only for $\boldsymbol{a}$ with $|\boldsymbol{a}| = 0$. And they hold also for $\boldsymbol{a}$ with $|\boldsymbol{a}| = 1$ if $e = p^+ - \frac{1}{2}$. The case for $e < p^+ - \frac{1}{2}$ is just a rewriting of Theorem 4.1 itself. □

[2] Note that such a formula is not well defined for an empty vector $\boldsymbol{p}$.

[3] The concept of pivotality is quite well known in voting theory. Some good articles that make use of it are (Martinelli, 2006; Mukhopadhaya, 2003; Persico, 2004).

[4] This is the case if the number of voters is odd, if it is even, then it is slightly different.

The general lesson, which is indeed similar to what we have seen for the previous chapters is that for a profile to be an equilibrium, the voter with the lowest competence must still *feel pivotal*, i.e. must have a considerable weight and a possible new voter should not feel pivotal enough in the case she switched to 1. Note however, that both in Chapter 2 and 3, the fact that the size of a profile was odd allowed us to drop the condition making sure that no agent would have switched from 0 to 1 (in this case: $M(\boldsymbol{p} + p^0_{max}(\boldsymbol{a})) - M(\boldsymbol{p}) \leq e$). Being odd implied that that condition was true. Here it is not the case anymore. There can be profiles with odd size and for which $M(\boldsymbol{p}) \geq M(\boldsymbol{p} - p^1_{min}(\boldsymbol{a})) + e$ holds that are not PNE. We shall look at the following example.

Example 4.2. *Let* $D = \langle N, e, \boldsymbol{p}_N \rangle$ *with* $N = \{1, 2, ..., 6\}$*,* $\boldsymbol{p}_N = (0.9, 0.6, 0.6, 0.6, 0.6, 0.6)$ *and* $e = 0.0004$*. If* $\boldsymbol{a} = (0, 1, 1, 1, 1, 1)$ *then agent 1 would want to vote (even if* $|\boldsymbol{a}|$ *is odd), as* $M(\boldsymbol{p}(\boldsymbol{a})) = 0.68$ *and* $M(\boldsymbol{p}(\boldsymbol{a'})) = 0.79$ *where* $\boldsymbol{a} = (\boldsymbol{a}_{-1}, i)$*. And, indeed,* $u_1(\boldsymbol{a}') - u_1(\boldsymbol{a}) = 0.11 - 0.01 = 0.1$.

Another consequence of Corollary 4.1 is that we also can be sure in this setting that there exists always at least one profile which is an equilibrium. Indeed, Corollary 4.1 shows clearly that for $e \geq max\{p_i | i \in N\} - \frac{1}{2}$ a profile $\boldsymbol{a} = (0, 0, ..., 0)$ is a PNE. And for $e < max\{p_i | i \in N\} - \frac{1}{2}$ we are sure that $\boldsymbol{a}$ with $\boldsymbol{p}(\boldsymbol{a}) = (max\{p_i | i \in N\})$ is a also a PNE.

For now, we cannot say anything more about the features of the equilibria. Indeed, unlike in the previous chapters the size of a profile does not help us in this case, i.e. there is not a one-to-one correspondence between the size of a profile and a certain overall accuracy. Of course, thanks to Theorem 4.1, it is potentially easy to compute all the equilibria, and look for the one with the highest accuracy. However, it is hard to connote them further. Nonetheless, in the next subsection we propose a conjecture that may give us some help in the study of this situation. Indeed, there seems to be a link between the accuracy of a group of voters with size $2k - 1$ and the accuracy of a group with size $2k$. We discuss there what advantages this conjecture could bring in terms of analysis of the equilibria.

Now, we shall turn our attention to strong pure Nash equilibria. Although Lemma 2.2 still holds, we must acknowledge the fact that this is not the case for Lemma 2.1. Indeed, recall that in Chapter 2, every agent had the same competence and consequently, just the number of voters mattered. Here, instead the identity of the voters is important. However, Theorem 4.2 is not so different from Theorem 2.2. Notably, as we did in Chapter 2 for each coalition K we check only the case for which $\boldsymbol{a}'_K$ is completely different from $\boldsymbol{a}_K$, i.e. for all $i \in K'$, $a_i \neq a'_i$. By doing so we are still sure that we can find a SPNE for the reasons discussed in Section 2.2. We denote with $\boldsymbol{p}_{max}$ the vector of competences that produces the highest accuracy in a certain decision structure $D = \langle N, e, \boldsymbol{p}_N \rangle$, i.e. $M(\boldsymbol{p}_{max}) \geq M(\boldsymbol{p})$ for all $\boldsymbol{p}$. Then, we shall start by proving a quite straightforward lemma.

Lemma 4.1. Let $D = \langle N, e, \boldsymbol{p}_N \rangle$ be a decision structure and let $\boldsymbol{a}$ be an action profile. Let $K \subseteq N$ be a coalition s.t. there exists $i \in K$ with $a_i = 0$. If K wants to deviate then a coalition K' s.t. $\boldsymbol{p}(\boldsymbol{a}'_{K'}, \boldsymbol{a}_{N\setminus K'}) = \boldsymbol{p}_{max}$ also wants to deviate.

Proof. Assume $u_j(\boldsymbol{a}'_K, \boldsymbol{a}_{N\setminus K}) > u_j(\boldsymbol{a})$ for all $j \in K$. Consider $i \in K$ s.t. $a_i = 0$. Thus, $u_i(\boldsymbol{a}'_K, \boldsymbol{a}_{N\setminus K}) = M(\boldsymbol{p}(\boldsymbol{a}'_K, \boldsymbol{a}_{N\setminus K})) - e \geq u_i(\boldsymbol{a})$. Since $M(\boldsymbol{p}_{max}) \geq M(\boldsymbol{p})$ for every vector $\boldsymbol{p}$ we get that $M(\boldsymbol{p}_{max}) - e \geq M(\boldsymbol{p}(\boldsymbol{a}'_K, \boldsymbol{a}_{N\setminus K})) - e$. Now consider the coalition K' that produces $\boldsymbol{p}_{max}$. If $j \in K'$ s.t. $a_j = 0$, then $u_j(\boldsymbol{a}'_{K'}, \boldsymbol{a}_{N\setminus K'}) = M(\boldsymbol{p}_{max}) - e \geq M(\boldsymbol{p}(\boldsymbol{a}'_K, \boldsymbol{a}_{N\setminus K})) - e > u_j(\boldsymbol{a})$. If $a_j = 1$, then $u_j(\boldsymbol{a}_{K'}, \boldsymbol{a}_{N\setminus K'}) = M(\boldsymbol{p}_{max}) \geq u_j(\boldsymbol{a})$. Thus, the coalition K' wants to change. $\square$

The idea is that if a coalition K that contains an agent that would go from the action 0 to the action 1 deviates then also the coalition that would produce the maximal accuracy deviates. Indeed, for that agent i would hold that $M(\boldsymbol{p}(\boldsymbol{a}'_K, \boldsymbol{a}_{N\setminus K})) - e \geq M(\boldsymbol{p}(\boldsymbol{a}))$. Consequently, $M(\boldsymbol{p}_{max}) - e \geq M(\boldsymbol{p}(\boldsymbol{a}))$ for all the people that would need to vote in $\boldsymbol{p}_{max}$. So, regardless who are the agents that would join a coalition producing the vector $\boldsymbol{p}_{max}$ if it is worth for one it is worth for everybody, since the value that each agent gives to the accuracy is the same for everybody. Now that we have this useful lemma we can proceed to our main result.

Theorem 4.2. Let $D = \langle N, e, \boldsymbol{p}_N \rangle$ be a decision structure. A profile $\boldsymbol{a}$ is an SPNE iff $M(\boldsymbol{p}_{max}) - M(\boldsymbol{p}(\boldsymbol{a})) \leq e \leq M(\boldsymbol{p}(\boldsymbol{a})) - M(\boldsymbol{p}(\boldsymbol{a}) - p^1_{min}(\boldsymbol{a}))$.

Proof. A profile $\boldsymbol{a}$ is an SPNE iff no coalition K would deviate by Definition 2.3. There are two cases. Either there exists $i \in K$ s.t. $a_i = 0$ or there exists not such i. By Lemma 2.2 we know that a coalition where there exists no i s.t. $a_i = 0$ would not deviate if the single agent j with $a_j = 1$ would not deviate. Similarly, by Lemma 4.1 we know that a coalition K where there exists $i \in K$ with $a_i = 0$ would not deviate if the coalition K' that produces the vector $\boldsymbol{p}_{max}$ does not deviate. So we need to check just these two cases. By Theorem 4.1 no voter deviates iff $e \leq M(\boldsymbol{p}(\boldsymbol{a})) - M(\boldsymbol{p}(\boldsymbol{a}) - p^1_{min}(\boldsymbol{a}))$. Now, consider a coalition $K \subseteq \{i \in N | a_i = 0\}$ s.t. $\boldsymbol{p}(\boldsymbol{a}'_K, \boldsymbol{a}_{N\setminus K}) = \boldsymbol{p}_{max}$. Coalition K would not deviate iff $M(\boldsymbol{p}_{max}) - e \leq M(\boldsymbol{p}(\boldsymbol{a}))$, i.e. iff $M(\boldsymbol{p}_{max}) - M(\boldsymbol{p}(\boldsymbol{a})) \leq e$. Thus, a profile $\boldsymbol{a}$ is an SPNE iff $M(\boldsymbol{p}_{max}) - M(\boldsymbol{p}(\boldsymbol{a})) \leq e \leq M(\boldsymbol{p}(\boldsymbol{a})) - M(\boldsymbol{p}(\boldsymbol{a}) - p^1_{min}(\boldsymbol{a}))$. $\square$

Notably, there is only one difference between this theorem and Theorem 2.2, which is a direct consequence of the framework: this theorem uses $\boldsymbol{p}_{max}$ instead of using the accuracy of the vector $\boldsymbol{p}_N$ (that would have corresponded to n voters in Chapter 2). Of course, this is because $M(\boldsymbol{p}_{max}) \geq M(\boldsymbol{p}_N)$.

A direct consequence of Definition 2.3 is that if a profile $\boldsymbol{a}$ is an SPNE it is also a PNE. As in Chapter 2 and Chapter 3, we are not sure that an SPNE exists. However, in this setting there is a more interesting detail. Remarkably, SPNEs may have different sizes. In Chapter 2 and 3 we saw that to be an SPNE a certain profile should have had only one size, if an SPNE existed. Here, instead, in the same game there can be vectors with different sizes that are both SPNE. Consider this scenario.

Example 4.3. *Let* $D = \langle N, e, \boldsymbol{p}_N \rangle$ *with* $N = \{1, ..., 7\}$, $\boldsymbol{p}_N = (0.72, 0.72, 0.72, 0.72, 0.7, 0.7, 0.7)$ *and* $e = 0.051$. *We have that* $\boldsymbol{p}^E = (0.72, 0.72, 0.72)$ *yields a PNE with* $M(\boldsymbol{p}^E) = 0.837$. *Similarly,* $\boldsymbol{a}$ *with* $\mathbf{p}(\boldsymbol{a}) = (0.72, 0.72, 0.7, 0.7, 0.7)$ *is also a PNE with* $M(\mathbf{p}(\boldsymbol{a})) = 0.847$. *In addition,* $\boldsymbol{p}_{max} = \boldsymbol{p}_N$ *with* $M(\boldsymbol{p}_N) = 0.888$. *Thus, a profile* $\boldsymbol{a}$ *can be an SPNE both if* $\mathbf{p}(\boldsymbol{a}) = \boldsymbol{p}^E$ *and if* $\mathbf{p}(\boldsymbol{a}) = (0.72, 0.72, 0.7, 0.7, 0.7)$. *Notably these two vectors have different sizes.*

Indeed, this is a result of the fact that a certain accuracy can be reached through different vectors (and different sizes). Instead, in Chapter 2 and 3 there was a one-to-one correspondence between size of a profile and overall accuracy.

We can conclude here the study of PNE. We provided a full characterization of PNEs and of SPNEs and we already discussed some of the salient features of the framework. As we mentioned, we dedicate the next subsection to a conjecture, and its consequences.

4.2.2 A conjecture

Recall that in Chapter 2 we proved that $M(2m-1, p) = M(2m, p)$ if $m \in \mathbb{N}$ (Proposition 2.1). Such a result, although not extremely complicated, allowed us to highlight the importance for a profile of being of an odd size (see Corollary 2.1 and 2.1 and Observation 2.1). We conjecture that something similar can be proven also in the case for heterogeneous competences. To the best our knowledge no proof of this conjecture can be found in the literature. Although the basic principle seems to be exactly the same one as for Proposition 2.1 we have been unable to prove it. Since such a conjecture, if proven to be true, may turn out to be important for a work like ours, we state it here and we sketch a possible way to approach the proof. Then, we briefly discuss which could be the consequences of such a proposition in the context of our research.

Conjecture 4.1. Let $\boldsymbol{p}$ be a vector with $|\boldsymbol{p}| = 2m - 1$ with $m \in \mathbb{N}$. If $p \leq p_i$ for each p_i that occurs in $\boldsymbol{p}$ then $M(\boldsymbol{p}) \geq M(\boldsymbol{p} + p)$.

We sketch a possible way of proving the proposition. In general, the flavour is very similar to the proof of Proposition 2.1. Given the vector $\boldsymbol{p}$ we shall consider the probability $P(\boldsymbol{p}, l)$, i.e. the probability for exactly l agents out of $|\boldsymbol{p}|$ to vote for the right answer. Indeed, it is easy to prove that $M(\boldsymbol{p} + p) \leq M(\boldsymbol{p})$ iff

$$pP(\boldsymbol{p}, \frac{|\boldsymbol{p}| - 1}{2}) \leq (1 - p)P(\boldsymbol{p}, \frac{|\boldsymbol{p}| + 1}{2}). \tag{4.8}$$

Indeed, this is very similar to the proof of Proposition 2.1. However, although we have proven Equation 4.8 for specific cases (for example for $|\boldsymbol{p}| = 3$), it is still unclear to us how it generalizes. For this reason, such a generalization is left for future work.

Proving Conjecture 4.1 would allow us to draw two important consequences out of it, which are quite similar to some of the consequences of Proposition 2.1 in Chapter 2. First of all, Conjecture 4.1 would have assured us that any profile $\boldsymbol{a}$ with $|\boldsymbol{a}| > 0$ which is a PNE is such that $|\boldsymbol{a}|$ is odd. This would actually resemble the scenario we described in Corollary 2.1 and Corollary 3.1. Indeed, the weight of the agent with the lowest competence in an even profile would be always equal or lower than 0 and consequently also lower than e.

Secondly, Conjecture 4.1 would also help to find a correspondence between the effort and the size of equilibria. This was, indeed, one of the most important results in Chapters 2 and 3: the smaller the effort, the larger the size of the mPNEs (see Figure 2.3 and Observation 2.1). However, even if we assumed Conjecture 4.1 to hold, the correspondence would not be immediately straightforward. We shall discuss briefly here how to find it.

