PROJECTIVE ABSOLUTENESS UNDER SACKS FORCING

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Abstract. We show that $\mathbf{\Sigma}^1_3$-absoluteness under Sacks forcing is equivalent to the Sacks measurability of every $\mathbf{\Delta}^1_2$ set of reals. We also show that Sacks forcing is the weakest forcing notion among all of the preorders which always add a new real with respect to $\mathbf{\Sigma}^1_3$ forcing absoluteness.

1. Introduction

In this paper we will concentrate on forcing absoluteness. Forcing absoluteness connects forcing with descriptive set theory, which is one of the main areas of set theory.

Forcing, which was introduced by Cohen [Coh63, Coh64], is a useful method to construct models of ZFC. He used it to show the independence of CH (Continuum Hypothesis) from ZFC (Zermelo-Fraenkel set theory with the Axiom of Choice) and that of AC (Axiom of Choice) from ZF (Zermelo-Fraenkel set theory). By using forcing, many propositions have been proved to be independent from ZFC or consistent with ZFC.

In forcing we use a transitive model M of ZFC (called a ground model), a preorder $\mathbb{P}$ in M (called a forcing notion), and a filter G of $\mathbb{P}$, which is generic in a certain sense (called a $\mathbb{P}$-generic filter over M), to construct the transitive model $M[G]$ of ZFC (called a generic extension). $M[G]$ is the smallest transitive model of ZFC such that $M \subset M[G]$ and $G \in M[G]$. Furthermore $M[G]$ is a model on whose properties we have some control in the ground model. These facts are called the generic model theorem and the forcing theorem respectively.

Since properties of generic extensions mainly depend on the combinatorial properties of the corresponding forcing notions, many forcing notions have been investigated. Typical examples are Cohen forcing, Hechler forcing, random forcing, amoeba forcing, and Sacks forcing (denoted by $\mathbb{C}$, $\mathbb{D}$, $\mathbb{B}$, $\mathbb{A}$, and $\mathbb{S}$ respectively). Generic filters of these

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forcing notions (except amoeba forcing) can be seen as reals. Such reals are called Cohen reals, Hechler reals, random reals, and Sacks reals respectively. In this way these forcing notions have something to do with reals. This is one of the reasons why forcing is a very useful method in descriptive set theory.

In descriptive set theory, we investigate the properties of 'definable' sets of reals. Usually we work on the real line or similar spaces. Our interest in subsets of such spaces is mainly in Borel sets, or in the more complicated ones called projective sets. Projective sets form the hierarchy called the projective hierarchy, which is defined with respect to their complexity (See Theorem 2.12). This hierarchy consists of the classes $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, and $\mathbf{\Delta}^1_n$, where n is a natural number. We concentrate on regularity properties of sets of reals. Regularity properties are 'nice' properties: a set with such a property can be approximated by a Borel set in some sense. Typical examples are the Baire property, Lebesgue measurability, and Sacks measurability. While a Lebesgue measurable set can be approximated by a Borel set in the measure-theoretical sense, a set with the Baire property can be approximated by a Borel set in the topological sense. A Sacks measurable set can be locally approximated by a perfect set (a closed set without isolated points).

Soon after the invention of forcing, it was realized that certain 'simple' statements were absolute between a ground model and a generic extension (i.e. the truth values of these statements were the same between them). Forcing absoluteness is this type of absoluteness between a ground model and a generic extension. More precisely, for a ground model M, a preorder $\mathbb{P}$, and a class of formulas Φ, M is Φ-$\mathbb{P}$-*absolute* if for any sentence ϕ in Φ with some parameters and any $\mathbb{P}$-generic filter G over M,

$$\phi \text{ is true in } M \iff \phi \text{ is true in } M[G].$$

If a statement is absolute under some forcing extension, we can use it to prove the statement by showing that it is true in the generic extension instead of the ground model. This is a useful method to prove theorems. This is why forcing absoluteness has been investigated for many years.

We can find a close relationship between forcing absoluteness under the above forcing notions and the above regularity properties. The following are typical examples:

Theorem 2.94 ([JS89, BJ95])**.**
The following are equivalent:

(1) $\mathbf{\Sigma}^1_3$-$\mathbb{C}$-absoluteness holds.
(2) Every $\mathbf{\Delta}^1_2$ set of reals has the Baire property.

(3) For any real x, there exists a Cohen real over $L[x]$.

Theorem 2.95 ([JS89, BJ95])**.**
The following are equivalent:

(1) $\mathbf{\Sigma}^1_3$-$\mathbb{B}$-absoluteness holds.
(2) Every $\mathbf{\Delta}^1_2$ set of reals is Lebesgue measurable.
(3) For any real x, there exists a random real over $L[x]$.

Theorem 2.96 ([Sol69, Jud93, BL99])**.**
The following are equivalent:

(1) $\mathbf{\Sigma}^1_3$-$\mathbb{D}$-absoluteness holds.
(2) Every $\mathbf{\Sigma}^1_2$ set of reals has the Baire property.
(3) For any real x, $\{c \mid c$ is a Cohen real over $L[x].\}$ is comeager.
(4) For any real x, there is a Hechler real over $L[x]$.

Theorem 2.97 ([Sol69, Jud93])**.**
The following are equivalent:

(1) $\mathbf{\Sigma}^1_3$-$\mathbb{A}$-absoluteness holds.
(2) Every $\mathbf{\Sigma}^1_2$ set of reals is Lebesgue measurable.
(3) For any real x, $\{r \mid r$ is a random real over $L[x].\}$ is of Lebesgue measure one.

Theorem 2.98 ([BJ95])**.**

(1) If $\mathbf{\Sigma}^1_4$-$\mathbb{D}$-absoluteness holds, then every $\mathbf{\Sigma}^1_3$ set of reals has the Baire property.
(2) Let n be a natural number with $n \geq 4$. If $\mathbf{\Sigma}^1_{n+1}$-$\mathbb{D}$-absoluteness and $\mathbf{\Sigma}^1_n$-$(\mathbb{D} * \mathbb{D})$-correctness hold, then every $\mathbf{\Sigma}^1_n$ set of reals has the Baire property.

Theorem 2.99 ([Bre93, BJ95])**.**

(1) If $\mathbf{\Sigma}^1_4$-$\mathbb{A}$-absoluteness holds, then every $\mathbf{\Sigma}^1_3$ set of reals is Lebesgue measurable.
(2) Let n be a natural number with $n \geq 4$. If $\mathbf{\Sigma}^1_{n+1}$-$\mathbb{A}$-absoluteness and $\mathbf{\Sigma}^1_n$-$\mathbb{A}$-correctness hold, then every $\mathbf{\Sigma}^1_n$ set of reals is Lebesgue measurable.

Uniformization is also important in considering the relationship between forcing absoluteness and regularity properties. Uniformization is a property of pointclasses, sets of sets of reals with certain properties. The uniformization property for a pointclass Γ is a choice principle as follows: any relation in Γ has a choice function also in Γ. By using the uniformization property for a part of the projective hierarchy, we can turn relations into functions without increasing the complexity of the objects.

The following is a typical example of the connection between uniformization, forcing absoluteness, and regularity properties:

Theorem 2.100 ([Woo82]).
Let n be a natural number with $n \geq 1$. Assume that $\mathbf{\Pi}^1_{2n-1} \restriction {}^\omega 2 \times {}^\omega 2$ has the uniformization property. If every $\mathbf{\Delta}^1_{2n}$ set of reals has the Baire property, then $\mathbf{\Sigma}^1_{2n+1}$-$\mathbb{C}$-absoluteness holds.

The following are our results:

Main Theorem 4.1.

(1) The following are equivalent:
  (a) $\mathbf{\Sigma}^1_3$-$\mathbb{S}$-absoluteness holds.
  (b) Every $\mathbf{\Delta}^1_2$ set of reals is Sacks measurable.
  (c) Every $\mathbf{\Sigma}^1_2$ set of reals is Sacks measurable.
  (d) For any real r, there is a real x such that x is not in $L[r]$.

(2) Suppose that $\mathbb{P}$ is a preorder which always adds a new real. Then $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness implies $\mathbf{\Sigma}^1_3$-$\mathbb{S}$-absoluteness.

Theorem 4.2.
Let n be a natural number with $n \geq 1$.

(1) If $\mathbf{\Sigma}^1_{n+1}$-$\mathbb{S}$-absoluteness holds, then every $\mathbf{\Delta}^1_n$ set of reals is Sacks measurable.

(2) Assume that $\mathbf{\Pi}^1_{2n-1} \restriction {}^\omega 2 \times {}^\omega 2$ has the uniformization property. If every $\mathbf{\Delta}^1_{2n}$ set of reals is Sacks measurable, then $\mathbf{\Sigma}^1_{2n+1}$-$\mathbb{S}$-absoluteness holds.

Note that the equivalence of (b), (c), and (d) of (1) in Main Theorem was already proved by Brendle and Löwe [BL99].

(1) in Main Theorem is an analogy of Theorems 2.94–2.97. (2) in Main Theorem states that Sacks forcing is the weakest forcing notion among all of the preorders which always add a new real with respect to $\mathbf{\Sigma}^1_3$ forcing absoluteness. (1) in Theorem is an analogy of Theorems 2.98 and 2.99. (2) in Theorem is an analogy of Theorem 2.100.

This paper consists of four sections. In the second section we will look at the definitions of and elementary facts about the basic notions. In the third section we will list some facts used in the proofs of our results. In the last section we will prove Main Theorem and Theorem.

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2. Basic Concepts

From now on, we will work in ZFC. We assume that readers are familiar with the elementary theories of forcing, general topology, and measure theory. (For basic definitions we will not mention, see [Kun80], [Kel75], and [Hal74].) Also, we follow from [Kun80] about basic notations. For example, ω is the set of natural numbers containing 0 as well as the least infinite ordinal. ω_1 is the least uncountable cardinal. We identify a natural number n with $\{0, \cdots, n-1\}$.

2.1. Borel sets, projective sets, the definability under second-order arithmetic, and Borel codes.

Notation 2.1.

- For a set X and a natural number m, X^m is the set of all m-tuples of elements of X. In particular, $X^0 = \{\emptyset\}$ where $\emptyset$ is the empty sequence.
- Let X be a set. ${}^{<\omega}X$ is the set of all finite sequences in X. Therefore,
$$^{<\omega}X = \bigcup_{m \in \omega} X^m.$$
- For sets X, Y, ${}^{X}Y$ is the set of all functions from X to Y.
- Let X, Y be sets and f be a function from X to Y.
  - $\mathrm{pr}_1 \colon X \times Y \to X$ is the first projection.
  - If f is injective, f^{-1} is the inverse function of f.
  - For a subset A of X, $f``A$ is the image of A by f.
  - For a subset B of Y, $f^{-1}``B$ is the preimage of B by f.
- L is the smallest transitive model of ZFC that contains all the ordinals.
- For a set X, $L[X]$ is the smallest transitive model of ZFC that contains all the ordinals and $L[X] \cap X$ as elements.
- DC, called the dependent choice, is the following statement:
  For any set X and any subset R of $X \times X$,

  if
$$(\forall x \in X)(\exists x' \in X)\ (x, x') \in R,$$
  then
$$(\exists f \colon \omega \to X)(\forall n \in \omega)\ (f(n), f(n+1)) \in R.$$

- V is the class of all sets.

Remark 2.2.

Suppose that X is a set of ordinals. Then, $L[X]$ is the smallest transitive model of ZFC that contains all the ordinals and X as elements.

Definition 2.3.

Let X be a topological space.

(1) X is a *Polish space* if X is a separable, completely metrizable space.
(2) X is *perfect* if X has no isolated points.

Example 2.4.

