Calculi
for
Constructive Communication

ILLC Dissertation Series 1994-4

For further information about ILLC-publications, please contact:

Institute for Logic, Language and Computation
Universiteit van Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam
The Netherlands
phone: +31 20 525 6090
fax: +31 20 525 5101
e-mail: illc@fwi.uva.nl

ITK Dissertation Series 1994-1

For further information about ITK-publications, please contact:

Instituut voor Taal- en Kennistechnologie
Katholieke Universiteit Brabant
Po Box 90153
5000 LE Tilburg
The Netherlands
phone: +31 13 66 3060
fax: + 31 13 66 2537
e-mail: itk@kub.nl

Calculi for Constructive Communication

A Study of the Dynamics of Partial States

Proefschrift ter verkrijging van de graad van doctor
aan de Katholieke Universiteit Brabant,
op gezag van de Rector Magnificus, Prof. Dr. L.F.W. de Klerk,
in het openbaar te verdedigen
ten overstaan van een door het college van dekanen
aangewezen commissie
in de aula van de Universiteit op dinsdag 14 juni 1994 te 16.15 uur

door

Johannes Ogier Maria Jaspars

geboren te Oegstgeest

Promotoren: Prof. dr. J.F.A.K. van Benthem
Prof. dr. H.C. Bunt

ISBN 90-74795-08-0

Cover-design: Mehran Zirak

Author's address: Centrum voor Wiskunde en Informatica, Kruislaan 413,
1098 SJ Amsterdam, The Netherlands
phone: +31 20 548 4080, e-mail: jaspars@cwi.nl

"La veritá non é in un solo sogno, ma in molti sogni"

'Il Fiori delle Mille e Una Notte'

"The truth is not in one dream, but in many dreams"

Pier Paolo Pasolini

Thanks to everybody who contributed to this thesis

Voor Ampa

Contents

Prologue

This dissertation presents a mathematical logical analysis of the infrastructure of *partial* worlds, and demonstrates how its model-theoretical treatment can be used for a constructive formalization of the dynamics of a group of reasoning and communicating agents. Our choice in favor of partial worlds as the basic semantic entity, which will be motivated elaborately in this introductory chapter, distinguishes our treatment from well-known proposals of formalization of epistemic dynamics and communication, such as [Jones 1983], [Appelt 1985] and [Cohen & Levesque 1990]. The latter theories have been founded on the classical principle of total or two-valued worlds and their underlying two-valued logic.

Because of this departure at the very basis of the model theory, we have spent much effort in the technical organization of the variety of logics on the basis of partial worlds. For this reason a complete part (II) of this thesis deals only with meta-theoretical issues. It presents a technical streamlining of completeness and decidability proof procedures. By means of a relatively small rearrangement of standard techniques, we will demonstrate that partial logics do not have to be much more troublesome, from a mathematical point of view, than their regular two-valued counterparts.

Besides this mathematical second half, the first part also appears technical at first sight. Nevertheless, the mathematics of part I is considerably less dense. It does not include long proofs, but is meant as a conscientious presentation of the relevant calculi and some prefab meta-theory, that prepares for part II. Part I may therefore appear to be somewhat stuffy to some of the readers. We have maintained this order, however, to guarantee the mathematical transparency of part II.

Following Gärdenfors' influential general view on epistemic dynamics [Gärdenfors 1988], we will first specify our means for static representation of information and then present the dynamics of such epistemic registrations. The static side of our model-theory consists of a straightforward partial variant of the *possible* *worlds semantics* of modal logic. Modal logics have been widely advocated to be used as epistemic logics since the work of Hintikka [Hintikka 1962] [Hintikka 1969]; and partial variants have emerged in epistemic logic in the last ten years, e.g. [Levesque 1984], [Lakemeyer 1991] and [Thijsse 1992].

The alternative aspect of partial possible worlds semantics on which this thesis will focus is its dynamics, which deviates from, or rather extends, the dynamics of ordinary two-valued possible worlds semantics. The dynamic perspective of classical possible worlds semantics, as proposed in formal linguistics [Stalnaker 1979], philosophy [Landman 1986] and logics of common sense reasoning [Veltman 1991], is purely *eliminative*. This means that in two-valued possible worlds semantics information grows through the elimination of (total) possibilities.

Partial possible worlds semantics adds a *constructive* component to this eliminative effect. This constructive dimension of epistemic dynamics is technically possible because the informational content of a partial world may grow, something which is impossible for a total two-valued world. The key issue of this thesis is to point out how such different ways of information flow can peacefully cohabit in the theory of partial possible worlds. On the basis of this construction-elimination dynamics we define relatively simple calculi for reasoning about interacting agents.

The contents of the thesis

We start with an extensive introductory chapter which unfolds our view on epistemic dynamics by means of interaction. This chapter is also meant as a compensation for the technocratic flavor of this dissertation. It gives the reader enough background information to bring the mathematics of part I and II to life. Of course, we will also try to sharpen the intuitions on the way, but this extensive introduction explains the basic motivations. It takes care for a thorough beginning and for convenient thumbing back.

The other reason for us to start with an informal introduction, is simply to give an explanation, beforehand, of the differences and philosophical advantages of partial logic with respect to two-valued semantics when it comes to a formal understanding of epistemic dynamic processes like communication.

The first two chapters of part I will hold on to the above-mentioned order of statics and dynamics. Chapter 2 discusses partial truth-assignments, their underlying calculus and, most importantly, their modal extensions, whereas chapter 3 presents different constructive extensions, which provide explicit inference for dynamics of modal (epistemic) information. The latter chapter ends with a logic (**Mud**) that describes the constructive and eliminative dynamics which we propagate.

The last chapter of part I, chapter 4, presents epistemic logical formalisms, based on the static partial modal formalism (**M**) of chapter 2 and the above mentioned dynamic system **Mud**. Besides axiomatic strengthenings, we also discuss additional expressive decoration for suitable interpretation of communicative actions. Essential linguistic ingredients for a dynamic theory of communication which will be formalized here are *intentional modalities* and the representation of *mutual epistemic information* of a group of agents. In section 1.4 of the introductory chapter we give an informal presentation of such modalities over partial worlds, and specify their use for different dynamic interpretations of simple communicative actions as assertions and questions. We will also demonstrate how different cooperation postulates and conversational maxims with regard to groups of communicating agents can be axiomatized. In chapter 4 we will specify the technical constraints which evolve from such pragmatic principles.

From the perspective of modal logic, the partial and constructive logics, which are to be presented in this thesis, are relatively new. Therefore, the thesis also contributes to the general knowledge of modal logic which has been a reason for us to dedicate a full part, part II, of the thesis to the meta-theory of these modal formalisms. The chapters 5 and 6 of part II are of course essential for the general setting of the thesis, as they prove the completeness and decidability of the logics which we present in part I. These two chapters present a generalization of the well-known *Henkin* method of proving completeness and decidability in conventional modal logic [Hughes & Cresswell 1984]. This generalization of the Henkin procedure facilitates accommodation of partial modal logics in the meta-theory of modal logic. Chapter 7 is meant as an initiative for the development of correspondence theory for partial modal logics. It has been incorporated merely as a contribution to general modal logic.

It will be obvious by now that this thesis is not heading for the one and only true logic of communication. What it does show is how a flexible epistemic dynamics can be defined on the basis of partial possible worlds. In fact, the only philosophical choice of this dissertation is our plea in favor of the earlier mentioned dynamics of partial worlds, which is a rather primitive fundamental preference. We leave it to linguists and philosophers in the field to support or reject different interpretations of communicative actions and principles of pragmatics. What we like to explain in section 1.4 and chapter 4 is how a variety of dynamic interpretations of such actions, and principles of pragmatic rationalism of communicating agents, can be stipulated in terms of partial modal formalisms.

We will try to build mathematical bridges from partial modal logic to theories of communication. The thesis should therefore be read as a study in applied logic, rather than as a contribution to philosophy or linguistics.

Chapter 1

Introduction

Most proposals for model-theoretic semantics of communicative actions, have been founded on modal logic [1] and its possible worlds semantics, for example [Jones 1983] and [Appelt 1985]. Such theories can be separated into a *static* and a *dynamic* component.

The static component concerns the epistemic information which the communicating partners possess at a certain point in time. Traditional possible worlds analyses for formal interpretation of the contents of epistemic propositional attitudes are briefly described in section 1.1.

The dynamic component of modal logical analyses of communication consists of a formalization of the way in which these epistemic information states are manipulated by *communicative actions.* A very important prerequisite of such a formal dynamic theory is a general structural specification of the way in which information states *may* change. In other words, we need to give a formal description of the freedom of the flow of information in a communicative setting.

Throughout the thesis we will hold on to this distinction, which follows Gärdenfors' influential view, as presented in [Gärdenfors 1988], on the construction of formal theories on changing epistemic information states. In fact, we will follow the line of thought of his book chronologically. First we present the static part, and then the additional dynamics. The distinction between statics and dynamics will be kept throughout the dissertation.

In this thesis we propose a combination of two traditional perspectives on information change. The first is the *eliminative* perspective, which has been propagated by various dynamic approaches to semantics and philosophy of natural language, e.g. [Stalnaker 1979] [Landman 1986], and more recently in dynamic model-theoretic approaches to common sense reasoning, e.g. [Veltman 1991]. The second view is the *constructive* analysis of information change which has evolved from model-theoretic interpretation of constructivistic philosophies of

[1] Modern standard texts on modal logics are [Chellas 1980], [Hughes & Cresswell 1984] and [Bull & Segerberg 1984].

the foundations of mathematical reasoning, e.g. [Fitting 1969]. These traditional interpretations of the dynamics of information are presented in section 1.2.

We will advocate a combination of these different dynamic views, in which we make a small, but particularly important, adaptation of the standard possible worlds model theory for static representation of epistemic information states. Instead of classical modal logics, which is most often employed for epistemic reasoning [Hintikka 1962] [Halpern 1986], we propose *partial* modal logics as in [Thijsse 1992] being a more suitable candidate for this static epistemic dimension. Partial modal logic arises from a partialization of ordinary possible worlds semantics. In general, this partialization presents a finer logical analysis of propositional attitudes [Barwise & Perry 1983]. Partial worlds may grow – become less partial – or may be eliminated in order to get rid of uncertainties. This roughly indicates how partial modal logic unfolds *both* a constructive and an eliminative dimension along which information flows [2]. Extending partial modal logics in this dynamic fashion gives rise to what we will call *constructive modal logics*. These logics are presented in section 1.3 as elementary constituents of logics of interaction.

From the viewpoint of dynamic semantics, the thesis focuses on one other important issue. Besides the two-dimensional dynamics, we wish to establish logics for reasoning about the simple epistemic dynamics of groups of interacting agents. Dynamic semantic theories are most often based on *single* agent analyses. They formally describe the way an interpreter of a language has to make up his mind given a certain input of consecutive sentences of this language. The reason for this limited epistemic setting of dynamic theories is that interpretation of text, described by the epistemic route of one virtual interpreter which reads a text, provides for formal comprehension of dynamic phenomena of natural language, such as anaphora and presuppositions. These dynamic appearances in natural language are of primary interest to natural language semanticists. Single agent interpretation suffices to get a good formal understanding of these phenomena, although complete dynamic interpretation of discourse requires a formalization of the above-mentioned multiple epistemic interchange [3].

Dynamic semantics for natural language interpretation originated from the fundamental insight that there should be a clear distinction between the pure static logical content of a proposition and its dynamic content, where the propo-

[2] Also, in the dynamic semantic perspective of Kamp's discourse representation theory [Kamp 1984] and Heim's file change semantics [Heim 1982], we find a small constructive component present. In these first order theories, variable assignments are taken to be finite. Existential statements enrich the domains of these 'partial' variable assignments. The constructivity in these theories is restricted to these sentences. Other propositions are taken to have a purely eliminative dynamic meaning. The role of constructivity in our approach is more dominant. Every consistent update may have a constructive effect.

[3] One might also claim that this multiple agent generalization is needed for interpretation of text. In fact, we deal with a one-way communication of writer and reader. Interpretation of sentences from the text may also depend on the knowledge that the reader has of the writer, and which may also change during the interpretation of the text. For a plea for such sender/receiver dynamics see [Bunt 1990b].

sition is considered as an utterance in a communicative context. According to standard formal semantics, the static interpretation is simply the set of worlds or situations in which this proposition is true. The dynamic interpretation should be *context sensitive.* The dynamic meaning of a proposition is then described by the way it modifies a given context.

In dynamic semantics, the context is nearly always restricted to its epistemic part, that is the epistemic states of interpreting agents. In this way, the meaning of an asserted proposition relies heavily on the propositions which have been asserted before.

The singular dynamics can only be partially satisfactory. Dynamic semantics of interaction obviously requires a multiple agent generalization of this perspective. If an agent a tells another agent b that "p" is the case, then this assertion not only changes the information state of b; for the speaker a this message also yields an epistemic switch. On a simplified account, this agent knows after the assertion "p" that the other agent, b, knows that "p". This information is particularly important for a to understand a continuation of the dialogue by b. The above mentioned epistemic switch is only a detail of the full epistemic 'force' of this assertion. Complete idealization of the full dynamic content of the message "p" from a to b is the change to a new state where "p" is *common* or *mutual knowledge* of a and b. This means that a and b know that "p", and that a and b know that a and b know that "p", and that a and b know that a and b know that a and b know that "p", etcetera. Of course, this interpretation depends on the unrestricted acceptance of a's information by b. Many other more sensible real life interpretations of a's message could be given. Many other context determining variables play a role. a might be blushing, turning up his nose or stammering. Moreover, dialogue roles like selling second-hand cars or teaching mathematics, and many other external influences might affect the interpretation of a's utterance. We have chosen for the reasonably safe position of a logician, who studies laboratory dialogues, and we therefore limit context sensitivity to individual attitudes.

Another important aspect of context, which is particularly important for dynamic modeling of conversation, is formed by *intentional* or *preferential* attitudes. They should be taken into account to obtain a proper understanding of communicative acts (e.g. [Searle 1983] [Bunt 1989] [Cohen & Levesque 1990]) [4]. They represent the communicating agents' personal views and preferences on the epistemic dynamics of interaction. All agents have a personal dynamic perspective on how a certain interactive setting may change their own epistemic state and those of the other interacting partners. Intention generates the communicative acts of an agent during interaction to a large extent. Agents try to establish epistemic configurations which match their personal preferences. An example which illustrates the importance of such preferential registration is the demand for formal comprehension of questions. If agent a asks b whether "q" is the case, then the questioner a enriches the mutual information of a and b

[4] For a recent survey of contextual parameters which are relevant for modeling communication see [Bunt 1994].

by a's intention to know whether ""q" holds or not. In an additional section of chapter 4 we show how intentions can be embedded in the framework of the dynamic epistemic logics of this thesis.

In the last section of this introduction we give an introductory exposition of how mutual belief reports and intentions can be embedded in the two-dimensional, i.e. constructive and eliminative, dynamic model-theory which we advocate. Our proposal follows roughly the line of the possible worlds analysis in [Cohen & Levesque 1990] of intentional attitudes.

A proper formalization of the 'subjectivistic' dynamic approach to semantics can be established by means of *relational* interpretation of propositions. The dynamic denotation of a proposition is taken to be a relation between in- and output states, which is in contrast to the classical static view according to which a proposition denotes a set of states. The dynamic relational interpretation of a proposition describes how an informational context, that is the input state, is changed by addition of the informational content of the proposition. We have chosen to incorporate both kinds of interpretations as mentioned above, and we will further motivate this in section 1.3. Doing so, we follow the earlier mentioned view on epistemic dynamics of Gärdenfors. The style of logic will be in line with the so-called dynamic modal logic of [van Benthem 1991b] and [de Rijke 1992] [5], where the distinction of statics and dynamics has been made explicit.

Summarizing, this introductory chapter consists of four sections. The first two sections present conventional proposals for static and dynamic epistemic reasoning respectively. The third section unfolds our motivations to prefer partial modal logics for epistemic representation and the two-dimensional dynamics of such epistemic states. The last section indicates how this dynamic epistemic formalism can be employed for modeling communication.

1.1 Modal logic and propositional attitudes

Modal logic, which has its roots in analytic philosophical studies on the concepts of necessity and contingency, has been used extensively as the basic formalism in the development of logical analyses of epistemic propositional attitudes like knowledge and belief [Hintikka 1962] [Lenzen 1978] [Halpern 1986]. Nowadays the modal approach to logical interpretation of epistemic reports is widely adhered to. Due to this development it has become an independent branch of applied modal logic called *epistemic logic*.

The invention of possible worlds semantics for modal logic, due to Carnap, Kanger and especially Kripke, has brought modal logic within sight of many other disciplines. Due to the pioneering work of Montague [Montague 1974] in the late sixties, modal logic has become influential in formal semantics of natural language. All kinds of intensional phenomena in natural language, such

[5] For an extensive general view on dynamic modal logic the reader is referred to de Rijke's thesis [de Rijke 1993].

as attitude reports, can be given a clear semantics in terms of possible worlds. Somewhat later, modal logic also made its entry in theoretical computer science (e.g. [Pratt 1980] [6]) and artificial intelligence (e.g. [Moore 1980] [7]).

Possible worlds semantics presents a simple and elegant model-theoretic analysis of reasoning with uncertainties. Every world can be seen as a state of information, which is linked to a given set of possible worlds by a so-called *accessibility relation*. Such a relation is meant to determine the intensional or modal information of the original world. In ordinary modal logic, a sentence or proposition is then said to be *necessarily* true if it holds in all accessible worlds, and it is *possibly* true if it holds in it at least one of the accessible worlds.

From the point of view of epistemic logic, the above-mentioned original world is taken to be the actual world, with its own extensional information, in which a certain agent lives. The related or accessible worlds, which are called *epistemic alternatives* in epistemic logic, represent the uncertainty of the agent. According to the agent, all alternatives can be the actual world. Subsequently, the only information that this agent is sure of is the information which is shared by all these epistemic alternatives. Certainty or knowledge as seen in ordinary epistemic logic therefore coincides with the interpretation of necessity in plain modal logic.

1.1. FIGURE.

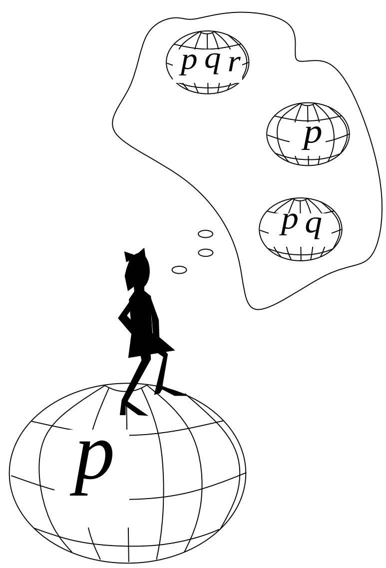


Figure 1.1 gives a partial illustration of the epistemological outlook on possible worlds semantics. The agent a knows that p, but does not know whether q.

[6] Pratt's modal logic has been baptized *propositional dynamic logic* (PDL). An extensive survey on this branch of modal logic can be found in [Harel 1984]. The original motivation was to establish a possible worlds interpretation of imperative programming languages. Also in process algebra, modal logical classifications have been found for certain calculi (e.g. [Stirling 1987]).

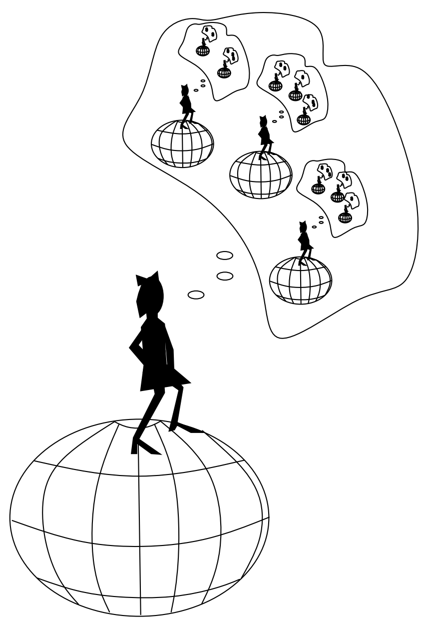
[7] Moore's logic is a combination of (a part of) dynamic logic and epistemic logic and is meant as a formal approach to robotics.

Different epistemic attitudes can be incorporated through scaling the epistemic alternatives. A very rough classification of possible worlds is the distinction between epistemic and doxastic alternatives. The latter collection is taken to be a subset of all the epistemic alternatives. Intuitively, this selection means that the agent takes this set to be more probable than the remaining epistemic alternatives. The agent is then said to believe a proposition if it holds in all the worlds of this selected set of doxastic alternatives. In the configuration in the figure 1.1, it might be the case that the agent a thinks that the p, q, r- and the p, q-world are more probable than the p-world. In this situation, the agent a does not know whether q, but nevertheless believes that q [8].

Knowledge differs logically from belief by the truth of its content [Hintikka 1962]. This can be understood in terms of possible world models as in figure 1.1 by including the reality to be one of the epistemic alternatives. This need not be the case for doxastic alternatives. Suppose once more, that the agent a's doxastic alternatives in figure 1.1 are the q-worlds. Consequently, a believes that q, but q is not true.

A full single agent possible worlds model also accounts for reflexive capacities with respect to an agent's personal knowledge. In terms of possible worlds, the agent meets herself thinking about the world in every accessible world. Incorporation of such introspective capacities can be accounted for by means of models which have the more general structure of the next figure.

1.2. FIGURE.



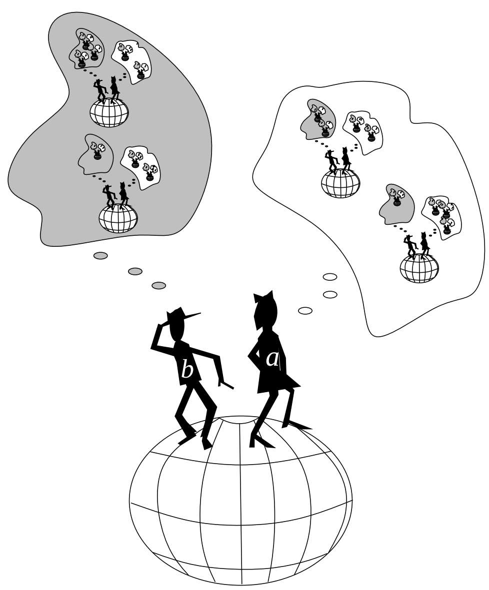
Introspection is usually taken to be so strong that an agent is fully certain about personal uncertainties. In fact, this *full introspection* restrains the class of

[8] In the literature on epistemic logic we find more fine-grained analyses of different attitudes, especially different degrees of belief. Full probabilistic ordering of worlds has been proposed and investigated in [Gärdenfors 1975]. Another approach to this differentiation is comparing the amount of epistemic alternatives which support a certain proposition [Lenzen 1980]. For an extensive survey on these extended modal formalisms see [van der Hoek 1992].

suitable models for interpreting epistemic attitudes radically. Let's say S is the set of alternatives. Full introspection means that the set of alternatives which is accessible from a world $s \in S$ must be S itself. For single agent epistemic logic, the one-layer models of figure 1.1 suffice. The set of 'second degree' worlds, or the set of worlds in the second inner 'mental clouds' in the picture above, is identical to the set of immediately accessible worlds. Therefore, for singular fully introspective logics, these long distance worlds do not have to be represented.

This simplification is only possible when we model single agent situations. The fully introspective capacity of being certain about one's own uncertainties can evidently not be extended to the uncertainties of somebody else. This means that in the alternatives of an agent a the alternatives of another agent b might very well fluctuate. The general picture of a model with two agents is presented in the following figure.

1.3. FIGURE.



Technically speaking, multiple agents can be accounted for by allowing more accessibility relations. The individualization of alternatives is simulated in figure 1.3 above through the different shades of the mental clouds. A formalization of this is a pair $\langle W, \{R_a\}_{a \in A} \rangle$ with W and A being non-empty sets of worlds and agents, respectively. Every R_a symbolizes the individual accessibility relation over the universe of worlds W, i.e. $R_a \subseteq W \times W$, of a specific agent a. This general framework is called a *possible worlds* or *Kripke frame* [9].

Full introspective capacity with respect to individual uncertainties such as we have mentioned earlier can now be formalized as follows:

$$\forall x, y, z \in W, \forall a \in A : R_a(x, y) \Rightarrow (R_a(x, z) \Leftrightarrow R_a(y, z)) \quad (1).$$ [10]

[9] A *Kripke model* is the result of addition of a truth assignment V to such a Kripke frame. Roughly speaking, it assigns information to all the possible worlds in the frame.

[10] The equivalence $\Leftrightarrow$ abbreviates the better known structural frame properties *transitivity* and *Euclidicity*. The former refers to the property which evolves when $\Leftrightarrow$ is replaced by $\Leftarrow$, and Euclidicity is caught through substitution of $\Rightarrow$ there.

This principle says that the set of accessible worlds of a given world x coincides with the set of accessible worlds of any accessible world of x. The first-order formula (1) can also be replaced by the shorter relational equation $R_a = R_a \circ R_a$ [11]. Fully introspective frames are a very specific subclass of all possible worlds frames. With regard to figure 1.3, full introspection can be understood as follows: every cloud of a certain shade contains the same set of worlds as all its inner clouds of the same shade.

For the formal interpretation of multiple attitudes, the set of accessibility relations needs to be extended proportionately again. Especially in modal logical theories of communication, such as [Jones 1983] and [Appelt 1985], the variety of proposed modalities to interpret different attitude reports is enormous. The most straightforward manner to interpret the variation of attitudes is to extend the general framework by extra indexing of the accessibilities: $\langle W, \{R_{a,i} \mid a \in A, i \in I\}\rangle$, where I is the assortment of attitudes. Modal logical analyses of communication pay most of their attention to analytic philosophical studies of the logical interplay of these attitudes, which semantically boils down to definitions and justifications of the structural interplay of the relations $R_{a,i}$ [12].

In this thesis we will not concern ourselves with investigations on the diversity of attitudes for a realistic interpretation of everyday dialogues. Instead, we will focus on a more fundamental question. Our main concern is to find an appropriate formal description of the *construction* of new possible worlds configurations by means of communication. As said earlier, we propagate an alternative view for the dynamics of possible worlds. Instead of exploring expressive wealth for fine-grained interpretation of all kinds of communicative actions, we try to limit this expressivity to an acceptable level, so that clear interpretations of simple actions can still be stipulated, without losing mathematical tractability. This sober attitude prevents us from debouching into an ocean of philosophical speculations.

A clear advantage of this technical inspection of the dynamics of limited epistemic information is that it has brought us natural and formally transparent interpretations of the other dynamic modalities which are of particular importance for a mathematical understanding of interaction. In section 1.3 we will illustrate how *dynamic modalities* can be given a clear semantics in terms of the dynamics of partial possible worlds. These modalities are called dynamic because they are interpreted in terms of the information change. In chapter 3 and 4 a precise formal specification of these interpretations is presented.

The epistemic part of our calculi of constructive communication, which are to be presented in chapter 4 and will be abbreviated as $\mathbf{C}^3$-calculi after the title of this dissertation, consists of only one modality. The incorporation of more

[11] The symbol $\circ$ denotes relational composition: $R \circ S(x, y) \Leftrightarrow \exists z \; : R(x, z) \;\&\; S(z, y)$.

[12] In figure 1.1 we already explained such a constraint to make a proper distinction between knowledge and belief. This simple constraint told us that the set of doxastic alternatives is taken to be a subset of the full set of epistemic alternatives. The logical consequence of this interplay of different alternatives yields that knowledge always implies belief.

epistemic attitudes disturbs the technical presentation and would only divert the reader from the key issues of the thesis. The epistemic modality which we discuss is *conviction*. Pure knowledge, as it implies the truth of its content, is taken to be too strong for interpretation of communication.

The formula $\Box_a\varphi$ denotes that the agent a is convinced that the proposition φ is the case [13]. Readers who are familiar with epistemic logics and who are used to differentiation of 'degrees of belief' may replace our notion of conviction by the strongest interpretation of belief, that is believing a proposition without doubting it. Whenever we speak of belief in the remainder of this text, we refer to this optimally strong doxastic attitude.

The variety of dynamic and preferential modalities, that is actions and intentions, is also limited up to a necessary but acceptable level. The dynamic dimension of the $\mathbf{C}^3$-calculi is restricted for the same reasons as the economic use of epistemic expressivity is.

1.2 Growth of information

From our point of view the most fundamental requisite of modal logics for communication is to establish a formal interpretation of the growth of information. In communication we deal with transfer of information between agents, and therefore we have to specify the way states of information, such as the epistemic possible worlds models in figure 1.3, grow during interaction.

A great deal of our investigations has been dedicated to the formalization of growth of information in partial possible worlds semantics. This research has led to the constructive modal logics of chapter 3 which constitute the underlying modal formalism of the family of $\mathbf{C}^3$-calculi. In this thesis we present the dynamic perspective that follows from our choice in favor of partial possible worlds semantics. All $\mathbf{C}^3$-logics will also contain logical equipment to reason about retractions of information.

Before switching to our own point of view, we discuss two traditional views on growth of information. The alternative perspective which we propose in section 1.3 is based on a combination of these traditional views.

Growth of information in classical modal logic

Despite the convenience and theoretical elegance of possible worlds interpretation of epistemic propositional attitudes, it suffers from an intrinsic unnatural property. Interpretation of attitudes like knowledge and belief as a necessity operator implies that little knowledge or belief corresponds to large models, and vice versa. Roughly speaking, the dimension of the content of such an attitude is taken to be inversely proportional to the degree of uncertainty which is represented by the quantity of the corresponding possible worlds. Minimalistic approaches to knowledge representation such as autoepistemic logic [Moore 1983] [Moore 1984], and circumscription of knowledge [Halpern & Moses 1984]

[13] In the case of single agent analysis in the next chapters 2 and 3 the index a is omitted.

illustrate most clearly this deficit of possible worlds semantics. A model where 'only p' is known [14], is called a *minimal* model for p. Highly conflicting to this designation and its underlying intuition, such a minimal model is particularly large in size. In fact it is the largest among the models where p is known. All different 'p-worlds' have to be contained to make the ignorance maximal.

It is not because of technical reasons that we are against large models for minimal interpretations, but we oppose to the total possible worlds perspective according to which ignorance is the same as *present* uncertainty. We believe that ignorance is only partially induced through uncertainty. This latter kind of ignorance we will call *active* ignorance. The other source of ignorance is simply lack of information. A distinction of uncertainty and absence of information is offered by the partial possible world semantics presented in the next section 1.3.

As a consequence of the possible worlds analysis of knowledge and belief, the only way to define enrichment of such attitudes is by means of elimination of uncertainty. Especially among natural language semanticists this 'destructive' approach towards cognitive dynamics has been propagated. Illustrative examples are Stalnaker's work on assertion [Stalnaker 1979], Heim's file change semantics [Heim 1982] and Landman's elimination models [Landman 1986]. Also in Veltman's recent analysis of default reasoning in terms of so-called update semantics [Veltman 1991] informational enrichment is taken to be purely eliminative.

Constructive logic and growth of information

An alternative to the eliminative approach to growth of information is offered by the kind of model-theory which emanated from semantic studies of many *constructive* logics. Opposite to the possible worlds in the models for classical modal logic the information states are taken to be *partial*. Intuitively, such a state may be interpreted as the most simplistic form of knowledge representation. Partiality refers to the assignment of truth values. The information which is left undefined, pictures the current ignorance of the agent.

To this static representation of information, by means of partial truth assignment, a simple dynamic component is added. The structure of these models is a special kind of Kripke frame: $\langle W, \leq \rangle$, with $\leq$ being a partial order [15] or a pre-order [16] over the collection of possible worlds W. Although these structures are a specific class of Kripke frames, the accessibility pattern of the models in section 1.1 must conceptually be distinguished clearly from the information order in the constructive models. This latter relation is meant to model the growth of information, while the accessibility patterns of Kripke frames in the previous section are meant for static modeling of beliefs on the basis of multiple uncertainties.

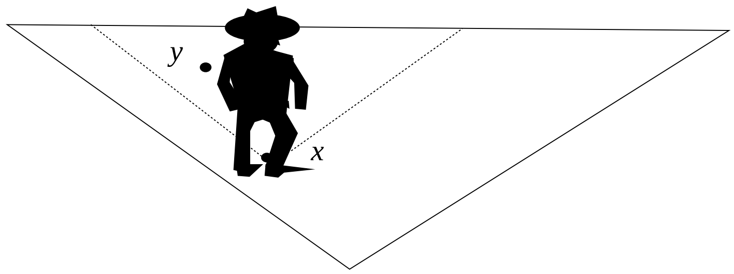
[14] By some specific agent.

[15] A partial order is a reflexive (every state is an extension of itself) transitive (all extensions of extensions of a state are also extensions of this state) anti-symmetric (if two states are extensions of one another then they must be identical) relation.

[16] A pre-order is a reflexive transitive relation.

If $x \leq y$, then y is said to be an *extension* or *enrichment* of x [17]. In other words, y is a possible continuation of x, which represents an information state of an agent on its way of acquiring knowledge.

1.4. FIGURE.



This incorporation of the extension relation as a primitive semantic parameter provides explicit reasoning about the growth of information in constructive logic. A simple example is the use of the negation in Heyting's *intuitionistic* logic [18]. In intuitionistic logic "not φ"means that the truth of the proposition φ can never be demonstrated or proved. In terms of the 'information states' semantics, this means that "not φ" is verified by a state x if and only if no extension of x verifies "φ". In figure 1.4, this means that in all points in the inner triangle φ does not hold.

Also the implication and universal quantification in intuitionistic logic are interpreted intensionally over enrichments of information states. The truth of an implication "if φ then ψ" means intuitionistically that a method which transfers any proof of φ into a proof of ψ is currently present. From an epistemic point of view this means that the knowledge of "if φ then ψ" refers to a situation, where enrichment of the current knowledge with φ automatically leads to knowing that also ψ holds. For the intuitionistic reading of universal quantifiers in its predicate logical version in terms of information states we refer to [Troelstra & van Dalen 1990].

We may have been given a paradoxical impression here. Little knowledge, also with respect to this constructive semantics, yields large models, as it generates a lot of possible extensions. Nevertheless, this impression is due to the resemblance of the technical equipment for dynamic and static modeling of information. The size of constructive models pictures the ways that information may grow. In other words, it represents the dynamic freedom. The size of the epistemic accessibility represents static registration of the currently active uncertainties.

An example of an application of constructive model-theory in formal semantics is so-called *data-semantics* of conditional sentences of Veltman [Veltman

---

[17] In order to guarantee that such an extension y enriches x indeed the truth assignment V, which defines a model on such an information frame $\langle W, \leq \rangle$, need to be monotonic over $\leq$. This means that if V determines a truth value for a certain proposition in x, then V also gives the same truth value to this proposition in y.

[18] Heyting presented a logical formalization of Brouwer's intuitionistic philosophy on the foundations of mathematics. For an extensive survey on Brouwer's philosophy see [van Stigt 1990]. For textbooks on the science of constructive mathematics which evolved from this philosophy see [Beeson 1985] or [Troelstra & van Dalen 1990].

1985]. Truth of a conditional is interpreted here just like the verification of the intuitionistic implication. This interpretation yields a plausible alternative to the rigorous material implication in classical logic.

The difference between Veltman's conditional logic and intuitionistic logic is the postulation of an explicit status to *falsity* as an additional truth value. This negative truth-value, which is not the same as the absence of truth because of the partiality of truth assignments, does not appear in the constructive philosophy of intuitionism. In this respect Veltman's conditional logic bears a closer resemblance to Nelson's *logic of constructible falsity* [Nelson 1949] [Nelson 1959] [19]. This logic extends intuitionistic logic with an extra negation which refers to the falsity of its argument. In Nelson's axiomatization of the logic of constructible falsity, this negation implies the intuitionistic negation and has therefore also been referred to as intuitionistic logic with strong negation [Gurevich 1977] [20]. Additional to the intuitionistic philosophy of 'proof as detection of truth', Nelson proposed *refutation* as a construction to determine falsity.

The semantics for Nelson's and Veltman's logic presents suitable formal equipment for extending partial logics for reasoning about the growth of information. In these partial logics a state is just a partial truth-value assignment. A proposition might be either true, false or left undefined by such a *partial valuation.* As we have already mentioned, and will further motivate in section 1.3, this partialization makes possible worlds more suitable for logical knowledge or belief representation (see also [Thijsse 1992] [van der Hoek, Jaspars & Thijsse 1993]). The locomotion of such static representations can be given a clear interpretation by means of the constructive extension order.

Of course, the above-mentioned constructive logics only describe the dynamics of a single epistemic alternative. This is not very surprising when we consider the philosophical motivations of Nelson's logic. It is meant to model mathematical knowledge. A partial state consists only of proofs and refutations, represented by the information which is verified and the information which is falsified by this state respectively. These mathematical constructions entail only information that is certain, and therefore, is liable to a very restricted dynamics. A logical consequence of this rigid interpretation is *persistence* [21] of information and the absence of *disjunctive uncertainty.*

Persistence simply means that once we have a proof or refutation of a proposition, we never can get rid of it. Semantically this means that once a classical truth-value has been assigned to a proposition, it will keep that truth-value in all informational extensions.

The absence of disjunctive uncertain information in constructive logics, such as intuitionistic logic and Nelson's logic, is a fundamental difference with clas-

[19] There are some fundamental differences between Veltman's conditional logic and Nelson's constructive logic. We will discuss them in chapter 3.

[20] In so-called four valued variants of Nelson's logic falsity does not imply the absence of truth. Such interpretations subsequently skip the inferential dominance of Nelson's negation above Heyting's negation (e.g. [López-Escobar 1972]).

[21] Veltman's conditional logic is not completely persistent. This is not very surprising because it is not meant to model mathematical reasoning.

sical logic. It means that once we have a proof of a proposition of the form "φ or ψ", we must have a proof of one of the disjuncts, "φ" or "ψ" [22]. A simple consequential divergence between these constructive logics and classical logic is its omission of the principle of *the excluded middle*, which says that for any proposition φ "φ or not φ" is true [23].

In order to deal with uncertainties in a constructive setting, we will combine the accessibility interpretation of possible worlds semantics and the constructive semantics of Nelson's and Veltman's logics. This means that we describe growth of information along two dimensions, both eliminative *and* constructive. Information can be gained through the elimination of uncertainty, just like the simple dynamics of classical modal logic, and through enrichment of the epistemic alternatives in the constructive way. This additional constructive dimension can be implemented by switching to partial states as epistemic alternatives. The rigid interpretation of truth and falsity by means of proof and refutation is replaced by available *evidence* and *counter-evidence* respectively. In the following section on the combination of partiality and modality we will present a more precise outline of this dynamic perspective.

1.3 Partial modal logic and its dynamics

The combination of modality and partiality has been an issue of extensive debate among natural language semanticists and philosophers of language since the presentation of situation semantics by Barwise and Perry [Barwise & Perry 1983]. Their work can be looked upon as a more cognitive approach to natural language understanding in reaction to the classical (onto-)logical formal semantics for natural language of Montague [Montague 1974] [24]. Despite the considerable deviation of the formal equipment of situation theory from standard logical approaches, the distinction with these latter approaches just comes down to a partialization ("It's a small world after all" cf. [Cresswell 1988]). Muskens has shown in his thesis that this partialization can be enforced without giving up the formally transparent Tarskian-Montagovian style of semantics [Muskens 1989b] [25].

[22] In Nelson's logic this also holds with respect to refutation of conjunctions. If "φ and ψ" has been refuted it must be the case that one of the conjuncts, "φ" or "ψ", has been refuted.

[23] In intuitionistic logic and in Nelson's logic *existential* uncertainty also vanishes. The proof of the existence of the assertion that a certain object has a given property is only correct, according to these constructive philosophies, if a fully identified witness which has this property can be given (for a formal explanation see [Troelstra & van Dalen 1990]).

[24] For more accessible texts on Montague's formal semantics see [Dowty, Wall & Peters 1981] or [Gamut 1991]

[25] Muskens introduced a relational theory of types [Muskens 1989a] as an alternative to the standard functional reading [Dowty, Wall & Peters 1981]. This (re-)interpretation facilitated the injection of partiality into the higher-order theory of Montague. In [Lapierre 1990] the reader finds a type theory on the basis of partial functions for a partialization of Montague semantics. In [Bunt 1990a] the reader finds a plea for the use of partiality for model-theoretic approaches to communication.

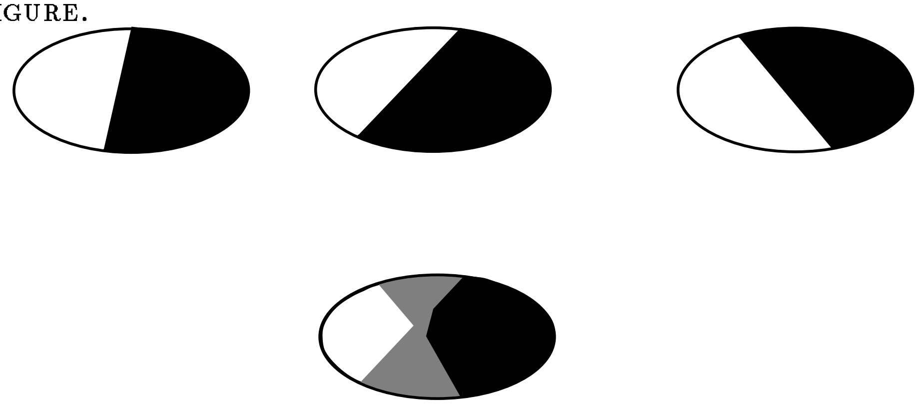
Especially for formal interpretation of propositional attitudes partial model theory seems to be more compatible with our intuitions. This does not only apply to the epistemic attitudes, which we discussed earlier, but extends to the much wider class of psychological verbs. Another class of attitudes which has been studied elaborately in partialized logical styles are perception reports [Barwise 1981] [Kamp 1983].

Why partial modal logic?

There is a large number of reasons to prefer partial above classical modal logic for interpretation of epistemic attitudes. Firstly, from our point of view the content of such an attitude is fully determined by actually *present* information. If in total possible world semantics a certain proposition is not present (true), then its negation is verified. This means that, if a proposition is absent in all the doxastic alternatives of a certain agent, then this agent believes the negation of this proposition.

The classical distillation of a state of belief out of given range of total doxastic alternatives is depicted below.

1.5. FIGURE.

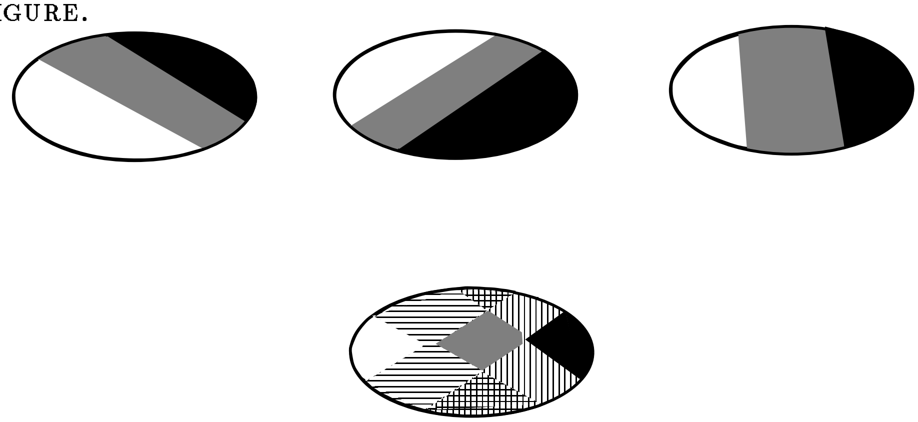


Suppose that the upper three worlds in the figure above are the doxastic alternatives of a certain agent. The white areas represent the true propositions. The complementary black areas represent the false propositions. The lower picture presents the resulting state of belief. Again the white and black represent true and false propositions which the agent believes to be actually true and false respectively. The grey area represents those propositions of which the agent is uncertain. He neither believes the truth nor the falsity of propositions which lie in the grey part. If a proposition lies in the black area of all three alternatives, the agent believes the negation of this proposition. Such a ridiculously strong inference from absent information does not fit with our intuitions.

In our view, negative information with regard to imaginary possibilities such as doxastic alternatives refers to actual rejection, and not to the absence of support. In this respect, our position coincides with Nelson's. In his logic of constructible falsity the presence of a refutation of a proposition represents its falsity. Our interpretation of conviction transmits this constructive analysis of falsity to multiple possible worlds models. An agent is convinced of the falsity of some proposition if all his doxastic alternatives contain counter-evidence against this proposition. The presence of counter-evidence is stronger than the mere absence of a proposition. Propositions which are undefined with respect to the set of doxastic alternatives of an agent, are not believed to be false by this agent. He does not reject this proposition, i.e. a doxastically plausible counter-model cannot be presented by this agent.

Truth of a proposition in a partial world means that evidence in favor of this proposition is locally available. An agent's conviction is then identified with the amount of information which is supported by means of the available evidence in all doxastic alternatives which this agent imagines to be possibly real. A partial possible worlds configuration is displayed in the next figure.

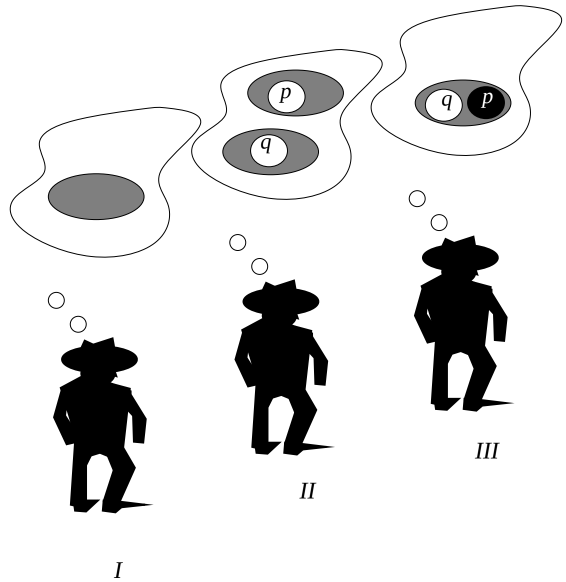
1.6. FIGURE.



The upper three worlds are partial alternatives of an agent. White and black refer to truth and falsity, respectively. The grey area represents the undetermined information. Neither evidence nor counter-evidence for such information is present. Falsity is no longer identical to the absence of truth. The lower figure represents the resulting belief state. Belief of the falsity of a proposition requires falsification of, or presence of counter-evidence against this proposition, with respect to all alternatives. This is represented by the black area. The striped areas represent contingently present information. Horizontal lines refer to information of which *some* evidence is present, and vertical stripes indicate information of which *some* counter-evidence is present. The densely grey area refers to information which is universally absent. Later on we will call this grey spot the *passive* disbelief of the agent.

The second argument in favor of a partial variant of possible worlds semantics depends heavily on the former motivation. It has already been mentioned in the previous section. In our view, the purely eliminative perspective on growth of information as a consequence of ordinary possible world semantics is limited. This destructive method of canceling alternatives only partially describes our means to extract epistemic or doxastic progress. The supplementary manner to gain information is constructive extension, which represents the widening of an agent's epistemic capacity. This second dynamic dimension is made possible by choosing in favor of partial worlds instead of total alternatives. A single partial world can grow by extending the amount of evidence and counter-evidence for propositions. The following picture illustrates this two dimensional dynamics of partial possible worlds.

1.7. FIGURE.



This picture represents one line of growing information of an agent, with an associated set of partial epistemic alternatives, in the dynamic space. In the first situation no information is available. The second situation arises from the first by persuasion of the agent that "p or q" must be the case. The third follows from updating the agent with the information that "p" is false. The second situation evolves from constructing worlds. The divergence into two worlds illustrates that a disjunctive message is *generating* uncertainty. Although information has been gained, the uncertainty has also increased. This effect reflects the growth of the epistemic range of the agent. In the third situation one of the doxastic alternatives has been eliminated and the remaining world has been extended with counter-evidence against "p". In this case the new information state is composed by means of construction *and* elimination.

We think that both construction and elimination of uncertainties, such as in figure 1.7, are essential to enforce growth of information. Generally speaking, the model-theoretic semantics of partial possible world models makes it possible to incorporate the more subtle collaboration of these two dynamic perspectives on cognitive progress.

Tearing the philosophies of total and partial possible worlds apart more roughly, we can associate the eliminative cognitive dynamics presented by total possible worlds to the medieval *homunculus* theory of cognitive development. According to this philosophy, the child differs only from adults in suffering from having wild fantasies. The task of its educators is to reduce these possibilities and lead it into the right direction. The two dimensional dynamic perspective offered by partial possible worlds semantics is closer to the environmentalist *tabula rasa* theories of Locke and Rousseau. In the beginning uncertainty is only latent and will arise from confrontation of the individual with its environment. The initial state of information is just the single empty world (see I in fig 1.7) reflecting the *tabula rasa* situation where no information is present, not even tautologies.

This logical *weakness* of partial modal logic illustrates yet another advantage of it. Believing nothing in classical modal logic corresponds to maximal possible worlds models. Every doxastic alternative is taken to be possibly real. Nevertheless, because of the complete informative status of all these possible worlds, the minimal state of belief is the same as taking all worlds to be possible, which entails that one believes all tautological information. In partial modal logic, this minimal belief state reduces to the tabula rasa single world situation of I in figure 1.7. This means that tautological information is even absent. Transposing once more this situation to theories of cognitive development, the newborn child is truly devoid of any extensional information.

Another argument in favor of partiality, which also relies on the distinction between falsity and absence of truth, is the distinction between two kinds of disbelief, which have been indicated earlier. Retrospection of figures 1.5 and 1.6 clarifies this distinction. In figure 1.5 we dealt with only one grey bulk of disbelief. In the resulting belief state of figure 1.6 different grey areas of disbelief appeared. In this latter figure, the full grey area represents information which is absent in all doxastic alternatives. The striped areas represent information which is present in at least one of the alternatives. This information indicates a certain *active* disbelief. The owner of this set of alternatives actively disbelieves the negation of this information, which means that he has a possible counter-model in mind. For example, the vertically striped area indicates information of which, in some of the alternatives, counter-evidence is present. In this situation the agent takes this counter-evidence to be possible, and therefore he actively disbelieves this information.

This refined analysis of disbelief offers a very good compromise in the discussion on the principle of *negative introspection*. This principle says, that if an agent a does not believe a proposition, then he also believes that he does not believe this proposition. Among philosophers this principle has often been rejected [Hintikka 1962] [Lenzen 1978]. In computer science, on the other hand, it has been widely propagated as a principle for formal reasoning about knowledge (e.g. [Moore 1983] [Halpern & Moses 1984]). The most dominant plea in favor of this epistemic principle is purely technical. It legitimates, in combination with the philosophically acceptable principle of positive introspection [26], a reduction to the simple structured models in case of single agent's modeling, which have been described earlier (see page 20 and 21). The distinction between active and passive disbelief shows that partial modal logic offers a settlement of this dispute. Negative introspection is acceptable when it is applied to active disbelief. If an agent has access to some contingently present counter-evidence of some proposition, he also believes that he has access to this feasible counter-model. However, we do not accept negative introspection with respect to passive disbelief. This refined justifiable partial acceptance of negative introspection admits the same elegant model-theoretical reduction to the full introspective models just like this principle in classical modal logic does. The strong correspondence result will be

[26] If a believes that φ then a believes that a believes that φ.

explained in chapter 4.

The last piece of propaganda we like to present in favor of partial modal logic for reasoning about epistemic attitudes is its lack of *contra-position.* Contra-position means that if φ implies ψ, then the falsity of ψ implies the falsity φ. This classical principle is simply caused by the classical interpretation of falsity as absence of truth. Abandonment of this complementary definition of falsity, like in partial logic, no longer entails contra-position [27].

The reader may wonder why such structural digression is fruitful. A radical example, which demonstrates the advantage of this 'logical decay' very sharply, can be given in terms of partial modal logic for representation of knowledge, instead of belief. As we have mentioned earlier, knowledge implies the truth of its content: "$\Box_a \varphi \Rightarrow \varphi$". In classical modal logic the contra-position of this 'veridicality' principle yields that the truth of a proposition implies the epistemic possibility of this proposition. In other words, everything which is true should be taken to be possibly true by the agents. The attribution of such a ridiculous width of intellect to agents does not automatically follow from implementation of the veridicality principle in partial modal logic. The corresponding constraint on total possible world models, which pictures this mental width (see figure 1.1), of taking the reality to be one of the epistemic alternatives, can be relativized in terms of partial possible world models to a more acceptable level as well. Model-theoretically, the veridicality principle of knowledge in partial modal logic requires that at least a part of reality appears as an epistemic alternative. A formal explanation of this correspondence is given in chapter 7. An epistemic logic which uses this veridicality axiom without its contra-position can be found in [van der Hoek, Jaspars & Thijsse 1993].

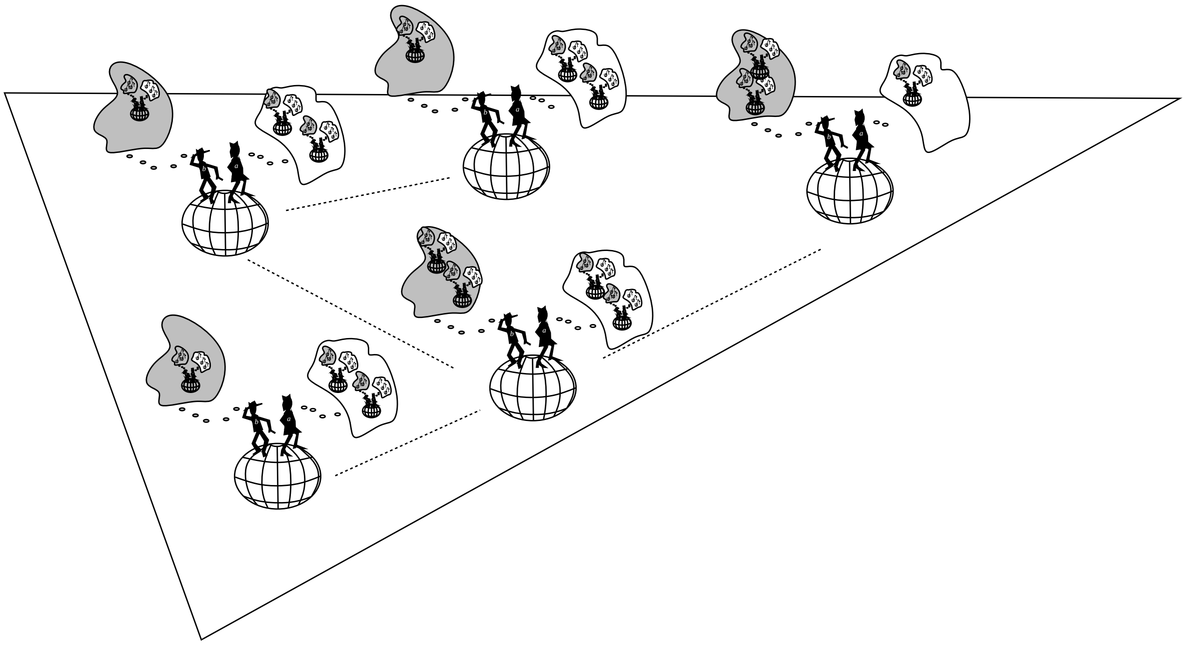
Models of epistemic dynamics

Let us now focus on the formalization of the dynamics of partial states. We wish to combine static partial possible worlds representations with the constructive information order as an additional dynamic parameter for reasoning about changes of such epistemic registrations. The clear difference is that we embed accessibility relations as well. In the case of multiple agents we deal with models of the format $\langle W, \{R_a\}_{a \in A}, \leq, V \rangle$, with the relations R_a as static individual epistemic accessibility relations, and $\leq$ the dynamic information structure. We take the relation to be fully introspective (see (1) on page 21) and *serial*: $\forall x \in W\ \exists y \in W : R_a(x, y)$ for all $a \in A$. This latter constraint makes sure that an agent's conviction is never inconsistent.

An illustration of the structure of these models is given by the following picture. The accessibility relation registers the mental clouds at each point in the structure. The dynamic dimension is represented by the horizontal plane, across which the epistemic configuration moves.

[27] In some partial logics the definition of implication or consequence has been defined in such a way that contra-position gets restored. An example is the definition of so-called double barreled consequence definitions [Blamey 1986] [Muskens 1989b].

1.8. FIGURE.



In constructive logics for mathematical reasoning, such as intuitionistic logic and Nelson's logic of constructible falsity, the extension order is a pure temporal order. Proofs and refutation are not retractable, and therefore changing information states leads only to enrichment. The modal formalisms which we will discuss do not advocate such a strict temporal interpretation of the information structure $\leq$. It might be the case that an agent needs to revise his conviction because of external persuasion of contrary facts. If $x \leq y$ then it is possible for the agents to move from situation x to situation y by extending their beliefs. Here we need to be careful with our terminology. By extending one's belief we mean that extensional information, i.e. non-epistemic or dynamic information, has been acquired. For example, it might be the case that in extension y an agent a has lost his belief about certain disbelief. This can be made clear by retrospection of figure 1.7. In situation III the agent has lost his active disbelief of the falsity of the proposition p.

In chapter 3 we will also embed the notion of retraction of doxastic information. If y is an extension of x, then agents may move from y to x by giving up extensional information.

In order to associate this dynamic status to the information order $\leq$, we need to define a structural interplay of the epistemic accessibility relation and $\leq$. A purely eliminative 'classical' interpretation of these models $\langle W, \{R_a\}_{a\in A}, \leq, V\rangle$ is established by taking the truth-value assignment V to be total and by requiring that if some state is an extension of another, then its uncertainty should be less. This constraint just means that if some world is accessible, it has always been accessible. In formal transcription,

$$\forall x, y, z \in W : x \leq y \ \&\ R_a(y, z) \implies R_a(x, z) \quad \text{for all } a \in A.$$

Constructive interpretation of these models is enforced by restraining the interplay of the *partial* truth-value assignment V and the information structure $\leq$. Just like in the model theory of constructive logic, the valuation function is taken to be monotonic over the information structure. The two-dimensional perspective is then laid upon the model structure through restraining the interplay of the accessibility relation and the information structure. If y is an enrichment of world x then everything which is accessible from y must be some extension of a world which is seen from x. The corresponding constraint is the following:

$$\forall x, y, z \in W, \forall a \in A : x \leq y \ \& R_a(y, z) \implies \exists z' \in W : R_a(x, z') \ \& \ z' \leq z^{28}.$$

This constraint precisely describes the requirements for growth of partial possible worlds which we sketched above. Every uncertainty must be an extension of some uncertainty in every poorer information state. In this constructive possible worlds semantics uncertainty is latent, while in the eliminative models all uncertainty in extensions must actually be present.

In terms of the conceptual presentation of figure 1.8, this constraint of the interplay of static accessibility and the dynamic or constructive extension order tells us how the mental clouds behave if we move forward in the dynamic plane.

Actions as up- and downdates

Communicative actions are to be interpreted in terms of the information structure $\leq$ in the constructive possible worlds models of the previous subsection. These actions switch cognitive states along this pattern.

The relational style of interpretation of information is currently influential among logicians and natural language semanticists. This *dynamic semantics*, as it is known among formal linguists, has reached a significant reputation through the work of Stalnaker on the formal interpretation of assertion [Stalnaker 1979], Kamp's discourse representation theory [Kamp 1984], Heim's file change semantics [Heim 1982], Barwise's dynamic theory of quantifiers and anaphora [Barwise 1987] and Groenendijk and Stokhof's dynamic predicate logic [Groenendijk & Stokhof 1991]. In mathematical logic we find the relational model theory in modern branches as *arrow logic* [van Benthem 1991a] and two-dimensional modal logic [Venema 1991].

In dynamic semantics the meaning of a proposition is taken to be a relation instead of a set of states such as in classical and in partial logic too. This relation reflects the mathematical denotation of such a proposition. Intuitive interpretation of such a relation is an update, a cognitive action such as assertion or revision, or an instruction or program. A proposition φ *is* the action which is performed when φ is added to some information state. This idea originally stems from operational semantics for imperative programming languages, e.g. [Goldblatt 1982].

In line with van Benthem and de Rijke's dynamic modal logic [van Benthem 1991b] [de Rijke 1992] we use both dynamic and static interpretation in one system. In the constructive possible world semantics which we have presented

[28] A shorter way to write this is $\leq \circ R_a \subseteq R_a \circ \leq$. Remember $\circ$ denotes relational composition.

above, dynamic meaning can be assigned to formulae by means of the information structure $\leq$, which represents the structural growth of epistemic information. With respect to a certain model $M = \langle W, \{R_a\}_{a \in A}, \leq, V\rangle$ the dynamic interpretation of a proposition φ is the set of pairs $\langle w, v\rangle$ of worlds such that v supports φ and $w \leq v$[29].. In other words, v is a φ-enrichment of w. Besides this positive enrichment interpretation, once again following the style of van Benthem's dynamic modal logic, we assign to every proposition φ a 'negative' retraction interpretation. This interpretation consists of pairs $\langle w, v\rangle$ such that $v \leq w$ and v does not contain φ. Informally speaking, v is a φ-impoverishment of w.

Explicit reasoning about the epistemic dynamics in the constructive possible worlds models which we discussed above is made possible by means of *action* or *dynamic* modalities. Every proposition φ corresponds to modal operators $[\varphi]_u$ and $[\varphi]_d$. The former modal operator ranges over states which extend the current state in such a way that they contain φ. The latter operator ranges over states which are poorer than the current one in such a way that they do not contain φ. In short, $[\varphi]_u$ ranges over φ-additions, and $[\varphi]_d$ over φ-retractions. $[\varphi]_u \psi$ is a proposition which means that ψ holds after any addition of φ, and analogously $[\varphi]_d \psi$ means that ψ holds after any retraction of φ [30].

This modal formulation of switching information states clearly distinguishes the dynamic relational interpretation from the static interpretation of propositions. Syntactic separation is enforced by the introduction of the dynamic modal operators, that correspond to the relational interpretation of the infix argument of such an operator. Normal propositions always refer to the static interpretation. The basic dynamic epistemic logic of this monograph, $\mathbf{C}^3$, combines individual modal operators with the dynamic *up*- and *downdate* operators [31].

1.4 Constructive communication

Now that we have provided the model-theoretic equipment for dynamic epistemic reasoning on the basis of deletion and construction of uncertainties, we want to introduce some essential additional expressivity for interpretation of communicative actions. The most important novelties are the representation of mutual beliefs and the interpretation of intentional attitudes. They are of

[29] There is nothing partial about this dynamic relation interpretation. For a partial logical style in pure relational semantics see e.g. [Krahmer 1994]

[30] Note that the meaning of $[\varphi]_u \psi$ coincides with the constructive interpretation of implication. Falsification of $[\varphi]_u \psi$ deviates from Nelson's judgement of the falsity of an implication. According to the last ideology an implication is false if the antecedent holds and the consequent fails. This means that falsification of implications, opposite to verification of implication, is defined extensionally. We follow Veltman's criterion of falsity assignment to conditionals. $[\varphi]_u \psi$ is false with respect to a certain state of information if and only if this state can be enriched with the truth of φ and falsity of ψ. Chapters 3 and 4 elaborate on this distinction.

[31] The names of up- and downdate may be a misleading. In most dynamic theories updates refer to minimal enrichments. $[\varphi]_u$ should then only range over minimal φ-enrichments. In our systems, an update is arbitrary. This means that a "p and q"-update is also taken to be a p-update and a q-update.

particular importance for the identification of natural interpretations of interactive behavior. They appear in most analytic studies of communication, such as speech act theory [Searle 1969] and analytic philosophical studies of conventions [Lewis 1969]. Modern logical approaches to communication which incorporate such operators are for example [Jones 1983] and [Appelt 1985].

Mutual beliefs

As we have already explained briefly, an important requirement for a suitable model-theoretic semantics for interaction is the interpretation of mutual beliefs. The full epistemic effect, or the dynamic meaning, of a (convincing) message φ from an agent a to an agent b is the addition of φ to the mutual belief of a and b. This means that even when b already had this information, there has been made some cognitive switch. The epistemic outcome is also that a believes that b believes that φ holds, and b believes that a believes that φ, etcetera. This infinite conjunction of epistemic information will be abbreviated by $\mathsf{C}_{\{a,b\}}\varphi$.

Interpretation of this operator is established through gluing the personal accessibility relation R_a and R_b together in arbitrary order. This operation is the *transitive closure* of the union of the relations R_a and R_b. This amounts to a new accessibility relation which we will call $R^t_{\{a,b\}}$[32].

$$R^t_{\{a,b\}} := \{\langle x,y\rangle \in W \times W \mid R_{x_1} \circ \ldots \circ R_{x_n}(x,y) \quad \text{for certain } n \in I\!N \setminus \{0\}, x_i \in \{a,b\}\}.$$

Note that this new accessibility relation establishes a suitable interpretation of the mutual belief operator in a possible worlds setting. If all worlds y with $R^t_{\{a,b\}}(x,y)$ are φ-worlds then both agents a and b believe that φ, they both believe that they both believe that φ, they both believe that they both believe that they both believe that φ, and so on.

The dynamic meaning of an assertion φ of agent a to another agent b can now be interpreted as a dynamic modal operator $[\mathsf{C}_{\{a,b\}}\varphi]_u$. The proposition $[\mathsf{C}_{\{a,b\}}\varphi]_u\psi$ expresses that ψ certainly holds after a has told b that φ is the case. It would be instructive to present such a mutual belief update by means of a picture in the style of the earlier mental cartoons. Because of the infinite nature of such updates, it is most problematic to display such an effect properly. However, the doxastic effect of ideal assertion can be comprehended easily by means of earlier illustrations. In terms of our illustration in figure 1.8, the definition of the mutual belief update of the two agents in this illustration with φ means that we move upwards to a situation where all mental clouds, that is all layers of embedded clouds as well, are filled with only φ-worlds.

The interpretation of assertion also requires the use of preconditions. A simple precondition of the assertion above is the sender's conviction of the content of the message φ. This can be seen as a certain qualitative precondition of the assertion. In formal pragmatics we find more of this kind of qualitative maxims of conversation [Grice 1975].

[32] This interpretation of mutual beliefs stems from [Halpern & Moses 1990].

Another important precondition is the requirement that a must think that her message possibly has *some* epistemic effect. Of course, if we stipulate such a strong update as the mutual belief change, this condition is a very weak requirement. Even if a believes that b believes that φ and b indeed believes that this proposition holds, then there is still some epistemic progress made by her assertion. For example, after the assertion, b also believes that a believes that φ. Such a change might be relevant. It might have been the case that b had tested a on φ before her assertion as an affirmation of sharing b's belief of φ. In chapter 4, we will present formalizations of the additional qualitative information requirements.

Of course, in real life communication the interpretation of assertion as a mutual belief update may be too idealistic. As we have said before, other context parameters might very well entail alternative interpretations. Instead of losing ourselves in philosophical speculations about stipulation of more realistic interpretations, we like to point out that a proper definition, and axiomatization, of mutual belief operators can be given in the constructive possible worlds semantics which we propagate. Many philosophers, linguists and computer scientists have found such mutual attitudes indispensable for formalization of interaction and cooperation [Lewis 1969] [Jones 1983] [Appelt 1985] [Halpern & Moses 1990] [Bunt 1990b].

In chapter 4 we will give a generalized formal definition of $\mathsf{C}_X\varphi$, which says that φ is mutually believed by X, for arbitrary sets $X \subseteq A$ of agents. Because the relation R^t_X is defined in terms of the accessibility relations, there is no need for more semantic equipment. Nevertheless, the implicit infinite conjunction of mutual belief yields a substantial complication of the axiomatization and its meta-theory (chapter 4 and chapter 6).

Intentional modalities

Boulomaic or intentional attitudes are particularly important for formal comprehension of communication [Appelt 1985] [Cohen & Levesque 1990]. Many pragmatic principles in speech act theory and prescriptions for cooperative behavior of interacting partners are defined on the basis of intentions [33].

The basic $\mathbf{C}^3$-logic, which will be discussed in chapter 4, does not contain such intentional operators, for the sake of technical transparency and gradual presentation. An additional section of this chapter is devoted to an extension of the semantic machinery in such a way that the intentional operator can be given a suitable interpretation. This interpretation is established by means of an additional semantic parameter P_a for every agent $a \in A$ to the constructive possible world models $\langle W, \{R_a\}_{a\in A}, \leq, V\rangle$, which denotes again a binary relation over the universe of possible worlds W. This relation P_a has a dynamic intentional meaning and is called a *preference relation* [34]. It is interpreted in

[33] For a modal-like approach to these matters see [Beun 1989].

[34] In the field of applied logic, preferential semantics refers to certain non-monotonic logics [Shoham 1988]. Preferential worlds or models are meant to cover lack of information. Instead of reasoning over all worlds like in classical logic, such preferential non-monotonic logic reasons

terms of the perspective of the future that the agent a has. It determines which of these *personal* possible future points are preferred by a. A proposition $[\mathsf{p}]_a\varphi$ then describes a situation x in which all a's preferred worlds agree on the truth of φ.

If only worlds have a preferential status, it does not tell us much about the intentions and the consequential behavior of agents. It tells us only what an agent wants. The agent may think that such a preferential world is not feasible, and therefore may leave the agent passive.

By an additional constraint we could attribute a subjective form of feasibility to these preferential world, so that they can be interpreted just like the *goal-worlds* in [Cohen & Levesque 1990]. To bring an agent to action, he must conceive the possible realizability of his preferences. Of course, such goals do not have to be factually realizable. In the two-valued modal formalisms of Cohen and Levesque, this realism is enforced simply by taking preferential worlds as a subset of doxastic alternatives, which brings along their *realism* principle $\Box_a\varphi \Rightarrow [\mathsf{p}]_a\varphi$. We will show how variations of this principle can be given in the partial modal logical style. In fact we will show that more realistic principles of realism can be encoded. To give a simple example, we do not wish to apply such a realism principle to belief of others. An agent may very well prefer situations where other agents have less information. Agents may even want to retract or revise beliefs of other agents. Such intentions are in fact one of the basic motives to interact. Our epistemic attitude is nevertheless taken to be so strong that agents never wish to revise or retract their own beliefs, which indicates a partial acceptance of the realism principle above.

In chapter 4 we will also discuss the interpersonal constraints on the preferential worlds and doxastic alternatives. An instructive example which models a certain integrity constraint is the following qualitative conversation principle $[\mathsf{p}]_a\Box_b\varphi \Rightarrow \Box_a\varphi$. If one aims at situations in which another agent believes φ, then one should believe it oneself [Grice 1975] [Beun 1989]. We will show what such a principle means for our kind of dynamic epistemic model theory. Other weakenings can be given for this principle.

A very important facility of embedding this preferential operator is the definition of a suitable simple interpretation of questions as communicative acts. The transfer of information generated by such an action implies that the receiver b of this question is now convinced that the sender a intends to receive a convincing argument in favor or against the content of the question. In fact, this preference of a becomes mutual belief of a and b. This means the epistemic effect of a's question to b is formally interpreted as a dynamic modal operator: $[\mathsf{C}_{\{a,b\}}[\mathsf{p}]_a(\Box_a\varphi \vee \Box_a\neg\varphi)]_u$ [35].

over a set of preferred models. In dynamic semantics such preferential relations also show up. For example Veltman's *expectation patterns* can be seen as a sort of preferential ordering of worlds [Veltman 1991]. This author proposes a dynamic interpretation of default rules in terms of updating such patterns. In [van Benthem, van Eijck & Frolova 1993] we find modal logical interpretations of such preferential updates, which they call *upgrades* as to distinguish them from 'hard' factual updates.

[35] $\Box_a\varphi \vee \Box_a\neg\varphi$ means that a knows whether φ. Of course, also in the case of questions more

It depends on the cooperation of b whether an answer follows. Such cooperative behavior can also be encoded in the logic. Let b be some servile agent, whose preferences consist only of situations of which he believes that the agent a prefers them as well. In this case, b should always generate an answer. The corresponding servility postulate is then $\Box_b[\mathsf{p}]_a\varphi \Leftrightarrow [\mathsf{p}]_b\varphi$. A more reasonable weakening of such maximal cooperation is the following: $\Box_b[\mathsf{p}]_a\varphi \Rightarrow [\mathsf{p}]_b\varphi \vee \Box_b\neg\varphi$ [36]. This means that b goes along with other agent's preferences as long as they are not contradictory with his own belief.

The preferential operator can also distinguish different degrees of assertion of information from one agent to another. In the previous subsection on mutual beliefs, the action of a telling b that φ has been interpreted as $[\mathsf{C}_{\{a,b\}}\varphi]_u$. A more skeptical interpretation would be $[\Box_b[\mathsf{p}]_a\Box_b\varphi]_u$, that is a switch to a situation where b is convinced that a intends to let b believe that φ. This means that real persuasion can be distinguished from telling.

In this subsection we have mixed up our vocabulary a little. In chapter 4 we distinguish intentions and preferences formally. Preferences are attitudes towards propositions, while intentions are defined as attitudes towards actions. Of course, there is a very close connection. If an action has been defined in terms of our dynamic epistemic model theory, we define an intention of an agent a with regard to such an action as the belief of a that the preconditions of this action are fulfilled, in combination with a's preference of situations where the epistemic effect, or dynamic denotation, of the utterance is verified. This formalizes the 'readiness' of the agent to perform a communicative action.

'Real worlds'

So far, we have only discussed subjective epistemic information. For this reason we have chosen for conviction instead of knowledge as epistemic modality. Knowledge implies the truth of its content. Such logical interplay of epistemic modalities and reality would entice us into discussions whether reality is a partial or a total world. We wish to resist such a philosophical seduction.

However, not choosing does not keep us from logical speculations. Incorporating a total reality does not lead to severe complications. It entails straightforward mixtures of classical and partial logics. Total worlds can be embedded easily, from a semantic point of view, in the constructive possible world semantics of the $\mathbf{C}^3$-logics. We may interpret the real world as a selection from the full universe of possible worlds. This leads to models of the form $\langle W, S, \{R_a\}_{a\in A}, \leq, V\rangle$ with S being the selected set of realities in W: $S \subseteq W$. This set has already implicitly been displayed in many of the preceding illustrations. In figure 1.8, the set S is simply the different globes on which the communicating agents stand. The particular role of this set is its interplay with the global valuation function V, which should locally taken to be total on S. Moreover, the local effect of V

cautious interpretations have been advocated (e.g. [Bunt 1989] and [Bunt 1990b]). Some of those proposals can be handled in our logical framework as well.

[36] $[\mathsf{p}]_b\varphi \vee \Box_b\neg\varphi$ means that b prefers φ-situations or believes that φ is false.

on the different members of S is identical. In chapter 4 we will discuss a logic, called $\mathbf{C}^{3R}$, which deals with such an additional physical outer world.

It is not only for philosophical reasons that additional modeling of total worlds is needed. In intelligent communication systems artificial realities play an important role. Especially for models of database querying through intelligent interfacing between the questioner and the system who is omniscient with respect to this artificial reality, which is modeled as a total world. Because the dialogue of users with such a system will only be about the information in the database, interpreting this as the artificial reality is legitimate (see e.g. [Bunt et al. 1984] [Ahn & Beun 1991]). Besides the totality of this artificial reality, the system, which is one of the agents in this epistemic setting, is completely informed about this reality. The only thing of which the system is not completely informed, is the information-state of the user. In this sense the information of the system might also grow (or shrink).

The way to interpret such interactive configurations in terms of constructive possible world semantics is through the same specification of a set of total worlds among the universe of possible worlds. This leads to the following models: $\langle W, S, \{R_\Omega, R_\mathbf{u}\}, \leq, V\rangle$, where Ω is the system and $\mathbf{u}$ the user (and $S \subseteq W$). R_Ω is the accessibility relation which determines the uncertainties of the system Ω. Ω's complete information about reality enforces:

$$\forall s \in S : R_\Omega(s,t) \Rightarrow t \in S.$$

Additional logical analysis of such configurations will be exhibited in a supplementary section of chapter 4. The corresponding system is called $\mathbf{C}^3_\Omega$.

Before deliberating on communication with completely omniscient agents, we wish to regain the humble position of the mathematician, and leave further speculation to the readers [37]. Instead of dwelling on fine-grained interpretations of all different kinds of utterances and their epistemic and pragmatic implications, we have chosen to present the mathematics of the dynamics of constructing *and deleting* worlds by means of communicative actions. This also implies that the pace of the forthcoming chapters will be considerably more careful and technically more attentive than that of this chapter.

[37] In the last section of chapter 4 we will present some ideas for further dressing up our epistemic dynamics, on the basis of other literature on epistemic logic and communication. We will meet some additional semantic parameters for establishing more fine-grained analyses of communicative actions.

Part I

Chapter 2

Partial Modal Logic

This chapter presents our logical means for static representation of modal information. First we present our partial possible worlds framework in section 2.1. In section 2.2 we present the corresponding derivational systems. Section 2.3 introduces a formal structural definition of 'growing' possible worlds. This section is meant as a bridge to the next chapter on constructive and dynamic modal logics for reasoning about the growth of modal information.

2.1 Partial worlds

Generally speaking, the most important aspect of partial logic is its model-theoretic semantics (see e.g. [Langholm 1988], [Blamey 1986] and [Barwise 1988]). As extensively argued in the previous chapter, the difference with ordinary classical logic originates from fundamental semantic motivations. In this thesis, we will only consider partial propositional and partial modal logics which differ from classical logic in the assignment of truth-values. To get the picture as clear as possible, we will start with the purely extensional partial propositional logic, which is based only on partial truth-assignments. Some interesting typical phenomena of partial semantics can be explained more easily in terms of this simple extensional semantics.

Partial valuations

The most elementary semantic entity in partial logic is the partial valuation. It partially assigns truth-values, 0 (*false*) and 1 (*true*), to a given set of propositional variables.

2.1. DEFINITION. A *partial valuation* V is a partial function which assigns truth-values to a given set of propositional variables $\mathbb{P}$. In order to distinguish partial functions from total functions we replace the normal functional arrow $\longrightarrow$ by

$\rightsquigarrow$. In short $V : I\!P \rightsquigarrow \{0,1\}$. The collection of all partial valuations is denoted by $\mathfrak{P}$.

The *domain* of $V \in \mathfrak{P}$, $\mathfrak{Dom}(V)$, is the set of all propositional variables which obtain a truth-value by V:

$$\mathfrak{Dom}(V) := \{p \in I\!P \mid V(p) = 1 \text{ or } V(p) = 0\}.$$

If $\mathfrak{Dom}(V) = I\!P$ then V is said to be *total.* $\mathfrak{T}$ denotes the set of all total valuations.

The following relations are considered to be of particular importance for partial logic. In the sequel of this work we will call them *information orders.*

2.2. DEFINITION.

$$V \sqsubseteq V' \quad \Leftrightarrow \quad p \in \mathfrak{Dom}(V) \Rightarrow V'(p) = V(p) \text{ for all } p \in I\!P,$$

$$V \sim V' \quad \Leftrightarrow \quad p \in \mathfrak{Dom}(V) \cap \mathfrak{Dom}(V') \Rightarrow V(p) = V'(p) \text{ for all } p \in I\!P,$$

$$V \sqsubseteq^d V' \quad \Leftrightarrow \quad \mathfrak{Dom}(V) \subseteq \mathfrak{Dom}(V').$$

The first relation is the most important one. It says that V' contains all the information of V and can therefore be understood as a possible enrichment of V. In short, we will call V' an *extension* of V whenever $V \sqsubseteq V'$. We will refer to the second relation as *coherence* of the pair V and V'. If $V \sqsubseteq^d V'$ we say that V' is *at least as large* as V.

2.3. OBSERVATION. The relation $\sqsubseteq$ is a partial order over $\mathfrak{P}$. The coherence relation $\sim$ is a symmetric reflexive relation over $\mathfrak{P}$, while the information order $\sqsubseteq^d$ pre-orders $\mathfrak{P}$, i.e. it is a reflexive transitive relation over the universe of partial valuations.

For these three relations over the universe of partial valuations the following equivalence holds:

$$V \sqsubseteq V' \iff V \sqsubseteq^d V' \;\&\; V \sim V'.$$

These information orders are typical notions of partial logic. With regard to total valuations these orders do not mean very much. The first two relations, $\sqsubseteq$ and $\sim$, reduce to the identity relation over the class of total valuations. The last relation, $\sqsubseteq^d$, expands into the universal relation over total valuations.

The formalization of these information orders establishes an interesting perspective when partial logic is used as technical equipment for uniform representation of stages of information. If V' is an extension of V, V' may be understood to be an informational enrichment of V. The coherence relation expresses the mutual compatibility of valuations. They can be taken as two parts of one common extension, namely the valuation which contains the joint information of such a pair. Such a union is denoted by the binary operator $\sqcup$:

$$V \sqcup V'(p) = \begin{cases} V(p) & \text{if } p \in \mathfrak{Dom}(V) \\ V'(p) & \text{if } p \in \mathfrak{Dom}(V') \\ \text{undefined} & \text{otherwise.} \end{cases}$$

This join of valuations is the unique minimal extension of a coherent pair of valuations:

$$\left.\begin{array}{l} V \sim V' \;\&\; V'' \sqsubseteq V \sqcup V' \;\& \\ V' \sqsubseteq V'' \;\&\; V \sqsubseteq V'' \end{array}\right\} \Longrightarrow \quad V'' = V \sqcup V'.$$

The relation $\sqsubseteq^d$ compares the informational size of two valuations. It does not consider truth-values, but only judges on the basis of the presence of truth-values. It compares the degree of determination of partial valuations.

Before elaborating on the specific properties of the presented partial semantics, we introduce the basic language with which we will be working.

2.4. DEFINITION. Let $I\!P$ be a non-empty enumerable set of propositional variables. The language $\mathcal{L}$ is the smallest superset of $I\!P$ such that

$$\varphi, \psi \in \mathcal{L} \Rightarrow (\neg\varphi), (\varphi \wedge \psi) \in \mathcal{L} \text{ and } \bot \in \mathcal{L}$$

These connectives are called *negation*, *conjunction* and *falsum* respectively. This is the basic language of this text. We avoid superfluous use of parentheses, and take binary connectives to dominate over unary connectives. For example $\neg\varphi \wedge \psi$ means $((\neg\varphi) \wedge \psi)$ and not $(\neg(\varphi \wedge \psi))$. The propositional variables $p \in I\!P$ are also called *atoms* or *atomic propositions*. A *literal* is an atom or the negation of an atom.

In the following sections we will use more connectives, let us say $c_1, .., c_n$ for the moment. The smallest superset extending $I\!P$ which is closed under these connectives and the connectives of $\mathcal{L}$ will be written as $\mathcal{L}^{c_1,..,c_n}$ [1]. An extension which is particularly relevant in the next subsections is $\mathcal{L}^{\Box}$, where $\Box$ refers to necessity. In some parts of the text we will also mention languages which do not incorporate all the basic connectives of $\mathcal{L}$. If some connective c is withdrawn, we specify this in a subscript: $\mathcal{L}_{-\bot}$ refers for example to the sublanguage of $\mathcal{L}$ without occurrences of the 0-ary connective $\bot$.

Furthermore, we will also use convenient abbreviations, like $\top := \neg\bot$ (verum), $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$ (disjunction) and $\Diamond\varphi := \neg\Box\neg\varphi$ (possibility).

In the sequel we will use the letters p, q, r, possibly with additional sub- or superscripts, as atoms. Greek undercast letters are used to denote arbitrary formulae. Greek capitals denote sets of formulae. We will often use the abbreviation $c\Gamma$ for a given unary connective c and a set of formulae Γ. $c\Gamma$ refers to the set $\{c\gamma \mid \gamma \in \Gamma\}$, and $c^{-}\Gamma$ denotes the set $\{\gamma \mid c\gamma \in \Gamma\}$, that is the set of all formulae γ which appear as $c\gamma$ in Γ. Repetitions of a unary connective c will be abbreviated by an exponential, e.g. $c^3\varphi = ccc\varphi$[2], and $c^{-n}\Gamma := \{\gamma \mid c^n\gamma \in \Gamma\}$.