To find such a correspondence we should focus our attention on *elite profiles*, i.e. profiles in which all the voters have a higher or equal competence than the non-voters. We call *elite vectors* the vectors that produce elite profiles. Notably, we make use of them also in the next section. Such profiles present some interesting features. First of all, we could be sure that if an elite profile of size k is a PNE, then all the elite profiles of size $l \leq k$ would also be PNEs, if l is odd. Indeed, this is what happens with n_{eq} in Chapter 2 (see Figure 2.3). This would be a direct consequence of Conjecture 4.1 and of the concept of weight. Relevantly, elite profiles are built by 'cutting the tail' of the vector $\boldsymbol{p}_N$. If we need an elite vector of length x we take the first x elements of $\boldsymbol{p}_N$. Thus, if the weight of the element x in the vector $\boldsymbol{p}$ with size x is higher than e, then also the weight of the element y in the vector $\boldsymbol{p}'$ with size $y < x$ is higher than e given that y is odd. This would bring us closer to the correspondence we were talking about. Indeed, if we consider the elite profile which is a PNE with the highest accuracy (we call it $\boldsymbol{p}^E$ from now on), we have that its size actually increases with the decrease of the effort in the same way we saw in Chapter 2 (recall that we are still working under the assumption that Conjecture 4.1 holds, although this is not proven). Consider the example.

Example 4.4. *Let* $D = \langle N, e, \boldsymbol{p}_N \rangle$ *a decision structure with* $N = \{1, ..., 14\}, \boldsymbol{p}_N = (0.87, 0.85, 0.85, 0.77, 0.76, 0.74, 0.74, 0.74, 0.7, 0.69, 0.65, 0.6, 0.6, 0.6)$*. Depending on the values for* e *we may have different value for the highest accuracy that can be reached by an elite profile which is an equilibrium. We shall have a look at Figure 4.1. Just to give an idea to the reader, we mention that* $|\boldsymbol{p}_{max}| = 11$.

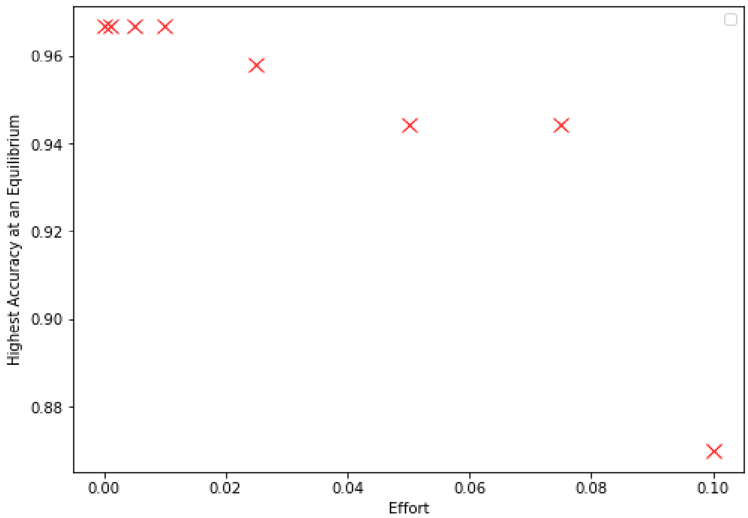

Figure 4.1: On the x-axis there are the different values for e and on the y-axis the overall group accuracy. The red cross represents the accuracy for $\boldsymbol{p}^E$ for different values of e.

Notably, we would be sure that any profile with a size smaller than $\boldsymbol{p}^E$ would have a lower accuracy, for how elite profiles would be built. However, the profile $\boldsymbol{a}$ which is an mPNE may also not be elite. And so, it may have a larger size and a higher accuracy than $\boldsymbol{p}^E$.

In the end, being able to prove effectively Conjecture 4.1 may provide our kind of research with two useful insights on the scenario, which would resemble the results we proved in Chapter 2 and Chapter 3. And, a good way to exploit such a scenario would be that of using elite vectors.

Now, we can put aside Conjecture 4.1, hoping to be able to prove it in future work and we shall briefly consider the social welfare of this game.

4.3 Social Welfare

In the present setting the function for the social welfare is extremely irregular and using elite profiles does not help enough. For this reason in this section we limit ourselves to define social welfare in this framework and discuss some of its features, in particular those that make our analysis especially hard. In order to consider the social welfare in this setting, we must, first of all, redefine its formula. Given a certain decision structure $D = \langle N, e, \boldsymbol{p}_N \rangle$, for a profile $\boldsymbol{a}$ the social welfare $sw : \boldsymbol{A} \to \mathbb{R}$ is so defined:

$$sw(\boldsymbol{a}) = n \cdot M(\boldsymbol{p}(\boldsymbol{a})) - |\boldsymbol{a}| \cdot e. \tag{4.9}$$

Notably, the social welfare entirely depends on the underlying vector of $\boldsymbol{a}$. So, we can rewrite the social welfare such that

$$sw(\boldsymbol{p}) = n \cdot M(\boldsymbol{p}) - |\boldsymbol{p}| \cdot e. \tag{4.10}$$

Now, in order to assess the quality of the equilibria we found, we would need to be able to find and characterize the profiles (or the profile – if it is just one) that produce the highest social welfare possible. A good step in this direction is noting that the vector that maximizes the social welfare is an elite vector. Indeed, an elite vector 'makes the most' out of the total effort spent. Recall, that an elite vector $\boldsymbol{p}$ is such that for all $\boldsymbol{p}'$ with $|\boldsymbol{p}'| = |\boldsymbol{p}|$, $\boldsymbol{p} \succ \boldsymbol{p}'$. And, consequently, $M(\boldsymbol{p}) \geq M(\boldsymbol{p}')$. For this reason, given a certain amount of effort, i.e. a number of voters, the corresponding elite profile is the one with the highest accuracy.

However, this is not enough to find the vector that maximizes sw. Indeed, if compared to the situation in Chapter 2 and Chapter 3 here we are provided with less information. In the previous chapters the function for the social welfare was such that, if constrained on profiles with odd size a local maximum would have also be a global maximum for all the function. This allowed us to draw inferences like Theorem 2.3 in the Chapter 2. In fact, if we managed to find a local maximum we could have been sure that that would have been a global maximum. Here, this is not the case. Consider the following example.

Example 4.5. *Let* $D = \langle N, e, \boldsymbol{p} \rangle$ *with* $\boldsymbol{p}_N = (0.9, 0.7, 0.7, 0.7, 0.7, 0.6, 0.6)$. *Consider the elite vectors:*

- $\boldsymbol{p}_0 = ()$
- $\boldsymbol{p}_1 = (0.9)$
- $\boldsymbol{p}_3 = (0.9, 0.7, 0.7)$
- $\boldsymbol{p}_5 = (0.9, 0.7, 0.7, 0.7, 0.7)$
- $\boldsymbol{p}_7 = (0.9, 0.7, 0.7, 0.7, 0.7, 0.6, 0.6)$

And, consequently the corresponding accuracy is:

- $M(\boldsymbol{p}_0) = 0.5$
- $M(\boldsymbol{p}_1) = 0.9$
- $M(\boldsymbol{p}_3) = 0.87$
- $M(\boldsymbol{p}_5) = 0.89$
- $M(\boldsymbol{p}_7) = 0.88$

As it may be noticed, size 1 *is a local maximum and and size* 5 *is a local maximum. At the same time,* 1 *is a global maximum, but* 5 *is not.*

Unfortunately, this prevents us to be able to characterize effectively our maximum. And consequently, we are also not able to look at how much a certain equilibrium is efficient. Indeed, since we do not have any 'local' information about the social welfare, it is hard to compare it with our information about equilibria. The irregularities that we showed in Example 4.5 are such that we prefer to stop our investigation here. Indeed, delving into the social welfare of such a framework may have required more advanced tools and would have probably not lead us to sufficiently interesting results. Instead, we dedicate the last part of this chapter to look into the possibility of delegation.

4.4 What if Agents Delegate?

So far, we just took into account the possibility for agents to either vote or abstain. This is indeed the most basic setting we could study. A natural expansion is to consider the possibility for agents to delegate. Forms of voting of this type are usually called *proxy systems*. These include the classical forms of representative democracy and also new voting structure like *liquid democracy*, which has drawn the attention of the political debate in the most recent years (Blum and Zuber, 2016; Brill, 2018; Paulin, 2020; Ramos, 2015). In particular, the basic idea of liquid democracy is that each agent is allowed to abstain, vote or delegate to any other agent. According to many researchers, such a proposal may at the same time take the best of both representative democracy and direct democracy. Indeed, it allows agents to express their vote when they think they are competent or to delegate and consequently not spend any effort in voting, if they do not care enough for the matter at stake. So, looking at liquid democracy in our framework seems quite a natural step.

However, we do not plan to give an account of what is the game-theoretic behavior of a structure in which agents are also allowed to delegate. Instead, our aim in this brief section is just to suggest that such a this direction is worth studying, as it presents already some very nice features. In order to do so, we will show some basic examples necessary to transmit our intuitions.

We shall briefly formalize the setting. We allow each agent to delegate to any other agent, as we saw being the case for liquid democracy. Consider the same decision structure as before $D = \langle N, e, \boldsymbol{p}_N \rangle$ and allow each agent i to be able to perform the following actions: $1, 0, del(1), ..., del(n)$ where $del(j)$ represents the action of delegating to j. Indeed, if agent i delegates to agent j we have that the power of i is now 0, and the power of j increases of 1. We call $pow(i)$ the power of i such that

$$pow(i) = \begin{cases} |\{j \in N | del(j) = i\}| + 1 & \text{if } a_i = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.11}$$

Notably, here we do not take into account the possibility for delegation of being transitive, i.e. if agent i delegates to j and j delegates to k, the agent k still just gets an increase of 1 in power and i's delegation is 'lost'. Now, in order to define properly the function $M(\boldsymbol{a})$ we denote with $N(\boldsymbol{a}) = \{i | a_i = 1\}$ the group of voters, with $pow(C)$ for $C \subseteq N(\boldsymbol{a})$ the power of a coalition, such that $pow(C) = \sum_{i \in C} pow(i)$. Eventually, let $WC = \{C \subseteq N | pow(C) \geq \frac{pow(N(\boldsymbol{a}))+1}{2}\}$ be the set of winning coalitions and $TC = \{C \subseteq N | pow(C) = \frac{pow(N(\boldsymbol{a}))}{2}\}$ the set of tie coalitions. Thus, the function $M : \boldsymbol{A} \to (0, 1)$ is so defined:

$$M(\boldsymbol{a}) = \begin{cases} \sum_{C \in WC} \prod_{i \in C} p_i \prod_{i \in N(\boldsymbol{a}) \setminus C} (1 - p_i) & \text{if } pow(\boldsymbol{a}) \text{ is odd} \\ \sum_{C \in WC} \prod_{i \in C} p_i \prod_{i \in N(\boldsymbol{a}) \setminus C} (1 - p_i) & \\ \quad + \frac{1}{2} \cdot \sum_{C \in TC} \prod_{i \in C} p_i \prod_{i \in N(\boldsymbol{a}) \setminus C} (1 - p_i) & \text{if } pow(\boldsymbol{a}) \text{ is even.} \end{cases} \tag{4.12}$$

Intuitively, this corresponds to looking at the coalitions of voters with power greater than half of the total power of voters and compute the probability for each coalition to exactly describe the set of people voting for the right answer if the total power is odd. And, if the total power is even, we add to that the probability that a coalition with power exactly equal to half of the total gets the correct answer multiplied by $\frac{1}{2}$. Similarly, we can assume that preforming the action 1 still costs an effort e and actions $0, del(1), ..., del(n)$ do not cost any effort. Consequently, the utility is so defined:

$$u_i(\boldsymbol{a}) = \begin{cases} M(\boldsymbol{a}) - e & \text{if } a_i = 1 \\ M(\boldsymbol{a}) & \text{otherwise.} \end{cases} \tag{4.13}$$

Indeed, it seems rational to think of the delegation process as something that allows a voter to save effort.

4.4.1 Some Examples and Some Observations

Given this formal framework we can observe some features of the process. First of all, delegation may allow a certain group to reach a higher value for the highest accuracy.

Example 4.6 (Delegation increases the maximal accuracy if agents' competences are different)**.** *Consider the delegation structure* $D = \langle N, e, \boldsymbol{p}_N \rangle$ *s.t.* $N = \{1, ..., 6\}$*,* $\boldsymbol{p}_N = (0.9, 0.75, 0.75, 0.7, 0.7, 0.6)$ *and* $e = 1$*. If we do not allow for delegation, i.e.* $A_i = \{1, 0\}$ *then the highest accuracy is reached when everybody votes but agent 6. Indeed, the group accuracy is 0.910. Instead, if we allow for delegation then we get that the highest accuracy is reached when everybody but agent 6 votes, and agent 6 delegates to agent 1. Indeed, this produces and accuracy of 0.916.*

Of course, this is not the case of every group. This can mostly happen when the differences among the competences of the agents are quite large. In fact, when the differences among the competences are small, delegation does not come into play, as it can not increase the accuracy of a profile.

Example 4.7 (Delegation does not increase the accuracy if the competences are similar)**.** *Consider a delegation structure* $D = \langle N, e, \boldsymbol{p}_N \rangle$*. If for all* $p_i = p$ *we have that distributing different weights over a certain group of voters either decreases the accuracy of the group or leaves it the same. The idea is that distributing weights basically decrease the number of coalitions that can win. And, if it does not increase the probability of every coalition of finding the truth that it is not beneficial. Instead, in other cases it may become useful.*

Although this seems quite a obvious result it represents an important step for the understanding of when and why delegations should be allowed. This is also why we introduced this part on the delegation here, since in order for delegation to be effective (i.e. to enhance the accuracy of a certain group of voters) we need the voters to have different levels of competence.

In addition, delegation may save effort and increase accuracy in maximal equilibria even if it does not produce a higher accuracy, as we can see in the next example.

Example 4.8 (Delegation saves effort)**.** *Consider the same delegation structure* $D = \langle N, e, \boldsymbol{p}_N \rangle$ *as in 4.6, s.t.* $N = \{1, ..., 6\}$*,* $\boldsymbol{p}_N = (0.9, 0.75, 0.75, 0.7, 0.7, 0.6)$ *and* $e = 0.01$*. If we do not allow for delegation, the profile* $\boldsymbol{a} = \{1, 1, 1, 1, 0\}$ *is an equilibrium, as* $u_5(\{1, 1, 1, 1, 0, 0\}) = 0.886 < 0.910 - 0.01 u_5(\boldsymbol{a})$ *and* $u_6(\{1, 1, 1, 1, 1, 1\}) = 0.882 - 0.01 < 0.910 = u_6(\boldsymbol{a})$*. Indeed, the group accuracy in* $\boldsymbol{a}$ *is 0.910 and the total effort* 0.05*. Instead, if we allow for delegation* $\boldsymbol{a}$ *is not an equilibrium anymore, because agent 6 would deviate to* $del(1)$

($M(\{1,1,1,1,del(1)\}) = 0.916$). Yet, similarly, $\{1,1,1,1,del(1)\}$ would also not be an equilibrium, as agent 5 would deviate to 0. Indeed, the profile $\boldsymbol{a}' = \{1,1,1,1,0,del(1)\}$ is an equilibrium: it has an overall accuracy of 0.922 and social welfare of 5.48 (total effort of 0.04). Relevantly, such a configuration has higher accuracy and higher social welfare than $\boldsymbol{a}$.