(1) The real line $\mathbb{R}$ is a perfect Polish space.
(2) Topologize ω by the discrete topology and topologize ${}^{\omega}\omega$ by the product topology. Then ${}^{\omega}\omega$ is a perfect Polish space. This space is called *Baire space*.
(3) Topologize 2 by the discrete topology and topologize ${}^{\omega}2$ by the product topology. Then ${}^{\omega}2$ is a perfect Polish space. This space is called *Cantor space*.
(4) Suppose that X is a Polish space and Y is a closed subspace of X. Then Y is also a Polish space.
(5) Suppose that X, Y are Polish spaces and topologize $X \times Y$ by the product topology. Then $X \times Y$ is also a Polish space. Moreover, if X is perfect, so is $X \times Y$.
(6) Let X be a perfect Polish space and m be a natural number with $m \geq 1$ and topologize X^m by the product topology. Then X^m is also a perfect Polish space.
(7) Suppose that X is a compact metrizable space and Y is a Polish space. Let $C(X, Y)$ be the set of all continuous functions from X to Y. Fix a compatible complete metric ρ on Y. Topologize $C(X, Y)$ by the sup metric induced by ρ. Then $C(X, Y)$ is a Polish space.

Definition 2.5.

Let X be a set and $\mathcal{A}$ be a subset of $\mathcal{P}(X)$. $\mathcal{A}$ is a *σ-algebra on X* if $\mathcal{A}$ satisfies the following conditions:

(1) $\emptyset$, X are in $\mathcal{A}$.
(2) For any A in $\mathcal{A}$, $X \setminus A$ is also in $\mathcal{A}$. Hence $\mathcal{A}$ is closed under complements of X.
(3) For any $\{A_n \in \mathcal{A} \mid n \in \omega\}$, $\bigcup_{n \in \omega} A_n$ is also in $\mathcal{A}$. Hence $\mathcal{A}$ is closed under countable unions.

Definition 2.6.

Let X be a topological space. The set of all σ-algebras on X containing all open sets in X is closed under arbitrary intersections. Therefore, the set

$$\mathbf{B} \restriction X = \bigcap\{\mathcal{A} \mid \mathcal{A} \text{ is a } \sigma\text{-algebra on } X \text{ containing all open sets in } X.\}$$

is the smallest σ-algebra containing all open sets in X. Elements of $\mathbf{B} \restriction X$ are called *Borel subsets of* X.

Definition 2.7.

Let X, Y be topological spaces.

(1) A function $f \colon X \to Y$ is a *Borel function* if for any Borel subset P of Y, f^{-1}“P is a Borel subset of X.
(2) A function $f \colon X \to Y$ is *Borel isomorphic* if f is bijective and both f and f^{-1} are Borel functions.

Remark 2.8.

Let X, Y be topological spaces and f be a function from X to Y. Then f is a Borel function iff for any open subset P of Y, f^{-1}“P is a Borel subset of X. Therefore, every continuous function is a Borel function.

The following theorem is basic and important:

Theorem 2.9 ([Kur58])**.**

Suppose that X, Y are perfect Polish spaces. Then there is a Borel isomorphic function f from X to Y. Therefore, the structure of the set of all Borel sets in a perfect Polish space is unique.

Definition 2.10.

Let X be a topological space and topologize X^m by the product topology for each natural number m with $m \geq 1$.

We will define $\mathbf{\Sigma}^1_n \restriction X^m$, $\mathbf{\Pi}^1_n \restriction X^m \subset \mathcal{P}(X^m)$ by induction on $1 \leq n < \omega$ for all m with $1 \leq m < \omega$ simultaneously.

$$\begin{aligned} P \in \mathbf{\Sigma}^1_1 \restriction X^m &\overset{\text{def}}{\iff} \text{There is a Borel subset } Q \text{ of } X^m \times X \\ &\qquad \text{such that } P = \mathrm{pr}_1\text{“}Q. \\ P \in \mathbf{\Pi}^1_1 \restriction X^m &\overset{\text{def}}{\iff} X \setminus P \in \mathbf{\Sigma}^1_1 \restriction X^m. \end{aligned}$$

For $n \geq 2$,

$$
\begin{aligned}
P \in \mathbf{\Sigma}^1_n \restriction X^m &\overset{\text{def}}{\iff} \text{There is a } Q \in \mathbf{\Pi}^1_{n-1} \restriction X^m \times X \\
&\phantom{\overset{\text{def}}{\iff}} \text{ such that } P = \mathrm{pr}_1{}^{\text{“}}Q. \\
P \in \mathbf{\Pi}^1_n \restriction X^m &\overset{\text{def}}{\iff} X \setminus P \in \mathbf{\Sigma}^1_n \restriction X^m.
\end{aligned}
$$

Then, for $n \geq 1$,

$$P \in \mathbf{\Delta}^1_n \restriction X^m \overset{\text{def}}{\iff} P \in \mathbf{\Sigma}^1_n \restriction X^m \text{ and } P \in \mathbf{\Pi}^1_n \restriction X^m.$$

We refer to members of $\mathbf{\Sigma}^1_n \restriction X$ as $\mathbf{\Sigma}^1_n$ *subsets of* X. Elements of $\mathbf{\Pi}^1_n \restriction X$, $\mathbf{\Delta}^1_n \restriction X$ are called in the analogous way. Also, if $X = {}^{\omega}2$, we refer to members of $\mathbf{\Sigma}^1_n \restriction X$ as $\mathbf{\Sigma}^1_n$ *sets of reals*. Elements of $\mathbf{\Pi}^1_n \restriction {}^{\omega}2$, $\mathbf{\Delta}^1_n \restriction {}^{\omega}2$ are called in the analogous way.

The following theorems and proposition are basic:

Theorem 2.11 ([Sus17, LS18])**.**
Suppose that X is a perfect Polish space. Then

$$\mathbf{\Delta}^1_1 \restriction X = \mathbf{B} \restriction X.$$

Theorem 2.12 ([Lus25, Sie25])**.**
Suppose that X is a perfect Polish space. Then the following inclusions hold:

$$
\begin{tikzcd}
& \mathbf{\Sigma}^1_n \restriction X \arrow[dr, phantom, sloped, "\subsetneq"] & & \cdots \\
\mathbf{\Delta}^1_n \restriction X \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & \mathbf{\Delta}^1_{n+1} \restriction X \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & \cdots \\
& \mathbf{\Pi}^1_n \restriction X \arrow[ur, phantom, sloped, "\subsetneq"] & & \cdots
\end{tikzcd}
$$

where n is a natural number with $n \geq 1$. This hierarchy is called the *projective hierarchy*. We refer to sets in $\mathbf{\Sigma}^1_n \restriction X$ for some n as *projective sets*.

Proposition 2.13.
Let X, Y be perfect Polish spaces and f be a Borel isomorphic function from X to Y. (The existence of such a function is ensured by Theorem 2.9.)
Then for any natural number n with $n \geq 1$ and any subset P of X, P is in $\mathbf{\Sigma}^1_n \restriction X$ iff $f{}^{\text{“}}P$ is in $\mathbf{\Sigma}^1_n \restriction Y$.
Therefore, the structure of $\mathbf{\Sigma}^1_n \restriction X$ for a perfect Polish space X is unique.
Also, we can deduce the same results for $\mathbf{\Pi}^1_n \restriction X$, $\mathbf{\Delta}^1_n \restriction X$.

From now on, we will identify $\mathcal{P}(\omega)$ with ${}^{\omega}2$ in a canonical way. Also when we call a *real*, it is an element of Cantor space. (Usually, we mean an element of Baire space by a real. But for simplicity, we will work on Cantor space and there are no essential differences in the following arguments.)

Remark 2.14.
Let x be a real. By the above identification and Remark 2.2, $L[x]$ is the smallest transitive model of ZFC that contains all the ordinals and x as elements.

Definition 2.15.
Consider the *second-order arithmetic structure* $\mathcal{A}^2 = \langle \omega, \mathcal{P}(\omega), \in, +, \cdot, 0, 1 \rangle$, where $\in$ is the relation between ω and $\mathcal{P}(\omega)$, $+$ is the addition on ω, $\cdot$ is the multiplication on ω, and 0, 1 are the constants in ω.

(1) We mean an $\mathcal{A}^2$*-formula* by a second-order formula of the language of $\mathcal{A}^2$.
(2) Let ϕ be an $\mathcal{A}^2$-formula and n be a natural number with $n \geq 1$.
  (a) ϕ is *arithmetical* if ϕ has no second-order quantifiers.
  (b) ϕ is a Σ^1_n*-formula* if there exists an arithmetical formula ψ such that

$$\phi \equiv \overbrace{\exists^1 \alpha_1 \forall^1 \alpha_2 \cdots Q\alpha_n}^{n\,\text{quantifiers}} \psi$$

  where $\exists^1$, $\forall^1$ are second-order quantifiers and Q is $\exists^1$ if n is odd, otherwise $\forall^1$.
  (c) ϕ is a Π^1_n*-formula* if there exists an arithmetical formula ψ such that

$$\phi \equiv \overbrace{\forall^1 \alpha_1 \exists^1 \alpha_2 \cdots Q\alpha_n}^{n\,\text{quantifiers}} \psi$$

  where Q is $\forall^1$ if n is odd, otherwise $\exists^1$.

Therefore, a Σ^1_n formula has n alternate second-order quantifiers beginning from $\exists^1$ and a Π^1_n formula has n alternate second-order quantifiers beginning from $\forall^1$.

Notation 2.16.
From now on, we abbreviate "$\mathcal{A}^2 \vDash \phi(\vec{r})$" to "$\phi(\vec{r})$" for a finite sequence of reals $\vec{r}$.

Definition 2.17.

For natural numbers m, n with m, $n \geq 1$ and a finite sequence $\vec{r}$ in ${}^{\omega}2$,

$$P \in \Sigma^1_n(\vec{r}) \upharpoonright ({}^{\omega}2)^m \overset{\text{def}}{\iff} \text{There is a } \Sigma^1_n\text{-formula } \phi \text{ such that } P = \{\vec{x} \in ({}^{\omega}2)^m \mid \phi(\vec{x}, \vec{r})\}.$$

$$P \in \Pi^1_n(\vec{r}) \upharpoonright ({}^{\omega}2)^m \overset{\text{def}}{\iff} \text{There is a } \Pi^1_n\text{-formula } \phi \text{ such that } P = \{\vec{x} \in ({}^{\omega}2)^m \mid \phi(\vec{x}, \vec{r})\}.$$

$$P \in \Delta^1_n(\vec{r}) \upharpoonright ({}^{\omega}2)^m \overset{\text{def}}{\iff} P \in \Sigma^1_n(\vec{r}) \upharpoonright ({}^{\omega}2)^m \text{ and } P \in \Pi^1_n(\vec{r}) \upharpoonright ({}^{\omega}2)^m.$$

We refer to members of $\Sigma^1_n(\vec{r}) \upharpoonright ({}^{\omega}2)^m$ as $\Sigma^1_n(\vec{r})$ *subsets of* $({}^{\omega}2)^m$. Elements of $\Pi^1_n(\vec{r}) \upharpoonright ({}^{\omega}2)^m$, $\Delta^1_n(\vec{r}) \upharpoonright ({}^{\omega}2)^m$ are called in the analogous way.

Also, when $m = 1$, we refer to members of $\Sigma^1_n(\vec{r}) \upharpoonright ({}^{\omega}2)^m$ as $\Sigma^1_n(\vec{r})$ *sets of reals*. Elements of $\Pi^1_n(\vec{r}) \upharpoonright ({}^{\omega}2)^m$, $\Delta^1_n(\vec{r}) \upharpoonright ({}^{\omega}2)^m$ are called in the analogous way.

Notation 2.18.