If a language $\mathcal{L}_S$ has been specified, and $\Gamma \subseteq \mathcal{L}_S$, then $(\Gamma)^{\complement}$ refers to the complement of Γ in $\mathcal{L}_S$: $(\Gamma)^{\complement} := \{\varphi \in \mathcal{L}_S \mid \varphi \notin \Gamma\} = \mathcal{L}_S \setminus \Gamma$. $\Gamma - \varphi$ and $\Gamma + \varphi$ are sometimes used as abbreviations of $\Gamma \setminus \{\varphi\}$ and $\Gamma \cup \{\varphi\}$, respectively. $Sub(\Gamma)$ refers to the set of subformulae of Γ.

---

[1] If we were more accurate, we would also have to indicate the arity of all these connectives. Instead of being formalistic, we guarantee that this parameter will always be clear from the context.

[2] $c^0\varphi = \varphi$.

At first sight, one might get the impression that this language $\mathcal{L}$ is not different from the language which is normally used in total (classical) propositional logic. However, the relative greater liberty of partial semantics creates more possibilities for truth-conditional interpretation of negation and conjunction. Let us first present our choices. In the jargon of partial logic, these choices are called the strong Kleene interpretations [Langholm 1988].

In choosing truth-conditions for partial logic, falsification has to be implemented explicitly, as falsity is no longer the same as absence of truth. Apart from the standard verification relation $\models$ between a valuation and a proposition, saying that such a proposition holds with respect to this valuation, a falsification relation $\dashv$ between valuations and propositions is defined. The compositional inductive clauses for determining truth and falsity of formulae with respect to a partial valuation V are specified in the following table.

2.5. TABLE.

$V \models p \Leftrightarrow V(p) = 1 \quad (p \in I\!P)$	$V \dashv p \Leftrightarrow V(p) = 0 \quad (p \in I\!P)$
$V \not\models \bot$	$V \dashv \bot$
$V \models \neg\varphi \Leftrightarrow V \dashv \varphi$	$V \dashv \neg\varphi \Leftrightarrow V \models \varphi$
$V \models \varphi \wedge \psi \Leftrightarrow V \models \varphi \ \& \ V \models \psi$	$V \dashv \varphi \wedge \psi \Leftrightarrow V \dashv \varphi \text{ or } V \dashv \psi$

The interpretation function $[\![\ ]\!]_{\mathfrak{P}} : \mathcal{L} \longrightarrow \mathfrak{P}$ is the function which assigns to every formula the set of partial valuations which verify it:

$$[\![\varphi]\!]_{\mathfrak{P}} = \{V \in \mathfrak{P} \mid V \models \varphi\}.$$

Note that the disjunction, which was introduced earlier as an abbreviation, obtains the intended truth-value assignment. It is true with respect to a partial valuation V if V verifies one of the disjuncts. Falsification of the disjunction by V occurs if both the disjuncts are falsified by V.

The language $\mathcal{L}$ has some important natural properties with respect to the structural information orders between valuations which were defined above. Some fundamental properties which are valid for propositional variables ($I\!P$) are inherited by all formulae.

2.6. THEOREM. Persistence of information:

$$V \sqsubseteq V' \quad \text{iff} \quad V \models \varphi \Rightarrow V' \models \varphi \text{ for all } \varphi \in \mathcal{L}.$$

Informational compatibility of coherent partial valuations:

$$V \sim V' \quad \text{iff} \quad V \models \varphi \Rightarrow V' \not\dashv \varphi \text{ for all } \varphi \in \mathcal{L}.$$

The union $V \sqcup V'$ of two coherent partial valuations V and V' contains the joint information of the two separate valuations:

$$\text{if } V \models \varphi \text{ or } V' \models \varphi \text{ then } V \sqcup V' \models \varphi \quad \text{for all } \varphi \in \mathcal{L}.$$ [3]

[3] Note that $V \sqcup V'$ may contain more information than the joint information of two coherent valuations V and V'. $V \sqcup V' \models p \wedge q$ does not guarantee $V \models p \wedge q$ and $V' \models p \wedge q$. Obviously,

Larger valuations contain more information:

$$V \sqsubseteq^d V' \quad \text{iff} \quad V \models \varphi \Rightarrow V' \models \varphi \text{ or } V' \dashv \varphi \text{ for all } \varphi \in \mathcal{L}.$$

The relation $\sqsubseteq^d$ preserves classical tautological information. A formula is classically tautological if it is verified by all total valuations, i.e. $\mathfrak{T} \subseteq [\![\varphi]\!]_{\mathfrak{P}}$. If φ is such a formula, then

$$V \sqsubseteq^d V' \;\&\; V \models \varphi \;\Rightarrow\; V' \models \varphi.$$

Proof. Proofs of all characterization results on these three information relations on partial valuations can be given by a straightforward induction on the construction of formulae in $\mathcal{L}$. We refer to [Langholm 1988] for proofs for the first two characterizations. The fourth result can be accomplished in the same manner. The third result follows from the first and the simple fact that $V \sqsubseteq V \sqcup V'$ and $V' \sqsubseteq V \sqcup V'$ for two coherent partial valuations V and V'. The last result is a simple consequence of the fourth, combined with the non-falsifiability of classical tautologies:

If $V \models \varphi$ for all $V \in \mathfrak{T}$ then $V' \not\dashv \varphi$ for all $V' \in \mathfrak{P}$[4].

This result is an immediate consequence of the persistence result for $\sqsubseteq$, and the fact that all partial valuations have a total extension: $\forall V' \in \mathfrak{P} \Rightarrow \exists V \in \mathfrak{T} : V \sqsubseteq V'$. ■

It is easy to see that with respect to total valuations the meaning of the negation and the conjunction is purely classical. The choices in case of undefined arguments of these connectives are open for debate and depend heavily on the intended application area of partial logics. For example, the negation which we used above means that its argument is false. If the argument is left undefined by a partial valuation, it also leaves the negation of this proposition undefined. So, a natural extension of the formalism above would be the addition of a so-called *weak negation*, which expresses that its argument is not true. We will use the symbol $\sim$ for this negation. The truth-value assignment looks as follows.

$$V \models {\sim}\varphi \Leftrightarrow V \not\models \varphi \quad \text{and } V \dashv {\sim}\varphi \Leftrightarrow V \models \varphi$$

In partial logic this connective is often omitted, because in a way it restores totality. In the general study of multiple valued logics – where partial logic is seen as the specific case of three valued logics – such a negation may not be ignored (see [Urquhart 1986]). Because of this perspective we will not exclude it, but keep it a little aside.

Also the conjunction we used is disputable. Its truth-conditional interpretation causes falsification of a conjunction if one of the conjuncts is falsified. The truth-value assignment of the other conjunct is overruled, and might therefore be undefined without having any influence. Applications of partial logic which make such falsification undesirable are conceivable. For example, partial logic may be used in order to permit truth-value assignment only to a class of 'comprehensible' propositions. In this case, one prefers dominance of undefinedness with respect to conjunctions. According to this conceptual analysis, the truth-value clauses for such a conjunction, which we will write as $\triangle$, look as follows:

---

the converse of this implication is a property which is not transferred from $I\!P$ to the full language $\mathcal{L}$.

[4] The converse of this observation is also valid [Thijsse 1992].

$$V \models \varphi \triangle \psi \iff V \models \varphi \ \& \ V \models \psi, \quad \text{and}$$

$$V \dashv \varphi \triangle \psi \iff \begin{cases} V \dashv \varphi \ \& \ (V \models \psi \text{ or } V \dashv \psi) & \text{, or} \\ V \dashv \psi \ \& \ (V \models \varphi \text{ or } V \dashv \varphi) & . \end{cases}$$

This conjunction, also called the weak Kleene conjunction, which expresses something stronger about its arguments than the conjunction $\wedge$, can be defined in terms of this latter conjunction and the strong negation in $\mathcal{L}$ in the following way:

$$\varphi \triangle \psi \ = \ \neg(\neg(\varphi \wedge \psi) \wedge \neg(\varphi \wedge \neg\varphi) \wedge \neg(\psi \wedge \neg\psi)).$$

The disjunctive duality $\triangledown$, the weak Kleene disjunction, whose verification requires verification of at least one of its arguments and definedness for both the disjuncts, has an easier definition in terms of the disjunction $\vee$:

$$\varphi \triangledown \psi \ = \ (\varphi \vee \psi) \wedge (\varphi \vee \neg\varphi) \wedge (\psi \vee \neg\psi).$$

In contrast with total valuations the language $\mathcal{L}$ is not *functionally complete* over partial valuations. Functional completeness means that not every connective with an extensional definition in terms of partial valuations can be rewritten by the connectives of $\mathcal{L}$. A simple example is the weak negation $\sim$.

Nevertheless, $\mathcal{L}$ defines a natural class of connectives [van Benthem 1984]. This class can be described as having classical interpretations with respect to total valuations, in the sense that truth-value assignment is fully guaranteed with respect to total valuations [5], and they preserve *persistence* of propositions. A proposition is persistent if its validity is never lost when information grows. In terms of partial valuations, a proposition φ is persistent if

$$V \models \varphi \ \& \ V \sqsubseteq V' \implies V' \models \varphi \qquad \text{for all } V, V' \in \mathfrak{P}.$$

A connective is then said to be persistence preserving if persistence of its arguments always yields a persistent proposition. We will not give further analysis of definability in partial propositional logic, as it is a subject somewhat outside the scope of this thesis[6].

The weak negation is typically not persistence preserving. To comprehend such behavior, consider the following simple illustration. The empty valuation, that is the valuation with the empty domain, verifies $\sim p$ for any propositional variable $p \in \mathbb{P}$. Evidently, every extension which assigns the truth-value 1 to such an atom p, does not verify $\sim p$. This means that $\sim p$ is not persistent, while p is.

[5] This property has been called closedness [van Benthem 1984] and also classical closure [Thijsse 1992].

[6] In [Blamey 1986] it has been proved that functional completeness with respect to preservation of persistence only can be achieved by adding 0-ary connective $\bigstar$ to the basic language $\mathcal{L}$ ($\mathcal{L}^{\bigstar}$). This proposition is always undefined: $V \not\models \bigstar$ and $V \not\dashv \bigstar$ for all $V \in \mathfrak{P}$. The language $\mathcal{L}^{\bigstar,\sim}$ is functionally complete with respect to the complete class $\mathfrak{P}$, e.g. [Langholm 1988].

We refer the interested reader to [Blamey 1986], [van Benthem 1984], [Langholm 1988] and [Thijsse 1992]. In this latter reference, the reader finds a full chapter dedicated to definability results in partial propositional logic.

Semantical consequence

The only formality with which we should deal before proceeding, is the notion of *valid consequence.*

2.7. DEFINITION. Let Γ and Δ be two sets of formulae in $\mathcal{L}$. We say that Δ is a valid $\mathfrak{P}$-consequence of Γ, whenever all partial valuations which support all elements of Γ also support at least one of the elements of Δ. The abbreviation of this relation is $\models_{\mathfrak{P}}$.

$$\Gamma \models_{\mathfrak{P}} \Delta \Longleftrightarrow \forall V \in \mathfrak{P} : (\forall \gamma \in \Gamma : V \models \gamma \Rightarrow \exists \delta \in \Delta : V \models \delta)$$

In terms of interpretations, this definition of validity comes down to

$$\Gamma \models_{\mathfrak{P}} \Delta \iff \bigcap_{\gamma \in \Gamma} [\![\gamma]\!]_{\mathfrak{P}} \subseteq \bigcup_{\delta \in \Delta} [\![\delta]\!]_{\mathfrak{P}} \text{ [7]}.$$

If for two formulae $[\![\varphi]\!]_{\mathfrak{P}} = [\![\psi]\!]_{\mathfrak{P}}$ then φ and ψ are *semantically equivalent*, i.e. have the same models. The abbreviation of this relation is $\varphi \approx_{\mathfrak{P}} \psi$.

There is some freedom here. The so-called *double barreled* consequence definition has also been used [Muskens 1989b]. This refers to a stricter notion of validity: "all models of Γ verify at least one of Δ *and* all models which falsify all formulae in Δ falsify at least one element of Γ". This notion of validity is propagated mainly because it structurally behaves better than our single-barreled definition. The underlying reason is that it restores contra-position: $\Gamma \models \Delta \Longrightarrow \neg\Delta \models \neg\Gamma$. Note that this does not hold for our definition of validity:

$$p \wedge \neg p \models_{\mathfrak{P}} q, \text{ but } \neg q \not\models_{\mathfrak{P}} \neg(p \wedge \neg p).$$

For a categorization of consequence relations and their axiomatizations in partial propositional and partial modal logic we refer the reader to [Thijsse 1992]. The definition which we gave above, which we will use throughout the thesis, is known in partial logic as the *strong consequence relation.*

Of course, the most remarkable non-$\mathfrak{P}$-validity is the *principle of the excluded middle*:

$$\emptyset \not\models_{\mathfrak{P}} \varphi \vee \neg\varphi.$$

Most other classical principles such as De Morgan laws (1), absorption (2), double negation (3), idempotence (4), distribution (5), commutativity (6), associativity (7) and 'ex falso' (8) are also $\mathfrak{P}$-valid principles.

[7] The most convenient understanding of this style of defining a consequence relation is probably obtained through interpretation of the left hand argument Γ as a big conjunction over its elements and the right hand argument Δ as a big disjunction.

2.8. TABLE.

(1)	$\neg(\varphi \wedge \psi) \approx_{\mathfrak{P}} \neg\varphi \vee \neg\psi$ $\neg(\varphi \vee \psi) \approx_{\mathfrak{P}} \neg\varphi \wedge \neg\psi$	(2)	$\varphi \wedge (\varphi \vee \psi) \approx_{\mathfrak{P}} \varphi$ $\varphi \vee (\varphi \wedge \psi) \approx_{\mathfrak{P}} \varphi$
(3)	$\neg\neg\varphi \approx_{\mathfrak{P}} \varphi$	(4)	$\varphi \vee \varphi \approx_{\mathfrak{P}} \varphi$ $\varphi \wedge \varphi \approx_{\mathfrak{P}} \varphi$
(5)	$\varphi \vee (\psi \wedge \chi) \approx_{\mathfrak{P}} (\varphi \vee \psi) \wedge (\varphi \vee \chi)$ $\varphi \wedge (\psi \vee \chi) \approx_{\mathfrak{P}} (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$	(6)	$\varphi \vee \psi \approx_{\mathfrak{P}} \psi \vee \varphi$ $\varphi \wedge \psi \approx_{\mathfrak{P}} \psi \wedge \varphi$
(7)	$\varphi \vee (\psi \vee \chi) \approx_{\mathfrak{P}} (\varphi \vee \psi) \vee \chi$ $\varphi \wedge (\psi \wedge \chi) \approx_{\mathfrak{P}} (\varphi \wedge \psi) \wedge \chi$	(8)	$\varphi \wedge \neg\varphi \models_{\mathfrak{P}} \psi$

In definition 2.7 above we presupposed that both arguments of the consequence relation were subsets of the restricted language $\mathcal{L}$. Of course, a similar definition can be given for syntactic extensions of $\mathcal{L}$. Such an extended use of the consequence relation will be specified by its subscript. For example, $\models_{\mathfrak{P}^\sim}$ refers to the consequence relation over $\mathfrak{P}$ for the language $\mathcal{L}^\sim$.

Partial Kripke models

In the introductory chapter 1 we already presented the basic parameters of a partial possible worlds or Kripke model. In this chapter and in the next chapter on constructive modal logics, we will use only one modality $\Box$. This means that only one accessibility relation – recall that this relation is the formal description of the collection of uncertainties with respect to a given world – suffices for suitable interpretation of modal formulae. Poly-modal formalisms, in order to cope with multiple agents, will be discussed later on in chapter 3 and chapter 4. For the presentation of the basic modal formalisms, expansion to multiple accessibility relations would only yield redundant syntax. It would not contribute to the technical profit of this chapter and the next one. In chapter 3 we discuss logics with mutual beliefs, where a multiple set of modalities (agents) is of course necessary.

For the sake of completeness we give the formalization of the above-mentioned models in the following definition.

2.9. DEFINITION. A *partial Kripke* or *partial possible worlds model* is a triple $M = \langle W, R, V \rangle$. W is a non-empty set of worlds, $R \subseteq W \times W$ is the accessibility relation and V is a global valuation function which assigns a partial valuation to every world: $V : W \longrightarrow \mathfrak{P}$. The class of all partial Kripke models is denoted by $\mathfrak{M}$.

The class of total Kripke models $M = \langle W, R, V \rangle$ is the subclass of $\mathfrak{M}$ with $\mathfrak{Dom}(V(w)) = I\!P$ for all $w \in W$. This class is denoted by $\mathfrak{K}$.

A (*Kripke*) *frame* is the accessibility structure of such a model, that is the universe of worlds together with the accessibility relation $\langle W, R \rangle$. A partial Kripke model $\langle W, R, V' \rangle$ with $V' : W \longrightarrow \mathfrak{P}$ is said to be a model on F.

Partial Kripke models allow us to interpret intensional formulae in $\mathcal{L}^\Box$ in a satisfactory way. The truth-value decomposition for the extensional connectives is precisely defined as for partial valuations. Only now, the semantic specification on the right-hand side of the forcing relation is a full partial Kripke model, instead of a single partial valuation. Besides that, we also specify the world, taken from the set of worlds in the model, which in fact assigns the truth-values.

2.10. TABLE. Let $M = \langle W, R, V\rangle$ and $w \in W$. The truth-value assignment with respect to w in M is given by the following inductive clauses[8].

$$M, w \models p \Leftrightarrow V(w)(p) = 1 \quad (p \in I\!P)$$
$$M, w \not\models \bot$$
$$M, w \models \neg\varphi \Leftrightarrow M, w \dashv \varphi$$
$$M, w \models \varphi \wedge \psi \Leftrightarrow M, w \models \varphi \text{ and } M, w \models \psi$$
$$M, w \models \Box\varphi \Leftrightarrow \forall w' : R(w, w') \Rightarrow M, w' \models \varphi$$

$$M, w \dashv p \Leftrightarrow V(w)(p) = 0 \quad (p \in I\!P)$$
$$M, w \dashv \bot$$
$$M, w \dashv \neg\varphi \Leftrightarrow M, w \models \varphi$$
$$M, w \dashv \varphi \wedge \psi \Leftrightarrow M, w \dashv \varphi \text{ or } M, w \dashv \psi$$
$$M, w \dashv \Box\varphi \Leftrightarrow \exists w' : R(w, w') \;\&\; M, w' \dashv \varphi$$

Above we formalized the meaning of our static modal operator, which will be employed in chapter 4 as our epistemic operator. Notice that this modal operator obtains the meaning which we have described in the introductory chapter. $\Box\varphi$ is verified if all worlds that are accessible (possible) verify φ, or epistemically speaking, contain evidence for φ. Falsification of such a proposition refers to active disbelief. An agent who has access to a counter-model, i.e. a world which falsifies φ, actively disbelieves this proposition.

2.11. DEFINITION. We say that the model M verifies a formula $\varphi \in \mathcal{L}^\Box$ if $M, v \models \varphi$ for all worlds $v \in W$. The frame $F = \langle W, R\rangle$ verifies a formula $\varphi \in \mathcal{L}^\Box$ iff all partial Kripke models over F verify φ.

The interpretation of a formula $\varphi \in \mathcal{L}^\Box$ with respect to $\mathfrak{M}$ is defined as follows:

$$[\![\varphi]\!]_{\mathfrak{M}} := \{\langle M, w\rangle \mid M, w \models \varphi\}.$$

The interpretation of a formula φ over a single model $M = \langle W, R, V\rangle \in \mathfrak{M}$ is the set of worlds in M which verify φ:

$$[\![\varphi]\!]_M = \{w \in W \mid M, w \models \varphi\}.$$

The interpretation of a formula $\varphi \in \mathcal{L}^\Box$ over a frame F is defined as

[8] The definitions and formats stem from [Thijsse 1990].

$$[\![\varphi]\!]_F := \{\langle M, w\rangle \mid M, w \models \varphi,\ M \text{ is a model on } F\}.$$

In general for any restricted class of partial Kripke models $\mathfrak{C} \subseteq \mathfrak{M}$

$$[\![\varphi]\!]_{\mathfrak{C}} := \{\langle M, w\rangle \mid M, w \models \varphi, M \in \mathfrak{C}\}.$$

Semantic equivalence of two formulae φ and ψ with respect to a class of partial Kripke models $\mathfrak{C}$, i.e. $[\![\varphi]\!]_{\mathfrak{C}} = [\![\psi]\!]_{\mathfrak{C}}$, is denoted by $\varphi \approx_{\mathfrak{C}} \psi$.

The possibility operator $\Diamond$ obtains its intended truth and falsity conditions. $\Diamond\varphi$ has been defined dually to $\Box\varphi$: $\neg\Box\neg\varphi$.

$$M, w \models \Diamond\varphi \ \Leftrightarrow\ \exists v \in W : R(w, v)\ \&\ M, v \models \varphi$$

$$M, w \mathrel{=\!\!|} \Diamond\varphi \ \Leftrightarrow\ \forall v \in W : R(w, v)\ \Rightarrow\ M, v \mathrel{=\!\!|} \varphi$$

2.12. DEFINITION. $\mathfrak{M}$-validity for pairs of sets of formulae from $\mathcal{L}^{\Box}$ obtains the following format:

$$\Gamma \models_{\mathfrak{M}} \Delta \iff \bigcap_{\gamma\in\Gamma} [\![\gamma]\!]_{\mathfrak{M}} \subseteq \bigcup_{\delta\in\Delta} [\![\delta]\!]_{\mathfrak{M}}.$$

In agreement with the vocabulary developed in the previous section, Δ is said to be an $\mathfrak{M}$-valid consequence of Γ. $\models_{\mathfrak{M}^{\sim}}$ denotes the same relation expanded over subsets of $\mathcal{L}^{\sim,\Box}$.

For a subclass of models $\mathfrak{C} \subseteq \mathfrak{M}$ we define $\mathfrak{C}$-validity, $\models_{\mathfrak{C}}$, in precisely the same way as above with $\mathfrak{C}$ substituted for $\mathfrak{M}$. Validity over a Kripke frame F, $\models_F$, is the same as validity over the class of all models in $\mathfrak{M}$ which have F as their underlying frame.

2.13. TABLE. A few important $\mathfrak{M}$-validities are listed below.

$\Box(\varphi \wedge \psi) \approx_{\mathfrak{M}} \Box\varphi \wedge \Box\psi$	$\Diamond\bot \approx_{\mathfrak{M}} \bot$
$\Diamond(\varphi \vee \psi) \approx_{\mathfrak{M}} \Diamond\varphi \vee \Diamond\varphi$	$\Box\top \approx_{\mathfrak{M}} \top$
$\Box\varphi, \Diamond\psi \models_{\mathfrak{M}} \Diamond(\varphi \wedge \psi)$	$\Box\varphi \vee \Box\psi \models_{\mathfrak{M}} \Box(\varphi \vee \psi)$
$\Box(\varphi \vee \psi) \models_{\mathfrak{M}} \Box\varphi, \Diamond\psi$	$\Diamond(\varphi \wedge \psi) \models_{\mathfrak{M}} \Diamond\varphi \wedge \Diamond\psi$

2.2 Derivation

In section 2.1 a multiple conclusion definition of validity has been presented. The system of inference for partial modal logic presented in this section does not diverge from this perspective. This amounts to a Gentzen sequential-like axiomatization [9], with the only difference that premises and conclusions are sets instead of sequences. This reduces the quantity of structural rules.

The sequential style of axiomatization has been chosen mainly because of two reasons. The first reason is that it shows very clearly the difference with classical modal logics, and the second reason is its calculational style of deduction, due

[9] Sequential calculi for modal logics are rare. We refer to [Wansing 1992b] and the bibliography there for different proposals.

to the equal tuning of the arguments of the inference relation. This relation orders the collection of sets of formulae. This latter argument will prove its benefit most evidently in the second part of this thesis which deals with meta-theoretical issues. Many definitions and meta-theoretical proofs can be presented concisely and in this sequential style. For example, the completeness proof of the axiomatization system, which will be presented below, can be shown by means of a short sequential calculation.

Sequential rules for partial modal logic

In the following three tables, 2.14, 2.15 and 2.16, we will present a sequential axiomatization of the minimal partial modal logic. This sequential axiomatization consists of a list of sequential rules of the form:

$$\frac{\Gamma_1 \vdash \Delta_1 \ ... \ \Gamma_n \vdash \Delta_n}{\Gamma_{n+1} \vdash \Delta_{n+1}} \quad \text{with } n \in I\!N \quad (1).$$

Γ_i and Δ_i are sets of formulae for all $i \in \{1, .., n+1\}$. The symbol $\vdash$ denotes the derivation relation between these sets of formulae. $\Gamma \vdash \Delta$ is called a *sequent*, Γ is the *assumption set* of this sequent and Δ its *conclusion set*. The fraction notation in (1) must be interpreted as a conditional. The sequents $\Gamma_i \vdash \Delta_i$ with $i \leq n$ are the conditions of the rule in (1), and $\Gamma_{n+1} \vdash \Delta_{n+1}$ is the consequence of this rule. If $n = 0$ then the set of conditions is empty. In this case the rule is said to be *axiomatic*. Because the arguments of the derivation relation are sets, the notations Γ, φ and Γ, Γ' refer to $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \Gamma'$, respectively.

Table 2.14 shows the structural rules, that is rules without linguistic specific properties such as connectives. The only relevant structural rules are the START rule and two monotonicity rules, L-MON and R-MON. The first one expresses the most trivial derivation step. It says that if an element of the conclusion set Δ also appears in the assumption set Γ then $\Gamma \vdash \Delta$. The monotonicity rules embody the freedom of extending both the assumption set and the conclusion set. Furthermore a CUT rule is present.

2.14. TABLE.

STRUCTURAL RULES

$\Gamma \vdash \Delta$ if $\Gamma \cap \Delta \neq \emptyset$ START

$$\frac{\Gamma \vdash \Delta \quad \Gamma \subseteq \Gamma'}{\Gamma' \vdash \Delta} \text{ L-MON} \qquad \frac{\Gamma \vdash \Delta \quad \Delta \subseteq \Delta'}{\Gamma \vdash \Delta'} \text{ R-MON}$$

$$\frac{\Gamma \vdash \varphi, \Delta \quad \Gamma', \varphi \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{ CUT}$$

The introduction rules for the connectives are separated into TRUE and FALSE rules. This somewhat unusual distinction is inspired by the partial model-theory of the preceding section. As falsity and truth are no longer interdefinable, we distinguish each introduction of a new connective in a false or a true proposition.

Syntactically, the former case of introduction simply means that the resulting connected proposition appears in the scope of a negation.

We present the TRUE rules first:

2.15. TABLE.

TRUE

$$\Gamma, \bot \vdash \Delta \quad \text{L-TRUE } \bot$$

$$\frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta} \quad \text{L-TRUE } \neg$$

$$\frac{\Gamma, \varphi, \psi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta} \quad \text{L-TRUE } \wedge \qquad \frac{\Gamma \vdash \varphi, \Delta \quad \Gamma' \vdash \psi, \Delta'}{\Gamma, \Gamma' \vdash \varphi \wedge \psi, \Delta, \Delta'} \quad \text{R-TRUE } \wedge$$

$$\frac{\Gamma \vdash \varphi, \neg\Delta}{\Box\Gamma \vdash \Box\varphi, \neg\Box\Delta} \quad \text{R-TRUE } \Box$$

The FALSE rules are the following:

2.16. TABLE.

FALSE

$$\Gamma \vdash \neg\bot, \Delta \quad \text{R-FALSE } \bot$$

$$\frac{\Gamma, \varphi \vdash \Delta}{\Gamma, \neg\neg\varphi \vdash \Delta} \quad \text{L-FALSE } \neg \qquad \frac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \neg\neg\varphi, \Delta} \quad \text{R-FALSE } \neg$$

$$\frac{\Gamma, \neg\varphi \vdash \Delta \quad \Gamma', \neg\psi \vdash \Delta'}{\Gamma, \Gamma', \neg(\varphi \wedge \psi) \vdash \Delta, \Delta'} \quad \text{L-FALSE } \wedge \qquad \frac{\Gamma \vdash \neg\varphi, \neg\psi, \Delta}{\Gamma \vdash \neg(\varphi \wedge \psi), \Delta} \quad \text{R-FALSE } \wedge$$

$$\frac{\Gamma, \neg\varphi \vdash \neg\Delta}{\Box\Gamma, \neg\Box\varphi \vdash \neg\Box\Delta} \quad \text{L-FALSE } \Box$$

Other sequential systems for partial logics

Instead of separating the introduction rules into TRUE and FALSE rules, we could also define two derivation relations. Apart from the ordinary relation $\vdash$, a second relation $\dashv$ can be defined. $\Gamma \dashv \Delta$ then means that at least one of the members of Δ is false, if all formulae in Γ hold. This gives a somewhat more elegant presentation of the false rules, i.e. without using the negation. For example, the reformulation of the FALSE introduction of the conjunction would then look as follows

$$\frac{\Gamma \dashv \varphi, \psi, \Delta}{\Gamma \dashv \varphi \wedge \psi, \Delta}.$$

Another stylish sequential system for partial logics has been proposed in [Fenstad, Langholm & Vespren 1992]. In this paper two-placed sequents are replaced by four-placed *quadrants*. Just like the additional $\dashv$-notation, falsity is used in the definition such that FALSE-rules do not need negations. An inference rule in this system has the following format:

$$\begin{array}{c|c} \Gamma_1 & \Sigma_1 \\ \hline \Delta_1 & \Theta_1 \end{array} \& \ldots \& \begin{array}{c|c} \Gamma_n & \Sigma_n \\ \hline \Delta_n & \Theta_n \end{array} \Longrightarrow \begin{array}{c|c} \Gamma' & \Sigma' \\ \hline \Delta' & \Theta' \end{array}.$$

The meaning of such a rule coincides with the following reformulation in our sequential style:

$$\frac{\Gamma_1, \neg\Delta_1 \vdash \Sigma_1, \neg\Theta_1 \;..\; \Gamma_n, \neg\Delta_n \vdash \Sigma_n, \neg\Theta_n}{\Gamma', \neg\Delta' \vdash \Sigma', \neg\Theta'}.$$

For example, L-FALSE $\Box$ corresponds to the following four-placed inference rule:

$$\begin{array}{c|c} \Gamma & \Sigma \\ \hline \Delta, \varphi & \Theta \end{array} \Longrightarrow \begin{array}{c|c} \Box\Gamma & \Diamond\Sigma \\ \hline \Diamond\Delta, \Box\varphi & \Box\Theta \end{array}.$$

An advantage of this four-placed variant is that the rules in the tables 2.14 – 2.16 can be formulated more symmetrically. The L-TRUE $\neg$ can be incorporated as a structural rule:

$$\begin{array}{c|c} \Gamma & \Sigma_1, \Sigma_2 \\ \hline \Delta & \Theta \end{array} \Longrightarrow \begin{array}{c|c} \Gamma & \Sigma_2 \\ \hline \Delta, \Sigma_1 & \Theta \end{array}.$$

Now, the other rules make up a nice dual system. The R-TRUE rules can be identified as North West rules (NW). The R-FALSE-rules reappear as NW- and SW-rules, and the L-TRUE- and L-FALSE-rules as NE- and SE-rules, respectively. For example, NW $\wedge$ and SE $\wedge$ obtain the same description:

$$\begin{array}{c|c} \Gamma, \varphi, \psi & \Sigma \\ \hline \Delta & \Theta \end{array} \Longrightarrow \begin{array}{c|c} \Gamma, \varphi \wedge \psi & \Sigma \\ \hline \Delta & \Theta \end{array} \quad \text{and}$$

$$\begin{array}{c|c} \Gamma & \Sigma \\ \hline \Delta & \Theta, \varphi, \psi \end{array} \Longrightarrow \begin{array}{c|c} \Gamma & \Sigma \\ \hline \Delta & \Delta, \varphi \wedge \psi \end{array}$$

The other two rules for conjunction introduction are also similar:

$$\begin{array}{c|c} \Gamma & \Sigma \\ \hline \Delta, \varphi & \Theta \end{array} \& \begin{array}{c|c} \Gamma' & \Sigma' \\ \hline \Delta', \psi & \Theta' \end{array} \Longrightarrow \begin{array}{c|c} \Gamma, \Gamma' & \Sigma, \Sigma' \\ \hline \Delta, \Delta', \varphi \wedge \psi & \Theta, \Theta' \end{array} \quad \text{and}$$

$$\begin{array}{c|c} \Gamma & \Sigma, \varphi \\ \hline \Delta & \Theta \end{array} \& \begin{array}{c|c} \Gamma' & \Sigma', \psi \\ \hline \Delta' & \Theta' \end{array} \Longrightarrow \begin{array}{c|c} \Gamma, \Gamma' & \Sigma, \Sigma', \varphi \wedge \psi \\ \hline \Delta, \Delta' & \Theta, \Theta' \end{array}.$$

Throughout the thesis we will stick to our explicit TRUE-FALSE distinction in sequential rules.

2.17. DEFINITION. A set of formulae Δ is $\mathbf{M}$-*derivable* from another set of formulae Γ whenever $\Gamma \vdash \Delta$ can be established by following a finite number of applications of the rules above (table 2.14 – 2.16). The corresponding relation is denoted as $\Gamma \vdash_M \Delta$, and is called an $\mathbf{M}$-sequent. $\mathbf{P}$-derivability refers to the relation between sets $\Gamma, \Delta \subseteq \mathcal{L}$, denoted by $\Gamma \vdash_P \Delta$, such that $\Gamma \vdash \Delta$ can be shown by a finite number of applications of the rules above with the exception of the $\Box$-introduction rules.

If $\varphi \vdash_M \psi$ and $\psi \vdash_M \varphi$ then we write $\varphi \equiv_M \psi$. The relation $\equiv_P$ is defined analogously.

In the sequel of this thesis we will skip this kind of definitions as they will always be in the same format as in the previous definition. Given a set of **S**-rules, **S**-derivability and **S**-sequent refer to this set of rules in the same way as these notions for **M** refer to the given **M**-rules.

Another important feature of definition 2.17 is its implicit finiteness.

2.18. OBSERVATION. If $\Gamma \vdash_M \Delta$ then there exist finite subsets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\Gamma' \vdash_M \Delta'$. This can be proved by a straightforward induction on the length of derivations. **M**-derivability is defined by making only a finite number of derivation steps.

This legalization of limiting sequents to the class of finite sequents is very practical for development of the meta-theory in part II. All the systems discussed throughout this thesis have this finiteness property. We will avoid superfluous reference to this property and freely apply it whenever it is necessary.

2.19. DEFINITION. Let **X** and **S** be two sequential derivation systems. **X** is said to be a *sequential extension*, or extension for short, of **S** if they both have the finiteness property as formulated in observation 2.18 and for all $\Gamma \vdash_S \Delta$ also $\Gamma \vdash_X \Delta$. If their associated languages also coincide, then **X** is said to be a *normal* extension of **S**.

Properties of P and M

The most urgent question is whether these systems are sound with respect to the validity notions $\models_{\mathfrak{P}}$ and $\models_{\mathfrak{M}}$ presented in section 2.1.

2.20. THEOREM. SOUNDNESS **P** AND **M**

For all $\Gamma, \Delta \subseteq \mathcal{L} : \Gamma \vdash_P \Delta \Longrightarrow \Gamma \models_{\mathfrak{P}} \Delta$, and

for all $\Gamma, \Delta \subseteq \mathcal{L}^{\Box} : \Gamma \vdash_M \Delta \Longrightarrow \Gamma \models_{\mathfrak{M}} \Delta$.

Proof. By a straightforward induction on the length of derivations. By way of illustration we demonstrate the soundness of the rule R-TRUE $\Box$ for **M**. Let this rule be applicable: for certain $\Gamma, \Delta \subseteq \mathcal{L}^{\Box}$ and $\varphi \in \mathcal{L}^{\Box}$ $\Gamma \vdash_M \varphi, \neg\Delta$, and therefore through application of this rule $\Box\Gamma \vdash_M \Box\varphi, \neg\Box\Delta$. We need to prove $\Box\Gamma \models_{\mathfrak{M}} \Box\varphi, \neg\Box\Delta$.

Suppose that $M, w \models \Box\gamma$ for all $\gamma \in \Gamma$ and $M, w \not\models \Box\varphi$. This latter assumption says that there exists a world v in M such that $M, v \not\models \varphi$. The former assumption yields $M, v \models \gamma$ for all $\gamma \in \Gamma$. Because $\Gamma \vdash \varphi, \neg\Delta$ and the induction hypothesis ($\Gamma \models_{\mathfrak{M}} \varphi, \neg\Delta$), we know that there exists $\delta \in \Delta$ such that $M, v \models \neg\delta$: whence $M, w \models \neg\Box\delta$. In short, $M, w \models \Box\varphi$ or $M, w \models \neg\Box\delta$ for certain $\delta \in \Delta$. Because $\langle M, w\rangle$ has been picked freely from $\bigcap_{\gamma\in\Gamma}[\![\Box\gamma]\!]_{\mathfrak{M}}$, we conclude $\Box\Gamma \models_{\mathfrak{M}} \Box\varphi, \neg\Box\Delta$. ■

The reader who is not familiar with sequential calculi is advised to check the soundness of different rules for himself.

The converse of these soundness results above, completeness of **M** and **P**, is less trivial. We will present the completeness results in chapter 5 in part II.

2.21. TABLE. Rules for the defined connectives $\top$, $\vee$ and $\Diamond$ are derivable by the **M**-rules. They can all be derived by means of the double negation rules L-FALSE $\neg$ and R-FALSE $\neg$ only.

$$\Gamma \vdash \top, \Delta \quad \text{R-TRUE } \top$$

$$\Gamma, \neg\top \vdash \Delta \quad \text{L-FALSE } \top$$

$$\frac{\Gamma, \varphi \vdash \Delta \quad \Gamma', \psi \vdash \Delta'}{\Gamma, \Gamma', \varphi \vee \psi \vdash \Delta, \Delta'} \text{ L-TRUE } \vee \qquad \frac{\Gamma \vdash \varphi, \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta} \text{ R-TRUE } \vee$$

$$\frac{\Gamma, \neg\varphi, \neg\psi \vdash \Delta}{\Gamma, \neg(\varphi \vee \psi) \vdash \Delta} \text{ L-FALSE } \vee \qquad \frac{\Gamma \vdash \neg\varphi, \Delta \quad \Gamma' \vdash \neg\psi, \Delta'}{\Gamma, \Gamma' \vdash \neg(\varphi \vee \psi), \Delta, \Delta'} \text{ R-FALSE } \vee$$

$$\frac{\Gamma, \varphi \vdash \Delta}{\Box\Gamma, \Diamond\varphi \vdash \Diamond\Delta} \text{ L-TRUE } \Diamond$$

$$\frac{\Gamma \vdash \neg\varphi, \Delta}{\Box\Gamma \vdash \neg\Diamond\varphi, \Diamond\Delta} \text{ R-FALSE } \Diamond$$

The $\Diamond$-notation in front of sets of formulae is more convenient than $\neg\Box$. Therefore, we most often use the following reformulation of R-TRUE $\Box$ and L-FALSE $\Box$:

$$\frac{\Gamma \vdash \varphi, \Delta}{\Box\Gamma \vdash \Box\varphi, \Diamond\Delta} \qquad \frac{\Gamma, \neg\varphi \vdash \Delta}{\Box\Gamma, \neg\Box\varphi \vdash \Diamond\Delta} \; .$$

As promised in the introduction of this section, the sequential axiomatization of partial propositional and modal logic illustrates very clearly the difference with classical modal logic. It simply comes down to the absence of a R-TRUE $\neg$-rule:

$$\frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta} \; .$$

This causes the absence of the law of the excluded middle: $\not\vdash_M \varphi \vee \neg\varphi$. A lot of other classically valid principles still hold. All the principles listed in table 2.8 and table 2.13 are quickly derivable. In the following example we demonstrate the distribution principles of conjunction over disjunction.

2.22. EXAMPLE. $(\varphi \wedge \psi) \vee (\varphi \wedge \chi) \vdash_P \varphi \wedge (\psi \vee \chi)$

1. $\varphi \vdash_P \varphi$ START
2. $\psi \vdash_P \psi, \chi$ START
3. $\chi \vdash_P \psi, \chi$ START
4. $\psi \vdash_P \psi \vee \chi$ R-TRUE $\vee$ (2)
5. $\chi \vdash_P \psi \vee \chi$ R-TRUE $\vee$ (3)

6. $\varphi, \psi \vdash_P \varphi \wedge (\psi \vee \chi)$ R-TRUE $\wedge$ (1,4)
7. $\varphi, \chi \vdash_P \varphi \wedge (\psi \vee \chi)$ R-TRUE $\wedge$ (1,5)
8. $\varphi \wedge \psi \vdash_P \varphi \wedge (\psi \vee \chi)$ L-TRUE $\wedge$ (6)
9. $\varphi \wedge \chi \vdash_P \varphi \wedge (\psi \vee \chi)$ L-TRUE $\wedge$ (7)
10. $(\varphi \wedge \psi) \vee (\varphi \wedge \chi) \vdash_P \varphi \wedge (\psi \vee \chi)$ L-TRUE $\vee$ (8,9)

$\varphi \wedge (\psi \vee \chi) \vdash_P (\varphi \wedge \psi) \vee (\varphi \wedge \psi)$

1. $\varphi \vdash_P \varphi$ START
2. $\psi \vdash_P \psi$ START
3. $\chi \vdash_P \chi$ START
4. $\varphi, \psi \vdash_P \varphi \wedge \psi$ R-TRUE $\wedge$ (1,2)
5. $\varphi, \chi \vdash_P \varphi \wedge \chi$ R-TRUE $\wedge$ (1,3)
6. $\varphi, \psi \vee \chi \vdash_P \varphi \wedge \psi, \varphi \wedge \chi$ L-TRUE $\vee$ (4,5)
7. $\varphi \wedge (\psi \vee \chi) \vdash_P \varphi \wedge \psi, \varphi \wedge \chi$ L-TRUE $\wedge$ (6)
8. $\varphi \wedge (\psi \vee \chi) \vdash_P (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$ R-TRUE $\vee$ (7)

These derivations also hold for $\vdash_M$, of course. All the **P**-rules are also **M**-rules.

Because of commutativity, associativity and idempotence of both the disjunction and the conjunction, the disjunction and conjunction over an arbitrary finite sets of formulae Γ is unambiguous modulo the equivalence relation $\equiv_M$. Throughout the thesis we will use the notation $\bigvee \Gamma$ and $\bigwedge \Gamma$ for this disjunction and conjunction, and treat them as ordinary formulae.

In the calculus which we presented we may replace finite right hand arguments by a disjunction, and finite left hand arguments by a conjunction.

2.23. OBSERVATION.

$$\Gamma \vdash_M \Delta \;\;\&\;\; \Delta \text{ finite} \Longrightarrow \Gamma \vdash_M \bigvee \Delta, \text{ and}$$

$$\Gamma \vdash_M \Delta \;\;\&\;\; \Gamma \text{ finite} \Longrightarrow \bigwedge \Gamma \vdash_M \Delta.$$

$\top$ and $\bot$ are taken to be the empty conjunction, $\bigwedge \emptyset$, and the empty disjunction, $\bigvee \emptyset$, respectively. Note that these definitions are correct with respect to the observation about finite assumption and conclusion sets above.

In order to demonstrate the use of modal rules we show two simple derivations.

2.24. EXAMPLE. We first give the simple derivation of the "modal ex falso": $\Diamond(\varphi \wedge \neg\varphi) \vdash_M \psi$.

1. $\varphi \vdash_M \varphi$ START
2. $\varphi, \neg\varphi \vdash_M \emptyset$ L-TRUE $\neg$ (1)
3. $\varphi \wedge \neg\varphi \vdash_M \emptyset$ L-TRUE $\wedge$ (2)
4. $\Diamond(\varphi \wedge \neg\varphi) \vdash_M \emptyset$ L-TRUE $\Diamond$ (3)
5. $\Diamond(\varphi \wedge \neg\varphi) \vdash_M \psi$ R-MON (4)

An important **M**-sequent is $\Box(\alpha \vee \beta) \vdash_M \Box\alpha, \Box\beta, \Diamond\alpha \wedge \Diamond\beta$. It tells us how to get rid of disjunctions inside the scope of a necessity operator. A six step derivation of this **M**-sequent is presented below.

1.	$\alpha \vdash_M \alpha, \beta$	START
2.	$\beta \vdash_M \alpha, \beta$	START
3.	$\alpha \vee \beta \vdash_M \alpha, \beta$	L-TRUE $\vee$ (1,2)
4.	$\Box(\alpha \vee \beta) \vdash_M \Box\alpha, \Diamond\beta$	R-TRUE $\Box$ (3)
5.	$\Box(\alpha \vee \beta) \vdash_M \Box\beta, \Diamond\alpha$	R-TRUE $\Box$ (3)
6.	$\Box(\alpha \vee \beta) \vdash_M \Box\alpha, \Box\beta, \Diamond\alpha \wedge \Diamond\beta$	R-TRUE $\wedge$ (4,5)

Remember that the sequential arguments, assumptions and conclusions are sets. The last inference step in the derivation above is therefore legitimate.

For additional mastering of the calculus, we advise the reader to derive the semantic validities listed in table 2.13 on page 52.

Adding the weak negation: $\mathbf{P}^{\sim}$ and $\mathbf{M}^{\sim}$

In the first subsection of this chapter we have discussed the weak negation, which expresses that its argument is not true. In partial logic, this absence of truth does not coincide with falsity. In our terminology, addition of the weak negation in the systems **P** and **M** leads to three-valued propositional and three-valued modal logic[10].

Establishing sequential calculi for axiomatization of $\models_{\mathfrak{P}^\sim}$ and $\models_{\mathfrak{M}^\sim}$ is not very hard. As mentioned earlier in section 2.1, the weak negation restores totality in a certain way. In terms of derivation this can be seen through its complete imitation of the classical negation: a R-TRUE $\sim$-rule is added. The other rules are simply the same as the ones presented for the strong negation.

2.25. TABLE.

$$\frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \sim\varphi \vdash \Delta}\ \text{L-TRUE} \sim \qquad \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \sim\varphi, \Delta}\ \text{R-TRUE} \sim$$

$$\frac{\Gamma, \varphi \vdash \Delta}{\Gamma, \neg \sim\varphi \vdash \Delta}\ \text{L-FALSE} \sim \qquad \frac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \neg \sim\varphi, \Delta}\ \text{R-FALSE} \sim$$

2.26. THEOREM. SOUNDNESS $\mathbf{P}^{\sim}$ AND $\mathbf{M}^{\sim}$

For all $\Gamma, \Delta \subseteq \mathcal{L}^{\sim} : \Gamma \vdash_{P^\sim} \Delta \implies \Gamma \models_{\mathfrak{P}^\sim} \Delta$, and

for all $\Gamma, \Delta \subseteq \mathcal{L}^{\Box,\sim} : \Gamma \vdash_{M^\sim} \Delta \implies \Gamma \models_{\mathfrak{M}^\sim} \Delta$.

Proof. Once again, by induction on the length of derivation. We demonstrate application of the rule R-TRUE $\sim$ here for the system $\mathbf{P}^{\sim}$. Suppose $\Gamma, \varphi \vdash_{P^\sim} \Delta$. The

[10] The first article on three-valued modal logic is [Segerberg 1965]. Other short essays on this issue are [Schotch et al. 1978] and [Morikawa 1989]. A longer article on many-valued modal logics can be found in [Fitting 1992].

induction hypothesis yields $\Gamma, \varphi \models_{\mathfrak{P}\sim} \Delta$ (3). We need to show that the derivation step $\Gamma \vdash_{P\sim} \sim\varphi, \Delta$ is sound: $\Gamma \models_{\mathfrak{P}\sim} \sim\varphi, \Delta$.

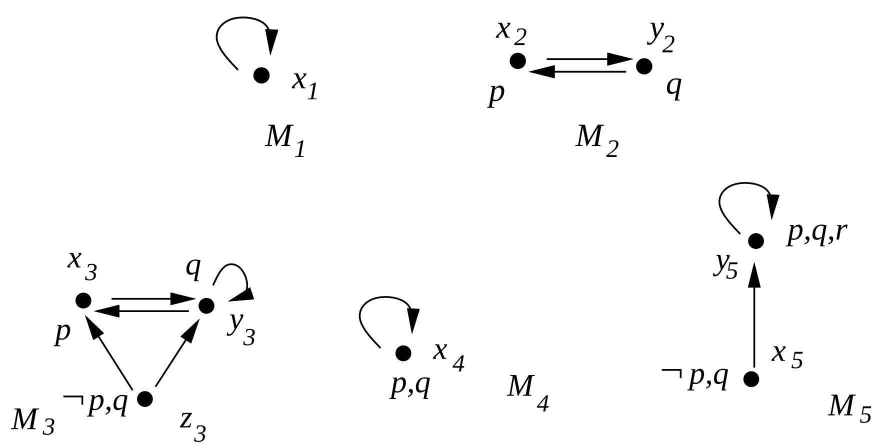
Let $V \models \gamma$ for all $\gamma \in \Gamma$ for certain $V \in \mathfrak{P}$. If $V \not\models \sim\varphi$ then $V \models \varphi$ and therefore, according to (3), $V \models \delta$ for certain $\delta \in \Delta$. This means, because of the arbitrary choice of V from $\bigcap_{\gamma\in\Gamma}[\![\gamma]\!]_{\mathfrak{P}}$, that $\Gamma \models_{\mathfrak{P}\sim} \sim\varphi, \Delta$. ■

2.3 Ordering partial possible worlds

The growth of modal information, as elaborately argued in the introduction chapter, is of particular importance for logical analysis of epistemic dynamic processes such as communication. This section is devoted to a structural definition of what it means for one partial possible world to extend another. In the next chapter we will develop supplementary linguistic tools which enable us to define inference systems to reason about growth and retraction of information.

In the first section of this chapter we have already defined an extension relation for partial valuations. This is a very straightforward definition of growth of extensional propositional information. A partial valuation V' contains all the information of another partial valuation V if the atomic content of V' contains the atomic content of V. An appropriate definition of such an extension relation among partial possible worlds is more complicated. Some intuition of this complication can be extracted from the following figure.

2.27. FIGURE.



With the exception of M_1 and M_4, each two models have a mutually different structure. It is not hard to grasp that x_4 should be an extension of x_1. Nevertheless, it turns out that all the worlds contain the information in x_1. For the extensional part of $\mathcal{L}^\Box$, $\mathcal{L}$, this can be seen immediately, because all the local truth-value assignments extend the empty local valuation of x_1.

Expansion of this conclusion to intensional information ($\mathcal{L}^\Box$) requires a closer exploration of the underlying frames of the different models.

The most instructive example from the illustration in figure 2.27 is a comparison between z_3 and x_5, because they have access to multiple worlds. With regard to their modal informational content x_5 turns out to be an enrichment of

z_3. The challenge is to give a structural reason why x_5 adopts all modal information of z_3. An explanation of this structural argument becomes most clear if we firstly demonstrate the transfer of truth-assignment in z_3 to formulae of the form $\Box\varphi$ with $\varphi \in \mathcal{L}$ to x_5.

Suppose $M_5, x_5 \not\models \Box\varphi$ with $\varphi \in \mathcal{L}$. This means that x_5 sees some world which does not verify φ. In other words, $M_5, y_5 \not\models \varphi$. Because z_3 has access to a world which has a smaller extensional content than y_5, namely y_3, we have made sure that $M_3, z_3 \not\models \Box\varphi$, because $M_3, y_3 \not\models \varphi$.

Suppose $M_3, z_3 \dashv \Box\varphi$ with $\varphi \in \mathcal{L}$. This means that $M_3, x_3 \dashv \varphi$ or $M_3, y_3 \dashv \varphi$. Because all these accessibilities of z_3, x_3 and y_3, have local valuations which are extended by the local valuation of y_5 in M_5 and $\varphi \in \mathcal{L}$ we obtain $M_5, y_5 \dashv \varphi$. Because x_5 sees y_5 in M_5, we also have $M_5, x_5 \dashv \Box\varphi$.

Generalization of this line of argumentation to arbitrary worlds w in M and w' in M', with respect to transfer of truth-values of $\Box\mathcal{L}$-formulae entails the following order.

$$\forall v \in W : R(w, v) \Rightarrow \exists v' \in W' : R'(w', v') \text{ and } V(v) \sqsubseteq V'(v), \text{ and}$$

$$\forall v' \in W' : R'(w', v') \Rightarrow \exists v \in W : R(w, v) \text{ and } V(v) \sqsubseteq V'(v).$$

The first requirement takes care of transfer of falsity of these formulae $\Box\varphi$ with $\varphi \in \mathcal{L}$, and the second forces adoption of truth of such formulae. If the relation above holds between w and w' and also $V(w) \sqsubseteq V'(w')$ then w' in M' is said to be an extension of degree 1 of w in M. This structural relation guarantees not only that all information of w about $\mathcal{L} \cup \Box\mathcal{L}$ is adopted by w'. In fact it amounts to the transfer of all information of modal depth not larger than 1, that is formulae in which no subformula appears in the scope of more than one modal operator.

2.28. DEFINITION. The modal depth of a formula $\varphi \in \mathcal{L}^\Box$, abbreviated by $md(\varphi)$, is defined by the following recursive definition:

$$md(p) = 0 \ (p \in I\!P) \qquad md(\bot) = 0$$

$$md(\neg\varphi) = md(\varphi) \qquad md(\varphi \wedge \psi) = \max\{md(\varphi), md(\psi)\}$$

$$md(\Box\varphi) = md(\varphi) + 1$$

In order to ensure full transfer of modal information of arbitrary modal depth we present the following recursive definition.

2.29. DEFINITION. Let $M = \langle W, R, V\rangle$ and $M' = \langle W', R', V'\rangle$ be a pair of partial Kripke models. A world w' in the model M' is said to be an *extension of degree* 0 of a world w in M iff $V(w) \sqsubseteq V'(w')$. This relation is abbreviated by $w \sqsubseteq^0_{M,M'} w'$. w' in M' is an *extension of degree* n, for $n > 0$, of w in M, $w \sqsubseteq^n_{M,M'} w'$, if

$$w \sqsubseteq^{n-1}_{M,M'} w',$$

$$\forall v \in W : R(w, v) \Rightarrow \exists v' \in W' : R'(w', v') \ \&\ v \sqsubseteq^{n-1}_{M,M'} v', \text{ and}$$

$\forall v' \in W' : R'(w', v') \Rightarrow \exists v \in W : R(w, v) \ \&\ v \sqsubseteq^{n-1}_{M,M'} v'$.[11]

2.30. OBSERVATION. If w' in M' is an extension of degree n of w in M for certain $n \in I\!N$ then also $w \sqsubseteq^{k}_{M,M'} w'$ for all $k \leq n$.

2.31. LEMMA. Let $M = \langle W, R, V\rangle$ and $M' = \langle W', R', V'\rangle$ be a pair of partial Kripke models, and let $w \in W$ and $w' \in W'$. For all $\varphi \in \mathcal{L}^{\Box}$

$$w \sqsubseteq^{md(\varphi)}_{M,M'} w' \Rightarrow \begin{cases} M, w \models \varphi & \Rightarrow \quad M', w' \models \varphi, \text{ and} \\ M, w \dashv \varphi & \Rightarrow \quad M', w' \dashv \varphi. \end{cases}$$

Proof. By induction on the construction of formulae of $\mathcal{L}^{\Box}$. The basic step, $\varphi = p \in I\!P$, is immediately obtained from the definition $w \sqsubseteq^{0}_{M,M'} w'$ and theorem 2.6. The case $\bot$ is trivial, and the other two 'extensional' connectives are immediate consequences of the induction hypothesis (in the case of conjunction observation 2.30 is needed as well).

Let $w \sqsubseteq^{md(\varphi)}_{M,M'} w'$ and $\varphi = \Box\varphi'$.

Suppose $M, w \dashv \Box\varphi'$. This means that $M, v \dashv \varphi'$ for certain $v \in W$ such that $R(w, v)$. By definition of $\sqsubseteq^{md(\varphi)}_{M,M'}$, we know that there also exists $v' \in W'$ such that $v \sqsubseteq^{md(\varphi)-1}_{M,M'} v'$. Because $md(\varphi') = md(\varphi) - 1$ and the induction hypothesis, we may conclude $M', v' \dashv \varphi'$ and also $M', w' \dashv \Box\varphi'$.

If $M', w' \not\models \Box\varphi'$ then there exists $v' \in W'$ such that $R'(w', v')$ and $M', v' \not\models \varphi'$. Analogously to the argument above, using the third clause in definition 2.29 and the induction hypothesis, we obtain $M, v \not\models \varphi'$ for certain $v \in W$ with $R(w, v)$. Consequently, $M, w \not\models \Box\varphi'$.

■

In general, the converse of this lemma does not hold. Nevertheless, it applies to wide classes of partial Kripke models. An example is the class of finitely branching models. This class consists of models in which every world has access to only a finite number of worlds.

2.32. LEMMA. Let $M = \langle W, R, V\rangle$ and $M' = \langle W', R', V'\rangle$ be a pair of partial Kripke models which are both finitely branching and let $w \in W$ and $w' \in W'$. If for all $\varphi \in \mathcal{L}^{\Box}$, with $md(\varphi) \leq n$, $M, w \models \varphi$ implies $M', w' \models \varphi$, then also $w \sqsubseteq^{n}_{M,M'} w'$.

Proof. By induction on the degree n. If $n = 0$ the result is simply a repetition of the persistence result in theorem 2.6.

Let $n > 0$, and suppose $w \not\sqsubseteq^{n}_{M,M'} w'$. This means that one of the three clauses in definition 2.29 does not hold. We need to show in all three cases the existence of a $\varphi \in \mathcal{L}^{\Box}$ with $md(\varphi) \leq n$ such that $M, w \models \varphi$ and $M', w' \not\models \varphi$.

If $w \not\sqsubseteq^{n-1}_{M,M'} w'$ then the induction hypothesis may be applied. It immediately guarantees the existence of a formula $\varphi \in \mathcal{L}^{\Box}$ such that $M, w \models \varphi$ and $M', w' \not\models \varphi$

[11] This recursive definition stems from [Jaspars 1991a]. In this article this definition has been employed to define minimal interpretation in $\mathfrak{M}$ of $\mathcal{L}^{\Box}$-formulae.

and $md(\varphi) < n$.

Suppose that there exists $v \in W$ such that $R(w,v)$ and for all $v' \in W'$ if $R'(w',v')$ then $v \not\sqsubseteq^{n-1}_{M,M'} v'$. This means, on account of the induction hypothesis, that for all v' with $R'(w',v')$ there exists $\varphi_{v'} \in \mathcal{L}^{\Box}$ such that $md(\varphi_{v'}) \leq n-1$, $M,v \models \varphi_{v'}$ and $M',v' \not\models \varphi_{v'}$. We define

$$\varphi := \bigwedge\{\varphi_{v'} \mid R'(w',v')\}.$$

This definition is legitimate, for M' is finitely branching. Moreover, $md(\varphi) < n$. Obviously, $M,v \models \varphi$, and therefore $M,w \models \Diamond\varphi$. On the other hand, if $R'(w',v')$ then $M',v' \not\models \varphi$, and so $M',w' \not\models \Diamond\varphi$. Note that $md(\Diamond\varphi) \leq n$.

Suppose there exists $v' \in W'$ such that $R'(w',v')$, and for all $v \in W$ if $R(w,v)$ then $v \not\sqsubseteq^{n-1}_{M,M'} v'$. The induction hypothesis guarantees the existence of a certain $\varphi_v \in \mathcal{L}^{\Box}$ such that $md(\varphi_v) \leq n-1$ and $M,v \models \varphi_v$ and $M',v' \not\models \varphi_v$ for all v with $R(w,v)$. Let

$$\varphi := \bigvee\{\varphi_v \mid R(w,v)\}.$$

M is finitely branching, and therefore φ is a formula with $md(\varphi) < n$. Clearly, $M,v \models \varphi$ for all v with $R(w,v)$. This yields $M,v \models \Box\varphi$. On the contrary, $M',v' \not\models \varphi$, and thus $M',w' \not\models \Box\varphi$ ($md(\Box\varphi) \leq n$).

■

The following picture shows that this lemma cannot be extended to the full class of partial Kripke models.

2.33. EXAMPLE. Let $\mathbb{P} = \{p_i \mid i \in \mathbb{N}\}$ and let $M = \langle W,R,V\rangle \in \mathfrak{M}$ with

$W = \{y_i \mid i \in \mathbb{N}\}$

$R = \{\langle y_0, y_i\rangle \mid i \neq 0\}$

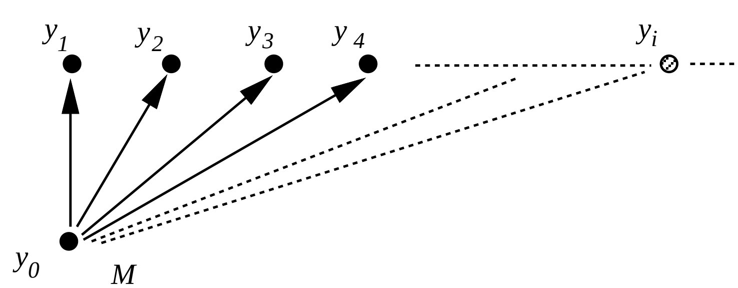
$$V(y_i)(p_j) = \begin{cases} 1 & \text{if } j \leq i \\ \text{undefined} & \text{otherwise.} \end{cases}$$

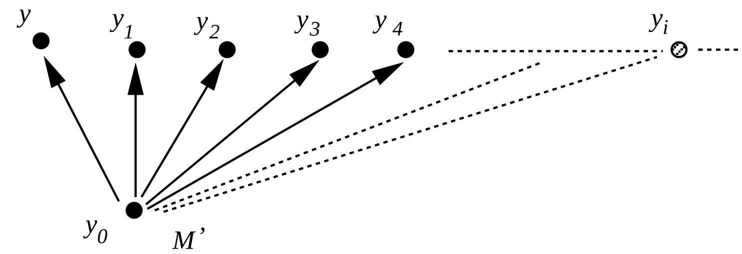
Furthermore, we define a second partial Kripke model $M' = \langle W',R',V'\rangle$:

$W' = W \cup \{y\}$ $\qquad V'(y_i)(p_j) = V(y_i,p_j)$

$R' = R \cup \{\langle y_0, y\rangle\}$ $\qquad V'(y)(p_j) = 1$ for all $p_j \in \mathbb{P}$.

The structures of these two models are displayed below.





It is not hard to verify, by an induction of the construction of formulae, that

$M, y_0 \models \varphi \iff M', y_0 \models \varphi$ (1).

We also find $y_0 \sqsubseteq^n_{M,M'} y_0$ for all $n \in I\!N$. Nevertheless, the converse does not hold. In particular, $y_0 \not\sqsubseteq^1_{M',M} y_0$. This property can easily be inferred from the simple fact that

$V'(y) \not\sqsubseteq V(y_i)$ for all $i \in I\!N \setminus \{0\}$.

This means that the right-to-left direction of the equivalence in (1) and this non-1-extension relation is a counter-example for the converse of lemma 2.31.

This counterexample of the converse of lemma 2.31 relies on the infinity of the set of atoms $I\!P$. The question arises whether restricting $I\!P$ to be finite helps to obtain a converse result of this lemma. This relative conversion of lemma 2.31 is indeed valid.

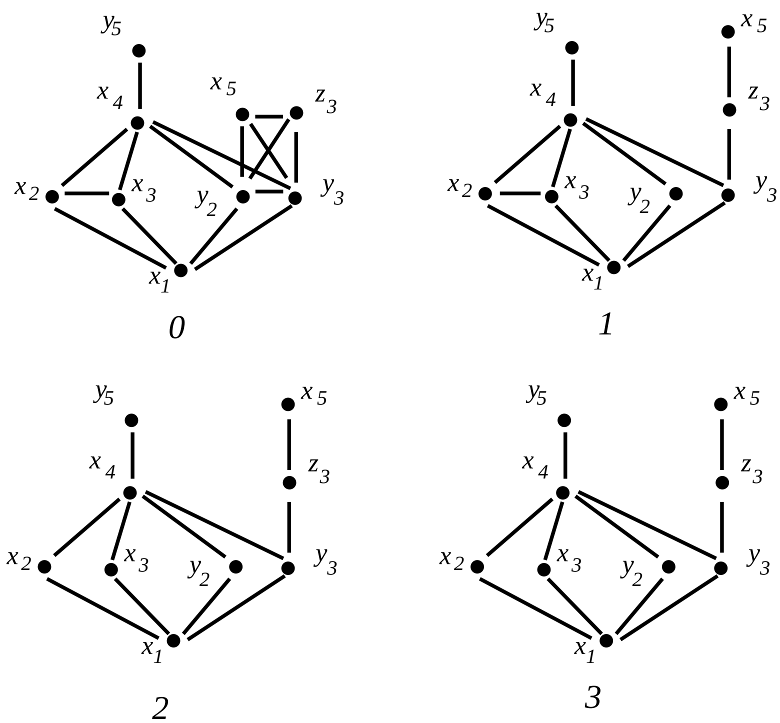
2.34. LEMMA. Let $M = \langle W, R, V\rangle$ and $M' = \langle W', R', V'\rangle$ be a pair of partial Kripke models, and let $w \in W$ and $w' \in W'$, and let $I\!P$, the set of atoms, be finite. If for all $\varphi \in \mathcal{L}^\Box$, with $md(\varphi) \leq n$, $M, w \models \varphi$ implies $M', w' \models \varphi$, then also $w \sqsubseteq^n_{M,M'} w'$.

Proof. We will only give a sketch of the proof. It can easily be deduced from the fact that the set of equivalence classes of semantically equivalent formulae with a given maximal modal depth n is finite if $I\!P$ is finite (see e.g. [Jaspars 1993]). Let $\Phi_n = \{\varphi_1, ..., \varphi_m\}$ be a set where every distinct equivalent class is represented by one of the φ_i's. So, $\forall \varphi \in \mathcal{L}^\Box : md(\varphi) \leq n \Rightarrow \exists \varphi_i \in \Phi_n : \varphi \equiv_{\mathfrak{M}} \varphi_i$.
Let $M, M' \in \mathfrak{M}$, possibly with infinite branches, and reconsider the induction steps in the proof of lemma 2.32. Of course, the first step immediately follows from the induction hypothesis again. The last two steps can now also be made by the finiteness of Φ_n. The formulae φ there can be constructed by taking $\varphi_{v'} \in \Phi_n$ in the first step, and $\varphi_v \in \Phi_n$ in the second. The resulting conjunction and disjunction are well-defined by the finiteness of Φ_n, despite the fact that w and w' may have infinitely accessible v and v'. ■

A similar converse result of lemma 2.31 can be obtained for a extension of $\mathcal{L}^\Box$ with infinite conjunctions and disjunctions. The proof of lemma 2.32 can be applied to obtain such a result.

In the following picture we present the result of application of the recursive definition 2.29 of gradual extension order to the worlds in figure 2.27.

2.35. FIGURE.



The numbers under the diagrams represent the degree of the extension order.

These diagrams must be interpreted in the way Hasse-diagrams are used for partial orders. If there exists a pure ascending path from a to b, then a is smaller than b. If there exists a horizontal path from a to b, then a is as large as b. Of course, the schematic presentation of these possible worlds is not completely correct. Additional labeling of the paths with model names would have been more accurate. As we have chosen different world names among the models, there is no danger of ambiguity here. In fact, we could have stopped after the extension order of degree 2. Deeper extension orders yield the same diagram.

Note that all these models are finitely branching, and thus lemma 2.31 and lemma 2.32 make sure that the order in the picture above coincides with the inclusion order of the informational contents up to the associated modal depth of these worlds. The last two diagrams are similar to the extension order for arbitrary extension order degree larger than 1. This means that an ordering of worlds according to their modal informational content coincides with the structure of these last two diagrams.

In the following subsection a more compact non-recursive definition is given on the basis of the widely employed notion of *bisimulation*. It redefines this structural description of growth of modal information in a more conventional way. Furthermore it creates a more general point of view, in the sense that also other information orders which we have introduced in section 2.1 can easily be raised up to the level of partial Kripke models.

The price of this redefinition by means of bisimulations is some loss of strength. Nevertheless, the results of the important lemmas 2.31 and 2.32 can be taken along. Only the result of lemma 2.34 is lost.

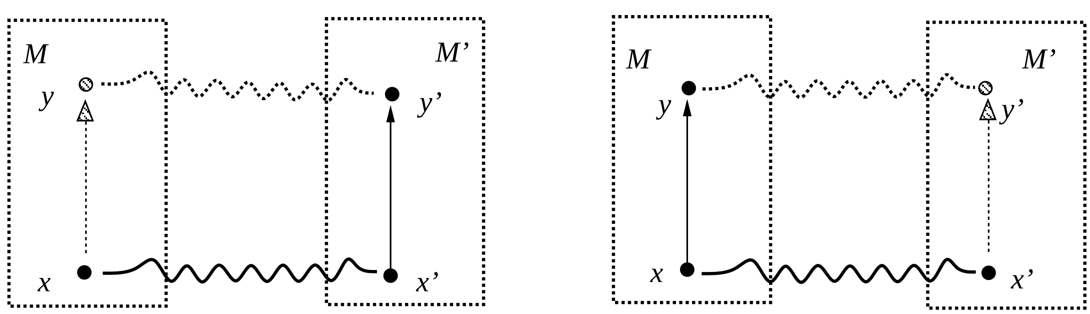
In part II, where bisimulations will reappear, the loss of strength of this reformulation is negligible. It turns out that the order of informational content and the structural bisimulation order coincide in the canonical model of **M**. As canonical modeling will be our most important means for establishing meta-theoretical results, such as completeness and correspondence results, the small difference between the earlier recursive definition 2.29 and the shorter and more workable bisimulation definition of the next subsection will not disturb us. In the sequel of this thesis we will stick to this bisimulation definition.

Bisimulations

Bisimulations are important meta-theoretical concepts in classical modal logic[12]. A bisimulation is a relation which links worlds to other worlds, regardless of their home models, such that the accessibility pattern of linked pairs is preserved. It can be seen as a two-way relational reformulation of the concept of homomorphism in mathematics. A homomorphism, in terms of Kripke models, is a function f from one Kripke model $M = \langle W, R, V\rangle$ to another $M' = \langle W', R', V'\rangle$ such that for every $\langle x, y\rangle \in R$ also $\langle f(x), f(y)\rangle \in R'$. This captures the functional perspective of accessibility structure preservation.

A bisimulation B is not defined as a map from one model to another, but as a relation between models M and M'. It intertwines pairs of homomorphism between M and M', of which one is going from M to M' and the other from M' to M. If $\langle x, y\rangle \in R$ and $B(x, x')$ then $\langle x', y'\rangle \in R'$ for certain y' with $B(y, y')$. Vice versa, if $\langle x', y'\rangle \in R'$ and $B(x, x')$ then there exists y such that $B(y, y')$ and $\langle x, y\rangle \in R$. In short, bisimulations capture the relational view on accessibility preservation in both directions. The following figure presents a schematic display of this situation.

2.36. FIGURE.



The vectors symbolize accessibility links, the zigzag lines denote a bisimulation. Black vectors, zigzag lines and points have a universal conditional meaning. The dashed variants have an existential denotation.

These relational views on structure preservation lead to the following definition

[12] The concept of bisimulation stems from process algebra (see e.g. [Hennessy 1988]). In modal logic we meet the same concept also as zigzag-correspondence [van Benthem 1985] [van Benthem 1991b]. In [van Benthem 1976] this concept already appeared as 'p-relation'.

of bisimulations.

2.37. DEFINITION. Let $M = \langle W, R, V\rangle$ and $M' = \langle W', R', V'\rangle$. A bisimulation between M and M' is a relation $B \subseteq W \times W'$ such that for all $x \in W$ and $y \in W'$

$$B(x,y) \implies \begin{cases} R(x,w) \implies \exists v \in W' : R'(y,v) \ \&\ B(w,v) \\ R'(y,v) \implies \exists w \in W : R(x,w) \ \&\ B(w,v) \end{cases} \quad (1)$$

If $\langle w,v\rangle \in B$ we say that w and v bisimulate by B. If B is not specified, this simply means that there exists a bisimulation through which they bisimulate. This relation is denoted by $w \bowtie_{M,M'} v$. The collection of all bisimulations between M and M' is abbreviated by $\mathfrak{Bis}(M, M')$. If B is a bisimulation between M and M itself, we say that B is a bisimulation on M. $\bowtie_M$ is the relation of bisimulating pairs in one model M.

The definition of bisimulation given above is based purely on frames and therefore we also speak of bisimulations between frames. In classical modal logic bisimulations often refer to a subclass of what we call bisimulations. In these definitions, e.g. [van Benthem 1991b], bisimulating pairs are taken to have identical local valuation as a structural description of worlds with identical modal informational content. As we plan to describe different relations between partial valuations we have chosen a more general position with our frame-based definition 2.37 above. This definition originates from [Stirling 1987].

2.38. OBSERVATION. A shorter reformulation of the requirement (1) in definition 2.37 of bisimulations can be given by the following relational equation:

$$B \circ R' \subseteq R \circ B \quad \text{and} \quad B^{-1} \circ R \subseteq R' \circ B^{-1}.$$

The symbol $\circ$ denotes composition of relations, while the superscript $^{-1}$ refers to the converse relation of its argument.

2.39. OBSERVATION. The following general principles hold for bisimulating pairs of possible worlds.

- $x \bowtie_M x$ for all x in the model M.
- $x \bowtie_{M,M'} y \Rightarrow y \bowtie_{M',M} x$ for all x in M and y in M'.
- $x \bowtie_{M,M'} y \ \&\ y \bowtie_{M',M''} z \Rightarrow x \bowtie_{M,M''} z$ for all x, y and z in M,M' and M''.

The first principle, reflexivity of $\bowtie_M$, is a simple consequence of the fact that the identity relation over the worlds in M is a bisimulation of M. The second symmetry principle holds because converting bisimulation yields a bisimulation in the other direction. The third transitivity principle is valid because the composition of two bisimulations yields a new bisimulation. This means that the relation $\bowtie_M$ is an equivalence relation.

2.40. EXAMPLE. All *dead ends*, that is worlds which do not have any accessible world, bisimulate. If x in $M = \langle W, R, V\rangle$ and y in $M' = \langle W', R', V'\rangle$ are dead ends, then $B = \{\langle x,y\rangle\}$ is a bisimulation. Application of the relational equations in observation 2.38 shows this immediately: $B \circ R' = R \circ B = \emptyset$ and $B^{-1} \circ R = R' \circ B = \emptyset$. Therefore, $x \bowtie_{M,M'} y$. On the other hand, dead ends *only* bisimulate with dead ends.

Bisimulations provide a very elegant machinery for transposing the informational orders on partial valuations in the preceding subsection into the universe of partial Kripke models $\mathfrak{M}$.

2.41. Definition. Let $M = \langle W, R, V\rangle$ and $M' = \langle W', R', V'\rangle$ be a pair of partial Kripke models. A world w in M is said to be extended by w' in M' if there exists $B \in \mathfrak{Bis}_{M,M'}$ such that $B(w, w')$ and

$$B(x,y) \implies V(x) \sqsubseteq V'(y) \quad \text{for all } x \in W, y \in W'.$$

This extension order relation is written as $w \sqsubseteq_{M,M'} w'$.
The coherence relation $\sim$ and the domain-inclusion relation $\sqsubseteq^d$ are transferred to $\mathfrak{M}$ in the same way. To obtain there definitions, substitute these other information orders for $\sqsubseteq$ in the definition above. Their abbreviations are $\sim_{M,M'}$ and $\sqsubseteq^d_{M,M'}$.

2.42. Example. Let us review the examples in figure 2.27 in order to further clarify the bisimulation definition. Consider x_3 in M_4 and y_5 in M_5. These worlds bisimulate through the bisimulation $B = \{\langle x_3, y_5\rangle, \langle y_3, y_5\rangle\}$.

$$\left.\begin{array}{l} B \circ R_5 = \{\langle x_3, y_5\rangle, \langle y_3, y_5\rangle\} \\ R_3 \circ B = \{\langle x_3, y_5\rangle, \langle y_3, y_5\rangle, \langle z_3, y_5\rangle\} \end{array}\right\} \Rightarrow B \circ R_5 \subseteq R_3 \circ B$$

$$\left.\begin{array}{l} B^{-1} \circ R_3 = \{\langle y_5, x_3\rangle, \langle y_5, y_3\rangle\} \\ R_5 \circ B^{-1} = \{\langle x_5, x_3\rangle, \langle x_5, y_3\rangle, \langle y_5, x_3\rangle, \langle y_5, y_3\rangle\} \end{array}\right\} \Rightarrow B^{-1} \circ R_3 \subseteq R_5 \circ B^{-1}$$

Because $V_3(x_3) \sqsubseteq V_5(y_5)$ and $V_3(y_3) \sqsubseteq V_5(y_5)$ we conclude $x_3 \sqsubseteq_{M_3,M_5} y_5$ and also $y_3 \sqsubseteq_{M_3,M_5} y_5$.
Our initial example, the structural explanation of adaptation of all modal information of z_3 by x_5, can be demonstrated by a small extension of the bisimulation above:

$$C = \{\langle z_3, x_5\rangle, \langle x_3, y_5\rangle, \langle y_3, y_5\rangle\}.$$

Notice that $V_3(z_3) \sqsubseteq V_5(x_5)$, $V_3(x_3) \sqsubseteq V_5(y_5)$ and $V_3(y_3) \sqsubseteq V_5(y_5)$. We need to prove additionally that this relation is a bisimulation in $\mathfrak{Bis}_{M_3,M_5}$. The following relational equations show this membership of C.

$C \circ R_5 = W_3 \times \{y_5\} = R_3 \circ C$, and

$C^{-1} \circ R_3 = W_5 \times \{x_3, y_3\} = R_5 \circ C^{-1}$.

This proves $z_3 \sqsubseteq_{M_3,M_5} x_5$.

In order to get more feeling for bisimulations, the reader is advised to try to find bisimulations which prove other extension relations in figure 2.35.

What is left to show is the correctness of the bisimulation definition of the extension order among possible worlds. According to lemma 2.31 we only need to prove that $\sqsubseteq_{M,M'}$ coincides with the extension order of arbitrary degree among possible worlds, which has been presented in definition 2.29.

2.43. THEOREM. Let $M = \langle W, R, V\rangle$ and $M' = \langle W', R', V'\rangle$ be a pair of partial Kripke models and let $w \in W$ and $w' \in W'$.

$$w \sqsubseteq_{M,M'} w' \implies \forall n \in I\!N : w \sqsubseteq^n_{M,M'} w'.$$

Proof. By induction on n. Suppose $w \sqsubseteq_{M,M'} w'$. This means that there exists a bisimulation $B \in \mathfrak{Bis}_{M,M'}$ such that $B(w, w')$ and for all $x \in W$ and $x' \in W'$ if $B(x, y)$ then $V(x) \sqsubseteq V'(x')$. This means that at least $V(w) \sqsubseteq V'(w')$ and therefore $w \sqsubseteq^0_{M,M'} w'$.
Let $n > 0$, and $R(w, v)$ for certain $v \in W$. According to the bisimulation definition, there exists $v' \in W'$ such that $B(v, v')$ and $R'(w', v')$. This also means that $v \sqsubseteq_{M,M'} v'$ by means of the bisimulation B. The induction hypothesis yields $v \sqsubseteq^{n-1}_{M,M'} v'$. Analogously for all $u' \in W'$ with $R'(w', u')$ there exists $u \in W$ such that $u \sqsubseteq^{n-1}_{M,M'} u'$ and $R(w, u)$. This bisimulation interplay of accessibilities of w and w' establishes $w \sqsubseteq^n_{M,M'} w'$. ■

The converse of this theorem does not hold in general. The following example illustrates this failure.

2.44. EXAMPLE. Consider the following two models $M = \langle W, R, V\rangle$, $M' = \langle W', R', V'\rangle \in \mathfrak{M}$ with

$W = \{w^i_j \mid i, j \in I\!N, j \leq i\} \cup \{0\}$

$W' = W \cup \{v_i\}_{i \in I\!N}$

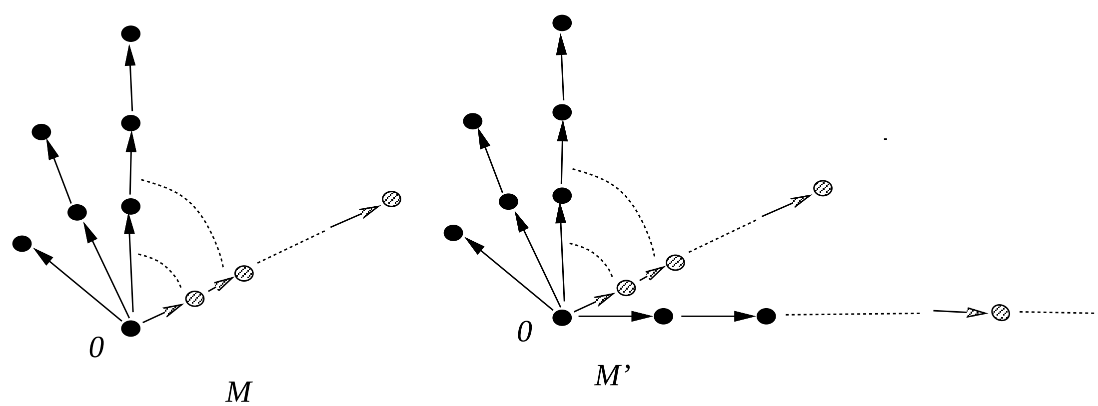
$R = \{\langle w^i_j, w^i_{j+1}\rangle \mid j < i\} \cup \{\langle 0, w^i_0\rangle \mid i \in I\!N\}$

$R' = R \cup \{\langle v_i, v_{i+1}\rangle \mid i \in I\!N\} \cup \{\langle 0, v_0\rangle\}$

$\mathfrak{Dom}(V(w)) = \emptyset$ for all $w \in W$

$\mathfrak{Dom}(V'(w')) = \emptyset$ for all $w' \in W'$

The following picture presents the structures of these two models.



It is not hard to prove that $0 \sqsubseteq^n_{M,M'} 0$ and $0 \sqsubseteq^n_{M',M} 0$ for all $n \in I\!N$. Nevertheless, we can show that $0 \not\sqsubseteq_{M,M'} 0$, and also $0 \not\sqsubseteq_{M',M} 0$. We can even prove that 0 in M and 0 in M' do not bisimulate.

Suppose that there is a $B \in \mathfrak{Bis}_{M,M'}$ such that $B(0,0)$. This means that $B \circ R' \subseteq R \circ B$ (1), and because $R'(0,v_0)$ we may infer that $R \circ B(0,v_0)$. In other words, there exists $i \in I\!N$ such that $B(w_0^i, v_0)$. By an iterative application of (1), we also find $B(w_i^i, v_i)$. This gives us the contradiction, for w_i^i is a dead world, while v_i is not.

In order to transfer the result of lemma 2.32 we only need the converse result for finitely branching partial Kripke models.

2.45. THEOREM. Let $M = \langle W, R, V\rangle$ and $M' = \langle W', R', V'\rangle$ be a pair of finitely branching partial Kripke models and $w \in W$ and $w' \in W'$. If $w \sqsubseteq^n_{M,M'} w'$ for all $n \in I\!N$, then also $w \sqsubseteq_{M,M'} w'$.