Apparently, delegation plays quite an important role in our type of games, as it seems to be able to generate fairly new and interesting equilibria. At the same time, it is not always the case that delegation can produce an mPNE with a higher accuracy than if delegation was not allowed. Two things can happen in this last case. First, it may simply be that delegating is not worth it for any of the agents. This usually occurs when the competence levels are all quite similar one to each other, as we mentioned. Secondly, it may happen that delegation creates instability, which ends up in producing a lower accuracy in the mPNE. Delegation increases the accuracy of a group of voters by privileging one of them, but then the weight of the voter with the lowest p consequently decreases and it may not be enough anymore for her to keep voting – because she would not be pivotal anymore. We shall consider an example of this situation.

Example 4.9 (Delegation may also harm)**.** *Consider a structure $D = \langle N, e, \boldsymbol{p}_N \rangle$ with $N = \{1,...,6\}$, $\boldsymbol{p}_N = (0.8, 0.75, 0.75, 0.7, 0.7, 0.6)$ and $e = 0.01$. Now, suppose to look for equilibria. Consider the profile $\boldsymbol{a} = \{1,1,1,1,1,0\}$. If we do not allow for delegation, this is an equilibrium, as $u_5(\{1,1,1,1,0,0\}) = 0.866 < 0.894 - 0.02 u_5(\boldsymbol{a})$ and $u_6(\{1,1,1,1,1,1\}) = 0.872 - 0.003 < 0.8940 = u_6(\boldsymbol{a})$. Indeed, the group accuracy in $\boldsymbol{a}$ is 0.894. Instead, if we allow for delegation and we assume that the cost of delegation is 0 we have that $\boldsymbol{a}$ is still a legitimate profile in the new framework but it is not an equilibrium anymore, as the profile $\{1,1,1,1,del(1),0\}$ is better for agent 5, indeed $u_5(\boldsymbol{a}) = 0.894 - 0.02 < 0.889 = u_5(\{1,1,1,1,del(1),0\})$. In fact, the profile $\{1,1,1,1,del(1),0\}$ is the equilibrium of this new game with delegation. However, the accuracy in this case is lower than the one for the game with no delegation.*

In a very similar vein, it may also be argued that delegation also brings up a cost (maybe the cost of identifying the delegate), and so, it may become less appealing.

4.4.2 Outlook

We believe that the previous examples and observations are enough to justify a more detailed look into the possibility of delegating, in particular, for this type of game. Indeed, as we pointed out delegation can increase the maximal accuracy of a group (and consequently the accuracy of the mPNE), and at the same time it can save some effort, still reaching the same overall accuracy, if compared to a situation where delegation is not allowed. However, at the same time it can also be useless, if the agents' levels of competence are all quite similar. Similarly, it can even produce equilibria with lower accuracy, if compared to non-delegation situation. In particular, the following questions seem to be highly relevant:

1. When is delegation useful and when it is not?[1]

2. How does the social welfare change if we allow for delegation?

3. What happens if also delegation has a cost?

Indeed, obtaining results about these kinds of games allows us to better understand when proxy voting, like liquid democracy, may increase the efficiency of a group in a setting where gathering information is costly. It has been proven that delegation may not directly increase the accuracy in many situations (Caragiannis and Micha, 2019; Kahng, Mackenzie, and A. Procaccia, 2021), but taking into account the effort may definitely turn the table in favour of it. So, although we will not spend more space in this thesis to look into this kind of structures we hope it will be addressed in the near future.

4.5 Summary, Discussion and Takeaways

The introduction of different levels of competence in the framework has a relevant impact on the picture we described in Chapter 2 and Chapter 3.

Irregularities. First of all, we could not prove here results as strong as in the previous chapters. The reason for this is that the function that computes the overall accuracy does not depend on the size of a profile, as we already mentioned. However, we characterized the equilibria, and we showed some interesting elements: for example, the fact that there can exist SPNEs with different sizes. In addition, we also discussed how Conjecture 4.1 may provide more insights on the structure (Subsection 4.2.2).

A stepping stone. This chapter can be regarded as a valuable stepping stone towards future research, as by proving these fundamental theorems we prepared the ground for many possible future directions. A first direction consists in the possibility of unifying this perspective and that of Chapter 3 in a new framework, by considering agents with different stakes and different levels of competences. The natural questions to answer would be: what kinds of agents would vote at the

[1] A good step in this direction has been done by (Zhang and Grossi, 2021).

equilibria with the largest size possible? Those with high competence or those with high stakes? If the agents with low competence have all high stakes would the decision structure produce a low social welfare? A second direction is the one we highlighted in Section 4.4. Delegation seems to be highly effective if we take into account the perspective of the effort. Indeed, whereas it can be seen as a process that reduces the overall accuracy, it also has the benefit of saving effort. Consequently, a full study of delegation under this point of view is in order.

5 Tracking Truth on Networks

So far we considered agents able to compute the final accuracy of the group and then decide how to behave, i.e. which action to choose. This is the result of many strong assumptions. First of all, we assumed that each agent is connected with every other member of the group and, secondly, that she can see the accuracy and the action profile of any agent whom she is connected with. In this chapter, we relax the first assumption, i.e. we allow every agent to be connected just with a portion of the rest of the group. To do so, we refer to an underlying *network* that links the agents. We look at the behaviour of the agents as a result of introducing such a network. This is an important step towards a realistic scenario. When groups of people are asked to find some ground truth, they may only be aware of the decisions of people in their immediate social circles. Consider, for example, asking all academic researchers around the world whether or not conferences will be in person next year. This is definitely a ground truth scenario. Each researcher may either expresses a vote or abstain. Quite obviously no researcher will get to know the choices of all the others. Often each person discusses the topic with no more than two or three friends.

We use Section 5.1 to present the new framework and to introduce some new notions. In Section 5.2 we look at some categories of graphs and in Section 5.3 we study the Nash equilibria there. Notably we do not take strong Nash equilibria into account. In Section 5.4, we integrate the analytical results with computer simulations that provide a more general overview of the game.

5.1 Preliminaries

We consider a group of agents $N = \{1, ..., n\}$ and a level of competence p which is equal for everybody. Again, every agent has the possibility of choosing between two actions: 0 (abstain) and 1 (vote). Let e represent the effort that is required to perform the action 1, i.e. 'vote'; choosing the action 0 does not yield any cost. We denote with $a_i \in \{0, 1\}$ the action of agent i and we call $\boldsymbol{A} = \{0, 1\}^n$ the space of all the possible action profiles. The accuracy of a group of x voters with accuracy p is computed following Equation 2.5.

In order to account for the fact that not every agents is able to see everybody else, we introduce an undirected graph $G = \langle N, E\rangle$ over the set N where E is the set of all the linked couples. If $\{x, y\} \in E$ we say that x and y are neighbors and that x sees y and viceversa. Note that $\{x, y\} \in E$ is a set – so order does not matter. We denote with $D(x)$ the set of all the agents that x 'sees': $D(x) = \{y \in N | \{x, y\} \in E\}$. And we call $d(x)$ the size of $D(x)$. In graph theory, $d(x)$ corresponds to the degree of connectedness of x (Jackson, 2010). Consequently, we denote with $D = \langle G, e, p\rangle$ the decision structure we will use in this chapter, where $G = \langle N, E\rangle$.

Clearly, our agents cannot decide which action to choose on the basis of the accuracy of the entire group (as they do not know it). For this reason, they have to use an heuristics, i.e. a local strategy that can help them in decide what to do in the absence of full information. There exist in our configuration many possible heuristics that can be chosen. We decided to use the one we describe here. In order to state it explicitly, we denote with $D_1(\boldsymbol{a}, x)$ the set of all the voters in a profile $\boldsymbol{a}$ such that they are either x herself or they are neighbors of x: $D_1(\boldsymbol{a}, x) = \{y \in N | a_i = 1 \text{ and either } y \in D(x) \text{ or } y = x\}$. Similarly, $d_1(\boldsymbol{a}, x)$ is the size of $D_1(\boldsymbol{a}, x)$. Every agent i has the following utility function:

$$u_i(\boldsymbol{a}) = M(d_1(\boldsymbol{a}, i), p) - a_i e. \tag{5.1}$$

Intuitively, since an agent i is not able anymore to assess the quality of her choice at the global level, she prefers to do it at the local level. This means that what is important is that she feels pivotal at the local level. Imagine to assist to a conversation among researchers about answering (through the expression of a vote) the question about conferences mentioned above. It is very likely that a person may feel more obliged to vote if the people she is talking to decided not to do so. This is the core of this utility function. Every agent is worried about whether among the people she knows there will be enough voters. Indeed, there may be a psychological correspondence that goes from the local perspective to the global perspective. The agent assumes that if she makes sure her circle of acquaintances has done its part, then she can also be confident that all the other groups did the same.

5.2 Graphs

A general study of our game on every type of graph is at this point premature. For this reason, we decided to work just with some specific categories of graphs, in the hope of being able in future work to expand our discussion of the topic.

However, as you will see the graphs we selected here present some interesting features already. For example, they allow for a comparison in terms of accuracy (and number of voters) between a more centralized network (like a *star*) and a more decentralized one (like a *regular graph* or a *grid*). We will analyze these differences further in the final section of this chapter; for now, we limit ourselves to introduce these types of graphs and define them formally. Although we just talk about graphs in our formal results we use interchangeably the term ‘graph’ and the term ‘network’ in the running text.

Definition 5.1. A graph $G = \langle N, E \rangle$ is a *star* with center n iff $E = \{\{1, n\}, ..., \{i, n\}, ..., \{n-1, n\}\}$.

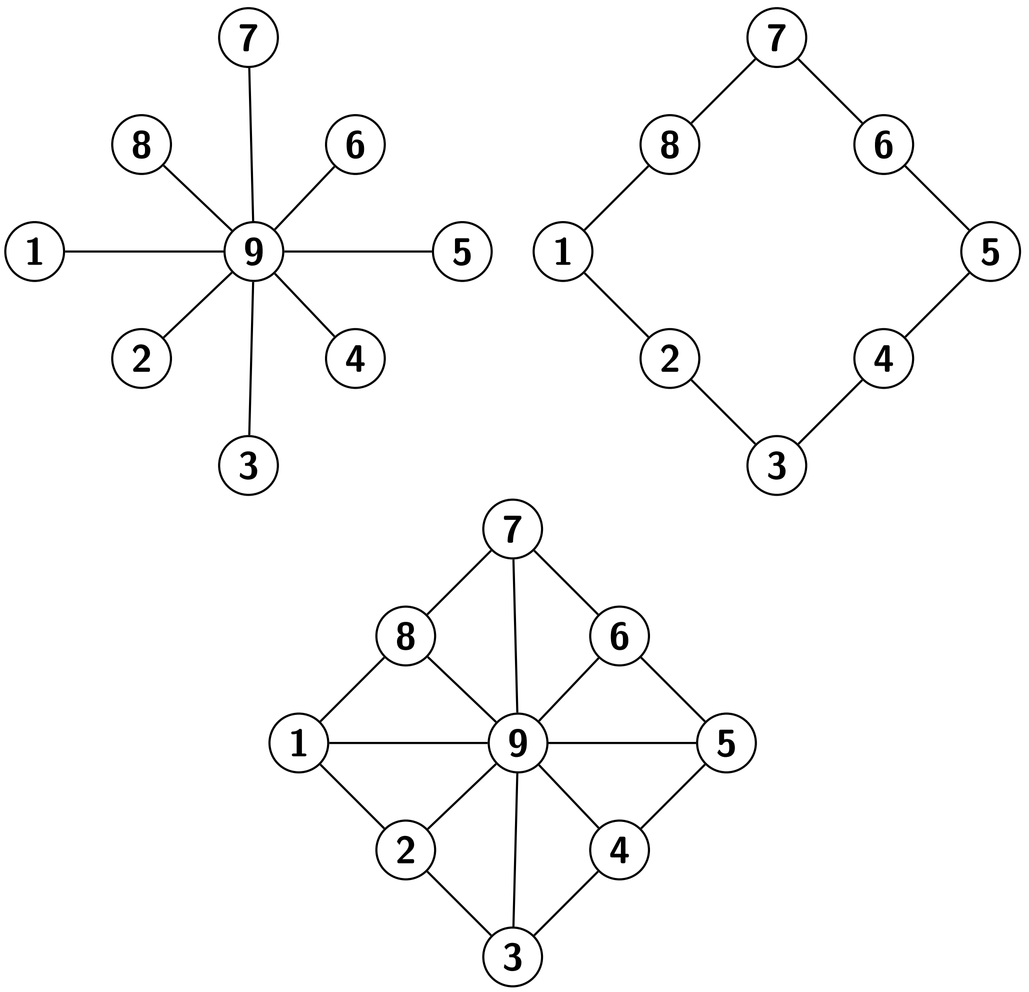

Figure 5.1: The first image is a star with $n = 9$. The second one a is ring, and the third one is a wheel with $n = 9$.

Definition 5.2. A graph $G = \langle N, E\rangle$ is a *ring* iff $E = \{\{1,2\},...,\{i-1,i\},\{i,i+1\},...,\{n-1,n\},\{n,1\}\}$.

Definition 5.3. A graph $G = \langle N, E\rangle$ is a *wheel* with center n iff $E = \{\{1,2\},..., \{i-1,i\},\{i,i+1\},...,\{n-1,1\}\} \cup \{\{1,n\},...,\{i,n\},...,\{n-1,n\}\}$.

Indeed, a wheel is exactly the composition of a ring and a star. We shall have a look at Figure 5.1, which collects these types of graphs. Notably, we assumed that the element in the center of the wheel and of the star is always agent n. This comes without loss of generality, and clearly our results apply to any possible star and wheel with different order. Our assumption allows us to manipulate these objects more efficiently. In addition, we also define formally grids and ℓ-regular graphs.

Definition 5.4. A graph $G = \langle N, E\rangle$ is a *ℓ-regular graph* iff E is such that for all $i \in N$, $d(i) = \ell$.

Notably, the following observation holds.

Observation 5.1. *A 2-regular graph $G = \langle N, E\rangle$ is the union $G = \langle\bigcup N_i, \bigcup E_i\langle$ of disjoint rings $\langle N_1, E_1\rangle, ..., \langle N_m, E_m\rangle$ where $\{N_i\}_{i=1}^m$ is a partition of N and $\{E_i\}_{i=1}^m$ is a partition of E.*

A ring itself is a 2-regular graph. Notably, if G is a 2-regular graph and it is connected then it corresponds to a ring.[1] If it is not connected then it corresponds to more than one ring. Notably, if in a 2-regular graph there is more than one ring, we place the agents such that if the first ring has k nodes, it is occupied by agent 1, 2, ..., k, similarly for ring 2 with j nodes, which will be occupied by agent $k+1, k+2, ..., k+j$ and so on.' Eventually, we define the *grids*.