For natural numbers m, n with m, $n \geq 1$ and a finite sequence $\vec{r}$ in ${}^{\omega}2$,

- $\mathrm{lh}(\vec{r})$ is the length of $\vec{r}$.
- For $i < \mathrm{lh}(\vec{r})$, r_i is the i-th coordinate of $\vec{r}$. Hence

$$\vec{r} = \langle r_0, \cdots, r_{\mathrm{lh}(\vec{r})-1} \rangle.$$

- If $\vec{r} = \langle r_0 \rangle$ (i.e. $\vec{r}$ is a sequence with $\mathrm{lh}(\vec{r}) = 1$), then $\Sigma^1_n(\vec{r}) \upharpoonright ({}^{\omega}2)^m$ is abbreviated to $\Sigma^1_n(r_0) \upharpoonright ({}^{\omega}2)^m$. $\Pi^1_n(\vec{r}) \upharpoonright ({}^{\omega}2)^m$, $\Delta^1_n(\vec{r}) \upharpoonright ({}^{\omega}2)^m$ are abbreviated in the same way.
- If $\vec{r} = \emptyset$ (i.e. the empty sequence), then $\Sigma^1_n(\vec{r}) \upharpoonright ({}^{\omega}2)^m$ is abbreviated to $\Sigma^1_n \upharpoonright ({}^{\omega}2)^m$. $\Pi^1_n(\vec{r}) \upharpoonright ({}^{\omega}2)^m$, $\Delta^1_n(\vec{r}) \upharpoonright ({}^{\omega}2)^m$ are abbreviated in the same way.

Remark 2.19.

A finite sequence of reals is coded by a real.

For natural numbers m, n with m, $n \geq 1$ and a finite sequence $\vec{r}$ in ${}^{\omega}2$ with $\mathrm{lh}(\vec{r}) > 0$, put $l = \mathrm{lh}(\vec{r})$ and define a bijective map $f \colon ({}^{\omega}2)^l \to {}^{\omega}2$ as follows:

$$f(\vec{x}) = y \ \text{ if } \ y(q \cdot l + i) = x_i(q)$$

where q, i are natural numbers with $i < l$.

Put $r' = f(\vec{r})$. Then,

$$\Sigma^1_n(r') = \Sigma^1_n(\vec{r}),\ \Pi^1_n(r') = \Pi^1_n(\vec{r}),\ \Delta^1_n(r') = \Delta^1_n(\vec{r}), \text{ and } L[r'] = L[\vec{r}].$$

The following proposition states the important relationship between projective sets and the definability under second-order arithmetic:

Proposition 2.20 ([Add59a]**).**
Let m, n be natural numbers with $m, n \geq 1$. Then

$$\begin{aligned}
\mathbf{\Sigma}^1_n \restriction ({}^\omega 2)^m &= \bigcup_{r \in {}^\omega 2} \Sigma^1_n(r) \restriction ({}^\omega 2)^m, \\
\mathbf{\Pi}^1_n \restriction ({}^\omega 2)^m &= \bigcup_{r \in {}^\omega 2} \Pi^1_n(r) \restriction ({}^\omega 2)^m, \\
\mathbf{\Delta}^1_n \restriction ({}^\omega 2)^m &= \bigcup_{r \in {}^\omega 2} \Delta^1_n(r) \restriction ({}^\omega 2)^m.
\end{aligned}$$

Definition 2.21.
Suppose that M, N are transitive models of ZF+DC with $M \subset N$. For a formula ϕ of the language of set theory, ϕ is *absolute between M and N* if for any $\vec{x} \in M$,

$$M \vDash \phi(\vec{x}) \iff N \vDash \phi(\vec{x}).$$

Theorem 2.22 ([Sol70]**).**
There is a Π^1_1 set of reals BC and a surjection $\pi \colon \mathrm{BC} \to \mathbf{B} \restriction {}^\omega 2$ such that for any transitive models M, N of ZF+DC with $M \subset N$, the following statements are absolute between them:

(1) c is in BC.
(2) c is in BC and x is in $\pi(c)$.
(3) c_1, c_2 are in BC and $\pi(c_1) = \pi(c_2)$.

In particular, if $c \in \mathrm{BC}^M$, then $c \in \mathrm{BC}^N$ and

$$\pi(c)^M = \pi(c)^N \cap M.$$

Elements of BC are called *Borel codes* and $\pi(c)$ is denoted by B_c for a Borel code c.

Therefore, we can consider a natural extension of Borel sets.

Definition 2.23.
Suppose that M, N are transitive models of ZF+DC with $M \subset N$ and B is a Borel set of reals in M.

$$B^N \overset{\text{def}}{=} B^N_c \text{ for any Borel code } c \text{ with } c \in M \text{ and } B^M_c = B.$$

By the above theorem, this definition is well-defined.

2.2. **Regularity properties.**

2.2.1. **The Baire property**.

Notation 2.24.

- For a topological space X and a subset P of X,
  - $\mathrm{Int}(P)$ is the interior of P in X.
  - $\mathrm{Cl}(P)$ is the closure of P in X.
- Let X, Y be sets.

$$X \triangle Y = (X \setminus Y) \cap (Y \setminus X).$$

  $X \triangle Y$ is called the symmetric difference between X and Y.
- Let S, T be sets of sentences of the language of set theory. $\mathrm{Con}(S)$ is the assertion that S is consistent. We say that S, T are equiconsistent if $\mathrm{Con}(S)$ is equivalent to $\mathrm{Con}(T)$.

Definition 2.25.
Let X be a set and I be a subset of $\mathcal{P}(X)$.

(1) I is an *ideal on* X if I satisfies the following conditions:
  (a) For any A in I and any subset B of A, B is also in I. Hence I is closed under subsets.
  (b) For any A, B in I, $A \cup B$ is also in I. Hence I is closed under finite unions.

(2) Let I be an ideal on X. I is a σ-*ideal* if for any $\{A_n \in I \mid n \in \omega\}$, $\bigcup_{n\in\omega} A_n$ is also in I. Hence I is closed under countable unions.

Usually, we consider an ideal on some set a set of small sets.

Definition 2.26.
For a topological space X and a subset P of X,

(1) P is *nowhere dense* if $\mathrm{Int}(\mathrm{Cl}(P)) = \emptyset$.
(2) P is *meager* if P is a union of a countable set of nowhere dense sets.
(3) P has the *Baire property* if there is an open subset O of X such that $P \triangle O$ is meager.
(4) P is *comeager* if $X \setminus P$ is meager.

Remark 2.27.
Suppose that X is a perfect T_1 space.

(1) Every singleton in X is nowhere dense.
(2) Every countable subset of X is meager. Moreover, the set of all meager sets in X is a σ-ideal on X. Therefore, meager sets are small sets and a set with the Baire property can be approximated by an open set in the topological sense.

(3) Every open subset of X has the Baire property. Moreover, the set of all subsets of X with the Baire property is the smallest σ-algebra containing all open subsets in X and all meager subsets of X. In particular, every Borel subset of X has the Baire property.

The following theorems are basic and important:

Theorem 2.28 ([Bai99])**.**
Let X be a completely metrizable space. Then every nonempty open set in X is not meager.

Theorem 2.29 ([Vit05, Ber08])**.**
Suppose that X is a perfect Polish space. Then there is a subset of X which does not have the Baire property.

Therefore, the Baire property cannot be trivial in a perfect Polish space.
Note that the last theorem needs the axiom of choice by the following theorem:

Theorem 2.30 ([She84, Rai84])**.**
Suppose that Con(ZFC) holds. Then Con(ZF + DC+ "Every set of reals has the Baire property.") holds.

The following theorems are important:

Theorem 2.31 ([LS23])**.**
Suppose that X is a perfect Polish space. Then every $\boldsymbol{\Sigma}^1_1$ subset of X has the Baire property.

Theorem 2.32 ([Göd38, Nov51])**.**
In L, "There is a Δ^1_2 set of reals which does not have the Baire property." holds. In particular, we cannot prove in ZFC that every Δ^1_2 set of reals has the Baire property.

2.2.2. **Lebesgue measurability**.

Notation 2.33.

- Let I be a set and $\langle (X_i,\, \mathfrak{B}_i,\, \mu_i) \mid i \in I \rangle$ be an I-sequence of probability spaces. $\left(\prod_{i \in I} X_i,\, \bigotimes_{i \in I} \mathfrak{B}_i,\, \bigotimes_{i \in I} \mu_i \right)$ is the completion of the product of $\langle (X_i,\, \mathfrak{B}_i,\, \mu_i) \mid i \in I \rangle$.
- $\mu_{\mathbb{R}}$ is Lebesgue measure on $\mathbb{R}$.
- For a set of ordinals X, sup$X$ is the supremum of X. Therefore, sup$X$ is the ordinal such that

  $$(\forall \alpha \in X)\ \alpha \leq \sup X \text{ and } (\forall \alpha < \sup X)\, (\exists \beta \in X)\ \alpha < \beta.$$

- Let X be a set. $|X|$ is the cardinality of X.
- Let κ be a cardinal.

$$2^\kappa = |{}^\kappa 2|.$$

Definition 2.34.
Consider the probability measure μ_2 on 2 such that

$$\mu_2(\{0\}) = \frac{1}{2},\ \mu_2(\{1\}) = \frac{1}{2}.$$

Put

$$\mu_{({}^\omega 2)} = \bigotimes_{n\in\omega} \mu_2.$$

This measure is also called the *Lebesgue measure on* ${}^\omega 2$.

(1) A subset P of ${}^\omega 2$ is *Lebesgue measurable on* ${}^\omega 2$ if P is in $\bigotimes_{n\in\omega} \mathcal{P}(2)$.
(2) A subset P of ${}^\omega 2$ is *null* if P is Lebesgue measurable on ${}^\omega 2$ and $\mu_{({}^\omega 2)}(P) = 0$.
(3) A subset P of ${}^\omega 2$ is of *Lebesgue measure one* if ${}^\omega 2 \setminus P$ is null.

The reason why we call $\mu_{({}^\omega 2)}$ Lebesgue measure is the following:

Proposition 2.35 (See [Lév02].)**.**
There is a Borel isomorphic function $\pi\colon {}^\omega 2 \to \mathbb{R}$ such that for any subset P of ${}^\omega 2$, P is Lebesgue measurable on ${}^\omega 2$ iff $\pi``P$ is Lebesgue measurable on $\mathbb{R}$.

By the above proposition, from now on, we will concentrate on the Lebesgue measurability on Cantor space.
The following theorem is analogous to Theorem 2.29:

Theorem 2.36 ([Vit05])**.**
There is a set of reals which is not Lebesgue measurable.

Similar to the Baire property, the above theorem needs the axiom of choice. For stating that, we see the definition of a strongly inaccessible cardinal.

Definition 2.37.
Let κ be an infinite cardinal.

(1) κ is *regular* if for any ordinal α with $\alpha < \kappa$ and any function $f\colon \alpha \to \kappa$, $\sup(f``\alpha) < \kappa$.
(2) Suppose that κ is uncountable.
κ is *strongly inaccessible* if κ is regular and for any cardinal γ with $\gamma < \kappa$, $2^\gamma < \kappa$.

A strongly inaccessible cardinal is a basic example of large cardinals. We cannot deduce Con(ZFC+"There is a strongly inaccessible cardinal.") from Con(ZFC).

Theorem 2.38 ([Sol70] and [She84, Rai84]).
The following are equiconsistent:

(1) ZFC+"There is a strongly inaccessible cardinal."
(2) ZF+DC+"Every set of reals is Lebesgue measurable."
(3) ZF+DC+"Every $\mathbf{\Sigma}^1_3$ set of reals is Lebesgue measurable."

By Theorem 2.30, 2.38, we can find one of the differences between the Baire property and Lebesgue measurability.