Proof. Let $w \sqsubseteq^n_{M,M'} w'$ for all $n \in I\!N$, and let $B = \{\langle x, x'\rangle \in W \times W' \mid x \sqsubseteq^n_{M,M'} x'$ for all $n \in I\!N\}$. Suppose that $B \notin \mathfrak{Bis}_{M,M'}$ (2). This means

$$\exists x \in W\ \exists x', y' \in W' : B(x,x')\ \&\ R'(x',y')$$
$$\&\ \forall y \in W : R(x,y) \Rightarrow \text{ not } B(y,y') \quad (3), \text{ or}$$
$$\exists x, y \in W\ \exists x' : B(x,x')\ \&\ R(x,y)$$
$$\&\ \forall y' \in W' : R(x',y') \Rightarrow \text{ not } B(y,y') \quad (4).$$

If (3) holds then for all y with $R(x,y)$ there exists $n \in I\!N$ such that $y \not\sqsubseteq^n_{M,M'} y'$ (5). Let k_y be the minimal natural number for which (5) holds for all such y with $R(x,y)$, and let k be the maximal natural number of the $k'_y s$. The finiteness of the set of accessible worlds from x makes sure that k is well-defined. Observation 2.30 ensures $y \not\sqsubseteq^k_{M,M'} y'$ for all y such that $R(x,y)$. This means $x \not\sqsubseteq^{k+1}_{M,M'} x'$, which contradicts $B(x,x')$. By an analogous argument, we can show that (4) also leads to a contradiction, and therefore (2) cannot be true, i.e. $B \in \mathfrak{Bis}_{M,M'}$.

Furthermore, note that $B(x,x')$ implies $V(x) \sqsubseteq V'(x')$ (take $n = 0$ in the definition of B). This shows $w \sqsubseteq_{M,M'} w'$, because $B(w,w')$. ■

2.46. COROLLARY. Lemma 2.31 and theorem 2.43 yield a persistence result for the extension order between possible worlds. If $M, M' \in \mathfrak{M}$ and w and w' are two worlds in M and M' respectively, then

$$w \sqsubseteq_{M,M'} w'\ \&\ M,w \models \varphi \ \Rightarrow\ M',w' \models \varphi \text{ for all } \varphi \in \mathcal{L}^{\Box}.$$

2.47. COROLLARY. Lemma 2.32 and theorem 2.43 yield the converse of this result for finitely branching partial Kripke models. If M and M' are two finitely branching partial Kripke models and w and w' are two worlds in M and M' respectively, then

$$(\forall \varphi \in \mathcal{L}^{\Box} : M,w \models \varphi \Rightarrow M',w' \models \varphi) \Longrightarrow w \sqsubseteq_{M,M'} w'.$$

In chapter 5 we will extend this latter corollary to the so-called canonical models. The result, lemma 5.26 on page 157, justifies the bisimulation definition of the extension order over partial possible worlds more evidently than the result above. For the sake of gradual presentation, we will come back to this issue in chapter 5 after the definition of these canonical models.

Appendix

2.4 Other information orders on worlds

By means of the definition of bisimulation we can also transfer the coherence relation $\sim$ and the domain-inclusion relation $\sqsubseteq_d$ to possible worlds in partial Kripke models. The question arises whether characterization results as in theorem 2.6 are preserved. As usually, the success of this transfer of information orders turns out to be only partly satisfactory.

The coherence relation

The success of the bisimulation transfer of the coherence relation is similar to the results which have been shown in the previous section for the extension order. Purely analogous to the procedure of proving preservation of $\mathcal{L}^{\Box}$-information over this bisimulation extension order, we can prove that two bisimulation coherent worlds do not contain mutually conflicting $\mathcal{L}^{\Box}$-information.

2.48. THEOREM. Let $M, M' \in \mathfrak{M}$ and let w and w' be two worlds in the models M and M' respectively. For all $\varphi \in \mathcal{L}^{\Box}$:

$w \sim_{M,M'} w' \;\&\; M, w \models \varphi \implies M', w' \not\dashv \varphi$[13].

Proof. By induction on the construction of formulae. Once again, only the $\Box$-step deserves some clarification.

Let $w \sim_{M,M'} w'$, by means of a bisimulation B between M and M', and $M, w \models \Box\varphi$. We need to prove that $M', w' \not\dashv \Box\varphi$.

Let v' be a world in M' such that $R'(w', v')$. This means $B \circ R'(w, v')$, because $B(w, w')$. Since $B \in \mathfrak{Bis}_{M,M'}$, we also have $R \circ B(w, v')$, or there exists $v \in M$ such that $R(w, v)$ and $B(v, v')$. The latter conclusion yields $v \sim_{M,M'} v'$. Evidently, $M, v \models \varphi$, which establishes $M', v' \not\dashv \varphi$ by the induction hypothesis. In other words, $R'(w', v') \Rightarrow M', v' \not\dashv \varphi$. This means $M', w' \not\dashv \Box\varphi$.

[13] By contra-position and the symmetry of the relation $\sim_{M,M'}$ we also have $w \sim_{M,M'} w' \;\&\; M, w \dashv \varphi \implies M', w' \not\models \varphi$. In fact, this is an equivalent reformulation.

■

Again, the converse of this theorem does not hold in general, but succeeds for finitely branching models.

2.49. THEOREM. Let $M, M' \in \mathfrak{M}$ both finitely branching and w and w' two worlds in M and M', respectively. If $M, w \models \varphi \Rightarrow M', w' \not\Dashv \varphi$ for all $\varphi \in \mathcal{L}^{\Box}$, then $w \sim_{M,M''} w'$.

Proof. Let $M, w \models \varphi \Rightarrow M', w' \not\Dashv \varphi$ for all $\varphi \in \mathcal{L}^{\Box}$ and let

$$B = \{\langle v, v' \rangle \in W \times W' \mid M, v \models \varphi \Rightarrow M', v' \not\Dashv \varphi\}.$$

Clearly $B(w, w')$ and $B(x, x') \Rightarrow V(x) \sim V'(x')$ (by theorem 2.6) for all x in M and $x' \in M'$. What we need to show is that this B is a bisimulation between M and M'. This ensures $w \sim_{M,M'} w'$.

Suppose that $B \notin \mathfrak{Bis}_{M,M'}$: $B \circ R' \not\subseteq R \circ B$ (1) or $B^{-1} \circ R \not\subseteq R' \circ B^{-1}$ (2).

If (1) were the case, then there exist u', v' in M' and $u \in M$ such that $B(u, u')$, $R'_{w'}(u', v')$ and for all $v \in W$ with $R(u, v)$ it does not hold that $B(v, v')$. By definition of B, this means that there exists a formula $\varphi_v \in \mathcal{L}^{\Box}$ for all $v \in W$ with $R(w, v)$ such that $M, v \models \varphi_v$ and $M', v' \Dashv \varphi_v$ for all such v. We define

$$\varphi = \bigvee_{R(u,v)} \varphi_v.$$

This formula is well-defined, because M is finitely branching. Clearly, $M, u \models \Box\varphi$ and $M', v' \Dashv \varphi$, and therefore also $M', u' \Dashv \Box\varphi$. This contradicts $B(u, u')$. In other words, (1) cannot be the case.

In order to prove that (2) also leads to a contradiction, we can use a similar argument. We leave this to the reader (see theorem 2.32 for analogy: use the finite branching of M', a big conjunction $\bigwedge$ and a $\Diamond$). ■

In the same way as for the extension order, we could also use a recursive definition of coherence as in definition 2.29. We only need to replace all occurrences of $\sqsubseteq$ and $\sqsubseteq^n_{M,M'}$, and we will end up with a suitable definition of $\sim^n_{M,M'}$. In an identical way we can prove that this definition strengthens the bisimulation definition.

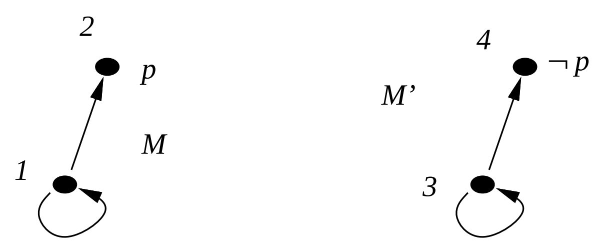
2.50. OBSERVATION. Let $M, M' \in \mathfrak{M}, \mathfrak{M}'$ and w and w' two worlds in M and M' respectively. If $w \sim_{M,M'} w'$ then $w \sim^n_{M,M'} w'$ for all $n \in \mathbb{N}$.

Again, the converse of this result does not hold in general. It holds for the class of finitely branching Kripke models. This means that the two theorems 2.48 and 2.49 can be repeated for this recursive definition of the coherence relation. It is also the case that if $\mathcal{L}^{\Box}$ only contains a finite set of propositional atoms, the recursive definition of the coherence relation precisely matches the informational compatibility of worlds. This rephrases lemma 2.34 for the coherence relation. We will not give technical details, as they can easily be collected from the analogous results for the extension order of the previous section.

The size of possible worlds

The situation for the transferred domain-inclusion relation, $\sqsubseteq^d_{M,M'}$, is more troublesome. For the relation $\sqsubseteq^d$ over $\mathfrak{P}$ we have found that verification of a formula by a partial valuation guarantees verification or falsification of that formula by all larger partial valuations. This does certainly not hold for the bisimulation version over worlds in $\mathfrak{M}$. The following picture presents a very simple counterexample.

2.51. FIGURE.
Consider the following partial Kripke models consisting of two worlds.



The relation $B = \{\langle 1,3\rangle, \langle 2,4\rangle\}$ is a bisimulation. Furthermore, $B(x,x') \Rightarrow V(x) \sqsubseteq^d_{M,M'} V'(x')$. Altogether, this means $1 \sqsubseteq^d_{M,M'} 3$. Nevertheless, $M,1 \models \Diamond p$ while $M',3 \not\models \Diamond p$ and $M',3 \not\dashv \Diamond p$.

An interesting question which remains to be answered, is whether an appropriate transfer definition of $\sqsubseteq^d$, which preserves the above-mentioned property, can be given. This is an issue which we leave for future research.

A result which still holds for the relation $\sqsubseteq^d_{M,M'}$ is preservation of tautological information.

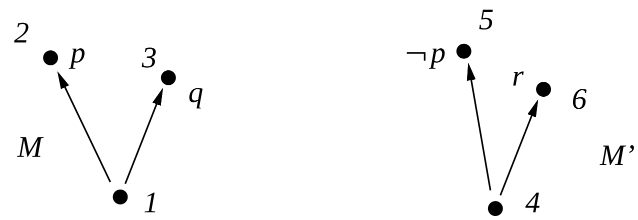
2.52. OBSERVATION. For all $\mathfrak{K}$-tautological $\varphi \in \mathcal{L}^\Box$, i.e. $\mathfrak{K} \subseteq [\![\varphi]\!]_\mathfrak{M}$, and for all $M, M' \in \mathfrak{M}$

$$w \sqsubseteq^d_{M,M'} w' \ \&\ M,w \models \varphi \implies M',w' \models \varphi.$$

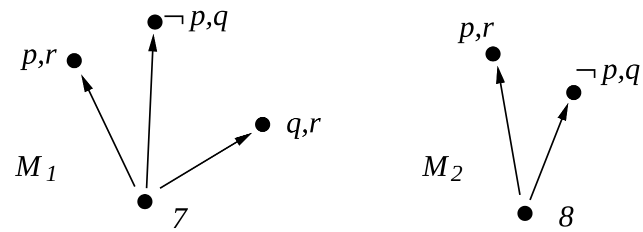
Joining possible worlds

Another structural loss when going from partial valuations, $\mathfrak{P}$, to partial Kripke models, $\mathfrak{M}$, is a proper definition of the join of possible worlds. In fact, a good technical definition cannot be given. It is not the case that pairs of coherent partial possible worlds always have a smallest common extension. A good illustration can be given by the following simple models.

2.53. FIGURE.
Suppose $I\!P = \{p,q,r\}$.



The worlds 1 in M and 4 in M' do not have a common smallest extension. This can be shown by the following extensions.



Both 7 in M_1 and 8 in M_2 are common extensions of the worlds 1 and 4 in the models in the first picture. Because all these models are finitely branching, this also means that the information of 1 and 4 is contained in 7 and 8. Furthermore, it can be seen immediately that $7 \not\sqsubseteq_{M_1,M_2} 8$ and $8 \not\sqsubseteq_{M_2,M_1} 7$. Furthermore, it can be shown that every common extension of the worlds 1 and 4 is an extension of at least one of 7 and 8. In this sense, 7 and 8 are minimal common extensions of 1 and 4. In other words, there exists no unique minimal common extension of 1 and 4.

Technically speaking, the problem of joining possible worlds boils down to the plurality of accessibilities [14]. The accessibility relations in Kripke models cause two worlds to be possibly structurally coherent in different ways, that is there might be different coherent bisimulations between two worlds. For example, in figure 2.53 we have two coherent bisimulations between 1 and 4: $B_1 = \{\langle 1,4\rangle, \langle 2,5\rangle, \langle 3,6\rangle\}$ and $B_2 = B_1 \cup \{\langle 3,5\rangle\}$. This leads to different ways of joining coherent pairs. Of course, we could fix such a coherent bisimulation B, and then speak of a B-join[15]. In figure 2.53, this leads to 7 as a B_2-join of 1 and 4, while 8 is the B_1-join of this pair.

2.54. DEFINITION. Let $M, M' \in \mathfrak{M}$ and let w and w' be two worlds in M and M' respectively, such that $w \sim_{M,M'} w'$ by means of a bisimulation B. The B-join is a world $w \sqcup_B w'$ in a model $M = \langle W \sqcup_B W', R \sqcup_B R', V \sqcup_B V'\rangle \in \mathfrak{M}$ such that

$$W = \{v \sqcup_B v' \mid B(v,v')\},$$

$$R \sqcup_B R'(v \sqcup_B v', u \sqcup_B u') \iff R(v,u) \text{ or } R'(v',u'), \text{ and}$$

[14] Of course, the problem disappears if accessibilities were taken to be partial functions: $\forall x,y,z : R(x,y) \;\&\; R(x,z) \;\Rightarrow\; y = z$.

[15] Or B-product, which is closer to the terminology of standard predicate logic.

$$V \sqcup_B V'(v \sqcup_B v') = V(v) \sqcup V'(v').$$

2.55. PROPOSITION. For every pair of worlds, which is coherent by means of a bisimulation B, its B-join is an extension.

Proof. Let $M, M' \in \mathfrak{M}$ and let w in M and w' in M' such that $w \sim_{M,M'} w'$ and let $B \in \mathfrak{Bis}_{M,M'}$. Define:

$$B_1(x, x \sqcup_B y) \text{ and } B_2(y, x \sqcup_B y) \text{ for all } x \text{ in } M \text{ and } y \text{ in } M'.$$

The two relations are bisimulation, because B is a bisimulation. Furthermore $V(x) \sqsubseteq V \sqcup_B V'(x \sqcup_B y)$ and $V'(y) \sqsubseteq V \sqcup_B V'(x \sqcup_B y)$ for all x in M and y in M'. This entails $w \sqsubseteq_{M,M'} w \sqcup_B w'$ and $w' \sqsubseteq_{M,M'} w \sqcup_B w'$. ■

2.56. COROLLARY. For every coherent bisimulation between two possible worlds w in $M \in \mathfrak{M}$ and w' in $M \in \mathfrak{M}$

$$M, w \models \varphi \text{ or } M', w' \models \varphi \implies M \sqcup_B M', w \sqcup_B w' \models \varphi.$$

From a conceptual point of view, this B-join is still not completely satisfactory. These B-joins do not have to be minimal extensions. In figure 2.53 we have shown that the possible bisimulation joins were both minimal, in the sense that for both these joins no smaller extension of 1 and 4 could be found. This minimality does not always hold. For example, take the models M and M' of figure 2.53 again and remove r from the world 6. We still have the same two coherent bisimulations, and the two products are the same as in figure 2.53 with r removed from all possible worlds. In this case, the three-world model is an extension of the four-world model. In other words, the four-world model is not minimal.

The way to establish this technically is to define *minimal coherence bisimulations*. Such bisimulations only link a world to a coherent partner if a smaller coherent alternative cannot be found.

2.57. DEFINITION. A *minimal bisimulation* B is a bisimulation for a pair of models $M, M' \in \mathfrak{M}$ such that

$$\forall v \text{ in } M\ \forall v', u' \text{ in } M' : B(v, v'), B(v, u') \ \&\ u' \sqsubseteq_{M'} v' \implies v' \sqsubseteq_{M'} u', \text{ and}$$

$$\forall v, u \text{ in } M\ \forall v' \text{ in } M' : B(v, v'), B(u, v) \ \&\ u \sqsubseteq_M v \implies v \sqsubseteq_M u.$$

It can be proved that such a bisimulation can be found for all coherent pairs of worlds. Furthermore, it can be shown that every coherence bisimulation B for a pair of worlds can be reduced to a minimal coherence bisimulation for this pair: $B' \subseteq B$, and $w \sqcup_{B'} w' \sqsubseteq_{M,M'} w \sqcup_B w'$. This entails that every common extension of a pair of worlds is an extension of some bisimulation join.

2.5 Fused partial modal logic

In [Jaspars 1991c] a so-called *fused* modal logic is proposed for the representation of inconsistent beliefs, based on an interpretation, which has been proposed by Rescher and Brandom in [Rescher & Brandom 1980], of inconsistent information by means of collections of 'fused' sets of truth assignments. In this fused modal logic, accessibility is taken to be a relation between worlds and non-empty sets of worlds. The underlying idea is that an agent confuses possible worlds.

The proposition $\Box\varphi$ is then said to be true if φ holds with respect to at least one of the worlds in each accessible 'confused' set. The effect of this modeling is a weakening of the modal strength of the logic, which disconnects the propositions $\Box\varphi \wedge \Box\psi$ and $\Box(\varphi \wedge \psi)$. The latter still implies the former, but not the other way around.

A partial version of this fused modal semantics is given by the following definition.

2.58. DEFINITION. A fused partial Kripke model is a triple $M = \langle W, R, V\rangle$ such that $W \neq \emptyset$, $R \subseteq W \times (\wp W \setminus \{\emptyset\})$ and $V : W \longrightarrow \mathfrak{P}$. The collection of fused partial Kripke models is denoted by $\mathfrak{FM}$.

The truth-value assignment of the propositional connective is defined as in the case of the partial modal logic in section 2.1. The modal operator is interpreted according to the following clauses.

2.59. TABLE.

$$M, w \models \Box\varphi \iff \forall W' \subseteq W : R(w, W') \Rightarrow (\exists w' \in W' : M, w' \models \varphi)$$

$$M, w \;=\!\!\mid \Box\varphi \iff \exists W' \subseteq W : R(w, W') \;\&\; (\forall w' \in W' : M, w' \;=\!\!\mid \varphi)$$

The system **FM** consists of the system **P** with additional restricted versions of the modal rules of **M**: R-TRUE $\Box$ and L-FALSE $\Box$. The sets Γ and Δ, which have been used as 'surroundings' for these rules in table 2.15 and table 2.16, are taken to be particularly small: one of them should be empty, while the other contains maximally one element. Formally speaking, $\#(\Gamma \cup \Delta) \leq 1$. This yields six instantiations of these rules.

2.60. TABLE.

R-TRUE $\Box$	L-FALSE $\Box$
$\vdash \varphi \Rightarrow \vdash \Box\varphi$	$\varphi \vdash \emptyset \Rightarrow \Diamond\varphi \vdash \emptyset$
$\varphi \vdash \psi \Rightarrow \Box\varphi \vdash \Box\psi$	$\varphi \vdash \psi \Rightarrow \Diamond\varphi \vdash \Diamond\psi$
$\vdash \varphi, \psi \Rightarrow \vdash \Box\varphi, \Diamond\psi$	$\varphi, \psi \vdash \emptyset \Rightarrow \Box\varphi, \Diamond\psi \vdash \emptyset$

Intermediate systems between **M** and **FM** can be given by changing the truth clause or the falsity clause for $\Box\varphi$ in table 2.59. If we take $\Box\varphi$ to be true iff φ holds in all worlds in all accessible sets, then the underlying calculus consists of less restricted versions of R-TRUE $\Box$ and L-FALSE $\Box$. Only the Δ in these rules in the tables 2.15 and 2.16 should be a singleton. This restriction to the modal rules of **M** are then the following:

$$\frac{\Gamma, \neg\varphi \vdash \psi}{\Box\Gamma, \neg\Box\varphi \vdash \Diamond\psi} \qquad \frac{\Gamma \vdash \varphi, \psi}{\Box\Gamma \vdash \Box\varphi, \Diamond\psi}$$

$$\frac{\Gamma, \neg\varphi \vdash \emptyset}{\Box\Gamma, \neg\Box\varphi \vdash \emptyset} \qquad \frac{\Gamma \vdash \varphi}{\Box\Gamma \vdash \Box\varphi}$$

If we change the falsification of $\Box\varphi$ into falsification of φ with respect to a world in at least one of the accessible sets, we end up with a system where the set Γ in the original formulation of the introduction rules for $\Box$ may only be a singleton.

$$\frac{\psi, \neg\varphi \vdash \Delta}{\Box\psi, \neg\Box\varphi \vdash \Diamond\Delta} \qquad \frac{\psi \vdash \varphi, \Delta}{\Box\psi \vdash \Box\varphi, \Diamond\Delta}$$

$$\frac{\neg\varphi \vdash \Delta}{\neg\Box\varphi \vdash \Diamond\Delta} \qquad \frac{\emptyset \vdash \varphi, \Delta}{\emptyset \vdash \Box\varphi, \Diamond\Delta}$$

Chapter 3

Constructive Modal Logic

In sections 2.1 and 2.3 we have demonstrated how partial worlds can be ordered on a structural basis. In this chapter we will present and investigate logics which use this type of information structures to reason explicitly about growth and loss of extensional and intensional information. The general aim of this chapter is to give a theoretical classification of the underlying formalisms which capture this dynamic reasoning. Because constructivity is the most characteristic dimension of the dynamics of these systems, we call them *constructive modal logics.* Such logics can be used, as we will see in chapter 4, for stipulating dynamic epistemic interpretation of communicative actions.

We start with a presentation of Nelson's logic of constructible falsity as a propositional basis of the constructive modal logics later on. This logic can be seen as a straightforward constructive extension of partial propositional logic. Its informational infrastructure is the simple extension order over partial valuations (see definition 2.2). Additional expressivity in the logic in order to capture this dynamics is obtained by a supplementary constructive implication.

The dynamics of Nelson's logic is only progressive, that is once information is obtained, it persists. This is not very surprising as Nelson's logic deals with the rigorous dynamics of present mathematical information. Such information persists. In the second section of this chapter we discuss simple extensions of Nelson's logic which contain 'non-persistent' pollution. The most obvious non-persistent extension is a system with an additional weak negation. It accommodates also reasoning about absent information. As a non-technical intermezzo, we will briefly describe different applications of such non-persistent extensions.

In the third section we discuss a logic which extends the freedom of information flow: it presents a Nelson-like system in which we also may move downwards. This means that information may also be retracted. These logics are meant to interpret the dynamics of information which is less rigorously anchored than proofs and refutations. As we deal with belief rather than mathematical knowledge, we will mainly focus on information which is only entailed by means of evidence and counter-evidence.

The last section of this chapter presents modal extensions of Nelson's logic which are based on the extension orders over partial possible worlds, which have been introduced in section 2.3. The most elegant modal extension is the peaceful confluence of the minimal partial modal logic **M** and Nelson's logic. Its semantics imitates exactly the bisimulation implementation in definition 2.41 on page 68 in $\mathfrak{M}$ of the extension order $\sqsubseteq$ over $\mathfrak{P}$.

From the epistemic point of view, this logic is not very interesting, as we have pointed out earlier in the first chapter, because all information is taken to be persistent. This means that information flows only along the constructive dimension. As propagated deliberately in chapter 1, we wish to add an eliminative dimension as well. From the perspective of epistemic dynamics, we capture deletion of epistemic alternatives as informational enrichment. Getting rid of uncertainties is a way of gaining information, and we will demonstrate how to combine this 'destructive' progress with the constructive locomotion of dynamic extensions of partial logic such as Nelson's logic. The proper way to capture this two-dimensional dynamics is to mitigate the structural constraints of growth of information. Technically, this wider interpretation of cognitive progress boils down to a reduction of the bisimulation requirements which we have met in definition 2.37 (page 67). We simply drop one of the two structural constraints in this definition and retain the characterization of the dynamics of construction and elimination as has been explained on page 34 in chapter 1. As we will see in this chapter and in chapter 5 in part II, this characteristic constraint enforces persistence preservation of the truth of the modal operator $\Box$ and not its falsity. In the epistemic terminology of chapter 1, belief about persistent information is persistent itself, while active disbelief does not have this property.

In the technical survey of this last section we present a modal extension of the 'up-and-down' generalization of the constructive semantics of Nelson's logic. This logic combines the above-mentioned non-persistent modal extension of Nelson's logic, with additional 'downdate' operators. This logic presents the basic modal logical equipment of the communication logics which are to be presented in the next chapter.

The constructive modal logics which we will meet below are relatively unknown. Modal extensions of Heyting's intuitionistic logic have been studied elaborately. Much of the techniques which have been used in the development of these latter logics, will be employed in this chapter for the presentation of modal extensions of Nelson's logic below (especially [[Božić & Došen 1984]). In the last subsection of the last section we give a brief outline of these intuitionistic modal logics.

3.1 Constructive logic

As may have become clear from the introductory chapter of this thesis, constructive logic is not a specific logic, but it rather refers to a class of logics. The underlying idea of constructivity refers to an *epistemological* analysis of truth, which highly contrasts with the *ontological* notion of truth in classical logic. A proposition, according to constructivists, is true whenever a construction is *present* which demonstrates it. Constructive truth is therefore related to our (human) capacities, which explains the underlying subjectivism of the constructivistic position. In classical logic truth is not related to subjects, but to a total reality, in which all propositions are either true or false. The crucial difference is that truth of a proposition has to be *demonstrated* according to the constructivists, while the standpoint of classical logic advocates that truth only needs to be *detected.*

This essential difference on the understanding of truth explains the validity of the principle of 'reductio ad absurdum' in classical logic, and its invalidity in constructive logic. According to the classical view, showing that the assumption 'not φ' leads to a contradiction, counts as a sound method to derive that φ must hold. The difference of constructive formalisms with respect to classical logic is the absence of this principle. Showing that every hypothetical construction which demonstrates 'not φ' leads to a contradiction, does not entail automatically a construction which demonstrates φ. In most axiomatic systems for constructive logics, this absence is most clear from the omission of the 'principle of the excluded third' (see [Troelstra & van Dalen 1990]): $\varphi \vee \neg\varphi$.

The divergence of different constructivistic philosophies can be understood as the dispute on the admissability of different constructions. The most well-known constructive logic adopted in mathematics is Heyting's formalization of Brouwer's intuitionism [Heyting 1956]. As a foundation of mathematics, the only construction that is essential for determining truth is *proof.* A proposition is true if and only if a proof is currently present.

Nelson's logic of constructible falsity

Another constructive logic, is Nelson's *logic of constructible falsity* [Nelson 1949] which should not be seen as a rival of Heyting's intuitionistic logic, but rather as an extension. Apart from the concept of proof, there exists a second construction in this formalism: *refutation*, which is introduced to account for extensional negative information. In other words, refutation is an independent mathematical construction to demonstrate the falsity of a proposition. This is an idea which can be traced back to [Kleene 1945].

In intuitionistic logic falsity does not have a semantic status. The truth of the negation of a proposition is explained as the presence of a method which shows that any proof of this proposition leads to a contradiction. This *intensional* explanation of negative information seems to be too limited to account for the only possible constructive denial of a proposition in mathematics. Many constructive falsifications in mathematical practice seem to be stronger than such intensional argumentation, since they have a much more direct extensional capacity. Many illustrations of such extensionally falsifying arguments in everyday mathematics can be found in the proof-refutation dialogues on geometry of [Lakatos 1976].

In Nelson's formulation of constructible falsity, refutation is indeed taken to be stronger than proving the absurdity of a proposition. For this reason, Nelson's logic has also been called *intuitionistic logic with strong negation* [Gurevich 1977]. It will be shown that this dominance of Nelson's negation over the intuitionistic negation is a simple side effect of excluding falsity and truth of one proposition[1].

Of course, we are not concerned with the foundations of mathematical reasoning, but we merely propagate Nelson's logic for technical reasons, as it is a natural simple constructive extension of partial logic. Much of the semantic techniques of Nelson's logic (e.g. [Thomason 1968] [Gurevich 1977] [Akama 1988]) can be used to implement dynamic extensions of partial logic, and also of partial modal logics in section 3.4.

The relatively easy model-theory of Nelson's logic incorporates truth, falsity and undefinedness of propositions. From the perspective of Nelson's constructivistic philosophy, a partial state represents a snap-shot of a mathematical reasoning agent on the way (see figure 1.4 on page 25). It registers an instantaneous set of proofs and refutations of a certain agent. The propositions which are assigned true and false represent the personal mathematical knowledge, that is his proofs and refutations, respectively. The undetermined part represents the agent's current ignorance.

The propositional language of Nelson's logic is the language $\mathcal{L}^{\rightarrow}$. By means of the implication $\rightarrow$ the agent reasons about his future. As in intuitionistic logic, a proof of a proposition of the form $\varphi \rightarrow \psi$ is considered to be a *method* to transform any hypothetical proof of φ into a proof of ψ. Such method can be thought of as a function which can be applied to every later hypothetical proof of φ and which has as outcome a proof of the conclusion ψ. In terms of possible worlds, $\varphi \rightarrow \psi$ is known by the agent, if all extensions of the current information state which contain φ, also contain ψ. Refutation of an implication is interpreted extensionally. It just means that the agent has a proof of the antecedent and a refutation of the consequent.

Nelson models

The conceptual semantics of Nelson's logic which has been illustrated above can be formalized by a certain class of partial Kripke models, which we will call Nelson models[2].

3.1. DEFINITION. A *Nelson* model is a triple $M = \langle W, \leq, V \rangle$, such that W is a non-empty set of worlds, $\leq$ is a pre-order over W, and V is a *monotonic* global

[1] Some weaker variants of Nelson's logic omit this dominance of the extensional strong negation, e.g. [López-Escobar 1972] [Pearce & Wagner 1990] and [Wansing 1992a].

[2] We avoid the longer name Kripke models for Nelson's logic. The use of Kripke's possible world semantics for Nelson's logic was introduced in [Thomason 1969], after Kripke's possible worlds analysis of intuitionistic logic (see [Fitting 1969]). Thomasson gave a completeness proof of a slightly different version of Nelson's predicate logical formulation of the logic of constructible falsity (the system **S** in [Nelson 1959]). In [Akama 1988] the reader finds a completeness proof of **S**. Akama used a *monotonic differentiation* of local domains of worlds, which means that if $x \leq y$ then the domain of x is contained in the domain of y. Thomasson chose a fixed domain of individuals for the total universe of worlds.

valuation function, i.e. $V : W \longrightarrow \mathfrak{P}$ such that for all $w, v \in W$ if $w \leq v$ then also $V(w) \sqsubseteq V(v)$. The class of all Nelson models is denoted by $\mathfrak{N}$.

It is immediately clear that $\mathfrak{N}$ is a proper subclass of $\mathfrak{M}$. From the point of view of this thesis, the relational pattern $\leq$ should nevertheless be separated sharply from the accessibility structure in ordinary partial Kripke models. The information structure $\leq$ is meant to describe the way information grows, and it is used to model the dynamic aspect of reasoning. The accessibility pattern in partial Kripke models defined in the previous chapter captures the set of uncertainties of an agent. It represents a static description of the belief of an agent. The combination of these two informational patterns has led to the constructive modal logics of section 3.4.

The syntactic means of Nelson's logic consist of the language $\mathcal{L}^{\rightarrow}$. The static connectives $\bot, \wedge$ and $\neg$ are interpreted in the same way as they have been interpreted in the previous chapter (see table 2.10 on page 51). Following Nelson's conceptual analysis above, the corresponding formal interpretation of the implication boils down to the following clauses:

3.2. TABLE.

$$M, w \models \varphi \rightarrow \psi \Leftrightarrow \forall w' \geq w : M, w' \models \varphi \Rightarrow M, w' \models \psi$$

$$M, w \mathrel{=\!\!|} \varphi \rightarrow \psi \Leftrightarrow M, w \models \varphi \ \& \ M, w \mathrel{=\!\!|} \psi$$

Note that only verification of the implication has a dynamic intensional reading.

Persistence of information

An important property of this interpretation of $\mathcal{L}^{\rightarrow}$ is the persistence of the full language with respect to the structural extension relation in Nelson models. This corresponds to the underlying philosophy of the structure of Nelson models. The persistence of information over $\leq$ should be seen as a technical guarantee of the infallibility of proofs and refutations. For model-theory of constructive mathematical reasoning this persistence is of course satisfactory, as the underlying constructions entail only hard information.

Technically speaking, the persistence result is a simple consequence of the persistence of $\mathcal{L}$ with respect to the extension relation over $\mathfrak{P}$ (theorem 2.6, page 46) and the monotonicity of global valuation functions in Nelson models. The preservation of persistence of the additional connective $\rightarrow$ is obvious as well. Falsity of $\varphi \rightarrow \psi$ is a simple extensional proposition, while truth of such an implication is a universal statement over the extensions of the current state. It is therefore preserved by these extensions, due to the transitivity of the information structure of extensions.

3.3. LEMMA. Let $M = \langle W, \leq, V \rangle$ be a Nelson model. For all $\varphi \in \mathcal{L}^{\rightarrow}$ and for all $w, w' \in W$:

$$M, w \models \varphi \ \& \ w \leq w' \implies M, w' \models \varphi.$$

A very important corollary of the persistence property in the observation above is the so-called *deduction property.*

3.4. LEMMA. For all $\Gamma \subseteq \mathcal{L}^{\rightarrow}$ and $\varphi, \psi \in \mathcal{L}^{\rightarrow}$:

$$\Gamma, \varphi \models_{\mathfrak{N}} \psi \iff \Gamma \models_{\mathfrak{N}} \varphi \rightarrow \psi^3.$$

Proof. The right-to-left direction of the equivalence is simply a consequence of the reflexivity of the information order in Nelson models. Let w in $M \in \mathfrak{N}$ be some Γ, φ-world. If also $\Gamma \models_{\mathfrak{N}} \varphi \rightarrow \psi$, we obtain $M, w \models \varphi \rightarrow \psi$. Because $w \leq w$ and $M, w \models \varphi$, we also have $M, w \models \psi$. In short, $\Gamma, \varphi \models_{\mathfrak{N}} \psi$.

The converse direction depends on the persistence of the full language $\mathcal{L}^{\rightarrow}$ over the information order in Nelson models. If $\Gamma, \varphi \models_{\mathfrak{N}} \psi$, and w is an arbitrary world which supports all Γ-formulae in some Nelson model M, and v is some extension of w in M which supports φ, then lemma 3.3 entails that v is also a Γ-world and therefore $M, v \models \psi$. Because v has been chosen as an arbitrary extension of w in M, we conclude $M, w \models \varphi \rightarrow \psi$, and so $\Gamma \models_{\mathfrak{N}} \varphi \rightarrow \psi$. ■

Yet another consequence of the persistence lemma is the so-called disjunction property, a well known phenomenon in constructive logic. It says that if a disjunction is tautological, then at least one of the disjuncts is tautological. This peculiarity contrasts sharply with classical logic. It can be proved easily by the following model-theoretic construction.

3.5. DEFINITION. Let $\mathfrak{F}_I = \{\langle W_i, \leq_i \rangle\}_{i \in I}$ be a collection of Nelson frames (I is some index-set). The *amalgamation frame* of $\mathfrak{F}_I$ is the Nelson frame $F^* = \langle W^*, \leq^* \rangle$ which consists of disjoint copies of the frames $\langle W_i, \leq_i \rangle$ and one additional world which is extended by every world in the family $\mathfrak{F}_I$. Technically,

$$W^* = \{w_i \mid w \in W_i\} \cup \{w^*\},$$

$$w_i \leq^* v_j \;\Leftrightarrow\; i = j \text{ and } w \leq_i v, \text{ and}$$

$$w^* \leq^* v \text{ for all } v \in W^*.$$

The new world w^* is called the *root* of the amalgamation frame.

The *amalgamation* of a collection of Nelson models $\mathfrak{M}_I = \{\langle W_i, \leq_i, V_i \rangle\}_{i \in I}$ is a Nelson model $M^* = \langle W^*, \leq^*, V^* \rangle$ such that $\langle W^*, \leq^* \rangle$ is the amalgamation frame of $\{\langle W_i, \leq_i \rangle\}_{i \in I}$ and for all $i \in I$ and $w \in W_i$ the valuation function V^* is identical to V_i and V^* assigns an empty valuation to the root. Formally,

$$V^*(w_i)(p) = V_i(w)(p) \text{ for all } p \in \mathbb{P}, \text{ and}$$

$$\mathfrak{Dom}(V^*(w^*)) = \emptyset.$$

3.6. OBSERVATION. Note that M^* is a Nelson model. The new information structure $\leq^*$ is a pre-order and V^* is clearly monotonic. The definition above has been taken from [Hughes & Cresswell 1984], which originates from well known techniques in the theory of classical Kripke models and can be traced back to [Lemmon & Scott 1977]. There's only a slight difference with our definition

[3] A definition of $\models_{\mathfrak{N}}$ has been given implicitly in definition 2.12 on page 52

above. We have chosen a fixed 'empty' valuation in the root, which is of course not possible in classical modal logic. In this case the definition of amalgamation allows every total valuation in the root. The clear relation with lemma 3.3 is that all information which is contained in the root persists in all models in the amalgamation:

$$M^*, w^* \models \varphi \Rightarrow M_i, w \models \varphi \text{ for all } i \in I,\ w \in W_i \text{ and } \varphi \in \mathcal{L}^{\rightarrow}.$$

This simple observation on amalgamations settles the disjunction property of Nelson's logic. The following lemma presents a general formulation.

3.7. LEMMA. DISJUNCTION PROPERTY
For all $\Delta \subseteq \mathcal{L}^{\rightarrow}$: $\emptyset \models_{\mathfrak{N}} \Delta \iff \exists \delta \in \Delta : \emptyset \models_{\mathfrak{N}} \delta$[4].

Proof. The $\Leftarrow$-direction of the proof is trivial. The $\Rightarrow$-direction can be demonstrated by observation 3.6 on amalgamations. Suppose that $\emptyset \not\models_{\mathfrak{N}} \delta$ for all $\delta \in \Delta$. This means that for all these $\delta \in \Delta$ there exists $M_\delta \in \mathfrak{N}$ and w_δ in M_δ such that $M_\delta, w_\delta \not\models \delta$. Let M^* be the amalgamation of these counter-models $\{M_\delta \mid \delta \in \Delta\}$ with root w^*. Observation 3.6 shows that $M^*, w^* \not\models \delta$ for all $\delta \in \Delta$, and so w^* is a non-Δ-world: $\emptyset \not\models_{\mathfrak{N}} \Delta$. ■

3.8. OBSERVATION. The result in lemma 3.7 only applies to tautological disjunctive information. This lemma does certainly not hold for non-empty assumption sets. Nevertheless, replacement of $\emptyset$ in lemma 3.7 is legitimate for certain $\Gamma \subseteq \mathcal{L}^{\rightarrow}$. For example, if Γ consists only of formulae of the form $\varphi \rightarrow \psi$, then also

$$\Gamma \models_{\mathfrak{N}} \Delta \iff \Gamma \models_{\mathfrak{N}} \delta \text{ for certain } \delta \in \Delta.$$

In fact, this property holds for all Γ which consists of formulae in $\mathcal{L}^{\rightarrow}$ where negations only appear immediately infront of atoms.

Proof. Let us give a sketch of the proof. It can be obtained by the amalgamation technique of defintion 3.5[5]. Suppose that Γ consists only of the above-mentioned set of formulae, and let $\Gamma \not\models_{\mathfrak{N}} \delta$ for all $\delta \in \Delta$, and consider the counter-worlds of these non-$\mathfrak{N}$-validities M_δ, w_δ, i.e. $\langle M_\delta, w_\delta\rangle \in \bigcap_{\gamma\in\Gamma} [\![\gamma]\!]_{\mathfrak{N}}$ and $M_\delta, w_\delta \not\models \delta$. Without loss of generality, we may assume that the worlds $\delta \in \Delta$ are strong generators of the models M_δ: $w_\delta \leq_\delta w$ for all w in M_δ. Let $M = \langle W^*, \leq^*, V\rangle$ be the model such that

$\langle W^*, \leq^*\rangle$ is the amalgamation frame of the frames of the models M_δ,

V coincides with V^* with respect to the M_δ-worlds, and

$$V(w^*)(p) = \begin{cases} 1 & \text{if } V(w_\delta) = 1 \text{ for all } \delta \in \Delta, \\ 0 & \text{if } V(w_\delta) = 0 \text{ for all } \delta \in \Delta, \text{ and} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

[4]In terms of disjunction, for all $\varphi, \psi \in \mathcal{L}^{\rightarrow}$: $\emptyset \models_{\mathfrak{N}} \varphi \vee \psi \Leftrightarrow (\emptyset \models_{\mathfrak{N}} \varphi \text{ or } \emptyset \models_{\mathfrak{N}} \psi)$.

[5]In [Jaspars 1991b] this amalgamation technique has been used for proving this stronger formulation of the disjunction property for fragments of the classical modal logic **S4**. In epistemic logic such disjunction properties are important in order to judge the so-called honesty of formulae [Halpern & Moses 1984] [van der Hoek, Jaspars & Thijsse 1993].

By, a simple induction on the construction of the restricted sublanguage, in which Γ is included, we can prove that $M, w^* \models \gamma$ for all $\gamma \in \Gamma$. Furthermore, by the persistence of all δ, we have $M, w^* \not\models \delta$ for all $\delta \in \Delta$. ■

Axioms for Nelson's logic

Axiomatization of $\models_{\mathfrak{N}}$ can be established by the addition of four rules for $\rightarrow$-introduction to the **P**-rules. We will call these rules the **N**-rules.

3.9. TABLE.

INTRODUCTION $\rightarrow$

$$\frac{\Gamma \vdash \varphi, \Delta \quad \Gamma', \psi \vdash \Delta'}{\Gamma, \Gamma', \varphi \rightarrow \psi \vdash \Delta, \Delta'} \text{ L-TRUE} \rightarrow \qquad \frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \rightarrow \psi} \text{ R-TRUE} \rightarrow$$

$$\frac{\Gamma, \varphi, \neg\psi \vdash \Delta}{\Gamma, \neg(\varphi \rightarrow \psi) \vdash \Delta} \text{ L-FALSE} \rightarrow \qquad \frac{\Gamma \vdash \varphi, \Delta \quad \Gamma' \vdash \neg\psi, \Delta'}{\Gamma, \Gamma' \vdash \neg(\varphi \rightarrow \psi), \Delta, \Delta'} \text{ R-FALSE} \rightarrow$$

Intuitionistic logic can also be formalized easily in terms of the rules of **P** and **N**. Its language is $\mathcal{L}^{\rightarrow;\vee}_{-\neg}$. Disappearance of the strong negation $\neg$ prohibits the use of FALSE rules for intuitionistic logic. On the other hand, it contains all the TRUE rules, with the exception of L-TRUE $\neg$, of course. Furthermore, it consists of the TRUE rules for $\vee$[6] and the TRUE rules for the constructive implication $\rightarrow$ above. In the sequel we will call this system **H**.

Note that for R-TRUE $\rightarrow$ we require that the conclusion set is a singleton. The classically valid $\Gamma, \varphi \vdash \psi, \Delta \Rightarrow \Gamma \vdash \varphi \rightarrow \psi, \Delta$ is unsound with respect to $\mathfrak{N}$-validity. This can be demonstrated by the simple observation that $p \models_{\mathfrak{N}} q, p$ but $\not\models_{\mathfrak{N}} p \rightarrow q, p$. As a simple counter-model, take the Nelson model M with two worlds, w and v, such that $w \leq v$, and let $\mathfrak{Dom}(V(w)) = \emptyset$ and $\mathfrak{Dom}(V(v)) = \{p\}$ with $V(v)(p) = 1$. Clearly $M \in \mathfrak{N}$, $M, w \not\models p \rightarrow q$ and $M, w \not\models p$.

3.10. THEOREM. SOUNDNESS **N**

For all $\Gamma, \Delta \subseteq \mathcal{L}^{\rightarrow}$: $\Gamma \vdash_N \Delta \Longrightarrow \Gamma \models_{\mathfrak{N}} \Delta$.

Proof. The soundness of the FALSE $\rightarrow$ rules is immediately obtained from the extensional falsity conditions of the implication. They coincide with the truth conditions of $\varphi \wedge \neg\psi$.

Soundness of R-TRUE $\rightarrow$ has been demonstrated above as the left-to-right direction of the deduction property (lemma 3.4).

What is left to show is the soundness of L-TRUE $\rightarrow$. Suppose

$$\Gamma, \psi \models_{\mathfrak{N}} \Delta \ (1) \quad \text{and} \quad \Gamma' \models_{\mathfrak{N}} \varphi, \Delta' \ (2),$$

and let w in $M \in \mathfrak{N}$ be a $\Gamma \cup \Gamma'$-world. Suppose furthermore that w is a non-$\Delta \cup \Delta'$-world. Because of (2), we obtain $M, w \models \varphi$ while (1) gives us $M, w \not\models \psi$. This shows $M, w \not\models \varphi \rightarrow \psi$. In other words, every $\Gamma \cup \Gamma'$-world which verifies $\varphi \rightarrow \psi$ must also verify at least one of the members of $\Delta \cup \Delta'$. In short,

[6] We need to include it explicitly because the strong negation is no longer present, and so disjunction can no longer be defined.

$$\Gamma \cup \Gamma' \cup \{\varphi \to \psi\} \models_{\mathfrak{N}} \Delta \cup \Delta'.$$

■

3.11. Observation. To make the L-TRUE → a bit more transparent, note that *modus ponens* is a simple consequence of this rule:

$$\varphi, \varphi \to \psi \vdash_N \psi.$$

Take $\Gamma = \varphi$, $\Delta' = \psi$ and $\Gamma' = \Delta = \emptyset$ in L-TRUE → in table 3.9.

A remarkable absence is contra-position for →:

$$\varphi \to \psi \not\vdash_N \neg\psi \to \neg\varphi \quad \text{and} \quad \neg\varphi \to \neg\psi \not\vdash_N \psi \to \varphi.$$

This can be demonstrated quite easily by two counter-models and the soundness result above.

Another remarkable weakness of **N** is

$$(\varphi \vee \neg\varphi) \to \bot \not\vdash_N \bot.$$

This reveals a basic difference between Nelson's negation and intuitionistic negation, which coincides with $\varphi \to \bot$. It can be derived easily that $((\varphi \vee (\varphi \to \bot)) \to \bot \vdash_N \bot$. This principle also holds in intuitionistic logic[7].

The consistency of $(\varphi \vee \neg\varphi) \to \bot$ in Nelson's logic is justified technically by the fact that informationally maximal elements of Nelson models do not have to be total. This means that an agent might be in a certain information stage such that a proof or refutation of φ is not even conceivable.

Nelson's explicit negation is stronger than this intuitionistic negation. This can be demonstrated easily through the following simple derivation in **N**.

1.	$\varphi \vdash_N \varphi$	START	3.	$\varphi, \neg\varphi \vdash_N \bot$	R-MON
2.	$\varphi, \neg\varphi \vdash_N \emptyset$	L-TRUE ¬	4.	$\neg\varphi \vdash_N \varphi \to \bot$	R-TRUE →

Typical principles which distinguish **N** from **H** are the de Morgan equivalences and the double negation principle. They do not hold for the intuitionistic negation.

3.2 Non-persistent variations

The weak negation

The Nelson models which have been introduced in the previous section have been used extensively to represent the semantics of information based logical formalisms. One area of application which we would like to mention is logic programming. Nelson semantics presents suitable interpretation of what is called the *explicit* negation among logic programmers. Apart from the very weak negation-as-failure, which refers to the non-derivability of a proposition, there is a natural demand for a negation which expresses that a logical database infers that something is *not* the case. Under the assumption that inference is interpreted as

[7] In fact all **N**-sequents with no occurrences of ¬ are derivable in intuitionistic logic.

provability, refutability offers a good symmetric concept for inference of negative facts. Therefore the strong negation $\neg$ in Nelson's logic has been proposed as a suitable candidate to capture this explicit use of negations (e.g. [Pearce & Wagner 1990]). The weak negation $\sim$, denoting that its argument is not true or proved, is then the natural candidate for the negation-as-failure [Wagner 1991]. Let us take a look at the weak negation once more. If $M \in \mathfrak{N}$ and w is world in M, then

$$M, w \models \sim \varphi \Leftrightarrow M, w \not\models \varphi \quad M, w \dashv \sim \varphi \Leftrightarrow M, w \models \varphi$$

Just as in partial logic, persistence with respect to the growth of information is lost once we introduce this weak negation. As we saw in the previous section, this means in terms of derivation that we have to give up the deduction property. In the sequential formulation which we presented above, this loss comes down to the unsoundness of R-TRUE $\rightarrow$. A demonstration of this unsoundness is given by the following simple example.

$$p \vee q, \sim p \models_{\mathfrak{N}^\sim} q \text{ but } \sim p \not\models_{\mathfrak{N}^\sim} (p \vee q) \rightarrow q$$ [8]

A counter-model is given by the two world model $M = \langle \{v, w\}, \leq, V \rangle$ with $v \leq w$ and $p \notin \mathfrak{Dom}(V(v))$ and $V(w)(p) = 1$. Clearly $M, v \models \sim p$, but $M, v \not\models (p \vee q) \rightarrow q$.

The deductive repair of the defeated right hand introduction of the constructive implication, R-TRUE $\rightarrow$, can be established by the following four rules, which are called the weak introduction rules for $\rightarrow$.

3.12. TABLE.

WEAK INTRODUCTION $\rightarrow$

$$\frac{\Gamma \vdash p, \Delta \quad p \in I\!P}{\Gamma \vdash \varphi \rightarrow p, \Delta} \text{ PERS } I\!P \qquad \frac{\Gamma \vdash \neg p, \Delta \quad p \in I\!P}{\Gamma \vdash \varphi \rightarrow \neg p, \Delta} \text{ PERS } \neg I\!P$$

$$\frac{\Gamma \vdash \psi \rightarrow \chi, \Delta}{\Gamma \vdash \varphi \rightarrow (\psi \rightarrow \chi), \Delta} \text{ PERS } \rightarrow$$

$$\frac{\Gamma, \varphi \vdash \psi}{\varphi \rightarrow \Gamma \vdash \varphi \rightarrow \psi} \text{ R-TRUE-WEAK } \rightarrow$$

In the last rule $\varphi \rightarrow \Gamma$ is used as an abbreviation of $\{\varphi \rightarrow \gamma \mid \gamma \in \Gamma\}$.

The system of all the **N**-rules, with R-TRUE $\rightarrow$ replaced by the four weak introduction rules for the implication, is called $\mathbf{N}^-$.

The new connective, the weak negation $\sim$, deductively acts the same way as the negation does in classical logic. The sequential rules are similar to the introduction rules for $\sim$ in the system $\mathbf{P}^\sim$ and $\mathbf{M}^\sim$. We list them once more below.

[8] $\models_{\mathfrak{N}^\sim}$ is $\mathfrak{N}$-validity extended for subsets of $\mathcal{L}^{\sim,\rightarrow}$.

$$\frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \sim \varphi \vdash \Delta} \quad \text{L-TRUE} \sim \qquad \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \sim \varphi, \Delta} \quad \text{R-TRUE} \sim$$

$$\frac{\Gamma, \varphi \vdash \Delta}{\Gamma, \neg \sim \varphi \vdash \Delta} \quad \text{L-FALSE} \sim \qquad \frac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \neg \sim \varphi, \Delta} \quad \text{R-FALSE} \sim$$

This explains formally the parasitic role of weak negation in constructive systems. Its own rules are maintained, while the original logic **N** has been affected. The system $\mathbf{N}^-$ with the additional rules above for weak negation is called $\mathbf{N}^\sim$.

3.13. THEOREM. SOUNDNESS $\mathbf{N}^\sim$
For all $\Gamma, \Delta \subseteq \mathcal{L}^{\sim,\rightarrow}$: $\Gamma \vdash_{N^\sim} \Delta \Longrightarrow \Gamma \models_{\mathfrak{N}^\sim} \Delta$.

Proof. We prove only the rule R-TRUE-WEAK $\rightarrow$. The other induction steps are left to the reader.

Suppose $\Gamma, \varphi \models_{\mathfrak{N}^\sim} \psi$ (4), and $M, w \models \varphi \rightarrow \gamma$ for all $\gamma \in \Gamma$. Let $v \geq w$ in M, such that $M, v \models \varphi$. This also yields $M, v \models \gamma$ for all $\gamma \in \Gamma$. According to (4) we may conclude $M, v \models \psi$. Because v has been chosen arbitrarily as an extension of w in M which verifies φ, we know that $M, w \models \varphi \rightarrow \psi$. This means $\varphi \rightarrow \Gamma \models_{\mathfrak{N}^\sim} \varphi \rightarrow \psi$. ■

3.14. OBSERVATION. Note that R-TRUE-WEAK $\rightarrow$ coincides with R-TRUE $\rightarrow$ if the assumption set Γ is empty: $\varphi \vdash_{N^\sim} \psi \Rightarrow \vdash_{N^\sim} \varphi \rightarrow \psi$.

A proposition φ does not longer mean the same as $\top \rightarrow \varphi$. The latter expresses that φ will always hold during the enrichment of information. Of course, these propositions are still semantic equivalent if $\varphi \in \mathcal{L}^{\rightarrow}$.

In general, $\sim \varphi$ has another meaning than $\top \rightarrow \sim \varphi$. The former proposition says that a proof of φ is currently missing. The latter proposition expresses the non-provability of φ, and therefore coincides with the intuitionistic negation of φ: $\varphi \rightarrow \bot$. This equivalence can be derived in $\mathbf{N}^\sim$ in a quite easy manner.

3.15. EXAMPLE. The following derivation illustrates one direction of the last equivalence: $\varphi \rightarrow \bot \vdash_{N^\sim} \top \rightarrow \sim \varphi$.

1.	$\bot \vdash_{N^\sim} \emptyset$	L-TRUE $\bot$
2.	$\varphi \vdash_{N^\sim} \varphi$	START
3.	$\varphi, \varphi \rightarrow \bot \vdash_{N^\sim} \emptyset$	L-TRUE $\rightarrow$ (1,2)
4.	$\varphi \rightarrow \bot \vdash_{N^\sim} \sim \varphi$	R-TRUE $\sim$ (3)
5.	$(\varphi \rightarrow \bot), \top \vdash_{N^\sim} \sim \varphi$	L-MON (4)
6.	$\top \rightarrow (\varphi \rightarrow \bot) \vdash_{N^\sim} \top \rightarrow \sim \varphi$	R-TRUE-WEAK $\rightarrow$ (5)
7.	$\varphi \rightarrow \bot \vdash_{N^\sim} \top \rightarrow (\varphi \rightarrow \bot)$	START and PERS $\rightarrow$
8.	$\varphi \rightarrow \bot \vdash_{N^\sim} \top \rightarrow \sim \varphi$	CUT (5,6)

Non-monotonic logic

In the field of non-monotonic logic, Nelson models have been re-introduced by Turner [Turner 1984]. The information order $\leq$ in the models, is presented as a *plausibility* relation, saying that a larger state models a plausible extension of a *current* theory. The underlying idea is to present a logic with an explicit *consistency*-operator M, such that default rules, such as in Reiter's original default logic [Reiter 1980][9], can be interpreted as normal inference rules. The original idea to use constructive logic for such a uniform logical analysis originates from Gabbay [Gabbay 1982]; in this article the same idea has been performed in intuitionistic logic[10].

The consistency operator M is added to the syntax of the logic, it refers to a situation where its argument can be consistently added to the current information. In terms of the plausibility relation it says that there exists a plausible extension of the current state which contains the argument of **M**. Formally, this looks as follows:

$$M,w \models \mathsf{M}\varphi \Leftrightarrow \exists v \geq w : M,v \models \varphi \qquad M,w \dashv \mathsf{M}\varphi \Leftrightarrow \forall v \geq w : M,v \dashv \varphi.$$

Clearly, this M-operator can be interpreted straightforwardly in $\mathbf{N}^{\sim}$. The language of Turner's logic has less expressive capacity than $\mathcal{L}^{\sim,\rightarrow}$. Nevertheless, a sequential system for this logic can easily be defined. We take the system **P** and add two modal rules for the M-operator, two rules for the persistence of literals and two rules for the persistence of propositions of the form $\neg\mathsf{M}\varphi$. The first two modal rules are the introduction rules for $\diamond$ in **M**, with $\diamond$ replaced by M and $\Box$ by $\neg\mathsf{M}\neg$. The persistence rules are the same PERS $I\!\!P$ and PERS $\neg I\!\!P$ of table 3.12, with $\varphi \rightarrow$ replaced by $\neg\mathsf{M}\neg$. The other persistence rules are

$$\frac{\Gamma\neg\mathsf{M}\varphi \vdash \Delta}{\Gamma\neg\mathsf{M}\mathsf{M}\varphi \vdash \Delta} \quad \text{and} \quad \frac{\Gamma \vdash \mathsf{M}\varphi, \Delta}{\Gamma \vdash \mathsf{M}\mathsf{M}\varphi, \Delta}.$$

Data semantics

Nelson models have also been employed for constructive analysis of natural language conditionals in Veltman's so-called data-semantics [Veltman 1981]. The implication has been given the same denotation as in Nelson's logic when it comes to verification. Falsification however is given a weaker intensional meaning, which is close to the verification clause of Turner's **M**-operator. In order to distinguish Veltman's implication from Nelson's, we use the symbol $\rightsquigarrow$. Interpretation of this implication comes down to

$$M,w \models \varphi \rightsquigarrow \psi \Leftrightarrow \forall v \geq w : (M,v \models \varphi \Rightarrow M,v \models \psi) \quad \text{and}$$

$$M,w \dashv \varphi \rightsquigarrow \psi \Leftrightarrow \exists v \geq w : (M,v \models \varphi \;\&\; M,v \dashv \psi).$$

Apart from the typical conditional implication data semantics uses a semantic constraint on the class of Nelson models. This requirement can be understood as a *refinability* constraint. It says that for every proposition φ in $\mathcal{L}^{\rightsquigarrow}$ and for every

[9] In default logic, default rules are taken to be applicable if it does not lead to an inconsistency with a current belief state with respect to the logic itself.

[10] Turner does not use the constructive implication, but the stronger Kleene implication: $\varphi \rightarrow \psi = \neg\varphi \vee \psi$.

world there exists an extension which determines a truth-value for φ. Formally, for all $M = \langle W, \leq, V\rangle$

$$\forall\varphi \in \mathcal{L}^{\leadsto}\ \forall w \in W\ \exists v \in W : w \leq v\ \&\ (M, v \models \varphi \text{ or } M, v \mathrel{=\!\!\mid} \varphi)\ ^{11}.$$

In section 5.4 we will give a complete axiomatization, called sequential data logic, for data semantics. The characteristic axiomatic addition provided by refinability is that $(\varphi \vee \neg\varphi) \leadsto \bot \vdash \bot$. In other words, a condition of the form $\varphi \vee \neg\varphi$ is always an empty assumption. As we have seen in observation 3.11, this principle does not hold in Nelson's logic.

3.3 Up and down

When we wish to model epistemic attitudes weaker than mathematical knowledge, the persistence of factual information is far too idealistic. As explained in chapter 1, we take a world to be an information carrier of evidence and counter-evidence, instead of proofs and refutations. This weaker interpretation of the underlying sources of truth and falsity, are not guaranteed to preserve their quality for life. Informally, this means that agents not only move upwards in the constructive direction of the structural extension order, but also may lose information and fall back.

This wider 'up-and-down' dynamic perspective has been propagated in artificial intelligence, as in the theory of *truth maintenance* [Doyle 1979], and in formal philosophy, as in the logic of theory change and belief contraction and revision [Alchourrón, Gärdenfors & Mackinson 1985] [Gärdenfors 1988].

Loss of information by an agent is not hard to model in terms of the constructive models of Nelson's logic $\mathfrak{N}$. Retraction of a proposition φ is an action, modeled as a relation in the opposite direction of the extension order, such that its output argument is a state where φ does not hold. This relation is denoted by a specific modal operator $[\varphi]_d$. The proposition $[\varphi]_d\psi$, which says that ψ always occurs after withdrawing φ from an information state w in a model $M \in \mathfrak{N}$, has the following truth-condition:

$$M, w \models [\varphi]_d\psi \iff \forall v \leq w : M, v \not\models \varphi \Rightarrow M, v \models \psi.$$

It simply says: "retraction of φ means ψ". In [van Benthem 1991b] and [de Rijke 1992] such kind of 'downdate-operators' have been defined in terms of a two-directional version of the classical modal logic **S4**.

Falsity of this downward implication $[\varphi]_d\psi$ with respect to such a state w, means that it is possible to withdraw φ in such a way that ψ is false with respect to the new state:

$$M, w \mathrel{=\!\!\mid} [\varphi]_d\psi \iff \exists v \leq w : M, v \not\models \varphi\ \ \&\ \ M, v \mathrel{=\!\!\mid} \psi.$$

[11] Veltman also stipulates another model-theoretic constraint which he calls *closedness*. This means that every chain over $\leq$ contains a maximal element. This stronger requirement is equivalent to refinability from the perspective of the underlying logic, i.e. they are completely axiomatized by the same system (data logic).

In other words, falsification of the downdate operators is contingently interpreted over φ-impoverishments of the actual state. The update-operators are defined in terms of the converse direction, but in the same intensional format. As said before, the constructive implication carries the status of this operator. Truth of a proposition $\varphi \rightarrow \psi$ with respect to a state s means that every φ-update of s leads to a state where ψ holds. The falsification of $\varphi \rightarrow \psi$ in Nelson's interpretation means the same as truth of $\varphi \wedge \neg\psi$. This interpretation does not seem the right candidate for falsification whenever we take $\neg(\varphi \rightarrow \psi)$ to be a proposition which tells us something about φ-updates. Just like the downdate-proposition, the falsity of the proposition that φ-updating a state s means ψ is the same as that there exists a φ-enrichment of s which falsifies ψ. This means that our update operators coincide with the conditional implication of Veltman's data semantics [Veltman 1985]. In other words, for $M \in \mathfrak{N}$ and w in M:

$$M, w \models [\varphi]_u \psi \iff \forall v \geq w : M, v \models \varphi \Rightarrow M, v \models \psi, \text{ and}$$

$$M, w \mathrel{=\!\!|} [\varphi]_u \psi \iff \exists v \geq w : M, v \models \varphi \;\&\; M, v \mathrel{=\!\!|} \psi.$$

We also use abbreviations for the 'possibility'-like dualities of these dynamic operators: $\langle\varphi\rangle_u = \neg[\varphi]_u\neg$ and $\langle\varphi\rangle_d = \neg[\varphi]_d\neg$. Furthermore $[\top]_u$ and $\langle\top\rangle_u$ are written as $[\ ]_u$ and $\langle\ \rangle_u$. The meaning of these operators are 'after every update' and 'after some update'. For $[\bot]_d$ and $\langle\bot\rangle_d$ we use $[\ ]_d$ and $\langle\ \rangle_d$. They stand for 'after every retraction' and 'after some retraction'.

The language which we will use is abbreviated by $\mathcal{L}^{\uparrow,\downarrow}$. If we were more consequent we should use here $\mathcal{L}^{[\ ]_u,[\ ]_d}$. Semantic consequence for $\mathcal{L}^{\uparrow,\downarrow}$ over the class of Nelson models $\mathfrak{N}$ is written as $\models_{\mathfrak{N}^{\uparrow,\downarrow}}$.

Minimal updates

Our use of the word update may be a bit misleading. In many texts on dynamic semantics an update is taken to be minimal, that is its changing effect should be as small as possible. Let us write such minimal updates as $[\varphi]^\mu_u\psi$. Truth of such a proposition with respect to a world w means that all smallest φ-enrichments of w verify ψ. Let's say, that $w \leq_\varphi v$ means that v is such a smallest φ-extension of w:

$$w \leq_\varphi v \Leftrightarrow w \leq v \;\&\; M, v \models \varphi \;\&\; M, u \not\models \varphi \text{ for all } u \text{ with } w \leq u < v.$$

The proper verification and falsification clause for such a minimal update are

$$M, w \models [\varphi]^\mu_u \psi \Leftrightarrow M, v \models \psi \text{ for all } v \text{ such that } w \leq_\varphi v, \text{ and}$$

$$M, w \mathrel{=\!\!|} [\varphi]^\mu_u \psi \Leftrightarrow M, v \mathrel{=\!\!|} \psi \text{ for certain } v \text{ such that } w \leq_\varphi v.$$

Conversely, we could define a relation $\geq_\varphi$ for the converse of the information structure, holding between worlds and their minimal φ-impoverishments. Minimal downdate-operators can then be interpreted in the same way.

The main reason for us to exclude these minimal variants of the dynamic operatros is merely technical. From a logical and meta-theoretical perspective these operators are far more complicated than our arbitrary up- and downdate operators. A hard question, which remains to be solved, is whether a complete axiomatization can be given, if we only add these minimal dynamic operators to the language $\mathcal{L}^{\uparrow,\downarrow}$. Our chances, as we may learn from classical modal approaches to minimal change, are probably much better if we would further extend the language. One way of doing so, is by incorporating the binary modal operators like Since and Until, which are used in temporal logic. Another opportunity lies in extending the relational expressivity of our language.

We have chosen to ignore these challenges. We think that the relatively weak evidence that updates, which arise from communicative actions, are indeed minimal, do not relate to the technical overload of abstract modal logic which would emerge from such an enterprise.

Anti-persistence

A non-persistent connective as the plausibility operator **M** of Turner reappears as $\langle\ \rangle_u$. The proposition $\langle\ \rangle_u\varphi$ is semantically equivalent with $\langle\varphi\rangle_u\top$, and simply means that the current information can consistently be updated with φ.

This type of information has a converse persistence property, which we will call *anti-persistence*. It means that it can never be lost when we switch to a poorer state. This sounds contradictory, but 'poorer' is a relative notion here. As explained earlier, it only means that factual information, i.e. literals, is retracted. Formally, anti-persistence of a formula φ means that for every $M = \langle W, \leq, V\rangle \in \mathfrak{N}$ and $v, w \in W$:

$$\text{if } M, w \models \varphi \ \text{ and } v \leq w \text{ then } M, v \models \varphi.$$

Obviously, all information of the form $\langle\varphi\rangle_u\psi$ is anti-persistent. The same holds for universal statements over downdates, i.e. information of the form $[\varphi]_d\psi$. Anti-persistence of a proposition φ can also be described as its equivalence with $[\ ]_d\varphi$.

Just like the disjunction property for persistent information, we can stipulate a *conjunction property* for anti-persistent information. The conjunction property says that if a set of anti-persistent formulae has no model, then there exists at least one member of this set which has no model.

3.16. LEMMA. CONJUNCTION PROPERTY

Let $\Gamma \subseteq \mathcal{L}^{\uparrow,\downarrow}$ be a set of anti-persistent formulae. Then

$$\Gamma \models_{\mathfrak{N}^{\uparrow,\downarrow}} \emptyset \iff \gamma \models_{\mathfrak{N}^{\uparrow,\downarrow}} \emptyset \text{ for certain } \gamma \in \Gamma.$$

Proof. The $\Leftarrow$-direction is immediate. The $\Rightarrow$-direction can be obtained by the same amalgamation construction as in the proof of the disjunction property for $\mathcal{L}^{\rightarrow}$ over $\mathfrak{N}$ (lemma 3.7). If all γ's have a model, then we can amalgamate these models into one model which verifies all $\gamma's$ in its root, because all these formulae are anti-persistent. In other words, this root is a Γ-world, and thus $\Gamma \not\models_{\mathfrak{N}^{\uparrow,\downarrow}} \emptyset$. ■

The relevance of anti-persistence will become clear in the following subsection, where explicit axioms have to be given to fix this property for certain fragments of the language. Anti-persistence need to be administered in the same way as we did for persistence in the logic $\mathbf{N}^-$.

The system ud

In this subsection we develop a sequential derivation system for up-and-down reasoning over Nelson models by means of the language $\mathcal{L}^{\uparrow,\downarrow}$. This system is called **ud**.

The first set of **ud**-rules which we will present is the above-mentioned book-keeping part of the logic. It formulates the persistence and anti-persistence property syntactically.

3.17. TABLE.

$$\frac{\Gamma \vdash p, \Delta \quad p \in I\!P}{\Gamma \vdash [\varphi]_u\, p, \Delta}\ \text{PERS } I\!P \qquad \frac{\Gamma \vdash \neg p, \Delta \quad p \in I\!P}{\Gamma \vdash [\varphi]_u \neg p, \Delta}\ \text{PERS } \neg I\!P$$

$$\frac{\Gamma \vdash [\psi]_u\, \chi, \Delta}{\Gamma \vdash [\varphi]_u\, [\psi]_u\, \chi, \Delta}\ \text{PERS } [\ ]_u \qquad \frac{\Gamma \vdash \langle\psi\rangle_d\, \chi, \Delta}{\Gamma \vdash [\varphi]_u\, \langle\psi\rangle_d\, \chi, \Delta}\ \text{PERS } \langle\ \rangle_d$$

Besides these persistence rules we need their contra-positional formulation as well. These rules are necessary because of our choice of falsity assignment for the operators $[\varphi]_u$ and $[\varphi]_d$.

$$\frac{\Gamma, \langle\psi\rangle_u\, \chi \vdash \Delta}{\Gamma, \langle\varphi\rangle_u\, \langle\psi\rangle_u\, \chi \vdash \Delta}\ \text{C-PERS } [\ ]_u \qquad \frac{\Gamma, [\psi]_d\, \chi \vdash \Delta}{\Gamma, \langle\varphi\rangle_u\, [\psi]_d\, \chi \vdash \Delta}\ \text{C-PERS } \langle\ \rangle_d$$

The following rules formulate the anti-persistence of the operators $\langle\varphi\rangle_u$ and $[\varphi]_d$, and their contra-positions.

$$\frac{\Gamma \vdash \langle\psi\rangle_u\, \chi, \Delta}{\Gamma \vdash [\varphi]_d\, \langle\psi\rangle_u\, \chi, \Delta}\ \text{A-PERS } \langle\ \rangle_u \qquad \frac{\Gamma \vdash [\psi]_d\, \chi, \Delta}{\Gamma \vdash [\varphi]_d\, [\psi]_d\, \chi, \Delta}\ \text{A-PERS } [\ ]_d$$

$$\frac{\Gamma, [\psi]_u\, \chi \vdash \Delta}{\Gamma, \langle\varphi\rangle_d\, [\psi]_u\, \chi \vdash \Delta}\ \text{C-A-PERS } \langle\ \rangle_u \qquad \frac{\Gamma, \langle\psi\rangle_d\, \chi \vdash \Delta}{\Gamma, \langle\varphi\rangle_d\, \langle\psi\rangle_d\, \chi \vdash \Delta}\ \text{C-A-PERS } [\ ]_d$$

The following set of rules uncover the part of the logic which is more inspiring. They present the TRUE and FALSE rules for the operators $[\varphi]_u$ and $[\varphi]_d$.

3.18. TABLE.

$$\frac{\Gamma \vdash \varphi, \Delta \quad \Gamma', \psi \vdash \Delta'}{\Gamma, \Gamma', [\varphi]_u\, \psi \vdash \Delta, \Delta'}\ \text{L-TRUE } [\ ]_u \qquad \frac{\Gamma, \varphi \vdash \psi, \neg\Delta}{[\varphi]_u\, \Gamma \vdash [\varphi]_u\, \psi, \neg[\varphi]_u\, \Delta}\ \text{R-TRUE } [\ ]_u$$

$$\frac{\Gamma, \varphi, \neg\psi \vdash \neg\Delta}{[\varphi]_u\, \Gamma, \neg[\varphi]_u\, \psi \vdash \neg[\varphi]_u\, \Delta}\ \text{L-FALSE } [\ ]_u \qquad \frac{\Gamma \vdash \varphi, \Delta \quad \Gamma', \vdash \neg\psi, \Delta'}{\Gamma, \Gamma' \vdash \neg[\varphi]_u\, \psi, \Delta, \Delta'}\ \text{R-FALSE } [\ ]_u$$

$$\frac{\Gamma, \varphi \vdash \Delta \quad \Gamma', \psi \vdash \Delta'}{\Gamma, \Gamma', [\varphi]_d\, \psi \vdash \Delta, \Delta'}\ \text{L-TRUE } [\ ]_d \qquad \frac{\Gamma \vdash \varphi, \psi, \neg\Delta}{[\varphi]_d\, \Gamma \vdash [\varphi]_d\, \psi, \neg[\varphi]_u\, \Delta}\ \text{R-TRUE } [\ ]_d$$

$$\frac{\Gamma, \neg\psi \vdash \varphi, \neg\Delta}{[\varphi]_d\, \Gamma, \neg[\varphi]_d\, \psi \vdash \neg[\varphi]_d\, \Delta}\ \text{L-FALSE } [\ ]_d \qquad \frac{\Gamma, \varphi \vdash \Delta \quad \Gamma' \vdash \neg\psi, \Delta'}{\Gamma, \Gamma' \vdash \neg[\varphi]_d\, \psi, \Delta, \Delta'}\ \text{R-FALSE } [\ ]_d$$

In the sequel of the text we will freely use $\langle\ \rangle_u$ and $\langle\ \rangle_d$-variants of these introduction rules. For example, R-TRUE $\langle\ \rangle_u$ refers to

$$\frac{\Gamma \vdash \varphi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \langle\varphi\rangle_u \psi, \Delta}.$$

Unraveling ud

The $[\ ]_u$- and $[\ ]_d$-introduction rules may look dazzling. However, after some meditation the beauty appears. The rule R-TRUE $[\ ]_u$ connects the right-hand introduction of the implication in **N** and the combinatorics of the right-hand introduction of the necessity operator in the minimal partial modal logic **M**. A somewhat weaker version of this rule, which will be called R-TRUE' $[\ ]_u$, clarifies this structural interpretation.

$$\frac{\Gamma, \varphi \vdash \psi, \Delta}{[\ ]_u \Gamma \vdash [\varphi]_u \psi, \langle\ \rangle_u \Delta} \quad \text{R-TRUE' } [\ ]_u .$$

The other rules in table 3.18 with a single condition, i.e. L-FALSE $[\ ]_u$, R-TRUE $[\ ]_d$ and L-FALSE $[\ ]_d$, are in fact permutation variants of R-TRUE $[\ ]_u$. In order to see this permutation variation, we list weaker marked versions of these rules as well.

$$\frac{\Gamma, \varphi, \neg\psi \vdash \Delta}{[\ ]_u \Gamma, \neg[\varphi]_u \psi \vdash \langle\ \rangle_u \Delta} \quad \text{L-FALSE' } [\ ]_u$$

$$\frac{\Gamma \vdash \varphi, \psi, \Delta}{[\ ]_d \Gamma \vdash [\varphi]_d \psi, \langle\ \rangle_d \Delta} \quad \text{R-TRUE' } [\ ]_d$$

$$\frac{\Gamma, \neg\psi \vdash \varphi, \Delta}{[\ ]_d \Gamma, \neg[\varphi]_d \psi \vdash \langle\ \rangle_d \Delta} \quad \text{L-FALSE' } [\ ]_d$$

We will refer to these weaker version of the R-TRUE and L-FALSE introduction of the up- and downdate operators when we will discuss the meta-theory of up-and-down logics in chapter 5 and 6 in part II. These rules can be derived by means of the following general principle.

3.19. PROPOSITION. Let $\alpha, \beta, \varphi \in \mathcal{L}^{\uparrow,\downarrow}$.

$$\text{If } \alpha \vdash_{ud} \beta \text{ then also } \begin{cases} [\beta]_u \varphi \vdash_{ud} [\alpha]_u \varphi, & \langle\alpha\rangle_u \varphi \vdash_{ud} \langle\beta\rangle_u \varphi, \\ [\alpha]_d \varphi \vdash_{ud} [\beta]_d \varphi, & \langle\beta\rangle_d \varphi \vdash_{ud} \langle\alpha\rangle_d \varphi \end{cases}$$

Proof. By way of illustration we show the first conclusion. It can be obtained by the following simple derivation.

1. $\alpha \vdash_{ud} \beta$	assumption
2. $\varphi \vdash_{ud} \varphi$	START
3. $\alpha, [\beta]_u \varphi \vdash_{ud} \varphi$	L-TRUE $[\ ]_u$ (1,2)
4. $[\alpha]_u [\beta]_u \varphi \vdash_{ud} [\alpha]_u \varphi$	R-TRUE $[\ ]_u$ (3)
5. $[\beta]_u \varphi \vdash_{ud} [\beta]_u \varphi$	START
6. $[\beta]_u \varphi \vdash_{ud} [\alpha]_u [\beta]_u \varphi$	PERS $[\ ]_u$ (5)
7. $[\beta]_u \varphi \vdash_{ud} [\alpha]_u \varphi$	CUT (6,4)

The other three conclusions can be obtained by substitution of the appropriate introduction rules and persistence or anti-persistence rules in the derivation rules above. For example, if L- and R-TRUE $[\ ]_u$ are replaced by L- and R-TRUE $[\ ]_d$, respectively, and PERS is replaced by A-PERS $[\ ]_d$ in the derivation above, one finds the derivation for the third conclusion in the proposition above. (The other two conclusions require some additional 'double negation' reasoning). ■

3.20. COROLLARY. Substitution of the trivial **ud**-sequents $\varphi \vdash_{ud} \top$ and $\bot \vdash_{ud} \varphi$ in proposition 3.19 yields the following sequents:

$[\ ]_u \psi \vdash_{ud} [\varphi]_u \psi \qquad [\ ]_d \psi \vdash_{ud} [\varphi]_d \psi$

$\langle\varphi\rangle_u \psi \vdash_{ud} \langle\ \rangle_u \psi \qquad \langle\varphi\rangle_d \psi \vdash_{ud} \langle\ \rangle_d \psi$

Application of CUT, the finiteness property of the system **ud** and corollary 3.20 establish the weaker marked version of the R-TRUE and L-FALSE introduction rules for the up- and downdate operators.

The rule L-TRUE $[\ ]_u$ yields again a modus ponens like variant: $[\varphi]_u \psi, \varphi \vdash_{ud} \psi$. The rules R-FALSE $[\ ]_u$, L-TRUE $[\ ]_d$ and R-FALSE $[\ ]_d$ produce permutation variants of this modus ponens (modulo double negation).

3.21. TABLE.

L-TRUE $[\ ]_d$	$[\varphi]_d \psi \vdash_{ud} \varphi, \psi$	R-FALSE $[\ ]_u$	$\varphi, \psi \vdash_{ud} \langle\varphi\rangle_u \psi$
		R-FALSE $[\ ]_d$	$\psi \vdash_{ud} \langle\varphi\rangle_d \psi, \varphi$

Some other important sequents are presented in the following examples. Most of them will reappear in the first two chapters of part II.

3.22. EXAMPLE.

SIMPLIFICATION OF $\langle\ \rangle_u$ AND $[\ ]_d$

$\langle\varphi\rangle_u \psi \equiv_{ud} \langle\psi\rangle_u \varphi \equiv_{ud} \langle\ \rangle_u (\varphi \wedge \psi)$

$[\varphi]_d \psi \equiv_{ud} [\psi]_d \varphi \equiv_{ud} [\ ]_d (\varphi \vee \psi)$

DUALITY PRINCIPLES

$\langle\ \rangle_u [\ ]_d \varphi \vdash_{ud} \varphi \qquad \varphi \vdash_{ud} [\ ]_u \langle\ \rangle_d \varphi$

$\langle\ \rangle_d [\ ]_u \varphi \vdash_{ud} \varphi \qquad \varphi \vdash_{ud} [\ ]_d \langle\ \rangle_u \varphi$

MODALITY REDUCTIONS

$[\ ]_u [\varphi]_u \psi \equiv_{ud} \langle\ \rangle_d [\varphi]_u \psi \equiv_{ud} [\varphi]_u \psi$

$[\ ]_d \langle\varphi\rangle_u \psi \equiv_{ud} \langle\ \rangle_u \langle\varphi\rangle_u \psi \equiv_{ud} \langle\varphi\rangle_u \psi$

$[\ ]_d [\varphi]_d \psi \equiv_{ud} \langle\ \rangle_u [\varphi]_d \psi \equiv_{ud} [\varphi]_d \psi$

$[\ ]_u \langle\varphi\rangle_d \psi \equiv_{ud} \langle\ \rangle_d \langle\varphi\rangle_d \psi \equiv_{ud} \langle\varphi\rangle_d \psi$

These duality principles are typical of 'back and forth'-modal systems. For example, they are used in temporal logics with 'past' and 'future' operators (see e.g. [van Benthem 1983]).

The equivalences and **ud**-sequents in the example 3.22 in 3.22 can be demonstrated by combining the persistence and anti-persistence rules with the simple 'modus ponens permutations' of table 3.21. An illustrative derivation of $\varphi \vdash_{ud} [\ ]_d \langle\ \rangle_u \varphi$ is presented below.

1.	$\vdash_{ud} \top$	R-TRUE $\top$
2.	$\top, \varphi \vdash_{ud} \langle\ \rangle_u \varphi$	R-FALSE $[\ ]_u$ in table 3.21
3.	$\varphi \vdash_{ud} \langle\ \rangle_u \varphi$	CUT (1,2)

4. $\langle\ \rangle_u \varphi \vdash_{ud} \langle\ \rangle_u \varphi$ START
5. $\langle\ \rangle_u \varphi \vdash_{ud} [\ ]_d \langle\ \rangle_u \varphi$ A-PERS $\langle\ \rangle_u$
6. $\varphi \vdash_{ud} [\ ]_d \langle\ \rangle_u \varphi$ CUT (3,5)

The first two equivalences in example 3.22 require application of the R-TRUE and L-FALSE introductions as well. Below we demonstrate a derivation of

$\langle\varphi\rangle_u \psi \vdash_{ud} \langle\psi\rangle_u \varphi$.

1. $\varphi, \psi \vdash_{ud} \langle\psi\rangle_u \varphi$ R-FALSE $[\ ]_u$ in table 3.21
2. $\langle\varphi\rangle_u \psi \vdash_{ud} \langle\varphi\rangle_u \langle\psi\rangle_u \varphi$ L-TRUE $\langle\ \rangle_u$
3. $\langle\psi\rangle_u \varphi \vdash_{ud} \langle\psi\rangle_u \varphi$ START
4. $\langle\varphi\rangle_u \langle\psi\rangle_u \varphi \vdash_{ud} \langle\psi\rangle_u \varphi$ C-PERS $\langle\ \rangle_u$
5. $\langle\varphi\rangle_u \psi \vdash_{ud} \langle\psi\rangle_u \varphi$ CUT (2,4)

Dutifully, we wind up with the soundness result for the system **ud**.

3.23. THEOREM. SOUNDNESS **ud**
For all $\Gamma, \Delta \subseteq \mathcal{L}^{\uparrow,\downarrow}$: $\Gamma \vdash_{ud} \Delta \Longrightarrow \Gamma \models_{\mathfrak{N}^{\uparrow,\downarrow}} \Delta$.

Proof. Soundness of the persistence and anti-persistence rules is straightforward. The L-TRUE $[\ ]_u$ rule is the same as L-TRUE $\rightarrow$ in **N**. R-TRUE $[\ ]_u$ does not completely coincide with R-TRUE $\rightarrow$. The conclusion side of the conditional sequent does not have to be a singleton. This additional facility is due to the way the update modality is falsified.

Just like all the other soundness proofs before, our strategy is a simple induction on the length of derivations. Below we present the result of the L-FALSE $[\ ]_d$ rule. The other cases are left to the suspicious, but diligent reader[12].