Definition 5.5. A graph G is a *$n \times m$-grid* iff their vertices correspond to the points in the plane with integer coordinates, x-coordinates being in the range 1,..., n, y-coordinates being in the range 1,..., m, and two vertices are connected by an edge whenever the corresponding points are at distance 1.

As we mentioned, we chose these graphs as they are sufficiently easy to analyze in our setting and at the same time they can already provide us with some useful insights about our game. For this reason, our analytical results will all be concerned about the above categories of graphs. And, in order to be coherent with those, also our simulation study of the game will mainly be about those.

[1] A graph $G = \langle N, E\rangle$ is said to be connected if every pair of vertices in the graph is connected. This means that there is a path between every pair of vertices.

5.3 Equilibria on Graphs

The characterization of pure Nash equilibria in different setting is the core of this work, as it provides us with good insights of agents' behaviour. In this section we characterize a pure Nash equilibrium on this new decision structure and we prove some theorems about equilibria on the categories of graphs we defined before. We also provide examples to explain more general intuitions about the game.

Notably, we can easily apply in this framework the results that we got about PNEs in Chapter 2, as we saw that the formula for the accuracy did not change. Indeed, the definition for a PNE is still Definition 2.1. We shall start with the following theorem.

Theorem 5.1. Let $D = \langle G, e, p \rangle$ be a decision structure and $\boldsymbol{a} \in \{0,1\}^n$ be an action profile. The profile $\boldsymbol{a}$ is a PNE iff for all i with $d_1(\boldsymbol{a}, i) = k_i$, $M(k_i + 1, p) - M(k_i, p) \leq e \leq M(k_i, p) - M(k_i - 1, p)$.

Proof. A profile $\boldsymbol{a}$ is a PNE iff for no agent i $u_i(\boldsymbol{a}) \geq u_i(a'_i, \boldsymbol{a}_{-i})$ by Definition 2.1. We proceed by first assuming that i is s.t. $a_i = 1$ and then assuming i s.t. $a_i = 0$. We show that i does not deivate in neither of the cases iff $M(k+1,p) - M(k,p) \leq e \leq M(k,p) - M(k-1,p)$. Notably the structure is exactly the same as Theorem 2.1, Theorem 3.1 and Theorem 4.1.

Consider an agent i s.t. $a_i = 0$. Such an agent would not deviate iff $u_i(\boldsymbol{a}) \geq u_i(1, \boldsymbol{a}_0)$ by Definition 2.1. By construction $u_i(\boldsymbol{a}) \geq u_i(1, \boldsymbol{a}_0)$ iff $M(d_1(\boldsymbol{a}, i), p) \geq M(d_1(\boldsymbol{a}, i)+1, p) - e$, which is equivalent to $M(d_1(\boldsymbol{a}, i)+1, p) - M(d_1(\boldsymbol{a}, i), p) \leq e$.

Now, consider an agent i s.t. $a_i = 1$. Such an agent would not deviate iff $u_i(\boldsymbol{a}) \geq u_i(0, \boldsymbol{a}_0)$ by Definition 2.1. By construction $u_i(\boldsymbol{a}) \geq u_i(1, \boldsymbol{a}_0)$ iff $M(d_1(\boldsymbol{a}, i), p) - e \geq M(d_1(\boldsymbol{a}, i) - 1, p)$, which is equivalent to $M(d_1(\boldsymbol{a}, i), p) - M(d_1(\boldsymbol{a}, i) - 1, p) \geq e$.

So, a profile $\boldsymbol{a}$ is a PNE iff $M(d_1(\boldsymbol{a}, i) + 1, p) - M(d_1(\boldsymbol{a}, i), p) \leq e \leq M(d_1(\boldsymbol{a}, i), p) - M(d_1(\boldsymbol{a}, i) - 1, p)$ for all $i \in N$. □

Relevantly, this is not different from what we saw in Theorem 2.1. And, indeed the assumption that $M(-1) = 0$ is used in the same way (see the discussion of Theorem 2.1). However, the presence of an underlying network introduces some very interesting effects at a very general level. Indeed, for example, in this case at an equilibrium there can be an agent i s.t. $d_1(\boldsymbol{a}, i) \neq 0$ and is even. Recall that this was not the case for the basic setting (see Corollary 2.1). We proceed with a very similar corollary to clarify the role of the effort in the game. This also helps us in understanding the differences between the present framework and the one we described in Chapter 2.

Corollary 5.1. Let $D = \langle G, e, p \rangle$ be a decision structure.

1. If $e > p - \frac{1}{2}$ a profile $\boldsymbol{a}$ is an equilibrium iff $|\boldsymbol{a}| = 0$.
2. If $e = p - \frac{1}{2}$ a profile $\boldsymbol{a}$ is an equilibrium iff $|\boldsymbol{a}| = 0$ or $\boldsymbol{a}$ is such that for all i, $d_1(\boldsymbol{a}, i) = 0$ or $d_1(\boldsymbol{a}, i) = 1$.
3. If $e < p - \frac{1}{2}$ a profile $\boldsymbol{a}$ is an equilibrium iff for all i either:
   - $M(d_1(\boldsymbol{a}, i), p) - M(d_1(\boldsymbol{a}, i) - 1, p) \geq e$, or
   - $a_i = 0$ and $M(d_1(\boldsymbol{a}, i) + 1, p) - M(d_1(\boldsymbol{a}, i), p) \leq e$.

Proof. The proof is a direct consequence of Theorem 5.1 and it proceeds in the same way as we saw in Chapter 2 for the proof of Corollary 2.1. We have a quick look at the case for $e < p - \frac{1}{2}$. If $M(d_1(\boldsymbol{a}, i), p) - M(d_1(\boldsymbol{a}, i) - 1, p)$ then $d_1(\boldsymbol{a}, i)$ is odd, which assures us that $M(d_1(\boldsymbol{a}, i) + 1, p) - M(d_1(\boldsymbol{a}, i), p) \leq e$ also holds. If $M(d_1(\boldsymbol{a}, i) + 1, p) - M(d_1(\boldsymbol{a}, i), p) \leq e$ then agent i would not deviate from 0 to 1 and so by Theorem 5.1 $\boldsymbol{a}$ is an equilibrium. □

The basic idea underlying this result is the same as in Chapter 2. An agent i who is voting deviates if her contribution to the final accuracy is not high enough. And similarly an agent who is not voting deviates if the contribution she would produce by voting is higher than the effort. What changes is that, in this scenario, the group that each agent considers is usually different from the group that another agent considers. Thus, it can easily happen that an agent thinks that she should vote because she sees in her neighborhood an even number of voters (and so with her it becomes odd), and another agent thinks that she should not vote, as she sees an odd number of voters around herself (and with her it would become even). We shall look at an example. However, we will discuss again this feature later, after we have seen some PNEs on graphs.

Example 5.1. *Let $D = \langle G, e, p\rangle$ such that $N = \{1, 2, 3, 4\}$, $p = 0.8$, $e = 0.01$ and $G = \langle N, E\rangle$ with $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$ as depicted in Figure 5.2. Clearly in this case neither a profile $\mathbf{a} = (1, 0, 1, 0)$ nor a profile $\mathbf{a'} = (0, 1, 0, 1)$ is a PNE. Indeed, in both the cases there would be an agent who is not voting that sees in her neighbourhood an even number of voters (and so her contribution would be 0). This would make her deviate as $M(3, p) - M(2, p) \geq e$. Instead, with $\mathbf{a}'' = (1, 1, 1, 1)$ we have that all the agents that vote also see an even number of other voters (see Figure 5.2) and so $d_1(\mathbf{a}'', i)$ is odd for everybody.*

Example 5.2. *Let $D = \langle G, e, p\rangle$ such that $N = \{1, ..., 5\}$, $p = 0.8$, $e = 0.01$ and $G = \langle N, E\rangle$ with $E = \{(1, 2), (2, 3), (3, 4), (4, 1), (3, 5\}$. In this case there exists no profile $\boldsymbol{a}$ s.t. $\boldsymbol{a}$ is a PNE. Notably, $d_1(\boldsymbol{a}, i)$ must be odd for everybody for $\boldsymbol{a}$ to be a PNE. Indeed, this is the only way to produce a positive contribution for all the voters. However, this is not possible in the graph.*

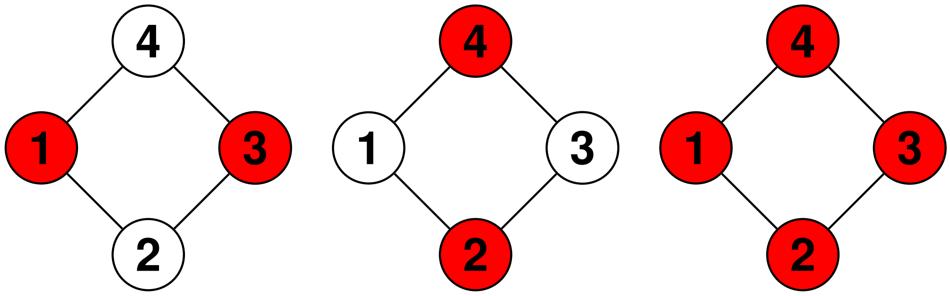

Figure 5.2: We use the red to indicate that the agent chooses the action 1. The first two profiles are not equilibria. The last one is.

Once we have settled down this first result we can start having a closer look to the different types of graphs. We characterize equilibria for cases where $e < p - \frac{1}{2}$. This is because, as we saw, if $e \geq p - \frac{1}{2}$ there can be just two types of equilibria. Neither of those are really relevant for us. Indeed, we are not really interested in studying situation in which agents are satisfied with nobody voting. Notably, these theorems are full characterization of equilibria on certain graphs. Indeed, they state clearly the sufficient and necessary conditions for a profile $\boldsymbol{a}$ to be an equilibrium.

Proposition 5.1. Let $D = \langle G, e, p \rangle$ be a decision structure such that $e < p - \frac{1}{2}$ and G is a *star*. A profile $\boldsymbol{a}$ is a PNE iff either

1. $\boldsymbol{a} = (0, 0, ..., 0, 1)$, or
2. $\boldsymbol{a} = (1, 1, ..., 1, 0)$ and $M(n, p) - M(n-1, p) \leq e$.

Proof. From right to left. Assume Condition 1. We have that $d_1(\boldsymbol{a}, i)$ is odd and $M(d_1(\boldsymbol{a}, i), p) - M(0, p) \geq e$ for all $i \in N$. Thus, by Theorem 5.1, $\boldsymbol{a}$ is a PNE. Assume Condition 2. We have that either $d_1(\boldsymbol{a}, n)$ is odd, or it is even and $M(d_1(\boldsymbol{a}, i) + 1, p) - M(d_1(\boldsymbol{a}, i), p) \leq e$. So, by Theorem 5.1 n would not deviate. For all $i \neq n$, $d_1(\boldsymbol{a}, i)$ is odd and $M(d_1(\boldsymbol{a}, i), p) - M(0, p) \geq e$. Thus, by Theorem 5.1, $\boldsymbol{a}$ is a PNE.

From right to left. Any profile $\boldsymbol{a}$ s.t. $a_j \neq a_k$ for $j \neq n, k \neq n$ would either produce $d_1(j)$ to be even or $d_1(k)$ to be even, as they both see n. So either j or k would deviate. Similarly, if $a_i = 1$ for all $i \in N$, then every element $i \neq n$ would deviate. If $a_i = 0$ for all i then each agent would deviate. Eventually, if $a_n = 0$, for all $j \neq n$ $a_j = 1$ and $M(n, p) - M(n-1, p) > e$, n would deviate. □

Intuitively, all the outside members should play the same action, otherwise there must be someone who would deviate. And, indeed, the actions of the outside members must be different from that of the inside one. Eventually, if $a_n = 0$ we must be sure that she does not deviate to 1 by imposing $M(n,p)-M(n-1,p) \leq e$. We prove now a similar result about wheels.

Proposition 5.2. Let $D = \langle G, e, p\rangle$ be a decision structure such that G is a *wheel*. If $p - \frac{1}{2} > e > M(3,p) - M(2,p)$ then a profile a is a PNE iff either

1. $a_n = 1$ and $a_i = 0$ for all $i \neq n$, or
2. $a_n = 0$, $a_{3m+1} = 0$, $a_{3m+2} = 0$, $a_{3m+3} = 1$ for some $m \in \mathbb{N}^+$, or
3. $a_n = 0$, $a_{2m} = 0$, $a_{2m+1} = 1$ for some $m \in \mathbb{N}^+$.

If $M(3,p) - M(2,p) \geq e$ then a is a PNE iff either:

1. $a_n = 1$ and $a_i = 0$ for all $i \neq n$, or
2. $a_n = 0$, $a_{3m+1} = 0$, $a_{3m+2} = 0$, $a_{3m+3} = 1$ for some $m \in \mathbb{N}^+$ and $M({}^{n-1}\!/_3 + 1, p) - M({}^{n-1}\!/_3, p) \leq e$, or
3. $a_n = 1$, $a_{3m+1} = 0$, $a_{3m+2} = 1$, $a_{3m+3} = 1$ for some $m \in \mathbb{N}^+$ and $M((2^{n-1}\!/_3) + 1, p) - M((2^{n-1}\!/_3), p) \leq e$, or
4. $a_n = 0$ and $a_i = 1$ for all $i \neq n$ and $M(n,p) - M(n-1,p) \leq e$, or
5. $a_n = 0$, $a_{2m} = 0$, $a_{2m+1} = 1$ for some $m \in \mathbb{N}^+$ and $M(\frac{n-1}{2} + 1, p) - M(\frac{n-1}{2}, p) \leq e$.

Proof. The proof is somewhat tedious, but the core idea consists of applying Theorem 5.1 and taking into account the different neighbors of each agent. We discuss once more the general intuitions of it later on. We look at the two settings.

- First, assume $M(1,p) - M(0,p) > e > M(3,p) - M(2,p)$. From right to left.
  1. Assume $\boldsymbol{a}$ is s.t. $a_n = 1$ and $a_i = 0$ for all $i \neq n$. Thus, $d_1(n) = 0$ and so n does not deviate. And for all i we have that $d_1(i) = 1$ and so they also do not deviate.
  2. Now, assume the second case. Note that for this to be possible $n-1$ must be multiple of 3. For each $a_i = 0$ we have that $d_1(\boldsymbol{a}, i) = 1$, and for each $a_j = 1$, $d_1(\boldsymbol{a}, j) = 1$, with $i, j \neq n$. For n we have that $d_1(n, \boldsymbol{a}) = \frac{n-1}{3}$. Thus, since $M({}^{n-1}\!/_3 + 1, p) - M({}^{n-1}\!/_3, p) < M(3,p) - M(2,p) \leq e$ $\boldsymbol{a}$ is a PNE by Theorem 5.1.