The following theorems are analogous to Theorem 2.31 and Theorem 2.32 respectively:

Theorem 2.39 ([Lus17]).
Every $\mathbf{\Sigma}^1_1$ set of reals is Lebesgue measurable.

Theorem 2.40 ([Göd38, Nov63]).
In L, "There is a Δ^1_2 set of reals which is not Lebesgue measurable." holds. In particular, we cannot prove in ZFC that every Δ^1_2 set of reals has the Baire property.

2.2.3. **Bernstein sets and Sacks measurability**.

Notation 2.41.

- Let X, Y be sets, f be a function from X to Y, and A be a subset of X. Put
$$f \restriction A = \{(x, f(x)) \in X \times Y \mid x \in A\}.$$
$f \restriction A$ is called the restriction of f to A.
- For s in ${}^{<\omega}2$, put
$$N_s = \{x \in {}^{\omega}2 \mid s \subset x\}.$$

Remark 2.42.
$\{N_s \mid s \in {}^{<\omega}2\}$ forms a basis for Cantor space.

Definition 2.43.
Let X be a nonempty set and T be a subset of ${}^{\omega}X$.

(1) T is a *tree on* X if for any element t of T and any subsequence s of t, s is also in T. Hence T is closed under subsequences. We call elements of T *nodes of* T.
(2) Let T be a tree on X.
    (a) For nodes s, t of T, s, t are *incompatible in* T if there are no nodes u of T such that s, t are subsequences of u.

(b) T is *perfect* if for any node t of T, there are two nodes u, v of T such that t is a subsequence of u, v and u, v are incompatible in T.

(c) Define $[T]$ as follows:

$$[T] \stackrel{\text{def}}{=} \{x \in {}^{\omega}X \mid (\forall n \in \omega)\ x \restriction n \in T.\}.$$

We call elements of $[T]$ *branches of* T.

(d) $\mathrm{stem}(T)$ is the maximal node t_0 of T such that for any node t of T, either $t_0 \subset t$ or $t \subset t_0$ holds.

Definition 2.44.

For a topological space X and a subset A of X,

(1) A is *perfect* if A is closed in X and A is perfect as a topological subspace of X.
(2) A is a *Bernstein set* if neither A nor $X \setminus A$ contains a perfect subset of X.

Remark 2.45.

Let X be a discrete topological space and topologize ${}^{\omega}X$ by the product topology.

(1) For any subset A of ${}^{\omega}X$, A is closed iff there is a tree T on X such that $A = [T]$. Therefore, there is a canonical correspondence between closed subsets of ${}^{\omega}X$ and trees on X.
(2) For any subset A of ${}^{\omega}X$, A is perfect iff there is a perfect tree T on X such that $A = [T]$. Therefore, there is a canonical correspondence between perfect subsets of ${}^{\omega}X$ and perfect trees on X.

Remark 2.46.

Suppose that X, Y are perfect Polish spaces and f is a Borel isomorphic function from X to Y. (The existence of such a function is ensured by Theorem 2.9.) Then for any subset A of X, A is a Bernstein subset of X iff $f``A$ is a Bernstein subset of Y. Therefore, a Borel isomorphic function between perfect Polish spaces preserves the property of being a Bernstein set.

The following theorem is analogous to Theorem 2.29, 2.36:

Theorem 2.47 ([Ber08])**.**

Suppose that X is a perfect Polish space. Then there is a subset of X which is a Bernstein set.

Remark 2.48.

Suppose that X is a perfect Polish space. If P is a Bernstein subset of X, then P does not satisfy the Baire property and the Lebesgue measurability.

By Theorem 2.30 and Remark 2.48, the following theorem holds:

Theorem 2.49.
Suppose that Con(ZFC) holds. Then Con(ZF + DC + "Every set of reals is not a Bernstein set.") holds.

Note that Theorem 2.47 needs the axiom of choice by the last theorem.
By Theorem 2.31, 2.39 and Remark 2.48, the following theorem holds:

Theorem 2.50.
Suppose that X is a perfect Polish space. Then every $\mathbf{\Sigma}^1_1$ subset of X is not a Bernstein set.

The following theorem is analogous to Theorem 2.32, 2.40:

Theorem 2.51 (See [BL99].)**.**
In L, "There is a Δ^1_2 set of reals which is a Bernstein set." holds. In particular, we cannot prove in ZFC that every Δ^1_2 set of reals is not a Bernstein set.

Definition 2.52.
For a set of reals P,

(1) P is *Sacks null* if for any perfect tree S on 2, there is a perfect tree S' on 2 such that $S' \subset S$ and $[S'] \cap P = \emptyset$.
(2) P is of *Sacks measure one* if for any perfect tree S on 2, there is a perfect tree S' on 2 such that $S' \subset S$ and $[S'] \subset P$.
(3) P is *Sacks measurable* if for any perfect tree S on 2, there is a perfect tree S' on 2 such that $S' \subset S$ and either $[S'] \cap P = \emptyset$ or $[S'] \subset P$ holds.

Note that some typical regularity properties can be expressed in the analogous way to the above definition.

Remark 2.53.
For a set of reals P,

(1) P has the Baire property iff for any s in ${}^{<\omega}2$, there exists a s' in ${}^{<\omega}2$ such that $s' \supset s$ and either $N_{s'} \cap P$ is meager or $N_{s'} \setminus P$ is meager.
(2) P is Lebesgue measurable iff for any Borel subset B of ${}^{\omega}2$ with a positive measure, there exists a Borel subset B' of ${}^{\omega}2$ with a

positive measure such that $B' \subset B$ and either $B' \cap P$ is null or $B' \setminus P$ is null.

The Sacks measurability coincides with the property of not being a Bernstein set in the following sense:

Remark 2.54.
Let n be a natural number with $n \geq 1$ and $\mathbf{\Gamma}$ denote one of the following pointclasses, $\mathbf{\Sigma}^1_n \restriction {}^\omega 2$, $\mathbf{\Pi}^1_n \restriction {}^\omega 2$, or $\mathbf{\Delta}^1_n \restriction {}^\omega 2$. Then the following are equivalent:

(1) Every set of reals in $\mathbf{\Gamma}$ is Sacks measurable.
(2) For any set of reals P in $\mathbf{\Gamma}$, there exists a perfect tree S on 2 such that either $[S] \cap P = \emptyset$ or $[S] \subset P$ holds.
(3) No sets of reals in $\mathbf{\Gamma}$ are Bernstein sets.

2.3. **Uniformization.**

Notation 2.55.
Let f be a function from some set to some set. $\mathrm{dom}(f)$ denotes the domain of f.

Definition 2.56.
Let X, Y be sets, P be a subset of $X \times Y$, and f be a function from a subset of X to Y.

f *uniformizes* P if for any $x \in \mathrm{dom}(f)$, $(x, f(x)) \in P$ and $\mathrm{dom}(f) = \mathrm{pr}_1 {}``P$. We call such an f a *choice function for P*.

If we use the axiom of choice, we can find a choice function for any subset of any product of two sets. But, the question is the complexity of such a choice function.

Definition 2.57.
Let X, Y be sets and Γ be a subset of $\mathcal{P}(X \times Y)$. Γ has the *uniformization property* if for any P in Γ, there exists a function f in Γ such that f uniformizes P.

In the above definition, our main concern is the case when X, Y are perfect Polish spaces and Γ is one of the following, $\mathbf{\Sigma}^1_n \restriction X \times Y$, $\mathbf{\Pi}^1_n \restriction X \times Y$, or $\mathbf{\Delta}^1_n \restriction X \times Y$, where n is a natural number with $n \geq 1$.

Remark 2.58.
Let n be a natural number with $n \geq 1$. Suppose that X, X', Y, Y' are perfect Polish spaces. (By Example 2.4, $X \times Y$, $X' \times Y'$ are also perfect Polish spaces.)

Then $\mathbf{\Sigma}^1_n \restriction X \times Y$ has the uniformization property iff $\mathbf{\Sigma}^1_n \restriction X' \times Y'$ has the uniformization property. Also, $\mathbf{\Pi}^1_n \restriction X \times Y$ has the uniformization property iff $\mathbf{\Pi}^1_n \restriction X' \times Y'$ has the uniformization property.

By the above remark, from now on, we concentrate when $X = Y = {}^{\omega}2$. Also, we consider when Γ is $\Sigma^1_n(\vec{r})$, $\Pi^1_n(\vec{r})$, or $\Delta^1_n(\vec{r})$, where $\vec{r}$ is a finite sequence of reals.

The following propositions are basic:

Proposition 2.59 (See [Mos80].)**.**
Let n be a natural number with $n \geq 1$ and $\vec{r}$ be a finite sequence of reals.

(1) Neither $\mathbf{\Delta}^1_n \restriction {}^{\omega}2 \times {}^{\omega}2$ nor $\Delta^1_n(\vec{r}) \restriction {}^{\omega}2 \times {}^{\omega}2$ can have the uniformization property.
(2) If $\mathbf{\Sigma}^1_n \restriction {}^{\omega}2 \times {}^{\omega}2$ has the uniformization property, then $\mathbf{\Pi}^1_n \restriction {}^{\omega}2 \times {}^{\omega}2$ cannot have the uniformization property and vice versa. The same result also holds for $\Sigma^1_n(\vec{r}) \restriction {}^{\omega}2 \times {}^{\omega}2$ and $\Pi^1_n(\vec{r}) \restriction {}^{\omega}2 \times {}^{\omega}2$.

Proposition 2.60 (See [Mos80].)**.**
Let n be a natural number with $n \geq 1$ and $\vec{r}$ be a finite sequence of reals.

(1) Suppose that $\Sigma^1_n \restriction {}^{\omega}2 \times {}^{\omega}2$ has the uniformization property. Then $\Sigma^1_n(\vec{r}) \restriction {}^{\omega}2 \times {}^{\omega}2$ has the uniformization property. The same result also holds for $\Pi^1_n \restriction {}^{\omega}2 \times {}^{\omega}2$ and $\Pi^1_n(\vec{r}) \restriction {}^{\omega}2 \times {}^{\omega}2$.
(2) Suppose that $\Sigma^1_n(\vec{r}) \restriction {}^{\omega}2 \times {}^{\omega}2$ has the uniformization property. Then $\mathbf{\Sigma}^1_n \restriction {}^{\omega}2 \times {}^{\omega}2$ has the uniformization property. The same result also holds for $\Pi^1_n(\vec{r}) \restriction {}^{\omega}2 \times {}^{\omega}2$ and $\mathbf{\Pi}^1_n \restriction {}^{\omega}2 \times {}^{\omega}2$.

Proposition 2.61 (See [Mos80].)**.**
Let n be a natural number with $n \geq 1$ and $\vec{r}$ be a finite sequence of reals. Suppose that $\Pi^1_n(\vec{r}) \restriction {}^{\omega}2 \times {}^{\omega}2$ has the uniformization property. Then $\Sigma^1_{n+1}(\vec{r}) \restriction {}^{\omega}2 \times {}^{\omega}2$ has the uniformization property. The same result also holds for $\mathbf{\Pi}^1_n \restriction {}^{\omega}2 \times {}^{\omega}2$ and $\mathbf{\Sigma}^1_{n+1} \restriction {}^{\omega}2 \times {}^{\omega}2$.

The following theorem is important:

Theorem 2.62 ([Kon39])**.**
Let $\vec{r}$ be a finite sequence of reals. Then $\Pi^1_1(\vec{r}) \restriction {}^{\omega}2 \times {}^{\omega}2$ and $\mathbf{\Pi}^1_1 \restriction {}^{\omega}2 \times {}^{\omega}2$ have the uniformization property. Therefore, $\Sigma^1_2(\vec{r}) \restriction {}^{\omega}2 \times {}^{\omega}2$ and $\mathbf{\Sigma}^1_2 \restriction {}^{\omega}2 \times {}^{\omega}2$ have the uniformization property.