Suppose $\Gamma, \neg\psi \models_{\mathfrak{N}^{\uparrow,\downarrow}} \varphi, \neg\Delta$ (5) and let w in $M \in \mathfrak{N}$ be a $[\varphi]_d\Gamma$-world. If $M, w \dashv [\varphi]_d \psi$ then there exists a φ-impoverishment of w in M, say v, such that $M, v \dashv \psi$. Because all φ-impoverishments of w are Γ-worlds, we know by (5) that $M, v \dashv \delta$ for certain $\delta \in \Delta$, which means $M, w \dashv [\varphi]_d \delta$. Summarizing this argumentation, we find $[\varphi]_d \Gamma, \neg[\varphi]_d \psi \models_{\mathfrak{N}^{\uparrow,\downarrow}} \neg[\varphi]_d \Delta$. ■

3.4 Constructive modal logics

The constructive logics of the preceding sections are simple constructive dynamic extensions of the minimal partial logic **P**. Their static part consists of single partial worlds, which are shortcoming for modeling the kind of epistemic dynamics which we have in mind. Just like in many epistemic logics, we take multiple possible worlds models to represent an epistemic state. As explained in chapter 1, such multiple worlds representation is meant to capture an instantaneous set of uncertainties.

[12] Somewhat easier to check are the weaker marked versions of the single conditioned rules, i.e. the R-TRUE and L-FALSE introductions.

Our choice in favor of partial instead of total two-valued possible worlds representation provides a constructive component to epistemic dynamics. In this section we introduce the most elementary constructive extension of the minimal partial modal logic **M** of chapter 2.

In the first paragraph we define the pure constructive extension of **M**. The second subsection discusses the weaker minimal dynamic extension of **M** on the basis of construction and elimination. In the third subsection we will introduce the system **Mud**; this is a combination of the up-and-down logic **ud** and **M** on the basis of the same construction-elimination dynamics. In the next chapter we will discuss a multiple agent epistemic logic which is built on this dynamic partial modal logic.

The system NM

From the viewpoint of deduction the most simple constructive extension of partial modal logic is the system **NM**. This system consists of both the rules of **M** and the TRUE- and FALSE-rules for the constructive implication $\rightarrow$ for **N**. The semantic part of this system is the class of $\mathfrak{NM}$-models. In this class of models there are two relations present; one takes care of the modal dimension, while the other ranges over the possible constructive extensions. All information persists on the basis of a pure constructive dynamics. The persistence is structurally enforced by an interrelation of the modal accessibility and the extension structure in $\mathfrak{NM}$-models. In such a model this latter relation is taken to be a bisimulation over the former relation. The monotonicity of the valuation function and corollary 2.46 guarantees persistence of the full language $\mathcal{L}^{\Box,\rightarrow}$.

3.24. DEFINITION. An $\mathfrak{NM}$-model is a quadruple $\langle W, R, \leq, V\rangle$ such that $W \neq \emptyset$, $R \subseteq W \times W$, $\leq$ is a pre-order over W and $V : W \longrightarrow \mathfrak{P}$ with

$$\leq \circ R \subseteq R \circ \leq \quad \text{and} \quad \geq \circ R \subseteq R \circ \geq,$$

$$\text{and } w \leq v \Rightarrow V(w) \sqsubseteq V(v) \ \text{ for all } w, v \in W.$$

Technically speaking, the relation $\leq$ is a bisimulation over the partial Kripke model $\langle W, R, V\rangle$. As we have seen in the previous chapter, this bisimulation constraint is not a precise characterization of the inclusion order between information contents of worlds. In chapter 5 we will prove that the bisimulation definition is yet satisfactory from the perspective of the canonical model. This means that from the axiomatic viewpoint, the bisimulation requirement of the $\mathfrak{NM}$-class is perfectly satisfactory.

3.25. OBSERVATION. For all $\varphi \in \mathcal{L}^{\Box,\rightarrow}$ and for all $M = \langle W, R, \leq, V\rangle \in \mathfrak{NM}$

$$(\ M, w \models \varphi \ \&\ w \leq v\ ) \Rightarrow M, v \models \varphi \text{ for all } w, v \in W.$$

The proof can be obtained immediately from corollary 2.46 on page 70. A soundness check of **NM** is left to the reader.

The disjunction property for constructive modal logics

A model-theoretic technical question which arises is whether the **NM** has the disjunction property. Just like **N** it is completely persistent. Nevertheless, a simple counterexample can be given.

3.26. EXAMPLE. Clearly $\models_{\mathfrak{NM}} \Diamond\top, \Box\bot$, but $\not\models_{\mathfrak{NM}} \Diamond\top$ and $\not\models_{\mathfrak{NM}} \Box\bot$. The following picture shows two very simple one-world counter-models for these two non-$\mathfrak{NM}$-tautologies:

Clearly $M, w \not\models \Diamond\top$ and $M', w' \not\models \Box\bot$, and $M, M' \in \mathfrak{NM}$.

What is the deeper reason of this failure? This can be clarified by a short retrospection of the amalgamation technique in the proof of lemma 3.7. It turns out that we cannot define a unique root for amalgamations of the richer $\mathfrak{NM}$-models, as can be seen from the picture above. We cannot define a world which is smaller than the two worlds in this figure. If a world is smaller then the leftmost world in the picture then it should be a dead-end as well. If a world is smaller than the rightmost world, it should at least have an accessible world.

A small extension of **NM** which has the disjunction property, is the system **NM** + **D**, with $\mathbf{D} = \Gamma \vdash \Delta \implies \Box\Gamma \vdash \Diamond\Delta$. This additional rule enforces seriality of the accessibility relation (see also chapter 4); in this case we can define a unique root. We simply take a world sharing the structure of the right-hand world in the picture above. Furthermore, just like for $\mathfrak{N}$-amalgamations, we take its local valuation to be empty. This world satisfies the bisimulation extension constraint for every world in an arbitrary serial $\mathfrak{NM}$-model. This means, on the basis of the persistence of the full language and the uniqueness of the amalgamation technique, that this system must have the disjunction property.

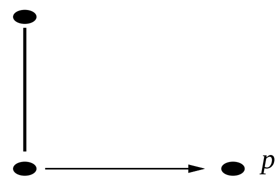
The system $\mathbf{NM}^\Box$

The system $\mathbf{NM}^\Box$ results from dropping the persistence claim for $\Diamond$. The underlying idea is that factual knowledge should behave in a conservative manner, while uncertainty might be eliminated. This means that the second claim for the $\mathfrak{NM}$-models is skipped: $\geq \circ R \subseteq R \circ \geq$. This wider class will be denoted by $\mathfrak{NM}^\Box$.

3.27. DEFINITION. A $\mathfrak{NM}^\Box$-model is a quadruple $\langle W, R, \leq, V\rangle$ such that $\langle W, R, V\rangle \in \mathfrak{M}$ and $\langle W, \leq, V\rangle \in \mathfrak{N}$ and $\leq \circ R \subseteq R \circ \leq$.

The interrelational constraint $\leq \circ R \subseteq R \circ \leq$ is the precise formal description of the construction-elimination dynamics which we have presented in chapter 1 (see page 34).

3.28. EXAMPLE. This example gives a simple illustration of the loss of persistence of the modal operator $\Diamond$. Consider the simple three world model in the following figure:



The arrow denotes the only accessibility in the model. The vertical line denotes the extension relation in the same way as in figure 2.27. Let furthermore the two leftmost worlds have an empty valuation, and let the right-hand world verify only p. It is not hard to show that the interrelational constraint on $\mathfrak{NM}^\Box$ holds, and furthermore the global valuation function is perfectly monotonic. Therefore, this model is a member of the class $\mathfrak{NM}^\Box$. The lower left world verifies $\Diamond p$, but its upper left extension does not. This pictures the loss of persistence of the proposition p by putting $\Diamond$ in front.

Validity over $\mathfrak{NM}^\Box$ is defined in the ordinary way. The corresponding notion of derivability is $\vdash_{NM^\Box}$. Like in the system $\mathbf{N}^\sim$ we have to replace R-TRUE by the weaker non-persistent R-TRUE-WEAK $\rightarrow$-rules. This makes the basic propositional language $\mathcal{L}$ and the constructive implication behave persistently. Syntactically, this means that all $\varphi \in \mathcal{L}^\rightarrow$ are $\mathbf{NM}^\Box$-equivalent to $\top \rightarrow \varphi$. In order to axiomatize also the persistence preservation of necessity $\Box$, we have to add one more rule. This can be formulated by a permission to distribute the necessity operator $\Box$ over the implication $\rightarrow$.

$$\frac{\Gamma \vdash \varphi \rightarrow \psi, \neg\Delta}{\Box\Gamma \vdash \Box\varphi \rightarrow \Box\psi, \neg\Box\Delta} \quad \text{R-DIS } \Box \rightarrow$$

3.29. OBSERVATION. The last rule R-DIS $\Box \rightarrow$ entails persistence preservation of $\Box$ indeed:

1.	$\Box\top \rightarrow \Box\varphi \vdash_{NM^\Box} \Box\top \rightarrow \Box\varphi$	START
2.	$\Box\top \rightarrow \Box\varphi \vdash_{NM^\Box} \top \rightarrow (\Box\top \rightarrow \Box\varphi)$	PERS $\rightarrow$ (1)
3.	$\vdash_{NM^\Box} \top$	R-TRUE $\top$
4.	$\vdash_{NM^\Box} \Box\top$	R-TRUE $\Box$ (3)
5.	$\Box\varphi \vdash_{NM^\Box} \Box\varphi$	START
6.	$\Box\top \rightarrow \Box\varphi \vdash_{NM^\Box} \Box\varphi$	L-TRUE $\rightarrow$ (4,5)
7.	$\top, \Box\top \rightarrow \Box\varphi \vdash_{NM^\Box} \Box\varphi$	L-MON (6)
8.	$\top \rightarrow (\Box\top \rightarrow \Box\varphi) \vdash_{NM^\Box} \top \rightarrow \Box\varphi$	R-TRUE-WEAK $\rightarrow$ (7)
9.	$\Box\top \rightarrow \Box\varphi \vdash_{NM^\Box} \top \rightarrow \Box\varphi$	CUT (2,8)
10.	$\top \rightarrow \varphi \vdash_{NM^\Box} \top \rightarrow \varphi$	START
11.	$\Box(\top \rightarrow \varphi) \vdash_{NM^\Box} \Box\top \rightarrow \Box\varphi$	R-DIS $\Box \rightarrow$ (10)

12. $\Box(\top \to \varphi) \vdash_{NM^\Box} \top \to \Box\varphi$ CUT (9,11)

This last $\mathbf{NM}^\Box$-sequent shows that if the persistence of a proposition φ can be derived, $\varphi \vdash_{NM^\Box} \top \to \varphi$, then also $\Box\varphi \vdash_{NM^\Box} \Box(\top \to \varphi)$ by R-TRUE $\Box$, and subsequently $\Box\varphi \vdash_{NM^\Box} \top \to \Box\varphi$ through application of CUT on 12 above and the latter $\mathbf{NM}^\Box$-sequent. This captures the persistence of $\Box\varphi$. The persistence preservation of $\vee$ and $\wedge$ can be derived in a much shorter way. The reader is invited to check the validity the following principle by himself.

$$\varphi \vdash_{NM^B} \top \to \varphi \;\&\; \psi \vdash_{NM^\Box} \top \to \psi \implies \varphi \vee \psi \vdash_{NM^\Box} \top \to (\varphi \vee \psi)$$

$$\varphi \vdash_{NM^B} \top \to \varphi \;\&\; \psi \vdash_{NM^\Box} \top \to \psi \implies \varphi \wedge \psi \vdash_{NM^\Box} \top \to (\varphi \wedge \psi)$$

What is left to be proved is the soundness of the logic $\mathbf{NM}^\Box$.

3.30. THEOREM. SOUNDNESS $\mathbf{NM}^\Box$

For all $\Gamma, \Delta \subseteq \mathcal{L}^{\Box,\to}$:

$$\Gamma \vdash_{NM^\Box} \Delta \Longrightarrow \Gamma \models_{\mathfrak{NM}^\Box} \Delta.$$

Proof. The soundness of all the rules of $\mathbf{N}^-$ and $\mathbf{M}$ follows immediately from earlier soundness results. What is left to show is the soundness of R-DIS $\Box \to$.

Suppose $\Gamma \models_{\mathfrak{NM}^\Box} \varphi \to \psi, \neg\Delta$ (6), and let w in $M = \langle W, R, \leq, V\rangle \in \mathfrak{NM}^\Box$ be a $\Box\Gamma$-world and a non-$\neg\Box\Delta$-world. These two latter assumption means that all worlds which are accessible from w in M are Γ-worlds and non-$\neg\Delta$-worlds. What is left to prove is $M, w \models \Box\varphi \to \Box\psi$.

If $M, w \not\models \Box\varphi \to \Box\psi$ we know that there exists an extension v of w in M such that $M, v \models \Box\varphi$ and $M, v \not\models \Box\psi$. This means that there exists at least one world u in M such that $R(v,u)$ and $M, u \not\models \psi$. Clearly, $(\leq \circ R)(w,u)$. The interrelational constraint for $\mathfrak{NM}^\Box$-models shows us that there exists u' in M such that $R(w,u')$ and $u' \leq u$ ($(R\circ \leq)(w,u)$). Because u is a φ-world ($R(v,u)$ and $M, v \models \Box\varphi$) and a non-ψ-world, we conclude $M, u' \not\models \varphi \to \psi$. This contradicts (6) ($u'$ is a Γ-world and a non-$\neg\Delta$-world), and therefore it must be the case that $M, w \models \Box\varphi \to \Box\psi$. ■

The disjunction property for $\mathbf{NM}^\Box$

With respect to the class $\mathfrak{NM}^\Box$ there exists an ultimate smallest model. It is the model as in the left-most figure in example 3.26. Notice that it is allowed for every world to have dead-end extensions, because they alway fulfill the single structural constraint: $\leq \circ R \subseteq R \circ \leq$. This does not automatically mean that $\mathbf{NM}^\Box$ has the disjunction property, since it is not fully persistent. Nevertheless, an amalgamation technique, on the basis of the minimal model given above, can be used to demonstrate the disjunction property for the persistent part.

3.31. PROPOSITION. Let $\Delta \subseteq \mathcal{L}^{\Box,\to}$ be a set of $\mathfrak{NM}^\Box$-persistent formulae. If $\models_{\mathfrak{NM}^\Box} \Delta$ then also $\models_{\mathfrak{NM}^\Box} \delta$ for certain $\delta \in \Delta$.

Up and down with uncertainties: Mud

The basic modal formalism which we employ for dynamic epistemic reasoning is based on the model class $\mathfrak{NM}^\Box$. This system, which is called **Mud**, is the modal extension of the propositional up and down system **ud**, but then interpreted over $\mathfrak{NM}^{\Box}$. This facilitates the kind of dynamic reasoning which we have in mind. Information grows through elimination of uncertainty, or possible worlds, and by constructive enrichment of these possibilities.

The language of the system **Mud** is $\mathcal{L}^{\Box,\uparrow,\downarrow} := \mathcal{L}^{\Box,[\ ]_u,[\ ]_d}$. The ordinary or static modal operator $\Box$ is interpreted in the normal way over the accessibility relation in the $\mathfrak{NM}^{\Box}$-models. The dynamic up- and downdate operators are interpreted over the structural extension order in this class of models, just like they have been interpreted for **ud** with respect to the Nelson class $\mathfrak{N}$. The corresponding notion of validity is written as $\models_{\mathfrak{NM}^{\Box,\uparrow,\downarrow}}$.

The system **Mud** consists of the **M**-rules, the **ud**-rules, an additional imitation of the rule R-DIS $\Box\ \to$ of the system $\mathbf{NM}^{\Box}$ to establish the persistence preservation of $\Box$, and an extra rule to capture the anti-persistence preservation of $\neg\Box$. This last rule is in fact a contra-positional variant of distribution of $\Box$ over updates. The following table presents these two 'new' **Mud**-rules.

3.32. TABLE.

$$\frac{\Gamma \vdash [\varphi]_u \psi, \neg\Delta}{\Box\Gamma \vdash [\Box\varphi]_u \Box\psi, \neg\Box\Delta}\ \text{DIS}\ \Box\ [\ ]_u \qquad \frac{\Gamma, \neg[\varphi]_u \psi \vdash \neg\Delta}{\Box\Gamma, \neg[\Box\varphi]_u \Box\psi \vdash \neg\Box\Delta}\ \text{C-DIS}\ \Box\ [\ ]_u$$

3.33. THEOREM. SOUNDNESS **Mud**

For all $\Gamma, \Delta \subseteq \mathcal{L}^{\Box,\uparrow,\downarrow}$: $\Gamma \vdash_{M_{ud}} \Delta \implies \Gamma \models_{\mathfrak{NM}^{\Box,\uparrow,\downarrow}} \Delta$.

Proof. The soundness of DIS $\Box\ [\ ]_u$ can be proved by an imitation of R-DIS $\Box \to$ in the system $\mathbf{NM}^{\Box}$ in the previous subsection. The soundness of all the other rules follows from earlier soundness results of **M** and **ud**. Here we illustrate the soundness of C-DIS $\Box\ [\ ]_u$.

Suppose that $\Gamma, \neg[\varphi]_u \psi \models_{\mathfrak{NM}^{\Box,\uparrow,\downarrow}} \neg\Delta$ (8), and let w in $M = \langle W, R, \leq, V\rangle \in \mathfrak{NM}^{\Box}$ be a $\Box\Gamma$-world and a non-$\neg\Box\Delta$-world. We need to show that $M, w \not\dashv [\Box\varphi]_u \Box\psi$.

Suppose that this is not the case. This implies the existence of a world $v \in W$ such that $M, v \models \Box\varphi$ and $M, v \dashv \Box\psi$. This shows that there exists $u \in W$ such that $(\leq \circ R)(w, u)$ $(R(v, u))$ with $M, u \models \varphi$ and $M, u \dashv \psi$. The interrelational constraint of the class $\mathfrak{NM}^{\Box}$ ensures $(R \circ \leq)(w, u)$, and so there exists $u' \in W$ such that $R(w, u')$ and $u' \leq u$. Clearly, $M, u' \dashv [\varphi]_u \psi$ (9). Because w is a $\Box\Gamma$-world and a non-$\neg\Box\Delta$-world, we know that u' must be a Γ-world and a non-$\neg\Delta$-world, and so (8) contradicts (9). This proves that $M, w \dashv [\Box\varphi]_u \Box\psi$ cannot be the case. ■

3.34. OBSERVATION. In order to make C-DIS $\Box\ [\ ]_u$ somewhat more transparent we present a '$\Diamond$-version' of this rule:

$$\frac{\Gamma, \langle\varphi\rangle_u \psi \vdash \Delta}{\Box\Gamma, \langle\Box\varphi\rangle_u \Diamond\psi \vdash \Diamond\Delta}\ \text{DIS}\ \Box\ \langle\ \rangle_u.$$

The persistence preservation of the modal operator $\Box$ can be shown by means of an analogous derivation to the one made in observation 3.29.

$$\varphi \vdash_{M_{ud}} [\ ]_u \varphi \Longrightarrow \Box\varphi \vdash_{M_{ud}} [\ ]_u \Box\varphi$$

By means of the rule C-DIS $\Box\ [\ ]_u$ we have settled the anti-persistence preservation of $\Diamond$. Anti-persistence of a proposition φ in syntactic terms means

$\varphi \vdash_{Mud} [\ ]_d \varphi$. What we need to show is that the anti-persistence of $\Diamond\varphi$ can be derived from the anti-persistence of φ.

1.	$\varphi \vdash_{Mud} [\ ]_d \varphi$	assumption
2.	$\varphi, \top \vdash_{Mud} [\ ]_d \varphi$	L-MON (1)
3.	$\langle\ \rangle_u \varphi \vdash_{Mud} \langle\ \rangle_u [\ ]_d \varphi$	L-TRUE $\langle\ \rangle_u$ (2)
4.	$\langle\ \rangle_u \varphi \vdash_{Mud} \varphi$	CUT (DUALITY,3)
5.	$\langle \Box\top \rangle_u \Diamond\varphi \vdash_{Mud} \Diamond\varphi$	DIS $\Box\langle\ \rangle_u$ (4)
6.	$\vdash_{Mud} \top$	L-TRUE $\top$
7.	$\vdash_{Mud} \Box\top$	R-TRUE $\Box$
8.	$\top \vdash_{Mud} \Box\top$	L-MON (7)
9.	$\langle\ \rangle_u \Diamond\varphi \vdash_{Mud} \Diamond\varphi$	proposition 3.19 (5,8)
10.	$\langle\ \rangle_u \Diamond\varphi \vdash_{Mud} \bot, \Diamond\varphi$	R-MON (9)
11.	$[\ ]_d \langle\ \rangle_u \Diamond\varphi \vdash_{Mud} [\ ]_d \Diamond\varphi$	L-TRUE $[\ ]_d$
12.	$\Diamond\varphi \vdash_{Mud} [\ ]_d \Diamond\varphi$	CUT (DUALITY,11)

Intuitionistic modal logics

The constructive modal logics which have been discussed in this chapter are relatively unknown. As far as we know, only [Routley 1974] discusses an **S4**-like extension of Nelson's logic.

Intuitionistic modal logic however, has a relatively rich history. In [Prior 1979] a kind of intuitionistic **S5** has already been discussed axiomatically. In [Bull 1965] one finds algebraic semantics for these kind of systems. In [Fischer Servi 1981] these logics have been provided a clear Kripke semantics. A general framework for intuitionistic modal logics in terms of possible world semantics has been proposed by Božić and Došen [Božić & Došen 1984] [Došen 1985] . In these papers different modal extensions of Heyting's logic **H** have been proposed for $\Box$ and $\Diamond$ separately. These are elegant extensions, just like **NM**, by full persistence requirements. The logic $\mathbf{H}^{\Box}$ consists of the system **H** with an additional restrictive use of the modal rule R-TRUE $\Box$ [13]:

$$\frac{\Gamma \vdash \varphi}{\Box\Gamma \vdash \Box\varphi}.$$

Its models coincide with $\mathfrak{NM}^{\Box}$. Only the falsity conditions are omitted, as falsity is not an intuitionistic concept. The logic $\mathbf{H}^{\Diamond}$ is the logic which consists of **H** with a restrictive use of the rule L-TRUE $\Diamond$:

$$\frac{\varphi \vdash \Delta}{\Diamond\varphi \vdash \Diamond\Delta}.$$

[13] The axiomatization of the intuitionistic modal logics in [Božić & Došen 1984] are in Hilbert-style.

Its corresponding semantics consists of constructive possible worlds models with the other bisimulation requirement as the additional structural constraint: $\geq \circ R \subseteq R \circ \geq$.

The combination of both systems, the logic $\mathbf{H}^{\Box\Diamond}$, evolves from enforcing the complete bisimulation constraint on its possible world models, just like for the class $\mathfrak{NM}$-models above. A sequential derivation system can be obtained by putting $\mathbf{H}$ and the two modal rules R-TRUE $\Box$ and L-TRUE $\Diamond$ together. From an intuitionistic point of view this logic is not satisfactory, because it lacks the disjunction property.

In [Plotkin & Stirling 1986] a somewhat different system has been proposed, by interpreting the necessity operator like a constructive intensional universal quantifier. In intuitionistic logic a universal quantified statement $\forall x\ \varphi(x)$ is taken to be true if there exists a function, which can be applied to every individual and then yields a proof for this object to have the property φ. Such an individual does not have to be constructed on the moment that the proof of $\forall x\ \varphi(x)$ has been found. The intensional status of such a function relies on possibly 'hypothetical' individuals in its domain. An analogous definition of truth of a proposition $\Box\varphi$ is then given by the truth of φ with respect to all future accessibilities. In our formalism, such interpretation of the necessity of a proposition φ coincides with $\top \to \Box\varphi$.

The models of this logic coincide with the models for $\mathbf{H}^{\Diamond}$, which guarantees $\Diamond$-persistence. This logic obeys the disjunction property.

The logic of Plotkin and Stirling has further been studied in Simpson's thesis [Simpson 1993][14]. On the basis of earlier combinations of intuitionistic logic and temporal logic [Ewald 1978] [Ewald 1986], an additional semantic constraint has been motivated there: $R\circ \leq \subseteq \leq \circ R$. This is a pure ideological matter. For a minimal intuitionistic modal logic this additional constraint does not lead to additional logical structure, while in the combination with temporal logic, where 'past'-operators range over the converse of the accessibility R, this additional constraint yields persistence for existential propositions about the past.

In [Wijesekera 1990] one finds an intuitionistic modal logic which does not use any model-theoretic relational constraints. Only the monotonicity of global valuations is maintained. Nevertheless, the full language is persistent by interpreting $\Box\varphi$ in the same way as the universal quantifier in intuitionistic predicate logic, while a proposition $\Diamond\varphi$ is interpreted as 'for ever possibly φ', which coincides with our $\top \to \Diamond\varphi$.

An interesting application of intuitionistic modal logics can be found in [Stirling 1987]. In this paper these kind of logics have been used to give a modal characterization of certain process algebras. Intuitionistic modal logics have also been proposed by for epistemic uses. Recent examples are [Aiello, Amati & Pirri 1991] and [Williamson 1992].

[14] This dissertation contains a detailed overview of intuitionistic modal logics.

Appendix

3.5 Translations of constructive logics

The issue which we want to discuss in this appendix section is how partial intensional systems can be translated into classical modal formalisms. A very well-known example of such a translation is Gödel's [Gödel 1933] embedding of intuitionistic logic into the classical modal logic **S4**. This system consists of the minimal classical modal logic **K** and two additional axioms: $\Box\varphi \vdash \varphi$ and $\Box\varphi \vdash \Box\Box\varphi$.

The first question which arises immediately, is whether such an embedding is also possible for Nelson's system **N**. We will show that Gödel's translation function can be straightforwardly extended for this purpose. In fact, we will define two translation functions, one for truth and one for falsity[15]. This section gives an elaborate presentation of the embedding result.

What about the other logics which have been dealt with in the previous sections of this chapter? After the **N**-**S4** result, we will shortly show that a very straightforward extension of the translation procedure encodes $\mathbf{N}^{\sim}$ satisfactorily into **S4**. Furthermore, we show a similar dichotomous translation for the up-and-down system of **ud** into the 'temporal' or 'back-and-forth' version of **S4** of [van Benthem 1991b]. Furthermore, we present how the constructive modal logics **NM** and $\mathbf{NM}^{\Box}$ can be interpreted into classical bi-modal logics of which one of the modalities is an **S4**-operator. Finally, we will show a similar result for **Mud** into a classical bi-modal logic with two 'temporal' **S4**-operators.

The proofs of the translation procedures are fully model-theoretic. We transform the models of the original logic into models of the embedding logic, and show that this transformation preserves counter-models, which means that if a counter-model can be given for a proposition φ in the original class, then the transformation of the counter-model is also counter-model of the translation of φ. This procedure is then completed by a converse transformation which also preserves counter-models with respect to the inverse translation.

[15] The same kind of dichotomous translation function has been given in [van Benthem 1986] for embedding Veltman's data logic into the modal system **S4.1**, i.e. **S4** + $\Box\Diamond\varphi \vdash \Diamond\Box\varphi$.

A very simple model transformation for going from partial to total modal logics, is the following notion of model completion.

3.35. DEFINITION. A *completion* of a partial Kripke model $M = \langle W, R, V\rangle$ is a total Kripke model $M^t = \langle W, R, V^t\rangle$ such that

$$V(w) \sqsubseteq V^t(w) \text{ and } V^t(w) \in \mathfrak{T} \text{ for all } w \in W.$$

3.36. OBSERVATION. Let $M = \langle W, R, V\rangle \in \mathfrak{M}$ and let $M^t \in \mathfrak{K}$ be a completion of M. For all $\varphi \in \mathcal{L}^{\Box}$:

$$M, w \models \varphi \Rightarrow M^t, w \models \varphi.$$

Proof. The identity relation I_W on W is a bisimulation between M and M^t, and for all $w \in W$ we have $V(w) \sqsubseteq V^t(w)$. According corollary 2.46 this means that $M, w \models \varphi \Rightarrow M^t, w \models \varphi$ for all $w \in W$. ■

Embedding N into S4

S4 embraces **N** in a natural way. The class of corresponding total Kripke models have the same frame-structure as **N**: pre-orders.

3.37. DEFINITION. An $\mathfrak{S}4$-model is a triple $\langle W, R, V\rangle \in \mathfrak{K}$ such that R is a pre-order on W.

Note that $\mathfrak{S}4$ is not a subset of $\mathfrak{N}$. The monotonicity constraint is not required.

Apart from the frame structural similarity, there are also clear formal resemblances of the use of the two logics. Nelson's logic represents a logic of proofs and refutations, while **S4** has been used for classical epistemic logics [Hintikka 1962]. The Gödel-translation of intuitionistic logic relates the formulation of proof in intuitionistic logic with the knowledge operator $\Box$ in **S4**. The following translation procedure extends this idea for Nelson's notion of refutation by an additional 'negative' translation function. The following table presents this dichotomy as two functions $\mathbf{t}^+, \mathbf{t}^- : \mathcal{L}^{\rightarrow} \longrightarrow \mathcal{L}^{\Box}$.

3.38. TABLE.

$\mathbf{t}^+(p) = \Box p \quad (p \in I\!\!P)$	$\mathbf{t}^-(p) = \Box\neg p \quad (p \in I\!\!P)$
$\mathbf{t}^+(\bot) = \bot$	$\mathbf{t}^-(\bot) = \top$
$\mathbf{t}^+(\neg\varphi) = \mathbf{t}^-(\varphi)$	$\mathbf{t}^-(\neg\varphi) = \mathbf{t}^+(\varphi)$
$\mathbf{t}^+(\varphi \wedge \psi) = \mathbf{t}^+(\varphi) \wedge \mathbf{t}^+(\psi)$	$\mathbf{t}^-(\varphi \wedge \psi) = \mathbf{t}^-(\varphi) \vee \mathbf{t}^-(\psi)$
$\mathbf{t}^+(\varphi \rightarrow \psi) = \Box(\neg\mathbf{t}^+(\varphi) \vee \mathbf{t}^+(\psi))$	$\mathbf{t}^-(\varphi \rightarrow \psi) = \mathbf{t}^+(\varphi) \wedge \mathbf{t}^-(\psi)$

Completion of models preserves counter-models for the inverse of the translation functions $\mathbf{t}^+$ and $\mathbf{t}^-$.

3.39. LEMMA. Let $M = \langle W, \leq, V\rangle \in \mathfrak{N}$ and let M^t be a completion of M. For all $\varphi \in \mathcal{L}^{\rightarrow}$:

$$M, w \models \varphi \Rightarrow M^t, w \models \mathbf{t}^+(\varphi) \quad \text{and} \quad M, w \mathrel{=\!\!\!|} \varphi \Rightarrow M^t, w \models \mathbf{t}^-(\varphi).$$

Proof. By induction on the construction of $\mathcal{L}^{\Box}$-formulae. The basic step follows immediately from the monotonicity of V.

> Suppose $M, w \models p$. This means also $V(v)(p) = 1$ for all $v \geq w$. Therefore $V^t(p) = 1$ for all $v \geq w$. This implies $M^t, w \models \Box p = \mathbf{t}^+(p)$. The $\dashv$-case runs analogously.

The step $\bot$ is immediate, and the cases $\neg$ and $\wedge$ follow instantaneously from the induction hypothesis. Also the $\dashv$-case for $\rightarrow$ is trivial. What is left to prove is the $\models$-case for $\rightarrow$.

> Suppose $M, w \models \varphi \rightarrow \psi$. This means $M, v \models \varphi \Rightarrow M, v \models \psi$ for all $v \geq w$. The induction hypothesis establishes $M^t, v \models \mathbf{t}^+(\varphi) \Rightarrow M^t, v \models \mathbf{t}^+(\varphi)$ for all $v \geq w$. The totality of M^t guarantees $M^t, v \models \neg\mathbf{t}^+(\varphi) \vee \mathbf{t}^+(\psi)$ for all $v \geq w$, and therefore $M^t, w \models \Box(\neg\mathbf{t}^+(\varphi) \vee \mathbf{t}^+(\psi)) = \mathbf{t}^+(\varphi \rightarrow \psi)$.

■

For preservation of counter-models in the other direction we use the following model transformation.

3.40. DEFINITION. The *constructification* of an $\mathfrak{S}4$-model $M = \langle W, \leq, V \rangle$ is a partial Kripke model $M^c = \langle W, \leq, V^c \rangle$ such that for all $p \in I\!P$:

$$V^c(w)(p) = \begin{cases} 1 & M, w \models \Box p \ , \\ 0 & M, w \models \Box\neg p \ , \text{ and} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

3.41. OBSERVATION. $M \in \mathfrak{S}4 \implies M^c \in \mathfrak{N}$ for all Kripke models M. Furthermore, an $\mathfrak{S}4$-model is always a totalization of its constructification.

Proof. Let $M = \langle W, \leq, V \rangle \in \mathfrak{S}4$. In order to show the first claim, we only need to show the monotonicity of V^c over $\leq$. Suppose $v \leq w$, and let $V^c(v)(p) = 1$. According to the definition of V^c, this means $M, u \models p$ for all $u \geq v$. If $u' \geq w$ then also $M, u' \models p$, by the transitivity of $\leq$ $(u' \geq v)$. This entails $M, w \models \Box p$ and $V^c(w)(p) = 1$. $V^c(v)(p) = 0 \Rightarrow V^c(w)(p) = 0$ can be proved in the same way. In other words, $V^c(v) \sqsubseteq V^c(w)$.

The second claim above can be proved by this monotonicity of V^c. We leave it to the reader. ■

3.42. LEMMA. Let $M = \langle W, \leq, V \rangle \in \mathfrak{S}4$. For all $\varphi \in \mathcal{L}^{\rightarrow}$ and $w \in W$

$$M, w \models \mathbf{t}^+(\varphi) \Rightarrow M^c, w \models \varphi \quad \text{and} \quad M, w \models \mathbf{t}^-(\varphi) \Rightarrow M^c, w \dashv \varphi.$$

Proof. By induction again. We give only a short demonstration of $\models$-case in the $\rightarrow$-step. All the other steps are straightforward.

> Suppose $M^c, w \not\models \varphi \rightarrow \psi$. This means that there exists $v \geq w$ such that $M, v \models \varphi$ and $M^c, v \not\models \psi$. The induction hypothesis tells us $M, v \not\models \mathbf{t}^+(\psi)$, while the previous lemma shows $M, v \models \mathbf{t}^+(\varphi)$ (M is a totalization of M^c). The totality of M implies $M, v \not\models \neg\mathbf{t}^+(\varphi) \vee \mathbf{t}^+(\psi)$, and therefore $M, w \not\models \Box(\neg\mathbf{t}^+(\varphi) \vee \mathbf{t}^+(\psi))$.

■

Because M in lemma 3.42 above is also a totalization of M^c, the converse of this lemma also holds, due to lemma 3.39. The following definitions present some simple transformations of models such that preservation of counter-models from the class $\mathfrak{N}$ to the embedding class $\mathfrak{S}4$ is guaranteed.

3.43. DEFINITION. The *doubling* of a Nelson model $M = \langle W, \leq, V\rangle$ is the Nelson model $M^{II} = \langle W^{II}, \leq^{II}, V^{II}\rangle$ with

$$W^{II} = \{w^+ \mid w \in W\} \cup \{w^- \mid w \in W\},$$

$$\leq^{II} = \{\langle w^x, v^y\rangle \mid w \leq v, x, y \in \{+,-\}\}$$

$$V^{II}(w^x) = V(w) \text{ for all } w \in W \text{ and } x \in \{+,-\}.$$

3.44. OBSERVATION. This doubling of a Nelson model M is indeed a Nelson model itself. The relation $\leq^{II}$ is a pre-order and V^{II} is monotonic over $\leq^{II}$. Furthermore, all splitted worlds in the doubling have to contain the same information as the original ones. Formally speaking, for all $\varphi \in \mathcal{L}^{\rightarrow}$:

$$M, w \models \varphi \;\Leftrightarrow\; M^{II}, w^+ \models \varphi \;\Leftrightarrow\; M^{II}, w^- \models \varphi.$$

3.45. DEFINITION. The *doubling completion* of a Nelson model $M = \langle W, \leq, V\rangle$ is a $\mathfrak{S}4$-model $M^{dc} = \langle W^{II}, R^{II}, V^{dc}\rangle$ such that $R^{II} = \leq^{II}$ and for all $p \in I\!P$

$$V(w^+)(p) = \begin{cases} V(w)(p) & \text{if } p \in \mathfrak{Dom}(V(w)) \\ 1 & \text{otherwise.} \end{cases}$$

$$V(w^-)(p) = \begin{cases} V(w)(p) & \text{if } p \in \mathfrak{Dom}(V(w)) \\ 0 & \text{otherwise.} \end{cases}$$

3.46. OBSERVATION. Doubling completion is a completion, and therefore, $M^{dc} \in \mathfrak{S}4$. Moreover, $(M^{dc})^c = M^{II}$.

The last equation in the observation above takes care of the above-mentioned preservation of counter-models.

3.47. THEOREM. For all $\varphi \in \mathcal{L}^{\rightarrow}$ we obtain $\models_{\mathfrak{N}} \varphi \Leftrightarrow \models_{\mathfrak{S}4} \mathbf{t}^+(\varphi)$.

Proof. Suppose $\not\models_{\mathfrak{S}4} \mathbf{t}^+(\varphi)$. This means that there exists a model $M \in \mathfrak{S}4$ with a non-$\mathbf{t}^+(\varphi)$-world w: $M, w \not\models \mathbf{t}^+(\varphi)$. Application of lemma 3.42 proves $M^c, w \not\models \varphi$, and because $M^c \in \mathfrak{N}$ we obtain $\not\models_{\mathfrak{N}} \varphi$.

Suppose $\not\models_{\mathfrak{N}} \varphi$. In other words $M, w \not\models \varphi$ for certain $M \in \mathfrak{N}$ and w in M. According to observation 3.44 this implies $M^{II}, w^+ \not\models \varphi$. Observation 3.46 proves that $(M^{dc})^c, w^+ \not\models \varphi$. Lemma 3.42 completes the argument: $M^{dc}, w^+ \not\models \mathbf{t}^+(\varphi)$, which means $\not\models_{\mathfrak{S}4} \mathbf{t}^+(\varphi)$ $(M^{dc} \in \mathfrak{S}4)$. ■

3.48. COROLLARY. This conclusion can also be generalized to sets of formulae. Let $\mathbf{t}^+(\Gamma) := \{\mathbf{t}^+(\gamma) \mid \gamma \in \Gamma\}$ if $\Gamma \subseteq \mathcal{L}^{\rightarrow}$. This definition establishes for all $\Gamma, \Delta \subseteq \mathcal{L}^{\rightarrow}$:

$$\Gamma \models_{\mathfrak{N}} \Delta \iff \mathbf{t}^+(\Gamma) \models_{\mathfrak{S}4} \mathbf{t}^+(\Delta).$$

In chapter 5 we will prove the completeness of **N** with respect to $\mathfrak{N}$-validity. From this forthcoming result, in combination with the translation result above, we conclude that the translation method also works for **N**-sequents:

$$\Gamma \vdash_N \Delta \iff \mathbf{t}^+(\Gamma) \vdash_S 4\mathbf{t}^+(\Delta).$$

Translating $\mathbf{N}^\sim$ and ud

A straightforward continuation of the translation of **N** into **S4** yields a similar result for the system $\mathbf{N}^\sim$. The simple extension of the functions $\mathbf{t}^+$ and $\mathbf{t}^-$ with two additional clauses: $\mathbf{t}^+(\sim\varphi) = \neg\mathbf{t}^+(\varphi)$ and $\mathbf{t}^-(\sim\varphi) = \mathbf{t}^+(\varphi)$ is satisfactory. It is not hard to verify that the *constructification* preserves $\mathbf{N}^\sim$-counter-models.

3.49. OBSERVATION. For all $M \in \mathfrak{S}4$ and all $\varphi \in \mathcal{L}^{\sim,\rightarrow}$:

$$M, w \models \mathbf{t}^+(\varphi) \ \Leftrightarrow\ M^c, w \models \varphi \text{ for all } w \text{ in } M.$$

A translation by means of a similar extension of the translation function $\mathbf{t}^+$ and $\mathbf{t}^-$ can be given for the system **ud**. The only difference is that we have to make the embedding logic somewhat wider as well. For this purpose we use the 'temporal' bi-directional version of **S4**. This system is called $\mathbf{S4}^2$, and has been employed for up-and-down dynamic modal logics in [van Benthem 1991b] and [de Rijke 1992]. This system has two modal operators: $\Box_{up}$ and $\Box_{down}$. The first operator is interpreted the same way as $\Box$ over $\mathfrak{S}4$. This is the universal 'upward' or 'future' operator. The 'downward' or 'past' operator is interpreted in the same way over the converse of the accessibility relation. In fact, it coincides with our $[\ ]_d$-operator. The derivation systems consists of **K** for the two modal operators, the two **S4**-axioms for $\Box_{up}$ and two temporal duality axioms: $\neg\Box_{up}\neg\Box_{down}\varphi \vdash \varphi$ and $\neg\Box_{down}\neg\Box_{up}\varphi \vdash \varphi$.

The translation function for embedding **ud** needs the following supplement.

3.50. TABLE.

$$\mathbf{t}^+([\varphi]_u\psi) = \Box_{up}(\neg\mathbf{t}^+(\varphi) \vee \mathbf{t}^+(\psi)) \qquad \mathbf{t}^-([\varphi]_u\psi) = \neg\Box_{up}(\neg\mathbf{t}^+(\varphi) \vee \neg\mathbf{t}^-(\psi))$$

$$\mathbf{t}^+([\varphi]_d\psi) = \Box_{down}(\mathbf{t}^+(\varphi) \vee \mathbf{t}^+(\psi)) \qquad \mathbf{t}^-([\varphi]_d\psi) = \neg\Box_{down}(\mathbf{t}^+(\varphi) \vee \neg\mathbf{t}^-(\psi))$$

3.51. OBSERVATION. For all $M \in \mathfrak{S}4$ and all $\varphi \in \mathcal{L}^{\uparrow,\downarrow}$:

$$M, w \models \mathbf{t}^+(\varphi) \ \Leftrightarrow\ M^c, w \models \varphi \text{ for all } w \text{ in } M.$$

Translations of constructive modal logics

The constructive modal logics **NM**, $\mathbf{NM}^\Box$ and **Mud** can be encoded in terms of classical bi-modal logics, of which one modal operator is **S4**-like in order to catch the constructive part of these systems. We write this latter operator as $\Box_{up}$ and the other as $\Box$. The structural interplay between these operators depends on the interrelation of accessibility and the information structure of the models of the logic which we wish to translate. The translation functions are straightforward extensions of the translations in the preceding subsections. The additional clauses are $\mathbf{t}^+(\Box\varphi) = \Box\mathbf{t}^+(\varphi)$ and $\mathbf{t}^-(\Box\varphi) = \neg\Box\mathbf{t}^+(\varphi)$.

The embedding of the fully persistent system **NM** is a classical bi-modal logic with two additional **S4**-axioms for $\Box_{up}$ and two axioms for catching the bisimulation restriction for the information structure of $\mathfrak{NM}$-models: $\Box\Box_{up}\varphi \vdash \Box_{up}\Box\varphi$

and $\Diamond_{up}\Box\varphi \vdash \Box\Diamond_{up}\varphi$. $\mathbf{NM}^{\Box}$ can be embedded into the system which drops the last axiom, and finally, **Mud** can be translated into the same strengthening of $\mathbf{S4}^2$. The following table pictures the precise embedding results.

3.52. TABLE.

NM	$\mathbf{K} + \mathbf{S4} + \Box\Box_{up} \Rightarrow \Box_{up}\Box + \Diamond_{up}\Box \Rightarrow \Box\Diamond_{up}$
$\mathbf{NM}^{\Box}$	$\mathbf{K} + \mathbf{S4} + \Box\Box_{up} \Rightarrow \Box_{up}\Box$
Mud	$\mathbf{K} + \mathbf{S4}^2 + \Box\Box_{up} \Rightarrow \Box_{up}\Box$

These translation results can be established by the same model-theoretic constructions as in the previous subsections.

Chapter 4

Constructive Communication

In this final chapter of part I we will present epistemic dynamic systems on the basis of the modal formalisms of the previous chapters. As argued in section 1.3, these systems are meant as formal calculi for logics of measuring the epistemic effects of communicative actions between agents. According to dynamic semanticists, the meaning of an utterance is the same as such an effect.

Most dynamic semantic theories deal with only one interpreting agent, that is they model the cognitive changes that an agent makes during the input of a text or message. In this chapter we present dynamic logics in a communicative setting. The basic dynamics of these logics is the elimination-construction dynamics of the constructive modal systems $\mathbf{NM}^{\Box}$ and **Mud** of the previous chapter. The dynamic meaning of a message is then no longer restricted to the cognitive effect on a single agent, but is captured as the full distributive effect on the group of communicating partners. As we already explained in chapter 1, even the interpretation of a simple assertion by an agent to another agent requires administration of the epistemic effect on the receiver *and* the sender.

We will start with a simple static epistemic logic. This modal formalism is a straightforward multiple agent, or *poly-modal*, extension of the elementary partial modal logic **M**. It is a full introspective extension of **M** with individualized modalities. This system is called $\mathbf{E}_A$, which is in fact a partial variant of the poly-modal extension of standard modal logic **KD45**.

The basic dynamic or constructive epistemic system is an up-down extension of this system $\mathbf{E}_A$. As a matter of fact it is a full introspective multiple agent variation of the system **Mud**. This system is called **CCC**, or shorter $\mathbf{C}^3$, after the title of this thesis. These systems $\mathbf{E}_A$ and $\mathbf{C}^3$ will be presented in section 4.1.

The other sections of this chapter present a family of $\mathbf{C}^3$-extensions. They contain extra technical facilities which are specially meant for interpretation of communicative actions. In section 4.2 we focus on the two essential additional modalities which have been discussed earlier in section 1.4. First, we establish an extension with additional mutual belief operators ($\mathbf{C}^{3*}$). These operators are meant to describe ideal transfer of information. The epistemic effect of an assertion φ of a sender a to a receiver b is then identified as the new mutual belief of the group $\{a, b\}$ of the information φ.

The second modal extension, which we will meet in section 4.3, contains intentional operators ($\mathbf{C}_i^3$). These modalities are essential for sensible comprehension of questions and also for more skeptical interpretation of assertions. This additional modality is interpreted over individual preferential states of information.

Such preferential modalities are needed to give more plausible interpretations of communicative actions. In the mutual belief interpretation of assertion given above, we have maximized the epistemic effect but totally ignored the intentional effect of the message. In real-life dialogues, transferred intentions are not less important than the epistemic conveyance. Communicative actions like questions illustrate this clearly. Even a pure formalistic approach as ours cannot ignore the importance of intentions, if we wish to assign a sensible dynamic denotation to actions which are in principle meant to convey the goals of the performer or to fulfill the goals of the receiver.

In fact, all communicative actions should be interpreted in terms of the motives of the dialogue partners. Even the simple assertion example above, could further be constrained by the precondition that the sender also intended to bring the message into the mutual belief of him and the receiver.

Of course, these kind of elaboration of the modal framework can become very entangling. We will try to avoid such confusion here, and we will restrict ourselves in pointing out how different dynamic denotations of communicative actions can be stipulated in terms of our partial modal framework.

In the last section we will discuss such different axiomatic extensions of the basic $\mathbf{C}^3$-calculi. These axioms are meant to capture different pragmatic principles of cooperative communicative behavior. They are postulates of the interrelation of intentions and beliefs, and we will show how such axioms enforce semantic constraints. They restrain the interplay of doxastic alternatives, the dynamic information structure and preferential worlds.

We will illustrate that the use of partial worlds for encoding pragmatic principles is an advantage. The structural flexibility of partial worlds makes it possible to formulate sensible weakenings of well-known postulates of cooperation between dialogue partners.

The last item of this section is devoted to two other $\mathbf{C}^3$-extensions, $\mathbf{C}^{3R}$ and $\mathbf{C}_\Omega^3$, where a non-empty set of total worlds is added to the partial possible world models. In $\mathbf{C}^{3R}$ these total worlds are meant to distinguish a real physical world as well. The system $\mathbf{C}_\Omega^3$ is meant to deal with one extraordinary agent Ω. This agent knows everything which holds in the real world, that is, he is omniscient with respect to ontological information. However, it may be ignorant with respect to the epistemic information which belongs to other communicating agents. This logic is meant as a contribution to the formalization of human-machine communication, or data-base querying, where such ontological omniscient capacity is often attributed to the machine. In such a case, the real world consists just of some database, of which the machine has complete knowledge, and the user wants to retrieve some information from this database by communication with the machine. Of course, this machine does not have complete knowledge of how much information the user has.

This final chapter of part I will be wound up with some general remarks and speculations about possible other directions for partial modal logics to achieve more fine-grained interpretations of communication.

4.1 Partial and constructive epistemic logics

Multiple agent epistemic reasoning on the basis of modal logics is normally accommodated through allowing several accessibility relations, one for each agent, in the possible worlds models. The underlying logics are called poly-modal logics, and contain multiple modal operators $\Box_a$ which are meant to keep track of the beliefs of an agent a.

Partial poly-modal logic

Let us first briefly present the minimal partial poly-modal logic. The following definition gives the formal picture of its semantics.

4.1. DEFINITION. A poly-modal partial Kripke model is a triple of the form $M = \langle W, \{R_a\}_{a\in A}, V\rangle$ with A being a non-empty finite set and $M_a := \langle W, R_a, V\rangle \in \mathfrak{M}$ for all $a \in A$. The collection of poly-modal Kripke models is denoted by $\mathfrak{M}_{\mathfrak{A}}$.

The set A should here be thought of as a set of agents[1]. The language of partial poly-modal logics is $\mathcal{L}_A$, which is an abbreviation of $\mathcal{L}^{\{\Box_a\}_{a\in A}}$. The proposition $\Diamond_a\varphi$ denotes $\neg\Box_a\neg\varphi$. For all non-empty subsets X of A we use $\Box_X\varphi$ as an abbreviation of $\bigwedge_{a\in X}\Box_a\varphi$. $\Diamond_X\varphi$ abbreviates $\neg\Box_X\neg\varphi$. $\Box_X\varphi$ means that all agents of the group X believe φ, while $\neg\Box_X\varphi$ means that at least one of the agents of this group has a counter-model of φ in mind (actively disbelieves φ).

4.2. TABLE. The modalities $\Box_a$ are all interpreted in the same way as the singular $\Box$-operator in terms of the single accessibility in the models in $\mathfrak{M}$:

$$M, w \models \Box_a\varphi \Leftrightarrow \forall v \in W : R_a(w, v) \Rightarrow M, v \models \varphi, \text{ and}$$

$$M, w \dashv \Box_a\varphi \Leftrightarrow \exists v \in W : R_a(w, v) \;\&\; M, v \dashv \varphi.$$

The corresponding derivation system, the minimal partial poly-modal system, is called $\mathbf{M}_A$.

4.3. TABLE. The minimal poly-modal system $\mathbf{M}_A$ is the system consisting of the rules for $\mathbf{P}$ and the individualized versions of the modal rules of the system $\mathbf{M}$:

$$\frac{\Gamma, \neg\varphi \vdash \neg\Delta \quad a \in A}{\Box_a\Gamma, \neg\Box_a\varphi \vdash \neg\Box_a\Delta}\ \text{L-FALSE}\ \Box_a \qquad \frac{\Gamma \vdash \varphi, \neg\Delta \quad a \in A}{\Box_a\Gamma \vdash \Box_a\varphi, \neg\Box_a\Delta}\ \text{R-TRUE}\ \Box_a$$

[1] Another application might be a partial variant of Pratt's propositional dynamic logic. In such a case, the set A stands for a set of atomic programs. See also observation 4.25 later on.

In the rules for the poly-modal formalisms in the sequel of this chapter, we will no longer specify $a \in A$ in the conditional part of the rule. If a appears in a rule, we take it to be an arbitrary member of A. For arbitrary non-empty sets of agents ($\subseteq A$) we will use X. For such a group of agents we will also use the following introduction rules of $\Box_X$. They are derivable in the system $\mathbf{M}_A$.

$$\frac{\Gamma, \neg\varphi \vdash \neg\Delta}{\Box_X\Gamma, \neg\Box_X\varphi \vdash \neg\Box_X\Delta}\ \text{L-FALSE}\ \Box_X \qquad \frac{\Gamma \vdash \varphi, \neg\Delta}{\Box_X\Gamma \vdash \Box_X\varphi, \neg\Box_X\Delta}\ \text{R-TRUE}\ \Box_X$$

We also make use of reformulations of these rules, where the $\neg\Box$ occurrences are replaced by $\Diamond$-operators. For example, $\Gamma, \varphi \vdash \Delta \Rightarrow \Box_X\Gamma, \Diamond_X\varphi \vdash \Diamond_X\Delta$ (L-TRUE $\Diamond_X$).

Static partial epistemic logic

The subclass of $\mathfrak{M}_{\mathfrak{A}}$ which is selected for static multiple agent epistemic representation is described formally in the following definition.

4.4. Definition. A partial epistemic model is a triple $M = \langle W, \{R_a\}_{a\in A}, V\rangle$ such that $M \in \mathfrak{M}_{\mathfrak{A}}$ and

$\forall x \in W\ \exists y \in W : R_a(x, y)$, and

$\forall x, y, z \in W : R_a(x, y) \Longrightarrow (R_a(x, z) \Leftrightarrow R_a(y, z))$ for all $a \in A$.

The first requirement is called *seriality*. The second constraint, which in fact summarizes transitivity and Euclidicity of the relations R_a, will be called *full introspection*. The class of partial epistemic models is denoted by $\mathfrak{E}_{\mathfrak{A}}$. The partial Kripke models with only a single accessibility relation, which satisfies the two constraints above, is called $\mathfrak{E}$.

Seriality takes care of the consistency of beliefs of all the agents: $\Box_a\varphi \wedge \Box_a\neg\varphi$ can never be verified in a serial model.

4.5. Observation. $[\![\Box_a\varphi \wedge \Box_a\neg\varphi]\!]_{\mathfrak{E}_{\mathfrak{A}}} = \emptyset$ and $\Box_a\varphi \models_{\mathfrak{E}_{\mathfrak{A}}} \Diamond_a\varphi$.

Technically speaking, full introspection means that if y is accessible from x, then the set of accessible worlds from y coincides with the set of worlds which are accessible from x. As we will see, it settles the axiomatic principles of positive introspection and negative introspection for the underlying derivational system $\mathbf{E}_A$.

4.6. Observation. $\Box_a\varphi \approx_{\mathfrak{E}_{\mathfrak{A}}} \Box_a\Box_a\varphi \approx_{\mathfrak{E}_{\mathfrak{A}}} \Diamond_a\Box_a\varphi$ and

$$\Diamond_a\varphi \approx_{\mathfrak{E}_{\mathfrak{A}}} \Diamond_a\Diamond_a\varphi \approx_{\mathfrak{E}_{\mathfrak{A}}} \Box_a\Diamond_a\varphi.$$

As mentioned in the introductory chapter, full introspection is philosophically tenable for partial modal logics. The proposition $\neg\Box_a\varphi$ means that the agent a has a counter-model of φ in mind, which differs from the classical interpretation that the agent a has access to some model where the truth of φ is absent. This stronger *active disbelief* interpretation in partial modal logic of $\neg\Box_a\varphi$, makes the conclusion $\Box_a\neg\Box_a\varphi$ acceptable, i.e. a believes that he has a counter-model in mind. The contra-position of this negative introspection, $\neg\Box_a\neg\Box_a\varphi \Rightarrow \Box_a\varphi$, which is logically independent of the principle of negative introspection in partial modal logic, is also accepted. It means that if a believes a proposition φ in one of his possible interpretations of the real world, then he must believe φ as well. An agent a cannot be uncertain about his own beliefs. This explains the $\mathfrak{E}_{\mathfrak{A}}$-equivalence of $\Diamond_a\varphi$ and $\Box_a\Diamond_a\varphi$ in observation 4.6.

Rules for static partial epistemic logic

The system $\mathbf{E}_A$ consists of $\mathbf{M}_A$ together with the following set of rules.

4.7. TABLE.

$$\frac{\Gamma \vdash \Delta}{\Box_a\Gamma \vdash \Diamond_a\Delta}\ \mathbf{D}$$

$$\frac{\Gamma \vdash \Box_a\Delta}{\Gamma \vdash \Box_a\Box_a\Delta}\ \mathbf{4}\text{-}\Box \qquad \frac{\Diamond_a\Gamma \vdash \Delta}{\Diamond_a\Diamond_a\Gamma \vdash \Delta}\ \mathbf{4}\text{-}\Diamond$$

$$\frac{\Gamma \vdash \Diamond_a\Delta}{\Gamma \vdash \Box_a\Diamond_a\Delta}\ \mathbf{5}\text{-}\Box \qquad \frac{\Box_a\Gamma \vdash \Delta}{\Diamond_a\Box_a\Gamma \vdash \Delta}\ \mathbf{5}\text{-}\Diamond$$

The first rule encodes the consistency of beliefs. It has been called **D** because it is an equivalent reformulation of the axiom **D**: $\Box_a\varphi \vdash \Diamond_a\varphi$, which is known from classical modal logic [Chellas 1980] [Hughes & Cresswell 1984]. Other equivalent formulations of **D** are $\Gamma \vdash \emptyset \Rightarrow \Box\Gamma \vdash \emptyset$ and $\emptyset \vdash \Delta \Rightarrow \emptyset \vdash \Diamond\Delta$.

The classical modal system **KD45** can be obtained from $\mathbf{E}_A$ and the classical rule R-TRUE $\neg$. So, $\mathbf{E}_A$ can be seen as a partial poly-modal variant of **KD45**.

The full system $\mathbf{E}_A$ is very strong. The **4**- and **5**-rules cause every iteration of modal operators of the same type a to be reducible to a formula with only one a-modal operator in front.

4.8. EXAMPLE.

$\Box_a\varphi \equiv_{E_A} \Box_a\Box_a\varphi \qquad \Box_a\varphi \equiv_{E_A} \Diamond_a\Box_a\varphi$

$\Diamond_a\varphi \equiv_{E_A} \Box_a\Diamond_a\varphi \qquad \Diamond_a\varphi \equiv_{E_A} \Diamond_a\Diamond_a\varphi$

We will use these equivalences far more often then the underlying sequential rules in table 4.7. In the sequel of the text we use a systematic abbreviation for the different directions of the equivalences above. If $C_{1,a}...C_{n,a}\varphi \vdash_{E_A} D_{1,a}...D_{m,a}\varphi$ for arbitrary $\varphi \in \mathcal{L}_A$ with $C_i, D_j \in \{\Box, \Diamond, \neg\Box\}$, then we use $C_1...C_n \Rightarrow D_1...D_m$ as a short reference. So, $\Box\Box \Rightarrow \Box$ refers to $\Box_a\Box_a\varphi \vdash_{E_A} \Box_a\varphi$. By way of illustration, we prove $\Box\Box \Rightarrow \Box$ in $\mathbf{E}_A$. The others equivalences are left to the reader.

1. $\Box_a\varphi \vdash_{E_A} \Box_a\varphi$ — START
2. $\Box_a\Box_a\varphi \vdash_{E_A} \Diamond_a\Box_a\varphi$ — **D** (1)
3. $\Diamond_a\Box_a\varphi \vdash_{E_A} \Box_a\varphi$ — **5**-$\Diamond$ (1)
4. $\Box_a\Box_a\varphi \vdash_{E_A} \Box_a\varphi$ — CUT (2,3)

In the examples above the strength of $\mathbf{E}_A$ has been shown in the way it reduces stacks of modalities of the same type. In the following proposition we show some important distributive effects of the strength of $\mathbf{E}_A$. These results show that much of the deductive capacity of classical **45**-logics are adopted by this partial variant.

4.9. EXAMPLE. $\Box_a(\Box_a\alpha \vee \beta) \equiv_{E_A} \Box_a\alpha \vee \Box_a\beta$

$$\Diamond_a(\Box_a\alpha \wedge \beta) \equiv_{E_A} \Box_a\alpha \wedge \Diamond_a\beta$$

Proof. The derivation runs the same as for classical **45**-logics. We present the derivation of the first equivalence below.

1.	$\Box_a\alpha \vdash_{E_A} \Box_a\alpha$	START
2.	$\beta \vdash_{E_A} \beta$	START
3.	$\Box_a\alpha \vee \beta \vdash_{E_A} \Box_a\alpha, \beta$	L-TRUE $\vee$ (1,2)
4.	$\Box_a(\Box_a\alpha \vee \beta) \vdash_{E_A} \Diamond_a\Box_a\alpha, \Box_a\beta$	R-TRUE $\Box_a$ (3)
5.	$\Diamond_a\Box_a\alpha \vdash_{E_A} \Box_a\alpha$	$\Diamond\Box \Rightarrow \Box$
6.	$\Box_a(\Box_a\alpha \vee \beta) \vdash_{E_A} \Box_a\alpha, \Box_a\beta$	CUT (4,5)
7.	$\Box_a(\Box_a\alpha \vee \beta) \vdash_{E_A} \Box_a\alpha \vee \Box_a\beta$	R-TRUE $\vee$ (6)

1.	$\beta \vdash_{E_A} \Box_a\alpha, \beta$	START
2.	$\beta \vdash_{E_A} \Box_a\alpha \vee \beta$	R-TRUE $\vee$ (1)
3.	$\Box_a\beta \vdash_{E_A} \Box_a(\Box_a\alpha \vee \beta)$	R-TRUE $\Box_a$ (2)
4.	$\Box_a\alpha \vdash_{E_A} \Box_a\alpha, \beta$	START
5.	$\Box_a\alpha \vdash_{E_A} \Box_a\alpha \vee \beta$	R-TRUE $\vee$ (4)
6.	$\Box_a\Box_a\alpha \vdash_{E_A} \Box_a(\Box_a\alpha \vee \beta)$	R-TRUE $\Box_a$ (5)
7.	$\Box_a\alpha \vdash_{E_A} \Box_a\Box_a\alpha$	$\Box \Rightarrow \Box\Box$
8.	$\Box_a\alpha \vdash_{E_A} \Box_a(\Box_a\alpha \vee \beta)$	CUT (6,7)
9.	$\Box_a\alpha \vee \Box_a\beta \vdash_{E_A} \Box_a(\Box_a\alpha \vee \beta)$	L-TRUE $\vee$ (3,8)

The other equivalence is left to the reader. They require the use of $\Diamond\Diamond \Rightarrow \Diamond$ and $\Diamond \Rightarrow \Box\Diamond$. ■

4.10. COROLLARY. Example 4.8 shows that iterations of the same type of modal operators can be reduced to single occurrences of such a modal operator. On the basis of example 4.9 we can even reduce every formula to an $\mathbf{E}_A$-equivalent formula having a so-called A-modal depth which is not larger than 1.

4.11. DEFINITION. The a-modal depth $md_a(\varphi)$ of a proposition $\varphi \in \mathcal{L}_A$ is determined through the following induction.

$md_a(p) = 0 \ (p \in I\!P)$	$md_a(\bot) = 0$
$md_a(\neg\varphi) = md(\varphi)$	$md_a(\varphi \wedge \psi) = \max\{md_a\varphi, md_a\psi\}$
$md_a(\Box_a\varphi) = md_a(\varphi) + 1$	$md_a(\Box_b\varphi) = 0 \ (b \neq a)$

The A-modal depth $md_A(\varphi)$ of a formula $\varphi \in \mathcal{L}_A$ is $\max_{a \in A} md_a \varphi$.

The formulae of A-modal depth 1 are formulae in which every modal operator of type a has only scope over formulae which are propositionally connected atoms and formulae of the form $\Box_b \varphi$ with $b \neq a$. This derivational strength of $\mathbf{E}_A$ is analogous to the classical **45**-logics.

Dynamic epistemic systems can be built from the basic constructive modal logics of chapter 3 in the same way as the static partial epistemic logic $\mathbf{E}_A$ has been constructed from the minimal partial modal logic $\mathbf{M}$ of chapter 2. We will shortly discuss epistemic extensions of the basic constructive modal logics and we will then present the system $\mathbf{C}^3$. On the basis of this system we will present some extensions for modeling communicative actions.

Constructive extensions of $\mathbf{E}_A$

4.12. DEFINITION. The poly-modal generalization of $\mathfrak{NM}$ is the class $\mathfrak{NM}_{\mathfrak{A}}$, which consists of models $M = \langle W, \{R_a\}_{a \in A}, \leq, V \rangle$ such that A is a non-empty finite set and $\langle W, R_a, \leq, V \rangle \in \mathfrak{NM}$ for all $a \in A$.
The poly-modal generalization of $\mathfrak{NM}^{\Box}$ is the class $\mathfrak{NM}^{\Box}_{\mathfrak{A}}$, which consists of models $M = \langle W, \{R_a\}_{a \in A}, \leq, V \rangle$ such that A is a non-empty finite set and $\langle W, R_a, \leq, V \rangle \in \mathfrak{NM}^{\Box}$ for all $a \in A$.

The corresponding minimal constructive poly-modal logics are the systems $\mathbf{NM}_{\mathrm{A}}$ and $\mathbf{NM}^{\Box}_{\mathrm{A}}$. The former derivation system is interpreted over the model class $\mathfrak{NM}_{\mathfrak{A}}$, and the latter over $\mathfrak{NM}^{\Box}_{\mathfrak{A}}$. The former system contains the rules of $\mathbf{N}$ and $\mathbf{M}_A$ together. The latter is the combination of $\mathbf{N}^-$ and $\mathbf{M}_A$ with an additional poly-modal version of R-DIS $\Box \rightarrow$:

$$\frac{\Gamma \vdash \varphi \rightarrow \psi, \neg\Delta}{\Box_a \Gamma \vdash \Box_a \varphi \rightarrow \Box_a \psi, \neg \Box_a \Delta} \quad \text{R-DIS } \Box_a \rightarrow.$$

For defining constructive epistemic systems we simply combine the systems $\mathbf{NM}_{\mathrm{A}}$ and $\mathbf{NM}^{\Box}_{\mathrm{A}}$ with the static system $\mathbf{E}_A$, respectively. The sum of these systems are called $\mathbf{NE}_{\mathrm{A}}$ and $\mathbf{NE}^{\Box}_{\mathrm{A}}$, respectively. Appropriate models can be defined by simply substituting $\mathfrak{E}$ for $\mathfrak{M}$ in definition 4.12 above.

4.13. DEFINITION.

$$\mathfrak{NE}_{\mathfrak{A}} = \{\langle W, \{R_a\}_{a \in A}, \leq, V \rangle \in \mathfrak{NM} \mid \forall a \in A : \langle W, R_a, V \rangle \in \mathfrak{E}\}$$

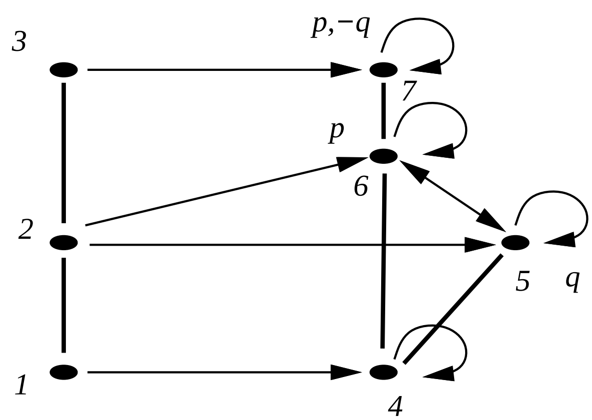
$$\mathfrak{C}^3 = \{\langle W, \{R_a\}_{a \in A}, \leq, V \rangle \in \mathfrak{NM}^{\Box} \mid \forall a \in A : \langle W, R_a, V \rangle \in \mathfrak{E}\}$$

$\mathfrak{NE}^{\Box}_{\mathfrak{A}}$ as a name for the latter model class would have been more in line with our nomenclature of model classes. The name $\mathfrak{C}^3$ has been chosen irregularly on purpose. It contains the basic epistemic and dynamic ingredients of the communication calculi which are going to be presented in this chapter. It is the essential model-theory which we have chosen for dynamic epistemic reasoning, with the information structure $\leq$ as the dynamic parameter and the epistemic accessibilities as the static representation of beliefs of the communicating agents.

This explains why it has been given the abbreviation of the title of this thesis as its name.

As already explained in the previous chapter, the choice of $\mathfrak{C}^3$ as a subclass of $\mathfrak{NM}^{\Box}_{\mathfrak{A}}$ provides a formal interpretation of the growth of epistemic information. It captures growth as a combination of construction and elimination, which is formalized by the interrelational constraint of the class $\mathfrak{NM}^{\Box}_{\mathfrak{A}}$ between the accessibility relation R_a and the information structure $\leq$ (see definition 3.27 on page 99).

4.14. EXAMPLE. The cognitive progress, on the basis of elimination and construction, of a single agent which has been depicted in figure 1.7 (page 30) can be formally interpreted in terms of the $\mathfrak{C}^3$-models. Let $M = \langle W, R, \leq, V \rangle$ be of the following form:



The vertical lines represent the information order $\leq$ as in figure 2.36 (page 66 and figure 3.28 (page 100). The epistemic accessibility relation is represented by the arrows. It is not hard to verify that $M \in \mathfrak{C}^3$.

The seriality of R can be seen immediately. All worlds have an outgoing arrow.

The full introspection of R can be scanned by checking whether all worlds which have an incoming arrow, starting from the same world, are mutually accessible. This also means that worlds with an incoming arrow have to be reflexive.

The valuation function in M is indeed monotonic.

What is left to prove is the construction-elimination dynamics of M: $\leq \circ R \subseteq R \circ \leq$. The reader may check for himself that

$$R\circ \leq = \leq \circ R \cup \{\langle 5, 7 \rangle\}.$$

Conclusion: $M \in \mathfrak{C}^3$. However, $M \notin \mathfrak{NC}$. The second bisimulation constraint does not hold for this model: $\geq \circ R \not\subseteq R\circ \geq$. We have $(\geq \circ R)(7,5)$, but not $(R\circ \geq)(7,5)$.

The model M' which describes the first cognitive step, i.e. the model M restricted to the worlds $1, 2, 4, 5, 6$ is a member of $\mathfrak{NC}$. This explains that this first step is purely constructive. The second step in M has an eliminative part: alternative 5 has been eliminated.

Up and down extension of $\mathbf{E}_A$

The basic dynamic epistemic derivation systems which we use is a combination of the modal version of the up and down logic of chapter 3 (**Mud**) and $\mathbf{E}_A$, which is interpreted over the model class $\mathfrak{C}^3$. This system is called $\mathbf{C}^3$. The only additional derivational adjustment is an individualized version of the rule DIS $\Box[\ ]_u$ and its contra-position C-DIS $[\ ]_u$.

4.15. TABLE.

$$\frac{\Gamma \vdash [\varphi]_u\,\psi, \neg\Delta}{\Box_a\Gamma \vdash [\Box_a\varphi]_u\,\Box_a\psi, \neg\Box_a\Delta} \quad \text{DIS } \Box_a[\ ]_u$$

$$\frac{\Gamma, \neg[\varphi]_u\,\psi \vdash \neg\Delta}{\Box_a\Gamma, \neg[\Box_a\varphi]_u\,\Box_a\psi \vdash \neg\Box_a\Delta} \quad \text{C-DIS } \Box_a[\ ]_u.$$

Latent belief and disbelief

In the introduction of this thesis we have distinguished two kinds of disbelief. Active disbelief of a proposition φ refers to a situation where a counter-model, or counter-world, of φ is epistemically accessible. Passive disbelief of φ refers to a situation where φ has no truth-value in the current set of epistemic alternatives. Yet another kind of disbelief can be described by means of a dynamic intensional reading. This is what we call *latent disbelief*, and it refers to a situation where at least one of the current epistemic alternatives can be extended to a counter-world of the content of this disbelief. For example, in 4.14 in stage 1, the agent latently disbelieves q.

Latent disbelief of a proposition φ by an agent a can be described in $\mathcal{L}_A^{\uparrow,\downarrow}$ as $\langle\neg\Box_a\varphi\rangle_u\top$. An $\mathfrak{C}^3$-equivalent formulation of this proposition of latent disbelief of φ is $\langle\ \rangle_u\neg\Box_a\varphi$. Informally speaking, telling the agent a that φ is possibly false would not lead necessarily lead to actual progress of his belief. If he was not aware of the proposition φ, this assertion only broadens his epistemic outlook.

In the same way we can describe *latent beliefs*. Latent belief of a proposition φ means that updating with a counter-world of φ is impossible, given the current range of epistemic alternatives. In other words, $\neg\Box_a\varphi$ is inconstructible. Formally speaking, this interpretation comes down to $[\neg\Box_a\varphi]_u\bot$[2].

In fact, this latent belief is a certain 're-classicalization' of our belief operator. Every classical propositional tautology is latently believed. In semantic terms, if a proposition $\varphi \in \mathcal{L}$ is verified by all total valuations, then φ is always latently believed. This property can also be strengthened to the classical variant $\mathbf{KD45}_A$ of $\mathbf{E}_A$.

[2] In terms of constructive logics, this is the intuitionistic negation of the proposition $\neg\Box_a\varphi$. This combination of the extensional negation of Nelson and Brouwer's intensional reading of negative information shows the use of their collaboration in $\mathbf{C}^3$.

4.16. OBSERVATION. If $\varphi \in \mathcal{L}^{\Box}$ and all total models in $\mathfrak{E}_{\mathfrak{A}}$ verify φ, then φ is always latently believed by all the agents: $[\neg\Box_a\varphi]_u\bot$:

$$\vdash_{KD45_A} \varphi \implies \vdash_{C^3} [\neg\Box_a\varphi]_u\bot.$$

In epistemic logic different approaches have been suggested for distinguishing these kinds of belief [Levesque 1984] [Fagin & Halpern 1988], better known as the distinction between *implicit* and *explicit* belief. The distinction has been proposed as an approach to the problem of *logical omniscience*, which is inherited by the classical modal approach towards logical belief representation as in the early work of Hintikka [Hintikka 1962]. This omniscience refers to the logical idealization of deductive closure of the beliefs of epistemic agents. Hintikka's philosophical defense of using this strength of classical modal logic was that his models were meant to represent implicit beliefs. The obvious suggestion is that we should find models of explicit belief, which avoid this logical omniscience.

In [Fagin & Halpern 1988] *awareness* has been introduced as a formal concept in order to establish a formal distinction of explicit and implicit belief. They present different proposals, of which the logic of *special awareness* comes close to partialization of classical Kripke models. They define an *awareness-function* on worlds. This function determines for every world a fixed domain of atoms in the accessible worlds. This means, that all worlds which are seen from the current situation are mutually just as partial.

Of course, this is certainly not the case in partial modal logics such as $\mathbf{E}_A$. However, by means of our description of latent beliefs and disbeliefs such awareness can be imitated. We could say that an agent a is aware of a proposition φ, if he has access to a world which determines a truth value for this proposition: $\Diamond_a\varphi \vee \Diamond_a\neg\varphi$. The notion of explicit belief of a proposition φ can then be captured as being aware of φ and the range of current doxastic alternatives cannot be extended with possible counter-evidence against φ. Formally,

$$\Diamond_a\varphi \wedge [\neg\Box_a\varphi]_u\bot.$$

The last conjunct coincides with our notion of latent belief above. In the terminology of awareness this latent belief might be seen as implicit belief: it indicates a certain triviality of its content. Nevertheless, the proposition φ may be outside the scope of the awareness of the agent a.

In section 4.3 we will use this notion of 'updatability' as latent disbelief again. Just like in Turner's constructive formulation of default logic, we use this notion of constructibility to specify weak preconditions. A proposition of the form $\langle\neg\Box_a\varphi\rangle_u\bot$, or $\mathbf{C}^3$-equivalently $\neg\Box_a[\ ]_u\neg\varphi$, tells us that the agent a attributes an informational content to the proposition φ. This kind of non-triviality of a proposition with respect to the agent a, will reappear as a requirement of contribution, which is one of the preconditions of this agent to communicate about this proposition.

Now that we have fully specified our logical means for dynamic epistemic reasoning, the remainder of this chapter is dedicated to additional decoration of $\mathbf{C}^3$ for dynamic interpretation of communicative actions.

4.2 Mutual beliefs

Mutual belief of a group of agents X that φ holds means that all members of the group X believe that φ, and all members of X believe that all members of X believe that φ, and all members of X believe that all members of X believe that all members of X believe that φ, and so on. Formally speaking, by the mutual belief of X that φ we refer to the set of formulae:

$$\{\Box_X^n \varphi \mid n \in \mathbb{N} \setminus \{0\}\} \quad (1).$$

Such mutual beliefs are particularly important for formal interpretation of communication. They represent a 'common ground' of a group of interacting agents. Such information can be used freely as presuppositions during communication. If a and b are members of a group X which mutually believe φ, then φ can be used as 'silent' background information when a and b are talking with each other. For example, if a and b are Dutchmen, then they may freely use the name 'Beatrix' without any explicit reference. However, if one of them were non-Dutch, then the user of this name should explain that Holland has a queen and that 'Beatrix' is her name.

Such mutual beliefs are not restricted to linguistic use. Also in theories of social behavior such belief representations show up. For planning activities, intelligent agents make use of all kind of social conventions, which are in fact mutual beliefs about the behavior of cooperating agents in a group. Many of our own social strategies utilize information which we have about the behavior of others [Lewis 1969].

The representation of mutual beliefs in (1) above can be seen as the most ideal acceptance of information of an interacting group X. In most situations a small finite part of the set in (1), i.e. up to a certain $n \in \mathbb{N} \setminus \{0\}$, seems to be enough for decision on new actions. Such a pragmatic upper bound to the epistemic nesting of information seems to relate to the importance or risks of the decisions which are to be made on the basis of this information.

An instructive example comes from [Halpern & Moses 1990] and deals with risky military decision making, which is known as the *Byzantine agreement problem.* They describe a situation where two collaborating armies want to attack a hostile town. The problem is that the two armies are at opposite sides of the town, and there is no way that the two generals in charge, let's call them a and b, can communicate face-to-face and reach agreement about the time of a joint attack. The only way to communicate is by sending a courier c through the town to the other side.

Let us say that general a is the initiator of this long distance conversation and sends c with the information p, which is the time of attack, to the other general b. If c reaches b safely, then b has got the information that p, but nevertheless this information is not sufficient for a decision of b to attack the town at p. He also needs to ensure a of the fact that the messenger c arrived, such that a is certain of the fact that b has the information p. So, he sends c back to a with the information $\Box_b p$. If c succeeds again, we have a new information state which contains $\Box_a p \wedge \Box_b p \wedge \Box_a \Box_b p$. Again, a needs to send c back with the information $\Box_a \Box_b p$, because b needs to be sure that a is sure that b will indeed attack at p.

This communication protocol never reaches a completely safe fulfilling of the preconditions of an attack. In more usual communicative situations where the risks of decisions on the basis of incoming messages are considerably lower, a partial fulfilling of a mutual belief precondition is taken to be sufficient. Even in face-to-face dialogues, we are never sure of reaching a state of mutual belief. When an assertion is transferred a simple nodding of the receiver, as a sign of affirmation or acceptance of the information, seems to be satisfactory to continue the conversation or undertake a certain new action. If communicating agents would insist on reaching real mutual beliefs by assertion, then any successful dialogue would lead to an infinite nodding procedure. Nevertheless, we wish to describe the ideal epistemic force of assertions and leave it to others to speculate on more realistic empirical approximation of such update effects which do not neglect the 'Byzantine noise'.

In the next subsection we will introduce the system $\mathbf{E}_A^*$ and its construction-elimination dynamic extension $\mathbf{C}^{3*}$. They contain additional mutual belief operators for every subset of agents. These systems are proper extensions of the epistemic systems of the previous section. The infinite set in (1) can not be expressed as a single proposition in their languages $\mathcal{L}_A$ and $\mathcal{L}_A^{\uparrow,\downarrow}$.

A formal interpretation of mutual beliefs

The system $\mathbf{E}_A^*$ is the static partial epistemic logic extended with additional modal operators $\Box_X^*$ for all $X \subseteq A$. Its language is called $\mathcal{L}_A^*$. A proposition of the form $\Box_X^* \varphi$ says that every proposition of the form $\Box_{a_1}\Box_{a_2}...\Box_{a_n}\varphi$ holds (if $n = 0$ this proposition equals φ) for all $a_i \in X$. These operators are interpreted by means of the reflexive transitive closure of the union of the individual accessibility relations of the different agents in X.

4.17. Definition. Let $M = \langle W, \{R_a\}_{a \in A}, V\rangle \in \mathfrak{E}_{\mathfrak{A}}$. We use the following notation for all non-empty subsets X of A:

$$R_X = \bigcup_{a \in X} R_a,$$

$$R_X^* = \{\langle x, y\rangle \in W^2 \mid \exists n \in I\!N : R_X^n(x,y)\} \text{ and}$$

$$R_X^t = R_X \circ R_X^*.$$

The last relation R_X^t is the *transitive closure* of R_X and R_X^* is the *reflexive transitive closure* of R_X.

4.18. Table. Let $M = \langle W, \{R_a\}_{a \in A}, V\rangle \in \mathfrak{E}_{\mathfrak{A}}$. The $\Box_X^*$-operators are interpreted with respect to M along the relation R_X^*.

$$M, w \models \Box_X^* \varphi \iff \forall v \in W : (R_X^*(w,v) \Rightarrow M, v \models \varphi)$$

$$M, w \,=\!\!\mid\, \Box_X^* \varphi \iff \exists v \in W : (R_X^*(w,v) \;\&\; M, v \,=\!\!\mid\, \varphi)$$

The falsity of $\Box^*_X$ with respect to a world w in a model $M \in \mathfrak{E}_{\mathfrak{A}}$, says that there exists a sequence $a_1, ..., a_n$ in X such that $\Box_{a_1} ... \Box_{a_n}\varphi$ is falsified with respect to w in M. Instead of $\neg\Box^*_X\neg\varphi$, we will also write $\Diamond^*_X\varphi$. Such a proposition says that φ holds in some world at some X-distance.

4.19. OBSERVATION. Notice that this interpretation is correct with respect to the earlier intuitive description. Let $M = \langle W, \{R_a\}_{a\in A}, V\rangle \in \mathfrak{E}_{\mathfrak{A}}$, and $w \in W$ and $X \subseteq A$.

$M, w \models \Box^*_X\varphi \iff M, w \models \Box_{a_1} ... \Box_{a_n}\varphi$ for all finite sequence $a_1, ..., a_n \in X$.

The falsification of $\Box^*_X\varphi$ obtains the interpretation which has been mentioned above:

$M, w \mathrel{=\!\!|} \Box^*_X\varphi \iff M, w \mathrel{=\!\!|} \Box_{a_1} ... \Box_{a_n}\varphi$ for certain sequence $a_1, ..., a_n \in X$.

Note that if $n = 0$ then $M, w \mathrel{=\!\!|} \varphi$.

4.20. DEFINITION. *Mutual belief* of a proposition φ among a group X of agents, is expressed by $\Box_X\Box^*_X\varphi$, which is abbreviated by $\mathsf{C}_X\varphi$.

4.21. OBSERVATION. Notice that $\mathsf{C}_X\varphi$ is interpreted in the same way as $\Box^*_X\varphi$, with R_X replaced by R^t_X in the truth-value conditions above.

4.22. OBSERVATION. Combination of observation 4.19 above and example 4.8 shows that

$$[\![\mathsf{C}_{\{a,b\}}\varphi]\!]_{\mathfrak{E}_{\mathfrak{A}}} = \bigcap_{\substack{\langle a_1,..,a_n\rangle\in\{a,b\}^n \\ a_i\neq a_{i+1}}} [\![\Box_{a_1}...\Box_{a_n}\varphi]\!]_{\mathfrak{E}_{\mathfrak{A}}} \quad (\text{with } n \geq 1).$$

The somewhat complicated subscript tells us that the sequences $a_1, .., a_n$ consist only of a's and b's, such that a and b occur alternately. This means that the mutual belief of a and b of φ is the same as that a believes that φ, b believes that φ, a believes that b believes that φ, b believes that a that φ, etcetera.