3. Now, assume the third case. Note that for this to be possible $n-1$ must be multiple of 2. Agent n would not deviate since $M({n-1}/{2}+1,p) - M({n-1}/{2},p) < M(3,p) - M(2,p) \leq e$. Any agent $a_i = 1$ would not deviate as $d_1(\boldsymbol{a}, i) = 1$. And, any agent $a_i = 0$ would not deviate as $d_1(\boldsymbol{a}, i) = 2$ and $M(3,p) - M(2,p) \leq e$.

From left to right. We prove that any profile that is different from those three is not a PNE.

(a) Any profile $\boldsymbol{a}$ s.t. $a_i = 1$ for all $i \neq n$ would result in all the outside agents deviating, because $d_1(i, \boldsymbol{a}) = 3$ but $M(3,p) - e < M(2,p)$ and so by Theorem 5.1 this is not a PNE.

(b) Any profile $\boldsymbol{a}$ s.t. $a_j \neq a_k$ for all $j \neq n$, $k \neq j$, $n \neq k$ and $\{j,k\} \in E$ with $a_n = 1$ implies that $d_1(\boldsymbol{a}, i) = 2$ and $a_i = 1$ for some i who would deviate to 0.

(c) Any profile $\boldsymbol{a}$ s.t. $a_n = 0$ and $a_j = a_k$ for some $k \neq j$ and $\{j,k\} \in E$ would result in two cases. First case: case (a). Second case: there exists $i \neq n$ such that $a_i = 0$. Thus, there must be an agent l such that l is voting and sees just another voter. Thus, l would deviate.

(d) Any profile $\boldsymbol{a}$ s.t. $a_n = 1$ and $a_j = a_k = 1$ for some $k \neq j$ and $\{j,k\} \in E$ would imply that $d_1(\boldsymbol{a}, n) \geq 3$. Thus n would deviate.

(e) Any profile $\boldsymbol{a}$ with $a_i = 0$ for all $i \in N$ would force every agent to deviate.

- Assume $e < M(3,p) - M(2,p)$. From right to left.

1. The first case is the same as before.
2. The second case is also similar, but this time we need to assume $M({n-1}/{3}+1,p) - M({n-1}/{3},p)$ as it is not granted anymore. Note that for this to be possible $n-1$ must be multiple of 3.
3. Assume the third case. Note that for this to be possible $n-1$ must be multiple of 3. If $i \neq n$ and $a_i = 1$ then $d_1(\boldsymbol{a}, i) = 3$. If $a_i = 0$ then $d_1(\boldsymbol{a}, i) = 3$. Thus, we have that $d_1(\boldsymbol{a}, n) = 2\frac{n-1}{3}$. This is even and from the assumption n would not deviate.
4. Assume the fourth case: $\boldsymbol{a}$ is s.t. $a_n = 0$ and $a_i = 1$ for all $i \neq n$ and $M(n,p) - M(n-1,p) \leq e$. Indeed, $d_1(n) = n-1$ and $d_1(i) = 0$ for all $i \neq n$.
5. Assume the fifth case. Note that for this to be possible $n-1$ must be multiple of 2. For all $i \neq n$ we have that $d_1(\boldsymbol{a}, i)$ is odd. For n it is even, but due to the assumption she would not deviate.

From left to right. We prove that every profile that is different from those three is not a PNE.

(a) Any profile $\boldsymbol{a}$ s.t. $a_i = 1$ for all i would force any outside member to deviate as $M(4, p) = M(3, p)$ and $d_1(\boldsymbol{a}, i) = 4$.

(b) Any profile $\boldsymbol{a}$ s.t. $a_i = 0$ for all i would force any outside member to deviate as $M(1, p) - M(0, p) > e$ and $d_1(\boldsymbol{a}, i) = 0$.

(c) Any profile $\boldsymbol{a}$ s.t. $a_n = 1$ and $a_j = a_k = 1$ for some $k \neq j$ and $\{j, k\} \in E$ would have three cases. First case: there exists no i with $a_i = 0$, and then it is case (a). Second case: $a_{3m+1} = 0$, $a_{3m+2} = 1$, $a_{3m+3} = 1$ for some $m \in \mathbb{N}^+$. If there is no condition on e, then n would deviate, otherwise we proved it is a PNE. Third case: there exists some i such that $d_1(\boldsymbol{a}, i) = 2$. And thus, she would deviate.

(d) Any profile $\boldsymbol{a}$ s.t. $a_n = 0$ and $a_j = a_k = 1$ for some $k \neq j$ and $\{j, k\} \in E$ would have two cases. First case: there exists no outside member with action 0. Then either we are in the last condition of the Proposition, or n would deviate. Second case: there exists an outside member with action 0. She would see 2 voters and then deviate.

(e) Any profile $\boldsymbol{a}$ s.t. $a_n = 1$ and $a_j = a_k = 0$ for some $k \neq j$ and $\{j, k\} \in E$ would have three cases. First case: there exists no i with $a_i = 1$, and then it is the first Condition and we proved that it is a PNE. Second case: we are in case (c). Third case: there exists an outside member that sees just one voter n and she would deviate.

(f) Any profile $\boldsymbol{a}$ s.t. $a_n = 0$ and $a_j = a_k = 0$ for some $k \neq j$ and $\{j, k\} \in E$ would have four cases. First case: there exists no i with $a_i = 1$, and then it is case (a). Second case: $a_n = 0$, $a_{3m+1} = 0$, $a_{3m+2} = 0$, $a_{3m+3} = 1$ for some $m \in \mathbb{N}^+$ and $M({}^{n-1}\!/_3 + 1, p) - M({}^{n-1}\!/_3, p) > e$ and n would deviate. Third case: there exists an outside member that does not see any voter. It is not a PNE. Fourth case: there are two oustide members that see each other and vote, and so it is case (d).

(g) Any profile $\boldsymbol{a}$ in $a_{2m} = 0$, $a_{2m+1} = 1$ and $a_n = 1$. Thus, an outside member that votes would deviate.

□

Notably, as it is explained in the proof of Proposition 5.2, a profile $\boldsymbol{a}$ s.t. $a_{3m+1} = 0$, $a_{3m+2} = 1$, $a_{3m+3} = 1$ can be chosen only if the underlying graph has a number of outside nodes that is a multiple of 3. And, we will see something similar in Theorem 5.2.

The proof we just saw can be summarized in few points. First of all, we must take note of the fact that a profile may not be an equilibrium for many reasons. To understand them better we must compute the value for n_{thr} as in Chapter 2 (Corollary 2.1) for the pair of p, e. If $n_{thr} \geq d(i)$ for all i then if a profile $\boldsymbol{a}$ does not satisfy the *odd-even dynamics*, it cannot be a PNE. We call *odd-even dynamics* the fact that for all i, $d_1(\boldsymbol{a}, i)$ must be odd. If $n_{thr} < d(i)$ there can be profiles that are equilibria even if they do not satisfy the odd-even dynamics for all their agents. If $d_1(\boldsymbol{a}, i) \geq n_{thr}$ and $d_1(\boldsymbol{a}, i)$ is odd with $a_i = 0$ a profile $\boldsymbol{a}$ can still be an equilibrium. And viceversa an agent i s.t. $d_1(\boldsymbol{a}, i)$ is odd, would deviate if $d_1(\boldsymbol{a}, i) > n_{thr}$. Such an 'exception' to the odd-even dynamics generates the need to check many cases in the characterization of the PNEs, as we saw in Proposition 5.2. However, in the end, this is the only exception, so for example, if there exists an i such that $d_1(\boldsymbol{a}, i)$ is even and $a_i = 1$, such a profile can never be an equilibrium. This is the case of the first graph of Figure 5.3. In the second situation, this even-odd dynamics is respected, but there is some i for which $d_1(\boldsymbol{a}, i) > n_{thr}$. In this case the effort of i is just not worth it (second graph of Figure 5.3. As you can see in proof of Proposition 5.2, the profiles that are not PNE when e is high may be profile in cases where e is lower, but only if the reason for them to not be PNE in the first place is not the un-satisfaction of the odd-even dynamics.

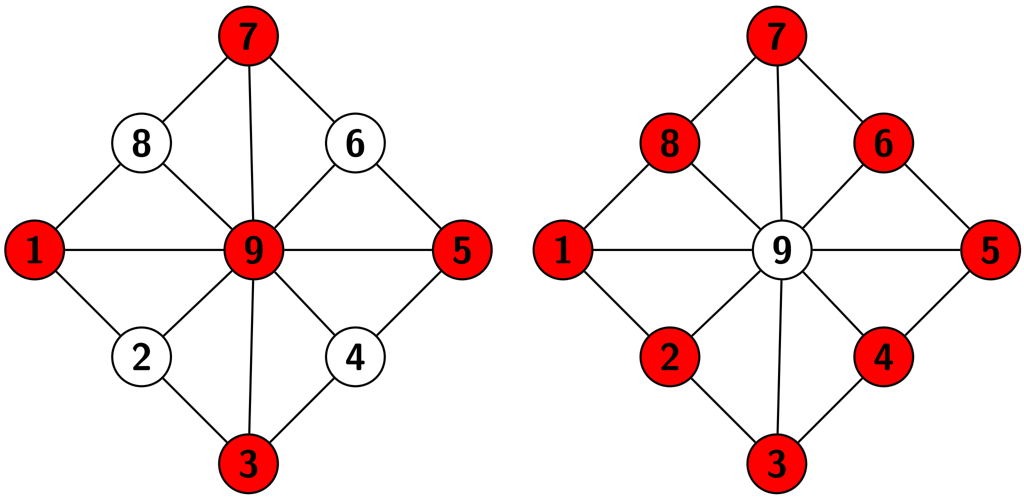

Figure 5.3: Two different profiles. The one on the left is not a PNE for any e. The one on the right is a PNE only if e is smaller than a certain value.

Now, we move on to the main result of the chapter, which provides us with insights about a general type of regular graphs.

Theorem 5.2. Let $D = \langle G, e, p \rangle$ be a decision structure with G a 2-regular graph. If $M(3, p) - M(2, p) < e$ a profile $\boldsymbol{a}$ is a PNE iff:

1. $a_{3m+1} = 0$, $a_{3m+2} = 0$, $a_{3m+3} = 1$ for some $m \in \mathbb{N}^+$.

If $M(3,p) - M(2,p) > e$ a profile $\boldsymbol{a}$ is a PNE iff either:

1. $a_{3m+1} = 0$, $a_{3m+2} = 0$, $a_{3m+3} = 1$ for some $m \in \mathbb{N}^+$, or

2. $a_i = 1$ for all $i \in N$.

Proof. Assume $M(3,p) - M(2,p) < e$. From right to left. Assume $\boldsymbol{a}$ is s.t. $a_{3m+1} = 1, a_{3m+2} = 0, a_{3m+3} = 0$ for some $m \in \mathbb{N}^+$. Thus, for all a_{3m+1} we have that $d_1(a_{3m+1}, \boldsymbol{a})) = 1$ and for all a_{3m+2} and a_{3m+3}, $d_1(a_{3m+2}, \boldsymbol{a})) = d_1(a_{3m+3}, \boldsymbol{a})) = 1$. Notably, this implies that G is such that all the rings which it is composed of have a number of nodes that is multiple of 3. From left to right: we take into account the other possible profiles.

- If $a_i = 1$ for all i then $d_1(\boldsymbol{a}, i) = 3$ and so i would deviate.
- If $a_i = a_{i+1} = 1$ and $a_{i+2} = 0$ for some i then $i + 1$ would deviate.
- If $a_i = a_{i+1} = a_{i+2} = 0$ for some i then $d_1(\boldsymbol{a}, i+1) = 0$ and so $i+1$ would deviate.
- If $a_i = a_{i+2} = 1$ and $a_{i+i} = 0$ for some i then $i + 1$ would deviate.

If $n \neq 3k$ then those above are the only possible options. If $n = 3k$ then there is the profile we already talked about.

Now assume $M(3,p) - M(2,p) \geq e$. The PNE that we already proved are given for granted. Assume $a_i = 1$ for all $i \in N$. Thus, for all i we have that $d_1(i, \boldsymbol{a}) = 1$. So, it is a PNE. Since nothing changed for all the other other possible profiles, the proofs go through in the same way. $\square$

This result is a characterization of equilibria on graph with two links. Notably, these kinds of regular graphs have always the same 'shape'. We may wonder if the features of regular graphs could allow us to just characterize equilibria on every type of regular graph fully. Although there seems to be some kind of connection between the two notions, an analytical result is left for future work. Indeed, it is true that by having always the same number of links such networks seem more easily to yield encouraging results in the odd-even dynamics. Yet, at the same time the fact that there can exist many different shapes for each k-regular network with $k > 2$ makes it difficult to be able to characterize equilibria generally. The following example may give us some insights about which way to follow for further research.

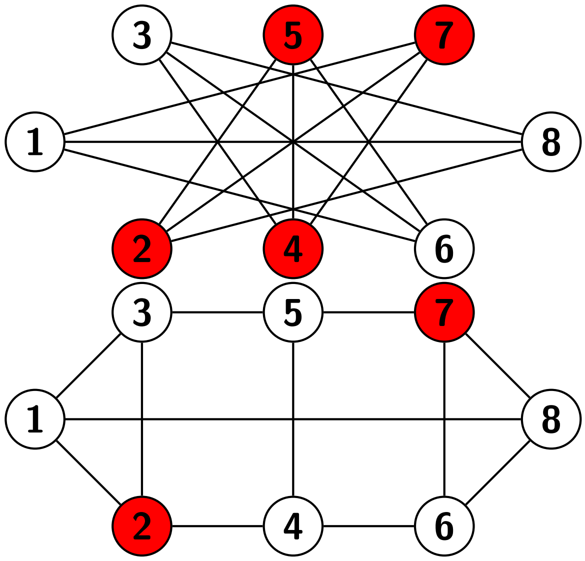

Figure 5.4: These two graphs are both 3-regular graphs. However, they present two different types of equilibria. In the first one the equilibria are many possible combinations of four agents alternating two and two. In the second only the two profiles of voters $\{2, 7\}$ and $\{3, 6\}$ are equilibria. Nonetheless, it is interesting to see that in both the cases the agents that vote must be a *vertex cover*.

Example 5.3. *Consider 3-regular graphs. In particular, consider the two graphs in Figure 5.4.*

The fact that in both the cases the equilibria are connected with the concept of vertex cover seems a good starting point to understand a possible correlation between regular graphs and equilibria. However, for now the problem is left open.

A last result on regular graphs is the following.

Proposition 5.3. Let $D = \langle G, e, p \rangle$ be a decision structure. Let G be an ℓ-regular graph. If ℓ is even and $M(l + 1, p) - e > M(l, p)$ then $\boldsymbol{a} = (1, 1, 1, ..., 1)$ is an equilibrium.

Proof. For each agent we have $d_1(i, \boldsymbol{a}) = l + 1$, and by Theorem 5.1, we know that i would not deviate. □

Eventually, we describe equilibria on grids. Notably, for a 3×3 we fully characterize equilibria. The idea is, again, to provide some ground for further research.

Proposition 5.4. Let $D = \langle G, e, p \rangle$ be a decision structure. Assume $M(5, p) - M(4, p) \geq e$. If G is a 3×3 grid (and we order the agents starting from the top left wlog) a profile $\boldsymbol{a}$ is a PNE iff $a_{2k+1} = 1$ and $a_{2k} = 0$ for $k \in \mathbb{N}$.