We cannot generalize Theorem 2.62 to higher pointclasses in ZFC.

Theorem 2.63 ([Lév65])**.**
We cannot prove in ZFC that every $\mathbf{\Pi}^1_2 \restriction {}^{\omega}2 \times {}^{\omega}2$ subset can be uniformized by some projective set.

We can generalize Theorem 2.62 to higher pointclasses under the existence of certain large cardinals.

Theorem 2.64 ([MS88, MS89] and [Mos71]).
Let n be a natural number with $n \geq 1$. Suppose that there exist $2n-1$ Woodin cardinals and a measurable cardinal above them. Then $\Pi^1_{2n+1} \restriction {}^\omega 2 \times {}^\omega 2$ and $\Sigma^1_{2n+2} \restriction {}^\omega 2 \times {}^\omega 2$ have the uniformization property.

Note that measurable cardinals and Woodin cardinals are typical examples of large cardinals.

2.4. Typical forcing notions and forcing absoluteness.

Notation 2.65.
Let $\mathbb{P}$ be a preorder.

- Suppose that G is a $\mathbb{P}$-generic filter over V. For any set X in $V[G]$ with $X \subset V$, $V[X]$ is the smallest transitive model of ZFC that contains all the sets in V and X as elements.
- $\mathbb{P} * \mathbb{P}$ is the two step iteration of $\mathbb{P}$.

Definition 2.66.

$$\mathbb{C} \stackrel{\text{def}}{=} {}^{<\omega}2.$$

For any p, q in $\mathbb{C}$, $p \leq q$ (i.e. p is stronger than or equal to q) if $p \supset q$.
This forcing notion is called *Cohen forcing*.

Remark 2.67.
Suppose that G is a $\mathbb{C}$-generic filter over V. Put

$$c = \bigcup_{p \in G} p.$$

Then, by the genericity of G, c is a real. Such a real is called a *Cohen real over V*.
On the other hand, G is constructed from c because

$$G = \{p \in \mathbb{C} \mid p \subset c\}.$$

Therefore, there is a canonical correspondence between Cohen reals over V and $\mathbb{C}$-generic filters over V.

Definition 2.68.
Let $\mathbb{P}$ be a preorder. $\mathbb{P}$ is *non-atomic* if for any condition p of $\mathbb{P}$, there are two conditions q, r of $\mathbb{P}$ such that q, $r \leq p$ and q, r are incompatible.

Remark 2.69.
Suppose that $\mathbb{P}$ is a forcing notion which always adds a new real (i.e. $\Vdash_{\mathbb{P}}$ "There is a real which is not in V".). Then $\mathbb{P}$ is non-atomic. (But the converse does not hold in general.)

The following characterization of Cohen forcing is useful and important:

Proposition 2.70 (Folklore. See [Kun80].)**.**
Suppose that $\mathbb{P}$ is a countable non-atomic forcing notion. Then $\mathbb{P}$ is forcing equivalent to Cohen forcing.

Remark 2.71.

(1) Cohen forcing is forcing equivalent to the following preorder $\mathbb{C}'$:

$$\mathbb{C}' = \{B \mid B \text{ is a non-meager Borel set of reals.}\}.$$

For B_1, B_2 in $\mathbb{C}'$, $B_1 \leq B_2$ if $B_1 \setminus B_2$ is meager.

(2) Let x be a real and M be a transitive model of ZF+DC. Then x is a Cohen real over M iff for any meager Borel set of reals B in M, x is not in B^V.

Definition 2.72.

$$\mathbb{D} \stackrel{\text{def}}{=} \omega \times {}^{\omega}\omega.$$

For $\langle m, f\rangle$, $\langle n, g\rangle$ in $\mathbb{D}$,

$\langle m, f\rangle \leq \langle n, g\rangle$ if $m \geq n, f \restriction n = g \restriction n,$ and $(\forall k \geq n) f(k) \geq g(k)$.

This forcing notion is called *Hechler forcing.*

Remark 2.73.

(1) Suppose that G is a $\mathbb{D}$-generic filter over V. Put

$$d = \bigcup \{f \restriction n \mid \langle n, f\rangle \in G\}.$$

Then, by the genericity of G, d is in ${}^{\omega}\omega$. Such a function is called a *Hechler real over V.*

On the other hand, G is constructed from d and V because

$$G = \{\langle n, f\rangle \in \mathbb{D} \cap V \mid f \restriction n \subset d\}.$$

Therefore, there is a canonical correspondence between Hechler reals over V and $\mathbb{D}$-generic filters over V.

(2) Suppose that d is a Hechler real over V. Then d is a dominating real over V, namely

$$(\forall x \in {}^{\omega}\omega \cap V)\,(\exists n \in \omega)\,(\forall k \geq n)\, d(k) \geq f(k).$$

Definition 2.74.

$$\mathbb{B} \stackrel{\text{def}}{=} \{B \mid B \text{ is a Borel subset of } {}^{\omega}2 \text{ with a positive measure.}\}.$$

For B_1, B_2 in $\mathbb{B}$, $B_1 \leq B_2$ if $B_1 \setminus B_2$ is null.

This forcing notion is called *random forcing.*

Remark 2.75.

Suppose that G is a $\mathbb{B}$-generic filter over V. Then, by the genericity of G, $\bigcap_{B \in G} B^{V[G]}$ is a singleton. Let r be the element of the singleton. Such a real is called a *random real over* V.

On the other hand, G is constructed from r and V in the following way:

$$G = \{B^V \mid r \in B^{V[r]}\}.$$

Therefore, there is a canonical correspondence between random reals over V and $\mathbb{B}$-generic filters over V.

Definition 2.76.

For a real number ε with $0 < \varepsilon < 1$, let

$\mathbb{A}_\varepsilon \stackrel{\text{def}}{=} \{U \mid U \text{ is a nonempty open subset of } {}^\omega 2 \text{ and } \mu_{({}^\omega 2)}(U) < \varepsilon.\}$.

For U_1, U_2 in $\mathbb{A}_\varepsilon$, $U_1 \leq U_2$ if $U_1 \supset U_2$.

Also, put

$$\mathbb{A} = \mathbb{A}_{\frac{1}{2}}.$$

$\mathbb{A}$ is called *amoeba forcing.*

Lemma 2.77 ([Tru88])**.**

Suppose that ε is a real number with $0 < \varepsilon < 1$.

Then $\mathbb{A}_\varepsilon$ is isomorphic to $\mathbb{A}$.

Remark 2.78.

Suppose that G is a $\mathbb{A}$-generic filter over V. Then, in $V[G]$, the set

$$\{r \in {}^\omega 2 \mid r \text{ is a random real over } V.\}$$

is of Lebesgue measure one.

Definition 2.79.

$$\mathbb{S} \stackrel{\text{def}}{=} \{S \mid S \text{ is a perfect tree on } 2.\}.$$

For S_1, S_2 in $\mathbb{S}$, $S_1 \leq S_2$ if $S_1 \subset S_2$.

This forcing notion is called *Sacks forcing.*

Remark 2.80.

Suppose that G is an $\mathbb{S}$-generic filter over V. Put

$$s = \bigcup \{\text{stem}(S) \mid S \in G\}.$$

Then, by the genericity of G, s is a real. Such a real is called a *Sacks real over* V.

On the other hand, G is constructed from s and V because

$$G = \{S \in \mathbb{S} \cap V \mid s \in [S]\}.$$

Therefore, there is a canonical correspondence between Sacks reals over V and $\mathbb{S}$-generic filters over V.

The following property is known as the minimality of Sacks forcing:

Theorem 2.81 ([Sac71])**.**
Suppose that s is a Sacks real over V. Then, in $V[s]$, for any set X such that X is not in V and $X \subset V$, $V[X] = V[s]$.

The following theorems are important:

Theorem 2.82 ([Sac71])**.**
Suppose that s is a Sacks real over V. Then, in $V[s]$, for any real s' which is not in V, s' is also a Sacks real over V.

Theorem 2.83 ([Bre00])**.**
Suppose that s is a Sacks real over V. Then, in $V[s]$, the set

$$\{s' \mid s' \text{ is a Sacks real over } V.\}$$

is of Sacks measure one.

Definition 2.84.
Let n be a natural number with $n \geq 1$, $\mathbb{P}$ be a preorder, and Γ be Σ^1_n or Π^1_n.

(1) $\boldsymbol{\Gamma}$-$\mathbb{P}$-*absoluteness* is the following statement:
For any Γ-formula ϕ and any finite sequence of reals $\vec{r}$,

$$\phi(\vec{r}) \text{ iff } \Vdash_{\mathbb{P}} \phi(\check{\vec{r}}).$$

(2) $\boldsymbol{\Gamma}$-$\mathbb{P}$-*correctness* is the following statement:
If G is a $\mathbb{P}$-generic filter over V, then for any Γ-formula ϕ and any real x in $V[G]$,

$$V[x] \vDash \phi(x) \text{ iff } V[G] \vDash \phi(x).$$

Remark 2.85.
Let n be a natural number with $n \geq 1$ and $\mathbb{P}$ be a preorder.

(1) By Remark 2.19, $\boldsymbol{\Sigma}^1_n$-$\mathbb{P}$-correctness implies $\boldsymbol{\Sigma}^1_n$-$\mathbb{P}$-absoluteness. Also, $\boldsymbol{\Pi}^1_n$-$\mathbb{P}$-correctness implies $\boldsymbol{\Pi}^1_n$-$\mathbb{P}$-absoluteness.
(2) $\boldsymbol{\Sigma}^1_n$-$\mathbb{P}$-correctness is equivalent to $\boldsymbol{\Pi}^1_n$-$\mathbb{P}$-correctness.
(3) In general, $\boldsymbol{\Sigma}^1_n$-$\mathbb{P}$-absoluteness is not equivalent to $\boldsymbol{\Pi}^1_n$-$\mathbb{P}$-absoluteness.

But, the following notion for a preorder is sufficient for the equivalence:

Definition 2.86.

Let $\mathbb{P}$ be a preorder.

(1) A function $\pi\colon \mathbb{P} \to \mathbb{P}$ is an *automorphism on* $\mathbb{P}$ if π is bijective and for any conditions p, q of $\mathbb{P}$, $p \leq q$ iff $\pi(p) \leq \pi(q)$.
(2) $\mathbb{P}$ is *weakly homogeneous* if for any conditions p, q of $\mathbb{P}$, there are conditions p', q' of $\mathbb{P}$ with $p' \leq p$, $q' \leq q$ and an automorphism π on $\mathbb{P}$ such that $\pi(p') = \pi(q')$.

Proposition 2.87.

Suppose that $\mathbb{P}$ is a weakly homogeneous preorder. Then for any natural number n with $n \geq 1$, $\boldsymbol{\Sigma}^1_n$-$\mathbb{P}$-absoluteness is equivalent to $\boldsymbol{\Pi}^1_n$-$\mathbb{P}$-absoluteness.

Example 2.88.

Cohen forcing, Sacks forcing are weakly homogeneous.

Remark 2.89.

Let n be a natural number with $n \geq 1$. By Proposition 2.87 and Example 2.88, $\boldsymbol{\Sigma}^1_n$-$\mathbb{C}$-absoluteness is equivalent to $\boldsymbol{\Pi}^1_n$-$\mathbb{C}$-absoluteness. The same result holds for Sacks forcing. Moreover, by Theorem 2.81, $\boldsymbol{\Sigma}^1_n$-$\mathbb{S}$-correctness is equivalent to $\boldsymbol{\Sigma}^1_n$-$\mathbb{S}$-absoluteness.

The following theorem is basic and important:

Theorem 2.90 ([Sho61])**.**

Suppose that M is a transitive model of ZF+DC and $\omega_1 \subset M$. Then every Σ^1_2-formula is absolute between M and V.