4.23. OBSERVATION. A negative property of $\models_{\mathfrak{E}_{\mathfrak{A}}}$ and $\models_{\mathfrak{E}_3}$ is their loss of *compactness*. This means there exist infinite $\Gamma \subseteq \mathcal{L}^*_A$ such that $\Gamma \models_{\mathfrak{E}_{\mathfrak{A}}} \varphi$, while for every finite $\Gamma' \subseteq \Gamma$ the proposition φ is not a valid $\mathfrak{E}_{\mathfrak{A}}$-consequence: $\Gamma' \not\models_{\mathfrak{E}_{\mathfrak{A}}} \varphi$. A simple example is obtained by taking Γ to be same as $\{\Box^n_X p \mid \emptyset \neq X \subseteq A\}$ and by substituting $\Box^*_X p$ for certain $p \in I\!P$. For all finite $\Gamma' \subseteq \Gamma$

$$\Gamma' \not\models_{\mathfrak{E}_{\mathfrak{A}}} \Box^*_X p.$$

The non-compactness of these logics indicate their meta-theoretical toughness, when we compare them to the logics which we have met in the preceding chapters. In the following subsection we will give derivation systems for their underlying calculi. All these calculi have the straightforward finiteness property, which means that they are incomplete with respect to the full class of sequents. Nevertheless, we obtain a satisfactory completeness result in chapter 6 for these systems. Their derivational capacity is still complete with respect to the class of finite sequents.

A derivation system for mutual beliefs

We need a considerable sequential extension of the basic static epistemic logic $\mathbf{E}_A$ in order to obtain a suitable axiomatization for the operators $\Box^*_X$. This system is called $\mathbf{E}^*_A$. $\mathbf{E}^*_A$ consists of $\mathbf{E}_A$ and the following additional rules for the $\Box^*_X$-operators.

4.24. TABLE.

RULES FOR $\Box^*_X$

$$\frac{\Gamma, \varphi \vdash \Delta}{\Gamma, \Box^*_X \varphi \vdash \Delta} \text{ L-TRUE } \Box^*_X$$

$$\frac{\Gamma, \varphi \vdash \Box_X \varphi, \neg\Delta \quad \Gamma' \vdash \varphi, \Delta'}{\Box^*_X \Gamma, \Gamma' \vdash \Box^*_X \varphi, \neg\Box^*_X \Delta, \Delta'} \text{ R-TRUE } \Box^*_X$$

$$\frac{\Gamma, \neg\Box_X \varphi \vdash \neg\varphi, \neg\Delta \quad \Gamma', \neg\varphi \vdash \Delta'}{\Box^*_X \Gamma, \Gamma', \neg\Box^*_X \varphi \vdash \neg\Box^*_X \Delta, \Delta'} \text{ L-FALSE } \Box^*_X$$

$$\frac{\Gamma \vdash \neg\varphi, \Delta}{\Gamma \vdash \neg\Box^*_X \varphi, \Delta} \text{ R-FALSE } \Box^*_X$$

$$\frac{\Gamma \vdash \Box^*_X \varphi, \Delta}{\Gamma \vdash \Box_X \Box^*_X \varphi, \Delta} \text{ R-TRUE } \Box_X \Box^*_X \qquad \frac{\Gamma, \neg\Box^*_X \varphi \vdash \Delta}{\Gamma, \neg\Box_X \Box^*_X \varphi \vdash \Delta} \text{ L-FALSE } \Box_X \Box^*_X$$

4.25. OBSERVATION. The system $\mathbf{M}^*_A$ consists of the rules of $\mathbf{M}_A$ and the rules for $\Box^*_X$ above. Apart from propositional tests, this system axiomatizes the minimal partial variant of Pratt's propositional dynamic logic [Pratt 1980]. The modal indices A should in this case be interpreted as a set of atomic programs. The $\Box^*_X$-operator refers then to the program which executes an arbitrary number of times the program sequence X.

4.26. OBSERVATION. The relatively easy rules L-TRUE $\Box^*_X$ and R-FALSE $\Box^*_X$ are used in the sequel of the text by a reformulation as $\Box^* \Rightarrow$ and $\Rightarrow \Diamond^*$ rules:

$$\Box^*_X \varphi \vdash_{E^*_A} \varphi \quad \Box^* \Rightarrow \quad \text{and} \quad \varphi \vdash_{E^*_A} \Diamond^* \varphi \quad \Rightarrow \Diamond^*.$$

Furthermore, we use $\Box^* \Rightarrow \Box\Box^*$ and $\Diamond\Diamond^* \Rightarrow \Diamond^*$ as names for $\Box^*_X \varphi \vdash_{E^*_A} \Box_X \Box^*_X \varphi$ and $\Diamond_X \Diamond^*_X \varphi \vdash_{E^*_A} \Diamond^*_X \varphi$. They are reformulations of R-TRUE $\Box_X \Box^*_X$ and L-FALSE $\Box_X \Box^*_X$, respectively.

The rules R-TRUE $\Box^*_X$ and L-FALSE $\Box^*_X$ look complicated at first sight. Reduction of the surrounding sets, the Γ's and the Δ's, helps to get a better comprehension. In fact, these rules are sequential formulations of induction principles.

4.27. OBSERVATION. $\varphi \vdash_{E^*_A} \Box_X \varphi \implies \varphi \vdash_{E^*_A} \Box^*_X \varphi$ R-IND and

$$\Diamond_X \varphi \vdash_{E^*_A} \varphi \implies \Diamond^*_X \varphi \vdash_{E^*_A} \varphi \quad \text{L-IND}$$

Proofs of these more pleasant induction formulations are obtained by taking $\Gamma = \Delta = \Delta' = \emptyset$ and $\Gamma' = \{\varphi\}$ in R-TRUE $\Box^*_X$, and $\Gamma = \Delta = \Gamma' = \emptyset$ and

$\Delta' = \{\varphi\}$ in L-FALSE $\Box^*_X$, respectively.

Important $\mathbf{E}^*_A$-sequents which capture this induction effect as well, are:

$\Box^*_X \Box_X \varphi, \varphi \vdash_{E^*_A} \Box^*_X \varphi$ R-IND' and

$\Diamond^*_X \varphi \vdash_{E^*_A} \varphi, \Diamond^*_X \Diamond_X \varphi$ L-IND'.

Proof. These induction principles can be obtained very easily from the R-TRUE $\Box^*_X$ and L-FALSE $\Box^*_X$. The following simple three steps derivation establishes the first $\mathbf{E}^*_A$-sequent.

1. $\Box_X \varphi, \varphi \vdash_{E^*_A} \Box_X \varphi$ START
2. $\varphi \vdash_{E^*_A} \varphi$ START
3. $\Box^*_X \Box_X \varphi, \varphi \vdash_{E^*_A} \Box^*_X \varphi$ R-TRUE $\Box^*_X$[3]

Other derivations are left to the reader. ■

4.28. OBSERVATION. Combination of the induction principles and the easier rules of observation 4.26 entails yet other important principles such as the equivalences $\Box^*_X \varphi \equiv_{E_A} \Box^*_X \Box^*_X \varphi$ and $\Diamond^*_X \varphi \equiv_{E_A} \Diamond^*_X \Diamond^*_X \varphi$, and the reversed version of the induction principles above: $\Box^*_X \varphi \vdash_{E^*_A} \varphi, \Box^*_X \Box_X \varphi$ and $\Diamond^*_X \Diamond_X \varphi, \varphi \vdash_{E^*_A} \Diamond^*_X \varphi$.

Proof.

1. $\Box^*_X \varphi \vdash_{E^*_A} \Box_X \Box^*_X \varphi$ $\Box^* \Rightarrow \Box\Box^*$
2. $\Box^*_X \varphi \vdash_{E^*_A} \Box^*_X \Box^*_X \varphi$ R-IND
3. $\Box^*_X \Box^*_X \varphi \vdash_{E^*_A} \Box^*_X \varphi$ $\Box^* \Rightarrow$ for $\Box^*_X \varphi$

From this "$\Box^* \equiv \Box^*\Box^*$-property we can derive $\Box^*_X \varphi \vdash_{E^*_A} \Box^*_X \Box_X \varphi$. $\Diamond^* \equiv \Diamond^*\Diamond^*$ establishes the contra-positional version of this latter sequent: $\Diamond^*_X \Diamond_X \varphi \vdash_{E^*_A} \Diamond_X \varphi$.

4. $\Box^*_X \varphi \vdash_{E^*_A} \varphi$ $\Box^* \Rightarrow$
5. $\Box_X \Box_X \Box^*_X \varphi \vdash_{E^*_A} \Box_X \Box_X \varphi$ 2× R-TRUE $\Box_X$ (4)
6. $\Box_X \Box^*_X \varphi \vdash_{E^*_A} \Box_X \Box_X \Box^*_X \varphi$ R-TRUE $\Box_X$ (1)
7. $\Box^*_X \varphi \vdash_{E^*_A} \Box_X \Box_X \Box^*_X \varphi$ CUT (1,6)
8. $\Box^*_X \varphi \vdash_{E^*_A} \Box_X \Box_X \varphi$ CUT (5,7)
9. $\Box^*_X \varphi, \Box_X \varphi \vdash_{E^*_A} \Box_X \Box_X \varphi$ L-MON (8)
10. $\Box^*_X \Box^*_X \varphi, \Box_X \varphi \vdash_{E^*_A} \Box^*_X \Box_X \varphi$ R-TRUE $\Box^*_X$ (9, $\Box_X \varphi \vdash_{E^*_A} \Box_X \varphi$)
11. $\Box^*_X \varphi, \Box_X \varphi \vdash_{E^*_A} \Box^*_X \Box_X \varphi$ CUT (2,10)
12. $\Box_X \Box^*_X \varphi \vdash_{E^*_A} \Box_X \varphi$ R-TRUE $\Box_X$ (4)
13. $\Box^*_X \varphi \vdash_{E^*_A} \Box_X \varphi$ CUT (1,12)
14. $\Box^*_X \varphi \vdash_{E^*_A} \Box^*_X \Box_X \varphi$ CUT (11,13)

■

4.29. OBSERVATION. The operators $\Box^*_X$ behave also as modal normal operators, that is the normal R-TRUE and L-FALSE for $\Box$ and $\Box_a$, can be derived in the system $\mathbf{E}^*_A$:

$\Gamma \vdash_{E_A^*} \varphi, \Delta \Rightarrow \Box_X^* \Gamma \vdash_{E_A^*} \Box_X^* \varphi, \Diamond_X^* \Delta$, and

$\Gamma, \varphi \vdash_{E_A^*} \Delta \Rightarrow \Box_X^* \Gamma, \Diamond_X^* \varphi \vdash_{E_A^*} \Diamond_X^* \Delta$.

Proof. We show the first implication below.

1.	$\Gamma \vdash_{E_A^*} \varphi, \Delta$	assumption
2.	$\Box_X \Gamma \vdash_{E_A^*} \Box_X \varphi, \Diamond_X \Delta$	R-TRUE $\Box_X$ (1)
3.	$\Box_X \Gamma, \varphi \vdash_{E_A^*} \Box_X \varphi, \Diamond_X \Delta$	L-MON (2)
4.	$\Box_X^* \Box_X \Gamma, \Gamma \vdash_{E_A^*} \Box_X^* \varphi, \Diamond_X^* \Delta, \Diamond_X \Delta$	R-TRUE $\Box_X^*$ (1,3)
5.	$\Box_X^* \Gamma, \Gamma \vdash_{E_A^*} \Box_X^* \varphi, \Diamond_X^* \Delta, \Delta$	$\Box^* \Box \Rightarrow \Box^*$ and $\Diamond^* \Rightarrow \Diamond^* \Diamond$
6.	$\Box_X^* \Gamma \vdash_{E_A^*} \Box_X^* \varphi, \Diamond_X^* \Delta$	$\Box^* \Rightarrow$ and $\Rightarrow \Diamond^*$

■

In the sequel of this text we will call these weaker version of $\Box_X^*$-introduction R-MOD-TRUE $\Box_X^*$ and L-MOD-FALSE $\Box_X^*$, respectively.

4.30. THEOREM. SOUNDNESS $\mathbf{E}_A^*$

For all $\Gamma, \Delta \subseteq \mathcal{L}_A^*$: $\Gamma \vdash_{E_A^*} \Delta \Rightarrow \Gamma \models_{\mathfrak{E}_\mathfrak{A}} \Delta$.

Proof. We will show the soundness of R-TRUE $\Box_X^*$. Suppose $\Gamma, \varphi \models_{\mathfrak{E}_\mathfrak{A}} \Box_X \varphi, \Delta$ (1) and $\Gamma \vdash_{E_A^*} \varphi, \Delta$ (2). Let $M = \langle W, \{R_a\}_{a \in A}, V \rangle \in \mathfrak{E}_\mathfrak{A}$ and $w \in W$ such that

$M, w \models \Box_X^* \gamma$ and $M, w \models \gamma'$ for all $\gamma \in \Gamma$ and $\gamma' \in \Gamma'$ (3), and

$M, w \not\models \neg \Box_X^* \delta$ and $M, w \models \delta'$ for all $\delta \in \Delta$ and $\delta' \in \Delta'$ (4).

The assumption (3) means that $M, v \not\dashv \delta$ for all $v \in W$ with $R_X^*(w, v)$. We will prove by an induction on n that

$\forall n \in I\!N\ \forall v \in W : R_X^n(w, v) \Rightarrow M, v \models \varphi$ (5).

This conclusion would establish $M, w \models \Box_X^* \varphi$.

If $n = 0$ the conclusion immediately follows from (2) and (4).

Let $n > 0$ and $R_X^n(w, v)$. This means that there exists a world $u \in W$ such that $R_X^{n-1}(w, u)$ and $R_X(u, v)$ (7). The induction hypothesis entails $M, u \models \varphi$. Combination of (1), (3) and this last conclusion entails $M, u \models \Box_X \varphi$, and therefore $M, v \models \varphi$ (7).

Because w has been chosen freely as $\Box_X^* \Gamma \cup \Gamma'$-world, we may conclude that the assumptions (1) and (2) lead to

$M, w \models \Box_X^* \varphi$ or

$M, w \dashv \Box_X^* \delta$ for certain $\delta \in \Delta$, or

$M, w \models \delta'$ for certain $\delta' \in \Delta'$.

In other words, (1) and (2) imply $\Box_X^* \Gamma, \Gamma' \models_{\mathfrak{E}_\mathfrak{A}} \Box_X^* \varphi, \neg \Box_X^* \Delta, \Delta'$. ■

Ideal assertion

As explained in the introduction of this chapter, we will take an idealistic position on the interpretation of assertions. The dynamic epistemic effect of an assertion is the mutual belief of the content of the assertion by the sender and the receiver, or group of receivers.

Furthermore, we need to stipulate a precondition for assertions. The simplest precondition of an assertion is that its sender believes the content himself. This establishes a first interpretation for assertion. We will write an assertion as an expression of the form $\boxed{a \ \ \mathsf{assert}\ \ \varphi \rangle X}$ with $a \in A$, $\emptyset \neq X \subseteq A$ and $\varphi \in \mathcal{L}_A^{\uparrow,\downarrow,*}$. The agent a is the sender, X is the non-empty group of receiving agents and φ is the content of the assertion. Such a communicative action is a complex dynamic operator. It takes a proposition $\psi \in \mathcal{L}_A^{\uparrow,\downarrow,*}$, to make a proposition $\boxed{a \ \ \mathsf{assert}\ \ \varphi \rangle X}\ \psi$, where ψ is a consequence of the assertion. Our first proposal for interpretation of such an assertion proposition is the following:

$$\boxed{a \ \ \mathsf{assert}\ \ \varphi \rangle X}\ \psi = \Box_a \varphi \wedge [\mathsf{C}_{\{a\}\cup X}\varphi]_u \psi.$$

Here we presumed that all members of the receiving group X are aware of the other receiving agents. This is typically not an interpretation of mass-communication[4].

In terms of this interpretation, the Byzantine protocol does not have finite success. Let φ_0 be the crucial proposition, and let $\varphi_i = \Box_a \varphi_{i-1}$ if i is even, and $\varphi_i = \Box_b \varphi_{i-1}$ if i is odd. For all $n \in \mathbb{N}$:

$$\not\vdash_{C^{3*}} \boxed{c \ \ \mathsf{assert}\ \ \varphi_n \rangle x_n} \ldots \boxed{c \ \ \mathsf{assert}\ \ \varphi_0 \rangle x_0}\ \mathsf{C}_{\{a,b\}}\varphi_0.$$

4.31. Observation. Another simple observation on this interpretation of assertion is the following:

$$\boxed{a \ \ \mathsf{assert}\ \ \varphi \rangle X}\ \psi, \Box_a \chi \vdash_{C^{3*}} \boxed{a \ \ \mathsf{assert}\ \ \varphi \wedge \chi \rangle X}\ \psi, \text{ and}$$

$$\neg\Box_a\varphi, \boxed{a \ \ \mathsf{assert}\ \ \varphi \rangle X}\ \psi \vdash_{C^{3*}} \bot.$$

The first $\mathbf{C}^3$-sequent can be put more general:

$$\chi \vdash_{C^{3*}} \varphi \Longrightarrow \boxed{a \ \ \mathsf{assert}\ \ \varphi \rangle X}\ \psi, \Box_a \chi \vdash_{C^{3*}} \boxed{a \ \ \mathsf{assert}\ \ \chi \rangle X}\ \psi.$$

An obvious non-$\mathbf{C}^3$-sequent is:

$$\boxed{a \ \ \mathsf{assert}\ \ \varphi \rangle X}\ \psi, \boxed{a \ \ \mathsf{assert}\ \ \chi \rangle X}\ \psi \not\vdash_{C^{3*}} \boxed{a \ \ \mathsf{assert}\ \ \varphi \vee \chi \rangle X}\ \psi.$$

Take $\varphi = p, \psi = q \in \mathbb{P}$ and let $b \in X$. If $\Box_a p$ and $\Box_a q$ both hold, then also

$$\boxed{a \ \ \mathsf{assert}\ \ p \rangle X}\ (\Box_b p \vee \Box_b q) \ \text{ and } \ \boxed{a \ \ \mathsf{assert}\ \ q \rangle X}\ (\Box_b p \vee \Box_b q).$$

Nevertheless, $\boxed{a \ \ \mathsf{assert}\ \ p \vee q \rangle X}\ (\Box_b p \vee \Box_b q)$ is certainly not guaranteed.

---

[4] If all agents in X were in isolation, the update effect would be $\bigwedge_{b \in X} \mathsf{C}_{\{a,b\}} \varphi$.

Bunt [Bunt 1989], in line with [Searle 1969], took the epistemic update effect of any communicative action to be identical to the mutual belief of the preconditions by the sender and receiver(s). This means that the meaning of such an action is fully determined by these preconditions and stipulates in the case of assertion a weaker interpretation.

$$\boxed{a \ \ \mathsf{assert} \ \ \varphi\rangle X} \ \psi = \Box_a\varphi \wedge [\mathsf{C}_{\{a\}\cup X}\Box_a\varphi]_u\psi.$$

Indeed, such an interpretation seems to be more realistic. The members of X rather learn that a believes φ rather than the content φ itself. So, in this case we do not have $\vdash_{C^{3*}} \boxed{a \ \ \mathsf{assert} \ \ \varphi\rangle X} \ \Box_X\varphi$, but still $\vdash_{C^{3*}} \boxed{a \ \ \mathsf{assert} \ \ \varphi\rangle X} \ \Box_X\Box_a\varphi$. Nevertheless, this interpretation is rather weak. The cautiousness of this interpretation does not cause any factual update of other believers.

A more 'pragmatic' interpretation of a φ-assertion can be stipulated by further strengthening of the precondition. A reasonable one is that the sender a keeps open the possibility that at least one of the receivers actively disbelieves φ. In terms of the $\mathbf{C}^3$-models, a has access to some possible world which can be enriched in such a way that at least one agent of the group X has access to a counter-model of φ. This additional constraint tells us in fact that a expects that his assertion has some informational effect on – or is a non-trivial update for – the group X. This contingent *'contribution'-requirement* with respect to the content φ of the assertion leads to the following interpretation:

$$\boxed{a \ \ \mathsf{assert} \ \ \varphi\rangle X} \ \psi = \Box_a\varphi \wedge \Diamond_a\langle\ \rangle_u\neg\Box_X\varphi \wedge [\mathsf{C}_{\{a\}\cup X}\varphi]_u\psi.$$

Somewhat stronger contribution conditions, like $\Box_a\langle\ \rangle_u\neg\Box_X\varphi$ or $\Diamond_a(\langle\ \rangle_u\Box_X\varphi \wedge\langle\ \rangle_u\neg\Box_X\varphi)$, seem reasonable as well. Note that the $\langle\ \rangle_u$ operators express a certain consistency with respect to worlds. Their interpretation coincides with M-operator of Turner (see page 89). Such a consistency claim gives the interpretation of a communicative action a certain 'default' flavor, which has also been advocated in [Perrault 1989] and [Beun 1989].

4.32. Observation. The contribution requirement above forbids trivial updates:

$$\vdash_{C^{3*}} \varphi \quad\Longrightarrow\quad \boxed{a \ \ \mathsf{assert} \ \ \varphi\rangle X} \ \psi \vdash_{C^{3*}} \bot.$$

Another weaker interpretation of the contribution-requirement is a replacement of this disbelief of X of φ by the falsity of the epistemic effect $\mathsf{C}_{\{a\}\cup X}\varphi$. It might be the case that a yet believes that all members of X believe that φ, but nevertheless wants to inform them of the fact that a shares this belief. So, yet a fourth interpretation is the following:

$$\boxed{a \ \ \mathsf{assert} \ \ \varphi\rangle X} \ \psi = \Box_a\varphi \wedge \Diamond_a\langle\ \rangle_u\neg\mathsf{C}_{\{a\}\cup X}\varphi \wedge [\mathsf{C}_{\{a\}\cup X}\varphi]_u\psi.$$

One might claim that this last interpretation is principally another action. We want to point out that a more cautious contribution requirement as above is just more tolerant, and deals satisfactorily with assertions which are meant to update only deeper layers of belief.

In fact, we will extend this style of interpretation structure for other communicative actions. If the intended epistemic effect of an action, performed by a, is χ, the contribution requirement equals $\Diamond_a\langle\ \rangle_u \neg\chi$. For appropriate specification of such an effect χ for different communicative action we need additional expressivity. This is the topic of the next section, which discusses intentional modalities for interpretation of actions like questions.

4.3 Intentions

Individual preferences of communicating agents are very important for interpretation of communicative actions. Such actions also express an intention of agents to reach new information states and the wish of cooperation by the other interactors in order to achieve this together.

These preferences are interpreted over a set of preferential worlds for every world in the model[5]. This approach has also been advocated in [Cohen & Levesque 1990]. A multiple world interpretation is meant to comprehend possible contradictory preferences. It might very well be the case that different preferences can not be fulfilled in one single world[6].

4.33. DEFINITION. A $\mathfrak{C}^{3i}$-model is a quintuple $\langle W, \{R_a\}_{a\in A}, \{P_a\}_{a\in A}, \leq, V\rangle$, where A is a non-empty finite set, and

$$\langle W, \{R_a\}_{a\in A}, \leq, V\rangle \in \mathfrak{C}^3,$$
$$\langle W, \{P_a\}_{a\in A}, V\rangle \in \mathfrak{M}_{\mathfrak{A}}.$$

The new relations P_a are the individual preference relations.

The arbitrary multiple world interpretation of preferential worlds is rather minimal. Below we will discuss different possible restrictions for the preference relations and their interplay with the doxastic accessibilities.

To begin with, we need an additional preference operator $[\mathsf{p}]_a$ for every agent $a \in A$. A proposition of the form $[\mathsf{p}]_a\varphi$ refers to situations where φ holds in all preferential worlds which belong to the agent a. In other words, a's preferences agree on the truth of φ. This additional operator is added to the language

[5] Preferences also appear in other parts of dynamic logic. An example is Veltman's definition of expectation patterns over worlds [Veltman 1991]. These expectation patterns are meant to interpret default information in a dynamic setting. Preferential worlds or models stem from the field of non-monotonic logic [Shoham 1988]. In this field, they are used to model reasoning on the basis of absent information, where only a limited non-monotonic consequence relation over the maximally preferential worlds is used to compensate this lack of information. The clear difference with our pure selection of sets of preferential worlds, is that these logics use the more delicate notion of orders on worlds. In the terminology of preferential semantics, we abstract away from these orders and act as if the maximal preferential worlds are determined beforehand. For a modal approach to preferential semantics see [van Benthem, van Eijck & Frolova 1993].

[6] In [Appelt 1985] the fused possible world semantics have been used in order to deal with incoherent preferences. See appendix 2.5 for a partial interpretation of fused possible world semantics.

$\mathcal{L}_A^{\uparrow,\downarrow}$. For this language we use the abbreviation $\mathcal{L}_A^{\uparrow,\downarrow,[\mathsf{p}]}$. The semantics of the preference operator in terms of the $\mathfrak{C}^{3i}$ is the following.

4.34. TABLE.

$$M, w \models [\mathsf{p}]_a\varphi \Leftrightarrow \forall v \in W : P_a(w, v) \;\Rightarrow\; M, v \models \varphi$$

$$M, w \dashv [\mathsf{p}]_a\varphi \Leftrightarrow \exists v \in W : P_a(w, v) \;\&\; M, v \dashv \varphi$$

Instead of $\neg[\mathsf{p}]_a\neg\varphi$, we will use $\langle\mathsf{p}\rangle_a\varphi$. It says that φ holds with respect to at least one of the situations which are preferred by the agent a.

The underlying system is very simple because we have left the preference relations completely free in definition 4.33. The system $\mathbf{C}_i^3$ is simply $\mathbf{C}^3$ with the minimal modal rules R-TRUE $\Box$ and L-FALSE $\Box$ for the operators $[\mathsf{p}]_a$. The system $\mathbf{C}_i^{3*}$ is the system $\mathbf{C}^{3*}$ with these additional rules for $[\mathsf{p}]_a$. Of course, such systems are much too liberal for reasoning about individual preferences. We now turn to possible constraints on the infrastructure of these preferential worlds.

Realistic preferences

We first consider constraints corresponding to principles of introspection. Just like for the epistemic modality we accept full introspection on the preferential worlds. In other words, every agent is completely certain about his own preferences. The set of preferential worlds of an agent a is identical with the set of preferential worlds of a in all his doxastic alternatives. Formally, for all $M = \langle W, \{R_a\}_{a\in A}, \{P_a\}_{a\in A}, V\rangle \in \mathfrak{C}^{3i}$ we take

$$\forall x, y, z \in W : R_a(x, y) \implies (P_a(x, z) \Leftrightarrow P_a(y, z)) \text{ for all } a \in A.$$

The corresponding dynamic epistemic inference system can be established by means of the following additional axioms to the systems $\mathbf{C}_i^3$ and $\mathbf{C}_i^{3*}$.

4.35. TABLE. $\Box_a[\mathsf{p}]_a\varphi \equiv \Diamond_a[\mathsf{p}]_a\varphi \equiv [\mathsf{p}]_a\varphi$

$$\Box_a\langle\mathsf{p}\rangle_a\varphi \equiv \Diamond_a\langle\mathsf{p}\rangle_a\varphi \equiv \langle\mathsf{p}\rangle_a\varphi$$

We do not venture to postulate these kind of doubling principles for the preference relation itself. Philosophically speaking, such axioms are pretty risky. If we would use the **45**-rules for the operator $[\mathsf{p}]_a$, then we would accept a perfect satisfaction of every agent with his own preferences. Such 'free will'-principles are disputable in many ways. For example, it expresses complete satisfaction of agents with their own basic biological and psychological motives. We therefore simply avoid such dangerous principles, and determine no pure preference constraints[7].

Further restriction of the interplay of doxastic and preferential worlds depends heavily on the intended application. For example, in [Cohen & Levesque 1990]

[7] Even an axiom $\vdash [\mathsf{p}]_a\top$, which means that every agent a has at least one preferential world, seems to be too strong. We do not exclude complete apathetic agents. Any reader who wishes to postulate a pure preference principle of the Geach format $\langle\mathsf{p}\rangle_a^k[\mathsf{p}]_a^l\varphi \vdash [\mathsf{p}]_a^m\langle\mathsf{p}\rangle_a^n$ is invited to stipulate a semantic constraint for preferential worlds. In chapter 7 this reader can find enough information to capture such a principle model-theoretically.

preferential operators are used as goal-worlds. This means that an agent attach a certain contingency to these worlds. In a two-valued classical modal logic, this comes down to the constraint that all goal-worlds should be doxastic alternatives as well: $P_a \subseteq R_a$. The corresponding axiom is $\Box_a\varphi \vdash [\mathsf{p}]_a\varphi$, which is called the *principle of realism* in [Cohen & Levesque 1990][8]. In partial or constructive modal logic, such a subset relation has to be encoded by an additional contra-position of the last axiom as well.

REALISM $\langle\mathsf{p}\rangle_a\varphi \vdash \Diamond_a\varphi$ & $\Box_a\varphi \vdash [\mathsf{p}]_a\varphi$

This principle is very strong. For example, if the agent a has a goal-world in which a believes a certain proposition φ, then a factually believes it:

$$\langle\mathsf{p}\rangle_a\Box_a\varphi \vdash \Diamond_a\Box_a\varphi \vdash \Box_a\varphi.$$

In other words, the complete use of the principle of realism implies a certain unwillingness of agents to learn new things. In [Cohen & Levesque 1990][9] stronger conceptions of goals, such as 'persistent goals' are incorporated to overcome this epistemic rigidity.

In partial modal logics it is possible to relativize such a principle of realism. As we have no contra-position, we can simply dump the unwanted negative part of this principle. We will also limit our use of the REALISM principle. A complete employment of $\Box_a\varphi \vdash [\mathsf{p}]_a\varphi$ would also lead to unwanted effects. For example, if it is applied to active disbelief, we obtain:

$$\neg\Box_a\varphi \vdash \Box_a\neg\Box_a\varphi \vdash [\mathsf{p}]_a\neg\Box_a\varphi.$$

It tells us that we prefer all our active disbeliefs unambiguously. Of course, this is not what we had in mind. Agents want to move upwards, and one way of doing so is deletion of doubts.

Another problem arises when the REALISM principle is applied to beliefs of others: $\Box_a\Box_b\varphi \vdash [\mathsf{p}]_a\Box_b\varphi$. This means that if an agent a believes something about b's beliefs, then a is completely satisfied with these beliefs of b. This extreme tolerance would remove much of the motives of agents to communicate. Of course, it should be possible to model intentions of agents to remove or revise beliefs of other agents. What is left of the principle of realism is the following cautious formulation.

CAUTIOUS REALISM $\Box_a\varphi \vdash [\mathsf{p}]_a\varphi$ for all $\varphi \in \mathcal{L}$.

This means that we only apply it to extensional information. This yields the following model-theoretic constraint for models $M = \langle W, \{R_a\}_{a\in A}, \{P_a\}_{a\in A}, \leq, V\rangle$:

$$\forall x, y : P_a(x, y) \implies \exists z : R_a(x, z) \ \&\ V(z) \sqsubseteq V(y).$$

It means that all preferential worlds of an agent have more atomic – and propositional – content than at least one of his doxastic alternatives. This does not

[8] Such realism towards your own goals can also be found in formal distinction between wishes and intentions [Appelt 1985] [Devlin 1991].

[9] They employ the logic **KD45** as the underlying epistemic logic. So, this inference can be made in their systems as well.

mean that all these alternatives are preserved in some preferential world. This removes the contra-positional effect of the REALISM postulate[10].

Preferences and the beliefs of others

Principles of cooperative behavior can formally be comprehended as constraints on the structural interplay between doxastic alternatives of an agent a and his preferences about the doxastic alternatives of other agents.

An important principle in formal pragmatics of natural language is Grice's maxim of quality [Grice 1975]. It says that, if an agent a aims at a situation where the agent b believes a certain proposition φ, then a should also be convinced of this information himself. In formal notation:

QUALITY 1 $[\mathsf{p}]_a \Box_b \varphi \vdash \Box_a \varphi$ for all $\varphi \in \mathcal{L}$.

A similar interpretation has also been stipulated in [Beun 1989]. Again, we avoid unwanted mixture of this principle with intensional information, and apply the rule only to extensional information. The corresponding constraint for a model $M = \langle W, \{R_a\}_{a \in A}, \{P_a\}_{a \in A}, \leq, V\rangle$ is the following

$$\forall x, y : R_a(x, y) \ \& \ R_a(x, z) \implies \exists z, z' : P_a(x, z) \ \& \ R_b(z, z') \ \& \ V(z') \sqsubseteq V(y).$$

The converse of this integrity constraint yields a kind of arrogant communicative attitude. It entails obtrusive agents: $\Box_a \varphi \vdash [\mathsf{p}]_a \Box_b \varphi$ for all $\varphi \in \mathcal{L}$. It says that if an agent a prefers worlds where everybody shares his beliefs. This arrogance postulate corresponds to the following semantic constraint

$$\forall x, y, z : P_a(x, y) \ \& \ R_b(y, z) \implies \exists z' : R_a(y, z') \ \& \ V(z') \sqsubseteq V(z).$$

Relativizing this principle of arrogance, by moving the negation in the conclusion to the right hand side yields a socially more acceptable postulate:

QUALITY 2 $\Box_a \varphi, \langle \mathsf{p} \rangle_a \Diamond_b \neg\varphi \vdash \bot$ for all $\varphi \in \mathcal{L}$.

In partial logic, this principle is independent of the pushy principle above. Its denotation is also much more sensible. It says that if an agents a believes φ then he may never prefer a situation where another agent has a counter-model of φ in mind. This postulate can semantically be characterized by the coherence relation $\sim$ on partial valuations. A simple substitution of $\sim$ for $\sqsubseteq$ in the 'arrogant' models above entails the satisfactory model-theoretic constraint[11].

Of course, a principle like QUALITY 2 is equivalent with the principle of arrogance in classical modal logic. The flexibility of partial modal logic, which separates such intuitively different principles, turns out to be an advantage by

[10] Chapter 7 focuses on this non-contra-positional partial modal logics.

[11] For modal correspondences of the coherence relation we refer the reader to chapter 7. In the same way we can relativize other principle. For example, the principle of cautious realism can be transformed into the principle of VERY CAUTIOUS REALISM: $\Box_a \varphi, \langle \mathsf{p} \rangle_a \neg\varphi \vdash \bot$ for all $\varphi \in \mathcal{L}$. The corresponding semantic constraint is obtained by replacing $\sqsubseteq$ by $\sim$ in the characteristics of models for CAUTIOUS REALISM.

its application to interpret rational communicative behavior. We therefore add it to our list of reasons to use partial modal logic instead of classical modal logic, which have been presented in section 1.3. Obviously, this advantage is a side effect of the freeness of contra-position of partial logics, which we already discussed in that section.

Ideal questions

An important facility of additional preferential semantics to the $\mathbf{C}^3$-style of logics, is that other communicative actions can be interpreted. Questions are such actions that require 'intentional' reasoning. Furthermore, more skeptical interpretations of assertions than the ultimate mutual belief updates in section 4.2, can be stipulated by means of preferential worlds. On the basis of preferential operators different utterances, classified by their epistemic and intentional force, can be distinguished semantically.

If we would assign to questions the same ideal dynamic interpretation as we have done for assertions, the epistemic effect of a question φ of a sender a to a group of receivers X is $[\mathsf{C}_{\{a\}\cup X}[\mathsf{p}]_a(\Box_a\varphi \vee \Box_a\neg\varphi)]_u$. This means that it becomes mutual belief of the group $\{a\} \cup X$ that a prefers to know the truth-value of the proposition φ. Preconditions of questions includes a weak contribution requirement, namely a's conceiving of the non-triviality of the utterance: $\Diamond_a\langle\ \rangle_u \neg\mathsf{C}_{\{a\}\cup X}[\mathsf{p}]_a(\Box_a\varphi \vee \Box_a\neg\varphi)$, and furthermore we require the non-triviality of a possible answer to the question: $\Diamond_a\langle\ \rangle_u\varphi \wedge \Diamond_a\langle\ \rangle_u\neg\varphi$. In short,

$$\boxed{a \;\; \mathsf{question} \;\; \varphi\rangle X} \;\; \psi = \Diamond_a\langle\ \rangle_u\varphi \wedge \Diamond_a\langle\ \rangle_u\neg\varphi \wedge \Diamond_a\langle\ \rangle_u\neg\chi \wedge [\chi]_u\psi$$

$$\text{with } \chi = \mathsf{C}_{\{a\}\cup X}[\mathsf{p}]_a(\Box_a\varphi \vee \Box_a\neg\varphi).$$

Of course, we could speculate on many other interpretations of questions, or distinguish different types of questions. An example here is a test action. Such a communicative action should have another dynamic meaning. Suppose that an agent b were testing a group X on the information φ. The epistemic effect is clearly different. The agent b wants to find out, for instance, whether one agent of the group X has information on the truth value of φ. In an ideal interpretation we get the following epistemic effect:

$$\chi := \mathsf{C}_{\{b\}\cup X}[\mathsf{p}]_b\rho \text{ with}$$

$$\rho := \Box_b(\bigvee_{a\in X}\Box_a\varphi \vee \bigvee_{a\in X}\Box_a\neg\varphi) \;\vee\; \Box_b\neg(\bigvee_{a\in X}\Box_a\varphi \vee \bigvee_{a\in X}\Box_a\neg\varphi)$$

The interpretation of this **test** action is then.

$$\boxed{b \;\; \mathsf{test} \;\; \varphi\rangle X} \;\; \psi = (\Box_b\varphi \vee \Box_b\neg\varphi) \wedge \Diamond_b\langle\ \rangle_u\neg\chi \wedge [\chi]_u\psi.$$

As mentioned above, we could also differentiate on assertions. Substitution of $[\mathsf{p}]_a\Box_X\varphi$ for φ in the epistemic effect of the ideal assertion interpretation in section 4.2 yields a much more skeptical interpretation for such assertions. For example, we could distinguish **informing** from **telling**.

We will not further speculate on different communicative actions. We think that deeper analysis would disturb the formal presentation here. We leave it to the reader to make up his own favorite interpretations. It is not our intention to claim the correctness of certain pragmatic postulates, but instead we have shown how such principles can be encoded, and that the machinery of partial and constructive modal logics is a sensible framework for doing so.

Intentions and preferences

A last remark which we should nevertheless make about our framework for communicative interpretation is the formal distinction between intentions and preferences. Somewhat misleading, we have used intentions in the title of this section, while we spoke only about preferences. Of course, the connection is very close, but because formal pragmatics speaks more about intentions than about preferences, a sharp definition of intention is still required.

We follow [Cohen & Levesque 1990] who define intentions in terms of preferential worlds. We take intentions to be defined over actions, such as **assert** and QUESTION. If an action is defined in terms of its preconditions and its dynamic effect, then we say that the intention of an agent a to perform such an action is the same as that a beliefs that the preconditions are fulfilled and prefers a state where the content of the dynamic effect of the action holds.

Let ACT be the set of communicative actions which the agent may use. And let $prec : ACT \times \mathcal{L}_A^{\uparrow,\downarrow,*,[\mathsf{p}]} \longrightarrow \mathcal{L}_A^{\uparrow,\downarrow,*,[\mathsf{p}]}$ and $epi : ACT \times \mathcal{L}_A^{\uparrow,\downarrow,*,[\mathsf{p}]} \longrightarrow \mathcal{L}_A^{\uparrow,\downarrow,*,[\mathsf{p}]}$ be the functions which specify for every action in ACT and every proposition $\varphi \in \mathcal{L}_A^{\uparrow,\downarrow,*,[\mathsf{p}]}$, which is conveyed by the action, its precondition(s) and its epistemic effect, respectively. If $\mathbf{act} \in ACT$, then

$$\boxed{a \ \ \mathsf{act} \ \ \varphi\rangle X} \ \psi = prec(\mathbf{act}, \varphi) \wedge [epi(\mathbf{act}, \varphi)]_u \psi.$$

The intention of an agent a to perform the action **act** with content φ, and with X as the set of receivers, can then be specified formally in the following way:

$$\Box_a prec(\mathbf{act}, \varphi) \wedge [\mathsf{p}]_a epi(\mathbf{act}, \varphi).$$

Most often, just as in many interpretations given above, one of the preconditions of an action is a contribution requirement. This makes sure that if an agent a intends to perform an action **action**, then a believes also its contribution requirement, which yields that intentions are meant to change the current information state.

4.4 Communication with the physical world

So far, we have only considered partial models in our dynamic epistemic formalisms. All relevant information for the proposals for interpretation of communicative action in the previous section is nested by modal epistemic and preference operators, and therefore refers to individual accessible worlds, such as doxastic and preferential worlds, and these worlds are partial, from our point of view.

But what about the real world? Is this a total world, or should we choose for a puritan position? In fact, in this thesis we will avoid philosophical deliberation on this topic. But still, we want to explain what it means technically if an outer total world would be accepted in a theory of partial possible worlds.

Outer worlds

For the model-theory, it is not hard to embed a specific total bi-valent reality. We simply add a set of total possible worlds, which perform the role of a changing physical world. Nevertheless, we take the factual information, i.e. literals, to have a constant truth-value. This choice leads to the following model class $\mathfrak{C}^{3\mathfrak{R}}$.

4.36. DEFINITION. A $\mathfrak{C}^{3\mathfrak{R}}$-model is a quintuple $\langle W, S, \{R_a\}_{a\in A}, \leq, V\rangle$ such that A is finite, and

$$\langle W, \{R_a\}_{a\in A}, \leq, V\rangle \in \mathfrak{C}^3,$$
$$\emptyset \neq S \subseteq W,$$
$$V(s) \in \mathfrak{T} \text{ for all } s \in S, \text{ and}$$
$$V(s)(p) = V(t)(p) \text{ for all } s, t \in S \text{ and } p \in I\!P.$$

The last requirement fixes the factual outer world. The notion of validity is different. We judge propositions of $\mathcal{L}_A^{\uparrow,\downarrow,*}$ only on the basis of the configuration of the real world.

$$[\![\varphi]\!]_{\mathfrak{C}^{3\mathfrak{R}}} = \{\langle M, s\rangle \mid M, s \models \varphi\}$$
$$\Gamma \models_{\mathfrak{C}^{3\mathfrak{R}}} \Delta \Leftrightarrow \bigcap_{\gamma\in\Gamma} [\![\gamma]\!]_{\mathfrak{C}^{3\mathfrak{R}}} \subseteq \bigcup_{\delta\in\Delta} [\![\delta]\!]_{\mathfrak{C}^{3\mathfrak{R}}}$$

The underlying system $\mathbf{C}^{3R}$ is a straightforward extension of $\mathbf{C}^3$, with some supplementary classical rules. The first additional rule is a partial acceptance of R-TRUE $\neg$ from classical logic.

4.37. TABLE. $$\frac{\Gamma, \varphi \vdash \Delta \quad \varphi \in \mathcal{L}^{\uparrow,\downarrow}}{\Gamma \vdash \neg\varphi, \Delta} \quad \text{R-TRUE } \neg \text{ FOR } \mathcal{L}^{\uparrow,\downarrow}.$$

This rule is sound because of the totality of the realities in the models. This means that the local extensional logic in these totalities is classical. Furthermore, the meaning of dynamic intensional information which does not contain individual modal, i.e. $\Box_a$- and $\langle \mathsf{p}\rangle_a$-, operators collapses into extensional information. In other words, they are equivalent to formulae in $\mathcal{L}$.

The other partial re-classicalization of the logic consists of strengthening the R-TRUE introduction of the update operator and the L-FALSE introduction of the downdate operator. These rules may be applied if the contextual sequential parameters are subsets of the sublanguage $\mathcal{L}^{\uparrow,\downarrow}$.

4.38. TABLE. $$\frac{\Gamma, \varphi \vdash \psi, \Delta \quad \Gamma, \Delta \subseteq \mathcal{L}^{\uparrow,\downarrow}}{\Gamma \vdash [\varphi]_u \psi, \Delta} \quad \text{R-STRONG-TRUE } [\;]_u$$

$$\frac{\Gamma, \neg\psi \vdash \varphi, \Delta \quad \Gamma, \Delta \subseteq \mathcal{L}^{\uparrow,\downarrow}}{\Gamma, \neg[\varphi]_d\psi \vdash \Delta} \quad \text{L-STRONG-FALSE } [\ ]_d$$

In fact, these rules re-establish the classical →-introduction for the update operator in purely $\mathcal{L}^{\uparrow,\downarrow}$ sequential contexts. Note furthermore that these rules also settle STRONG versions for L-FALSE and R-TRUE for updates and downdates, respectively. We furthermore obtain a complete extensional flattening of the language $\mathcal{L}^{\uparrow,\downarrow}$ in $\mathcal{L}$.

4.39. OBSERVATION. For all $\varphi \in \mathcal{L}^{\uparrow,\downarrow}$

$$\varphi \equiv_{C^{3R}} \langle\ \rangle_u\varphi \equiv_{C^{3R}} [\ ]_u\varphi \quad [\varphi]_u\psi \equiv_{C^{3R}} \neg\varphi \vee \psi \quad [\varphi]_d\psi \equiv_{C^{3R}} \varphi \vee \psi$$

$$\varphi \equiv_{C^{3R}} \langle\ \rangle_d\varphi \equiv_{C^{3R}} [\ ]_d\varphi \quad \langle\varphi\rangle_u\psi \equiv_{C^{3R}} \varphi \wedge \psi \quad \langle\varphi\rangle_d\psi \equiv_{C^{3R}} \neg\varphi \wedge \psi$$

Extensionally omniscient machines

A more practical use of total realities is human-machine communication, especially in intelligent systems of information retrieval and database querying. In such a dialogue configuration we deal, in principle, with two communicating agents: the system and the user. The reality is the factual internal data of the machine. The system knows what is in this database, but typically has incomplete knowledge of the user's information. In computer terminology: the database is total, while the usermodel is incomplete. Furthermore, the machine is also omniscient with regard to its own data. In the possible worlds terminology, this means that the doxastic worlds of Ω contain the same factual information as the database, i.e. they are realities. This leads to the following subclass of $\mathfrak{C}^{3\mathfrak{R}}$.

4.40. DEFINITION. A $\mathfrak{C}^3_\Omega$ model is a $\mathfrak{C}^{3\mathfrak{R}}$-model $M = \langle W, S, \{R\}_{a\in A}, \leq, V\rangle$ with $A = \{\Omega, \mathbf{u}\}$ and

$$\forall s \in S : R_\Omega(s,t) \Rightarrow t \in S.$$

The system $\mathbf{C}^3_\Omega$ is similar to $\mathbf{C}^{3R}$, with a bit more of classical freedom. The rules R-TRUE ¬, R-TRUE-STRONG $[\ ]_u$ and L-FALSE-STRONG $[\ ]_d$ may now be applied to all formulae of $\mathcal{L}^{\uparrow,\downarrow}_\Omega$.

Furthermore, due to the partial omniscience of Ω, we acquire an equivalence of φ, $\Diamond_\Omega\varphi$ and $\Box_\Omega\varphi$ for the same part of the language. In the sequential style, this boils down to the following rules:

$$\frac{\Gamma, \neg\varphi \vdash \Delta \quad \varphi \in \mathcal{L}^{\uparrow,\downarrow}_\Omega}{\Gamma \vdash \Box_\Omega\varphi, \Delta} \qquad \frac{\Gamma \vdash \varphi, \Delta \quad \varphi \in \mathcal{L}^{\uparrow,\downarrow}_\Omega}{\Gamma, \neg\Box_\Omega\varphi \vdash \Delta}$$

4.41. OBSERVATION. $\varphi \equiv_{C^3_\Omega} \Diamond_\Omega\varphi \equiv_{C^3_\Omega} \Box_\Omega\varphi$ for all $\varphi \in \mathcal{L}^{\uparrow,\downarrow}_\Omega$.

Proof. We only prove the first equivalence. Let $\varphi \in \mathcal{L}^{\uparrow,\downarrow}_\Omega$.

1. $\varphi \vdash_{C^3_\Omega} \varphi$ START
2. $\vdash_{C^3_\Omega} \neg\varphi, \varphi$ R-TRUE ¬ (1)
3. $\Diamond_\Omega\varphi \vdash_{C^3_\Omega} \varphi$ table above (2)

4.	$\Box_\Omega\varphi \vdash_{C^3_\Omega} \Diamond_\Omega\varphi$	**D** (1)
5.	$\varphi, \neg\varphi \vdash_{C^3_\Omega} \emptyset$	L-TRUE $\neg$ (1)
6.	$\varphi \vdash_{C^3_\Omega} \Box_\Omega\varphi$	left rule in table above (5)
7.	$\varphi \vdash_{C^3_\Omega} \Diamond_\Omega\varphi$	CUT (4,6)

The other equivalences are left to the reader. The equivalence of $\Diamond_\Omega\varphi$ and $\Box_\Omega\varphi$ can be obtained by using the rules above in combination with the rules of introspection, i.e. the **45**-rules. ■

The rules for $\mathbf{C}^3_\Omega$ lead to the same extensional flattening of $\mathcal{L}^{\uparrow,\downarrow}_\Omega$ into $\mathcal{L}$ as which we have found for $\mathcal{L}^{\uparrow,\downarrow}$ in $\mathbf{C}^{3R}$ in observation 4.39.

4.5 Conclusions and reflections

In this chapter we have shown how the machinery of partial modal logics and their constructive extensions can be employed for distributive dynamic interpretations of communicative actions. As argued in the introductory chapter, we prefer to combine constructive and eliminative dynamics in one uniform logical framework. Above, we have illustrated how such a two-dimensional dynamics can be used in a general setting of theories of communication.

We have kept our presentation free from severe discussions on the legitimacy of different kinds of interpretations of communicative actions. It has been our aim to show that partial and constructive modal logics can be used for describing such interpretations. Furthermore, the logical weakness of these systems turned out to be advantageous for relativizing pragmatic principles.

As a consequence, we have broken off our outline rather abruptly. This has been done on purpose. Further philosophical speculation would simply reach above the limits of the mathematical skies of this thesis. The point of return has come, and we will get back to the mathematics of these dynamic logics over partial states in the forthcoming part II.

As we have focussed on the alternative prospects of partial worlds for dynamic logics, we did not investigate many relevant additional tools for modeling communication which are known from classical modal logics. We have only focussed on essential extensions by means of mutual belief and preferential operators.

As mentioned in chapter 1, further scaling of modal information would contribute to theories of communication in order to assign degrees to attitudes. In classical settings we find many of such formalisms. For example, assigning probabilities to worlds, such as proposed in different classical modal settings [Gärdenfors 1975] [van der Hoek 1992], is both intuitively appealing and could be of importance to theories of communication. Dynamic interpretations of communicative actions could then be decomposed in terms of the reliability of information as well.

In [van Benthem, van Eijck & Frolova 1993] different suggestions have been given to incorporate ordered preferential semantics in a modal logical style as well. Such an approach for partial modal logic could be very useful for stipulating more fine-grained interpretations of preferences and intentions in combination with the construction-elimination dynamics of partial modal logics.

From a wider perspective, there are many other more general challenges for our dynamic theory of partial states. First order logical extensions of partial and constructive modal logic would be of particular interest[12]. Other relevant extensions are temporal extensions. These kinds of extensions would supply enough equipment for partial logic to imitate expressive classical systems such as in [Cohen & Levesque 1990][13].

Another important issue with respect to theories of communication is the reliability of agents, instead of the above-mentioned differentiation of the quality of information. We could relate the epistemic force of an utterance to the authority that a receiver assigns to the sender with respect to this utterance. Interpretation of this kind of reliabilities calls for more relational semantics. Besides static and dynamic interpretation of propositions, we could incorporate an epistemic interpretation of a proposition φ which induces an *expertise order* over the set A of agents. The epistemic force of an utterance φ from a sender a to a receiver b would then depend on the reliability orderings that a and b associate to φ. Similar ideas can be found in so-called *belief dependency* logics as in [Zhisheng Huang 1991] and [Zhisheng Huang & van Emde Boas 1993].

A recent approach to communication which seems to be very promising is the use of *constructive type theories*, which has been propagated in [Ahn 1992] for usermodeling in man-machine communication. In a more general style [van Benthem 1993] advocates this direction for epistemic logic. These type systems are originally meant for automated proof checking , e.g. [de Bruyn 1980]. They consist of powerful typed λ-calculi which are meant for representation of propositions *and* their proofs. This means that these systems have an essentially richer information structure than standard logics. They keep track of the arguments of propositions and this creates more possibilities to interpret and generate communicative behavior. However, just like the constructive propositional logics which have been discussed in the thesis, these systems are too rigid for common sense modeling as they are meant for reasoning about proofs. Furthermore, they model only *one* mathematical reasoner. In [Borghuis 1993] the reader finds modal epistemic extensions of these λ-calculi, which are meant to overcome this loneliness and rigidity.

As the logics which we have presented are not widely known, and some of them are unknown, we will withdraw to their meta-theory. All the possible extensions which we have mentioned above are therefore taken to be challenges for future research. Instead, we wish to give a solid mathematical analysis of the basic formalisms of this thesis, which are also meant to support such future explorations.

---

[12] For a survey on first order partial modal logics see [Huertas 1994].

[13] Another aspect of communication which is relevant for modal approaches to communication is normative behavior. In [Weigand 1993] the reader finds a survey on the use of *deontic logic* for normative interaction.

Part II

Chapter 5

Completeness

This chapter introduces a uniform method to derive completeness results for the basic logics which were presented in the preceding chapters of this part. Completeness is the converse result of soundness. We have interpreted a sequential derivation system **S** in terms of a consequence relation $\models_{\mathfrak{S}}$ over a class of models $\mathfrak{S}$. The symbolic formulation of completeness of this system with respect to $\mathfrak{S}$ looks as follows:

$$\Gamma \models_{\mathfrak{S}} \Delta \implies \Gamma \vdash_S \Delta \quad \text{for all } \Gamma, \Delta \subseteq \mathcal{L}_S.$$

The method which we will use is based on the well-known Henkin- or canonical style of completeness proving. We refer the reader to [Hughes & Cresswell 1984] and [Chellas 1980] for systematic presentation of this style for classical modal logics. As we will find out in the course of this chapter, we need to revise some of the known basic definitions of this Henkin style in order to let the machinery run fluently for partial logics.

The first section presents this modified, often even generalized, basic equipment. The second section justifies these modifications, in the sense that the Henkin method properly fixes completeness for the extensional and intensional basic systems **P** and **M**. It shows that partial systems do not behave worse than classical propositional and modal logic when it comes to proving completeness.

In the third section the reader finds the same method applied to constructive propositional and modal logics. These latter systems need some special attention because of their particular frame characteristics, that is the interplay of accessibility and the information pattern which we have met in section 3.4 on the semantics of these logics. The last section focuses on completeness for the up-and-down systems. Accomplishment of their completeness is simply an imitation of the results which are to be presented in the third section. The supplemental expressivity for judgement of the logical consequences of retraction of information requires some adaptation, though the needed extra technical effort is small.

Completeness theorems of the mutual belief logics $\mathbf{E}^*_A$ and $\mathbf{C}^{3*}$ are postponed until the following chapter on finite models. Due to the non-compactness of such mutual belief logics, we need stronger meta-theoretical means. This is due to the implementation of the common belief operator. In line with well-known completeness proofs for modal logics with reflexive transitive closure operators [Kozen & Parikh 1981] [Halpern & Moses 1992] , such as the mutual belief operator, we exhibit a completeness result with respect to finite models. This may seem queer, as it is a stronger result than just ordinary completeness. The price is of course a somewhat weaker completeness result, which applies only to finite sequents.

We will no longer dwell upon issues of later chapters here. We turn instead to the basic principles needed to develop the technical equipment for a solid meta-theory of partial intensional logics.

5.1 Saturated sets

In classical logic the notion of *maximally consistent* sets of formulae is the essential ingredient of Henkin-style completeness proofs. In classical modal logic such maximally consistent sets have the same function [Hughes & Cresswell 1984]. Such a set of formulae is *consistent* with respect to the underlying logic, and it is *maximal* in the sense that it does not have proper consistent extensions. The utility of maximally consistent sets is that they enable us to get grip on the semantic entities, like valuations in propositional logic, worlds in modal logic or interpretations and assignments in predicate logic, by syntactic means. A completeness theorem is then accomplished by making two steps.

One of these steps justifies this semantic use of maximally consistent sets. It guarantees that the elements of such a set coincide with those formulae which it verifies. This result, which is called the *truth lemma* of a logic, tells us that maximally consistent sets behave like worlds.

The other step, which is normally made first, is called the *Lindenbaum lemma*, and states that every consistent set can be extended to a maximally consistent set. This result is normally established by adding as much information to a given consistent set as long as one can, i.e. without losing consistency. The limit of this construction turns out to be a maximally consistent set.

The completeness recipe is the following. On account of the Lindenbaum lemma we obtain that whenever a set of formulae Δ is not a conclusion set of a set of formulae Γ, there must exist a maximally consistent extension of $\Gamma \cup \neg\Delta$, let us say Γ^*. The truth lemma tells us that Γ^* verifies all members of Γ, but none of Δ. Therefore Γ^*, interpreted as a world, provides a counter-example proving that Δ is not a valid consequence of Γ.

The requirement of maximal consistency turns out to be too strict for selecting sets of formulae in order to simulate the Henkin procedure for partial logics properly. We have to be more liberal in accepting sets of formulae as valuations or worlds. It could be the case that a formula φ is neither verified nor falsified. If we want to imitate such gaps of truth-assignments by means of sets of formulae, then these sets do not necessarily have to be maximal. The following concept of *saturated set* defines how such sets should behave if we want to simulate partial states. Original definitions of saturated sets can be traced back to [Aczel 1968] and [Thomason 1968] [1]. Our definition of saturated sets of formulae, which we present in the sequential style of the preceding chapters, boils down to maximal consistency for total logics. In other words, saturation is not an adjustment of maximal consistency, but is rather a generalization. We present its definition in such a way that the forthcoming results can be applied just as easily to total logics as to the partial logics which we focus on.

5.1. DEFINITION. Let **S** be a certain sequential derivation system, and let $\mathcal{L}_S$ be its language.

S is *consistent* iff $\emptyset \nvdash_S \emptyset$.

A set of formulae $\Gamma \subseteq \mathcal{L}_S$ is said to be **S**-*consistent*, whenever $\Gamma \nvdash_S \emptyset$.

A set of formulae $\Gamma \subseteq \mathcal{L}_S$ is said to be **S**-*saturated* whenever for all $\Delta \subseteq \mathcal{L}_S$:

$\Gamma \vdash_S \Delta \Rightarrow \Delta \cap \Gamma \neq \emptyset$.

The collection of all **S**-saturated sets will be denoted by $\mathfrak{Sat}_S$ in the sequel of the text.

The criterion of saturation is the converse of the START rule. This can be seen as the most basic step of inference in the sequential systems of part I. In this sense, a saturated set can be seen as an ideal assumption set. Every consequence set is already represented in the assumption set by means of at least one of its elements.

5.2. OBSERVATION. An **S**-saturated set is always **S**-consistent, because whenever a set Γ is not **S**-consistent, then $\Gamma \vdash_S \emptyset$, but $\emptyset \cap \Gamma = \emptyset$.

Maybe the definition of **S**-consistency is not immediately clear. The best way to explain its definition is that whenever $\emptyset$ is derivable from a set Γ, then everything must be derivable from Γ by the rule R-MON: $\Gamma \vdash_S \Delta$ for all $\Delta \subseteq \mathcal{L}_S$. Of course, not every system contains this rule, but in this paper it is contained by all the derivational systems which were put on the stage in part I. We prefer this style of definitions because it does not make use of linguistic specific properties such as connectives.

In [Aczel 1968] saturated sets are presented as saturated *theories*. The word theory indicates that they are deductively closed. This automatically follows from our sequential definition, when we substitute singletons for the right hand sequence:

$\Gamma \vdash_S \delta \Longrightarrow \delta \in \Gamma$.

If finite **S**-conclusion sets might be replaced by disjunctions, which is the case in the systems that we discuss here, we also obtain

$\Gamma \vdash_S \varphi \vee \psi \Longrightarrow \varphi \in \Gamma$ or $\psi \in \Gamma$

[1] In these articles saturated sets are used to give completeness proofs for intuitionistic predicate logic.

for **S**-saturated sets. Sets of formulae which satisfy this property are called *prime* in [Aczel 1968]. Sets which are consistent, prime and deductively closed are called saturated there. Whenever **S** satisfies $\Gamma \vdash_S \varphi \vee \psi \Leftrightarrow \Gamma \vdash_S \varphi, \psi$, our definition coincides with Aczel's. Again, for reasons of linguistic independence[2], and also its conciseness, we prefer our sequential definition.

This definition of saturated sets generalizes the notion of maximal consistency in classical logic indeed, because whenever a set is saturated with respect to the sequential system for classical logic, the rule R-TRUE $\neg$, i.e. $\Gamma, \varphi \vdash \Delta \Longrightarrow \Gamma \vdash \neg\varphi, \Delta$, forces it to be maximal as well. Suppose that Γ is such a saturated set, related to a sequential derivation system for classical logic. Clearly $\Gamma, \varphi \vdash \varphi$, and therefore also $\Gamma \vdash \neg\varphi, \varphi$. This entails for all formulae φ either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$. If Γ had a proper saturated extension Δ, then there exists $\varphi \in \Delta$ such that $\varphi \notin \Gamma$. Given the earlier observation, this entails $\neg\varphi \in \Gamma$, and because $\Gamma \subset \Delta$ also $\neg\varphi \in \Delta$. This entails $\Delta \vdash \neg\varphi$, and by L-FALSE $\neg$ and the double negation property in classical logic, we have $\Delta, \varphi \vdash \emptyset$. $\varphi \in \Delta$ makes us conclude $\Delta \vdash \emptyset$, which contradicts the consistency of Δ.

This observation also applies to systems with the weak negation $\sim$ such as $\mathbf{P}^{\sim}$ and $\mathbf{M}^{\sim}$. In short, $\mathbf{P}^{\sim}$- and $\mathbf{M}^{\sim}$-saturation coincides with maximal $\mathbf{P}^{\sim}$- and $\mathbf{M}^{\sim}$-consistency respectively.

Saturation lemmas

The construction of a maximally consistent set out of a given consistent set, which proves the above-mentioned Lindenbaum lemma in classical logic, is most often carried out without limitations of building materials. Maximality is just the final stage of piling up arbitrary formulae, with the only restriction that we are not allowed to give up the consistency. In partial logic the construction of saturated sets is often more restricted in the sense that certain formulae are simply prohibited from being used as such building material. Especially in proving truth lemmas for partial modal formalisms, as we will see in the completeness proof procedure for **M**, saturated sets have to be built inside another set. Such a limiting set is normally given in advance as an upper bound of the construction. The generalization of the Lindenbaum lemma which we will present, guarantees a successful construction whenever this upper bound is rich enough to intersect all sequences which are derivable from the set with which we start the construction. The following definition gives the precise prescriptions of such upper bounds. We will call these upper bounds *saturators*.

5.3. DEFINITION. Let **S** be a sequential derivation system and $\mathcal{L}_S$ its language. $\Lambda \subseteq \mathcal{L}_S$ is an **S**-*saturator* of a set $\Gamma \subseteq \mathcal{L}_S$ whenever for all $\Delta \subseteq \mathcal{L}_S$:

$$\Gamma \vdash_S \Delta \Rightarrow \Delta \cap \Lambda \neq \emptyset.$$

We will call Γ an **S**-*saturant* of Λ. We abbreviate this relation between Γ and Λ by $\Gamma \trianglelefteq_S \Lambda$.

[2] Such dependence is relevant if we wish to arrange completeness for systems with a $\vee$-free language. An illustrative example is a partial logic with weak Kleene conjunction $\triangle$ and the strong negation, as we have discussed in chapter 2 (page 48).

5.4. OBSERVATION. Note that every **S**-saturant Γ must also be **S**-consistent: $\Gamma \not\vdash_S \emptyset$. The reader should be aware of the possible **S**-inconsistency of **S**-saturators. A simple example is the full language $\mathcal{L}$ which is a **P**-saturator of every **P**-consistent set. In general we obtain

$$\Gamma \trianglelefteq_S \mathcal{L}_S \Leftrightarrow \Gamma \text{ is } \mathbf{S}\text{-consistent.}$$

If **S** contains the L-MON rule and $\Gamma \trianglelefteq_S \Lambda$, then we also have $\Gamma' \trianglelefteq_S \Lambda$ for every subset Γ' of Γ. Note furthermore that $\Gamma \trianglelefteq_S \Lambda'$ for all $\Lambda' \supseteq \Lambda$, whenever $\Gamma \trianglelefteq_S \Lambda$.

The definition of saturator expresses a relative richness with respect to the deductive range of its saturants. The meaning of an **S**-sequent $\Gamma \vdash_S \Delta$, which we presented in part I, was the guarantee that if all elements of the assumption set Γ hold, some $\delta \in \Delta$ also holds. In this regard, a saturator Λ of Γ is rich enough to select at least one the members of Δ.

The generalization of the classical Lindenbaum lemma, which we will baptize as the *bounded saturation lemma*, shows that the richness of a saturator is sufficient to guarantee that a saturated extension can be found for every saturant. This result applies to all the systems which contain the START rule, the two monotonicity rules and the CUT rule, hence to all the systems which we discussed in the earlier chapters. In fact, this forthcoming result identifies the definition of saturator as being the precise requirement for an upper bound to contain a saturated extension. Notice that the converse of this bounded saturation lemma is a trivial statement. If Λ is not a saturator of Γ, then it cannot possibly contain a saturated extension of Γ (L-MON).

Let $\Gamma \trianglelefteq_S \Lambda$. Our aim is to find a $\Gamma^* \in \mathfrak{Sat}_S$ such that $\Gamma \subseteq \Gamma^* \subseteq \Lambda$. The procedure to obtain this result is by adding only formulae from Λ to Γ in such a way that Λ does not have to give up its role as **S**-saturator. The construction is more careful then the construction of maximally consistent sets in classical logic, as this proposed addition procedure implies the maintenance of consistency according to observation 5.4.

The following lemma shows that such additions to Γ are always possible from its **S**-conclusion sets of the saturant.

5.5. LEMMA. Let **S** be a sequential derivation system which contains the CUT rule. If $\Gamma \trianglelefteq_S \Lambda$ and $\Gamma \vdash_S \Delta$ for certain finite set $\Delta \subseteq \mathcal{L}_S$, then there exists $\delta \in \Delta$ such that $\Gamma + \delta \trianglelefteq_S \Lambda$.

Proof. Let $\Gamma \trianglelefteq_S \Lambda$ and $\Gamma \vdash_S \Delta$ with Δ finite, and suppose that $\Gamma + \delta \not\trianglelefteq_S \Lambda$ for all $\delta \in \Delta$. This means that for all $\delta \in \Delta$ there exists $\Sigma_\delta \subseteq \mathcal{L}_S$ such that

$$\Gamma, \delta \vdash_S \Sigma_\delta \quad \text{and} \quad \Sigma_\delta \cap \Lambda = \emptyset.$$

Let $\Sigma := \bigcup_{\delta \in \Delta} \Sigma_\delta$. L-MON yields $\Gamma, \delta \vdash_S \Sigma$ for all $\delta \in \Delta$. Application of CUT to this last **S**-sequent and the assumption $\Gamma \vdash_S \Delta$ yields $\Gamma \vdash_S \Delta - \delta, \Sigma$. Repetition of CUT-application for all δ's eliminates the complete Δ from the last **S**-sequent. In short,

$\Gamma \vdash_S \Sigma$. Because $\Gamma \trianglelefteq_S \Lambda$ we conclude $\Sigma \cap \Lambda \neq \emptyset$. This contradicts that $\Sigma_\delta \cap \Lambda = \emptyset$ for all $\delta \in \Delta$. ■

This result is responsible for the success of the proposed construction of **S**-saturated sets which we had in mind.

5.6. LEMMA. BOUNDED SATURATION LEMMA (BSL)

Suppose **S** is a sequential derivation system containing the structural rules START, L-MON, R-MON and CUT. If $\Lambda \subseteq \mathcal{L}_S$ be an **S**-saturator of $\Gamma \subseteq \mathcal{L}_S$, then Λ contains an **S**-saturated set Γ^* such that $\Gamma \subseteq \Gamma^*$. In formal transcription:

$$\forall\, \Gamma, \Lambda \subseteq \mathcal{L}_S : \ \Gamma \trianglelefteq_S \Lambda \quad \Rightarrow \quad \exists\, \Gamma^* \in \mathfrak{Sat}_S : \ \Gamma \subseteq \Gamma^* \subseteq \Lambda$$

Proof. Let $\Gamma \trianglelefteq_S \Lambda$ and let $\{\varphi_i\}_{i \in \mathbb{N}}$ be an enumeration of Λ. We define the following sequence of subsets of $\mathcal{L}_S$

$$\Gamma_0 := \Gamma$$

$$\Gamma_{n+1} := \begin{cases} \Gamma_n + \varphi_n & \text{if } \ \Gamma_n + \varphi_n \trianglelefteq_S \Lambda \\ \Gamma_n & \text{otherwise.} \end{cases}$$

Furthermore we take $\Gamma^* \subseteq \mathcal{L}_S$ to be the limit of this sequence:

$$\Gamma^* := \bigcup_{n \in \mathbb{N}} \Gamma_n.$$

$\Gamma \subseteq \Gamma^* \subseteq \Lambda$ is immediately clear from the definition of Γ^* above. Another direct consequence of the construction above is $\Gamma_n \trianglelefteq_S \Lambda$ for all $n \in \mathbb{N}$. What is left to show is $\Gamma^* \in \mathfrak{Sat}_S$.

Suppose $\Gamma^* \vdash_S \Delta$. We need to prove $\Gamma^* \cap \Delta \neq \emptyset$. The assumption set can be reduced to a finite sequence $\gamma_1, .., \gamma_m$ in Γ^* such that $\gamma_1, .., \gamma_m \vdash_S \Delta$. Because every member of Γ^* is a member of some Γ_i, this means that there exists Γ_k such that $\{\gamma_1, .., \gamma_m\} \subseteq \Gamma_k$ (take for example $k = \max_{i \in \{1,..,n\}} \Gamma_{n_i}$ where $\{\Gamma_{n_i}\}_{i=1}^m$ is a subsequence of $\{\Gamma_n\}_{n \in \mathbb{N}}$ with $\gamma_i \in \Gamma_{n_i}$), and thus $\Gamma_k \vdash_S \Delta$ according L-MON. Since $\Gamma_k \trianglelefteq_S \Lambda$, we also have $\Delta \cap \Lambda \neq \emptyset$. Because $\Delta \subseteq \mathcal{L}_S$ has been picked arbitrarily as an **S**-conclusion set of Γ^* we have $\Gamma^* \trianglelefteq_S \Lambda$. This conclusion, combined with lemma 5.5, guarantees the existence of a formula $\delta \in \Delta$ such that

$$\Gamma^* + \delta \trianglelefteq_S \Lambda.$$

This result also ensures that $\Gamma_n + \delta \trianglelefteq_S \Lambda$ for all $n \in \mathbb{N}$ whenever $\Gamma^* \vdash_S \Delta$, because all these sets are subsets of the limit set Γ^* (observation 5.4). Obviously, $\delta \in \Lambda$, which means that there exists $l \in \mathbb{N}$ such that $\varphi_l = \delta$. Because $\Gamma_l + \varphi_l \trianglelefteq_S \Lambda$, we know that $\delta \in \Gamma_{l+1}$ by the inductive definition of the sequence $\{\Gamma_n\}_{n \in \mathbb{N}}$. We conclude $\delta \in \Gamma^*$, and so $\Gamma^* \cap \Delta \neq \emptyset$. This establishes the desired result: $\Gamma^* \in \mathfrak{Sat}_S$. ■

Also the proof of this Lindenbaum-like lemma is a generalization of the normal proof of the classical Lindenbaum lemma. In order to obtain maximally consistent sets out of consistent sets, we add φ_n to Γ_n in the n-th construction step of such a maximally consistent set whenever it is consistent with Γ_n. In this proof the parameter Λ is taken to be constantly the full language $\mathcal{L}_S$. According to the last remark in observation 5.4, this simplification turns the addition test in the inductive construction of Γ^* into an ordinary consistency check, and subsequently presents the classical proof of the Lindenbaum lemma for maximally consistent sets. This pictures the subtlety of the procedure in BSL, when we compare it to the relatively rough construction in the classical Lindenbaum lemma.

BSL turns out to be useful in proving completeness results on the basis of canonical Henkin models, whenever we have to look for saturated sets in a certain direction. Such manipulation from above is particularly relevant in the completeness proof for **M**, but also for extensions of constructive logics with additive non-persistent connectives such as $\mathbf{N}^{\sim}$ and $\mathbf{NM}^{\Box}$.

Lemma 5.6 has been given the name bounded saturation lemma because it is an equivalent formulation of the so-called *saturation lemma*, which is widely used for proving completeness of partial and constructive logics [Aczel 1968] [Thomason 1968] [Veltman 1985] [Troelstra & van Dalen 1990] [Thijsse 1992][3]. It says that if a formula φ is not derivable from an assumption set Γ, then there exists a saturated set which contains Γ but not φ. A sequential reformulation of this result comes down to the existence of a saturated set Σ such that $\Gamma \subseteq \Sigma$ and $\Sigma \cap \Delta = \emptyset$ for all $\Gamma, \Delta \subseteq \mathcal{L}_S$ with $\Gamma \not\vdash_S \Delta$. The equivalence of these two lemmas can easily be deduced from an observation made by Elias Thijsse, which relates the notion of saturator and non-derivability.

5.7. PROPOSITION. If **S** is a sequential derivation system which contains the R-MON rule, then

$$\Gamma \trianglelefteq_S \Lambda \quad \Leftrightarrow \quad \Gamma \not\vdash_S \Lambda^{\mathsf{c}}.$$

Proof. $\Rightarrow$: Suppose $\Gamma \vdash_S \Lambda^{\mathsf{c}}$. Clearly $\Lambda^{\mathsf{c}} \cap \Lambda = \emptyset$, and therefore $\Gamma \not\trianglelefteq_S \Lambda$.
$\Leftarrow$: Suppose $\Gamma \not\trianglelefteq_S \Lambda$. This means that there exists $\Delta \subseteq \mathcal{L}_S$ such that $\Gamma \vdash_S \Delta$ and $\Delta \cap \Lambda = \emptyset$, or in other words, $\Delta \subseteq \Lambda^{\mathsf{c}}$. Hereupon, R-MON entails $\Gamma \vdash_S \Lambda^{\mathsf{c}}$. ■

The precise formulation of the saturation lemma is given below.

5.8. LEMMA. SATURATION LEMMA

Let **S** be a derivational system as in BSL.

$$\forall\, \Gamma, \Delta \subseteq \mathcal{L}_S : \ \ \Gamma \not\vdash_S \Delta \quad \Rightarrow \quad \exists \Sigma \in \mathfrak{Sat}_S : \ \Gamma \subseteq \Sigma \ \&\ \Sigma \cap \Delta = \emptyset.$$

Proof. Suppose $\Gamma \not\vdash_S \Delta$. Proposition 5.7 shows that $\Gamma \trianglelefteq_S \Delta^{\mathsf{c}}$. BSL proves the existence of an **S**-saturated set Σ such that $\Gamma \subseteq \Sigma \subseteq \Delta^{\mathsf{c}}$, or in other words, $\Gamma \subseteq \Sigma$ and $\Sigma \cap \Delta = \emptyset$. ■

The proof above shows us that BSL implies the saturation lemma. A demonstration that the saturation lemma is equivalent to BSL for systems with the structural rules which have been mentioned in these lemmas, can also be obtained with the help of Thijsse's proposition 5.7. If $\Gamma \trianglelefteq_S \Lambda$ then also $\Gamma \not\vdash_S \Lambda^{\mathsf{c}}$,

[3] In [Thijsse 1992] this saturation lemma has been called generalized Lindenbaum lemma.

and thereupon the saturation lemma entails an **S**-saturated set Σ such that $\Gamma \subseteq \Sigma$ and $\Sigma \cap \Lambda^{\complement} = \emptyset$, or in other words $\Gamma \subseteq \Sigma \subseteq \Lambda$.

Our understanding of saturated sets is now sufficient to prove the relevant truth lemmas for the systems which were introduced in the preceding chapters, and the demonstration of their consequential completeness.

5.2 Completeness of P and M

It is easy to show that the propositional system **P** is complete with respect to $\mathfrak{P}$-validity, by means of the more conventional saturation lemma [Thijsse 1992]. Suppose $\Gamma \not\vdash_P \Delta$ for certain $\Gamma, \Delta \subseteq \mathcal{L}$. This means that there exists a **P**-saturated set Γ^* such that $\Gamma^* \cap \Delta = \emptyset$. With every **P**-saturated set Σ we associate a partial valuation V_Σ which is defined completely by the atomic content of Σ:

$$V_\Sigma(p) = 1 \Leftrightarrow p \in \Sigma \quad \text{and} \quad V_\Sigma(p) = 0 \Leftrightarrow \neg p \in \Sigma \quad \text{for all } p \in I\!P.$$

5.9. Observation. Truth lemma **P**

$$V_\Sigma \models \varphi \Leftrightarrow \varphi \in \Sigma \quad \text{and} \quad V_\Sigma \mathrel{=\!\!|} \varphi \Leftrightarrow \neg\varphi \in \Sigma \quad \text{for all } \varphi \in \mathcal{L}.$$

This is the truth lemma formulated for **P**. Besides truth we also refer to falsity ($\mathrel{=\!\!|}$) in its presentation above. The formulation is equivalent with the left conjunct only as a consequence of the truth conditional meaning of the negation $\neg$. The reason to state the result in this way, is that it makes things easier during the inductive proof procedure, which is based on the construction of formulae. In the case of **P** this proof is completely straightforward. To show how it works, we present the falsification step for $\wedge$.

> Suppose $\neg(\varphi \wedge \psi) \in \Sigma$. Because $\neg\varphi \vdash_P \neg\varphi$ and $\neg\psi \vdash_P \neg\psi$ we obtain through application of R-FALSE $\wedge$ $\neg(\varphi \wedge \psi) \vdash_P \neg\varphi, \neg\psi$. Since Σ is **P**-saturated, we obtain $V_\Sigma \mathrel{=\!\!|} \varphi \wedge \psi$ by application of the induction hypothesis ($V_\Sigma \mathrel{=\!\!|} \varphi$ or $V_\Sigma \mathrel{=\!\!|} \psi$, because $\neg\varphi \in \Sigma$ or $\neg\psi \in \Sigma$). The other way around is instantaneously obtained by the induction hypothesis and the definition of falsification of conjunctions by V_Σ.

Because $\Gamma^* \cap \Delta = \emptyset$, we know, by means of the truth lemma above, that $V_{\Gamma^*} \not\models \delta$ for all $\delta \in \Delta$, while $V_{\Gamma^*} \models \gamma$ for all $\gamma \in \Gamma$, for $\Gamma \subseteq \Gamma^*$. Therefore, $\Gamma \not\models_{\mathfrak{P}} \Delta$. For sake of presentation, we give a formal transcription of the completeness result in the following theorem.

5.10. Theorem. Completeness **P**
For all $\Gamma, \Delta \subseteq \mathcal{L}$: $\Gamma \models_{\mathfrak{P}} \Delta \implies \Gamma \vdash_P \Delta$.

In classical modal logics the maximally consistent sets are normally assembled as worlds in one Kripke model [Hughes & Cresswell 1984]. This is the so-called *canonical* or *Henkin* model of the logic. This is what we will do as well in the case of saturated sets with regard to partial Kripke models.

5.11. DEFINITION. The **M**-canonical model is the triple $M_M = \langle \mathfrak{Sat}_M, R_M, V_M \rangle$ with

$$R_M(\Gamma, \Delta) \iff \begin{cases} \Box\varphi \in \Gamma \Rightarrow \varphi \in \Delta & \text{, and} \\ \varphi \in \Delta \Rightarrow \Diamond\varphi \in \Gamma & \text{for all } \varphi \in \mathcal{L}^{\Box}; \end{cases}$$

$$V_M(\Gamma)(p) = 1 \Leftrightarrow p \in \Gamma \quad \text{and} \quad V_M(\Gamma)(p) = 0 \Leftrightarrow \neg p \in \Gamma \quad \text{for all } p \in \mathbb{P}.$$

The definition of V_M makes sure that the proof of the truth lemma succeeds for the propositional variables and the extensional connectives $\bot$, $\neg$ and $\wedge$. Note that this function is well-defined, for $p \in \Sigma$ implies $\neg p \notin \Sigma$ for all $\Sigma \in \mathfrak{Sat}_M$. This is a simple consequence of the consistency of saturated sets (observation 5.4).

The accessibility relation R_M has been defined in such away that it enables us to prove the truth lemma for the intensional connective $\Box$. We can simplify this definition:

$$R_M(\Gamma, \Delta) \Leftrightarrow \Box^-\Gamma \subseteq \Delta \subseteq \Diamond^-\Gamma.$$

This reformulation explicitly states that accessible saturated sets should be contained in a given upper bound, which is imposed by the saturated sets from which this set is accessible. The intuitive idea behind this upper bound is the requirement that possible or accessible worlds should never contain more information than the information which is determined as being possible by the original world (set).

BSL will be of help in respecting these upperbounds, whenever we look after particular accessible saturated sets. In general, the essence of proving completeness for intensional partial systems on the basis of BSL, most often boils down to finding satisfactory saturators. Also in the truth lemma of **M**, which is presented below, BSL facilitates the argumentation[4].

5.12. LEMMA. TRUTH LEMMA **M**

For all $\Gamma \in \mathfrak{Sat}_M$ and $\varphi \in \mathcal{L}^{\Box}$:

$$M_M, \Gamma \models \varphi \Leftrightarrow \varphi \in \Gamma \quad \text{and} \quad M_M, \Gamma \dashv \varphi \Leftrightarrow \neg\varphi \in \Gamma.$$

Proof. The proof runs, as usual, by induction on the construction of formulae. We skip the extensional cases $p \in \mathbb{P}$, $\bot$, $\neg$ and $\wedge$. They follow immediately from the definition of V_M and the induction hypothesis. What is left is an exposition of the induction step of the intensional $\Box$-operator. This part of the proof is established by the right choice of the canonical accessibility relation R_M.

Suppose $\Box\varphi \in \Gamma$.

The definition of the accessibility relation R_M guarantees $\varphi \in \Delta$ for all $\Delta \in \mathfrak{Sat}_M$ such that $R_M(\Gamma, \Delta)$. This entails, on account of the induction hypothesis, that

$$R_M(\Gamma, \Delta) \Rightarrow M_M, \Delta \models \varphi \quad \text{for all } \Delta \in \mathfrak{Sat}_M.$$

According to the truth condition of $\Box\varphi$, this means $M_M, \Gamma \models \Box\varphi$.

Take $\neg\Box\varphi \notin \Gamma$.

[4] Earlier completeness proofs for **M** were troublesome and lengthy, both in Henkin style [Thijsse 1992] and also on the basis of normal forms [Jaspars 1993]

As in the induction step above, the definition of R_M and the induction hypothesis gives us the desired result: $M_M, \Gamma \not\dashv \Box\varphi$

The somewhat more difficult cases are:

$$M_M, \Gamma \models \Box\varphi \Rightarrow \Box\varphi \in \Gamma \quad \text{and} \quad \neg\Box\varphi \in \Gamma \Rightarrow M_M, \Gamma \dashv \Box\varphi.$$

Below we will demonstrate by means of the modal sequential derivation rules of **M**, R-TRUE $\Box$, L-FALSE $\Box$ and the monotonicity rules that the following two claims hold

$(i) \quad \Box\varphi \notin \Gamma \quad \Rightarrow \quad \Box^{-}\Gamma \trianglelefteq_M (\Diamond^{-}\Gamma - \varphi)$, and

$(ii) \quad \neg\Box\varphi \in \Gamma \quad \Rightarrow \quad (\Box^{-}\Gamma + \neg\varphi) \trianglelefteq_M \Diamond^{-}\Gamma.$

If $\Box\varphi \notin \Gamma$, application of the bounded saturation lemma to the first claim (i) guarantees the existence of a $\Delta \in \mathfrak{Sat}_M$ such that $R_M(\Gamma, \Delta)$ and $\varphi \notin \Delta$. The induction hypothesis yields $M_M, \Delta \not\models \varphi$, and subsequently $M_M, \Gamma \not\models \Box\varphi$.