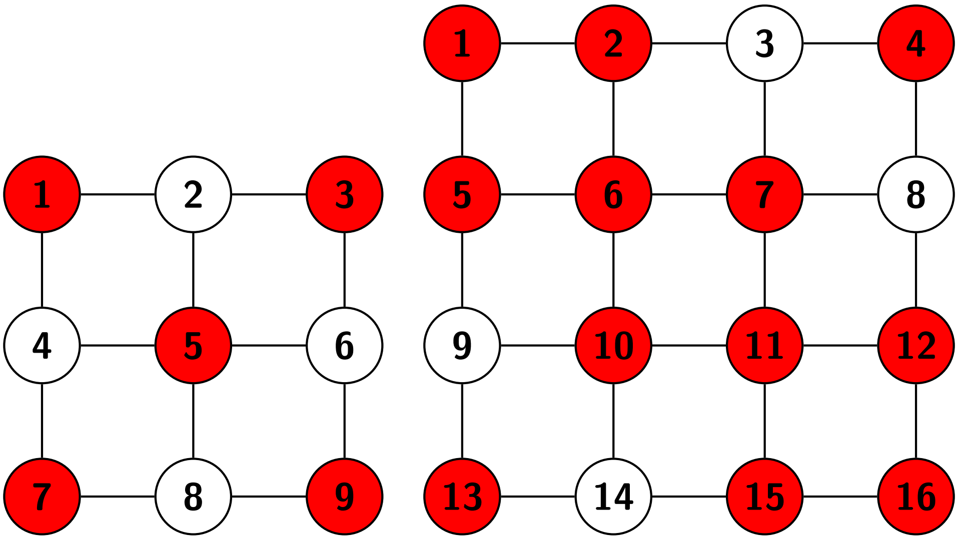

Figure 5.5: Profiles of Proposition 5.4.

Figure 5.5 illustrates the PNE.

Proof. As in Proposition 5.2 the proof just is just a rigorous application of Theorem 5.1. Assume $\boldsymbol{a} = (1, 0, 1, 0, 1, 0, 1, 0, 1)$. Thus, every odd agent is voting and sees an odd number of people voting around her and every agent who is not voting sees an even number of people. Now, we prove that every other profile is not a PNE.

- Assume $a_1 = 1$. Thus, either $a_4 = a_2 = 0$ or $a_4 = a_2 = 1$.
  - Assume $a_4 = a_2 = 1$. Thus, either $a_7 = a_8 = 1$ or $a_7 = a_8 = 0$.
    * Assume $a_7 = a_8 = 1$. Thus, $a_5 = 1$ and $a_6 = 1$. Consequently, $a_3 = a_9 = 1$. Thus, 6 would deviate.
    * Assume $a_7 = a_8 = 0$. Thus, $a_5 = 0$ and $a_9 = a_3 = 1$. So, $a_6 = 1$. But then a_3 would deviate.
  - Assume $a_4 = a_2 = 0$. Thus, either $a_8 = a_7 = 1$ or $a_8 = a_7 = 0$.
    * Assume $a_7 = a_8 = 1$. Thus, $a_5 = 1$ and $a_6 = 1$. Consequently, $a_3 = a_9 = 1$. Thus, 6 would deviate.
    * Assume $a_7 = a_8 = 0$. Thus, $a_5 = 0$ and $a_9 = a_3 = 1$. So, $a_6 = 1$. But then a_3 would deviate.
- Assume $a_1 = 0$. Thus, wlog $a_2 = 1$ and $a_4 = 0$. Consequently, either $a_3 = 1$ or $a_3 = 0$.

- Assume $a_3 = 1$. So, $a_6 = 1$. Thus, either $a_9 = a_8 = 1$ or $a_9 = a_8 = 0$.
  - Assume $a_9 = a_8 = 1$. Thus, $a_5 = 0$ and $a_7 = 0$. Thus, a_2 would deviate.
  - Assume $a_9 = a_8 = 0$. Thus, $a_5 = 1$ and $a_7 = 1$. Thus, 8 would deviate.
- Assume $a_3 = 0$. So, $a_6 = 0$. Thus, either $a_9 = a_8 = 1$ or $a_9 = a_8 = 0$.
  - Assume $a_9 = a_8 = 1$. Thus, $a_5 = 0$ and $a_7 = 0$. Thus, a_2 would deviate.
  - Assume $a_9 = a_8 = 0$. Thus, $a_5 = 1$ and $a_7 = 1$. Thus, 8 would deviate.

□

Our result for 3×3 grids is again just a starting point. A general characterization of PNEs on grids seems out of reach at the moment. However, we can also have a look at other examples for bigger grids.

Example 5.4. *Let $D = \langle G, e, p \rangle$ s.t. G is a 4×4-grid and assume $M(5,p) - M(4,p) \geq e$. Thus, the profile $\boldsymbol{a} = (1,1,0,1,1,1,1,0,0,1,1,1,1,0,1,1)$ is a PNE (see again Figure 5.5. Now, let $D = \langle G, e, p \rangle$ s.t. G is a 5×5-grid and again assume $M(5,p) - M(4,p) \geq e$. Thus, the profile $\boldsymbol{a} = (1,0,1,0,1,0,1,1,1,0,1,1,1,1,1,0,1,1,1,0,1,0,1,0,1)$ is a PNE. These results are just obtained by applying Theorem 5.1.*

Although our analysis is not exhaustive, it provides good insights about how to proceed in order to find solutions for this type of games. First of all, it is more than enough to grasp how the process of finding equilibria works. In addition, it characterizes them on some very basic graphs, giving us already an idea of some more general finding. For example, compare the type of equilibria that can be reached on a *star* and on a 2*-regular graph*. Interestingly, in a star, which can be considered a more *centralized* type of graph, either everybody (except the central agent) votes or just one person votes. The two equilibria are quite extreme and relevantly not depending on e as soon as $e < p - \frac{1}{2}$. Instead, in a 2-regular graph the equilibria that can be found are more dependent on the effort and less ‘extreme’. The distribution of agents choosing action 1 should be more regular. In order to complement these analytical results with a deeper analysis of the system we also decided to look at the so-called *iterated best-response dynamics*.

5.4 Iterated Best-Response Dynamics

The framework and the utility we defined above are such that each agent can make her choice based on the actions of her neighbors. Indeed, for every configuration that is not a PNE there exists an agent i that wants to deviate. However, if that agent deviates (by updating her action) it may easily be that there is another agent, who was satisfied in the beginning, who wants to change her action, and so on. The process of updating a choice based on the actions of the other agents is called *best response dynamics*. It can be iterated, following this process, which is called *iterated best response dynamics (IBRD)*:

1. we start from a configuration $\boldsymbol{a}$,
2. at every turn, we ask the agents if they want to switch to another action and we update $\boldsymbol{a}$, and
3. the process finishes when no agent wants to deviate anymore.

Of course, IBRD can be a useful tool to find an equilibrium. When nobody wants to update her position anymore, we can be sure that the situation that has been reached is a PNE. Unfortunately, computing processes of IBRD may be extremely long. For this reason we made use of a computer simulation. The program is written so to allow the user to set different parameters and allow for different environments choices. The language we used is Python. Through such a simulation we manage to study the IBRD of the initial game on networks of different types. Notably, this process gives us two types of insights. The first one is about the process itself. For example, looking at the simulation we can get an idea of how much easy is for a certain group to reach an equilibrium through this mechanism. As we show in Subsection 5.4.2 both the number of the links and the number of agents impact the results in this case. The second insight is about the equilibria that can be found. By employing a simulation we can more easily generalize some of the results about the equilibria we had in the first section and expand them.

Subsection 5.4.1 is dedicated to study the simulation when we assume same level of competence and same stakes for everybody. In this case, we just look at the IBRD in terms of individuals. In Subsection 5.4.2, we look at when a simulation reaches an equilibrium. Subsection 5.4.3 studies what happens if we introduce agents with different stakes. In Subsection 5.4.4, we allow the agents to update their actions in coalitions.

5.4.1 Simulating the Basic Setting

Each agent is described with the two features p, e, which are the same for everyone, and is placed on a graph G. Thus, the program simulates a process of IBRD, by following this schema.

- Every agent is assigned the action 1.

- At each step, agents are asked to update their action.
- At a certain step (set in the initial simulation) the process interrupts.

The first thing that we consider important studying is the way in which agent activate and are asked to update their choices. We can have three possible ways.

- **Simultaneous Activation**.
  - In order, from agent 1 each agent evaluates which action to choose to improve her situation.
  - Thus, in order from agent 1, each agent updates her position.
- **Ordered Activation**.
  - In order from agent 1.
  - Agent i evaluates which action to choose to improve her situation. Thus, she updates her position.
  - Then, agent $i + 1$ does the same and so on...
- **Random Activation**
  - At each turn, a sequence of numbers is created. Then agent activates following that sequence.
  - Agent i evaluates which action to choose to improve her situation. Thus, she updates her position.
  - Then, agent j does the same and so on...

Indeed, the activation process matters. We will focus more on it in the next subsection.

Before moving on we shall have a quick look at how the rest of the simulation is built. Each simulation is assigned many parameters. First of all, we set the number of agents n, the value for the effort e and the value for the competence p. Thus, we also set the type of graph (or the graph itself) in which agents are placed. As it can be seen in Subsection 5.4.4 there is also the possibility to change the stakes of each agent (or to randomly generate the value for them between a certain interval). If it is not said otherwise the value for the stakes is set to 1 for everybody. This is indeed the equivalent of the basic setting. Eventually, there is the possibility to allow agents to move together, i.e. in coalitions. This is not permitted unless said otherwise (see Subsection 5.4.5). In addition, every run has usually 100000 steps if not said otherwise. The reason for this will become apparent in Subsection 5.4.2. Eventually, when we have to extract a value out of a certain configuration we run that same configuration 50 times and then we make an average. Relevantly, the average is done taking into account only the results coming from simulations that reached an equilibrium. As we will see in Subsection 5.4.2 with certain parameters this happen almost all the times.

In addition, we briefly sketch how agents evaluate actions and update them. Each agent i has two possible actions $\{1, 0\}$ and can see the actions of her neighbors. Let $\boldsymbol{a}_{-i}$ be the profile describing the action of the rest of the agents. The agent compares $u_i(1, \boldsymbol{a}_{-i})$ and $u_i(0, \boldsymbol{a}_{-i})$ and choose which one she prefers.

5.4.2 How To Reach an Equilibrium

We consider the three activation patterns: simultaneous, ordered and random. We show that the best suited for our scope is the *random* one. We show that both the simultaneous and the ordered one may be unable to find an equilibrium, and incur in a periodical cycle even in very basic situations. Consider the example:

Example 5.5. *Consider $D = \langle G, e, p\rangle$ with $N = \{1, 2\}$, $E = \{(1, 2)\}$ and $e < M(1, p) - \frac{1}{2}$. Assume we start from the configuration $\boldsymbol{a} = (0, 0)$. At the first step both the agents would deviate to 1. We update simultaneously. Thus, $\boldsymbol{a}_1 = (1, 1)$. Consequently, they update again: $\boldsymbol{a}_2 = (0, 0)$ and the cycle goes on. Indeed, in this case our process would never be able to find the only two PNEs: $\boldsymbol{a}' = (1, 0)$ and $\boldsymbol{a}'' = (1, 0)$.*

However, the ordered activation can also incur in a similar pattern.

Example 5.6. *Consider $D = \langle G, e, p\rangle$ with $N = \{1, 2, ..., 5\}$, $E = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 2)\}$ and $e < M(3, p) - M(2, p)$. We shall have a look at the dynamics starting from the configuration $\boldsymbol{a}_0 = (1, 1, 1, 1, 1, 1)$.*

1. $a_1 = (0, 0, 0, 0, 0, 1)$

2. $a_2 = (1, 1, 0, 1, 1, 1)$

3. $a_3 = (0, 0, 0, 0, 0, 1)$

4. ...

Again, we incur in a periodical pattern that does not allow us to find the only PNE, i.e. $\boldsymbol{a} = (0, 1, 1, 1, 1, 1)$. We represent the dynamics in Figure 5.6.

As it stands, the random activation seems the best option we have. Indeed, by activating agents in a random order, it escapes loops and it can more easily find an equilibrium. Consider the case of Example 5.6. Assume we are in the situation

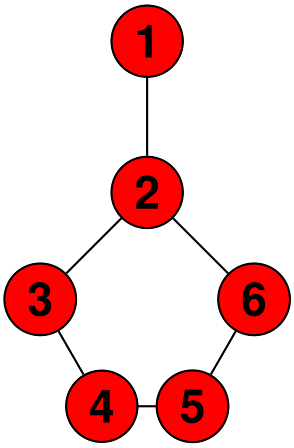

Figure 5.6: The ordered activation in this case produces a loop that goes through the graphs represented here.

$\boldsymbol{a} = (1, 1, 0, 1, 1, 1)$. If we activate the agents in this order $3, 4, 5, 6, 1, 2$ we obtain the equilibrium. Of course, this is just a possibility over many others. So, the key to obtain an equilibrium is to execute every round of activation as many times as possible. We call, from now on, each round a *step*.

However, although in the basic cases the random activation is also able to find an equilibrium if present, this is not true in all the cases. It is definitely interesting to look at when this happens and when it does not. In order to do so, we take into consideration a first sets of results from our simulation (see Figure 5.7). We consider the following decision structure $D = \langle G, e, p\rangle$ with $N = \{1, 2, ..., 20\}$, $p = 0.7$, $e = 0.02$ and with different sets for E such that the graph is always an ℓ-regular graph with $\ell \in \{3, 5, 7, 9, 11, 13, 15, 17\}$. And then we considered another decision structure $D = \langle G, e, p\rangle$ with $N = \{1, 2, ..., k\}$, $p = 0.7$, $e = 0.02$ with G being a 5-regular graph and with k varying among $\{10, 20, 30, 40, 50\}$. For each of these different values we run 50 simulations of 100000 steps and count what percentage of them landed on an equilibrium. As you can see in the first plot the percentage goes relevantly down when the number of agents passes the value of 20. In the second plot the percentage goes slightly down when the number of links passes 11.