The following corollary is the start line of the investigation on forcing absoluteness:

Corollary 2.91.

For any preorder $\mathbb{P}$, $\boldsymbol{\Sigma}^1_2$-$\mathbb{P}$-absoluteness, $\boldsymbol{\Pi}^1_2$-$\mathbb{P}$-absoluteness, $\boldsymbol{\Sigma}^1_2$-$\mathbb{P}$-correctness, and $\boldsymbol{\Pi}^1_2$-$\mathbb{P}$-correctness hold.

The following theorem is also basic and important:

Theorem 2.92 ([Göd40, Add59b])**.**

(1) The statement "x is a real and x is in L" is equivalent to a Σ^1_2-formula under ZF+DC.
(2) The statement "x, y are reals and x is in $L[y]$" is equivalent to a Σ^1_2-formula under ZF+DC.

The following corollary states that the above corollary is optimal in ZFC:

Corollary 2.93.
The statement "There is a real which is not in L" is equivalent to a Σ^1_3 sentence under ZF+DC. In particular, if $\mathbb{P}$ is a forcing notion which always adds a new real, then in L, $\boldsymbol{\Sigma}^1_3$-$\mathbb{P}$-absoluteness fails.

2.5. **Known results on forcing absoluteness.**

There is a close relationship between forcing absoluteness and regularity properties.

Theorem 2.94 ([JS89, BJ95])**.**
The following are equivalent:
(1) $\boldsymbol{\Sigma}^1_3$-$\mathbb{C}$-absoluteness holds.
(2) Every $\boldsymbol{\Delta}^1_2$ set of reals has the Baire property.
(3) For any real x, there exists a Cohen real over $L[x]$.

Theorem 2.95 ([JS89, BJ95])**.**
The following are equivalent:
(1) $\boldsymbol{\Sigma}^1_3$-$\mathbb{B}$-absoluteness holds.
(2) Every $\boldsymbol{\Delta}^1_2$ set of reals is Lebesgue measurable.
(3) For any real x, there exists a random real over $L[x]$.

Theorem 2.96 ([Sol69, Jud93, BL99])**.**
The following are equivalent:
(1) $\boldsymbol{\Sigma}^1_3$-$\mathbb{D}$-absoluteness holds.
(2) Every $\boldsymbol{\Sigma}^1_2$ set of reals has the Baire property.
(3) For any real x, $\{c \mid c$ is a Cohen real over $L[x].\}$ is comeager.
(4) For any real x, there is a Hechler real over $L[x]$.

Theorem 2.97 ([Sol69, Jud93, BL99])**.**
The following are equivalent:
(1) $\boldsymbol{\Sigma}^1_3$-$\mathbb{A}$-absoluteness holds.
(2) Every $\boldsymbol{\Sigma}^1_2$ set of reals is Lebesgue measurable.
(3) For any real x, $\{r \mid r$ is a random real over $L[x].\}$ is of Lebesgue measure one.

The following theorems are generalizations of the above theorems:

Theorem 2.98 ([BJ95])**.**
(1) If $\boldsymbol{\Sigma}^1_4$-$\mathbb{D}$-absoluteness holds, then every $\boldsymbol{\Sigma}^1_3$ set of reals has the Baire property.
(2) Let n be a natural number with $n \geq 4$. If $\boldsymbol{\Sigma}^1_{n+1}$-$\mathbb{D}$-absoluteness and $\boldsymbol{\Sigma}^1_n$-$(\mathbb{D} * \mathbb{D})$-correctness hold, then every $\boldsymbol{\Sigma}^1_n$ set of reals has the Baire property.

Theorem 2.99 ([BJ95])**.**

(1) If $\mathbf{\Sigma}^1_4$-$\mathbb{A}$-absoluteness holds, then every $\mathbf{\Sigma}^1_3$ set of reals is Lebesgue measurable.
(2) Let n be a natural number with $n \geq 4$. If $\mathbf{\Sigma}^1_{n+1}$-$\mathbb{A}$-absoluteness and $\mathbf{\Sigma}^1_n$-$\mathbb{A}$-correctness hold, then every $\mathbf{\Sigma}^1_n$ set of reals is Lebesgue measurable.

Theorem 2.100 ([Woo82]).
Let n be a natural number with $n \geq 1$. Assume that $\mathbf{\Pi}^1_{2n-1} \restriction {}^\omega 2 \times {}^\omega 2$ has the uniformization property. If every $\mathbf{\Delta}^1_{2n}$ set of reals has the Baire property, then $\mathbf{\Sigma}^1_{2n+1}$-$\mathbb{C}$-absoluteness holds.

3. Facts

In this section, some known facts we will use in the proof of our theorems are listed.

Notation 3.1.

- For a set X and finite sequences s and t in X, $s^\frown t$ is the concatenation of s and t. Hence
$$s^\frown t = \langle s_0, \ \cdots, \ s_{\mathrm{lh}(s)-1}, \ t_0, \ \cdots, \ t_{\mathrm{lh}(t)-1} \rangle.$$
- For a metric space (X, d) and a subset A of X,
$$\mathrm{diam}(A) = \sup\{d(x, y) \mid x, y \in A\}.$$

Fact 3.2 ([BL99]).
The following are equivalent:
(1) Every $\mathbf{\Delta}^1_2$ set of reals is Sacks measurable.
(2) Every $\mathbf{\Sigma}^1_2$ set of reals is Sacks measurable.
(3) For any real r, there is a real x such that x is not in $L[r]$.

Fact 3.3 ([Sol69]).
Let r be a real and P be a $\Sigma^1_2(r)$ subset of ${}^\omega 2 \times {}^\omega 2$. Then either $P \subset L[r]$ or P contains a perfect subset.

Fact 3.4 ([Sac71]).
Suppose that $\langle S_t \mid t \in {}^{<\omega}2 \rangle$ is a sequence of perfect trees on 2 such that
(1) for any t_1, t_2 in ${}^{<\omega}2$ with $t_1 \subset t_2$, $S_{t_1} \supset S_{t_2}$,
(2) for any t in ${}^{<\omega}2$, $[S_{t^\frown\langle 0\rangle}] \cap [S_{t^\frown\langle 1\rangle}] = \emptyset$,
(3) for any t in ${}^{<\omega}2$, $\mathrm{lh}(\mathrm{stem}(S_t)) \geq \mathrm{lh}(t)$.

Put
$$C = \bigcup_{m \in \omega} \bigcap_{t \in {}^m 2} [S_t].$$
Then, C is a perfect subset of ${}^\omega 2$ with $C \subset [S_\emptyset]$.

4. Proofs of Theorems

Main Theorem 4.1.

(1) The following are equivalent:
  (a) $\mathbf{\Sigma}^1_3$-$\mathbb{S}$-absoluteness holds.
  (b) Every $\mathbf{\Delta}^1_2$ set of reals is Sacks measurable.
  (c) Every $\mathbf{\Sigma}^1_2$ set of reals is Sacks measurable.
  (d) For any real r, there is a real x such that x is not in $L[r]$.

(2) Suppose that $\mathbb{P}$ is a preorder which always adds a new real. Then $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness implies $\mathbf{\Sigma}^1_3$-$\mathbb{S}$-absoluteness.

Proof.
(1) By Fact 3.2, it suffices to show that (a) $\Leftrightarrow$ (d).

First, we will show that $(a) \Rightarrow (d)$, but this is taken care of by the proof of (2). In fact, in the proof of (2), we will show that $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness implies (d). Since Sacks forcing adds a Sacks real over V, it satisfies the assumption about $\mathbb{P}$ in (2). (Of course, we will not use (a) $\Rightarrow$ (d) in the proof of (2).)

Next, we will show that $(d) \Rightarrow (a)$.

Suppose that $\mathbf{\Sigma}^1_3$-$\mathbb{S}$-absoluteness fails and we will derive a contradiction.

Then there is a Σ^1_3-formula ϕ and a finite sequence of reals $\vec{r}$ such that

$$\phi(\vec{r}) \not\Leftrightarrow \Vdash_{\mathbb{S}} \phi(\check{\vec{r}}).$$

First, we will show that $\phi(\vec{r})$ implies $\Vdash_{\mathbb{S}} \phi(\check{\vec{r}})$. Let ψ be the Π^1_2-formula such that $\phi \equiv \exists^1 \alpha_1 \psi$. Since $\phi(\vec{r})$ holds, there exists a real x such that $\psi(x, \vec{r})$. By Theorem 2.90, for any $\mathbb{S}$-generic filter G over V, $V[G] \vDash \psi(x, \vec{r})$, which implies $V[G] \vDash \phi(\vec{r})$. Hence $\Vdash_{\mathbb{S}} \phi(\check{\vec{r}})$ holds.

Therefore,

$$\Vdash_{\mathbb{S}} \phi(\check{\vec{r}}), \text{ but } \neg\phi(\vec{r}).$$

Let θ be the Σ^1_1-formula such that $\phi \equiv \exists^1 \alpha_1 \forall^1 \alpha_2 \theta$.

Suppose that s is a Sacks real over V. Then, by $\Vdash_{\mathbb{S}} \phi(\check{\vec{r}})$, in $V[s]$, there is a real s' such that for any real y, $\theta(s', y, \vec{r})$. By $\neg\phi(\vec{r})$ and Theorem 2.90, s' is not in V. Therefore, by Theorem 2.81 and Theorem 2.82, $V[s'] = V[s]$ and s' is also a Sacks real over V. Hence in $V[s']$, for any real y, $\theta(s', y, \vec{r})$. By the forcing theorem, there is an S in $\mathbb{S}$ such that

$$S \Vdash_{\mathbb{S}} \text{“}\big(\forall y \in {}^{\omega}2\big)\, \theta(\dot{s}, y, \check{\vec{r}})\text{”}, \tag{$*$}$$

where $\dot{s}$ is a canonical name for a Sacks real.

Next, by $\neg\phi(\vec{r})$, for any real x, there is a real y such that $\neg\theta(x, y, \vec{r})$. Since $\{(x, y) \mid \neg\theta(x, y, \vec{r})\}$ is a $\Pi^1_1(\vec{r})$ subset of ${}^\omega 2 \times {}^\omega 2$, by Theorem 2.62, there is a $\Pi^1_1(\vec{r})$ function $f\colon {}^\omega 2 \to {}^\omega 2$ such that for any real x, $\neg\theta(x, f(x), \vec{r})$.

We will approximate $f \restriction [S]$ by some Borel function whose domain is a perfect subset. Since S can be seen as a real, $f \restriction [S]$ is in $\mathbf{\Pi}^1_1 \restriction {}^\omega 2 \times {}^\omega 2$. By Proposition 2.20, there is a real r' such that $f \restriction [S] \in \Pi^1_1(r') \restriction {}^\omega 2 \times {}^\omega 2$. By Fact 3.3, $f \restriction [S] \subset L[r']$ or $f \restriction [S]$ contains a perfect subset. By Remark (d), there is a real x such that x is not in $L[r']$. Since there is a canonical bijection from ${}^\omega 2$ to $[S]$, such a real is also in $[S]$ and such a set is in $f \restriction [S]$, hence $f \restriction [S] \not\subseteq L[r']$. Therefore, $f \restriction [S]$ contains a perfect subset g. Note that g is a function since f is a function. Put $C = \mathrm{dom}(g)$. Then $\mathrm{pr}_1 \restriction g\colon g \to C$ is surjective and continuous. Since g is a function, it is also injective. Moreover, since g is compact and C is Hausdorff, it is homeomorphism. Therefore, C is perfect because g is perfect. Hence g is a Borel function which is a restriction of f to a perfect subset, as we desired.