Responding to the second claim (ii), the bounded saturation lemma guarantees the existence of a $\Delta \in \mathfrak{Sat}_M$ such that $R_M(\Gamma, \Delta)$ and $\neg\varphi \in \Delta$ whenever $\neg\Box\varphi \in \Gamma$. This Δ enables us to use the induction hypothesis, and then conclude $M_M, \Gamma \dashv \Box\varphi$.

The claims (i) and (ii) above can be demonstrated through two simple **M**-derivations.

Suppose $\Box\varphi \notin \Gamma$.

1. $\Box^{-}\Gamma \vdash_M \Sigma \implies$ R-MON
2. $\Box^{-}\Gamma \vdash_M \varphi, \Sigma - \varphi \implies$ R-TRUE $\Box$
3. $\Box\Box^{-}\Gamma \vdash_M \Box\varphi, \Diamond(\Sigma - \varphi) \implies$ L-MON[5]
4. $\Gamma \vdash_M \Box\varphi, \Diamond(\Sigma - \varphi)$

Because $\Gamma \in \mathfrak{Sat}_M$ and $\Box\varphi \notin \Gamma$ we know that there exists $\sigma \in \Sigma - \varphi$ such that $\Diamond\sigma \in \Gamma$. Reformulation of this result gives us $\Sigma \cap (\Diamond^{-}\Gamma - \varphi) \neq \emptyset$. Because Σ have been chosen arbitrarily as an **M**-conclusion set of $\Box^{-}\Gamma$, this result establishes $\Box^{-}\Gamma \trianglelefteq_M \Diamond^{-}\Gamma - \varphi$.

Let $\neg\Box\varphi \in \Gamma$.

1. $\Box^{-}\Gamma, \neg\varphi \vdash_M \Sigma \implies$ L-FALSE $\Box$
2. $\Box\Box^{-}\Gamma, \neg\Box\varphi \vdash_M \Diamond\Sigma \implies$ R-MON[6]
3. $\Gamma \vdash_M \Diamond\Sigma$

Because $\Gamma \in \mathfrak{Sat}_M$ this result entails $\Diamond^{-}\Gamma \cap \Sigma \neq \emptyset$, and subsequently $\Box^{-}\Gamma + \neg\varphi \trianglelefteq_M \Diamond^{-}\Gamma$, by the arbitrariness of Σ as a conclusion set of $\Box^{-}\Gamma + \neg\varphi$.

■

5.13. THEOREM. COMPLETENESS OF **M**

For all $\Gamma, \Delta \subseteq \mathcal{L}^{\Box}$: $\quad \Gamma \models_{\mathfrak{M}} \Delta \Longrightarrow \Gamma \vdash_M \Delta$.

Proof. $\Gamma \nvdash_M \Delta \;\Rightarrow\;$ saturation lemma

$\exists\Sigma \in \mathfrak{Sat}_M : \Gamma \subseteq \Sigma \;\&\; \Sigma \cap \Delta = \emptyset \;\Rightarrow\;$ truth lemma **M**

[5] $(\Box\Box^{-}\Gamma = \{\Box\varphi \mid \Box\varphi \in \Gamma\} \subseteq \Gamma)$.
[6] $(\neg\Box\varphi \in \Gamma$ and $\Box\Box^{-}\Gamma \subseteq \Gamma)$.

$\exists \Sigma \in \mathfrak{Sat}_M \; \forall \gamma \in \Gamma \; \forall \delta \in \Delta : \; M_M, \Sigma \models \gamma \; \& \; M_M, \Sigma \not\models \delta \;\; \Rightarrow \;\; \Gamma \not\models_{\mathfrak{M}} \Delta.$ ∎

The proof of the final completeness result is fully analogous to the completeness of **P** as a corollary of its truth lemma. Henceforth we will skip proofs like the one above, because we will continuously use the same procedure.

A little reflection on the proof of the truth lemma for **M** shows that we have made minimal use of specific properties of the system **M**. In fact the argumentation can be copied for any extension **X** of **M**. If $\mathcal{L}_X = \mathcal{L}^\Box$ for such an extension, then the truth lemma of **X** with respect to the **X**-*canonical model* has already been accomplished by the proof for **M**. This model is simply defined as $M_X = \langle \mathfrak{Sat}_X, R_X, V_X \rangle$ with $R_X = R_M \restriction \mathfrak{Sat}_X$ and $V_X = V_M \restriction \mathfrak{Sat}_X$. The reader may convince himself of the fact that all references to **M**-saturated sets may freely be replaced by **X**-saturated sets.

5.14. OBSERVATION. Suppose that **X** is an extension of **M** with $\mathcal{L}_X = \mathcal{L}^\Box$ (that is $\Gamma \vdash_M \Delta \;\Rightarrow\; \Gamma \vdash_X \Delta$ for all $\Gamma, \Delta \subseteq \mathcal{L}^\Box$). For all $\Gamma \in \mathfrak{Sat}_X$ and $\varphi \in \mathcal{L}^\Box$:

$$M_X, \Gamma \models \varphi \Leftrightarrow \varphi \in \Gamma \quad \text{and} \quad M_X, \Gamma \dashv \varphi \Leftrightarrow \neg\varphi \in \Gamma.$$

The economic use of logical sources in the truth lemma 5.12 entails more generalization, especially about the structural behavior of the modal rules, R-TRUE $\Box$ and L-FALSE $\Box$. These two modal rules and the two monotonicity rules were the only rules which have been employed in order to establish the desired result for the modal operator $\Box$.

5.15. OBSERVATION. The justification of the $\Box$-steps in the proof of the truth lemma 5.12 illustrates that for every system **S**, which contains these two modal rules and the structural rules of **M**, if $\Gamma \in \mathfrak{Sat}_S$ and $\varphi \in \mathcal{L}_S$ then

$$\varphi \notin \Box^-\Gamma \;\Rightarrow\; \exists \Delta \in \mathfrak{Sat}_S : \quad \Box^-\Gamma \subseteq \Delta \subseteq \Diamond^-\Gamma \;\&\; \varphi \notin \Delta, \text{ and}$$

$$\varphi \in \Diamond^-\Gamma \;\Rightarrow\; \exists \Theta \in \mathfrak{Sat}_S : \quad \Box^-\Gamma \subseteq \Theta \subseteq \Diamond^-\Gamma \;\&\; \varphi \in \Theta.$$

We did not use the structure of the canonical model M_M which it has on account of the other connectives. This means that a completeness result with respect to $\mathfrak{M}$-validity for a system $\mathbf{M} - \wedge$ with $\mathcal{L}_{M-\wedge} = \mathcal{L}^\Box_{-\wedge}$, which consists of the **M**-rules except the rules for $\wedge$, can immediately be distillated from the truth lemma 5.12.

In fact all extensional connectives can be thrown out. This radical linguistic impoverishment leads to an ultimate minimal modal logic. It contains the structural rules and R-TRUE $\Box$ and L-TRUE $\Diamond$. Let us call this system **Mod**. Its language $\mathcal{L}_{Mod}$ is the smallest superset of $I\!\!P$ such that for every $\varphi \in \mathcal{L}_{Mod}$ also $\Box\varphi \in \mathcal{L}_{Mod}$ and $\Diamond\varphi \in \mathcal{L}_{Mod}$ (the definition $\Diamond = \neg\Box\neg$ is no longer possible, because the negation has been dropped as well). Provision of the regular semantics to this restricted language [7] leads immediately to a completeness result on the basis of the same arguments as in the truth lemma of **M** and the observation

[7] $M = \langle W, R, V \rangle \in \mathfrak{M}, w \in W$: $M, w \models p \Leftrightarrow V(w)(p) = 1$, $M, w \models \Box\varphi \Leftrightarrow M, v \models \varphi$ for all v such that $R(w, v)$, and $M, w \models \Diamond\varphi \Leftrightarrow M, v \models \varphi$ for certain v such that $R(w, v)$. Note that we no longer have to use falsity definitions.

5.15 above. Only a slight modification of the '$\neg\Box\varphi \in \Gamma$'- to a '$\Diamond\varphi \in \Gamma$'-step has to be made. [8]

Completeness of $\mathbf{P}^{\sim}$ and $\mathbf{M}^{\sim}$

These observations above also apply to the extensions of **P** and **M** with a weak negation. The truth lemmas for these logics are really straightforward. Only the weak negation step of the induction of the proof of the truth lemmas need to be checked. We leave this to the reader. In fact, in the case of **M**, the earlier canonical model can be used. It is not hard to see that $M_M = M_{M^{\sim}}$.

5.3 Completeness of constructive modal logics

In this section we present completeness proofs for constructive extensions of **P** and **M**. Linguistically the difference with these basic systems is the constructive implication. In chapter 3 we have seen that this implication is interpreted by means of an intensional information structure. Accomplishment of completeness for systems like **N**, $\mathbf{N}^{\sim}$, **NM** and $\mathbf{NM}^{\Box}$ with respect to their corresponding model classes consequently requires some deeper analysis for finding proper canonical interpretation of this information structure. As truth lemmas are quickly obtainable by the earlier results of the previous section, most of the work in this section is spent on justification of this canonical interpretation. This structural choice must be made in such a way that the canonical models are inhabitants of the proper classes. Otherwise, such canonical models would not serve as a uniform counter-model for non-sequents.

We will start this survey by an illustration of how a canonical interpretation of this information structure can be implemented such that the completeness of **N** and its most elementary non-persistent variation $\mathbf{N}^{\sim}$ can be accomplished in the Henkin style of the preceding sections of this chapter.

Completeness of N and $\mathbf{N}^{\sim}$

As an example of a restricted class of partial Kripke models we have presented Nelson models $M = \langle W, \leq, V\rangle$ for interpretation of Nelson's logic of constructible falsity. The two characteristic requirements for such models as a suitable semantics for constructive logics were that the relation $\leq$ had to be a pre-order and the valuation function needed to be monotonic with respect to this information order.

As a matter of fact, our final choice for a canonical information structure, which is forthcoming in definition 5.20, is closely related to the canonical interpretation of accessibility in classical modal logic. Only a lower bound is required.

[8] Actually, this completeness is also valid with respect to total Kripke models. In this case, the lower-upper-bound definition of the accessibility relation in the canonical model has to be used as well, due to the absence of a negation.

In the next section on up-and-down logic, a lower-upper bound definition for information structures reappears.

It turns out that for constructive systems with a fully persistent language the order of set inclusion can be used. Below we give the completeness procedure for **N**.

5.16. DEFINITION. The **N**-canonical model is the triple $M_N = \langle \mathfrak{Sat}_N, \subseteq, V_N \rangle$, where V_N is precisely the same valuation function as we defined for the **M**-canonical model.

5.17. OBSERVATION. $M_N \in \mathfrak{N}$.

Proof. $\mathfrak{Sat}_N$ is not empty, $\subseteq$ is a pre-order and V_N is surely monotonic. ■

The demonstration of the truth lemma is short. We only expose the induction step for the constructive implication.

5.18. LEMMA. TRUTH-LEMMA **N**

Let Γ be an **N**-saturated set, and let $\varphi \in \mathcal{L}^{\rightarrow}$.

$$M_N, \Gamma \models \varphi \Leftrightarrow \varphi \in \Gamma \text{ and } M_N, \Gamma \dashv \varphi \Leftrightarrow \neg\varphi \in \Gamma$$

Proof. Again, we leave the induction steps for the $\mathcal{L}$-connectives. They are immediate consequences of the definition of V_N and the induction hypothesis. This is also the case for $\neg(\varphi \rightarrow \psi) \in \Gamma$ as the negation 'extensionalizes' the implication into a conjunction $\varphi \wedge \neg\psi$.

$$M_N, \Gamma \dashv \varphi \rightarrow \psi \Leftrightarrow M_N, \Gamma \models \varphi \ \& \ M_N, \Gamma \dashv \psi \Leftrightarrow$$

$$\varphi, \neg\psi \in \Gamma \Leftrightarrow \neg(\varphi \rightarrow \psi) \in \Gamma$$

This last step is fully backed up by L- and R-FALSE $\rightarrow$.

In order to make the verification step for implications, we need the information order in the **N**-canonical model, i.e. the order of set inclusion.

Let $\varphi \rightarrow \psi \in \Gamma$. %

This means that for all $\Delta \supseteq \Gamma$ with $\varphi \in \Delta$ also $\psi \in \Delta$, because also $\varphi \rightarrow \psi \in \Delta$ and $\varphi, \varphi \rightarrow \psi \vdash_N \psi$. By induction for all $\Delta \supseteq \Gamma$ we obtain $M_N, \Delta \models \varphi \Rightarrow M_N, \Delta \models \psi$. We conclude $M_N, \Gamma \models \varphi \rightarrow \psi$.

Suppose $\varphi \rightarrow \psi \notin \Gamma$. %

This also gives us $\Gamma \nvdash_N \varphi \rightarrow \psi$ and therefore (R-TRUE $\rightarrow$) $\Gamma, \varphi \nvdash_N \psi$. Upon this, the saturation lemma yields an **N**-saturated set Δ such that $\Gamma \cup \{\varphi\} \subseteq \Delta$ with $\psi \notin \Delta$. Induction gives us $M_N, \Delta \models \varphi$ and $M_N, \Delta \not\models \psi$, and in conclusion $\Gamma \subseteq \Delta$ entails $M_N, \Gamma \not\models \varphi \rightarrow \psi$.

■

5.19. THEOREM. COMPLETENESS **N**

For all $\Gamma, \Delta \subseteq \mathcal{L}^{\rightarrow}$: $\Gamma \models_{\mathfrak{N}} \Delta \implies \Gamma \vdash_N \Delta$.

Admission of non-persistence means that the inclusion order cannot be used as a canonical information structure any longer. As we have seen in chapter 3 non-persistent variations of constructive logic do not preserve the logic **N**. They add some axioms for the new connectives, but rule out the right hand introduction of implication, because deduction property is lost. This also leads to a weaker canonical information order. For the non-persistent variations of Nelson logic, but also for the logic $\mathbf{NM}^{\Box}$, we use the relation $\Subset$ between saturated sets. Intuitively $\Gamma \Subset \Delta$ says that all information in Γ which is persistent with respect to information orders in models for constructive logic also appears in Δ. As we have only syntactic means at our disposal to implement this definition we use the following classification of persistent information of a set of formulae Γ:

$$\mathbf{p}\Gamma = \{\varphi \in \Gamma |\ \varphi \vdash_{N^{\sim}} \top \rightarrow \varphi\}.$$

For a constructive system **C**, we also specify this in the notation of $\Subset$ and **p** by subscripts.

5.20. Definition. Let Γ and Δ be **C**-saturated sets.

$$\Gamma \Subset_C \Delta \Longleftrightarrow \mathbf{p}_C\Gamma \subseteq \Delta.$$

The converse relation $\Subset_C^{-1}$ is abbreviated by $\Supset_C$.

5.21. Observation. The proposition $\top \rightarrow \varphi$ means that φ holds in every extension of the current state. $\top \rightarrow$ is therefore a special instance of a necessity operator. This means this lower bound of the canonical interpretation of the information order, is principally the same as $\Box^{-}\Gamma$ in the definition of R_M. An upper bound, like $\Diamond^{-}\Gamma$ as in definition 5.11, is not needed here. In the next section, where we prove completeness of up-and-down logic, an upper bound reappears. This coincides with the anti-persistence of the left argument.

Notice furthermore that the definition of $\Subset$ is identical to the normal inclusion order $\subseteq$ if the full language $\mathcal{L}_C$ is persistent. This indicates that $\Subset$ can be employed as a standard canonical information order for constructive systems. For example, the canonical model for **N** in definition 5.16. Also for the modal extensions of the constructive logics which we presented in the earlier chapters, this information order is implemented.

5.22. Definition. Let **C** be a non-modal extension of $\mathbf{N}^{-}$. The **C**-canonical model is the triple $M_C = \langle \mathfrak{Sat}_C, \Subset_C, V_C \rangle$, where V_C is precisely defined as in definition 5.16 with the restricted domain $\mathfrak{Sat}_C$.

If **C** is a normal extension of $\mathbf{NM}^{-} = \mathbf{M} + \mathbf{N}^{-}$, then its canonical is the quadruple $M_C = \langle Sat_C, R_C, \Subset_C, V_C \rangle$ where R_C is the imitation of R_M in definition 5.11 and V_C is defined as in the definition of the canonical valuation function above (with the domain $\mathfrak{Sat}_C$).

These definitions are a sufficient preparation for a short presentation of the truth lemma for the most simple non-persistent variation of Nelson's logic: $\mathbf{N}^{\sim}$.

5.23. Lemma. Truth lemma $\mathbf{N}^{\sim}$

For all $\Gamma \in \mathfrak{Sat}_{N^{\sim}}$ and $\varphi \in \mathcal{L}^{\sim,\rightarrow}$

$$M_{N^{\sim}}, \Gamma \models \varphi \Leftrightarrow \varphi \in \Gamma \quad \text{and} \quad M_{N^{\sim}}, \Gamma \dashv \varphi \Leftrightarrow \neg\varphi \in \Gamma.$$

Proof. Only the verification step of the implication needs some clarification. Falsification of $\rightarrow$ is the same as in the proof of the truth lemma of **N**.

Suppose $\varphi \rightarrow \psi \in \Gamma$.

This is the easy part. Clearly $\top \rightarrow (\varphi \rightarrow \psi) \in \Gamma$ by PERS $\rightarrow$. So, if $\Delta \Supset_{N^{\sim}} \Gamma$ then also $\varphi \rightarrow \psi \in \Delta$. L-TRUE $\rightarrow$ guarantees upon this:

$\forall \Delta \Supset_{N^{\sim}} \Gamma : \ \varphi \in \Delta \Rightarrow \psi \in \Delta.$

By means of the induction hypothesis, we conclude

$\forall \Delta \Supset_{N^{\sim}} \Gamma : M_{N^{\sim}}, \Delta \models \varphi \Rightarrow M_{N^{\sim}}, \Delta \models \psi.$

This conclusion yields $M_{N^{\sim}}, \Gamma \models \varphi \rightarrow \psi$.

Now consider, $\varphi \rightarrow \psi \notin \Gamma$.

This assumption also yields $\mathbf{p}_{N^{\sim}}\Gamma \not\vdash_{N^{\sim}} \varphi \rightarrow \psi$, because $\mathbf{p}_{N^{\sim}}\Gamma \subseteq \Gamma$ and L-MON. Since $\mathbf{p}_{N^{\sim}}\Gamma = \{\varphi \mid \top \rightarrow \varphi \in \Gamma\}$ (observation 5.21) and R-TRUE-WEAK $\rightarrow$, we obtain $\mathbf{p}_{N^{\sim}}\Gamma, \varphi \not\vdash_{N^{\sim}} \psi$. Application of the saturation lemma gives us the desired Δ: $\mathbf{p}_{N^{\sim}}\Gamma \subseteq \Delta \ \& \ \varphi \in \Delta \ \& \ \psi \notin \Delta$. The induction hypothesis proves

$M_{N^{\sim}}, \Delta \models \varphi$ and $M_{N^{\sim}}, \Delta \not\models \psi$.

Because $\Delta \Supset_{N^{\sim}} \Gamma$, this conclusion entails $M_{N^{\sim}}, \Gamma \not\models \varphi \rightarrow \psi$.

■

5.24. THEOREM. COMPLETENESS $\mathbf{N}^{\sim}$

For all $\Gamma, \Delta \subseteq \mathcal{L}^{\sim,\rightarrow}$: $\Gamma \models_{\mathfrak{N}^{\sim}} \Delta \Rightarrow \Gamma \vdash_{N^{\sim}} \Delta$.

Establishing truth lemmas for the basic constructive logics turned out to be a relatively easy job. In fact, both truth lemmas, lemma 5.18 and lemma 5.23, above can fully be adopted by the logics **NM** and $\mathbf{NM}^{\Box}$. The completeness result only demands for a demonstration that their canonical models belong to the proper class: $M_{NM} \in \mathfrak{NM}$ and $M_{NM^{\Box}} \in \mathfrak{NM}^{\Box}$. The characteristic property of the class $\mathfrak{NM}$ is that the information order in a given model $M = \langle W, R, \leq, V\rangle \in \mathfrak{NM}$ is a bisimulation over the accessibility frame of the model $\langle W, R, V\rangle$. Henceforth, what has to be proved in the sequel, to derive the completeness for **NM**, is

$$\forall \Gamma, \Gamma', \Delta' \in \mathfrak{Sat}_{NM} : \quad \Gamma \subseteq \Gamma' \ \& \ R_{NM}(\Gamma', \Delta') \ \Rightarrow$$
$$\exists \Delta \in \mathfrak{Sat}_{NM} : R_{NM}(\Gamma, \Delta) \ \& \ \Delta \subseteq \Delta' \quad (1), \text{ and}$$

$$\forall \Gamma, \Gamma', \Delta \in \mathfrak{Sat}_{NM} : \quad \Gamma \subseteq \Gamma' \ \& \ R_{NM}(\Gamma, \Delta) \ \Rightarrow$$
$$\exists \Delta' \in \mathfrak{Sat}_{NM} : R_{NM}(\Gamma', \Delta') \ \& \ \Delta \subseteq \Delta' \quad (2)^{9}.$$

Reaching a completeness result for the system $\mathbf{NM}^{\Box}$ with respect to $\mathfrak{NM}^{\Box}$-validity is obtained by a demonstration that the equation (1), with replacement of $\subseteq$ by $\Subset_{NM^{\Box}}$, holds for $\mathfrak{Sat}_{NM^{\Box}}$.

In the following subsection we will dwell upon such extension orders in canonical models in order to establish the classification of these canonical models as inhabitants of $\mathfrak{NM}$ and $\mathfrak{NM}^{\Box}$, respectively[10].

[9] Remember that $\Subset_{NM} = \subseteq$.

[10] In chapter 7 we will also mention other information orders which are based on bisimulations

Extension orders in canonical models

A convenient procedure to establish the bisimulation result for $\subseteq$ in the canonical model of a system of some **M**-extension **X**, consists of showing that for all $\Gamma, \Gamma' \in \mathfrak{Sat}_X$ if $\Gamma \subseteq \Gamma'$ then

$$\forall \Delta' \in \mathfrak{Sat}_X : \quad R_X(\Gamma', \Delta') \;\Rightarrow\; \Box^-\Gamma \trianglelefteq_X (\Diamond^-\Gamma \cap \Delta') \quad (3) \;\text{ and}$$

$$\forall \Delta \in \mathfrak{Sat}_X : \quad R_X(\Gamma, \Delta) \quad \Rightarrow\; (\Box^-\Gamma' \cup \Delta) \trianglelefteq_X \Diamond^-\Gamma' \; (4)\,.$$

Such results would indeed, in combination with the bounded saturation lemma, recognize $\subseteq$ as a bisimulation on $\langle \mathfrak{Sat}_X, R_X \rangle$. BSL responds to the first claim that there exists a $\Delta \in \mathfrak{Sat}_X$ such that $\Box^-\Gamma \subseteq \Delta \subseteq (\Delta' \cap \Diamond^-\Gamma)$, which can be rewritten in the intended form: $R_X(\Gamma, \Delta)$ and $\Delta \subseteq \Delta'$. With respect to the second claim, BSL yields a $\Delta' \in \mathfrak{Sat}_X$ such that $(\Box^-\Gamma' \cup \Delta) \subseteq \Delta' \subseteq \Diamond^-\Gamma'$, or in other words $R_X(\Gamma', \Delta')$ and $\Delta \subseteq \Delta'$. The following lemma presents a more general result.

5.25. LEMMA. Let **X** be an extension of **M**, and suppose Γ is an **X**-saturated set and $\Lambda \subseteq \mathcal{L}_X$.

If $\Box^-\Gamma \trianglelefteq_X \Lambda$ then there exists an **X**-saturated set Δ such that

$$\Box^-\Gamma \subseteq \Delta \subseteq \Diamond^-\Gamma \quad \text{and} \quad \Delta \subseteq \Lambda.$$

If $\Lambda \trianglelefteq_X \Diamond^-\Gamma$ there exists an **X**-saturated set Δ such that

$$\Box^-\Gamma \subseteq \Delta \subseteq \Diamond^-\Gamma \quad \text{and} \quad \Lambda \subseteq \Delta.$$

A full formal transcription of these results looks like this:

$$\forall \Gamma \in \mathfrak{Sat}_X \; \forall \Lambda \subseteq \mathcal{L}_X : \begin{cases} \Box^-\Gamma \trianglelefteq_X \Lambda \Rightarrow \exists \Delta \in \mathfrak{Sat}_X : R_X(\Gamma, \Delta) \;\&\; \Delta \subseteq \Lambda \; (1), \\ \Lambda \trianglelefteq_X \Diamond^-\Gamma \Rightarrow \exists \Delta \in \mathfrak{Sat}_X : R_X(\Gamma, \Delta) \;\&\; \Lambda \subseteq \Delta \; (2). \end{cases}$$

Proof. To begin with, we prove the saturation result (1) above. Suppose $\Box^-\Gamma \trianglelefteq_X \Lambda$, and $\Box^-\Gamma \vdash_X \Theta$ for certain finite set $\Theta \subseteq \mathcal{L}_X$. Combination of these two assumptions yields $\Theta \cap \Lambda \neq \emptyset$. Separation of Θ into a Λ- and a non-Λ-component set entails the following reformulation: $\Box^-\Gamma \vdash_X \Theta \cap \Lambda, \Theta \setminus \Lambda$. Application of R-TRUE $\vee$ entails

$$\Box^-\Gamma \vdash_X \bigvee \Theta \setminus \Lambda, \Theta \cap \Lambda.$$

Thereupon, application of R-TRUE $\Box$ and L-MON successively yields

$$\Gamma \vdash_X \Box(\bigvee \Theta \setminus \Lambda), \Diamond(\Theta \cap \Lambda).$$

The **X**-saturation of Γ guarantees that either $\Box(\bigvee \Theta \setminus \Lambda) \in \Gamma$ or that there exists $\theta \in \Theta \cap \Lambda$ such that $\Diamond\theta \in \Gamma$. In the former case we have $\Box^-\Gamma \vdash_X \Theta \setminus \Lambda$. This does not reconcile with $\Box^-\Gamma \trianglelefteq_X \Lambda$, because the conclusion set $\Theta \setminus \Lambda$ obviously does not intersect Λ. This means that $\Theta \cap \Lambda \cap \Diamond^-\Gamma \neq \emptyset$. Because Θ has been picked arbitrarily as a finite **X**-conclusion set of $\Box^-\Gamma$, we have made sure by the latter observation that

$$\Box^-\Gamma \trianglelefteq_X \Lambda \cap \Diamond^-\Gamma.$$

---

in canonical models. Such results are particularly important for finding good model-theoretic characteristics for axiomatic extensions of the basic partial modal logic **M**.

Application of the bounded saturation lemma gives us the desired **X**-saturated set Δ with $\Box^-\Gamma \subseteq \Delta \subseteq \Diamond^-\Gamma$ and $\Delta \subseteq \Lambda$ (1).

Now we will prove the result (2) above. Suppose $\Lambda \trianglelefteq_X \Diamond^-\Gamma$ and $\Box^-\Gamma, \Lambda \vdash_X \Sigma$ for certain $\Sigma \subseteq \mathcal{L}_X$. The latter assumption implies, on account of L-TRUE $\wedge$, the existence of a finite subset $\Lambda' \subseteq \Lambda$ such that

$$\Box^-\Gamma, \bigwedge \Lambda' \vdash_X \Sigma.$$

L-MON and L-TRUE $\Diamond$ transform this **X**-sequent into

$$\Gamma, \Diamond(\bigwedge \Lambda') \vdash_X \Diamond\Sigma.$$

Because $\Lambda \vdash_X \bigwedge \Lambda'$ and $\Lambda \trianglelefteq_X \Diamond^-\Gamma$ we know that $\Diamond(\bigwedge \Lambda') \in \Gamma$. This result tells us that the **X**-sequent above is the same as $\Gamma \vdash_X \Diamond\Sigma$. The **X**-saturation of Γ guarantees thereupon $\Sigma \cap \Diamond^-\Gamma \neq \emptyset$. As we have chosen Σ as an arbitrary **X**-conclusion set of $\Box^-\Gamma \cup \Lambda$, we may conclude that

$$\Box^-\Gamma \cup \Lambda \trianglelefteq_X \Diamond^-\Gamma.$$

According the bounded saturation lemma this last result means that there exists an **X**-saturated set Δ such that $\Box^-\Gamma \subseteq \Delta \subseteq \Diamond^-\Gamma$ and $\Lambda \subseteq \Delta$. ■

Note that we have used the presence of the disjunction and conjunction rules to obtain the result of lemma 5.25. The proof above might be seen as a generalization of the results (*i*) and (*ii*) in the proof of the truth lemma 5.12 on page 149 for **M**. The single argument φ there is replaced by finite sets of formulae. To derive a similar conclusion, these multiple arguments have to be compressed by means of disjunctions and conjunctions, respectively. In this sense, our means to prove the completeness of constructive modal logics are not as pure as in the case of partial modal logics, and therefore, general transposition of completeness results for sublanguages are no longer obtainable from our forthcoming completeness results for **NM** and $\mathbf{NM}^\Box$. So, a technical challenge which remains is to show whether lemma 5.25 still holds for systems which evolve from retracting the conjunction and its introduction rules. We leave this issue as an open question.

An important corollary of lemma 5.25 is that the order of set inclusion is a bisimulation over the canonical accessibility pattern $\langle \mathfrak{Sat}_X, R_X \rangle$ for every **M**-extension **X**.

5.26. COROLLARY. The inclusion relation $\subseteq$ is a bisimulation over the 'canonical accessibility' structure $\langle \mathfrak{Sat}_X, R_X \rangle$ for every **M**-extension **X**.

Proof. Let $\Gamma, \Gamma', \Delta' \in \mathfrak{Sat}_X$ such that $\Gamma \subseteq \Gamma'$ and $R_X(\Gamma', \Delta')$. Obviously, $\Box^-\Gamma \trianglelefteq_X \Delta'$, because the last set is **X**-saturated. Moreover, $\Box^-\Gamma \subseteq \Box^-\Gamma' \subseteq \Delta'$. Application of lemma 5.25 entails an **X**-saturated set Δ such that $R_X(\Gamma, \Delta)$ and $\Delta \subseteq \Delta'$. This argumentation shows

$$(\subseteq \circ R_X) \subseteq (R_X \circ \subseteq) \quad (1).$$

Suppose $\Gamma, \Gamma', \Delta \in \mathfrak{Sat}_X$ such that $\Gamma \subseteq \Gamma'$ and $R_X(\Gamma, \Delta)$. Clearly, $\Delta \trianglelefteq_X \Diamond^-\Gamma'$, because Δ is **X**-saturated and $\Delta \subseteq \Diamond^-\Gamma \subseteq \Diamond^-\Gamma'$. Lemma 5.25 makes sure that there exists a $\Delta' \in \mathfrak{Sat}_X$ such that $R_X(\Gamma', \Delta')$ and $\Delta \subseteq \Delta'$. We conclude

$(\supseteq \circ R_X) \subseteq (R_X \circ \supseteq)$ (2).

Findings (1) and (2) show that $\subseteq$ is a bisimulation over $\langle \mathfrak{Sat}_X, R_X \rangle$. ■

5.27. OBSERVATION. Lemma 5.12 and corollary 5.26 prove that the converse formulation of corollary 2.46 (page 70 holds for **X**-canonical models, for every normal **M**-extension.

A question, which arises from the perfect match of informational contents of worlds in canonical models of **M**-extensions and the bisimulation order, is whether functional completeness results can be achieved for our language $\mathcal{L}^{\Box}$ with respect to these canonical models on the basis on the bisimulation extension order. We think that bisimulation reformulations of information orders may be useful to obtain such definability results for partial modal logics similar to those of partial propositional logics[11]. As said in chapter 2 such questions lie outside the scope of the thesis, but the technical observations of this subsection may support such future investigations.

Completeness of NM and NM$^{\Box}$

Corollary 5.26 of lemma 5.25 immediately shows that the canonical model for the logic **NM** is an inhabitant of the intended class $\mathfrak{NM}$.

5.28. LEMMA. $M_{NM} \in \mathfrak{NM}$.

Proof. Because $\mathcal{L}^{\Box,\rightarrow}$ is persistent with respect to the information structure in $\mathfrak{NM}$-models, we know that $\Subset_{NM} = \subseteq$ by observation 5.21. Corollary 5.26 shows that $\subseteq$ is a bisimulation over $\langle \mathfrak{Sat}_{NM}, R_{NM} \rangle$. The monotonicity of the valuation function is accounted for in the same way as in the **N**-canonical model (see proof of lemma 5.17). ■

5.29. THEOREM. COMPLETENESS **NM**

$\forall \Gamma, \Delta \subseteq \mathcal{L}^{\Box,\rightarrow} : \ \Gamma \models_{\mathfrak{NM}} \Delta \ \Rightarrow \ \Gamma \vdash_{NM} \Delta.$

Proof. The truth lemma is simply the same as for **M** and **N**. By means of this truth lemma every non-**NM**-sequent has M_{NM} as a counter-model, and the corollary above shows that this model is in the proper class in $\mathfrak{NM}$, and therefore it is an appropriate counter-model. ■

Demonstration of $M_{NM^{\Box}} \in \mathfrak{NM}^{\Box}$, which would give us the completeness theorem for $\mathbf{NM}^{\Box}$, needs a little modification of corollary 5.26. The extension relation does not coincide with $\subseteq$, but with the less restricted relation $\Subset$. Here, the extra modal rule DIS $\Box \rightarrow$ comes on the stage.

5.30. LEMMA. $M_{NM^{\Box}} \in \mathfrak{NM}^{\Box}$.

[11] A study of definability for partial modal logic on the basis of bisimulation can be found in [Thijsse 1992].

Proof. We need to prove that $(\Subset_{NM^\Box} \circ R_{NM^\Box}) \subseteq (R_{NM^\Box} \circ \Subset_{NM^\Box})$. This result can be obtained by showing that

$$\Box^-\Gamma \trianglelefteq_{NM^\Box} \Lambda \quad \text{with} \quad \Lambda := \Delta' \cup (\mathbf{p}_{NM^\Box}\mathcal{L}^{\Box,\rightarrow})^{\complement} \quad (5),$$

whenever $\Gamma \Subset_{NM^\Box} \Gamma'$ and $R_{NM^\Box}(\Gamma', \Delta')$ for certain $\Gamma' \in \mathfrak{Sat}_{NM^\Box}$. Indeed, lemma 5.25 would guarantee the existence of a $\Delta \in \mathfrak{Sat}_{NM^\Box}$ such that $R_{NM^\Box}(\Gamma, \Delta)$ and $\Delta \subseteq \Lambda$, which entails $\mathbf{p}_{NM^\Box}(\Delta) \subseteq \Delta'$ ($\Delta \Subset_{NM^\Box} \Delta'$).

Suppose $\Box^-\Gamma \vdash_{NM^\Box} \Sigma$ for some finite $\Sigma \subseteq \mathcal{L}^{\Box,\rightarrow}$. We need to prove $\Sigma \cap \Lambda \neq \emptyset$. We separate two complementary cases.

(i): If $\Sigma \cap (\mathbf{p}_{NM^\Box}\mathcal{L}^{\Box,\rightarrow})^{\complement} \neq \emptyset$, then we immediately have intersection with the premeditated saturator Λ.

(ii): If $\Sigma \cap (\mathbf{p}_{NM^\Box}\mathcal{L}^{\Box,\rightarrow})^{\complement} = \emptyset$, then $\Sigma \subseteq \mathbf{p}_{NM^\Box}\mathcal{L}^{\Box,\rightarrow}$, or in other words, for all $\sigma \in \Sigma$:

$$\sigma \vdash_{NM^\Box} \top \rightarrow \sigma.$$

Because $\top \rightarrow \varphi \vdash_{NM^\Box} \top \rightarrow (\varphi \vee \psi)$, we also conclude for all these $\sigma \in \Sigma$:

$$\sigma \vdash_{NM^\Box} \top \rightarrow (\bigvee \Sigma).$$

Iterative application of CUT yields

$$\Box^-\Gamma \vdash_{NM^\Box} \top \rightarrow (\bigvee \Sigma).$$

By R-TRUE $\Box$ and observation 3.29 on page 100, which captures the persistence preservation of $\Box$ in terms of $\mathbf{NM}^\Box$-deduction, we see that

$$\Gamma \vdash_{NM^\Box} \top \rightarrow \Box(\bigvee \Sigma).$$

This last result shows that $\Box(\bigvee \Sigma) \in \mathbf{p}_{NM^\Box}(\Gamma)$, which also implies $\Box(\bigvee \Sigma) \in \Gamma'$ and $\bigvee \Sigma \in \Delta'$. Because the last set is $\mathbf{NM}^\Box$-saturated we have made sure that $\Sigma \cap \Delta' \neq \emptyset$, and of course also $\Sigma \cap \Lambda \neq \emptyset$.

These complementary arguments (i) and (ii) show the validity of (5). ■

5.31. THEOREM. COMPLETENESS $\mathbf{NM}^\Box$

$\forall \Gamma, \Delta \subseteq \mathcal{L}^{\Box,\rightarrow} : \ \Gamma \models_{\mathfrak{NM}^\Box} \Delta \ \Rightarrow \ \Gamma \vdash_{NM^\Box} \Delta.$

Proof. The truth lemma can be obtained by copying the implication step from $\mathbf{N}^\sim$. The modal steps are precisely as in $\mathbf{M}$, because $\mathbf{NM}^\Box$ is an $\mathbf{M}$-extension. Lemma 5.30 above shows that the $\mathbf{NM}^\Box$-canonical model, whose universal potential as a countermodel for every non-$\mathbf{NM}^\Box$-sequent is recognized by its truth lemma, lives in the intended class: $\mathfrak{NM}^\Box$. ■

5.4 Completeness of up and down logics

Canonical models for ud-extensions

A suitable definition of the canonical information structure for the up-and-down systems, i.e. **ud**-extensions, can be given by an addition of an upperbound restriction to the implementation of the canonical information structure of the constructive logics in the previous section. This upper bound can be stipulated by means of anti-persistent information.

5.32. DEFINITION. Let $\mathbf{X}$ be an extension of **ud**, and let $\Gamma \in \mathfrak{Sat}_X$. The $\mathbf{X}$-persistent part of Γ, $\mathbf{p}_X\Gamma$, is the set $\{\varphi \in \Gamma \mid \varphi \vdash_X [\ ]_u\varphi\}$. The anti-persistent part of Γ, $\mathbf{ap}_X\Gamma$, is the set $\{\varphi \in \Gamma \mid \varphi \vdash_{ud} [\ ]_d\varphi\}$.

The $\mathbf{X}$-canonical information structure is the relation $\ll_X$ over the collection of $\mathbf{X}$-saturated sets. This relation holds between two $\mathbf{X}$-saturated sets Γ and Δ if and only if the persistent part of Γ is contained by Δ *and* the anti-persistent part of Δ is contained by Γ. This is the above-mentioned additional upper bound. Formally,

$$\Gamma \ll_{ud} \Delta \Leftrightarrow \mathbf{p}_X\Gamma \subseteq \Delta \ \&\ \mathbf{ap}_X\Delta \subseteq \Gamma.$$

The following list presents some important properties of operators $\mathbf{p}_X$ and $\mathbf{ap}_X$ for **ud**-extensions $\mathbf{X}$. Let $\Gamma, \Delta \in \mathfrak{Sat}_X$.

5.33. TABLE.

$\mathbf{p}_X\Gamma \subseteq \Gamma$	$\mathbf{ap}_X\Gamma \subseteq \Gamma$
$\mathbf{p}_X\Gamma \subseteq \Delta \Leftrightarrow \mathbf{p}_X\Gamma \subseteq \mathbf{p}_X\Delta$	$\mathbf{ap}_X\Gamma \subseteq \Delta \Leftrightarrow \mathbf{ap}_X\Gamma \subseteq \mathbf{ap}_X\Delta$
$\mathbf{p}_X\mathbf{p}_X\Gamma = \mathbf{p}_X\Gamma$	$\mathbf{ap}_X\mathbf{ap}_X\Gamma = \mathbf{ap}_X\Gamma$
$\mathbf{p}_X\Gamma = \{\varphi \in \Gamma \mid \langle\ \rangle_d\varphi \vdash_X \varphi\}$	$\mathbf{ap}_X\Gamma = \{\varphi \in \Gamma \mid \langle\ \rangle_u\varphi \vdash_X \varphi\}$

The last two properties follow from $\varphi \vdash_X [\ ]_u\varphi \Leftrightarrow \langle\ \rangle_d\varphi \vdash_X \varphi$ and $\varphi \vdash_X [\ ]_d\varphi \Leftrightarrow \langle\ \rangle_u\varphi \vdash_X \varphi$. These equivalences can immediately be obtained by the PERS and A-PERS rules in the **ud**-tables in chapter 3, table 3.17. The other properties immediately follow from the definitions of $\mathbf{p}_X$ and $\mathbf{ap}_X$, respectively.

5.34. OBSERVATION. From the last two properties in table 5.33 we also learn that the canonical information structure of **ud**-extensions is in fact a reformulation of the canonical accessibility structure of **M**-extensions.

$$\Gamma \ll_X \Delta \Leftrightarrow [\ ]_u{}^-\Gamma \subseteq \Delta \subseteq \langle\ \rangle_u{}^-\Gamma, \text{ and}$$

$$\Gamma \ll_X \Delta \Leftrightarrow [\ ]_d{}^-\Delta \subseteq \Gamma \subseteq \langle\ \rangle_d{}^-\Delta.$$

Proof. Let $\Gamma \ll_X \Delta$. We prove the $\Rightarrow$-direction of the first equivalence.

Suppose $[\ ]_u\varphi \in \Gamma$. Because $[\ ]_u\varphi \vdash_X [\ ]_u[\ ]_u\varphi$, we obtain $[\ ]_u\varphi \in \mathbf{p}_X\Gamma$ and therefore $[\ ]_u\varphi \in \Delta$. Because $[\ ]_u\varphi \vdash_X \varphi$, also $\varphi \in \Delta$.

Suppose $\varphi \in \Delta$. This assumption entails $\langle\ \rangle_u\varphi \in \Delta$, because $\varphi \vdash_X \langle\ \rangle_u\varphi$. Furthermore $\langle\ \rangle_u \in \mathbf{ap}_X\Delta$, because $\langle\ \rangle_u\langle\ \rangle_u\varphi \vdash_X \langle\ \rangle_u\varphi$ (last property of $\mathbf{ap}_X$ in table 5.33). We conclude $\langle\ \rangle_u\varphi \in \Gamma$.

The $\Rightarrow$-direction of the second claim above, can be proved by using the persistence and anti-persistence rules of the **ud**-calculus, $[\ ]_d\varphi \vdash_X \varphi$ and $\varphi \vdash_X \langle\ \rangle_d\varphi$.

The $\Leftarrow$-directions can be obtained by a similar use of the modality reduction principles of the **ud**-calculus. For example, if $[\ ]_u{}^-\Gamma \subseteq \Delta$, and $\varphi \in \mathbf{p}_X\Gamma$, then also $[\ ]_u\varphi \in \Gamma$ and therefore $\varphi \in \Delta$. ■

Our reason for preferring the definition of $\ll_X$ in 5.32 above is merely technical, and not yet important. In the next chapter on finite models, it turns out that this definition of the canonical information structure is very practical to find finite counter-models for finite non-**ud**-sequents.

5.35. DEFINITION. Let **X** be a normal **ud**-extension. The **X**-canonical model M_X is the triple $\langle \mathfrak{Sat}_X, \ll_X, V_X\rangle$ with $V(\Gamma)(p) = 1(0) \Leftrightarrow (\neg)p \in \Gamma$ for all $p \in I\!P$ and $\Gamma \in \mathfrak{Sat}_X$.

5.36. OBSERVATION. $\ll_X$ is a pre-order over $\mathfrak{Sat}_X$ and V_X is monotonic over $\mathfrak{Sat}_X$.

Proof. The reflexivity of $\ll_X$ follows from the first two properties in table 5.33. Transitivity of $\ll_X$ follows from the second line of this table. Monotonicity of V_X is an immediate consequence of $p \vdash_X [\ ]_u p$ and $\neg p \vdash_X [\ ]_u \neg p$ for all $p \in I\!P$ (PERS $(\neg)$ $I\!P$). ■

In the forthcoming completeness proofs of up- and down-logics we use the following slight reformulation of the rules of R-TRUE and L-FALSE introduction of the update and downdate operators.

5.37. OBSERVATION. If **X** is some normal **ud**-extension, and $\Gamma, \Delta \subseteq \mathcal{L}_X$, $\varphi, \psi \in \mathcal{L}_X$, then

1. $\mathbf{p}_X\Gamma, \varphi \vdash_X \psi, \mathbf{ap}_X\Delta \quad \Rightarrow \quad \Gamma \vdash_X [\varphi]_u\psi, \Delta$,
2. $\mathbf{p}_X\Gamma, \varphi, \psi \vdash_X \mathbf{ap}_X\Delta \quad \Rightarrow \quad \Gamma, \langle\varphi\rangle_u\psi \vdash_X \Delta$,
3. $\mathbf{ap}_X\Gamma \vdash_X \varphi, \psi, \mathbf{p}_X\Delta \quad \Rightarrow \quad \Gamma \vdash_X [\varphi]_d\psi, \Delta$, and
4. $\mathbf{ap}_X\Gamma, \psi \vdash_X \varphi, \mathbf{p}_X\Delta \quad \Rightarrow \quad \Gamma, \langle\varphi\rangle_d\psi \vdash_X \Delta$.

These alternative rules can be derived from the weaker versions of the above-mentioned rules in **ud**, which we have found in section 3.3 on page 95: R-TRUE' $[\ ]_u$ & $[\ ]_d$ and L-FALSE' $[\ ]_u$ & $[\ ]_d$. The rules above can be derived immediately by means of CUT and the finiteness property of **ud**.

The completeness of ud

From observation 5.36 we conclude immediately the structural appropriateness of the **ud**-canonical model: $M_{ud} \in \mathfrak{N}$. What is left to prove is the relevant truth lemma.

5.38. LEMMA. TRUTH LEMMA **ud**

For all $\Gamma \in \mathfrak{Sat}_{ud}$ and $\varphi \in \mathcal{L}^{\uparrow,\downarrow}$:

$M_{ud}, \Gamma \models \varphi \Leftrightarrow \varphi \in \Gamma \quad$ and $\quad M_{ud}, \Gamma \dashv \varphi \Leftrightarrow \neg\varphi \in \Gamma$.

Proof. Again, we skip the basic step and the proofs of the extensional connectives. For the intensional connectives, there are four cases which are nearly immediately obtainable from the definition of $\ll_{ud}$. Analogously to the $\Box$-step in lemma 5.12, these four 'easy' cases are:

$(i)\quad [\varphi]_u\,\psi \in \Gamma \Rightarrow M_{ud}, \Gamma \models [\varphi]_u\,\psi,$

$(ii)\quad M_{ud}, \Gamma \dashv [\varphi]_u\,\psi \Rightarrow \neg[\varphi]_u\,\psi \in \Gamma,$

$(iii)\quad [\varphi]_d\,\psi \in \Gamma \Rightarrow M_{ud}, \Gamma \models [\varphi]_d\,\psi,$

$(iv)\quad M_{ud}, \Gamma \dashv [\varphi]_d\,\psi \Rightarrow \neg[\varphi]_d\,\psi \in \Gamma.$

We will demonstrate the first and the last step. The two others are left to the reader.

$[\varphi]_u\,\psi \in \Gamma \implies ([\varphi]_u\,\psi \vdash_{ud} [\ ]_u\,[\varphi]_u\,\psi$, example 3.22)

$\forall\Delta \gg_{ud} \Gamma : [\varphi]_u\,\psi \in \Delta \implies (\varphi, [\varphi]_u\,\psi \vdash_{ud} \psi)$

$\forall\Delta \gg_{ud} \Gamma : \varphi \in \Delta \Rightarrow \psi \in \Delta \implies$ (induction hypothesis)

$\forall\Delta \gg_{ud} \Gamma : M_{ud}, \Delta \models \varphi \Rightarrow M_{ud}, \Delta \models \psi \implies M_{ud}, \Gamma \models [\varphi]_u\,\psi.$

$\neg[\varphi]_d\,\psi \notin \Gamma \implies (\neg[\varphi]_d\,\psi \vdash_{ud} [\ ]_u\,\neg[\varphi]_d\,\psi$, second line of table 5.33)

$\forall\Delta \ll_{ud} \Gamma : \neg[\varphi]_d\,\psi \notin \Delta \implies (\varphi \vdash_{ud} \neg[\varphi]_d\,\psi, \neg\psi$, example 3.22)

$\forall\Delta \ll_{ud} \Gamma : \varphi \notin \Delta \Rightarrow \neg\psi \notin \Delta \implies$ (induction hypothesis)

$\forall\Delta \ll_{ud} \Gamma : M_{ud} \not\models \varphi \Rightarrow M_{ud} \not\dashv \psi \implies M_{ud}, \Gamma \not\dashv [\varphi]_d\,\psi.$

The completing converse results of these four 'easy' cases are consequences of the following sequential statements, in combination with the bounded saturation lemma.

$(v)\quad [\varphi]_u\,\psi \notin \Gamma \quad \Rightarrow \quad \mathbf{p}_{ud}\Gamma + \varphi \trianglelefteq_{ud} \Gamma \cup (\mathbf{ap}_{ud}\mathcal{L}^{\uparrow,\downarrow})^{\complement} - \psi$

$(vi)\quad \neg[\varphi]_u\,\psi \in \Gamma \quad \Rightarrow \quad \mathbf{p}_{ud}\Gamma + \varphi + \neg\psi \trianglelefteq_{ud} \Gamma \cup (\mathbf{ap}_{ud}\mathcal{L}^{\uparrow,\downarrow})^{\complement}$

$(vii)\quad [\varphi]_d\,\psi \notin \Gamma \quad \Rightarrow \quad \mathbf{ap}_{ud}\Gamma \trianglelefteq_{ud} \Gamma \cup (\mathbf{p}_{ud}\mathcal{L}^{\uparrow,\downarrow})^{\complement} - \varphi - \psi$

$(viii)\quad \neg[\varphi]_d\,\psi \in \Gamma \quad \Rightarrow \quad \mathbf{ap}_{ud}\Gamma + \neg\psi \trianglelefteq_{ud} \Gamma \cup (\mathbf{p}_{ud}\mathcal{L}^{\uparrow,\downarrow})^{\complement} - \varphi$

These saturation relations may seem complicated statements. The following simple derivations explain why they lead to immediate success. For sake of briefness we only prove that the claims (v) and $(viii)$ give us the desired results: $(v) \Rightarrow M_{ud}, \Gamma \not\models [\varphi]_u\,\psi$ and $(viii) \Rightarrow M_{ud} \dashv [\varphi]_d\,\psi$.

$(v) \implies_{\mathrm{BSL}} \exists\Delta \in \mathfrak{Sat}_{ud} : \mathbf{p}_{ud}\Gamma \subseteq \Delta \subseteq \Gamma \cup (\mathbf{ap}_{ud}\mathcal{L}^{\uparrow,\downarrow})^{\complement}\ \&\ \varphi \in \Delta\ \&\ \psi \notin \Delta \implies$

$\Gamma \ll_{ud} \Delta\ \&\ M_{ud}, \Delta \models \varphi\ \&\ M_{ud}, \Delta \not\models \Delta \implies M_{ud}, \Gamma \not\models [\varphi]_u\,\psi.$

The second step in this formal transcription of the proof is correct, due to the induction hypothesis and $\Delta \subseteq \Gamma \cup (\mathbf{ap}_{ud}\mathcal{L}^{\uparrow,\downarrow})^{\complement} \Rightarrow \mathbf{ap}_{ud}\Delta \subseteq \mathbf{ap}_{ud}\Gamma \Rightarrow \mathbf{ap}_{ud}\Delta \subseteq \Gamma$ (table 5.33).

$(viii) \implies_{\mathrm{BSL}} \exists\Delta \in \mathfrak{Sat}_{ud} : \mathbf{ap}_{ud}\Gamma \subseteq \Delta \subseteq \Gamma \cup (\mathbf{p}_{ud}\mathcal{L}^{\uparrow,\downarrow})^{\complement}\ \&\ \varphi \notin \Delta\ \&\ \neg\psi \in \Delta \implies$
$\Delta \ll_{ud} \Gamma\ \&\ M_{ud}, \Delta \not\models \varphi\ \&\ M_{ud}, \Delta \dashv \psi \implies M_{ud}, \Gamma \dashv [\varphi]_d\,\psi.$

Again, the second step consists of application of the induction hypothesis and the rightmost property of the second line in table 5.33.

The proofs of $(vi) \Rightarrow M_ud, \Gamma \dashv [\varphi]_u\,\psi$ and $(vii) \Rightarrow M_{ud}, \Gamma \not\models [\varphi]_d\,\psi$ are left to the reader.

What is left to show is the validity of the claims (v) - $(viii)$. Once again, we prove the first and the last claim. The other two can be reproduced through mere imitation.

Suppose $[\varphi]_u \psi \notin \Gamma$.

Let $\Sigma \subseteq \mathcal{L}^{\uparrow,\downarrow}$ such that $\mathbf{p}_{ud}\Gamma, \varphi \vdash_{ud} \Sigma$. We need to prove that

$\Sigma \cap (\Gamma \cup (\mathbf{ap}_{ud}\mathcal{L}^{\uparrow,\downarrow})^{\mathtt{C}} - \psi) \neq \emptyset$.

If $\Sigma \cap (\mathbf{ap}_{ud}\mathcal{L}^{\uparrow,\downarrow})^{\mathtt{C}} - \psi \neq \emptyset$, then we are done. So, suppose $\Sigma \cap (\mathbf{ap}_{ud}\mathcal{L}^{\uparrow,\downarrow})^{\mathtt{C}} - \psi = \emptyset$, which is the same as $\Sigma \subseteq \mathbf{ap}_{ud}\mathcal{L}^{\uparrow,\downarrow} + \psi$. In other words, *all* non-ψ-elements of Σ are anti-persistent, i.e. $\mathbf{ap}_{ud}(\Sigma - \psi) = \Sigma - \psi$. The re-styling of R-TRUE $[\ ]_u$ in observation 5.37 establishes the following derivation:

1. $\mathbf{p}_{ud}\Gamma, \varphi \vdash_{ud} \Sigma - \psi, \psi$ R-MON
2. $\Gamma \vdash_{ud} \Sigma - \psi, [\varphi]_u \psi$ observation 5.37 and $\mathbf{ap}_{ud}(\Sigma - \psi) = \Sigma - \psi$.

Because $\Gamma \in \mathfrak{Sat}_{ud}$, the last **ud**-sequent above and the assumption $[\varphi]_u \psi \notin \Gamma$ entail $(\Sigma - \psi) \cap \Gamma \neq \emptyset$, and therefore also $\Sigma \cap (\Gamma \cup (\mathbf{ap}_{ud}\mathcal{L}^{\uparrow,\downarrow})^{\mathtt{C}} - \psi) \neq \emptyset$.

Suppose $\neg[\varphi]_d \psi \in \Gamma$.

Let $\Sigma \subseteq \mathcal{L}^{\uparrow,\downarrow}$ with $\mathbf{ap}_{ud}\Gamma + \neg\psi \vdash_{ud} \Sigma$. We need to prove that

$\Sigma \cap (\Gamma \cup (\mathbf{p}_{ud}\mathcal{L}^{\uparrow,\downarrow})^{\mathtt{C}} - \varphi) \neq \emptyset$ (2).

If $\Sigma \cap ((\mathbf{p}_{ud}\mathcal{L}^{\uparrow,\downarrow})^{\mathtt{C}} - \varphi) \neq \emptyset$, then we have immediately our desired result. So, let $\Sigma \subseteq \mathbf{p}_{ud}\mathcal{L}^{\uparrow,\downarrow} + \varphi$. This means that $\mathbf{p}_{ud}(\Sigma - \varphi) = \Sigma - \varphi$. The following derivation settles this complementary case.

1. $\mathbf{ap}_{ud}\Gamma, \neg\psi \vdash_{ud} \Sigma - \varphi, \varphi$ R-MON
2. $\Gamma, \neg[\varphi]_d \psi \vdash_{ud} \Sigma - \varphi$ observation 5.37, and $\mathbf{p}_{ud}(\Sigma - \varphi) = \Sigma - \varphi$
3. $\Gamma \vdash_{ud} \Sigma - \varphi$ $\neg[\varphi]_d \psi \in \Gamma$

Because $\Gamma \in \mathfrak{Sat}_{ud}$, we conclude $\Sigma \cap (\Gamma - \varphi) \neq \emptyset$ which also establishes (2).

■

5.39. THEOREM. COMPLETENESS **ud**

For all $\Gamma, \Delta \subseteq \mathcal{L}^{\uparrow,\downarrow}$: $\Gamma \models_{\mathfrak{N}^{\uparrow,\downarrow}} \Delta \Longrightarrow \Gamma \vdash_{ud} \Delta$.

Note that the truth lemma of **ud**, just like the truth lemma of the minimal partial modal logic **M**, can be used for any normal extension. Furthermore, the induction steps only call upon the corresponding introduction rules. This means that the completeness of systems which evolve from removing connectives or operators immediately follow from the syntactically puritan proof of the truth lemma above. For example, a complete system for only update-reasoning over Nelson models is simply the system which consists of all the rules without the ones which mention downdate operators. The only little modification which we should make beforehand is a redefinition of anti-persistent information in terms of updates. Table 5.33 shows that a re-styling like $\mathbf{ap}_X\Gamma := \{\varphi \in \Gamma \mid \langle\ \rangle_u \varphi \vdash_X \varphi\}$ for all $\Gamma \subseteq \mathcal{L}_X$ is satisfactory. This system is called **u**.

Sequential data logic

A system which is an update system over Nelson models is Veltman's data semantics. As we have mentioned in section 3.1, this conditional logic uses Nelson models with the refinability constraint: for all formulae there exists an extension of the current state which either falsifies or verifies this formula. Let's write Veltman's conditional $\varphi \rightsquigarrow \psi$ as $[\varphi]_u\psi$. Their interpretations completely coincide.

The additional axiom to **u** to capture the refinability constraint, in terms of our update operator, is

$$\frac{\Gamma, \varphi \vdash \Delta}{[\ ]_u\Gamma \vdash \neg[\ ]_u\varphi, \langle\ \rangle_u\Delta} \quad \text{SDL-R-FALSE } [\ ]_u$$

Let's call this system **sdl**, i.e. $\mathbf{u}$ + SDL-R-FALSE $[\ ]_u$, an abbreviation of *sequential data logic*. In terms of persistence and anti-persistence, the rule SDL-R-TRUE $[\ ]_u$ above can be rephrased as follows:

$$\mathbf{p}_{sdl}\Gamma, \varphi \vdash_{sdl} \mathbf{ap}_{sdl}\Delta \implies \Gamma \vdash_{sdl} \neg[\ ]_u\varphi, \Delta.$$

The canonical model is the triple $M_{sdl} = \langle \mathfrak{Sat}_{sdl}, \ll_{sdl}, V_{sdl}\rangle$ where $\ll_{sdl}$ is defined as $\ll_X$ in definition 5.32, with the definition of the anti-persistent part replaced by its reformulation above. V_{sdl} is defined as all previous global canonical valuation functions. Clearly $M_{sdl} \in \mathfrak{N}$ (observation 5.33), and the truth lemma for **sdl** can be obtained through a pure imitation of lemma 5.38. What is left to show is that M_{sdl} is a model of data semantics, i.e. for all $\varphi \in \mathcal{L}^{\uparrow}$ every **sdl**-saturated set has an extension in M_{sdl} which determines the truth-value of φ.

5.40. OBSERVATION. For all $\Sigma \in \mathfrak{Sat}_{sdl}$ and $\varphi \in \mathcal{L}^{\uparrow}$ there exists $\Theta \in \mathfrak{Sat}_{sdl}$ such that $\Sigma \ll_{sdl} \Theta$, and either $\varphi \in \Theta$ or $\neg\varphi \in \Theta$.

Proof. Let $\Sigma \in \mathfrak{Sat}_{sdl}$ and $\varphi \in \mathcal{L}^{\uparrow}$. Because $\vdash_{sdl} \langle\ \rangle_u\varphi, \langle\ \rangle_u\neg\varphi$, we know that $\langle\ \rangle_u\varphi \in \Sigma$ or $\langle\ \rangle_u\neg\varphi \in \Sigma$. In the former case, it can be proved that $\mathbf{p}_{sdl}\Sigma + \varphi \trianglelefteq_{sdl} \Sigma \cup (\mathbf{ap}_{sdl}\mathcal{L}^{\uparrow})^{\complement}$, just like claim (*vi*) in the proof of lemma 5.38. This means that there exists $\Theta \in \mathfrak{Sat}_{sdl}$ such that $\Sigma \ll_{sdl} \Theta$ with $\varphi \in \Theta$.

The latter case establishes $\neg\varphi \in \Theta$ for certain $\Theta \in \mathfrak{Sat}_{sdl}$ by the same argumentation, with φ replaced by $\neg\varphi$. ■

5.41. COROLLARY. **sdl** is a complete axiomatization of strong consequence relation over models of data semantics.

The completeness of Mud

The **Mud**-canonical model is $M_{Mud} = \langle \mathfrak{Sat}_{Mud}, R_{Mud}, \ll_{Mud}, V_{Mud}\rangle$, with R_{Mud} being the canonical accessibility relation over $\mathfrak{Sat}_{Mud}$ (see definition 5.11) and $\ll_{Mud}$ is the canonical information structure of **ud**-extensions over $\mathfrak{Sat}_{Mud}$. V_{Mud} is the ordinary canonical global valuation function.

5.42. LEMMA. $M_{Mud} \in \mathfrak{NM}^{\Box}$.

Proof. We need to show that $\ll_{Mud} \circ R_{Mud} \subseteq R_{Mud} \circ \ll_{Mud}$. This cannot be shown as easily as the same interrelational constraint for the $\mathbf{NM}^{\Box}$-canonical model (lemma 5.30). The additional difficulty here is the 'anti-persistence'-part of the definition of $\ll_{Mud}$.

Let $\Gamma, \Gamma', \Delta' \in \mathfrak{Sat}_{Mud}$ such that $\Gamma \ll_{Mud} \Gamma'$ and $R_{Mud}(\Gamma', \Delta')$. We have to demonstrate the existence of a $\Delta \in \mathfrak{Sat}_{Mud}$ such that $\Delta \ll_{Mud} \Delta'$ and $R_{Mud}(\Gamma, \Delta)$. This can be enforced by means of the following claim:

$$\Box^{-}\Gamma \cup \mathbf{ap}_{Mud}\Delta' \trianglelefteq_{Mud} (\Delta' \cup (\mathbf{p}_{Mud}\mathcal{L}^{\Box,\uparrow,\downarrow})^{\complement}) \cap \Diamond^{-}\Gamma \quad (5).$$

This claim yields the desired **Mud**-saturated set Δ indeed. BSL shows that there exists such a $\Delta \in \mathfrak{Sat}_{Mud}$ such that $\Box^{-}\Gamma \subseteq \Delta \subseteq \Diamond^{-}\Gamma$ and $\mathbf{ap}_{Mud}\Delta' \subseteq \Delta$ and $\Delta \subseteq \Delta' \cup (\mathbf{p}_{Mud}\mathcal{L}^{\Box,\uparrow,\downarrow})^{\complement}$. According to earlier observations in table 5.33, this last conclusion demonstrates $\mathbf{p}_{Mud}\Delta \subseteq \Delta'$. In short, $R_{Mud}(\Gamma, \Delta)$ and $\Delta \ll_{Mud} \Delta'$.

What is left to prove is the validity of (5), or formally,

$$\Box^{-}\Gamma \cup \mathbf{ap}_{Mud}\Delta' \vdash_{Mud} \Sigma \implies \Sigma \cap (\Delta' \cup (\mathbf{p}_{Mud}\mathcal{L}^{\Box,\uparrow,\downarrow})^{\complement}) \cap \Diamond^{-}\Gamma \neq \emptyset \quad (6).$$

Suppose $\Box^{-}\Gamma \cup \mathbf{ap}_{Mud}\Delta' \vdash_{Mud} \Sigma$, and let $\Sigma_1 = \Sigma \cap \Diamond^{-}\Gamma$ and $\Sigma_2 = \Sigma \setminus \Diamond^{-}\Gamma$. Furthermore, we define $\sigma_2 = \bigvee \Sigma_2$.

To start with, we claim $\Sigma_1 \neq \emptyset$ (7).

This claim can be proved by the finiteness property of **Mud**. In combination with L-TRUE $\wedge$, this property makes sure that there exists a finite sequence $\delta_1, .., \delta_n \in \mathbf{ap}_{Mud}\Delta'$ such that

$$\Box^{-}\Gamma, \delta_1 \wedge ... \wedge \delta_n \vdash_{Mud} \Sigma \quad (8).$$

Let $\delta := \delta_1 \wedge ... \wedge \delta_n$. According to table 5.33 we also may conclude $\delta \in \mathbf{ap}_{Mud}\mathcal{L}^{\Box,\uparrow,\downarrow}$ and $\Diamond\delta \in \mathbf{ap}_{Mud}\mathcal{L}^{\Box,\uparrow,\downarrow}$. Because $\delta \in \Delta'$, and $R_{Mud}(\Gamma', \Delta')$, we obtain $\Diamond\delta \in \mathbf{ap}_{Mud}\Gamma'$. This yields $\Diamond\delta \in \Gamma$, for $\Gamma \ll_{Mud} \Gamma'$ ($\mathbf{ap}_{Mud}\Gamma' \subseteq \Gamma$). This result, in combination with L-TRUE $\Diamond$ and L-MON, transforms (8) into

$$\Gamma \vdash_{Mud} \Diamond\Sigma \quad (9).$$

Because $\Gamma \in \mathfrak{Sat}_{Mud}$, we find $\Sigma \cap \Diamond^{-}\Gamma = \Sigma_1 \neq \emptyset$.

If $\Sigma_1 \cap (\mathbf{p}_{Mud}\mathcal{L}^{\Box,\uparrow,\downarrow})^{\complement} \neq \emptyset$, then we are done. So, let $\Sigma_1 \subseteq \mathbf{p}_{Mud}\mathcal{L}^{\Box,\uparrow,\downarrow}$. In this complementary case, we need to prove that $\Sigma_1 \cap \Delta' \neq \emptyset$. Remember $\Sigma_1 = \Diamond^{-}\Gamma \cap \Sigma$.

Because of the finiteness property of **Mud**, and $\alpha \in \Box^{-}\Gamma, \beta \in \Box^{-}\Gamma \Rightarrow \alpha \wedge \beta \in \Box^{-}\Gamma$, there exists $\varphi \in \Box^{-}\Gamma$ such that

$$\mathbf{ap}_{Mud}\Delta', \varphi \vdash_{Mud} \Sigma_1, \sigma_2.$$

Because $\Sigma_1 \subseteq \mathbf{p}_{Mud}\mathcal{L}^{\Box,\uparrow,\downarrow}$ and observation 5.37, we know that

$$\Delta', \langle\sigma_2\rangle_d\,\varphi \vdash_{Mud} \Sigma_1 \quad (10).$$

Suppose that $\langle\sigma_2\rangle_d\,\varphi \notin \Delta'$. Since $\langle\sigma_2\rangle_d\,\varphi \in \mathbf{p}_{Mud}\mathcal{L}^{\Box,\uparrow,\downarrow}$, also $\Box\langle\sigma_2\rangle_d\,\varphi \in \mathbf{p}_{Mud}\mathcal{L}^{\Box,\uparrow,\downarrow}$, and therefore, $\Box\langle\sigma_2\rangle_d\,\varphi \notin \Gamma$. Because $\varphi \vdash_{Mud} \langle\sigma_2\rangle_d\,\varphi, \sigma_2$, we obtain the sequent $\Box\varphi \vdash_{Mud} \Box\langle\sigma_2\rangle_d\,\varphi, \Diamond\Sigma_2$. This proves $\Gamma \vdash_{Mud} \Box\langle\sigma_2\rangle_d\,\varphi, \Diamond\Sigma_2$, and $\Sigma_2 \cap \Diamond^{-}\Gamma \neq \emptyset$. This conclusion contradicts the definition of Σ_2, and so, it must be the case that $\langle\sigma_2\rangle_d\,\varphi \in \Delta'$. This reduces (10) to $\Delta' \vdash_{Mud} \Sigma_1$, and therefore, $\Sigma_1 \cap \Delta' \neq \emptyset$, since $\Delta' \in \mathfrak{Sat}_{Mud}$. ■

5.5 Completeness of epistemic logics

Completeness proofs for the basic static and dynamic logic , $\mathbf{E}_A$ and $\mathbf{C}^3$, are really straightforward. Truth lemmas do not have to be proved, because the truth lemmas for their underlying modal formalism, **M** and **Mud**, respectively, suffice (e.g. see lemma 5.14). In this case, the plurality of epistemic operators does not complicate matters. The only thing which we need to check is whether their canonical models have the desired frame properties: serial full introspective accessibilities. A simple technical affirmation is presented in the next subsection.

A completeness result for the totally free preferential extension of $\mathbf{C}^3$, $\mathbf{C}^3_i$, is just as simple. More difficult are the strengthenings by additional principles, like REALISM and the cooperative QUALITY principles. For such completeness results we also need a canonical definition of the coherence relation. We postpone this definition and the completeness results of such additional communication postulates to our last chapter 7, because in this chapter we deal with frame characteristics for so-called Geach extensions of partial modal logics. Relativized formulations of these extensions come quite close to the above-mentioned communication principles, and subsequently we can prove completeness for these systems straightforwardly in that chapter.

Other epistemic systems of interest in chapter 4 were the systems with supplementary 'real worlds': $\mathbf{C}^{3R}$ and $\mathbf{C}^3_\Omega$. Many of the earlier techniques in this chapter can be used to establish completeness for these systems as well. The only difference is that a proper definition a unique canonical model is not possible here. We need to define a bundle of canonical models. Still, the proof procedure does not deviate from our Henkin procedure in the previous sections.

As have been yet announced in the introduction of this chapter, the completeness proofs of the mutual belief systems $\mathbf{E}^*_A$ and $\mathbf{C}^{3*}$ are postponed until the next chapter (section 6.4).

The completeness of $\mathbf{E}_A$ and $\mathbf{C}^3$

5.43. LEMMA. $M_{E_A} \in \mathfrak{E}_{\mathfrak{A}}$ and $M_{C^3} \in \mathfrak{C}^3$.

Proof. We need to show that the $\mathbf{E}_A$-canonical model is serial and full introspective. We shortly present these relational requirements for $(R_{E_A})_a$[12].

Seriality:

Let $\Gamma \in \mathfrak{Sat}_{E_A}$, and suppose $\Box_a^- \Gamma \vdash_{E^A} \Sigma$. The rule **D** establishes $\Gamma \vdash_{E^A} \Diamond_a \Sigma$. This means $\Diamond_a^- \Gamma \cap \Sigma \neq \emptyset$, which implies $\Box_a^- \Gamma \trianglelefteq_{E^A} \Diamond_a^- \Gamma$. BSL shows that $\exists \Delta \in \mathfrak{Sat}_{E_A} : (R_{E^A})_a(\Gamma, \Delta)$.

Full introspection:

Suppose $(R_{E^A})_a(\Gamma, \Delta)$. We need to show $(R_{E^A})_a(\Gamma, \Theta) \Leftrightarrow (R_{E^A})_a(\Delta, \Theta)$ for all $\Theta \in \mathfrak{Sat}_{E_A}$ (1). This can demonstrated easily by means of the introspection axioms in example 4.8 on page 115.

$$\left.\begin{array}{l} \Box_a\varphi \in \Gamma \Rightarrow \Box_a\Box_a\varphi \in \Gamma \Rightarrow \Box_a\varphi \in \Delta \\ \Diamond_a\varphi \notin \Gamma \Rightarrow \Diamond_a\Diamond_a\varphi \notin \Gamma \Rightarrow \Diamond_a\varphi \notin \Delta \end{array}\right\} \Longrightarrow ((R_{E^A})_a(\Delta, \Theta) \Rightarrow (R_{E^A})_a(\Gamma, \Theta))$$

[12] $(R_{E_A})_a$ is the indivudialization of the canonical accessibility relation R_M: $(R_{E_A})_a(\Gamma, \Delta) \Leftrightarrow \Box_a^- \Gamma \subseteq \Delta \subseteq \Diamond_a^- \Gamma$, expanded over the collection $\mathfrak{Sat}_{E^A}$.

$$\left.\begin{array}{l}\Box_a\varphi \in \Delta \Rightarrow \Diamond_a\Box_a\varphi \in \Gamma \Rightarrow \Box_a\varphi \in \Gamma \\ \Diamond_a\varphi \notin \Delta \Rightarrow \Box_a\Diamond_a\varphi \notin \Gamma \Rightarrow \Diamond_a\varphi \notin \Gamma\end{array}\right\} \Longrightarrow ((R_{E^A})_a(\Gamma,\Theta) \Rightarrow (R_{E^A})_a(\Delta,\Theta))$$

In a complete analogous manner one can prove the seriality and full introspection of $(R_{C^3})_a$. ■

5.44. THEOREM. $\mathbf{E}_A$ is complete with respect to the class $\mathfrak{E}_{\mathfrak{A}}$. $\mathbf{C}^3$ is complete for $\mathfrak{C}^3$.

Completeness of systems with realities: $\mathbf{C}^{3R}$ and $\mathbf{C}^3_\Omega$

As already mentioned above, a unique canonical model choice is not possible for the mixture systems of classical and partial modal logics such as $\mathbf{C}^{3R}$ and $\mathbf{C}^3_\Omega$. For each of these systems we define a collection of canonical models.

5.45. DEFINITION. A $\mathbf{C}^{3R}$-canonical model is a quintuple

$M^{\mathbb{P}'}_{C^{3R}} = \langle \mathfrak{Sat}_{C^3}, \mathfrak{R}, \{(R_{C^3})_a\}_{a\in A}, \ll_{C^3}, V_{C^3}\rangle$, with

$\mathbb{P}' \subseteq \mathbb{P}$ and $\mathfrak{R} = \{\Sigma \in \mathfrak{Sat}_{C^{3R}} \mid \Sigma \cap \mathbb{P} = \mathbb{P}'\}$.

This definition shows that the full $\mathbf{C}^3$-canonical model is employed. Because we need canonical models with different selected realities, the unique canonical choice is not possible for this system. In fact, there are $2^{\#\mathbb{P}}$ $\mathbf{C}^{3R}$-canonical models.

5.46. OBSERVATION. The membership of $\mathfrak{R}$ can also be identified by the global valuation function:

$$[\ V_{C^3}(\Sigma) \in \mathfrak{T}\ \&\ (V_{C^3}(\Sigma)(p) = 1\ )\ \Leftrightarrow\ p \in \mathbb{P}'] \quad\Longleftrightarrow\quad \Sigma \in \mathfrak{R}.$$

First, we need to prove that these models are all $\mathfrak{C}^{3R}$-models.

5.47. LEMMA. $M^{\mathbb{P}'}_{C^{3R}} \in \mathfrak{C}^{3R}$ for all $\mathbb{P}' \subseteq \mathbb{P}$.

Proof. $V_{C^3}(\Sigma) \in \mathfrak{T}$ for all $\Sigma \in \mathfrak{R}$, because $\vdash_{C^{3R}} p, \neg p$ for all $p \in \mathbb{P}$, which means $p \in \Sigma$ or $\neg p \in \Sigma$. Note that for all $\Sigma \in \mathfrak{R}$: $V_{C^3}(\Sigma)(p) = 1 \Longleftrightarrow p \in \mathbb{P}'$.

$\mathfrak{R} \subseteq \mathfrak{Sat}_{C^3}$, because $\mathbf{C}^{3R}$ is an extension of $\mathbf{C}^3$. Furthermore, $\mathfrak{R}$ is clearly non-empty. Suppose that $\Sigma \ll_{C^3} \Theta$ for certain $\Sigma \in \mathfrak{R}$. We need to show that $\Theta \in \mathfrak{R}$. $\Sigma \ll_{C^3} \Theta$ entails $\mathbf{p}_{C^3}\Sigma \subseteq \Theta$. Because $p \in \Sigma \Rightarrow p \in \mathbf{p}_{C^3}\Sigma$ and $\neg p \in \Sigma \Rightarrow \neg p \in \mathbf{p}_{C^3}\Sigma$, we immediately conclude $V_{C^3}(\Sigma) = V_{C^3}(\Theta)$ (1). [1mm] What is left to prove is $\Theta \in \mathfrak{Sat}_{C^{3R}}$ (2). In combination with (1), we have immediately $\Theta \in \mathfrak{R}$.

Suppose $\Theta \vdash_{C^{3R}} \Xi$. We need to show $\Xi \cap \Theta \neq \emptyset$. Without loss of generality we may assume that both sets are finite, due to the finiteness property of $\mathbf{C}^{3R}$. We define the following abbreviations.

$\Theta' := \Theta \cap \mathcal{L}^{\uparrow,\downarrow}$ $\qquad\qquad$ $\theta := \bigwedge(\Theta \setminus \Theta')$

$\Xi' := \Xi \cap \mathcal{L}^{\uparrow,\downarrow}$ $\qquad\qquad$ $\xi := \bigvee(\Xi \setminus \Xi')$

Clearly, $\Theta', \theta \vdash_{C^{3R}} \xi, \Xi'$. Application of R-TRUE-STRONG $[\ ]_u$ yields

$\Theta' \vdash_{C^{3R}} [\theta]_u\,\xi, \Xi'$.

Because $\Sigma \ll_{C^3} \Theta$, we have $\Theta' \subseteq \Theta \subseteq \langle\ \rangle_u{}^{-}\Sigma$. By $\langle\ \rangle_u \varphi \vdash_{C^{3R}} \varphi$ for all $\varphi \in \mathcal{L}^{\uparrow,\downarrow}$, we infer $\langle\ \rangle_u{}^{-}\Sigma \cap \mathcal{L}^{\uparrow,\downarrow} = \Sigma \cap \mathcal{L}^{\uparrow,\downarrow}$. This means $\Theta' \subseteq \Sigma$, because $\Theta' \subseteq \mathcal{L}^{\uparrow,\downarrow}$. So, $\Sigma \vdash_{C^{3R}} [\theta]_u \xi, \Xi'$, and because $\Sigma \in \mathfrak{Sat}_{C^{3R}}$ we obtain:

$[\theta]_u \xi \in \Sigma$ (3) or $\Xi' \cap \Sigma \neq \emptyset$ (4).

Suppose (3) were the case. This implies $[\theta]_u \xi \in \Theta$ because $[\theta]_u \xi \in \mathbf{p}_{C^3}\Sigma$. Furthermore, $\theta \in \Theta$, and therefore also $\xi \in \Theta$. By definition of ξ and Θ's saturation, we find $\Xi \cap \Theta \neq \emptyset$.

Assume (4), and let $\zeta \in \Xi' \cap \Sigma$. Because $\zeta \in \mathcal{L}^{\uparrow,\downarrow}$, it is also $\mathbf{C}^{3R}$-persistent: $\zeta \vdash [\ ]_u \zeta$. Furthermore, $[\ ]_u \zeta \in \mathbf{p}_{C^3}\mathcal{L}_A^{\uparrow,\downarrow}$, and so $[\ ]_u \zeta \in \Theta$, and of course also $\zeta \in \Theta$. Finally, $\Xi' \subseteq \Theta$ entails $\Xi \cap \Theta \neq \emptyset$.

Both (3) and (4) entail $\Theta \cap \Xi \neq \emptyset$, and this confirms (2).

■

Second, we need the truth lemma for this system. This can be obtained instantaneously from the truth lemma of **Mud** and $\mathbf{C}^3$. Because $\mathfrak{R} \subseteq \mathfrak{Sat}_{C^3}$, we find

$$M_{C^{3R}}^{\mathbb{P}'}, \Sigma \models \varphi \iff M_{C^3}, \Sigma \models \varphi \iff \varphi \in \Sigma \quad (1).$$

This simple observation, together with lemma 5.47, establishes the completeness of $\mathbf{C}^{3R}$.

5.48. THEOREM. $\mathbf{C}^{3R}$ is complete with respect to $\mathfrak{C}^{3\mathfrak{R}}$-validity.

Proof. Suppose $\Gamma \not\vdash_{C^{3R}} \Delta$. This means that there exists $\Sigma \in \mathfrak{C}^{3\mathfrak{R}}$ such that $\Gamma \subseteq \Sigma$ and $\Delta \cap \Sigma = \emptyset$. Let $\mathbb{P}' = \Sigma \cap \mathbb{P}$. Clearly Σ in $M_{C^{3R}}^{\mathbb{P}'}$. Furthermore, according to (1) above, Σ provides a Γ-world, which verifies none of the members of Δ. ■

A completeness proof for $\mathbf{C}_\Omega^3$ can be found in the same fashion.

5.49. DEFINITION. A $\mathbf{C}_\Omega^3$-canonical model is a quintuple

$M_{C^{3R}}^{\mathbb{P}'} = \langle \mathfrak{Sat}_{C^3}, \mathfrak{R}, \{(R_{C^3})_a\}_{a \in \{\Omega, \mathbf{u}\}}, \ll_{C^3}, V_{C^3} \rangle$, with

$\mathbb{P}' \subseteq \mathbb{P}$ and $\mathfrak{R} = \{\Sigma \in \mathfrak{Sat}_{C^{3\Omega}} \mid \Sigma \cap \mathbb{P} = \mathbb{P}'\}$.

The only difference with the definition of the $\mathbf{C}^{3R}$-canonical model is that $\mathfrak{R}$ is the selection of $\mathbf{C}_\Omega^3$-saturated sets with the same atomic content. The only essential specific property which has to be checked is whether the following typical $\mathbf{C}_\Omega^3$-constraint holds:

5.50. LEMMA. $R_\Omega(\Sigma, \Theta)$ & $\Sigma \in \mathfrak{R} \implies \Theta \in \mathfrak{R}$.

Proof. Suppose $(R_{C_\Omega^3})_\Omega(\Sigma, \Theta)$ for certain $\Sigma \in \mathfrak{R}$. Totality of $V_{C^3}(\Theta)$ follows immediately from the simple $\mathbf{C}_\Omega^3$-sequents $\vdash_{C_\Omega^3} \Box_\Omega p, \neg p$ and $\vdash_{C_\Omega^3} \Diamond_\Omega p, \neg p$ for all $p \in \mathbb{P}$. So, what is left to prove is $\Theta \in \mathfrak{Sat}_{C_\Omega^3}$.

Suppose $\Theta \vdash_{C_\Omega^3} \Xi$ for certain $\Xi \subseteq \mathcal{L}_A^{\uparrow,\downarrow}$, and define Θ', Ξ', θ and ξ in the same way as in the proof of lemma 5.47, but with $\mathcal{L}^{\uparrow,\downarrow}$ replaced by $\mathcal{L}_\Omega^{\uparrow,\downarrow}$. Analogously with this proof, we obtain:

$\Theta' \vdash_{C^3_\Omega} [\theta]_u \xi, \Xi.$

Because $(R_{C^3_\Omega})_\Omega(\Sigma, \Theta)$, we know that $\Theta' \subseteq \diamond^-_\Omega \Sigma \cap \mathcal{L}^{\uparrow,\downarrow}_\Omega$. Subsequently, $\diamond_\Omega \varphi \vdash_{C^3_\Omega} \varphi$ yields $\Theta' \subseteq \Sigma \cap \mathcal{L}^{\uparrow,\downarrow}_\Omega$, and therefore $\Sigma \cap \mathcal{L}^{\uparrow,\downarrow}_\Omega \vdash_{C^3_\Omega} [\theta]_u \xi, \Xi'$. Application of R-TRUE-STRONG $\Box_\Omega$ entails $\Sigma \vdash_{C^3_\Omega} \Box_\Omega[\theta]_u \xi, \Xi'$, and also

$\Box_\Omega[\theta]_u \xi \in \Sigma$ (1) or $\Xi' \cap \Sigma \neq \emptyset$ (2).