When the number of agents increases the simulation is less likely to land on an equilibrium simply because the number of possible configuration skyrockets. This relevantly diminishes the chance of the program to find a ‘right one’. However, we cannot exclude the possibility that certain types of regular network just do not have a equilibrium. Instead, with the number of links, the reason why the percentage is lower is not entirely clear. A good guess seems to be the fact that

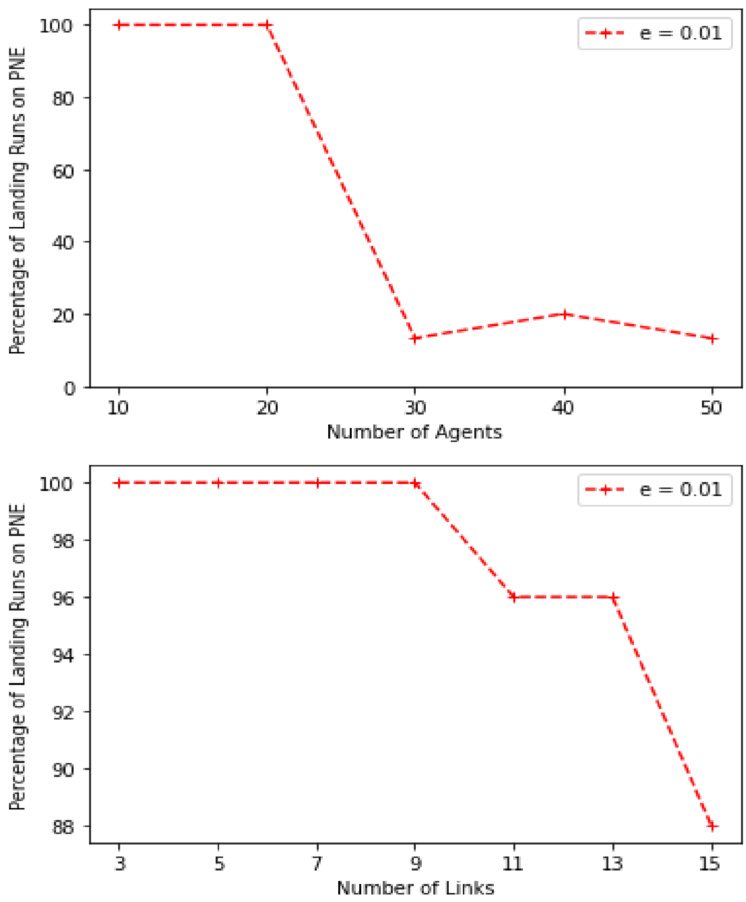

Figure 5.7: Percentages of runs that reach an equilibrium with different number of links in a regular graph on the left. And, percentages of runs that reach an equilibrium with different number of agents in a 5-regular graph on the right.

having more links increases the interdependency between the nodes. Indeed, if each node is just connected with 5 other nodes, if nothing changes in those 4 also the action of the agent in that node remains the same, whereas if it is connected to 15 it is more likely that a change in a node will make her change her choice. Thus, it seems reasonable to consider *highly inter-connected graphs* quite more instable. In addition, we can also have a closer look at how the internal dynamics of a run is structured. We can say that the simulation spends 'most of its time' in profiles where the number of voters stays always in a certain interval. See Figure 5.8 for an illustration.

Depending on the number of runs, we are either able to find an equilibrium or not. Indeed, by increasing the length of the runs you can increase the likelihood

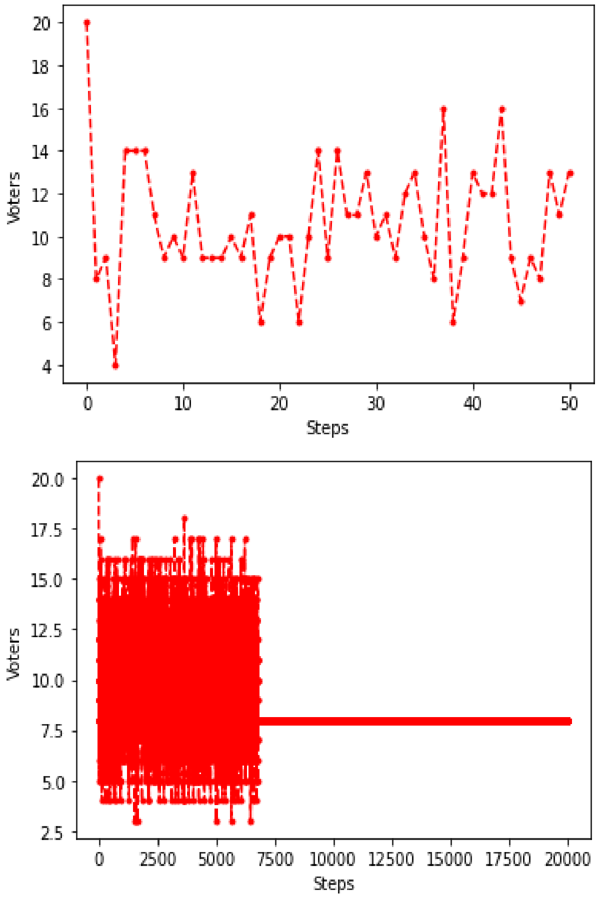

Figure 5.8: On the above plot you can see how the number of voters at each step changes. Clearly it has not reached the equilibrium yet. On the plot below you can see the same simulation after 20000 steps. Around step number 7000 the simulation reached an equilibrium.

of finding an equilibrium if the equilibrium is in a certain set of data (mainly that in which the simulation 'spends most of its time').

At the same time, of course, the number of steps influences quite heavily the computation time. For this reason, from now on we run every simulation for 100000 steps. By doing so, and by mainly employing simulations with a number of agents around 20, we can be sure that almost always our runs land on an equilibrium. From now on, every time we make an experiment with certain parameters to assess a certain value, we run the same simulation 50 times, and we extract this value by averaging on all the simulations that reached an equilibrium (almost all of them everytime). Indeed, as we stated before, we are mainly interested in looking at the results at the equilibria.

To validate the results we obtain with our computer simulation we run some initial simulations on complete networks and we compared our results with those we obtained analytically. Indeed, our analytical results of the first three chapters can be considered as if the graph was a complete network. The results matched.

5.4.3 More Connections Less Voters

As we mentioned, networks play a role in the size of the equilibria. We briefly discussed this matter taking into account the analytical proofs. Yet, what about more complex networks? We show by means of our simulation that the more the links in a network the lower becomes the number of voters at the equilibrium.

In order to do so, we prepare a simulation with a decision structure with $N = \{1, ..., 20\}$, $p = 0.7$, $e = 0.1$ on a regular graph with ℓ links, where $\ell \in \{3, 5, 7, 9, 11, 13, 15, 17, 19\}$. In order to be sure that such result is not dependent on the competence level, we repeat the same experiment for another structure $D = \langle\langle N, E\rangle, 0.8, e\rangle$. As you can see from Figure 5.9, the higher is the number of links the lower the number of voters becomes. Such a mechanism has quite a straightforward explanation. The basic concept depends indeed on Theorem 2.1 and Observation 2.1. For a decision structure $D = \langle N, e, p\rangle$ we have that the maximal equilibrium (n_{eq}) is reached at 15 people voting. However, we also have equilibria at $n_1 = 1, 3, ..., 13$. Now imagine that there are only 5 people voting in the entire graph. What is the chance that one of the agents that is not voting sees one of them? The answer depends on the number of links. If the number of links is high, then it is also likely that each agent of the graphs sees at least one voter. And, that configuration is likely to be an equilibrium. If the number of links is low, it is quite unlikely. Thus, in the second case 5 will not be an equilibrium, as those who do not see any voter and are not voting themselves would want to deviate. And, the same argument can be repeated for 7 voters, 9 voters and so on. However, as it can be seen, the difference is mainly between 10 and 19 number of links. Indeed, when the number of links is lower than ten the differences between the numbers of voters are not so significant.

For a decision structure $D = \langle N, e, p\rangle$ like the one we just saw, the maximum social welfare is reached at 19 voters, and as we know, the function for the social welfare is convex.[1] This means that the higher the number of voters the higher the social welfare. And, consequently, the lower the number of links, the higher the social welfare. And something similar can be said for $D = \langle\langle N, E\rangle, 0.8, e\rangle$.

[1] See Chapter 2 for an explanation of this point.

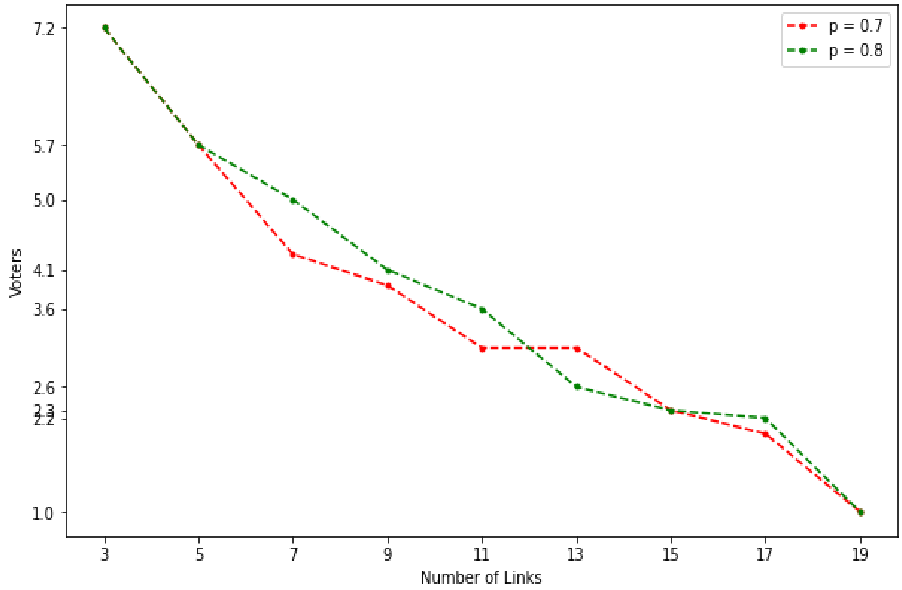

Figure 5.9: On the x-axis you can see the different number of links of each graph and on the y-axis you can see the average final number of voters for simulations with that number of links and two different levels of competences ($p \in \{0.7, 0.8\}$).

5.4.4 Those Who Care and Those Who Do not, in a Simulation

In Chapter 3, we introduced the possibility for agents to have different stakes at play. We may wonder what happens if we introduce different stakes on network games as well. We mainly try to answer the following questions.

1. If the average value for the stakes increases does also the number of voters increase?
2. Are the people voting those with, generally, higher stakes than the average?

Indeed, these two were the important takeaways of Chapter 3. Not surprisingly, the answer seems to be yes to both the questions. The way we test these hypotheses is by assigning to each agent in the simulation a value for the stakes that it is drawn from a normal distribution with center in a certain value $\bar{b}$.

The simulation is built in the exact same way we saw in Subsection 5.4.1. However this time each agent i is assigned with three different values p, e, b_i, and, consequently, the utility of each agent is so defined:

$$u_i(\boldsymbol{a}) = b_i(M(d_1(\boldsymbol{a}, i), p)) - a_i e. \tag{5.2}$$

Thus, the action choice is computed starting from this formula.

The More People Care the More People Vote. To test the answer to the first question we set up simulations with 20 agents on a 7-regular graph, with $\overline{b} \in \{10, 20, 30, 40, 50, 60\}$ and $e = 0.49$. For each value of $\overline{b}$ we run 50 simulations for 100000 steps and we extract a value for the final number of voters. Thus, for each value $\overline{b}$ we average over the values for n_1 of the simulations that reached an equilibrium (almost all of them) and we obtain a precise value n_1 for it. Again, we repeat the experiment for $p = 0.8$. Quite interestingly, the higher is the value for $\overline{b}$, the higher is the value for n_1, as it is shown in Figure 5.10. This is in line with what we found about these kind of games in complete networks analytically. And, of course, it is also quite in line with our intuitions about the process. Since the agents have higher stakes, we expect them to be more inclined to vote and make the effort.

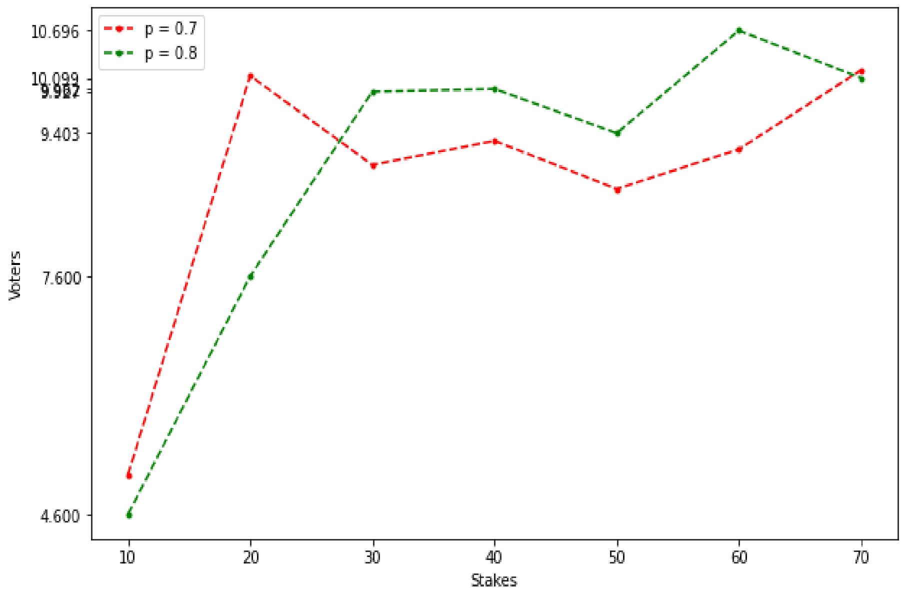

Figure 5.10: On the x-axis there are the different values for b and on the y-axis the average number of voters at the equilibrium for simulations with the respective bs.

The Voters Care. To test the answer to the second question we set up simulations with 20 agents on an ℓ-regular graph, with $\overline{b} = 50$, $e = 0.49$ and with $\ell \in \{7, 11, 15, 19\}$. For each value of ℓ we run 50 simulations for 100000 steps and we extract a value for $\overline{b}_1$, i.e. the average stakes of the people that are voting, for each of them. Thus, for each value ℓ we average over the values for $\overline{b}_1$ of the simulations that reached an equilibrium and we obtain a precise value $\overline{b}_1$ for it. Quite interestingly, these values were all higher than the mean ($b = 50$), as it is

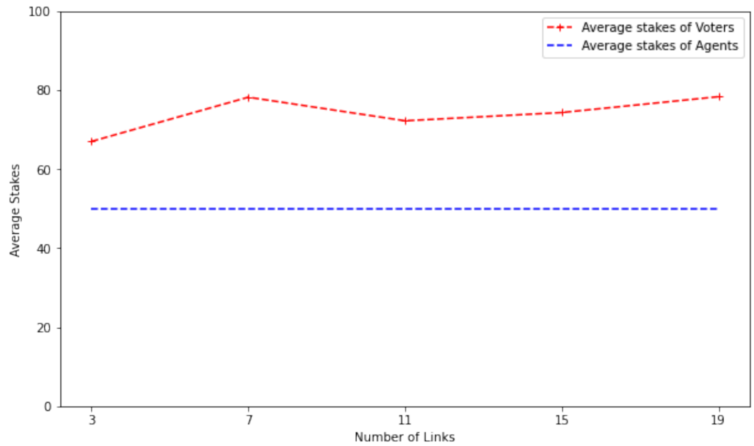

Figure 5.11: On the x-axis you can see the number of links of each network. On the y-axis the average stakes of the voters (red) and of the agents in general (blue).

shown in Figure 5.10.

This seems to be in line with our intuitions and our analytical results. In fact, it is reasonable to expect people with high stakes to vote. To get a better grasp of the motivation for this we can look at the example.