By Remark 2.45, there are perfect trees S' on 2 and T on 2×2 such that $C = [S']$ and $g = [T]$. Note that $S' \subset S$. Take a Sacks real s'' over V with s'' in $[S']$. Since for any real x, $\neg\theta(x, f(x), \vec{r})$ and $g \subset f$, the following statements hold in V:

$$\forall x \forall y \Big((x, y) \in [T] \to \neg\theta(x, y, \vec{r}) \Big),$$
$$\forall x \in [S']\, \exists y \Big((x, y) \in [T] \Big).$$

Since a perfect tree on 2 and a perfect tree on 2×2 can be seen as reals, the first statement is equivalent to a Π^1_1-formula with parameters T and $\vec{r}$ and the second statement is equivalent to a Π^1_2-formula with parameters S' and T. Therefore, by Theorem 2.90, the above statements also hold in $V[s'']$. Since s'' is in $[S']$,

$$V[s''] \models \text{“}\big(\exists y \in {}^\omega 2\big)\, \neg\theta(s'', y, \vec{r})\text{”},$$

which contradicts with $(*)$.

(2) Suppose that $\mathbb{P}$ is a preorder which always adds a new real. By (d) $\Rightarrow$ (a), it suffices to show that (d).

Take any real r. By the assumption about $\mathbb{P}$,

$$\Vdash_{\mathbb{P}} \text{“}(\exists x \in {}^\omega 2)\ x \notin L[\check{r}]\text{”}.$$

By Theorem 2.92, the above statement is equivalent to a Σ^1_3-formula with a parameter r. By $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness, it also holds in V. Hence we obtained (d). ∎

Theorem 4.2.
Let n be a natural number with $n \geq 1$.

(1) If $\mathbf{\Sigma}^1_{n+1}$-$\mathbb{S}$-absoluteness holds, then every $\mathbf{\Delta}^1_n$ set of reals is Sacks measurable.
(2) Assume that $\mathbf{\Pi}^1_{2n-1} \restriction {}^{\omega}2 \times {}^{\omega}2$ has the uniformization property. If every $\mathbf{\Delta}^1_{2n}$ set of reals is Sacks measurable, then $\mathbf{\Sigma}^1_{2n+1}$-$\mathbb{S}$-absoluteness holds.

Proof.
(1) Take any $\mathbf{\Delta}^1_n$ set of reals P. By Remark 2.54, it suffices to show that there exists a perfect tree S on 2 such that either $[S] \cap P = \emptyset$ or $[S] \subset P$ holds.

By Proposition 2.20, there is a Σ^1_n-formula ϕ, a Π^1_n-formula ψ, and a real r such that

$$\forall x \in {}^{\omega}2 \ \Big(\phi(x, r) \leftrightarrow \psi(x, r)\Big), \tag{$**$}$$
$$P = \{x \in {}^{\omega}2 \mid \phi(x, r)\}.$$

Note that $(**)$ is equivalent to a Π^1_{n+1}-formula with a parameter r. By Remark 2.89 and $\mathbf{\Sigma}^1_{n+1}$-$\mathbb{S}$-absoluteness, $\mathbf{\Pi}^1_{n+1}$-$\mathbb{S}$-absoluteness holds. Therefore, for any Sacks real s over V, $(**)$ also holds in $V[s]$.

Let s be a Sacks real over V.

Claim 1.
Let Φ be a formula of the language of set theory. If $V[s] \vDash \Phi(s)$, then there is a perfect tree S on 2 in $V[s]$ such that for any real x in $[S] \cap V[s]$, $V[s] \vDash \Phi(x)$.

Proof of Claim 1.
By the forcing theorem, there is an S' in $\mathbb{S} \cap V$ such that

$$S' \Vdash_{\mathbb{S}} \Phi(\dot{s}).$$

By Theorem 2.83, there is a perfect tree S on 2 in $V[s]$ with $S \subset S'$ such that for any real s' in $[S] \cap V[s]$, s' is a Sacks real over V. Then by Theorem 2.81 and $S' \Vdash_{\mathbb{S}} \Phi(\dot{s})$, $V[s'] = V[s]$ and $V[s'] \vDash \Phi(s')$. Hence for any real x in $[S] \cap V[s]$, $V[s] \vDash \Phi(x)$. □

Suppose that $V[s] \vDash \phi(s, r)$. Then by Claim 1, there exists a perfect tree S on 2 in $V[s]$, for any real x in $[S] \cap V[s]$, $V[s] \vDash \phi(x, \vec{r})$.

On the other hand, suppose that $V[s] \vDash \neg\phi(s, \vec{r})$. Then by Claim 1, there exists a perfect tree S on 2 in $V[s]$, for any real x in $[S] \cap V[s]$, $V[s] \vDash \neg\phi(x, \vec{r})$.

Hence in $V[s]$,

"There is a perfect tree S on 2 such that

either $\big((\forall x \in [S]\big)\ \phi(x, r)$ or $\big(\forall x \in [S]\big)\ \neg\phi(x, r)$ holds."

By $(**)$ in $V[s]$, the above statement is equivalent to

"There is a perfect tree S on 2 such that

either $\big(\forall x \in [S]\big)\ \psi(x, r)$ or $\big(\forall x \in [S]\big)\ \neg\phi(x, r)$ holds."

Since we took s arbitrarily,

$\Vdash_{\mathbb{S}}$ "There is a perfect tree S on 2 such that

either $\big(\forall x \in [S]\big)\ \psi(x, r)$ or $\big(\forall x \in [S]\big)\ \neg\phi(x, r)$ holds.".

This is equivalent to a Σ^1_{n+1}-formula with a parameter r. Therefore, by $\mathbf{\Sigma}^1_{n+1}$-$\mathbb{S}$-absoluteness, the above statement also holds in V.

Since $P = \{x \in {}^{\omega}2 \mid \phi(x, r)\}$, there exists a perfect tree S on 2 such that either $[S] \subset P$ or $[S] \cap P = \emptyset$ holds.

This is what we desired.

(2) We will show that for any $k \leq 2n + 1$, $\mathbf{\Sigma}^1_k$-$\mathbb{S}$-absoluteness holds by induction on k. The case $k = 1$ or 2 is done by Corollary 2.91. Suppose that $k \geq 3$.

Suppose that $\mathbf{\Sigma}^1_k$-$\mathbb{S}$-absoluteness fails and we will derive a contradiction.

Then there is a Σ^1_k-formula and a finite sequence of reals $\vec{r}$ such that

$$\phi(\vec{r}) \not\Leftrightarrow \Vdash_{\mathbb{S}} \phi(\check{\vec{r}}).$$

First, we will show that $\phi(\vec{r})$ implies $\Vdash_{\mathbb{S}} \phi(\check{\vec{r}})$. Let ψ be the Π^1_{k-1}-formula such that $\phi \equiv \exists^1 \alpha_1 \psi$. Since $\phi(\vec{r})$, there exists a real x such that $\psi(x, \vec{r})$. By the induction hypothesis and Remark 2.89, $\mathbf{\Pi}^1_{k-1}$-$\mathbb{S}$-absoluteness holds. Hence $\Vdash_{\mathbb{S}} \psi(\check{x}, \check{\vec{r}})$ and then $\Vdash_{\mathbb{S}} \phi(\check{\vec{r}})$.

Therefore,

$$\Vdash_{\mathbb{S}} \phi(\check{\vec{r}}),\ \text{but}\ \neg\phi(\vec{r}).$$

Let θ be the Σ^1_{k-2}-formula such that $\phi \equiv \exists^1 \alpha_1 \forall^1 \alpha_2 \theta$.

Suppose that s is a Sacks real over V. Then, by $\Vdash_{\mathbb{S}} \phi(\check{\vec{r}})$, in $V[s]$, there is a real s' such that for any real y, $\theta(s', y, \vec{r})$. By $\neg\phi(\vec{r})$ and $\mathbf{\Pi}^1_{k-1}$-$\mathbb{S}$-absoluteness, s' is not in V. Therefore, by Theorem 2.81 and Theorem 2.82, $V[s'] = V[s]$ and s' is also a Sacks reals over V. Hence in $V[s']$, for any real y, $\theta(s', y, \vec{r})$. By the forcing theorem, there is an S in $\mathbb{S}$ such that

$$S \Vdash_{\mathbb{S}} \text{"}\big(\forall y \in {}^{\omega}2\big)\, \theta(\dot{s}, y, \check{\vec{r}})\text{"}. \tag{$***$}$$

Next, by $\neg\phi(\vec{r})$, for any real x, there is a real y such that $\neg\theta(x, y, \vec{r})$. Since $\{(x, y) \mid \neg\theta(x, y, \vec{r})\}$ is a $\Pi^1_{k-2}(\vec{r})$ subset of ${}^\omega 2 \times {}^\omega 2$, $k-2 \leq 2n-1$, and $\boldsymbol{\Pi}^1_{2n-1} \restriction {}^\omega 2 \times {}^\omega 2$ has the uniformization property, there is a $\boldsymbol{\Pi}^1_{2n-1}$ function $f \colon {}^\omega 2 \to {}^\omega 2$ such that for any real x, $\neg\theta(x, f(x), \vec{r})$.

We will approximate $f \restriction [S]$ by some Borel function whose domain is a perfect subset.

Claim 2.

Suppose that $\langle N_l' \mid l \in \omega \rangle$ is an enumeration of a basis for Cantor space. Then for any perfect tree S' on 2, there are a perfect tree S'' on 2 with $S'' \subset S'$ and a natural number l such that for any x in $[S']$, $f(x)$ is in N_l'.

Proof of Claim 2.

Suppose that the above statement fails. Then there is a perfect tree S' on 2 such that for any perfect tree S'' on 2 with $S'' \subset S'$ and any natural number l, there is a real x in $[S'']$ such that $f(x)$ is not in N_l'.

For any natural number l, put $P_l = f^{-1}$“N_l'. Then, since f is a $\boldsymbol{\Pi}^1_{2n-1}$ subset, P_l is a $\boldsymbol{\Delta}^1_{2n}$ set of reals for any l. By the assumption about the Sacks measurability and the above condition which S' satisfies, we can construct $\langle S_l \mid l \in \omega \rangle$ such that

(1) for any l, S_l is a perfect tree on 2,
(2) for any l, $S_l \supset S_{l+1}$,
(3) for any l, $[S_l] \cap P_l = \emptyset$.

Since $[S_l]$ is compact for any l, we can take a real x in $\bigcap_{l \in \omega} [S_l]$.

By the construction of $\langle S_l \mid l \in \omega \rangle$, for any l, x is not in P_l, hence $f(x)$ is not in N_l'. But this is a contradiction because $\{N_l' \mid l \in \omega\}$ is a basis for Cantor space. $\square$

Fix a compatible complete metric d on Cantor space. By Claim 2, we can construct $\langle S_t \mid t \in {}^{<\omega}2 \rangle$ and $\langle l_t \mid t \in {}^{<\omega}2 \rangle$ such that

(1) for any t in ${}^{<\omega}2$, S_t is a perfect tree on 2 and l_t is a natural number,
(2) $S_\emptyset \subset S$,
(3) for any t_1, t_2 in ${}^{<\omega}2$ with $t_1 \subset t_2$, $S_{t_1} \supset S_{t_2}$,
(4) for any t in ${}^{<\omega}2$, $[S_{t^\frown\langle 0 \rangle}] \cap [S_{t^\frown\langle 1 \rangle}] = \emptyset$,
(5) for any t in ${}^{<\omega}2$ and any x in $[S_t]$, $f(x) \in N_{l_t}$,
(6) for any t in ${}^{<\omega}2$, $\mathrm{lh}(\mathrm{stem}(S_t)) \geq \mathrm{lh}(t)$,
(7) for any t in ${}^{<\omega}2$, $\mathrm{diam}(N_{l_t}) < 2^{-\mathrm{lh}(t)}$,

by induction on $\mathrm{lh}(t)$.