Suppose (1) holds. This means $[\theta]_u \xi \in \Theta$, and because $\theta \in \Theta$, and by the definition of ξ, we find $\Theta \cap \Xi \neq \emptyset$.

Take (2) to be the case, and let $\zeta \in \Xi' \cap \Sigma$. Because $\zeta \vdash_{C^3_\Omega} \Box_\Omega \zeta$ ($\zeta \in \mathcal{L}^{\uparrow,\downarrow}_\Omega$), we have $\Box_\Omega \zeta \in \Sigma$, and therefore $\zeta \in \Theta$.

Both (1) and (2) yield the desired result $\Theta \cap \Xi \neq \emptyset$, which means $\Theta \in \mathfrak{Sat}_{C^3_\Omega}$. ■

5.51. THEOREM. $\mathbf{C}^3_\Omega$ is complete with respect to $\mathfrak{C}^3_\Omega$.

5.6 Conclusions

This chapter has shown that establishing completeness results for partial intensional logics does not have to be much more complicated than for classical intensional logics. The generalization of maximally consistent sets to our definition of saturated sets in order to build suitable canonical models has turned out to be fruitful.

The complication of establishing completeness results on the basis of these canonical models for partial intensional logics is that in many truth lemmas saturated sets need to be located below given upper bounds. A simple example is the $\Box$-step in the proof of the truth lemma of **M** (page 149).

This problem of constructing saturated sets below an upper bound can be reduced to saturation equations. If an upper bound is a saturator with respect to the initial set from which we start the construction, then we know that the construction will be successful, due to our bounded saturation lemma (page 146). The sequential style of definition of the notions of saturation and saturators, is thus shown to be practical in this respect. Many saturation equations were easily solvable by the use of our sequential rules. In case of the systems **M** we managed to give a completeness proof which is much simpler than earlier proofs [Thijsse 1992] [Jaspars 1993]. This technique has been extended successfully to the new constructive modal logics of chapter 3.

In the next chapter we will show how these techniques can be used to construct sets which are saturated up to a certain (finite) subset of formulae. On the basis of these further investigations decidability results can be established for the logics of part I. Furthermore, we are able to prove the completeness for finite sequents for the non-compact mutual belief logics $\mathbf{E}_A$ and $\mathbf{C}^{3*}$.

Appendix

5.7 Completeness of fused partial modal logic

In section 2.5 we have introduced a partial modal logic **FM**, the so-called fused partial modal logic, which is weaker than the minimal partial modal logic **M**. Below, we will give a brief exposition of the completeness proof for this 'non-normal' partial modal logic.

5.52. DEFINITION. The **FM**-canonical model is defined to be the triple $M_{FM} = \langle \mathfrak{Sat}_{FM}, R_{FM}, V_{FM} \rangle$ with

$R_{FM}(\Gamma, \mathfrak{G}) \Leftrightarrow \Box^{-}\Gamma \subseteq \bigcup \mathfrak{G}, \bigcap \mathfrak{G} \subseteq \Diamond^{-}\Gamma$, and

$V(\Gamma)(p) = 1(0) \Leftrightarrow (\neg)p \in \Gamma$ for all $\Gamma \in \mathfrak{Sat}_{FM}$ and $\mathfrak{G} \subseteq \mathfrak{Sat}_{FM}$.

5.53. OBSERVATION. $M_{FM} \in \mathfrak{FM}$.

Proof. Suppose $R_{FM}(\Gamma, \mathfrak{G})$. Of course $\vdash_{FM} \Box\top$, and therefore $\top \in \Box^{-}\Gamma$. This means $\top \in \bigcup \mathfrak{G}$, and so, $\mathfrak{G} \neq \emptyset$. ■

5.54. LEMMA. For all $\Gamma \in \mathfrak{Sat}_{FM}$ and $\varphi \in \mathcal{L}^{\Box}$:

$M_{FM}, \Gamma \models \varphi \Leftrightarrow \varphi \in \Gamma$ and $M_{FM}, \Gamma \models\!\!| \varphi \Leftrightarrow \neg\varphi \in \Gamma$.

Proof. By induction on the construction of formulae. The steps of the **P**-connectives can be made as immediate as before. We only elaborate on the $\Box$-steps.

Suppose $\Box\varphi \in \Gamma$.

If $R_{FM}(\Gamma, \mathfrak{G})$ then $\varphi \in \bigcup \mathfrak{G}$, which means that there exists $\Delta \in \mathfrak{G}$ such that $\varphi \in \Delta$. The induction hypothesis establishes $M_{FM}, \Delta \models \varphi$. Because $\mathfrak{G}$ has been chosen as an arbitrary accessible collection of **FM**-saturated sets, we conclude that $M_{FM}, \Gamma \models \Box\varphi$.

Suppose $\Box\varphi \notin \Gamma$.

Let $\mathfrak{G} := \{\Delta \in \mathfrak{Sat}_{FM} \mid \varphi \notin \Delta\}$. Because $\nvdash_{FM} \Box\varphi$, also $\nvdash_{FM} \varphi$ (table 2.60) and therefore $\mathfrak{G} \neq \emptyset$ (saturation lemma).

Suppose $\Box\psi \in \Gamma$. Because $\Box\varphi \notin \Gamma$, we know that $\Box\psi \not\vdash_{FM} \Box\varphi$, and subsequently $\psi \not\vdash_{FM} \varphi$ (table 2.60). The saturation lemma shows that there must exist $\Delta_\psi \in \mathfrak{G}$ such that $\psi \in \Delta_\psi$, and therefore, $\psi \in \bigcup\mathfrak{G}$. In other words,

$\Box^-\Gamma \subseteq \bigcup\mathfrak{G}$ (1).

Suppose $\Diamond\chi \notin \Gamma$. Because $\Box\varphi \notin \Gamma$, we conclude $\not\vdash_{FM} \Box\varphi, \Diamond\chi$, and therefore, $\not\vdash_{FM} \varphi, \chi$ (table 2.60). The saturation lemma shows that there exists $\Delta_\chi \in \mathfrak{G}$ such that $\chi \notin \Delta_\chi$, which means $\chi \notin \bigcap\mathfrak{G}$. In short,

$\bigcap\mathfrak{G} \subseteq \Diamond^-\Gamma$ (2).

The results (1) and (2) prove that $R_{FM}(\Gamma, \mathfrak{G})$. The induction hypothesis tells us that $M_{FM} \not\models \varphi$ for all $\Delta \in \mathfrak{G}$. This entails $M_{FM}, \Gamma \not\models \Box\varphi$.

If $\neg\Box\varphi \notin \Gamma$, we obtain $\neg\Box\varphi \notin \Gamma$ immediately through the definition of R_{FM} and the induction hypothesis.

Suppose $\neg\Box\varphi \in \Gamma$.

Clearly, $\neg\varphi \in \Diamond^-\Gamma$. Let $\mathfrak{F} := \{\Delta \in \mathfrak{Sat}_{FM} \mid \neg\varphi \in \Delta\}$. The **FM**-consistency of $\neg\varphi$ shows that $\mathfrak{F}$ is non-empty[13].

Take $\Box\psi \in \Gamma$. Because $\Box\psi, \Diamond\neg\varphi \not\vdash_{FM} \emptyset$, we know that $\psi, \neg\varphi \not\vdash_{FM} \emptyset$. The saturation lemma proves that there exists $\Delta_\psi \in \mathfrak{F}$ such that $\psi \in \Delta_\psi$, or shorter $\psi \in \mathfrak{F}$. Because $\Box\psi \in \Gamma$ have been picked randomly, we conclude

$\Box^-\Gamma \subseteq \bigcup\mathfrak{F}$ (3).

Let $\Diamond\chi \notin \Gamma$. This means $\Diamond\neg\varphi \not\vdash_{FM} \Diamond\chi$, and therefore, $\neg\varphi \not\vdash_{FM} \chi$. The saturation lemma shows the existence of a $\Delta_\chi \in \mathfrak{F}$ such that $\chi \notin \Delta_\chi$. This means $\chi \notin \bigcap\mathfrak{F}$. The arbitrariness of $\chi \notin \Diamond^-\Gamma$ shows

$\bigcap\mathfrak{F} \subseteq \Diamond^-\Gamma$ (4).

The results (3) and (4) prove $R_{FM}(\Gamma, \mathfrak{F})$, while the induction hypothesis entails $M_{FM}, \Delta \dashv \varphi$ for all $\Delta \in \mathfrak{F}$. The combination of these conclusions yields $M_{FM}, \Gamma \dashv \Box\varphi$.

■

Completeness for the two 'intermediate' systems which have also been discussed shortly in section 2.5 can be obtained by combination of the method above and the procedure of the truth-lemma for **M** (lemma 5.12).

[13] $\Diamond\neg\varphi \not\vdash_{FM} \emptyset \;\Rightarrow\; \neg\varphi \not\vdash_{FM} \emptyset$ (table 2.60).

Chapter 6

Finite Models

In this chapter we will continue our search for counter-models. The only difference with the previous chapter is a stringent restriction on our search space. Our quest is to find *finite* counter-models. Such finite counter-models appear of importance to us for two basic reasons.

First, we wish to establish decidability results in addition to the completeness results of the previous chapter for the systems which have been discussed in this thesis.

Second, due to the mutual belief operators in the system $\mathbf{E}_A^*$ and $\mathbf{C}^{3*}$, we need to acquire a procedure for finding finite counter-models for these systems in order to derive their completeness. Just like propositional dynamic logic [Kozen & Parikh 1981] and mutual knowledge and belief extensions of classical polymodal logics [Halpern & Moses 1992], where also (reflexive) transitive closures of accessibility relations are employed, the extra complications caused by the infinite nature of these operators turn out to be manageable by means of finite imitations of their canonical models. This procedure settles the completeness of these systems only up to finite sequents. But as we have seen earlier such a restriction is necessary, since these logics are non-compact (observation 4.23, page 123).

Of course, we need to establish enough meta-theoretical insights to justify the above-mentioned finite limitations. This enterprise is the key issue of this chapter.

6.1 Restricted saturated sets

Proving decidability of a modal logic is most often achieved by showing that it enjoys the so-called *finite model property* (FMP) [Hughes & Cresswell 1984][1].

[1] For an extensive treatment of the finite model property in classical modal logics see [de Jongh & Veltman 1988].

This means that for every finite non-sequent a finite counter-model can be found. The combination of a completeness result and FMP establishes subsequently a decidability result for finite sequents.

In this chapter we will focus on the so-called *strong* version of the finite model property. This means that a fixed upper bound of the size of the counter-model of a given non-sequent can be stipulated beforehand, that is in terms of the size of the assumption and conclusion set. This implies that if we want to know whether $\Gamma \vdash_S \Delta$, with Γ and Δ finite, holds for a system **S**, then we only have to check a finite space of models for finding a counter-model. If in this restricted finite class of models such a counter-model is not found, then we know that $\Gamma \vdash_S \Delta$. This strong FMP immediately establishes the decidability result, without calling upon the completeness of **S**.

We will use much of the techniques of the previous chapter to establish finite model properties. We make use of so-called *filtrations* of the canonical models which have been presented in chapter 5. We construct *filtrated* canonical models from sets which are saturated up to a given finite subset of formulae, which contains only a set of 'relevant' formulae with respect to a given non-sequent. In general, these restricted sets of formulae are fairly small extensions of the set of subformulae of the assumption and conclusion set of the non-sequent.

In the following section we will show that the main saturation lemmas of section 5.1 also apply to these limited saturated sets. FMP for **M**, **N** and **ud** can be deduced quite easily from these new saturation lemmas. In the second section we will show that proving FMP for constructive modal logics requires somewhat richer filtrations. The last section is dedicated to completeness proofs for finite sequents in $\mathbf{E}^*_A$ and $\mathbf{C}^{3*}$, which also employ these enriched filtrations.

Saturation lemmas for restricted sets

6.1. Definition. Let **S** be a sequential derivation system, with $\mathcal{L}_S$ its language, and $\Phi \subseteq \mathcal{L}_S$. A set $\Gamma \subseteq \Phi$ is said to be **S**-Φ-saturated if for all $\Delta \subseteq \Phi$

$$\Gamma \vdash_S \Delta \implies \Gamma \cap \Delta \neq \emptyset$$

The set of all **S**-Φ-saturated sets is written as $\mathfrak{Sat}^{\Phi}_S$. $\Lambda \subseteq \mathcal{L}_S$ is said to be an **S**-Φ-saturator of $\Gamma \subseteq \Phi$ if for all $\Delta \subseteq \Phi$

$$\Gamma \vdash_S \Delta \implies \Lambda \cap \Delta \neq \emptyset.$$

We denote this relation by $\Gamma \trianglelefteq^{\Phi}_S \Lambda$. Γ is said to be an **S**-Φ-saturant of Λ.

6.2. Observation. Note that all **S**-Φ-saturated sets are subsets of Φ. This does not have to be the case for **S**-Φ-saturators. All **S**-Φ-saturated sets are **S**-consistent. We list some properties below of the definitions above, which are for a great deal reformulations of the simple observation 5.4 (page 145) on the general definitions of saturated sets, saturators and saturants.

$$\Gamma \trianglelefteq^{\Phi}_S \Lambda \Rightarrow \Gamma \text{ is } \mathbf{S}\text{-consistent.}$$

$$\Gamma \trianglelefteq^{\Phi}_S \Lambda \ \&\ \Gamma' \subseteq \Gamma \Rightarrow \Gamma' \trianglelefteq^{\Phi}_S \Lambda.$$

$$\Gamma \trianglelefteq^{\Phi}_S \Lambda \ \&\ \Lambda \subseteq \Lambda' \Rightarrow \Gamma \trianglelefteq^{\Phi}_S \Lambda'.$$

$\Gamma \trianglelefteq_S^{\Phi} \Lambda \;\&\; \Psi \subseteq \Phi \Rightarrow \Gamma \cap \Psi \trianglelefteq_S^{\Psi} \Lambda.$

$\emptyset \in \mathfrak{Sat}_S^{\emptyset} \Leftrightarrow \mathbf{S}$ is consistent.

There is a strong connection between these restricted saturated sets and normal saturated sets.

6.3. PROPOSITION. Let $\mathbf{S}$ be a derivational system which contains all the structural rules of table 2.14.

For every $\mathbf{S}$-saturated set Σ and every set $\Phi \subseteq \mathcal{L}_S$ the set $\Sigma \cap \Phi$ is $\mathbf{S}$-Φ-saturated:

$\forall \Phi \subseteq \mathcal{L}_S : \Sigma \in \mathfrak{Sat}_S \Rightarrow \Sigma \cap \Phi \in \mathfrak{Sat}_S^{\Phi}.$

This observation can be formulated somewhat sharper:

$\forall \Phi, \Psi \subseteq \mathcal{L}_S : \Phi \subseteq \Psi \;\&\; \Sigma \in \mathfrak{Sat}_S^{\Psi} \Rightarrow \Sigma \cap \Phi \in \mathfrak{Sat}_S^{\Phi}.$

If $\Phi \subseteq \mathcal{L}_S$ and $\Sigma \in \mathfrak{Sat}_S^{\Phi}$ then there exists $\Sigma^* \in \mathfrak{Sat}_S$ such that $\Sigma^* \cap \Phi = \Sigma$. This can be made more general in similar terms as the result above: if $\Phi \subseteq \Psi \subseteq \mathcal{L}_S$ and $\Sigma \in \mathfrak{Sat}_S^{\Phi}$, then there exists $\Sigma' \in \mathfrak{Sat}_S^{\Psi}$ such that $\Sigma' \cap \Phi = \Sigma$. Formally,

$$\forall \Phi, \Psi \subseteq \mathcal{L}_S : \Phi \subseteq \Psi \;\&\; \Sigma \in \mathfrak{Sat}_S^{\Phi} \;\Rightarrow \\ \exists \Sigma' \in \mathfrak{Sat}_S^{\Psi} : \Sigma' \cap \Phi = \Sigma \quad (1).$$

Proof. The first part of the proposition is really straightforward by using the L-MON rule. The second part requires some explanation. Let $\Phi \subseteq \Psi \subseteq \mathcal{L}_S$, let $\Sigma \in \mathfrak{Sat}_S^{\Phi}$ and $\Lambda := (\Phi)^{\mathsf{C}} \cup \Sigma$. Clearly, $\Sigma \trianglelefteq_S \Lambda$. The bounded saturation lemma 5.6 shows that there exists $\Sigma^* \in \mathfrak{Sat}_S$ such that $\Sigma \subseteq \Sigma^* \subseteq \Lambda$ and $\Sigma^* \cap \Phi = \Sigma$. The first item of the proposition proves $\Sigma^* \cap \Psi \in \mathfrak{Sat}_S^{\Psi}$, and furthermore $\Sigma \subseteq \Sigma^* \cap \Psi$. So, $\Sigma' := \Sigma^* \cap \Psi$ is a fulfilling choice for the consequence in (1): $\Sigma \subseteq \Sigma' \subseteq \Lambda$. ■

The bounded and the ordinary saturation lemma are now easily obtained from the observations made in the proposition above.

6.4. LEMMA. BOUNDED SATURATION LEMMA FOR FILTRATIONS

Let $\mathbf{S}$ be a sequential derivation system such as in BSL and $\Phi \subseteq \mathcal{L}_S$. If $\Lambda \subseteq \Phi$ is an $\mathbf{S}$-Φ-saturator of a set $\Gamma \subseteq \Phi$, then it also contains an $\mathbf{S}$-Φ-saturated extension of Γ. Formally speaking,

$\forall \Gamma, \Phi, \Lambda \subseteq \mathcal{L}_S : \Gamma \trianglelefteq_S^{\Phi} \Lambda \Rightarrow \exists \Delta \in \mathfrak{Sat}_S^{\Phi} : \Gamma \subseteq \Delta \subseteq \Lambda.$

Proof. The proof is very short. We give a formal presentation below.

$\Gamma \trianglelefteq_S^{\Phi} \Lambda \Rightarrow \Gamma \trianglelefteq \Lambda \cup (\Phi)^{\mathsf{C}} \Rightarrow \exists \Delta^* \in \mathfrak{Sat}_S : \Gamma \subseteq \Delta^* \subseteq \Lambda \Rightarrow$

$\exists \Delta \in \mathfrak{Sat}_S^{\Phi} : \Gamma \subseteq \Delta \subseteq \Lambda \quad (\text{take } \Delta := \Delta^* \cap \Psi).$

■

This filtration version of BSL is also equivalent to a relativized formulation of the saturation lemma 5.8 (page 147). It can be illustrated by means of an adaptation of proposition 5.7.

6.5. PROPOSITION. Let **S** be a sequential system which contains R-MON, and let Γ, Φ, Λ be subsets of $\mathcal{L}_S$.

$$\Gamma \trianglelefteq^{\Phi}_{S} \Lambda \Leftrightarrow \Gamma \not\vdash_S \Phi \setminus \Lambda.$$

Proof. $\Rightarrow$: $\Gamma \vdash_S \Phi \setminus \Lambda \Rightarrow \Gamma \not\trianglelefteq^{\Phi}_{S} \Lambda$ (because $(\Phi \setminus \Lambda) \subseteq \Phi$, and $(\Phi \setminus \Lambda) \cap \Lambda = \emptyset$).
$\Leftarrow$: $\Gamma \not\trianglelefteq^{\Phi}_{S} \Lambda \Rightarrow \exists \Delta \subseteq \Phi : \Gamma \vdash_S \Delta \ \& \ \Delta \cap \Lambda = \emptyset$. Because $\Delta \subseteq \Phi \setminus \Lambda$, we also have $\Gamma \vdash_S \Phi \setminus \Lambda$. ■

6.6. LEMMA. SATURATION LEMMA FOR FILTRATIONS

Let **S** be a sequential derivation system as in BSL. Let $\Phi \subseteq \mathcal{L}_S$ and let Γ and Δ be two subsets of Φ. If $\Gamma \not\vdash_S \Delta$ then there exists a set Σ which is **S**-Φ-saturated, with $\Gamma \subseteq \Sigma$ and $\Sigma \cap \Delta = \emptyset$.

Proof. A proof can be obtained from proposition 6.5 in the same way as lemma 5.8 has been deduced from Thijsse's proposition 5.7. The full equivalence of this lemma and lemma 6.4 can also be obtained by proposition 6.5 and an analogous argumentation for SL $\Rightarrow$ BSL as in the previous chapter. ■

6.2 First decidability results

As already mentioned in the introduction of this chapter, we are aiming at suitable finite restrictions of the language such that from selecting the saturated sets inside this restricted set a finite 'canonical-like' model can be constructed. This model is meant as a finite counter-model of a given finite non-sequent. In the case of **M** this construction is relatively easy. The following definition describes our syntactic needs for establishing the FMP of **M**.

6.7. DEFINITION. Let Φ be a subset of $\mathcal{L}^{\Box}$. The *truth-value division* of Φ is the set

$$\overline{\Phi} := Sub(\Phi) \cup \neg Sub(\Phi).$$

The *modal assimilation* of Φ is the set

$$\Phi^{\Box} := \Box\overline{\Phi} \cup \Diamond\overline{\Phi} \cup \overline{\Phi}.$$

In classical modal logic the simple restriction $Sub(\Phi)$ is most often employed for implementation of filtrated canonical models. It turns out that $\neg Sub(\Phi)$ need to be added for imitation of such filtrations for partial modal logics. This is a direct consequence of explicit definitions of falsity in partial model-theory.

The supplementary syntactic material which we obtain through modal assimilation of such a restricted set of formulae Φ is particularly useful for defining an appropriate and technically manageable accessibility structure of the filtrated **M**-canonical model. An important property of this modal assimilation $\Phi^{\Box}$ of a set of formulae Φ, is the regularity of the modal content of $\Phi^\Box$ with respect to the subformulae of the set Φ: $\Box^-\Phi^\Box = \Diamond^-\Phi^\Box = \overline{\Phi}$.

The decidability of M

The **M**-Φ-canonical model is defined as follows.

6.8. DEFINITION. Let $\Phi \subseteq \mathcal{L}^\Box$. The **M**-$\Phi$-canonical model is the triple $M = \langle \mathfrak{Sat}_M^{\Phi^\Box}, R_M^\Phi, V_M^\Phi \rangle$ with

$$R_M^\Phi(\Sigma,\Theta) \Leftrightarrow \ \forall\varphi \in \overline{\Phi} : \begin{cases} \Box\varphi \in \Sigma \Rightarrow \varphi \in \Theta, \text{ and} \\ \varphi \in \Theta \Rightarrow \Diamond\varphi \in \Sigma. \end{cases}$$

$$V_M^\Phi(\Sigma)(p) = 1(0) \Leftrightarrow (\neg)p \in \Sigma \text{ for all } \Sigma, \Theta \in \mathfrak{Sat}_M^\Psi \text{ and } p \in I\!P.$$

The syntactic richness of $\Phi^\Box$ provides nearly a complete imitation of the truth lemma for **M** (lemma 5.12, page 149) for this filtrated canonical model with respect to the subformulae of Φ. Only simple earlier observations on restricted saturated sets, like proposition 6.3 and lemma 6.4, have to be used for the finishing touch of this successful transposition.

Note that the filtrated version of the canonical accessibility relation is somewhat different from the ordinary canonical definition. This modification is really needed. The left argument, Σ, only needs to recognize possibilities, that is formulae of the form $\Diamond\varphi$, which are contained in $\Phi^\Box$. As a matter of fact, we will employ this definition for filtrated canonical accessibilities in the sequel. Furthermore, we will use the following convenient abbreviation:

$$\Sigma \subseteq^\Phi \Theta \iff \Sigma \cap \overline{\Phi} \subseteq \Theta \quad \text{and} \quad \Sigma =^\Phi \Theta \iff \Sigma \cap \overline{\Phi} = \Theta.$$

For every **M**-extension, we use

$$R_S^\Phi(\Sigma,\Theta) \iff \Box^-\Sigma \subseteq^\Phi \Theta \subseteq^\Phi \Diamond^-\Sigma.$$

Below, we will prove a restricted truth-lemma for the subformulae of Φ with respect to the model M_M^Φ.

6.9. LEMMA. For all Σ in M_M^Φ and for all $\varphi \in Sub(\Phi)$:

$$M_M^\Phi, \Sigma \models \varphi \Leftrightarrow \varphi \in \Sigma \quad \text{and } M_M^\Phi, \Sigma \dashv \varphi \Leftrightarrow \neg\varphi \in \Sigma.$$

Proof. By induction on the construction of formulae. The proof in the case of the extensional connectives is as immediate as before[2]. Also the steps $\Box\varphi \in \Sigma \Rightarrow M_M^\Phi, \Sigma \models \Box\varphi$ and $M_M^\Phi, \Sigma \dashv \Box\varphi \Rightarrow \neg\Box\varphi \in \Sigma$ are direct consequences of the definition of R_M^Φ. The converse directions can be obtained from the truth-lemma for **M**, proposition 6.3 and lemma 6.4.

Suppose $\Box\varphi \in Sub(\Phi)$ and $\Box\varphi \notin \Sigma$.

Let $\Theta \subseteq \overline{\Phi}$. If $\Box^-\Sigma \vdash_M \Theta$, then also $\Box^-\Sigma \vdash_M \varphi, \Theta - \varphi$. As in the truth-lemma of **M**, we may conclude $\Sigma \vdash_M \Box\varphi, \Diamond(\Theta - \varphi)$, and therefore $\Theta \cap (\Diamond^-\Sigma - \varphi) \neq \emptyset$. This step is completely legitimate, because $\Diamond\Theta \subseteq \Phi^\Box$. This conclusion proves

[2] The $\neg$-step is facilitated by the fact that all negations of subformulae are present in $\Phi^\Box$.

$\Box^{-}\Sigma \trianglelefteq^{\overline{\Phi}}_{M} (\Diamond^{-}\Sigma - \varphi)$.

Therefore, BSL for restricted sets guarantees the existence of some $\Delta \in \mathfrak{Sat}^{\overline{\Phi}}_{M}$ with $\Box^{-}\Sigma \subseteq \Delta \subseteq \Diamond^{-}\Sigma$ and $\varphi \notin \Delta$. Proposition 6.3 then yields a $\Delta' \in \mathfrak{Sat}^{\Phi^{\Box}}_{M}$ with $\Delta' =^{\Phi} \Delta$. By definition of R^{Φ}_{M}, we conclude $R^{\Phi}_{M}(\Sigma, \Delta')$, and because $\varphi \in \overline{\Phi}$, we find $\varphi \notin \Delta'$. The induction hypothesis entails $M^{\Phi}_{M}, \Delta' \not\models \varphi$ and $M^{\Phi}_{M}, \Sigma \not\models \Box\varphi$.

The step $\neg\Box\varphi \in \Sigma \Rightarrow M^{\Phi}_{M}, \Sigma \dashv \Box\varphi$ can be accounted for by a similar imitation of the corresponding part of the proof of the truth lemma of **M**. We leave the details to the reader. ■

6.10. OBSERVATION. By a similar generalization as made in observation 5.15, we conclude that for every **M**-extension **S** and all $\Phi \subseteq \mathcal{L}_S, \Psi \supseteq \Phi^{\Box}$, $\Sigma \in \mathfrak{Sat}^{\Psi}_{S}$ and $\varphi \in Sub(\Phi)$ that

$$\varphi \notin \Box^{-}\Sigma \implies \exists\Theta \in \mathfrak{Sat}^{\Psi}_{S} : R^{\Phi}_{S}(\Sigma, \Theta) \ \&\ \varphi \notin \Theta, \text{ and}$$

$$\varphi \in \Diamond^{-}\Sigma \implies \exists\Theta \in \mathfrak{Sat}^{\Psi}_{S} : R^{\Phi}_{S}(\Sigma, \Theta) \ \&\ \varphi \in \Theta.$$

6.11. THEOREM. The system **M** has the finite model property.

Proof. Suppose $\Gamma \not\vdash_M \Delta$ with $\Gamma, \Delta \subseteq \mathcal{L}^{\Box}$ finite. Because $\Gamma \cup \Delta \subseteq (\Gamma \cup \Delta)^{\Box}$ we know that there exists $\Sigma \in \mathfrak{Sat}^{(\Gamma\cup\Delta)^{\Box}}_{M}$ such that $\Gamma \subseteq \Sigma$ and $\Sigma \cap \Delta = \emptyset$. This means, according to the filtration version of the truth lemma for **M** above, that Σ in the model $M^{\Gamma\cup\Delta}_{M}$ is a Γ-world which verifies none of the Δ-members. For all $\gamma \in \Gamma$ and $\delta \in \Delta$

$$M^{\Gamma\cup\Delta}_{M}, \Sigma \models \gamma \ \&\ M_{\Gamma\cup\Delta}, \Sigma \not\models \delta.$$

Because $M^{\Gamma\cup\Delta}_{M}$ is finite, **M** has FMP. ■

Let $\Gamma \not\vdash_M \Delta$ for finite $\Gamma, \Delta \subseteq \mathcal{L}^{\Box}$. Now, consider models of the form $\langle W, R, V\rangle$ with $\emptyset \neq W \subseteq \wp(\Gamma\cup\Delta)^{\Box}$, and let R and V be defined in the same way as R^{Φ}_{M} and V^{Φ}_{M} expanded over the elements of W. This class consists of $2^{3(\#Sub(\Gamma\cup\Delta))} - 1$ members, and M^{Φ}_{M} is one of these members. Consecutively checking these models on the existence of a counter-world is therefore a successful sound and complete finite procedure. This argumentation demonstrates the earlier strong version of FMP.

6.12. THEOREM. The system **M** is decidable for finite sequents.

In the sequel of this chapter we use the same kind of abbreviations for the forthcoming definitions of the filtrated canonical information structures.

$$\Gamma \Subset^{\Phi}_{S} \Delta \iff \mathbf{p}_S\Gamma \subseteq^{\Phi} \Delta \text{ and}$$

$$\Gamma \ll^{\Phi}_{S} \Delta \iff \mathbf{p}_S\Gamma \subseteq^{\Phi} \Delta \ \&\ \mathbf{ap}_S\Delta \subseteq^{\Phi} \Gamma.$$

Decidability of N and ud

For a proof of FMP of **N** we only need the **N**-$\overline{\Phi}$-saturated sets for a given restricted set of formulae Φ. The **N**-Φ-canonical model is $M^{\Phi}_{N} = \langle \mathfrak{Sat}^{\overline{\Phi}}_{N}, \subseteq^{\Phi}, V^{\Phi}_{N}\rangle$[3] with V^{Φ}_{N} being the restriction of V_N to $\mathfrak{Sat}^{\overline{\Phi}}_{N}$. For Nelson's logic, the restricted truth

[3]Notice $\Subset^{\Phi}_{N} = \subseteq^{\Phi}$.

lemma can be transferred from its general truth lemma (lemma 5.18, page 153). Its FMP and decidability follows from this restricted truth lemma in a similar way as the decidability of **M** follows from lemma 6.9.

6.13. THEOREM. **N** has the FMP and is decidable for finite sequents.

A decidability result for the system **ud** can be found by the same means. We define the **ud**-Φ-canonical model $M^{\Phi}_{ud} = \langle \overline{\mathfrak{Sat}^{\Phi}_{ud}}, \ll^{\Phi}_{ud}, V^{\Phi}_{ud}\rangle$ in the same fashion as M^{Φ}_{N} above.

6.14. LEMMA. $M^{\Phi}_{ud} \in \mathfrak{N}$.

Proof. The monotonicity of V^{Φ}_{ud} follows immediately from the PERS-rules for literals.
Suppose $\Gamma \ll^{\Phi}_{ud} \Delta \ll^{\Phi}_{ud} \Theta$ for a triple $\Gamma, \Delta, \Theta \in \overline{\mathfrak{Sat}^{\Phi}_{ud}}$. Let $\varphi \in \mathbf{p}_{ud}\Gamma$. This means $\varphi \vdash_{ud} [\;]_u \varphi$, and also $\varphi \in \mathbf{p}_{ud}\Delta$ and $\varphi \in \Theta$. In short, $\mathbf{p}_{ud}\Gamma \subseteq \Theta$. By a similar argument we may conclude $\Theta \subseteq \mathbf{ap}_{ud}\Theta$. Altogether, $\Gamma \ll^{\Phi}_{ud} \Theta$, or in other words, $\ll^{\Phi}_{ud}$ is transitive.
Reflexivity of $\ll^{\Phi}_{ud}$ is trivial. We conclude that $\ll^{\Phi}_{ud}$ is a pre-order. ■

The truth lemma for a set of subformulae Φ with respect to M^{Φ}_{ud} can be proved by an imitation of the general truth lemma for **ud** (page 161). This observation establishes the decidability result for finite **ud**-sequents.

6.15. THEOREM. The system **ud** has the FMP and is decidable for finite sequents.

In the FMP proof of **ud** our definition of $\ll_{ud}$ appears to be advantageous. If the more natural candidate of observation 5.34 (page 160) would have been used, we would have to revise the definition of the canonical extension order for the filtrated canonical models in a rigorous way. Such a procedure would divert us from the general FMP strategy of this chapter. We try to enforce specific structural properties of filtrated canonical models by modification of the filtering set only. The clearest advantage of this straightness is that earlier results for canonical structures, which have been found in the previous chapter, can be used in FMP proofs as well. This means that we will stick with the definitions of R^{Φ}_{S}, $\in^{\Phi}_{S}$ and $\ll^{\Phi}_{S}$ above and have convenient transfer of results in chapter 5.

6.3 Richer filtrations

Proving the finite model property for systems which are interpreted in terms of a more structured semantics, or have more expressivity than **M** may be proportionately more complicated than the relatively easy proof of FMP for **M**, **N** and **ud** in the previous section.

To start with, just like the completeness proofs for **M**-extensions, we need to show that an appropriate finite counter-model for a given finite non-sequent can be found in the proper model class. Sometimes the straightforward filtrations of the canonical model, as used in the previous section, does not guarantee satisfaction of the right model-theoretic constraints of the system in hand.

As already mentioned above, we try to find finite counter-models by appropriately enriching the filtrations. Such enrichments have to ensure preservation of the structural properties found for the canonical models of chapter 5. It might very well be the case that the syntactically poor filtrations of the previous sections are not 'preservative' in this sense. A more specific reason to use richer filtrations, which will become clear in the next section, consists of some important structural additional properties which we will need for proving the completeness of the more complicated mutual belief logics $\mathbf{E}_A^*$ and $\mathbf{C}^{3*}$.

Decidability of constructive modal logics

An important result of the preceding chapter has been lemma 5.25 on page 156. This lemma proved the bisimulation property for the inclusion relation over the **M**-canonical model.

Furthermore, this result facilitated the proofs of many bisimulation-like constraints of canonical models for **NM**, $\mathbf{NM}^\Box$ and **Mud**. In order to maintain our style of FMP-proving, we need a similar result for restricted saturated sets in order to bring along FMP for these systems as well.

As we saw during the proof of lemma 5.25 we needed the disjunction and conjunction for obtaining this structural result. The following definition is meant as a new extra closure condition of filtering sets, in order to apply a filtrated version lemma 5.25. Just like we added negations and modal operators to the subformulae of a subset of formulae Φ for the **M**-Φ-canonical model ($\Phi^\Box$), we also wish to employ disjunction and conjunction in the same manner.

6.16. DEFINITION. Suppose that $\Phi \subseteq \mathcal{L}_S$ for some sequential system **S** with $\mathcal{L}^\Box \subseteq \mathcal{L}_S$. The *conjunctive assimilation* of Φ is the set

$$\Phi^\wedge := \{\varphi_1 \wedge ... \wedge \varphi_n \mid \varphi_i \in \overline{\Phi}, \text{ and } \varphi_i = \varphi_j \Leftrightarrow i = j\}.$$

The *disjunctive assimilation* of Φ is the set

$$\Phi^\vee := \{\varphi_1 \vee ... \vee \varphi_n \mid \varphi_i \in \overline{\Phi}, \text{ and } \varphi_i = \varphi_j \Leftrightarrow i = j\}.$$

The *modal cover* of Φ is the set

$$\Phi^\boxplus := \Box\Phi^\vee \cup \Diamond\Phi^\wedge \cup \overline{\Phi}.$$

The following lemma rephrases lemma 5.25 for restricted saturated sets.

6.17. LEMMA. Let **S** be an **M**-extension, and let $\Phi, \Lambda \subseteq \mathcal{L}_S$ and $\Gamma \in \mathfrak{Sat}_S^{\Phi^\boxplus}$. The following filtration version of lemma 5.25 holds.

$$\Box^-\Gamma \trianglelefteq_S^{\Phi^\boxplus} \Lambda \implies \exists\Delta \in \mathfrak{Sat}_S^{\Phi^\boxplus} : R_S^\Phi(\Gamma, \Delta) \text{ and } \Delta \subseteq^\Phi \Lambda.$$

$$\Lambda \trianglelefteq_S^{\Phi^\boxplus} \Diamond^-\Gamma \implies \exists\Delta \in \mathfrak{Sat}_S^{\Phi^\boxplus} : R_S^\Phi(\Gamma, \Delta) \text{ and } \Lambda \subseteq^\Phi \Delta.$$

Proof. We only prove the first item of the lemma. It is an imitation of the proof of lemma 5.25, and the second item can be obtained by a similar application of the dual result in lemma 5.25.

Let $\Box^-\Gamma \trianglelefteq_S^{\Phi^\boxplus} \Lambda$. This also means $\Box^-\Gamma \cap \overline{\Phi} \trianglelefteq_S^{\overline{\Phi}} \Lambda$. Analogously to the proof of lemma 5.25 in the previous chapter, we can prove

$$\Box^-\Gamma \cap \overline{\Phi} \trianglelefteq_S^{\overline{\Phi}} \Diamond^-\Gamma \cap \Lambda \quad (1).$$

This claim yields the desired result. An application of BSL for restricted sets shows that there exists $\Delta' \in \mathfrak{Sat}_S^{\overline{\Phi}}$ with

$$\Box^-\Gamma \subseteq \Delta' \subseteq \Diamond^-\Gamma \quad \text{and} \quad \Delta' \subseteq \Lambda.$$

Proposition 6.3 thereupon guarantees the existence of a $\Delta \in \mathfrak{Sat}_S^{\Phi^\boxplus}$ with $\Delta =^\Phi \Delta'$, and so

$$\Box^-\Gamma \subseteq^\Phi \Delta \quad \text{and} \quad \Delta \subseteq^\Phi \Diamond^-\Gamma \quad \text{and} \quad \Delta \subseteq^\Phi \Lambda.$$

What is left to prove is the claim (1) above. Suppose $\Box^-\Gamma \cap \overline{\Phi} \vdash_S \Theta$ for certain $\Theta \subseteq \overline{\Phi}$. We need to show $\Theta \cap \Diamond^-\Lambda \cap \Lambda \neq \emptyset$. We give a short formal transcription below.

$$\Box^-\Gamma \cap \overline{\Phi} \vdash_S \Theta \implies \Box^-\Gamma \cap \overline{\Phi} \vdash_S \Theta \setminus \Lambda, \Theta \cap \Lambda \implies$$

$$\Box^-\Gamma \cap \overline{\Phi} \vdash_S \bigvee \Theta \setminus \Lambda, \Theta \cap \Lambda \implies \Gamma \vdash_S \Box(\bigvee \Theta \setminus \Lambda), \Diamond(\Theta \cap \Lambda) \implies (2)$$

$$\Diamond(\Theta \cap \Lambda) \cap \Gamma \neq \emptyset \implies \Theta \cap \Lambda \cap \Diamond^-\Gamma \neq \emptyset \quad (3).$$

Because $\Theta \setminus \Lambda \subseteq \overline{\Phi}$, we obtain $\Box(\bigvee \Theta \setminus \Lambda) \in \Phi^\boxplus$. $\Box^-\Gamma \cap \overline{\Phi} \cap (\Theta \setminus \Lambda) = \emptyset$ implies $\Box(\bigvee \Theta \setminus \Lambda) \notin \Gamma$ $(\Box^-\Gamma \trianglelefteq_S^{\Phi^\boxplus} \Lambda)$. So, because $\Gamma \in \mathfrak{Sat}_S^{\Phi^\boxplus}$, the implication in (2) holds. Note that the modal cover definition is required to legitimate this step.

The final conclusion (3) establishes claim (1). ■

6.18. COROLLARY. The proof above shows that the conclusions of lemma 6.17 can also be obtained through replacing the requirements $\Box^-\Gamma \trianglelefteq_S^{\Phi^\boxplus} \Lambda$ and $\Lambda \trianglelefteq_S^{\Phi^\boxplus} \Diamond^-\Gamma$, by $\Box^-\Gamma \cap \overline{\Phi} \trianglelefteq_S^{\overline{\Phi}} \Lambda$ and $\Lambda \cap \overline{\Phi} \trianglelefteq_S^{\overline{\Phi}} \Diamond^-\Gamma$, respectively.

As mentioned earlier, lemma 6.17 is of great importance for establishing forthcoming decidability and completeness results. Just like lemma 5.25, it facilitates searching saturated sets. Most often the result of lemma 6.17 reduces the search for a saturated set to a fairly easily provable saturation relation. The following definition presents the Φ-filtrated canonical models of the constructive modal logics **NM** and $\mathbf{NM}^\Box$.

6.19. DEFINITION. Let $\mathbf{S} \in \{\mathbf{NM}, \mathbf{NM}^\Box\}$. The canonical **S**-model filtrated by Φ is the quadruple $M_S^{\overline{\Phi}} = \langle \mathfrak{Sat}_S^{\Phi^\boxplus}, R_S^\Phi, \in_S^{\overline{\Phi}}, V_S^\Phi \rangle$. All the definitions of the different canonical model-theoretic parameters, can be found in the previous section.

FMP and decidability of **NM** can easily be obtained. The only thing we need to show is that $M_{NM}^{\overline{\Phi}}$ is an $\mathfrak{NM}$-model.

6.20. LEMMA. $M_{NM}^{\overline{\Phi}} \in \mathfrak{NM}$.

Proof. V^{Φ}_{NM} is clearly monotonic. Furthermore, we need to check the two bisimulation conditions $(\subseteq^{\Phi} \circ R^{\Phi}_{NM}) \subseteq (R^{\Phi}_{NM} \circ \subseteq^{\Phi})$ and $(\supseteq^{\Phi} \circ R^{\Phi}_{NM}) \subseteq (R^{\Phi}_{NM} \circ \supseteq^{\Phi})$. They can be obtained immediately from lemma 6.17 and corollary 6.18. If $\Gamma, \Gamma', \Delta' \in \mathfrak{Sat}^{\Phi^{\boxplus}}_{NM}$ with $\Gamma \subseteq^{\Phi} \Gamma'$ and $R^{\Phi}_{NM}(\Gamma', \Delta')$, then also $\Box^{-}\Gamma \cap \overline{\Phi} \trianglelefteq^{\overline{\Phi}}_{NM} \Delta'$. Corollary 6.18 gives us the desired Δ: $R^{\Phi}_{NM}(\Gamma, \Delta)$ and $\Delta \subseteq^{\Phi} \Delta'$. This establishes the first bisimulation constraint. The second can immediately be found by the same corollary and the simple saturation equation

$$\Gamma \subseteq^{\Phi} \Gamma' \ \&\ R^{\Phi}_{NM}(\Gamma, \Delta) \implies \Delta \cap \overline{\Phi} \trianglelefteq^{\overline{\Phi}}_{NM} \Diamond^{-}\Gamma'.$$

■

The restricted truth lemma for **NM** can be proved in the same way as its ordinary truth lemma. Note that $\Phi^{\Box} \subseteq \Phi^{\boxplus}$, which is important for the $\Box$-steps in the proof of this restricted truth-lemma. This result combined with lemma 6.20 yields a strong FMP result and the consequential decidability of **NM**.

6.21. THEOREM. The system **NM** has the FMP and is decidable for finite sequents.

A decidability result for $\mathbf{NM}^{\Box}$ can be established by means of the same procedure.

6.22. LEMMA. $M^{\Phi}_{NM^{\Box}} \in \mathfrak{NM}^{\Box}$ for all $\Phi \subseteq \mathcal{L}^{\Box,\rightarrow}$.

Proof. Again, the monotonicity of $V^{\Phi}_{NM^{\Box}}$ is evident. The only structural constraint can be obtained by application of corollary 6.18 and the saturation equation:

$$\Gamma \Subset^{\Phi}_{NM^{\Box}} \Gamma' \ \&\ R^{\Phi}_{NM^{\Box}}(\Gamma', \Delta') \implies \Box^{-}\Gamma \cap \overline{\Phi} \trianglelefteq^{\overline{\Phi}}_{NM^{\Box}} \Delta' \cup (\mathbf{p}_{NM^{\Box}}\mathcal{L}^{\Box,\rightarrow})^{\mathsf{C}}.$$

This validity of this equation can be observed in the same way as in the proof of the structural adequacy of the $\mathbf{NM}^{\Box}$-canonical model, lemma 5.30 (page 158). An inspection of this proof learns us that the closure step $\Sigma \subseteq \overline{\Phi} \implies \Box(\bigvee \Sigma) \in \Phi^{\boxplus}$ is again required. We leave it to the reader to check the precise justification of this transfer. ■

A restricted truth lemma for $\mathbf{NM}^{\Box}$ can be obtained in the same way as for **NM**.

6.23. THEOREM. $\mathbf{NM}^{\Box}$ has the FMP and is decidable for finite sequents.

Establishing a decidability result for **Mud** by means of the completeness proof requires an enrichment of the filtrations which have not been used for **NM** and $\mathbf{NM}^{\Box}$ above. The proof of lemma 5.42, page 164, shows that we have also used the downdate operator to establish the structural adequacy of the **Mud**-canonical model: $M_{Mud} \in \mathfrak{NM}^{\Box}$. The following filtration legalizes a similar inference for filtrated canonical models.

6.24. DEFINITION. The down-closure of a set Φ is the set

$$\Phi^{d} = \{\langle\varphi\rangle_{d}\psi \mid \varphi \in \Phi^{\vee}, \psi \in \Phi^{\wedge}\} \ \cup\ \overline{\Phi}.$$

The set $\Phi^{\boxplus_d}$ denotes the set which will be used as a filtration for **Mud**. It is defined as follows:

$$\Phi^{\boxplus_d} = \Box\Phi^d \cup \Diamond\Phi^\wedge \cup \overline{\Phi}.$$

The **Mud**-Φ-canonical model for a set Φ is the model $\langle \mathfrak{Sat}^{\Phi^{\boxplus_d}}_{Mud}, R^\Phi_{Mud}, \ll^\Phi_{Mud}, V^\Phi_{Mud}\rangle$. The definition of the accessibility relation and the global valuation function are the same as in all the earlier filtrated canonical models. The information structure $\ll^\Phi_{Mud}$ is the same as $\ll_{ud}$ in M^Φ_{ud}.

6.25. Lemma. $M^\Phi_{Mud} \in \mathfrak{NM}^\Box$.

Proof. A complete imitation of the proof of lemma 5.42, page 164, can be given by the richness of the filtration. The central claim (6) there needs to be replaced by the following reformulation. If $\Gamma \ll^\Phi_{Mud} \Gamma'$ and $R^\Phi_{Mud}(\Gamma', \Delta')$ then

$$\Box^-\Gamma \cap \mathbf{ap}_{Mud}\Delta' \cap \overline{\Phi} \trianglelefteq^{\overline{\Phi}}_{Mud} (\Delta' \cup (\mathbf{p}_{ud}\mathcal{L}^{\uparrow,\downarrow})^{\complement}) \cap \Diamond^-\Gamma.$$

Analogously to this proof, this reformulation establishes the desired $\Delta \in \mathfrak{Sat}^{\Phi^{\boxplus_d}}_{Mud}$: $R^\Phi_{Mud}(\Gamma, \Delta)$ and $\Delta \ll^\Phi_{Mud} \Delta'$.

A pointwise inspection of the proof of lemma 5.42 shows that step (7) can be obtained by $\Delta \subseteq \overline{\Phi} \Rightarrow \Diamond(\bigwedge \Delta) \in \Phi^{\boxplus_d}$. Reaching conclusion (10) does not require any filtration richness, and the final conclusion can be recaptured by the closure step $\Sigma, \Theta \subseteq \overline{\Phi} \Rightarrow \Box\langle\bigvee \Sigma\rangle_d \bigwedge \Theta \in \Phi^{\boxplus_d}$. ■

6.26. Theorem. The system **Mud** has the FMP and is decidable for finite sequents.

6.4 The completeness of mutual belief systems

As mentioned earlier, completeness proofs for the mutual belief systems require finite filtrations of their canonical models. The structural properties of the accessibility relation, i.e. seriality and full introspection, need suitable enrichments of the filtrations which have been employed for the decidability proof of **M**. The following definition presents different filtering sets which we will use for the filtrated canonical models in this section.

6.27. Definition. Let $a \in A$ and $\Phi \subseteq \mathcal{L}_S$ for some extension **S** of $\mathbf{E}_A$. The set $\Phi^{\Box_a}$ is the modal assimilation of Φ for the modal operator $\Box_a$. The set $\Phi^{\boxplus_a}$ refers to the modal cover of Φ under $\Box_a$. Furthermore, we define

$$\Phi^{\Box_A} := \bigcup_{a\in A} \Phi^{\Box_a} \qquad \Phi^{\boxplus_A} := \bigcup_{a\in A} \Phi^{\boxplus_a}.$$

$\Phi^{\boxtimes_A}$ is the smallest superset of $Sub(\Phi)$ such that

$$\varphi \in \Phi^{\boxtimes_A} \implies \neg\varphi \in \Phi^{\boxtimes_A}, \text{ and}$$

$$\varphi \in \Phi^{\boxtimes_A} \implies \Box_a\varphi \in \Phi^{\boxtimes_A} \text{ for all } a \in A.$$

The filtration which we will employ for the forthcoming completeness proof of $\mathbf{E}_A^*$ is the set $(\Phi^{\boxtimes_A})^{\boxplus_A}$. This set is infinite for all non-empty Φ, but it is logically finite whenever Φ is finite. If Φ is finite then the $\mathbf{E}_A^*$-Φ-canonical model is finite as well. Due to the strong equivalences of $\mathbf{E}_A^*$, which have been listed in example 4.8 on page 115, we can easily prove that every element of $\Phi^{\boxtimes_A}$ is $\mathbf{E}_A^*$-equivalent with some member of $\Phi^{\Box_A}$. This also means that every member of the complex filtration $(\Phi^{\boxtimes_A})^{\boxplus_A}$ is $\mathbf{E}_A^*$-equivalent with some member of the finite set $(\Phi^{\Box_A})^{\boxplus_A}$.

6.28. Observation. For all $\varphi \in \Phi^{\boxtimes_A}$ there exists $\varphi' \in \Phi^{\Box_A}$ such that $\varphi \equiv_{E_A^*} \varphi'$. For all $\varphi \in (\Phi^{\boxtimes_A})^{\boxplus_A}$ there exists $\varphi' \in (\Phi^{\Box_A})^{\boxplus_A}$ such that $\varphi \equiv_{E_A^*} \varphi'$.

Proof. By an easy induction on the closure principles in the definition of $\Phi^{\boxtimes_A}$ in definition 6.27, and the modality reduction principles of $\mathbf{E}_A$ in example 4.8. ■

Because $\Phi^{\Box_A} \subseteq \Phi^{\boxtimes_A}$, the former set can be seen as a finite representation of the latter, as long as Φ is finite itself. For the same reasons the set $(\Phi^{\Box_A})^{\boxplus_A}$ is a finite representation of $(\Phi^{\boxtimes_A})^{\boxplus_A}$ for all finite Φ. This ensures that $\mathfrak{Sat}_{EAs}^{(\Phi^{\boxtimes_A})^{\boxplus_A}}$ that is finite whenever Φ is finite.

The completeness of $\mathbf{E}_A^*$

Let us first give a formal description of the $\mathbf{E}_A^*$-Φ-canonical model for $\Phi \subseteq \mathcal{L}_A^*$.

6.29. Definition. Let $\Phi \subseteq \mathcal{L}_A^*$. The $\mathbf{E}_A^*$-Φ-canonical model is the triple

$$M_{E_A^*}^{\Phi} = \langle \mathfrak{Sat}_{E_A^*}^{(\Phi^{\boxtimes_A})^{\boxplus_A}}, \{R_{E_A^*}^{\Phi^{\boxtimes_A}}\}_{a \in A}, V_{E_A^*}^{\Phi} \rangle,$$

with the following accessibility relation

$$(R_{E_A^*}^{\Phi^{\boxtimes_A}})_a(\Sigma, \Theta) \Leftrightarrow \forall \varphi \in \Phi^{\boxtimes_A} : \begin{cases} \Box_a \varphi \in \Sigma \implies \varphi \in \Theta & \text{, and} \\ \varphi \in \Theta \implies \Diamond_a \varphi \in \Sigma & . \end{cases}$$

$V_{E_A^*}^{\Phi}$ is defined as in the other filtrated canonical models of the previous sections of this chapter.

A prerequisite for the adequacy of $M_{E_A^*}^{\Phi}$ which should be demonstrated next, is its membership of the class $\mathfrak{E}_{\mathfrak{A}}$. This means that we need to show the seriality and the full introspection of $(R_{E_A^*}^{\Phi^{\boxtimes_A}})_a$ for all $a \in A$. As mentioned earlier this follows immediately from the richness of $\Phi^{\boxtimes_A}$ and by a complete imitation of the same lemma of the structural adequacy of the $\mathbf{E}_A$-canonical model (lemma 5.43, page 166).

6.30. Lemma. $M_{E_A^*}^{\Phi} \in \mathfrak{E}_{\mathfrak{A}}$.

A useful side effect of the enrichment $\Phi^{\boxtimes_A}$ is that if $\Box_X^* \varphi \in \Phi^{\boxtimes_A}$, then also $\Box_a \Box_X^* \varphi \in \Phi^{\boxtimes_A}$ for all $a \in A$ and $\varphi \in \mathcal{L}_A^*$. This is a typical closure condition for filtrations of systems with modalities for reasoning about reflexive transitive closures of accessibilities [Fischer & Ladner 1979] [Kozen & Parikh 1981] [Halpern

& Moses 1992]. The use of this closure is that formulae of the form $\Box^*_X\varphi$ are preserved under accessibility. The following lemma presents a formalization of this effect.

6.31. LEMMA. Let $\Sigma, \Theta \in \mathfrak{Sat}^{(\Phi^{\boxtimes_A})^{\boxplus_A}}_{E^*_A}$ and $k \in \mathbb{N}$.

$$\text{If } \Box^*_X\varphi \in \Sigma \text{ and } (R^{\Phi^{\boxtimes_A}}_{E^*_A})^k_X(\Sigma,\Theta) \text{ then } \Box^*_X\varphi \in \Theta, \text{ and}$$

$$\text{if } \neg\Box^*_X\Phi \in \Theta \text{ and } (R^{\Phi^{\boxtimes_A}}_{E^*_A})^k_X(\Sigma,\Theta) \text{ then } \neg\Box^*_X\varphi \in \Sigma.$$

Proof. The proof is by induction on k. If $k = 0$ then the result is trivial, for $(R^{\Phi^{\boxtimes_A}}_{E^*_A})^0(\Sigma,\Theta) \Leftrightarrow \Sigma = \Theta$.

Let $k > 0$ and suppose $(R^{\Phi^{\boxtimes_A}}_{E^*_A})^k(\Sigma,\Theta)$.

This means that there exists $\Theta' \in \mathfrak{Sat}^{(\Phi^{\boxtimes_A})^{\boxplus_A}}_{E^*_A}$ such that $R^{\Phi^{\boxtimes_A}}_{E^*_A}(\Theta',\Theta)$ and also $(R^{\Phi^{\boxtimes_A}}_{E^*_A})^{k-1}(\Sigma,\Theta')$. The induction hypothesis implies that $\Box^*_X\varphi \in \Theta'$. Because $\Box^*_X\varphi \vdash_{E^*_A} \Box_X\Box^*_X\varphi$, we also know that $\Box_a\Box^*_X\varphi \in \Theta'$ for all $a \in X$. By definition of $(R^{\Phi^{\boxtimes_A}}_{E^*_A})_X$ and because $\Box^*_X\varphi \in \Phi$[4], we have $\Box^*_X\varphi \in \Theta$.

A proof of the second claim in the lemma can be obtained analogously by the simple fact that if $\neg\Box^*_X\varphi \in \Phi^{\boxtimes_A}$ then also $\Diamond_a\neg\Box^*_X\varphi \in \Psi$, for all $a \in A$. The details are omitted. ■

Our information on $M^{\Phi}_{E^*_A}$ developed so far, suffices for the proof of the truth lemma of $\mathbf{E}^*_A$ with respect to $M^{\Phi}_{E^*_A}$. As usual the proof runs by an induction on the construction of formulae. The only new and nasty part is the $\Box^*_X$-step. One side of this step runs easily on the basis of lemma 6.31. The completion of this step, which is the difficult direction of the $\Box^*_X$-case, follows roughly the proof in [Halpern & Moses 1992] for mutual belief and knowledge extensions of classical (poly-)modal logics. The basic ideas of that proof can be traced back to [Kozen & Parikh 1981].

Of course, much of this procedure needed to be revised for installation in our partial poly-modal logic. It turns out that the general definition of saturated sets, is somewhat harder to handle in this proof. Nevertheless, earlier important findings, particularly lemma 5.25 and its filtrated version lemma 6.17, are of great help for this modification. Notice that the latter lemma applies to the filtration set $(\Phi^{\boxtimes_A})^{\boxplus_A}$, because $\overline{\Phi^{\boxtimes_A}} = \Phi^{\boxtimes_A}$.

6.32. LEMMA. Let $\Phi \subseteq \mathcal{L}^*_A$. For all $\varphi \in Sub(\Phi)$ and $\Gamma \in \mathfrak{Sat}^{(\Phi^{\boxtimes_A})^{\boxplus_A}}_{E^*_A}$:

$$M^{\Phi}_{E^*_A},\Gamma \models \varphi \Leftrightarrow \varphi \in \Gamma \quad \text{and} \quad M^{\Phi}_{E^*_A},\Gamma \Dashv \varphi \Leftrightarrow \neg\varphi \in \Gamma.$$

Proof. By induction of the construction of the subformulae of the set Φ. Only the

[4] For all $\Box^*_X\varphi \in \Psi$ also $\Box^*_X\varphi \in \Phi$.

$\Box^*_X$-step needs to be accounted for. The other cases are identical to the FMP-proof of $\mathbf{M}$ in section 6.2.

Suppose $\Box^*_X \varphi \in \Gamma$.

Lemma 6.31 shows that if $(R^{\Phi^{\boxtimes A}}_{E^*_A})^*_X(\Gamma, \Theta)$ then $\Box^* \varphi \in \Theta$ for all $\Theta \in \mathfrak{Sat}^{(\Phi^{\boxtimes A})^{\boxplus A}}_{E^*_A}$. Because $\Box^*_X \varphi \vdash_{E^*_A} \varphi$, we also have $\varphi \in \Theta$ for all $\Theta \in \mathfrak{Sat}^{(\Phi^{\boxtimes A})^{\boxplus A}}_{E^*_A}$ such that $(R^{\Phi^{\boxtimes A}}_{E^*_A})^*_X(\Gamma, \Theta)$. The induction hypothesis yields $M^{\Phi}_{E^*_A}, \Theta \models \varphi$ for all such Θ, and therefore $M^{\Phi}_{E^*_A}, \Gamma \models \Box^*_X \varphi$.

Suppose $M^{\Phi}_{E^*_A}, \Gamma \models \Box^*_X \varphi$.

To begin with we need some abbreviations. The following definitions are particularly important for the proof procedure:

$$\mathfrak{G} := \{\Sigma \in \mathfrak{Sat}^{(\Phi^{\boxtimes A})^{\boxplus A}}_{E^*_A} \mid M^{\Phi}_{E^*_A}, \Sigma \models \Box^*_X \varphi\}$$
$$\varphi_\Sigma := \bigwedge(\Sigma \cap \Phi^{\Box_A})$$
$$\alpha := \bigvee_{\Sigma \in \mathfrak{G}} \varphi_\Sigma$$

Note that the finiteness of $\Phi^{\Box_A}$ is required here, otherwise the conjunction φ_Σ would not be well-defined. This formula can be seen as a finite representation of the $\Phi^{\boxtimes_A}$-content of Σ.

The proof consists of three essential claims:

(3) $\alpha \vdash_{E^*_A} \varphi$,

(4) $\Sigma \in \mathfrak{G} \Rightarrow \Sigma \vdash_{E^*_A} \alpha$, and

(5) $\alpha \vdash_{E^*_A} \Box_X \alpha$

Claim (5) settles $\alpha \vdash_{E^*_A} \Box^*_X \alpha$ (R-IND on page 124). This yields $\alpha \vdash_{E^*_A} \Box^*_X \varphi$, by means of claim (4), and so, $\Box^*_X \alpha \vdash_{E^*_A} \Box^*_X \varphi$ (R-MOD-TRUE $\Box^*_X$). Claim (3) shows that $\Gamma \vdash_{E^*_A} \alpha$, and therefore $\Box^*_X \varphi \in \Gamma$. What is left to be proved are the three claims (3), (4) and (5).

The first two claims are trivial. The induction hypothesis yields $\varphi \in \Sigma$ for all $\Sigma \in \mathfrak{G}$. Because φ appears in all conjuncts φ_Σ for all $\Sigma \in \mathfrak{G}$, we obtain $\Sigma \in \mathfrak{G} \Rightarrow \varphi_\Sigma \vdash_{E^*_A} \varphi$. L-TRUE $\vee$ subsequently yields $\alpha \vdash_{E^*_A} \varphi$ (3).

For all $\Sigma \in \mathfrak{Sat}^{(\Phi^{\boxtimes A})^{\boxplus A}}_{E^*_A}$ also $\Sigma \vdash_{E^*_A} \varphi_\Sigma$. So, if $\Sigma \in \mathfrak{G}$ then also $\Sigma \vdash_{E^*_A} \alpha$.

Proving (5) is the nasty part of the proof. Suppose that (5) is not the case, i.e. $\alpha \not\vdash_{E^*_A} \Box_X \alpha$. This means, according the saturation lemma, that there exists $\Sigma^* \in \mathfrak{Sat}_{E^*_A}$ such that $\alpha \in \Sigma^*$ and $\Box_X \alpha \notin \Sigma^*$. This last conclusion shows that there exists $a \in X$ such that $\Box_a \alpha \notin \Sigma^*$. An appropriate poly-modal reformulation of observation 5.15 on page 151 shows that there exists another $\mathbf{E}^*_A$-saturated set Θ^* such that

$$\Box^-_a \Sigma^* \subseteq \Theta^* \subseteq \Diamond^-_a \Sigma^* \quad \text{and} \quad \alpha \notin \Theta^* \quad (6).$$

Because $\Sigma^* \vdash_{E^*_A} \alpha$, and by definition of α, we know that there exists $\Sigma \in \mathfrak{G}$ such that $\varphi_\Sigma \in \Sigma^*$. This also means $\Sigma \cap \Phi^{\boxtimes_A} \subseteq \Sigma^*$. Since $\Box^-_a(\Sigma \cap \Phi^{\boxtimes_A}) =$

$\Box_a^- \Sigma \cap \Phi^{\boxtimes_A} \subseteq \Box_a^- \Sigma^* \subseteq \Theta^*$ ($\Box_a^- \Phi^{\boxtimes_A} = \Phi^{\boxtimes_A}$). and $\Theta^* \in \mathfrak{Sat}_{E_A^*}$, we know that $\Box^- \Sigma \cap \Phi^{\boxtimes_A} \trianglelefteq^{\Phi}_{E_A^*} \Theta^*$. Lemma 6.17, and its reformulation in corollary 6.18 shows that there exists $\Theta \in \mathfrak{Sat}_{E_A^*}^{(\Phi^{\boxtimes_A})^{\boxplus_A}}$ such that $(R_{E_A^*}^{\Phi^{\boxtimes_A}})_a(\Sigma, \Theta)$ and $\Theta \cap \Phi^{\boxtimes_A} \subseteq \Theta^*$. This last conclusion shows that $\Theta \not\vdash_{E_A^*} \alpha$[5], and so $\Theta \notin \mathfrak{G}$, according to (4). This last conclusion implies

$$M_{E_A^*}^{\Phi}, \Theta \not\models \Box_X^* \varphi,$$

and so there exists $\Xi \in \mathfrak{Sat}_{E_A^*}^{(\Phi^{\boxtimes_A})^{\boxplus_A}}$ such that $(R_{E_A^*}^{\Phi^{\boxtimes_A}})_X^*(\Theta, \Xi)$ and $M_{E_A^*}^{\Phi}, \Xi \not\models \varphi$. Because $(R_{E_A^*}^{\Phi^{\boxtimes_A}})_a(\Sigma, \Theta)$ and $a \in X$, also $(R_{E_A^*}^{\Phi^{\boxtimes_A}})_X^*(\Sigma, \Xi)$. This conclusion entails $M_{E_A^*}^{\Phi}, \Sigma \not\models \Box_X^* \varphi$, which contradicts $\Sigma \in \mathfrak{G}$.

This contradiction means that $\alpha \vdash_{E_A^*} \Box_X \alpha$ (5) must hold.

The step $\neg\Box_X^* \varphi \notin \Gamma \Rightarrow M_{E_A^*}^{\Phi}, \Gamma \not\dashv \Box_X^* \varphi$ can be derived by means of the second result in lemma 6.31 in the same way as in the earlier $\Box_X^* \varphi \in \Gamma$-case has been inferred from the first of this lemma. We leave the details to the reader. The converse of this implication can be found through some dualization of the proof of $\Box_X^* \varphi \notin \Sigma \;\Rightarrow\; M_{E_A^*}^{\Phi}, \Sigma \not\models \Box_X^* \varphi$ above.

Suppose $M_{E_A^*}^{\Phi}, \Gamma \not\dashv \Box_X^* \varphi$.

We use the following three abbreviations.

$$\mathfrak{F} := \{\Sigma \in \mathfrak{Sat}_{E_A^*}^{(\Phi^{\boxtimes_A})^{\boxplus_A}} \mid M_{E_A^*}^{\Phi}, \Sigma \not\dashv \Box_X^* \varphi\}$$

$$\varphi_\Sigma^{\mathsf{C}} := \bigvee(\Phi^{\Box_A} \setminus \Sigma)$$

$$\beta := \bigwedge_{\Sigma \in \mathfrak{F}} \varphi_\Sigma^{\mathsf{C}}$$

This last induction step of the truth lemma will be established by proving the following claims.

(7) $\neg\varphi \vdash_{E_A^*} \beta$

(8) $\Sigma \in \mathfrak{F} \Rightarrow \Sigma \not\vdash_{E_A^*} \beta$

(9) $\Diamond_X \beta \vdash_{E_A^*} \beta$.

These three results suffice indeed. Claim (9) yields $\Diamond_X^* \beta \vdash_{E_A^*} \beta$ (L-IND), and (7) entails $\neg\Box_X^* \varphi \vdash_{E_A^*} \Diamond_X^* \beta$ (L-MOD-FALSE $\Box_X^*$). Combining of these conclusions by means of CUT gives us $\neg\Box_X^* \varphi \vdash_{E_A^*} \beta$. Because $\Gamma \in \mathfrak{F}$, the second claim (8) yields $\neg\Box_X^* \varphi \in \Gamma$.

Claim (7) follows from the induction hypothesis: $\neg\varphi \notin \Sigma$ for all $\Sigma \in \mathfrak{F}$ and therefore $\neg\varphi$ appears in all the disjuncts $\varphi_\Sigma^{\mathsf{C}}$ for all $\Sigma \in \mathfrak{F}$, and so $\neg\varphi \vdash_{E_A^*} \varphi_\Sigma^{\mathsf{C}}$ for these $\Sigma \in \mathfrak{F}$. In combination with R-TRUE $\wedge$, this establishes $\neg\varphi \vdash_{E_A^*} \beta$.

Because all members of $\mathfrak{F}$ are $\mathbf{E}_A$-$(\Phi^{\boxtimes_A})^{\boxplus_A}$-saturated, we know that $\Sigma \not\vdash_{E_A^*} \varphi_\Sigma^{\mathsf{C}}$ for all $\Sigma \in \mathfrak{F}$ ($\Phi^{\boxtimes_A} \setminus \Sigma \cap \Sigma = \emptyset$). This result establishes $\Sigma \not\vdash_{E_A^*} \beta$ for all $\Sigma \in \mathfrak{F}$ (8).

---

[5] $\Theta^* \not\vdash_{E_A^*} \alpha \Rightarrow \forall\Sigma \in \mathfrak{G} : \Theta^* \not\vdash_{E_A^*} \varphi_\Sigma \Rightarrow \forall\Sigma \in \mathfrak{G}\ \exists\sigma \in \Sigma \cap \Phi^{\boxtimes_A} : \Theta^* \not\vdash_{E_A^*} \sigma \Rightarrow \forall\Sigma \in \mathfrak{G}\ \exists\sigma \in \Sigma \cap \Phi^{\boxtimes_A} : \Theta \not\vdash_{E_A^*} \sigma \Rightarrow \Theta \not\vdash_{E_A^*} \varphi_\Sigma$ for all $\Sigma \in \mathfrak{G} \;\Rightarrow\; \Theta \not\vdash_{E_A^*} \alpha$.

Suppose $\Diamond_X\beta \nvdash_{E_A^*} \beta$. An application of the saturation lemma shows that there exists a $\Sigma^* \in \mathfrak{Sat}_{E_A^*}$ such that $\Diamond_X\beta \in \Sigma^*$ and $\beta \notin \Sigma^*$. Observation 5.15 shows that there exists another $\Theta^* \in \mathfrak{Sat}_{E_A^*}$ such that for certain $a \in X$:

$\Box_a^- \Sigma^* \subseteq \Theta^* \subseteq \Diamond_a^- \Sigma^*$ with $\beta \in \Theta^*$ (10).

From (10) it follows that $(R_{E_A^*}^{\Phi^{\boxtimes_A}})_a(\Sigma^* \cap (\Phi^{\boxtimes_A})^{\boxplus_A}, \Theta^* \cap (\Phi^{\boxtimes_A})^{\boxplus_A})$[6]. Furthermore, $\Sigma \cap (\Phi^{\boxtimes_A})^{\boxplus_A} \nvdash_{E_A^*} \beta$, which means that there exists $\Sigma \in \mathfrak{F}$ such that

$\Sigma^* \cap (\Phi^{\boxtimes_A})^{\boxplus_A} \nvdash_{E_A^*} \varphi_\Sigma^{\complement}$ (11).

This conclusion yields $(\Sigma^* \cap (\Phi^{\boxtimes_A})^{\boxplus_A}) \cap (\Phi^{\boxtimes_A} \setminus \Sigma) = \emptyset$, and therefore, $\Sigma^* \cap \Phi^{\boxtimes_A} \subseteq \Sigma$. Lemma 6.17 shows that there exists $\Theta \supseteq \Theta^* \cap \Phi^{\boxtimes_A}$ such that $(R_{E_A^*}^{\Phi^{\boxtimes_A}})_a(\Sigma, \Theta)$. Because $\Theta^* \cap \Phi^{\boxtimes_A} \vdash_{E_A^*} \beta$[7] we obtain $\Theta \vdash_{E_A^*} \beta$. Hereupon, (8) yields $\Theta \notin \mathfrak{F}$. By definition of $\mathfrak{F}$, this conclusion entails

$M_{E_A^*}^{\Phi}, \Theta \dashv \Box_X^* \varphi$.

In other words, there exists $\Xi \in \mathfrak{Sat}_{E_A^*}^{(\Phi^{\boxtimes_A})^{\boxplus_A}}$ such that $(R_{E_A^*}^{\Phi^{\boxtimes_A}})_X^*(\Theta, \Xi)$ with $M_{E_A^*}^{\Phi}, \Xi \dashv \varphi$. Because $(R_{E_A^*}^{\Phi^{\boxtimes_A}})_a(\Sigma, \Theta)$ and $a \in X$, we conclude $M_{E_A^*}^{\Phi}, \Sigma \dashv \Box_X^* \varphi$. This contradicts the fact that $\Sigma \in \mathfrak{F}$.

The contradiction above shows that (9), i.e. $\Diamond_X\beta \vdash_{E_A^*} \beta$, must hold.

■

This truth-lemma shows the completeness of $\mathbf{E}_A^*$ for finite sequents with respect to the model class $\mathfrak{E}_{\mathfrak{A}}$. Furthermore, due to the finiteness of the countermodels, such as $M_{E_A^*}^{\Phi}$ above, we have caught the decidability of $\mathbf{E}_A^*$ with respect to finite sequents.

6.33. THEOREM. COMPLETENESS of $\mathbf{E}_A^*$
For all finite $\Gamma, \Delta \subseteq \mathcal{L}_A^*$: $\Gamma \models_{\mathfrak{E}_{\mathfrak{A}}} \Delta \Rightarrow \Gamma \vdash_{E_A^*} \Delta$.

Proof. If $\Gamma \nvdash_{E_A^*} \Delta$ then by the truth lemma above and the saturation for filtrations $M_{E_A^*}^{\Gamma\cup\Delta}, \Sigma \models \gamma$ and $M_{E_A^*}^{\Gamma\cup\Delta}, \Sigma \not\models \delta$, for all $\gamma \in \Gamma$ and $\delta \in \Delta$. Or shortly, $\Gamma \not\models_{\mathfrak{E}_{\mathfrak{A}}} \Delta$. ■

An immediate consequence of the finiteness of $M_{E_A^*}^{\Phi}$ for finite $\Phi \subseteq \mathcal{L}_A^*$ establishes the decidability result immediately.

6.34. THEOREM. The system $\mathbf{E}_A^*$ has the FMP is decidable for finite sequents.

The completeness of $\mathbf{C}^{3*}$

A completeness and decidability result for $\mathbf{C}^{3*}$ can be obtained by a filtration which combines the **Mud**- and the $\mathbf{E}_A^*$-closures. We take instead of $(\Phi^{\boxtimes_A})^{\boxplus_A}$ for a finite $\Phi \subseteq \mathcal{L}_A^{*,\uparrow,\downarrow}$, the set $(\Phi^{\boxtimes_A})^{\boxplus_{d,A}}$, which denotes the following complicated filtration:

[6] $\Sigma^* \cap (\Phi^{\boxtimes_A})^{\boxplus_A}, \Theta^* \cap (\Phi^{\boxtimes_A})^{\boxplus_A} \in \mathfrak{Sat}_{E_A^*}^{(\Phi^{\boxtimes_A})^{\boxplus_A}}$, according to proposition 6.3.

[7] By definition of β and the fact that $\Theta^* \in \mathfrak{Sat}_{E_A^*}$.

$$\Box(\Phi^{\boxtimes_A})^d \cup \Diamond(\Phi^{\boxtimes_A})^\wedge \cup \Phi^{\boxtimes_A}.$$

The $\mathbf{C}^{3*}$-Φ-canonical model obtains the following definition:

$$M^{\Phi}_{C^{3*}} = \langle \mathfrak{Sat}_{C^{3*}}^{(\Phi^{\boxtimes_A})^{\boxplus_{d,A}}}, R^{\Phi^{\boxtimes_A}}_{C^{3*}}, \ll^{\Phi}_{C^{3*}}, V^{\Phi}_{C^{3*}} \rangle.$$

The truth lemma for the subformulae of such a finite Φ can be obtained by an imitation of the proof of the truth lemma of $\mathbf{E}^*_A$. The structural claim $M^{\Phi}_{C^{3*}} \in \mathfrak{C}^{3*}$ can be obtained by an imitation of lemma 6.25.

6.35. THEOREM. The system $\mathbf{C}^{3*}$ is complete for finite sequents, has the FMP and is decidable for finite sequents.

6.5 Conclusions and reflections

We have seen that the basic results of chapter 5 could be transposed to restricted saturated sets. Doing so, we have established decidability results for the basic formalisms of part I on the basis of finitely filtrated canonical models. In some cases we needed to extend the filtering sets in such a way that earlier results of chapter 5 could be employed in a proper way.

Furthermore, we proved completeness results for finite $\mathbf{E}_A$- and $\mathbf{C}^{3*}$-sequents. An imitation of classical proofs for this type of systems could be given. Additional insights on saturated sets which we obtained in the previous chapter had to be used to guarantee the correctness of this imitation.

Of course, our decidability results are very general mathematical results. Apart from the finiteness, these results do not tell us very much about the complexity of possible decision procedures. This may be disappointing, because we may have made the impression with our plea for more 'realism' by means of partialization in chapter 1 that lower technical complexity results would support this ideology. However, as mentioned in section 1.2, we do not argue against the use of large models for technical reasons. Our principal argument to prefer partial modeling was to distinguish two different forms of negative information: falsity and absence of truth. In epistemic logic, this distinction makes it possible to separate different kinds of disbelief. In this respect, the absence of lower complexity results does not bother us, and it certainly does not violate our arguments in chapter 1 for using partial possible worlds semantics.

Of course, we do not wish to neglect the issue. We think that for many partial modal logics well-known complexity upper bounds of their classical counterparts can be used[8]. In fact, simple embedding results such as in the appendix of chapter 3 settle such classical upper bounds for constructive logics. For example, the systems $\mathbf{N}$ and $\mathbf{N}^{\sim}$ are PSPACE-hard. This result can be obtained immediately by the PSPACE-hardness of **S4** [Ladner 1977] and the linearity of the translation. We have the impression that often things do not get worse in partial logic, and take this conjecture as a starting point for further analysis of partial intensional logics.

[8] For an extensive survey on the complexity of classical modal logics, see [Spaan 1993].

Chapter 7

A Bit of Correspondence Theory

In this chapter we will discuss a regular class of axiomatic extensions of **M** and define and demonstrate corresponding model-theoretic characterizations.

In classical modal logic, so-called *correspondence theory* is one of the main research issues [van Benthem 1985]. The purpose of this study is to find precise correspondences between modal axiomatic extensions of the classical minimal system **K** and classes of possible worlds models. Most attention has been devoted to classes of models defined on the basis of a frame-condition. We already defined validity over frames in chapter 2. The *characteristic* class of frames $\mathfrak{F}$ of a modal logic **S** is the collection of frames which verify all **S**-sequents.

$$\mathfrak{Char}(S) := \{F \in \mathfrak{F} \mid \Gamma \vdash_S \Delta \Rightarrow \Gamma \models_F \Delta\}$$

This characteristic class is the maximal class of frames to which soundness of **S** holds.

Well known axioms like $\mathbf{T} = \Box\varphi \vdash \varphi$, $\mathbf{4} = \Box\varphi \vdash \Box\Box\varphi$, $\mathbf{5} = \Diamond\varphi \vdash \Box\Diamond\varphi$, $\mathbf{B} = \varphi \vdash \Box\Diamond\varphi$ and $\mathbf{G} = \Diamond\Box\varphi \vdash \Box\Diamond\varphi$ can be characterized by well-defined classes of frames. For example, **T** corresponds to reflexive frames, and **B** corresponds to symmetric frames. All these logics find such a nice characterization because they are all members of conveniently characterizable class of modal logics, namely axiomatic extensions of the form:

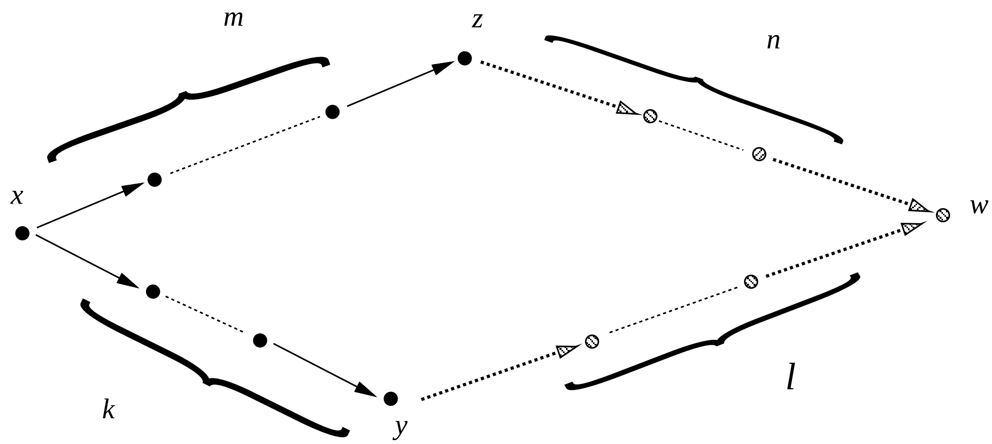
$$\Diamond^k\Box^l\varphi \vdash \Box^m\Diamond^n\varphi \quad \text{with } k,l,m,n \in I\!N.$$

This class of axioms is denoted as $\mathbf{G}^{k,l}_{m,n}$ and are called Geach axioms. The indices refer to the corresponding number of $\Box$- and $\Diamond$-iterations above. So, for example $\mathbf{T} = \mathbf{G}^{0,1}_{0,0}$, $\mathbf{4} = \mathbf{G}^{0,1}_{2,0}$ and $\mathbf{5} = \mathbf{G}^{1,0}_{1,1}$. All these logics can be characterized by the class of frames with an accessibility relation R such that

$$\forall x,y,z : R^k(x,y) \;\&\; R^m(x,z) \Rightarrow \exists w : R^l(y,w) \;\&\; R^n(z,w).$$

The following figure depicts this general relational frame constraint. The black vectors have a universal meaning and the dashed vectors should be interpreted as being existential.

7.1. FIGURE.



A stronger correspondence result is the so-called *frame-completeness*. It expresses a maximal semantic utility of the characteristic frame restriction for a given logic. It means that a counter-model of any non-sequent can be found in this characteristic class:

$$\Gamma \models_{\mathfrak{Char}(S)} \Delta \iff \Gamma \vdash_S \Delta^1.$$

All Geach extensions of the minimal classical modal logic **K** are frame-complete (in classical possible world semantics). This means that, for every non-$\mathbf{G}^{k,l}_{m,n}+\mathbf{K}$-sequent, there exists a total counter-model over a certain frame in the characteristic class.

Such frame completeness of a **K**-extension **S** can most often be shown by the *canonicity* of this system **S** [Hughes & Cresswell 1984]. This means that the underlying frame of the canonical model is a member of the characteristic class of frames: $\langle \mathfrak{Sat}_S, R_S \rangle \in \mathfrak{Char}(S)$. Together with the truth-lemma of **S**, which can be extracted straightforwardly from the truth lemma of the minimal system **K**, this result guarantees frame completeness.

The following sections discuss different Geach-style extensions of **M**. The first section discusses the normal extensions as presented above. It turns out that these extensions have the same frame characteristics, but frame completeness is most often lost. The second section discusses how completeness can be restored through combination of the Geach characteristics of figure 7.1 with the extension relation. The third section presents a correspondence result for weaker Geach-like axiom rules $\Diamond^k \Box^l \varphi, \Diamond^m \Box^n \neg\varphi \vdash \emptyset$. Their characteristic conditions can be caught by means of a frame definition and the bisimulation coherence order which we have defined in section 2.4.

As may become clear from the title of this chapter, the contents of this chapter do not include a generally exhaustive investigation of correspondence theory of partial modal logics. It rather presents some ideas of using information orders

[1] Frame completeness officially is the left-to-right direction of this equivalence. The other direction is the soundness of **S** with respect to its own characteristic class. The definition $\mathfrak{Char}(S)$ brings along this soundness.

over worlds in combination with pure frame conditions. Roughly speaking, the technical findings of this chapter establish completeness results for wide classes of partial modal logics. For example, completeness results for the logics which evolved from pragmatic principles, and have been proposed in sections 4.2 and 4.3, can be obtained quite easily from the results of this chapter. Furthermore, the use of different bisimulation orders allows us to capture different subtle variations of Geach-like axioms. For example, the use of the coherence order allow us to capture structural model-theoretical conditions for the above-mentioned weakenings of the Geach-axioms.

In the last section of this chapter we will shortly speculate on further results. On the basis of the combinatorial results in terms of frame-conditions and information orders, we see that partial systems asks for fundamental insights on the interaction of accessibility structures and constraints on valuations[2].