Example 5.7. *Assume that two agents i and j are both connected with the same five other agents. Assume that $a_i = a_j = 0$ and that four out of the other five agents vote. Let $b_i > b_j$ such that $b_i(M(5,p) - M(4,p)) > e > b_j(M(5,p) - M(4,p))$. Thus, we have that a_i would deviate going for action 1, whereas a_j would not.*

5.4.5 Let the Coalitions Move!

The structure of the simulation we used so far is such that each agent chooses her best option and updates it by herself. This is the typical approach of a game theoretic analysis that looks into pure Nash equilibrium. However, as we argued in Chapter 2, looking at how coalitions may move is equally important for our setting. Indeed, in Chapter 2 we found that the only possible SPNE are the profiles that are mPNE, but it is not always the case that an SPNE profile exists. Here, we extend our simulation to also include the possibility for coalitions to deviate (see Chapter 2.2). The main aim of such an extension is to show that it is almost impossible to find a strong equilibrium for games on network through our simulation. Consequently, the structure of the simulation becomes the following.

1. At an odd step, agents have the possibility of updating their choice individually in the same way we saw before (random activation).

2. At an even step, coalitions are allowed to deviate, i.e. the agents in the coalitions may update their actions at the same time.

   (a) At random a certain coalition K of agents that are not voting is chosen, and they evaluate if they would prefer to vote.

   (b) Each agent i evaluates the function u_i in a new profile defined following the rules for Definition 2.3. This means that he compares $u_i(\boldsymbol{a})$ and $u_i(\boldsymbol{a}_K, \boldsymbol{a}_{N\setminus K})$, i.e. whether she would prefer to switch to 1 if also all the other members switched to 1.

   (c) If for all agents $i \in K$, $u_i(\boldsymbol{a}_K, \boldsymbol{a}_{N\setminus K}) > u_i(\boldsymbol{a})$, all the agents in the coalition switch to 1.

   (d) Then another coalition K' is chosen...

Notably, our simulation does not take into account the possibility for coalitions that include non-voters to move. Indeed, this choice has been made for two reasons. First of all, being able to restrict the set of possible coalitions allowed us to speed up our simulation. Secondly, as we saw in Chapter 2, there is no reason for a coalition formed only by voters in $\boldsymbol{a}$ to deviate together if nobody would deviate alone from $\boldsymbol{a}$ (Lemma 2.2). So, we do not consider coalitions of just voters, as they are not needed to find an equilibrium. At the same time we can be sure that if a coalition K that deviates is formed both by voters and by non-voters, then there also exists a coalition K' formed just by voters that would want to deviate (Lemma 2.1). And, so in order to find an equilibrium we can restrict our search to coalitions of non-voters deviating to action 1.

Now, to get an idea of how such a simulation behaves we can have a look at the plots in Figure 5.12. The decision structure that is taken into account is $D = \langle G, e, p\rangle$ with $N = \{1, ..., 20\}$, $p = 0.7$, $e = 0.01$, and G being a 7-regular graph. As you can see, when the agents update individually the resulting number of voters is always quite low (odd steps). Instead, when agents are allowed to update together, the number of voters reaches 20 quite often. The underlying mechanism is the following: if all the voters were to update in coalitions they would all vote as they would all benefit, but then in the situation where they all vote, everybody realizes that her contribution is not really needed and she ‘backs off’. Indeed, such dynamics is continuously at work and it seems that such game does not have an equilibrium. Although we cannot be sure completely of this result, as our tool is just computational, it is true that our simulation does not find any stability and that the process seems to be quite periodical. We can compare two long runs for the same decision structure, allowing agents to update in delegations in the first case and not allowing it in the second in Figure 5.13. Remarkably, in the case with coalitions no stability is reached.

This result seems to be quite in line with what we proved in Theorem 2.2. Although, it is easy for a group to find a PNE, it is not easy for a group to find a SPNE as the conditions are more strict. Indeed, in any situation either single agents may want to deviate from 1 to 0 (this is the case for odd steps in Figure 5.12) or the 'big coalition' is not satisfied (even steps in Figure 5.12).

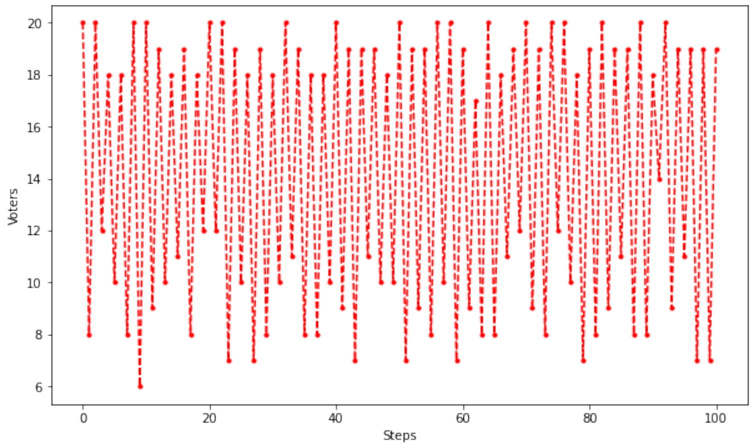

Figure 5.12: A run of a simulation with coalitions allowed to move. On the x-axis the steps and on the y-axis the number of voters at the end of each step.

5.5 Summary, Discussion and Takeaways

This chapter presents many interesting results and at the same time has a great potential for an expansion.

Discussion of the Results. First of all, it gives an idea of what impact networks could have on certain voting situations. Although quite a lot has been said about how networks influence the social dynamics of groups, the literature about the interaction between voting theory and networks is not so vast. Indeed, we discussed some of the works that go in this direction in the section about related literature in Chapter 1. In particular, in this case the impact of networks is quite relevant. Thanks to our simulation, we showed how in regular graphs the higher is the number of links and the lower is the number of voters on average at an equilibrium. This is extremely interesting as it a mechanism that is completely independent from the value of the effort. Although our model is still quite idealistic, it provides us with an indication of what kinds of network are more efficient in terms of accuracy. It seems quite reasonable to assume that groups that are poorly connected will probably be quite accurate. This finding may even be regarded as a possible explanation of the *voter paradox* (Aidt, 2000). A rational agent would not vote if she had full information about the entire group. Instead, she may be willing to vote (which is indeed what usually happens for a large part of citizens in general elections, for example) if she did not have full information. At the same time our analytical results warn us about another feature: centralized networks may produce very different equilibria. This is what we saw in Proposition 5.1. This is of course, extremely risky, because the overall accuracy at an equilibrium may fluctuate quite relevantly. In addition, through our simulation we also made sure that some of the results we proved in Chapter 3 still hold on networks. Indeed, regardless of the number of links the people more interested are always those who vote. Eventually, also looking at the coalition dynamics is highly rewarding, as it exemplifies a mechanism that we just sketched in Chapter 2.

Future work. Eventually, we shall have a look at which are the possible directions of research that can be taken starting from the results of this chapter. A natural step is, of course, to extend the results we obtained in the first section to other types of graphs. A theorem taking into account a more general category of graphs may definitely be useful for the field. And, for example, as we mentioned in Section 5.3 the concept of vertex cover may be fundamental. On the other hand, once the simulation is set, it is quite easy to assess the influence of different factors on the IBRD process. For example, allowing agents to delegate, following the structure described in Chapter 4 may shed some light on how the creation of equilibria goes in that case. Similarly, it may also be useful to look at other types of networks to see if the general link between the connectedness of a graph and the number of voters at an equilibrium holds also there.

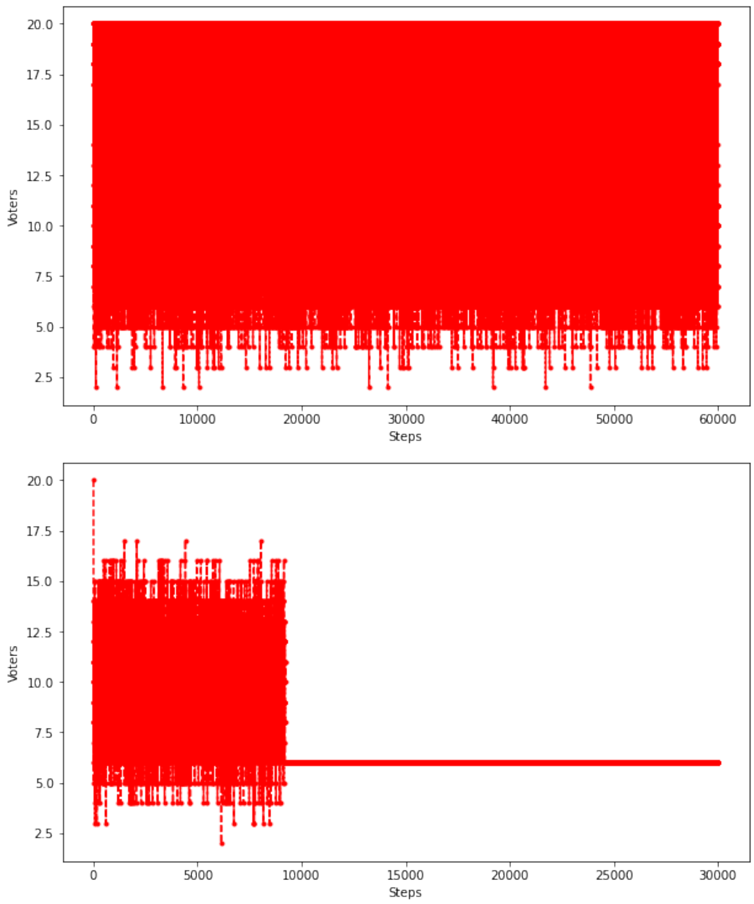

Figure 5.13: Two runs of the same decision structure with agents allowed to deviate in coalitions in the plot above and not allowed to do so in the plot below. Notably, the number of the steps above is two times the number of steps in the plot below as, like this, the number of ‘individual’ steps is the same. In the plot above the simulation never reaches a stability, and you can see the number of voters keeps fluctuating. In the plot below the simulation reaches the stability after around 7000 steps.

6 Conclusions

In this thesis we presented a systematic approach to the problem of costly information in group decision making. We started from de Condorcet's framework and by introducing the notion of effort we accounted for the fact that agents cannot become competent without paying any cost.

Summary of the results. In Chapter 2, we characterized the (weak and strong) equilibria of this game and we showed through our analytical results which claims can be made about these kinds of situations. We proved that larger groups are not always better at taking decisions (although they are never worse). And, similarly, we proved that the level of competence that agents can reach is relevant for the accuracy of the group, regardless of its size. Eventually, we also explained the difference between the 'individual perspective' and the 'collective one' and we showed that a group is more efficient in terms of social welfare when it has fewer members.

In Chapters 3 and 4 we generalized some of these initial results and we looked at the impact of two possible variations on the theorems of Chapter 2: the possibility for agents of having different stakes in the matter and of reaching different levels of competences. In both the frameworks we characterized the equilibria and we highlighted the effects of these generalizations on the results of Chapter 2. We proved that in the case of different stakes the only way to reach a maximal PNE is to have all the agents with the higher stakes voting. And, using similar formal results we also laid the ground for a comparison in terms of social welfare of different allocations of stakes. In addition, we showed that in the case of different competences there can exist strong equilibria with different sizes. Indeed, such results can be considered an integration of those obtained in Chapter 2.

Eventually, in Chapter 5 we introduced the notion of social networks. We used them to account for the possibility of agents of not being aware of other people's choices and we characterized our equilibria on some categories of graphs. We showed that highly centralized types of graphs are likely to produce quite 'extreme' results in terms of number of voters, as either almost everybody votes or just one agent does so. Instead, regular graphs seem to present more diverse equilibria. Again, we looked at how the effort influenced the agents' behaviour. Eventually, we complemented our analytical results on graphs with the study of a simulation of best-response dynamics. With the help from that we consolidated some of the results we obtained in Chapters 2, 3 and 4 on networks. In addition, we also showed how the degree of connectivity on regular graph increases the number of voters at equilibria and consequently the overall group accuracy.

Besides these more specific results, this thesis makes two main contributions. First of all, it studied the truth tracking performance of groups where agents are also allowed to completely abstain and free-ride. This gives the reader a clear idea of the basic features of many situations of decision making in groups: equilibria, social welfare, price of anarchy. Our approach provides mostly normative insights: following some of the results groups can be designed so to minimize negative effects, e.g. a too high price of anarchy. In addition, our this also revealed the importance of what we can call the *odd-even dynamics*, i.e. the fact that being odd is often a requirement for the size of equilibria (see, for example, Corollary 2.1 or Corollary 3.1). Indeed, 'being the fifth wheel usually pays off' (at least more than being the fourth one). Secondly, this thesis, in Chapter 5, puts together our simple perspective on voting theory and a study of social networks. As we mentioned in the introduction, this is a field which is quickly becoming more and more important and we believe that with this work we made a little step further. For example, the odd-even dynamics becomes even more interesting if studied on a network (see, for example, Corollary 5.1).

Lastly, as we highlighted in many of the final sections of chapters, this thesis has the virtue of opening up quite a good number of new possible directions of research. In each of the chapters we already took into account the more specific extensions for a certain setting: we dedicate the next paragraph to briefly discuss the more general ones.

Future Work. There are many possible ways in which our framework can be expanded and our results generalized. First of all, it seems natural to work a bit more in order to find a complete proof for Conjecture 4.1. Indeed, we are quite confident that such a proof exists and although we did not have enough time to work on it properly, we believe it can be found. Then, once that is settled down we may explore formally what are the consequences of it.

Secondly, it is reasonable not to assume that each agent has full information of other agents' choices and features. Each agent may be assigned a certain probability of knowing other people's actions and similarly she may be able to see other people's levels of competence with a certain imprecision. This may lead, for example, to an increase of the number of voters. Indeed, if an agent does not know either the competences or the actions of the other agents with certainty, she may be more inclined to make an effort to do her part.

A second possible direction consists in extending the study on networks. This can be done in two possible ways. One possibility is to follow the direction we already sketched here and try to prove more general results on network and at the same time make a more diffused use of simulations. As we mentioned in Chapter 5 there are some intuitions that seem promising in the study of the odd-even dynamics of graphs and that can be developed further in order to account for more general cases. Instead, the second possible direction would require a little change of perspective. Indeed, it would be extremely interesting to study what would happen in case another type of heuristics would have been chosen. What if agents follow other agents' behaviour, instead of trying to counterbalance it in the way we assumed here? A comparison between possible ways of coping with imperfect information on a network represents a necessary step in this framework.

Eventually, a third line of research is that of unifying the three different variations we introduced in Chapter 3, 4 and 5 and, with the help of a simulation, look at more complex interactions in this setting. In addition, such a framework would be highly suited for an investigation of the role of delegation. As we highlighted in Chapter 4, a system like liquid democracy has one of its strengths exactly in the fact that it allows citizens to freely choose what to do at each vote. Indeed, the agent may be willing to spend effort in a decision or not. And, consequently, to evaluate the efficiency of such a system is very important to consider at the same time the overall accuracy of a group and the effort that each agents makes.

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