PRODUCTS OF TOPOLOGICAL MODAL LOGICS

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Approved for the University Committee on Graduate Studies.

Dedication

This thesis is dedicated to my late mother Emina and my father Milivoj who taught me the meaning of courage and dedication.

Abstract

This thesis is about logic of space. In it we use various techniques of modal logic and topology to devise a class of increasingly stronger logics of space. The underlying intuition is that for lots of applications spatial intuition and spatial reasoning seem basic. And this not only in applications such as guiding robots or automated vehicles through real three-dimensional space, but also for such diverse applications as reasoning about knowledge, processing and updating of information. The thesis makes some initial steps in understanding the structure of space with efficient languages of modal logic, with an ultimate aim of applying them to cognitive settings.

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Contents

Chapter 1

Introduction

This thesis is about modal languages for spatial structures, and some new logics to which these give rise. But does space need a new logic? After all, since the Greeks, geometry has done fine as a mathematical theory, and we know a lot about its properties, both at the level of formal derivations and of meta-properties of axiom systems. And the 20th century has even added further mathematical theories of space, such as topology, and mathematical morphology, based on linear algebra. All this is true, and yet it seems of interest to describe spatial structures by the simplest possible logical formalisms bringing out their combinatorics in a perspicuous, and perhaps even low-complexity manner. The guiding example for us here is Tarski's analysis of topology in a modal language, with modal box as an interior operator, showing that the modal logic **S4** is the complete and decidable logic of at least this basic structure. Tarski and McKinsey [56] also showed that **S4** is the logic of specific structures such as the reals with their standard order topology, discovering tight connections between Kripke models and patterns of subsets on metric spaces like the reals.

Interest in modal logics for reasoning about space is picking up these days. To mention just a few sources, there is the work on topological spaces by the Georgian School of L. Esakia [34], [33] and his students (G. and N. Bezhanishvili, D. Gabelaia [20], [4], [21], [39]), which is being extended these days to richer hybrid languages allowing for reference to specific named points and other expressive extensions. Another strand is the pioneering work by Mints [58] on simplifications of the original Tarski-McKinsey proof, which has in turn inspired work in Amsterdam and New Mexico ([4],[1], [20]). Another line concerns extensions of the basic topological framework with continuous maps, adding a temporal, or dynamic logic dimension to the system ([30], [31], [32], [6], [48], and also the forthcoming chapter by Mints and Kremer in [2]).

Related again to the original topological setting is the epistemic-topological framework of [59]. And there are many other modal approaches, including logics of affine and projective geometries ([66], [7], [8]). Much of this work will be collected in the forthcoming *Handbook of Spatial Reasoning* [2], edited by M. Aiello, J. van Benthem, and I. Pratt. Of course, there are also other possible logical approaches, using first-order languages in the tradition of Tarski's *Elementary Geometry*, or higher-order ones. For these, the same Handbook is a good source (cf. the chapters by Pratt, Balbiani, Goranko & Vakarelov, and Andréka & Németi), while Kerdiles' Thesis [46] is a nice sample of spatial analysis in a minimal framework of Peircean 'conceptual graphs'.

In the general area of modal logics of space, the specific topic of this thesis can be described as follows. To us, spatial structure only comes into its own with at least two dimensions, and one very natural way of creating two-dimensional structures is by the formation of products. Product constructions for logical purposes have been proposed for many different reasons (cf. [12]), and there is quite some literature on this subject in modal logic (cf. [38]), sparked off largely by Gabbay and Shehtman [37]. But the main emphasis so far has been on using products of Kripke models with binary accessibility relations, as a sort of minimal way of combining information from different dimensions. Instead, we shall be mainly interested in products of topological spaces with various topologies on them. To us this seems an interesting area for experimenting with new spatial languages. At the same time, however, our approach also generalizes that of Gabbay and Shehtman, and it even provides a more flexible setting for combination of modal logics.

We now turn to the description of our specific topics and results.

Chapter 2, *Modal Logics of Products of Topologies*, introduces the horizontal and vertical topologies on the product of topological spaces and studies their relationship with the standard product topology. Our main contributions are:

1. A new definition of topological products with horizontal and vertical topologies introduced on the Cartesian product as a generalization of the Gabbay-Shehtman product construction for Kripke frames,

2. A completeness result showing that the minimal logic of this class of topological spaces is the fusion logic $\mathbf{S4} \oplus \mathbf{S4}$. Thus, in the topological product setting, various 'interference principles' that hold in the relational case and turn out to be notoriously difficult to analyze computationally are absent.

3. The systematic correspondence analysis of additional interaction axioms and their topological and set-theoretic content,

4. It is shown that the modal logic of products of topological spaces with horizontal and vertical topologies is the fusion $\mathbf{S4} \oplus \mathbf{S4}$.

5. We axiomatize the modal logic of products of topological spaces with horizontal, vertical, and standard product topologies.

6. We prove that both of these logics are complete for the product of rational numbers $\mathbb{Q} \times \mathbb{Q}$ with the appropriate topologies.