7.1 Geach extensions of M

As mentioned above, the frame characteristics are precisely the same as for the Geach extensions of **K**.

7.2. THEOREM. Every system $\mathbf{G}^{k,l}_{m,n} + \mathbf{M}$ has the same characteristic frame class as $\mathbf{G}^{k,l}_{m,n} + \mathbf{K}$: the class of $\mathbf{G}^{k,l}_{m,n}$-frames. In short,

$$\mathfrak{Char}(\mathbf{G}^{k,l}_{m,n} + \mathbf{K}) = \mathfrak{Char}(\mathbf{G}^{k,l}_{m,n} + \mathbf{M}).$$

Proof. This characterization result can be obtained freely from the corresponding proof for $\mathbf{G}^{k,l}_{m,n} + \mathbf{K}$ [Hughes & Cresswell 1984]. Classical frames are the same as frames in our partial possible worlds semantics. This classical proof does not appeal to properties of the system **K**, only the Geach axioms themselves are utilized. This ensures that these proofs can be used here as well. Let us give an outline of the proof.

Suppose F is a $\mathbf{G}^{k,l}_{m,n}$-frame. Let $M = \langle W, R, V\rangle$ be a partial Kripke model on F and w a world in M such that $M, w \models \Diamond^k \Box^l \varphi$. Semantical decomposition of the modal operators in front of φ entails the existence of a world $v \in W$ such that $R^k(w, v)$, and that all $u \in W$ with $R^l(v, u)$ verify φ: $M, u \models \varphi$ (1). Because of the Geach frame condition, we know that if $R^m(w, t)$ then there exists $s \in W$ such that $R^l(v, s)$ and $R^n(t, s)$. Conclusion (1) yields $M, s \models \varphi$, and therefore $M, t \models \Diamond^n \varphi$, and subsequently, due to the arbitrariness of t as a world in M with accessibility distance m from w, $M, w \models \Box^m \Diamond^n \varphi$. M was chosen freely on F and w in M. This yields $\Diamond^k \Box^l \varphi \models_F \Box^m \Diamond^n \varphi$.

Let $F = \langle W, R\rangle$ be a frame outside the $\mathbf{G}^{k,l}_{m,n}$-class. This means that there exists a triple of worlds x, y, z in F such that

$$R^k(x, y) \ \&\ R^m(x, z) \ \&\ (\forall w \in W : R^l(y, w) \Rightarrow \text{ not } R^n(z, w)) \ (2).$$

Take $M = \langle W, R, V\rangle$ with

$$V(s)(p) = \begin{cases} 1 & \text{if } R^l(y, s) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

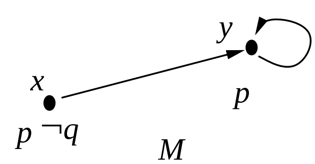
[2] See also [Rodenburg 1986] and [Plotkin & Stirling 1986].

Clearly $M, y \models \Box^l p$ and $M, x \models \Diamond^k \Box^l p$. Because of (2) and the definition of V for the input p, we know that all worlds which have an accessibility distance n from z do not define p and therefore $M, z \not\models \Diamond^n p$. This yields $M, x \not\models \Box^m \Diamond^n p$. We have found a counter-model for $\mathbf{G}^{k,l}_{m,n}$ on an arbitrary non-$\mathbf{G}^{k,l}_{m,n}$-frame. ■

Frame incompleteness

The optimism due to the easy transfer of the characterization given above cannot be hold on to when frame completeness comes on the stage. In fact, according to the formulation of frame completeness above, these Geach extensions are most often frame incomplete. A simple illustration can be given by the logic $\mathbf{T}$ + $\mathbf{M}$, which we will call $\mathbf{T}^\Box$ in the sequel. In this system we have the following non-sequent $\varphi \not\vdash_{T^\Box} \Diamond\varphi$. The following model in the figure below illustrates this weakness of the system $\mathbf{T}^\Box$.

7.3. FIGURE.



In this depicted model, $y \sqsubseteq_M x$ through the bisimulation $\{\langle x,x\rangle, \langle y,x\rangle, \langle y,y\rangle\}$. If $M, x \models \Box\varphi$ then $M, y \models \varphi$, and because of $y \sqsubseteq_M x$ and corollary 2.46 (page 157) we conclude $M, x \models \varphi$. This means that the world x in M satisfies all $\mathbf{T}^\Box$-sequents. Nevertheless $M, x \models \neg q$ but $M, x \not\models \Diamond\neg q$, and therefore

$\varphi \not\vdash_{T^\Box} \Diamond\varphi$[3] (3).

Application of theorem 7.2 shows that the $\mathfrak{Char}(\mathbf{T}^\Box)$ is the class of reflexive frames. It is not hard to see that every world in a reflexive model which verifies φ also supports $\Diamond\varphi$, and so

$\varphi \models_{\mathfrak{Char}(T^\Box)} \Diamond\varphi$ (4).

Combination of (3) and (4) shows that $\mathbf{T}^\Box$ is frame incomplete according to the definition of frame completeness in the definition on page 192.

Figure 7.3 already suggests the structural correspondence between the axiom $\mathbf{T}$ and a world like x in the displayed model. It is surely not reflexive and it is neither 'reflexive-like' in the sense that it has access to some world which has an identical informational content. Nevertheless it 'sees a part of itself': $R(x,y)$ and $y \sqsubseteq_M x$. In a trivial way y does the same. Models which bear this property are called *small reflexive* in the sequel. In theorem 7.5 below a completeness result for the system $\mathbf{T}^\Box$ with respect to this wider class of models is given. This theorem adds also a completeness result for the 'contra-positional'

[3] In epistemic logic with $\Box$ interpreted as a knowledge operator this non-derivability is desirable. Of course all knowledge should imply truth (e.g. [Hintikka 1962]and [Lenzen 1978]). However we do not want that everything which holds should be taken to be possibly true by every agent. Simple incomplete awareness of agents prohibits this unnatural broadness of mind. A logic of knowledge which uses the axiom $\mathbf{T}^\Box$ but not its contra-position can be found in [van der Hoek, Jaspars & Thijsse 1993].

system $\mathbf{T}^{\diamond} := \mathbf{M} + \varphi \vdash \diamond\varphi$. This logic is complete with respect to the class of *big reflexive* models. In these models every world has access to an extension of itself. In order to avoid confusion, we give a formal definition of these two classes below.

7.4. DEFINITION. A model $M = \langle W, R, V\rangle$ is said to be small reflexive if

$$\forall x \in W\ \exists y \in W : R(x,y)\ \&\ y \sqsubseteq_M x.$$

Such a model is said to be big reflexive if

$$\forall x \in W\ \exists y \in W : R(x,y)\ \&\ x \sqsubseteq_M y.$$

These classes are abbreviated by $\mathfrak{T}^{\square}$ and $\mathfrak{T}^{\diamond}$, respectively.

7.5. THEOREM. $\mathbf{T}^{\square}$ is sound and complete with respect to $\mathfrak{T}^{\square}$. $\mathbf{T}^{\diamond}$ is sound and complete with respect to $\mathfrak{T}^{\diamond}$.

Proof. Soundness of these logics can be understood immediately on the basis of the persistence result for $\sqsubseteq_M$ (corollary 2.46).

Completeness can be obtained in the same manner as in the procedure to which we already have referred in the introductory part of this chapter, namely by showing that the canonical model of these systems shares the structural condition. What has to be proved are the following two claims:

$$M_{T^{\square}} \in \mathfrak{T}^{\square}\ (5) \quad \text{and} \quad M_{T^{\diamond}} \in \mathfrak{T}^{\diamond}\ (6).$$

Lemma 5.25 on page 156 provides the essential equipment for proving the claims above. Note that if $\Gamma \in \mathfrak{Sat}_{T^{\square}}$ then $\square^{-}\Gamma \subseteq \Gamma$, and also $\square^{-}\Gamma \trianglelefteq_{T^{\square}} \Gamma$. Application of lemma 5.25 yields a $\mathbf{T}^{\square}$-saturated set Δ such that

$$R_{T^{\square}}(\Gamma,\Delta) \text{ and } \Delta \subseteq \Gamma.$$

This latter conclusion also entails $\Delta \sqsubseteq_{M^{T^{\square}}} \Gamma$, according to corollary 5.26 on page 157. Because Γ has been picked arbitrarily in $\mathfrak{Sat}_{T^{\square}}$ we know that $M_{T^{\square}} \in \mathfrak{T}^{\square}$.

If $\Gamma \in \mathfrak{Sat}_{T^{\diamond}}$ then $\Gamma \subseteq \diamond^{-}\Gamma$, and also $\Gamma \trianglelefteq_{T^{\square}} \diamond^{-}\Gamma$. Lemma 5.25 successively yields a $\mathbf{T}^{\diamond}$-saturated set Δ such that

$$R_{T^{\square}}(\Gamma,\Delta) \text{ and } \Gamma \subseteq \Delta.$$

We conclude $M_{T^{\diamond}} \in \mathfrak{T}^{\diamond}$ by corollary 5.26. ■

Intermediate worlds

In [Thijsse 1992] one finds a completeness proof for the system $\mathbf{M}\ +\ \square\varphi \vdash \varphi\ +\ \varphi \vdash \diamond\varphi$ with respect to the class of reflexive partial Kripke models. In fact this is a frame complete system. Of course, reflexivity is a stronger condition than small and big reflexivity together. This restriction of the smaller class of reflexive models can be understood easily by means of theorem 7.6 below. This theorem states that worlds in partial Kripke models are insensitive for adding and removing so-called *intermediate worlds*. Whenever a world w in a partial Kripke model M has two accessible worlds v and u, then every intermediate world x of v and u in M, which simply means $v \sqsubseteq_M x \sqsubseteq_M u$, can be taken to be accessible from w as well, without changing the informational content of w.

7.6. THEOREM. Let $M = \langle W, R, V\rangle \in \mathfrak{M}$, and take $w \in W$ such that for certain $v, u \in W$: $R(w, v)$ and $R(w, u)$. Suppose $x \in M$ such that $v \sqsubseteq_M x \sqsubseteq_M u$, and let $M' = \langle W, R', V\rangle$ such that $R' = R \cup \{\langle w, x\rangle\}$. Then for all $\varphi \in \mathcal{L}^\Box$ we obtain

$$M, w \models \varphi \Longleftrightarrow M', w \models \varphi.$$

Proof. By induction on the construction of formulae. The $\Box$ and $\neg\Box$-step are simply obtained by the persistence result for $\sqsubseteq_M$ (corollary 2.46). ■

In a model which is both big and small reflexive, every world has access to a larger and a smaller world. This means that the original world is an intermediate world of itself, and could therefore be taken to be accessible as well, without loosing or gaining information. In this straightforward manner, we are able to transform every model which is both big and small reflexive into a reflexive one. In canonical models for **M**-extensions, according to the definition of accessibility, all intermediate worlds are accessible. This means that the canonical model of $\mathbf{M} + \Box\varphi \vdash \varphi + \varphi \vdash \Diamond\varphi$ is reflexive.

7.7. COROLLARY. The system $\mathbf{M} + \Box\varphi \vdash \varphi + \varphi \vdash \Diamond\varphi$ is frame complete, that is, complete with respect to its characteristic frame class, the reflexive frames.

7.2 A completeness result for $\mathbf{G}^{k,l}_{m,n} + \mathbf{M}$

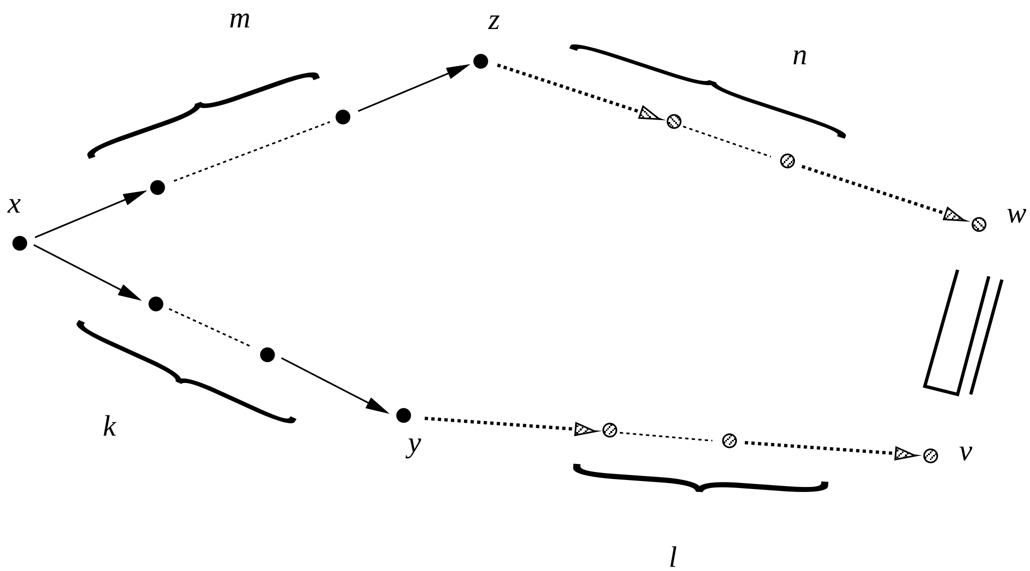
In this subsection we present a uniform correspondence result for the Geach extensions of **M**. As above, we define its model-theoretic characteristics by means of a combination of constraints on the accessibility and the extension order over possible worlds together. The correspondence theorem for this class is a generalization of the empirical study on the logic $\mathbf{T}^\Box$ and $\mathbf{T}^\Diamond$ in the previous section.

7.8. THEOREM. The logic $\mathbf{M} + \mathbf{G}^{k,l}_{m,n}$ is sound and complete with respect to the collection of models $M = \langle W, R, V\rangle$ with

$$\forall x, y, z \in W : R^k(x, y) \ \&\ R^m(x, z) \implies \exists v, w \in W :\ R^l(y, v) \ \&\ R^n(z, w) \ \&\ v \sqsubseteq_M w.$$

This class of models will be abbreviated by $\mathfrak{G}^{k,l}_{m,n}$. The following figure pictures this relational requirement for partial Kripke models to be in this specific class.

7.9. FIGURE.



The only difference with the classical result is that the 'end-points' v and w do not have to be equal. The only requirement is that that w extends v.

Some meditation is needed to grasp this correspondence result. Some rehearsal is given by the following examples. In the table below, the left column lists some well-known Geach extensions of $\mathbf{M}$. The right column gives the corresponding constraint for a model $M = \langle W, R, V\rangle$ to be in the proper characteristic class.

7.10. TABLE.

$\mathbf{M} + \mathbf{B}$ $(\varphi \vdash \Box\Diamond\varphi)$	$\forall x\ \exists y, z : R(x,y)\ \&\ R(y,z)\ \&\ x \sqsubseteq_M z$
$\mathbf{M} + \mathbf{5}$ $(\Diamond\varphi \vdash \Box\Diamond\varphi)$	$\forall x, y, z : R(x,y)\ \&\ R(x,z) \implies \exists w : R(z,w)\ \&\ y \sqsubseteq_M w.$
$\mathbf{M} + \mathbf{4}$ $(\Box\varphi \vdash \Box\Box\varphi)$	$\forall x, y, z : R(x,y)\ \&\ R(y,z) \implies \exists w : R(x,w)\ \&\ w \sqsubseteq_M z$
$\mathbf{M} + \Diamond\Diamond\varphi \vdash \Diamond\varphi$	$\forall x, y, z : R(x,y)\ \&\ R(y,z) \implies \exists w : R(x,w)\ \&\ z \sqsubseteq_M w$

In agreement with our vocabulary of the two different forms of quasi-reflexivity in the previous section , a proper christening of these four properties above would be *small symmetry*, *small Euclidicity*, *small* and *big transitivity respectively.*

The remainder of this subsection is dedicated to a proof of theorem 7.8. The soundness result is, once again, a direct result of the persistence of the complete language $\mathcal{L}^{\Box}$ over $\sqsubseteq_M$.

Proof. (SOUNDNESS) Suppose $M = \langle W, R, V\rangle \in \mathfrak{G}^{k,l}_{m,n}$ for certain natural numbers k, l, m and n, and suppose $M, x \models \Diamond^k\Box^l\varphi$. This means that there exists $y \in W$ such that $R^k(x,y)$ and $M, y \models \Box^l\varphi$, This entails for every $v \in W$ with $R^l(y,v)$ $M, v \models \varphi$. Suppose that a world $z \in W$ has an m-accessibility distance from x: $R^m(x,z)$. Since $M \in \mathfrak{G}^{k,l}_{m,n}$, we know that there exists $w \in W$ such that $R^n(z,w)$ and $v \sqsubseteq_M w$. The persistence result for $\sqsubseteq_M$ brings along that $M, w \models \varphi$, and also $M, z \models \Diamond^n\varphi$. Because z have been chosen as an arbitrary world living m accessibility steps from x, we obtain $M, x \models \Box^m\Diamond^n\varphi$. Because $M \in \mathfrak{G}^{k,l}_{m,n}$ has been picked at random, and x as an arbitrary world in M, we know that

$$\Diamond^k \Box^l \varphi \models_{\mathfrak{G}^{k,l}_{m,n}} \Box^m \Diamond^n \varphi.$$

■

The completeness proof is accomplished by proving that $M_{G^{k,l}_{m,n}} \in \mathfrak{G}^{k,l}_{m,n}$. This is certainly not a straightforward result. The following generalization of lemma 5.26 will help us. It facilitates reasoning about saturated sets with a given finite accessibility distance.

7.11. LEMMA. Suppose Γ and Δ are two **X**-saturated sets, where **X** is an extension of **M**.

$\Box^{-k}\Gamma \subseteq \Delta \;\Rightarrow\; \exists \Delta' \in \mathfrak{Sat}_X : \Delta' \subseteq \Delta \;\&\; R^k_X(\Gamma, \Delta')$ (7), and

$\Delta \subseteq \Diamond^{-k}\Gamma \;\Rightarrow\; \exists \Delta' \in \mathfrak{Sat}_X : \Delta' \supseteq \Delta \;\&\; R^k_X(\Gamma, \Delta')$ (8), for all $k \in I\!N$.

Proof. The proof runs by induction on the index k, that is, the **X**-canonical accessibility distance. If $k = 0$, then the result is immediately obtained, because $\Box^0\Gamma = \Diamond^0\Gamma = \Gamma$. Substitute Γ for Δ' to get the desired result for the fulfillment of both claims (7) and (8) in the case that $k = 0$.

To prove the induction step, we separate the two claims. We start by proving (7).

Let $k > 0$ and $\Box^{-k}\Gamma \subseteq \Delta$ with $\Gamma, \Delta \in \mathfrak{Sat}_X$.

To start with, we need a bit of syntactic dressing in order to make the procedure more digestable:

$\Delta_n = \Box^n\Delta \cup (\Box^n\mathcal{L}^\Box)^\complement$ for all $n \in I\!N$.

This set Δ_n consists of all formulae which are not of the form $\Box^n\varphi$ and of all formulae of the form $\Box^n\delta$ with δ originating from Δ. We claim that substitution of $k-1$ in this definition entails an **X**-saturator of $\Box^-\Gamma$.

$\Box^-\Gamma \trianglelefteq_X \Delta_{k-1}$ (9).

This result and the induction hypothesis establish (7). Application of lemma 5.25 to claim (9) yields a $\Theta \in \mathfrak{Sat}_X$ such that $R_X(\Gamma, \Theta)$ and $\Theta \subseteq \Delta_{k-1}$. By definition of Δ_{k-1} we infer $\Box^{-(k-1)}\Theta \subseteq \Delta$. According to the induction hypothesis there exists a $\Delta' \in \mathfrak{Sat}_X$ such that $R^{k-1}_X(\Theta, \Delta')$ and $\Delta' \subseteq \Delta$. Because $R_X(\Gamma, \Theta)$, we also obtain $R^k_X(\Gamma, \Delta')$ and consequently (7) holds.

What remains to be demonstrated is claim (9). To show the validity of this claim, let $\Box^-\Gamma \vdash_X \Sigma$ for certain $\Sigma \subseteq \mathcal{L}^\Box$. We need to prove $\Sigma \cap \Delta_{k-1} \neq \emptyset$.

To begin with we immediately conclude $\Sigma \neq \emptyset$, for if $\Box^-\Gamma \vdash_X = \emptyset$ then $\Box^-\Gamma \vdash_X \bot$, and by application of R-TRUE $\Box$ and L-MON also $\Gamma \vdash_X \Box\bot$. Because $\Box\bot \vdash_X \Box^k\bot$, we would also get $\Gamma \vdash_X \Box^k\bot$, and so $\bot \in \Delta$, which contradicts the **X**-consistency of Δ.

If $\Sigma \cap (\Box^{k-1}\mathcal{L}^\Box)^\complement \neq \emptyset$ then also $\Sigma \cap \Delta_{k-1} \neq \emptyset$ by the definition of Δ_{k-1}.

What is left to show is $\Sigma \cap \Delta_{k-1} \neq \emptyset$ whenever $\emptyset \neq \Sigma \subseteq \Box^{k-1}\mathcal{L}^\Box$. In other words, we need to prove the intersection with Δ_{k-1} for all non-empty conclusion sets Σ of $\Box^-\Gamma$ which consist only of formulae of the form $\Box^{k-1}\varphi$.

This means that there exists $\Box^{k-1}\varphi_1, ..., \Box^{k-1}\varphi_m$ in Σ such that

$$\Box^{-}\Gamma \vdash_X \Box^{k-1}\varphi_1, ..., \Box^{k-1}\varphi_m.$$

Because $\Box^{k-1}\varphi_i \vdash_X \Box^{k-1}(\varphi_1 \vee ... \vee \varphi_m)$ for all $i \in \{1, .., m\}$, we may reform the last conclusion into

$$\Box^{-}\Gamma \vdash_X \Box^{k-1}(\varphi_1 \vee ... \vee \varphi_m).$$

Application of R-TRUE $\Box$ and L-MON yields

$$\Gamma \vdash_X \Box^{k}(\varphi_1 \vee ... \vee \varphi_m).$$

This last **X**-sequent entails $\varphi_1 \vee ... \vee \varphi_m \in \Delta$, and because this last set is **X**-saturated, we know that there exists $i \in \{1, .., m\}$ such that $\varphi_i \in \Delta$ and therefore $\Box^{k-1}\varphi_i \in \Delta_{k-1}$. Because this $\Box^{k-1}\varphi$ stems from Σ, we have made sure that $\Sigma \cap \Delta_{k-1} \neq \emptyset$.

Proving (8), the dual result of (7), comes down to a similar procedure. Once again, lemma 5.25 paves the way. Let $k > 0$ and suppose $\Delta \subseteq \Diamond^{k-1}\Gamma$ for $\Gamma, \Delta \in \mathfrak{Sat}_X$.

To obtain the desired result we will show that

$$\Diamond^{k-1}\Delta \trianglelefteq_X \Diamond^{-}\Gamma \text{ (10)}.$$

Lemma 5.25 subsequently shows that there exists a $\Theta \in \mathfrak{Sat}_X$ such that $R_X(\Gamma, \Theta)$ and $\Diamond^{k-1}\Delta \subseteq \Theta$. This latter conclusion entails $\Delta \subseteq \Diamond^{-(k-1)}\Theta$, which brings along, according to the induction hypothesis, an **X**-saturated set $\Delta' \supseteq \Delta$ such that $R_{k-1}(\Theta, \Delta')$. This proves (8) because $R_X(\Gamma, \Theta)$ entails $R^k_X(\Gamma, \Delta')$.

To complete the proof, we need to show that $\Sigma \cap \Diamond^{-}\Gamma \neq \emptyset$ for any $\Sigma \subseteq \mathcal{L}^{\Box}$ with $\Diamond^{k-1}\Delta \vdash_X \Sigma$. To demonstrate the validity of this implication, and the consequential correctness of the claim (10) above, we presuppose the validity of the latter **X**-sequent.

This **X**-sequent tells us that there exists a finite set $\delta_1, .., \delta_l$ such that

$$\Diamond^{k-1}\delta_1, ..., \Diamond^{k-1}\delta_l \vdash_X \Sigma.$$

Applying CUT an l number of times to this **X**-sequent and using the simple **X**-sequent $\Diamond^{k-1}(\delta_1 \wedge ... \wedge \delta_l) \vdash_X \Diamond^{k-1}\delta_i$ (for all $i \in \{1, .., l\}$) entails

$$\Diamond^{k-1}(\delta_1 \wedge ... \wedge \delta_l) \vdash_X \Sigma.$$

Upon this **X**-sequent application of L-TRUE $\Diamond$ yields

$$\Diamond^{k}(\delta_1 \wedge ... \wedge \delta_l) \vdash_X \Diamond\Sigma.$$

Of course $\delta_1 \wedge ... \wedge \delta_l \in \Delta$ because Δ is **X**-saturated, and therefore we conclude (L-MON) $\Diamond^{k}\Delta \vdash_X \Diamond\Sigma$. Furthermore, $\Diamond^{k}\Delta \subseteq \Gamma$ for $\Delta \subseteq \Diamond^{-k}\Gamma$, which means $\Gamma \vdash_X \Diamond\Sigma$. This last conclusion brings us the final result $\Sigma \cap \Diamond^{-}\Gamma \neq \emptyset$, because $\Gamma \in \mathfrak{Sat}_X$.

■

7.12. COROLLARY. A simple corollary, which is important for the remainder of this chapter, is the observation that this lemma combined with the bounded saturation lemma settles the following conclusion.

$$\Box^{-k}\Gamma \trianglelefteq_X \Lambda \Rightarrow \exists\Sigma \in \mathfrak{Sat}_X : R^k_X(\Gamma, \Sigma) \ \& \ \Sigma \subseteq \Lambda, \text{ and}$$

$\Lambda \trianglelefteq_X \Diamond^{-k}\Gamma \Rightarrow \exists \Sigma \in \mathfrak{Sat}_X : R^k_X(\Gamma, \Sigma) \ \& \ \Lambda \subseteq \Sigma$ for all $\Gamma \in \mathfrak{Sat}_X$, $k \in I\!N$.

This result is a full generalization of lemma 5.25.

Proof. If $\Box^{-k}\Gamma \trianglelefteq_X \Lambda$ for $\Gamma \in \mathfrak{Sat}_X$ then there exists $\Delta \in \mathfrak{Sat}_X$ such that $\Box^{-k}\Gamma \subseteq \Delta \subseteq \Lambda$ (BSL). Application of lemma 7.11 brings us the first implication above. The second claim can be obtained analogously. ■

These results facilitate reasoning about arbitrary accessibility distances in the canonical models of the Geach extensions of **M**. The modal metrical equipment of lemma 7.11 gives us enough formal understanding of these canonical models to prove their membership of the associated Geach classes, which have been depicted in figure 7.9 on page 197. The following lemma presents a formal demonstration and establishes the completeness theorem 7.8.

7.13. LEMMA. $M_{G^{k,l}_{m,n}} \in \mathfrak{G}^{k,l}_{m,n}$.

Proof. Suppose that $R^k_{G^{k,l}_{m,n}}(\Gamma, \Delta)$ and $R^m_{G^{k,l}_{m,n}}(\Gamma, \Theta)$ for a certain triple Γ, Θ and Δ, of $\mathbf{G}^{k,l}_{m,n}$-saturated sets.

Suppose $\Box^{-l}\Delta \vdash_{G^{k,l}_{m,n}} \Sigma$ for a certain finite set of formulae Σ. This entails, by l time application of R-TRUE $\Box$ and L-MON, that $\Delta \vdash_{G^{k,l}_{m,n}} \Box^l(\bigvee \Sigma)$, and therefore $\Diamond^k\Box^l(\bigvee\Sigma) \in \Gamma$, because $R^k_{G^{k,l}_{m,n}}(\Gamma, \Delta)$. The $\mathbf{G}^{k,l}_{m,n}$-axiom entails $\Box^m\Diamond^n(\bigvee\Sigma) \in \Gamma$. Because $R^m_{G^{k,l}_{m,n}}(\Gamma, \Theta)$, we know that $\Diamond^n(\bigvee\Sigma) \in \Theta$. $\Diamond$-distribution over $\vee$, and the fact that Θ is $\mathbf{G}^{k,l}_{m,n}$-saturated, makes us conclude

$\Diamond^n\sigma \in \Theta$ for certain $\sigma \in \Sigma$.

This means that $\Sigma \cap \Diamond^{-n}\Theta \neq \emptyset$. In other words, $\Diamond^{-n}\Theta$ turns out to be an $\mathbf{G}^{k,l}_{m,n}$-saturator of $\Box^{-l}\Delta$. The bounded saturation lemma 5.6 tells us thereupon that there exists an $\mathbf{G}^{k,l}_{m,n}$-saturated set Ξ such that

$\Box^{-l}\Delta \subseteq \Xi \subseteq \Diamond^{-n}\Theta$.

Lemma 7.11 finishes the job. Since $\Box^{-l}\Delta \subseteq \Xi$, we know that there exists a $\mathbf{G}^{k,l}_{m,n}$-saturated set $\Xi_1 \subseteq \Xi$ such that $R^l(\Delta, \Xi_1)$. Because $\Xi \subseteq \Diamond^{-n}\Theta$, we may also conclude that there exists a $\mathbf{G}^{k,l}_{m,n}$-saturated set $\Xi_2 \supseteq \Xi$ such that $R^n_{G^{k,l}_{m,n}}(\Theta, \Xi_2)$. Obviously, $\Xi_1 \subseteq \Xi_2$. On account of corollary 5.26, we know that $\sqsubseteq_{M_{G^{k,l}_{m,n}}}$ coincides with $\subseteq$ over $\mathfrak{Sat}_{\mathbf{G}^{k,l}_{m,n}}$, and so $M_{G^{k,l}_{m,n}} \in \mathfrak{G}^{k,l}_{m,n}$. ■

The completeness result in theorem 7.8 can be given by the well-known procedure on the basis of the saturation lemma 5.8.

7.3 Correspondences by coherence

In section 4.3 we have met different pragmatic principles which restrain the interplay of preferential worlds and doxastic alternatives. These model-theoretic constraints were also defined on the basis of the bisimulation extension order. An apparent advantage of partial modal logic is its lack of contra-position for 'reasonable' weakening of this type of principles. An example of such a weakening is the CAUTIOUS reformulation of the REALISM principle. The correspondence for this weakening is that all preferential worlds are extensions of some doxastic alternative. This result can be obtained immediately by a poly-modal generalization of our correspondence result for the Geach axioms in the preceding section. For every poly-modal axiom of the form

$$\Diamond_{a_1} ... \Diamond_{a_k} \Box_{b_1} ... \Box_{b_l} \varphi \vdash \Box_{c_1} ... \Box_{c_m} \Diamond_{d_1} ... \Diamond_{d_n} \varphi$$

we find a similar complete correspondence as for the singular partial Geach logics. If $M = \langle W, \{R_a\}_{a \in A}, V \rangle \in \mathfrak{M}_{\mathfrak{A}}$, then it corresponds to the axiom above if and only if

$$\forall x, y, z \; R_{a_1} \circ .. \circ R_{a_k}(x, y) \;\&\; R_{c_1} \circ .. \circ R_{c_m}(x, z) \implies \\ \exists v, w \; R_{b_1} \circ .. \circ R_{b_l}(y, v) \;\&\; R_{d_1} \circ .. \circ R_{d_n}(z, w) \;\&\; v \sqsubseteq_M w \quad (*)$$

The right correspondence of the CAUTIOUS REALISM principle can immediately be obtained from the proper substitution in the possible worlds constraint above. Take $\Box_{b_1} = [\mathsf{p}]_a$ and $\Box_{c_1} = \Box_a$. To get the precise model-theoretic correspondence, replace R_{b_1} by R_a and R_{c_1} by P_a in (*) above.

Freedom of contra-position in partial modal logic is caused by the absence of the classical rule R-TRUE $\neg$. This means that the weakenings of axiomatic rules can be enforced by moving formulae from the conclusion set in negated form to the assumption set. An example of this alternative kind of weakening, which has also been presented in section 4.3, is obtained by changing the arrogant principle $\Box_a \varphi \vdash [\mathsf{p}]_a \Box_b \varphi$ into $\Box_a \varphi, \langle \mathsf{p} \rangle_a \Diamond_b \neg \varphi \vdash \emptyset$ (QUALITY 2).

As have been claimed in section 4.3, this principle has a possible worlds correspondence on the basis of the coherence relation, instead of the extension order. Such correspondences arise by the kind of negation-to-the-left weakenings of Geach axioms. In this section we will find that the axiom rules $\mathbf{rG}^{k,l}_{m,n} := \Diamond^k \Box^l \varphi, \Diamond^m \Box^n \neg \varphi \vdash \emptyset$ correspond to the models which are defined by replacing the extension order by the coherence relation in the Geach correspondence in the previous chapter.

7.14. DEFINITION. The class $\mathfrak{rG}^{k,l}_{m,n}$ consists of models $M = \langle W, R, V \rangle$ such that

$$\forall x, y, z \in W : R^k(x, y) \;\&\; R^m(x, z) \implies \\ \exists v, w : R^l(y, v) \;\&\; R^n(z, w) \;\&\; v \sim_M w.$$

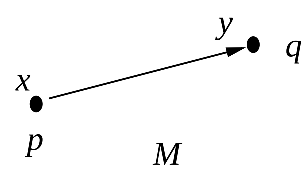
A poly-modal generalization of this result, which can be obtained by substitution of $\sim_M$ for $\sqsubseteq_M$ the relational equation (*), establishes the correspondence result for QUALITY 2, by taking $k = 0$, $l = 1$, $m = 2$, $n = 0$ and $\Box_{b_1} = \Box_a$, $\Diamond_{c_1} = \langle \mathsf{p} \rangle_a$ and $\Diamond_{c_2} = \Diamond_b$.

This section is dedicated to a proof of the Geach-like correspondence result for the logics $\mathbf{rG}^{k,l}_{m,n} + \mathbf{M}$. To begin with, we give a simple illustration to get the feeling of coherence correspondences.

7.15. EXAMPLE. Let $\mathbf{A} := \mathbf{rG}^{1,0}_{0,0} = \varphi, \Diamond\neg\varphi \vdash \emptyset$. In classical modal logic this axiom rule is equivalent with $\Diamond\varphi \vdash \varphi$, which corresponds to models where worlds only may see themselves:

$$\forall x : R(x,y) \implies x = y.$$

In partial modal logic this last axiom implies $\mathbf{A}$: $\varphi, \Diamond\neg\varphi \vdash_{G^{1,0}_{0,0}+M} \emptyset$, but not the other way around $\Diamond\varphi \not\vdash_A \varphi$. A simple counter-model is depicted in the following figure.



It is left to the reader to show that this model is indeed a model which satisfies all rules of $\mathbf{A}$. Clearly it violates $\mathbf{G}^{1,0}_{0,0}$ because $M, x \models \Diamond q$, but $M, x \not\models q$. This system $\mathbf{A}$ corresponds to partial Kripke models in which every world only has access to worlds which are coherent with itself. Let us call this model class $\mathfrak{A}$. The formal definitions of the characteristic class looks as follows:

$$M = \langle W, R, V\rangle \in \mathfrak{A} \quad \text{iff} \quad \forall x, y \in W : R(x,y) \Rightarrow x \sim_M y.$$

The soundness of this system with respect to the class $\mathfrak{A}$ is obvious from the characteristic coherence theorem 2.48 If $M, x \models \Diamond\neg\varphi$ for certain $M \in \mathfrak{A}$ and x in M, then there exists y in M such that $R(x,y)$ and $M, y \dashv \varphi$. Because we know $x \sim_M y$, theorem 2.48 entails $M, x \not\models \varphi$.

The completeness of $\mathbf{A} + \mathbf{M}$ with respect to the class $\mathfrak{A}$ in the example above, can be determined by checking $M_A \in \mathfrak{A}$. This result can be established after some elementary inspection of the coherence relation in canonical models (example 7.17 below).

Canonical coherence

Completeness of the logics $\mathbf{rG}^{k,l}_{m,n}$ comes in two technical lemmas on the coherence relation in the canonical models of partial modal logics. To start with, we will prove an analogy of the structural result $\sqsubseteq_{M_X} = \subseteq$ for normal extensions $\mathbf{X}$ of $\mathbf{M}$ for the relation $\sim_{M_X}$, i.e. the structural coherence relation over the $\mathbf{X}$-canonical model. By means of this result, we obtain a similar justification of our structural description of coherence as the one which we have found for the structural extension order in corollary 5.26. In the case of the coherence order, we claim that two $\mathbf{X}$-saturated sets are coherent if and only if there is no formula φ which is contained in one of the sets and appears in negated form, $\neg\varphi$, in the other.

7.16. LEMMA. Let $\mathbf{X}$ be a normal extension of $\mathbf{M}$, and let $M_X = \langle \mathfrak{Sat}_X, R_X, V_X\rangle$ be its canonical model. For all $\Gamma, \Delta \in \mathfrak{Sat}_X$:

$$\Gamma \sim_{M_X} \Delta \iff \Gamma \cap \neg\Delta = \emptyset$$

Proof. The left-to-right direction of this equivalence follows from the truth-lemma for **X**-canonical models and theorem 2.48. The converse direction of the equivalence can be proved by a procedure which is analogously to the proof of corollary 5.26.

Let $\mathcal{C}$ be the relation which links *all* pairs of **X**-saturated sets which are mutually coherent with respect to their content.

$$\mathcal{C}(\Gamma, \Delta) \overset{def}{\iff} \Gamma \cap \neg\Delta = \emptyset.$$

We will prove that this relation is a bisimulation over M_X:

$$\mathcal{C} \circ R_X \subseteq R_X \circ \mathcal{C} \quad \text{and} \quad \mathcal{C}^{-1} \circ R_X \subseteq R_X \circ \mathcal{C}^{-1} \ (11).$$

The relation $\mathcal{C}$ over M_X is symmetric due to the "double negation" rules, i.e. FALSE rules for $\neg$[4]. This implies $\mathcal{C} = \mathcal{C}^{-1}$, and therefore we only need to check one of these interrelational claims in (11). We pick the left one.

Suppose Γ, Δ, Θ is a triple of **X**-saturated sets such that $\mathcal{C}(\Gamma, \Delta)$ and $R_X(\Gamma, \Theta)$. What we need to prove is that there exists an **X**-saturated set Σ such that $R_X(\Delta, \Sigma)$ and $\mathcal{C}(\Theta, \Sigma)$. This can be accomplished by the use of lemma 5.25 and a proof of that $\Box^-\Delta$ is an **X**-saturant of $(\neg\Theta)^{\complement}$:

$$\Box^-\Delta \trianglelefteq_X (\neg\Theta)^{\complement} \ (12).$$

If this claim holds then lemma 5.25 entails an **X**-saturated set Σ such that $R_X(\Delta, \Sigma)$ and $\Sigma \subseteq (\neg\Theta)^{\complement}$. Indeed, such a Σ is fulfilling, because the last conclusion is just another way of saying $\mathcal{C}(\Theta, \Sigma)$.

The only requirement which remains to be demonstrated is claim (12) above.

Suppose $\Box^-\Delta \vdash_X \Xi$ for a finite set $\Xi \subseteq \mathcal{L}^{\Box}$. We need to show $\Xi \cap (\neg\Theta)^{\complement} \neq \emptyset$.

Suppose that $\Xi \cap (\neg\Theta)^{\complement} = \emptyset$, or in other words, $\Xi \subseteq \neg\Theta$. This means that Ξ is of the form $\{\neg\theta_1, .., \neg\theta_n\}$ with $\theta_i \in \Theta$ for all $i \in \{1, .., n\}$[5]. By R-TRUE $\Box$ and L-MON we derive

$$\Delta \vdash_X \Box(\neg\theta_1 \vee ... \vee \neg\theta_n)^6.$$

A bit of (partial) propositional reasoning yields

$$\Delta \vdash_X \Box(\neg(\theta_1 \wedge ... \wedge \theta_n)).$$

Let $\theta := \theta_1 \wedge ... \wedge \theta_n$. Because Δ and Θ are **X**-saturated sets we conclude $\Box\neg\theta \in \Delta$ and $\theta \in \Theta$. The last observation yields $\neg\Box\neg\theta \in \Gamma$, since $R_X(\Gamma, \Theta)$. This means that $\neg\Delta \cap \Gamma \neq \emptyset$, which contradicts $\mathcal{C}(\Gamma, \Delta)$. This makes us conclude that $\Xi \cap (\neg\Theta)^{\complement} \neq \emptyset$.

The bisimulation result of $\mathcal{C}$ combined with the simple observation that $\mathcal{C}(\Gamma, \Delta) \Rightarrow V_X(\Gamma) \sim V_X(\Delta)$ for all $\Gamma, \Delta \in \mathfrak{Sat}_X$ affirms $\Gamma \cap \neg\Delta \Rightarrow \Gamma \sim_{M_X} \Delta$. ■

7.17. EXAMPLE. On the basis of this lemma we can immediately show that the system **A** in example 7.15 is complete with respect to $\mathfrak{A}$-validity. A simple proof of $M_A \in \mathfrak{A}$ together with the truth lemma 5.14 suffice.

Suppose that Γ and Δ are two **A**-saturated sets such that $R_A(\Gamma, \Delta)$, which

[4] If $\Gamma \cap \neg\Delta = \emptyset$ and $\alpha \in \Delta$, then $\neg\neg\alpha \in \Delta$ and hence $\neg\alpha \notin \Gamma$. This means $\Delta \cap \neg\Gamma = \emptyset$.

[5] The limiting case is $n = 0$ which means that $\Xi = \emptyset$.

[6] Remember that the empty disjunction is $\bot$.

means $\Box^{-}\Gamma \subseteq \Delta \subseteq \Diamond^{-}\Gamma$. Suppose $\alpha \in \Delta$. This entails $\Diamond\alpha \in \Gamma$, and by the special **A**-axiom this brings along $\neg\alpha \notin \Gamma$[7]. Because $\alpha \in \Delta$ have been chosen arbitrarily we have $\Delta \cap \neg\Gamma = \emptyset$. Lemma 7.16 above makes sure that $\Gamma \sim_{M_X} \Delta$. Therefore, $M_A \in \mathfrak{A}$.

Our second lemma provides a tool for detecting coherence on arbitrary finite accessibility distances in the canonical models of partial modal logics. It consists of an result for the canonical coherence relation which is analogous to the result which we have found in the previous section for the extension relation, lemma 7.11. This result can be established easily by means of corollary 7.12 and the previous lemma of this subsection.

7.18. LEMMA. Let **X** be a normal extension of **M**. If $\Gamma, \Delta \in \mathfrak{Sat}_X$ then

$$\Box^{-k}\Gamma \cap \neg\Delta = \emptyset \Rightarrow \exists\Delta' \in \mathfrak{Sat}_X : R^k_X(\Gamma, \Delta') \ \&\ \Delta' \sim_{M_X} \Delta, \text{ and}$$

$$\neg\Delta \cap \Diamond^{-k}\Gamma = \emptyset \Rightarrow \exists\Delta' \in \mathfrak{Sat}_X : R^k_X(\Gamma, \Delta') \ \&\ \Delta' \sim_{M_X} \Delta \text{ for all } k \in I\!N.$$

Proof. The result has been formulated somewhat too elaborate. The second claim is simply equivalent with the first because $\neg\Delta \cap \Diamond^{-k}\Gamma = \emptyset$ if and only if $\Box^{-k}\Gamma \cap \neg\Delta = \emptyset$.

Suppose $\Box^{-k}\Gamma \cap \neg\Delta = \emptyset$ for a pair Γ, Δ of **X**-saturated sets. We claim

$$\Box^{-k}\Gamma \trianglelefteq_X (\neg\Delta)^{\mathbf{C}} \quad (13).$$

Corollary 7.12 shows that if (13) holds, then the existence of a $\Delta' \in \mathfrak{Sat}_X$ such that $R^k_X(\Gamma, \Delta')$ and $\Delta' \subseteq (\neg\Delta)^{\mathbf{C}}$, or in other words $\Delta' \cap \neg\Delta = \emptyset$. According lemma 7.16 this latter conclusion is the same as $\Delta' \sim_{M_X} \Delta$. A proof of (13) would therefore be adequate.

Let $\Box^{-k}\Gamma \vdash_X \Sigma$ for certain $\Sigma \subseteq \mathcal{L}^{\Box}$, and suppose $\Sigma \cap (\neg\Delta)^{\mathbf{C}} = \emptyset$. This means $\Sigma \subseteq \neg\Delta$. This means there exists $\delta_1, .., \delta_n \in \Delta$ such that

$$\Box^{-k}\Gamma \vdash_X \neg\delta_1, ..., \neg\delta_n.$$

Due to R-TRUE $\Box$, L-MON and the de Morgan principles of **M**, we may rewrite this **X**-sequent by

$$\Gamma \vdash_X \Box^k\neg(\delta_1 \wedge ... \wedge \delta_n).$$

This means that $\Box^k\neg(\delta_1 \wedge ... \wedge \delta_n) \in \Gamma$ and because, $\Box^{-k}\Gamma \cap \Delta = \emptyset$, we conclude $\delta_1 \wedge ... \wedge \delta_n \notin \Delta$. This last conclusion contradicts the **X**-saturation of Δ, because $\Delta \vdash_X \delta_1 \wedge ... \wedge \delta_n$ (for all $i \in \{1, .., n\} : \delta_i \in \Delta$), and thus $\Sigma \cap (\neg\Delta)^{\mathbf{C}} \neq \emptyset$. ■

A completeness result for $\mathbf{rG}^{k,l}_{m,n} + \mathbf{M}$

The two results in the previous subsection establish the completeness result for the logics $\mathbf{rG}^{k,l}_{m,n} + \mathbf{M}$ – $\mathbf{rG}^{k,l}_{m,n}$ for short – with respect to the model class $\mathfrak{rG}^{k,l}_{m,n}$. To obtain this result, we use our saturation equation style for proving that that the $\mathbf{rG}^{k,l}_{m,n}$-canonical model is contained in the class $\mathfrak{rG}^{k,l}_{m,n}$.

[7] $\Diamond\neg\neg\alpha \in \Gamma$.

7.19. LEMMA. $M_{rG^{k,l}_{m,n}} \in \mathfrak{r}\mathfrak{G}^{k,l}_{m,n}$.

Proof. Suppose $\Gamma, \Delta, \Theta \in \mathfrak{Sat}_{rG^{k,l}_{m,n}}$ such that $R^k_{rG^{k,l}_{m,n}}(\Gamma, \Delta)$ and $R^m_{rG^{k,l}_{m,n}}(\Gamma, \Theta)$. We have to demonstrate the existence of a pair of $\mathbf{rG}^{k,l}_{m,n}$-saturated sets Ξ_1 and Ξ_2 such that $R^l_{rG^{k,l}_{m,n}}(\Delta, \Xi_1)$, $R^n_{rG^{k,l}_{m,n}}(\Theta, \Xi_2)$ and $\Xi_1 \sim_{M_{rG^{k,l}_{m,n}}} \Xi_2$.

To start with, we prove the following claim:

$$\Box^{-l}\Delta \trianglelefteq_{rG^{k,l}_{m,n}} (\neg\Box^{-n}\Theta)^{\complement} \quad (15).$$

Suppose that (15) is not the case. This yields a set $\Sigma \subseteq \mathcal{L}^{\Box}$ such that $\Box^{-l}\Delta \vdash_{rG^{k,l}_{m,n}} \Sigma$ (16) and $\Sigma \cap (\neg\Box^{-n}\Theta)^{\complement} = \emptyset$. In other words, $\Sigma \subseteq \neg\Box^{-n}\Theta$. This inclusion means that Σ consists only of negated formulae $\neg\varphi$ with $\Box^n\varphi \in \Theta$. This conclusion $\Box^{-l}\Delta \vdash_{rG^{k,l}_{m,n}} \Sigma$ can be rephrased by

$$\Box^{-l}\Delta \vdash_{rG^{k,l}_{m,n}} \neg\varphi_1 \vee .. \vee \neg\varphi_{n'} \text{ for certain } \{\Box^n\varphi_i\}_{i=1}^{n'} \subseteq \Theta.$$

Let $\varphi = \varphi_1 \wedge .. \wedge \varphi_{n'}$. Clearly $\Box^{-l}\Delta \vdash_{rG^{k,l}_{m,n}} \neg\varphi$. This yields $\Delta \vdash_{rG^{k,l}_{m,n}} \Box^l\neg\varphi$, and therefore, $\Diamond^k\Box^l\neg\varphi \in \Gamma$. Application of the $\mathbf{rG}^{k,l}_{m,n}$-axiom ensures $\Diamond^m\Box^n\varphi \notin \Gamma$. Because $\Theta \subseteq \Diamond^{-m}\Gamma$, we conclude $\Box^n\varphi \notin \Theta$. This contradicts $\Box^n\varphi_i \in \Theta$ for all $i \in \{1, .., n'\}$, because $\{\Box^n\varphi_i\}_{i-1}^{n} \vdash_{rG^{k,l}_{m,n}} \Box^n\varphi$. So, the saturation relation in (15) must hold.

Application of BSL to the result of (15) shows the existence of a $\Xi \in \mathfrak{Sat}_{rG^{k,l}_{m,n}}$ such that $\Box^{-l}\Delta \subseteq \Xi \subseteq (\neg\Box^{-n}\Theta)^{\complement}$. The first inclusion relation also entails $\Box^{-l}\Delta \cap \neg\Xi = \emptyset$, because $\neg\Xi \cap \Xi = \emptyset$. Lemma 7.18 thereupon guarantees the existence of a $\Xi_1 \in \mathfrak{Sat}_{rG^{k,l}_{m,n}}$ such that $R^l_{rG^{k,l}_{m,n}}(\Delta, \Xi_1)$ and $\Xi_1 \subseteq \Xi$. A bit of 'double-negation' reasoning also shows $\neg\Xi \cap \Box^{-n}\Theta = \emptyset$, and therefore (lemma 7.18), there must exist a $\Xi_2 \in \mathfrak{Sat}_{rG^{k,l}_{m,n}}$ such that $R^n_{rG^{k,l}_{m,n}}(\Theta, \Xi_2)$ and $\Xi_2 \subseteq \Xi$.

Because Ξ_1 and Ξ_2 have a common $\mathbf{rG}^{k,l}_{m,n}$-saturated extension Ξ, we immediately conclude $\Xi_1 \cap \neg\Xi_2 = \emptyset$ ($\neg\Xi \cap \Xi = \emptyset \Rightarrow \neg\Xi \subseteq \Xi_1 = \emptyset \Rightarrow \Xi_1 \cap \neg\Xi_2 = \emptyset$). To wind up with, lemma 7.16 shows that $\Xi_1 \sim_{M_{rG^{k,l}_{m,n}}} \Xi_2$. ■

7.20. THEOREM. $\mathbf{rG}^{k,l}_{m,n}$ is sound and complete with respect to $\mathfrak{r}\mathfrak{G}^{k,l}_{m,n}$.

7.4 Reflections

In this chapter we have shown a clear and strong model-theoretic correspondence for the class of Geach extensions of the minimal partial modal logic. Unlike the Geach extensions of minimal classical modal logic, these correspondences are not purely frame oriented. In a very straightforward manner Geach extensions of **M** are in general not frame complete according to the classical definition of this concept (page 192). This means that we cannot find completeness results with respect to model-classes sharing the characteristic frame conditions of these logics. These conditions are the same as the frame characteristics which correspond to the Geach extensions of **K**. A contrasting poverty of partial Geach logics with respect to their classical counterparts is the absence of contra-position. This deficit causes the following inequality for most of the partial Geach logics:

$$\mathbf{G}^{k,l}_{m,n} + \mathbf{M} \neq \mathbf{G}^{m,n}_{k,l} + \mathbf{M}.$$

Nevertheless, they share the same frame characterization (see figure 7.1 on page 192). This insight brings frame incompleteness immediately on the carpet for most of these partial Geach logics.

The deeper semantic reason of this failure is that there is nothing partial in the definition of Kripke frames. The only partial parameter of our possible worlds semantics is the assignment of truth-values to worlds. The classical definition of frames brings along

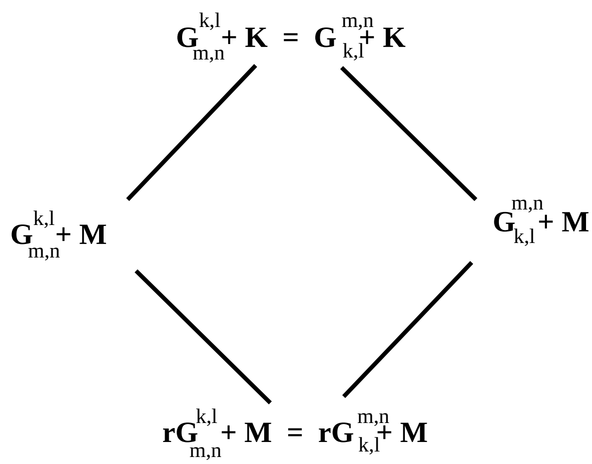
$$\Diamond^m \Box^n \varphi \models_{\mathfrak{Char}(G^{k,l}_{m,n})} \Box^k \Diamond^l \varphi.$$

Partiality forbids this kind of contra-positional switching. The logical independence of these contra-positional systems demands for a different style of correspondence between model-theory and modal axioms.

Besides restraining accessibilities, we have presented some equipment to structure the global valuation functions over these accessibilities as well. Conditions that intertwine information orders such as the extension order and the coherence relation with the accessibility relation turned out to be very successful. We have shown that this kind of model correspondences, gave us uniform completeness results for the class of Geach extensions of $\mathbf{M}$ by means of the extension order between possible worlds. We found a similar result for the weaker relativized form of Geach extensions of $\mathbf{M}$ by using the coherence relation instead.

All these characterizations are weakenings of the classical Geach frame characteristics. The classes of logic which we discussed in this chapter can be ordered according their deductive strength in the following systematic way.

7.21. FIGURE.



An intriguing question which arises from the correspondence analysis of this final chapter is whether the logics $\mathbf{G}^{k,l}_{m,n} + \mathbf{G}^{m,n}_{k,l} + \mathbf{M}$ are frame complete. Is addition of the contra-position of the Geach axiom strong enough to give a complete axiomatization of the frame condition of classical Geach logics? In [Thijsse 1992] such a result has been positively conjectured. This optimism arose from inspection of only simple Geach-logics with particularly low indices $(k, .., n)$. And indeed, from the simple observation which we have made in section 7.1 on intermediate worlds, we can recapture some frame completeness results for lowly indexed Geach logics. One of these results which have already been outlined is the frame completeness for the logic $\mathbf{T}^{\Box} + \mathbf{T}^{\Diamond}$. By a similar analysis we also find that we may reform big and small transitivity as normal transitivity. The same holds for symmetry and Euclidicity. In fact, by using theorem 7.6, we can formulate relatively large partial confirmation of Thijsse's conjecture. For example, as long as $l = 0$, we obtain a general frame-completeness result.

7.22. OBSERVATION. The logic $\Diamond^k\varphi \vdash \Box^m\Diamond^n\varphi \ + \ \Diamond^m\Box^n\varphi \vdash \Box^k\varphi \ + \ \mathbf{M}$ is frame complete. In other words, it is complete with respect to the class of frames $\langle W, R\rangle$ with

$$\forall x, y, z : R^k(x,y) \ \& \ R^m(x,z) \implies R^n(z,y).$$

Proof. Application of the correspondence result of theorem 7.8 to the separate Geach axioms in question yield: $M = \langle W, R, V\rangle \in \mathfrak{G}^{k,0}_{m,n} \cap \mathfrak{G}^{m,n}_{k,0}$ iff

$$R^k(x,y) \ \& \ R^m(x,z) \Rightarrow \exists v, w : R^n(z,v) \ \& \ R^n(z,w) \ \& \ v \sqsubseteq_M y \sqsubseteq_M w.$$

An easy generalization of theorem 7.6 for arbitrary accessibility distances shows that every such model can be reformed by taking $R^n(z,y)$ in the equation above as well. ■

Many other similar results might be recaptured for larger parts of the Geach class. Nevertheless, we are still very suspicious with regard to Thijsse's conjecture. It is still unclear to us that for every Geach axiom, the contra-positional addition yields a reduction to general frame completeness. It certainly needs far more technical insights than our simple result on intermediate worlds.

A very easy way of restoring pure frame correspondences for Geach-style logics is by adding them to the fully persistent minimal constructive modal logic **NM** of section 3.4. The frame correspondence of Geach logics over **NM** can be caught simply by replacing the bisimulation extension order by the information structure $\leq$ in our correspondence result of theorem 7.8. Still, the valuations interfere with such frame conditions by the postulated monotonicity constraint over information orders in these constructive possible world models[8].

Another interesting question is whether our results on partial Geach logics can be put more general. Is it possible to transfer more general correspondence theorems from classical modal logic, such as the powerful Sahlqvist theorem [Sahlqvist 1975]? We take this question as a challenge for future research. As modestly stated at the beginning of this chapter, this survey has been meant as an eye-opener. We state that additional information ordering to frame correspondences is a useful method to capture partial modal logics model-theoretically.

[8] For an extensive survey of correspondences on the basis of 'monotonic frames' many insights can be extracted from Rodenburg's thesis on intuitionistic correspondence theory [Rodenburg 1986].

References

P. Aczel [1968], "Saturated intuitionistic theories," in *Contributions to Mathematical Logic*, H. Schmidt, K. Schütte & H. Thiele, eds., North Holland, Amsterdam, 1–13.

R. Ahn [1992], "A type-theoretical approach to communication," *Think Quarterly* 1, 25–30.

R. Ahn & R.J. Beun [1991], "Voortgangsverslag Integratieproject DenK: Augustus 1990 – Januari 1991 (in Dutch)," Institute for Language Technology and AI (ITK), Research Memo 6, Tilburg.

L. Aiello, G. Amati & F. Pirri [1991], "Intuitionistic modal calculus for autoepistemic logic," in *Proceedings of the First World Conference on the Fundamentals of AI*, M. de Glas & D.M. Gabbay, eds., Angkor Publishing Company, Paris, 3–16.

S. Akama [1988], "Constructive predicate logic with strong negation and model theory ," *Notre Dame Journal of Formal Logic* 29, 18–27.

C.E. Alchourrón, P. Gärdenfors & D. Mackinson [1985], "On the logic of theory change: partial meet functions for contraction and revision," *Journal of Symbolic Logic* 50, 510–530.

D.E. Appelt [1985], *Planning English Sentences*, Studies in Natural Language Processing, Cambridge University Press, Cambridge.

J. Barwise [1981], "Scenes and other situations," *Journal of Philosophy* 78, 369–397.

J. Barwise [1987], "Noun phrases, generalized quantifiers and anaphora," in *Generalized Quantifiers*, P. Gärdenfors, ed., Reidel, Dordrecht.

J. Barwise [1988], *The Situation in Logic*, CSLI Lecture Notes #17, Stanford University, Stanford.

J. Barwise & J. Perry [1983], *Situations and Attitudes*, MIT Press, Cambridge (USA).

M.J. Beeson [1985], *Foundations of Constructive Mathematics: Metamathematical Studies*, Ergebnisse der Mathematik und ihrer Grenzgebiete: A Series of Modern Surveys in Mathematics #3.6, Springer Verlag, Berlin.

J.F.A.K. van Benthem [1976], "Modal Correspondence Theory," University of Amsterdam, PhD.-thesis, Amsterdam.

J.F.A.K. van Benthem [1983], *The Logic of Time*, Reidel, Dordrecht.

J.F.A.K. van Benthem [1984], *A Manual of Intensional Logic*, CSLI Lecture Notes #1, Stanford University, Stanford.

J.F.A.K. van Benthem [1985], *Modal Logic and Classical Logic*, Bibliopolis, Naples.

J.F.A.K. van Benthem [1986], "Partiality and non-monotonicity," *Logique et Analyse* 29, 225–247.

J.F.A.K. van Benthem [1991a], *Language in Action*, Studies in Logic and the Foundations of Mathematics #130, North Holland, Amsterdam.

J.F.A.K. van Benthem [1991b], "Logic and the Flow of Information," Institute for Logic, Language and Computation (ILLC), ILLC Prepublication Series LP-91-10, Amsterdam.

J.F.A.K. van Benthem [1993], "Reflections on epistemic logic," Manuscript , To appear in a special issue of *Logique et Analyse*.

J.F.A.K. van Benthem, D.J.N. van Eijck & A. Frolova [1993], "Changing Preferences," CWI Computer Science / Dpt. of Software Technology, Technical Report CS-R9310, Amsterdam.

R.J. Beun [1989], "The Recognition of Declarative Questions in Dialogues," PhD-Thesis, Tilburg University, Tilburg University.

S. Blamey [1986], "Partial Logic," in *Handbook of Philosophical Logic, vol. III*, D. Gabbay & F. Guenthner, eds., Reidel, Dordrecht, 1–70.

T. Borghuis [1993], "Interpreting modal natural deduction in type theory," in *Diamonds and Defaults*, M. de Rijke, ed., Synthese Library #229, Kluwer Academic Publishers, Dordrecht, 67–102.

M. Božić & K. Došen [1984], "Models for normal intuitionistic modal logic," *Studia Logica* XLIII, 15–43.

N.G. de Bruyn [1980], "A survey of the project Automath," in *To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalisms*, Seldin & Hindley, eds., Academic Press.

R. Bull & K. Segerberg [1984], "Basic modal logic," in *Handbook of Philosophical Logic*, D. Gabbay & F. Guenthner, eds. #II, Reidel, Dordrecht.

R.A. Bull [1965], "A modal extension of intuitionist logic," *Notre Dame Journal of Formal Logic* VI, 142–146.

H.C. Bunt [1989], "Information dialogues as communicative actions in relation to partner modelling and information processing," in *The Structure of Multimodal Dialogue*, M.M. Taylor, F. Néel & D.G. Bouwhuis, eds., North Holland, Amsterdam.

H.C. Bunt [1990a], "Modular incremental modelling of belief and intention," in *Proceedings of the Second Workshop on User-modelling and Dialogue Systems*, University of Hawaii, Honolulu, 160–189.

H.C. Bunt [1990b], "DIT – Dynamic interpretation in text and dialogue," in *Second Symposium on Logic and Language*, L. Kálmán & L. Pólos, eds., Akademiai Kiadó, Budapest.

H.C. Bunt [1994], "Dimensions of context and dialogue control," *Think Quarterly* 3.

H.C. Bunt, R.J. Beun, F.J.H. Dols, J.A. van der Linden & G.O. thoe Schwartenberg [1984], "The TENDUM dialogue system and its theoretical basis," in *IPO Annual Progress Report 19*, Institute for Perception Research, Eindhoven, 105–113.

B.F. Chellas [1980], *Modal Logic: An Introduction*, Cambridge University Press, Cambridge.

P.R. Cohen & H.J. Levesque [1990], "Intention is choice with commitment," *Artificial Intelligence* 42, 213–261.

M. Cresswell [1988], *Semantical Essays: Possible Worlds and Their Rivals*, Kluwer Academic publishers, Dordrecht.

K. Devlin [1991], *Logic and Information*, Cambridge University Press, Cambridge.

K. Došen [1985], "Models for stronger normal intuitionistic modal logics," *Studia Logica* XLIV, 39–70.

D.R. Dowty, R.E. Wall & S. Peters [1981], *Introduction to Montague Semantics*, Reidel, Dordrecht.

J. Doyle [1979], "A truth maintenance system," *Artificial Intelligence* 12, 231–272.

W.B. Ewald [1978], "Time, Modality and Intuitionism," PhD-thesis, Oxford.

W.B. Ewald [1986], "Intuitionistic tense and modal logic," *Journal of Symbolic Logic* 51.

R. Fagin & J.Y. Halpern [1988], "Belief, awareness and limited reasoning," *Artificial Intelligence* 34, 39–76.

J.E. Fenstad, T. Langholm & E. Vespren [1992], "Representations and interpretations," in *Computational Linguistics and Formal Semantics*, M. Rosner & R. Johnson, eds., Studies in Natural Language Processing, Cambridge University Press, Cambridge, 31–95.

M.J. Fischer & R.E. Ladner [1979], "Propositional dynamic logic of regular programs," *Journal of Computer and System Sciences* 18, 194–211.

G. Fischer Servi [1981], "Semantics for a class of intuitionistic modal calculi," in *Italian Studies in the Philosophy of Sciences*, M.L. Dalla Chiara, ed., Reidel, Dordrecht, 59–72.

M.C. Fitting [1969], *Intuitionistic Logic Model Theory and Forcing*, Studies in Logic and the Foundations of Mathematics, North Holland, Amsterdam.

M.C. Fitting [1992], "Many-valued modal logics," *Fundamenta Informaticae* 15, 235–254.

D.M. Gabbay [1982], "Intuitionistic basis for non-monotonic logic," in *Proceedings of The 6th Conference on Automated Deduction*, Lecture Notes in Computer Science #138, Springer Verlag, Heidelberg, 260–273.

L.T.F. Gamut [1991], *Logic, Language and Meaning*, University of Chicago Press, Chicago.

P. Gärdenfors [1975], "Qualitative probability as an intensional logic," *Journal of Philosophical Logic* 4, 171–185.

P. Gärdenfors [1988], *Knowledge in Flux: Modelling the Dynamics of Epistemic States*, MIT Press, Cambridge, MA.

K. Gödel [1933], "Eine Interpretation des Intuitionistischen Aussagenkalküls," *Ergebnisse eines Mathematischen Kolloquiums* 4, 34–40.

R. Goldblatt [1982], *Axiomatising the Logic of Computer Programming*, Lecture Notes in Computer Science #130, Springer Verlag, Berlin.

H.P. Grice [1975], "Logic and conversation," in *Speech Acts, Syntax and Semantics*, P.Cole & J. Morgan, eds. #III: Speech Acts, Academic Press, New York.

J. Groenendijk & M. Stokhof [1991], "Dynamic Predicate Logic," *Linguistics and Philosophy* 14, 39–100.

Y. Gurevich [1977], "Intuitionistic logic with strong negation," *Studia Logica* 36, 49–59.

J.Y. Halpern [1986], "Reasoning about knowledge: an overview," in *Theoretical Aspects of Reasoning About Knowledge: The 1986 Conference*, J.Y. Halpern, ed., Morgan Kaufmann, Los Altos.

J.Y. Halpern & Y.O. Moses [1984], "Towards a theory of knowledge and ignorance: a preliminary report ," in *Proceedings of the AAAI Workshop on Non-Monotonic Reasoning*, New Paltz, NY, 125–143.

J.Y. Halpern & Y.O. Moses [1990], "Knowledge and common knowledge in a distributed environment," *Journal of the Association for Computing Machinery* 37, 549–587.

J.Y. Halpern & Y.O. Moses [1992], "A guide to the completeness and complexity of modal logics of knowledge and belief," *Artificial Intelligence* 54, 319–379.

D. Harel [1984], "Dynamic logic," in *Handbook of Philosophical Logic, Vol. II:*, D. Gabbay & F. Guenther, eds., Reidel, Dordrecht.

I. Heim [1982], "The Semantics of Definite and Indefinite Noun Phrases," University of Massachusetts, Ph.D. Thesis, Amherst.

M. Hennessy [1988], *Algebraic Theory of Processes*, MIT Press Series in the Foundations of Computer Science #VI, MIT Press, Cambridge, MA.

A. Heyting [1956], *Intuitionism: An Introduction*, Studies in Logic and The Foundations of Mathematics , North Holland, Amsterdam.

J. Hintikka [1962], *Knowledge and Belief: An Introduction to the Logic of the Two Notions*, Cornell University Press, Ithaca N.Y..

J. Hintikka [1969], *Models for Modalities: selected essays*, Reidel, Dordrecht.

W. van der Hoek [1992], "Modalities for Reasoning about Knowledge and Quantities," Ph.D. Thesis, Free University Amsterdam, Amsterdam.

W. van der Hoek, J. Jaspars & E. Thijsse [1993], "Honesty in partial logic ," RUU Dpt of Computer Science, Technical Report RUU-CS-93-32, Utrecht, To appear in the Proceedings of 9th Conference on the Principles of Knowledge Representation (KR'94).

A. Huertas [1994], "Lógica Modal de Predicados y Lógica-clásica heterogénea Trivalorada," Universidad Autónoma de Barcelona: Departamento de Matemàticas, Ph-D. Thesis, Barcelona.

G.E. Hughes & M.J. Cresswell [1984], *A Companion to Modal Logic*, Methuen, New York.

J.O.M. Jaspars [1991a], "Theoretical Circumscription in Partial Modal Logic," in *Logics in AI / JELIA'90*, J. van Eijck, ed., Lecture Notes in Artificial Intelligence #478, Springer Verlag, Heidelberg, 303–318.

J.O.M. Jaspars [1991b], "A generalization of stability and its application to circumcription of positive introspective knowledge," in *Computer Science Logic: Proceedings 4th Workshop, CSL'90*, E. Börger, H. Kleine Büning, M.M. Richter & W. Schönfeld, eds., Lecture Notes in Computer Science #533, Springer Verlag, Heidelberg, 289–299.

J.O.M. Jaspars [1991c], "Fused modal logic and inconsistent belief," in *Proceedings of the First World Conference on the Fundamentals of AI*, M. de Glas & D.M. Gabbay, eds., Angkor Publishing Company, Paris, 267–275.

J.O.M. Jaspars [1993], "Normal forms in partial modal logic," in *Proceedings of the XXXVIII Banach Center Semester on Algebraic Methods in Logic and Their Computer Science Applications*, C. Rauszer, ed., Polish Academy of Sciences, Warsaw, 37–50.

A.J.I. Jones [1983], "Communication and Meaning: An Essay in Applied Modal Logic," Dordrecht.

D. de Jongh & F. Veltman [1988], "Intensional Logic," Lecture Notes, Dpt. of Philosophy, University of Amsterdam.

H. Kamp [1983], "A Scenic Tour through the Land of Naked Infinitives," Manuscript.

H. Kamp [1984], "A theory of truth and semantic representation," in *Truth, Interpretation and Information*, J. Groenendijk, Th. Jansen & M. Stokhof, eds., Foris Publications, Dordrecht.

S.C. Kleene [1945], "On the interpretation of intuitionistic number theory," *Journal of Symbolic Logic* 10, 109–124.

D. Kozen & R. Parikh [1981], "An elementary proof of the completeness of PDL," *Theoretical Computer Science* 14, 113–118.

E. Krahmer [1994], "Partial Dynamic Predicate Logic," Institute for Language Technology and AI (ITK), Research Report 48, Tilburg.

R.E. Ladner [1977], "The computational complexity of provability in systems of modal logic," *SIAM Journal on Computing* 6, 467–480.

I. Lakatos [1976], *Proofs and Refutations: The Logic of Mathematical Discovery*, Cambridge University Press, Cambridge.

G. Lakemeyer [1991], "A computionally attractive first-order logic of belief," in *Logics in AI/JELIA'90*, J. van Eijck, ed., Lecture Notes in Artificial Intelligence #478, 333–347.

F. Landman [1986], *Towards a Theory of Information: The Status of Partial Objects in Semantics* , Groningen Amsterdam Studies in Semantics #6, Foris, Dordrecht.

T. Langholm [1988], *Partiality, Truth and Persistence*, CSLI Lecture Notes #15, Stanford University, Stanford.

S. Lapierre [1990], "A Function Partial Semantics for Intensional Logic," Institute for Logic, Language and Computation (ILLC), ILLC Prepublication Series LP-90-12, Amsterdam.

E.J. Lemmon & D.S. Scott [1977], *The 'Lemmon Notes': An Introduction to Modal Logic*, Blackwell, Oxford.

W. Lenzen [1978], "Recent work in epistemic logic," *Acta Philosophica Fennica* 30, 1–219.

W. Lenzen [1980], *Glauben, Wissen und Warscheinlichkeit*, Springer Verlag, Heidelberg.

H.J. Levesque [1984], "A logic of implicit and explicit belief," in *Proceedings of Sixth Conference of the American Association of Artificial Intelligence*, Morgan Kaufmann, Los Altos, 198–202.

D.K. Lewis [1969], *Convention*, Harvard University Press, Cambridge MA.

E.G.K. López-Escobar [1972], "Refutability and elementary number theory," *Indagationes Mathematicae* 32, 362–374.

R. Montague [1974], in *Formal Philosophy, selected papers of Richard Montague*, R. Thomason, ed., Yale University Press, New Haven.

R.C. Moore [1980], "Reasoning About Knowledge and Action," SRI International, Technical note, Menlo Park, CA.

R.C. Moore [1983], "Semantical considerations on non-monotonic reasoning," in *Proceedings of the Eighth International Joint Conference on Artificial Intelligence*, Morgan Kaufmann, Los Altos, 272–279.

R.C. Moore [1984], "Possible world semantics for autoepistemic logic," in *Proceedings of the AAAI Workshop on Non-Monotonic Reasoning*, New Paltz, NY, 344–354.

O. Morikawa [1989], "Some modal logics based on a three valued logic," *Notre Dame Journal of Formal Logic* 30, 130–137.

R.A. Muskens [1989a], "A relational formulation of the theory of types," *Linguistics and Philosophy* 12, 325–346.

R.A. Muskens [1989b], "Meaning and Partiality," University of Amsterdam, PhD.-thesis.

D. Nelson [1949], "Constructible falsity," *Journal of Symbolic Logic* 14, 16–26.

D. Nelson [1959], "Negation and seperation of concepts in constructive systems ," in *Constructivity in Mathematics*, A. Heyting, ed., Studies in Logic and The Foundations of Mathematics, North Holland, Amsterdam, 208–225.

D. Pearce & G. Wagner [1990], "Reasoning with negative information: Strong negation in logic programs," *Acta Philosophica Fennica* 49, 430–453.

C.R. Perrault [1989], "An application of default logic to speech act theory," in *Intentions and Communication*, J. Morgan & M.E. Pollack, eds., MIT Press, Cambridge MA.

G. Plotkin & C. Stirling [1986], "A framework for intuitionistic modal logics," in *Theoretical Aspects of Reasoning About Knowledge: The 1986 Conference*, J.Y. Halpern, ed., Morgan Kaufmann, Los Altos, 399–406.

V. Pratt [1980], "Application of modal logic to programming," *Studia Logica* 39, 257–274.

A.N. Prior [1979], *Time and Modality*, Greenwood Press, Westport, Connecticut.

R. Reiter [1980], "A logic for default reasoning," *Artificial Intelligence Journal* 13, 81–132.

N. Rescher & R. Brandom [1980], *The Logic of Inconsistency*, Basil Blackwell, London.

M. de Rijke [1992], "A System of Dynamic Modal Logic," Institute for Logic, Language and Computation (ILLC), ILLC Prepublication Series LP-92-08, Amsterdam , Appears also in Journal of Philosophical Logic.

M. de Rijke [1993], "Extending Modal Logic," Institute for Logic, Language and Computation (ILLC), PhD.-thesis, ILLC-dissertation series 93-4, Amsterdam.

P.H. Rodenburg [1986], "Intuitionistic Correspondence Theory," University of Amsterdam, PhD.-thesis.

R. Routley [1974], "Semantical analyses of propositional systems of Fitch and Nelson," *Studia Logica* 33, 283–298.

H. Sahlqvist [1975], "Completeness and correspondence in the first and second order semantics of modal logics," in *Proceedings of the 3rd Scandinavian Logic Symposium*, North Holland, Amsterdam.

P.K. Schotch, J.B. Jensen, P.F. Larsen & E.J. McLellan [1978], "A note on three-valued modal logic," *Notre Dame Journal of Formal Logic* XIX, 63–68.

J.R. Searle [1969], *Speech Acts*, Cambridge University Press, Cambridge.

J.R. Searle [1983], *Intentionality: An Essay in the Philosophy of Mind*, Cambridge University Press, Cambridge.

K. Segerberg [1965], "Some modal logics based on three-valued logic," *Theoria* 33, 53–71.

Y. Shoham [1988], *Reasoning About Change: Time and Causation From the Standpoint of Artificial Intelligence*, The MIT Press Series in Artificial Intelligence, MIT Press, Cambridge (MA).

A.K. Simpson [1993], "The Proof Theory and Semantics of Intuitionistic Modal Logic," PhD-thesis, Edinburgh.

E. Spaan [1993], "Complexity of Modal Logics," Institute for Logic, Language and Computation (ILLC), PhD.-thesis, Amsterdam.

R.C. Stalnaker [1979], "Assertion," in *Syntax and Semantics Vol.9: Pragmatics*, P. Cole, ed., Academic Press, New York, 315–332.

W.P. van Stigt [1990], *Brouwer's Intuitionism*, Studies in The History and Philosophy of Mathematics #2, North Holland, Amsterdam.

C. Stirling [1987], "Modal logics for communicating systems," *Theoretical Computer Science* 49, 311–347.

E.G.C. Thijsse [1990], "Partial propositional and modal logic; the overall theory," in *Proceedings of the Seventh Amsterdam Colloquium*, M. Stokhof & L. Torenvliet, eds., ITLI, Amsterdam, 555–579.

E.G.C. Thijsse [1992], in *Partial Logic and Knowledge Representation*, Eburon, Delft, PhD-Thesis, Tilburg University.

R.H. Thomason [1968], "On the strong semantical completeness of the intuitionistic predicate logic," *Journal of Symbolic Logic* 33:1, 1–7.

R.H. Thomason [1969], "A semantical study of constructible falsity," *Zeitschrift für Mathematischen Logik und Grundlagen der Mathematik* 15, 247–257.

A.S. Troelstra & D. van Dalen [1990], *Constructivism in Mathematics vol. I*, Studies in Logic and The Foundations of Mathematics #121, North Holland, Amsterdam.

R. Turner [1984], *Logics for Artificial Intelligence*, Ellis Horwood, Chicester.

A. Urquhart [1986], "Many valued logics," in *Handbook of Philosophical Logic III*, D. Gabbay & F. Guenther, eds., Reidel, Dordrecht.

F. Veltman [1981], "Data semantics," in *Formal Methods in the Study of Language, Vol. 2*, J. Groenendijk, Th. Janssen & M. Stokhof, eds., Mathematical Center Tracts, Amsterdam.

F. Veltman [1985], "Logics for Conditionals," University of Amsterdam, Ph.-D. thesis, Amsterdam.

F. Veltman [1991], "Defaults in Update Semantics," Institute for Logic, Language and Computation (ILLC), ILLC Prepublication Series LP-91-02, Amsterdam, Appears also in Journal of Philosophical Logic.

Y. Venema [1991], "Many-Dimensional Modal logic," Department of Mathematics and Computer Science, University of Amsterdam, Ph.D.-thesis, Amsterdam.

G. Wagner [1991], "Logic programming with strong negation and inexact predicates ," *Journal of Logic and Computation* 1, 835–859.

H. Wansing [1992a], "The Logic of Information Structures," PhD-thesis, Freie Universität Berlin, Berlin.

H. Wansing [1992b], "Sequent calculi for normal modal logics," Institute for Logic, Language and Computation (ILLC), ILLC Prepublication Series LP-92-12, Amsterdam.

H. Weigand [1993], "Deontic aspects of communication," in *Deontic Logic in Computer Science: Normative System Specification*, J-J. Ch. Meyer & R.J. Wieringa, eds., John Wiley & Sons, Chicester, 259–273.

D. Wijesekera [1990], "Constructive modal logics I," *Annals of Pure and Applied Logic* 50, 271–301.

T. Williamson [1992], "On intuitionistic modal epistemic logic," *Journal of Philosophical Logic* 21, 63–89.

L. Zhisheng Huang [1991], "Logics for belief dependence," in *Computer Science Logic: Proceedings 4th Workshop, CSL'90*, E. Börger, H. Kleine Büning, M.M. Richter & W. Schönfeld, eds., Lecture Notes in Computer Science #533, Springer Verlag, Heidelberg, 274–288.

L. Zhisheng Huang & P. van Emde Boas [1993], "Information Acquisition from Multiple-Agent Resources: Abstract," Institute for Logic, Language and Computation (ILLC), ILLC Prepublication Series LP-93-08, Amsterdam.

Samenvatting

De doelstelling van het proefschrift is tweeledig. Enerzijds biedt het een mathematisch logische behandeling van partiële modale logica's en anderzijds laat het zien dat deze logica's geschikt zijn voor het beschrijven van groei en afname van informatie door middel van communicatieve handelingen. Deze twee doeleinden zijn ook onderscheiden door middel van twee verschillende delen, die beiden bestaan uit een drietal hoofdstukken. Het eerste deel presenteert de logische apparatuur om dynamische interpretatie van communicatieve handelingen te geven (hoofdstukken 2, 3 en 4). Het tweede gedeelte behandelt een aantal belangrijke abstract mathematische thema's met betrekking tot deze voorgestelde logica's (hoofdstukken 5, 6 en 7). De twee delen worden voorafgegaan door een algemene inleiding (hoofdstuk 1).

Een korte inhoud van de verschillende hoofdstukken wordt hieronder gegeven.

De **inleiding** geeft een aantal argumenten waarom de combinatie van partialiteit en modaliteit interessant is voor algemene dynamische semantiek, en in het byzonder voor de formele semantiek van communicatieve handelingen. De twee belangrijkste argumenten zijn dat onwaarheid onderscheiden wordt van afwezigheid van waarheid en dat partialiteit en modaliteit een dynamiek toestaat van *constructie* en *eliminatie* van mogelijke werelden.

Hoofdstuk 2 introduceert de minimale partiële modale logica. Allereerst zetten we haar eenvoudige semantiek voor, en vervolgens definiëren we een kort sequentensysteem voor deze logica. Daarna beschrijven we hoe informatieordeningen over partiële mogelijke werelden gedefinieerd kunnen worden. **Hoofdstuk 3** laat zien hoe de extensie-orde, zoals gedefinieerd in hoofdstuk 2, gebruikt kan worden om constructief-dynamische uitbreidingen van partiële modale logica te definiëren. **Hoofdstuk 4** beschrijft verschillende "epistemische" uitbreidingen van de constructieve modale logica's van hoofdstuk 3. Deze zijn bedoeld om dynamische interpretaties te stipuleren voor eenvoudige communicatieve handelingen. Naast axiomatische versterkingen worden ook talige uitbreidingen gedefinieerd, zoals logica's met additionele "gemeenschappelijkgeloofs"-operatoren en preferentiële operatoren.

Met dit laatste hoofdstuk is deel I afgerond. Deel II biedt een wiskundige analyse van de systemen zoals die beschreven zijn in deel I.

Hoofdstuk 5 laat zien dat door een relatief eenvoudige abstractie van de notie van maximale consistentie uit de klassieke (modale) logica toegankelijke volledigheidsbewijzen voor de partiële modale logica te geven zijn. In **hoofdstuk 6** tonen we beslisbaarheid aan op basis van de eindige modeleigenschap voor de logica's uit deel I. Bovendien bewijzen we volledigheid voor eindige sequenten voor de niet-compacte logica's met "gemeenschappelijk-geloofs"-operatoren. **Hoofdstuk 7** heeft een minder directe betekenis met betrekking tot deel I. Hierin geven we een opzet voor correspondentie-theorie voor partiële modale logica. We laten zien hoe informatie-ordeningen met relationele frame-karakteristieken interacteren in partiële mogelijke wereldensemantiek als het gaat om het vinden van correspondenties voor modale axioma's.

Previous titles in the ILLC Dissertation Series:

Transsentential Meditations; Ups and downs in dynamic semantics
Paul Dekker
ILLC Dissertation series, 1993-1

Resource Bounded Reductions
Harry Buhrman
ILLC Dissertation series, 1993-2

Efficient Metamathematics
Rineke Verbrugge
ILLC Dissertation series, 1993-3

Extending Modal Logic
Maarten de Rijke
ILLC Dissertation series, 1993-4

Studied Flexibility
Herman Hendriks
ILLC Dissertation series, 1993-5

Aspects of Algorithms and Complexity
John Tromp
ILLC Dissertation series, 1993-6

The Noble Art of Linear Decorating
Harold Schellinx
ILLC Dissertation series, 1994-1

Generating Uniform User-Interfaces for Interactive Programming Environments
Jan Willem Cornelis Koorn
ILLC Dissertation series, 1994-2

Process Theory and Equation Solving
Nicoline Johanna Drost
ILLC Dissertation series, 1994-3