(For the last condition, we only need to choose a subsequence $\langle N_l' \mid l \in \omega\rangle$ of $\langle N_l \mid l \in \omega\rangle$ such that $\{N_l' \mid l \in \omega\}$ is a basis for Cantor space and for any l, $\mathrm{diam}(N_l') < 2^{-\mathrm{lh}(t)}$.)

Put

$$C = \bigcap_{m\in\omega} \bigcup_{t\in {}^m2} [S_t].$$

Then, by Fact 3.4, C is perfect and $C \subset [S]$.

Define $h\colon {}^\omega 2 \to C$ as follows:

$$h(z) = x \text{ if for any } m \in \omega,\ x \in [S_{z\restriction m}].$$

Then h is a homeomorphism.

Define $g\colon C \to {}^\omega 2$ as follows:

$$g(x) = y \text{ if for any } m \in \omega,\ y \in N_{l_{h^{-1}(x)\restriction m}}.$$

Then g is a continuous function on C.

We will show that $g \subset f$. Take any real x in C. Then for any natural number m, $x \in [S_{h^{-1}(x)\restriction m}]$. By the fifth condition about $\langle S_t \mid t \in {}^{<\omega}2\rangle$ and $\langle l_t \mid t \in {}^{<\omega}2\rangle$, $f(x) \in N_{l_{h^{-1}(x)\restriction m}}$ for any m. By the definition of g, $f(x) = g(x)$. Hence g is the desired one.

Since g is continuous, g is closed. By Remark 2.45, there is a perfect tree S' on 2 and a tree T on 2×2 such that $C = [S']$ and $g = [T]$. Note that $S' \subset S$. Take a Sacks real s'' over V with s'' in $[S']$. Since for any real x, $\neg\theta(x,\, f(x),\, \vec{r})$ and $g \subset f$, the following statements hold in V:

$$\forall x \forall y \Big((x,\, y) \in [T] \to \neg\theta(x,\, y,\, \vec{r}) \Big),$$
$$\forall x \in [S']\, \exists y\ \Big((x,\, y) \in [T] \Big).$$

Since a perfect tree on 2 and a tree on 2×2 can be seen as reals, the first statement is equivalent to a Π^1_{k-2}-formula with parameters T and $\vec{r}$ and the second statement is equivalent to a Π^1_2-formula with parameters S' and T. Therefore, by $\mathbf{\Pi}^1_{k-2}$-$\mathbb{S}$-absoluteness and Theorem 2.90, the above statements also hold in $V[s'']$.

Since s'' is in $[S']$,

$$V[s''] \models \text{“}\big(\exists y \in {}^\omega 2\big)\, \neg\theta(s'',\, y,\, \vec{r})\text{”},$$

which contradicts with $(*{*}*)$. ■

Remark 4.3.

When $n = 1$, the last proof is another proof for (d) $\Rightarrow$ (a) of (1) in Main Theorem.

References

[Add59a] John W. Addison, Jr., *Separation principles in the hierarchies of classical and effective descriptive set theory*, Fund. Math. **46** (1959), 123–135.

[Add59b] John W. Addison, Jr., *Some consequences of the axiom of constructibility*, Fund. Math. **46** (1959), 337–357.

[Bai99] René Baire, *Sur les fonctions de variables réelles*, Ann. Matematica Pura ed Applicata, Serie III^a (1899), no. 3, 1–123 (French).

[Ber08] Felix Bernstein, *Zur theorie der trigonometrischen Reihe*, Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematische-Physische Klass **60** (1908), 325–338 (German).

[BJ95] Tomek Bartoszyński and Haim Judah (Jaime I. Ihoda), *Set theory. On the structure of the real line*, A K Peters Ltd., Wellesley, MA, 1995.

[BL99] Jörg Brendle and Benedikt Löwe, *Solovay-type characterizations for forcing-algebras*, J. Symbolic Logic **64** (1999), no. 3, 1307–1323.

[Bre93] Jörg Brendle, *Amoeba-absoluteness and projective measurability*, J. Symbolic Logic **58** (1993), no. 4, 1284–1290.

[Bre00] Jörg Brendle, *How small can the set of generics be?*, Logic Colloquium '98 (Prague), Lect. Notes Log., vol. 13, Assoc. Symbol. Logic, Urbana, IL, 2000, pp. 109–126.

[Coh63] Paul J. Cohen, *The independence of the continuum hypothesis.*, Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 1143–1148.

[Coh64] Paul J. Cohen, *The independence of the continuum hypothesis. II*, Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 105–110.

[Göd38] Kurt Gödel, *The consistency of the Axiom of Choice and of the Generalized Continuum-Hypothesis*, Proc. Natl. Acad. Sci. USA **24** (1938), 556–557, Reprinted in 26–27 of [Göd90].

[Göd40] Kurt Gödel, *The consistency of the Axiom of Choice and of the Generalized Continuum-Hypothesis with the Axioms of Set Theory*, Ann. Math. Studies, vol. 3, Princeton University Press, Princeton, N.J., 1940, Reprinted in 33–101 of [Göd90].

[Göd90] Kurt Gödel, *Collected works. Vol. II*, The Clarendon Press Oxford University Press, New York, 1990, Publications 1938–1974, Edited and with a preface by Solomon Feferman.

[Hal50] Paul R. Halmos, *Measure Theory*, D. Van Nostrand Company, Inc., New York, N. Y., 1950.

[Hal74] Paul R. Halmos, *Measure Theory*, Springer-Verlag, New York Haidelberg Berlin, 1974, Reprint of [Hal50], Graduate Texts in Mathematics No. 18.

[JS89] Haim Judah (Jaime I. Ihoda) and Saharon Shelah, Δ^1_2*-sets of reals*, Ann. Pure Appl. Logic **42** (1989), no. 3, 207–223.

[Jud93] Haim Judah (Jaime I. Ihoda), *Absoluteness for projective sets*, Logic Colloquium '90 (Helsinki, 1990), Lecture Notes Logic, vol. 2, Springer, Berlin, 1993, pp. 145–154.

[Kec95] Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995.

[Kel55] John L. Kelley, *General topology*, D. Van Nostrand Company, Inc., Toronto-New York-London, 1955.

[Kel75] John L. Kelley, *General topology*, Springer-Verlag, New York, 1975, Reprint of [Kel55], Graduate Texts in Mathematics, No. 27.

[Kon39] Motokichi Kondô, *Sur l'uniformisation des complémentaires analytiques et les ensembles projectifs de la seconde classe.*, Japan J. Math. **15** (1939), 197–230 (French).

[Kun80] Kenneth Kunen, *Set theory. An introduction to independence proofs*, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland Publishing Co., Amsterdam, 1980.

[Kur58] Casimir Kuratowski, *Topologie. Vol. I*, Monografie Matematyczne, Tom 20, Państwowe Wydawnictwo Naukowe, Warsaw, 1958 (French), 4ème éd.

[Lév65] Azriel Lévy, *Definability in axiomatic set theory. I*, Logic, Methodology and Philos. Sci. (Proc. 1964 Internat. Congr.), North-Holland, Amsterdam, 1965, pp. 127–151.

[Lév79] Azriel Lévy, *Basic set theory*, Springer-Verlag, Berlin, 1979.

[Lév02] Azriel Lévy, *Basic set theory*, Dover Publications Inc., Mineola, NY, 2002, Reprint of [Lév79].

[LS18] Nicolas Lusin and Wacław Sierpiński, *Sur quelques proprietés des ensembles* (A), Bull. Int. Acad. Sci. Cracovie, Série A (1918), 35–48 (French).

[LS23] Nicolas Lusin and Wacław Sierpiński, *Sur un ensemble non mesurable B*, Journal de Mathématiques Pures et Appliquées **2** (1923), no. 9, 53–72 (French), Reprinted in 504–519 of [Sie75].

[Lus17] Nicolas Lusin, *Sur la classification de M. Baire*, C. R. Acad. Sci. Paris **164** (1917), 91–94 (French).

[Lus25] Nicolas Lusin, *Sur les ensembles non mesurables B et l'emploi de la diagonale Cantor*, C. R. Acad. Sci. Paris **181** (1925), 95–96 (French).

[Mos71] Yiannis N. Moschovakis, *Uniformization in a playful universe*, Bull. Amer. Math. Soc. **77** (1971), 731–736.

[Mos80] Yiannis N. Moschovakis, *Descriptive set theory*, Studies in Logic and the Foundations of Mathematics, vol. 100, North-Holland Publishing Co., Amsterdam, 1980.

[MS88] Donald A. Martin and John R. Steel, *Projective determinacy*, Proc. Nat. Acad. Sci. U.S.A. **85** (1988), no. 18, 6582–6586.

[MS89] Donald A. Martin and John R. Steel, *A proof of projective determinacy*, J. Amer. Math. Soc. **2** (1989), no. 1, 71–125.

[Nov51] Petr S. Novikov, *On the uncontradictability of certain propositions of the descriptive theory of sets*, Trudy Mat. Inst. Steklov., v. 38, Trudy Mat. Inst. Steklov., v. 38, Izdat. Akad. Nauk SSSR, Moscow, 1951, Translated in [Nov63], pp. 279–316 (Russian).

[Nov63] Petr S. Novikov, *On the consistency of some propositions of the descriptive theory of sets*, Amer. Math. Soc. Transl. (2) **29** (1963), 51–89.

[Rai84] Jean Raisonnier, *A mathematical proof of S. Shelah's theorem on the measure problem and related results*, Israel J. Math. **48** (1984), no. 1, 48–56.

[Sac71] Gerald E. Sacks, *Forcing with perfect closed sets*, Axiomatic Set Theory (Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967), Amer. Math. Soc., Providence, R.I., 1971, pp. 331–355.

[She84] Saharon Shelah, *Can you take Solovay's inaccessible away?*, Israel J. Math. **48** (1984), no. 1, 1–47.

[Sho61] J. R. Shoenfield, *The problem of predicativity*, Essays on the foundations of mathematics, Magnes Press, Hebrew Univ., Jerusalem, 1961, pp. 132–139.

[Sie25] Wacław Sierpiński, *Sur une classe d'ensembles*, Fund. Math. **7** (1925), 237–243 (French), Reprinted in 571–576 of [Sie75].

[Sie75] Wacław Sierpiński, *Oeuvres choisies*, PWN–Éditions Scientifiques de Pologne, Warsaw, 1975 (French), Tome II: Théorie des ensembles et ses applications, travaux des années 1908–1929, Publié par les soins de Stanisław Hartman et al.

[Sol69] Robert M. Solovay, *On the cardinality of Σ^1_2 sets of reals*, Foundations of Mathematics (Symposium Commemorating Kurt Gödel, Columbus, Ohio, 1966), Springer, New York, 1969, pp. 58–73.

[Sol70] Robert M. Solovay, *A model of set-theory in which every set of reals is Lebesgue measurable*, Ann. of Math. (2) **92** (1970), 1–56.

[Sus17] M Suslin, *Sur une definition des ensembles mesurables B sans nombres transfinis*, C. R. Acad. Sci. Paris **164** (1917), 88–91 (French).

[Tru88] J. K. Truss, *Connections between different amoeba algebras*, Fund. Math. **130** (1988), no. 2, 137–155.

[Vit05] Giuseppe Vitali, *Sul problema della misura dei gruppi di punti di una retta*, Gamberinie Parmeggiani, Bologna, Tip., 1905 (Italian).

[Woo82] W. Hugh Woodin, *On the consistency strength of projective uniformization*, Proceedings of the Herbrand symposium (Marseilles, 1981) (Amsterdam), Stud. Logic Found. Math., vol. 107, North-Holland, 1982, pp. 365–384.

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