The material in this chapter is based largely on two papers: [16] and [52]. The first one has been coauthored with J. van Benthem, G. Bezhanishvili, and B. ten Cate and is in the process of review by *Studia Logica*. The latter one is with B. Löwe and will appear in The *Logic Journal of the IGPL*.

Chapter 3 investigates what happens when we introduce a linear ordering in our topological spaces. This corresponds to the natural idea of looking in a space along some direction. Our approach combines ideas from temporal logic ([62], [26], [44]) with topological semantics, but with the following new twist. Some earlier attempts at studying this combination have, as in the above, considered products of relational models, witness Venema's well-known modal 'compass logic' [71]. The result is high modal expressive power (leading to undecidability), and on the other hand poor topological expressive power. Other authors have added an explicit order to the topology: Shehtman is an important example [64]. This also leads to logics of clear spatial interest, but hard to axiomatize: cf. [21]. Our new proposal is to strike a middle ground, and have modal operators whose interpretation combines the order and the topology, stating that a proposition is true in an open neighborhood along a certain direction. Moreover, as in Chapter 2, we show how to lift this to a family of modalities in products of ordered topological spaces. Our main results are

1. A complete axiomatization of the the logic of generalized order topologies on our models in this language on one topo-dimension,
2. A complete axiomatization in one dimension for $\mathbb{Q}$,
3. A complete axiomatization in one dimension for $\mathbb{N}$,
4. A completeness result for the specific structure $\mathbb{Q}\times\mathbb{Q}$ viewed as a product space,
5. A completeness result for the specific structure $\mathbb{N}\times\mathbb{N}$ viewed as a product space.

We believe that our arguments will also settle the complete logic of order topologies, but this still remains to be clarified.

Finally, we make a perhaps surprising turn. Topological semantics in its earlier phases has always served a dual purpose. On the one hand, it described spatial structures, but on the other, it also served as a modelling for intuitionistic logic, and hence as a sort of epistemic semantics. In Chapter 4, we take this idea further and introduce topological semantics for epistemic logic, generalizing the usual Hintikka-style relational models [43]. By itself, this idea has also been pursued in [59], but they use a somewhat idiosyncratic language, instead of going all out as we do, identifying agents with topologies on some given space. Our generalization turns out to have several new consequences. One is that we can model the various different senses of common knowledge proposed in Barwise, for which no satisfactory modelling existed.

Another is that we can discuss combinations of groups of agents in greater generality than has been possible so far.

Our results include:

1. A new analysis of modal fixed-point procedures in topological semantics,
2. A completeness result for a multi-agent topological semantics,
3. Introduction of several new kinds of topological 'agents'.

This chapter is based on the paper *The Geometry of Knowledge* with J. van Benthem that appeared in [22].

Finally in the concluding chapter, we point out some further related results that we have obtained in the area and several fruitful directions for further research. The main results we mention are:

1. Some decidability questions for the Topo-Compass Logic of Chapter 3,
2. A generalization of products of modal logics into a hybrid logic setting,
3. A completeness transfer result similar to that in the relational setting,
4. An undecidability result for a temporal logic in a hybrid product setting.

Our hope is that this thesis makes a contribution to the recent upsurge of interest in both the logic of space, and the products of modal logics.

Chapter 2

Modal Logics for Products of Topologies

We introduce the horizontal and vertical topologies on the product of topological spaces, and study their relationship with the standard product topology. We show that the modal logic of products of topological spaces with horizontal and vertical topologies is the fusion $\mathbf{S4} \oplus \mathbf{S4}$. We axiomatize the modal logic of products of topological spaces with horizontal, vertical, and standard product topologies. We prove that both of these logics are complete for the product of rational numbers $\mathbb{Q} \times \mathbb{Q}$ with the appropriate topologies

2.1 Introduction

The study of products of Kripke frames and their modal logics was initiated by Shehtman [63]. A systematic study of multi-dimensional modal logics of products of Kripke frames can be found in Gabbay and Shehtman [37], and for an up to date account of the most important results in the field we refer to Gabbay et al. [38]. We recall that for given two frames $\mathcal{F} = \langle W, S \rangle$ and $\mathcal{G} = \langle V, T \rangle$, the 'horizontal' and 'vertical' relations on the product $W \times V$ are defined as follows.

$$(w, v)R_1(w', v') \text{ iff } wSw' \text{ and } v = v'$$
$$(w, v)R_2(w', v') \text{ iff } w = w' \text{ and } vTv'$$

Amongst many other results, Gabbay and Shehtman proved that if L_1 and L_2 are modal logics complete with respect to frame classes $\mathbb{F}_1$ and $\mathbb{F}_2$ defined by universal Horn conditions and closed under taking disjoint unions, then the logic $L_1 \times L_2$ of the class of products

$$\mathbb{F}_1 \times \mathbb{F}_2 = \{\langle W \times V, R_1, R_2\rangle : \langle W, S\rangle \in \mathbb{F}_1 \text{ and } \langle V, T\rangle \in \mathbb{F}_2\}$$

is axiomatized by the fusion $L_1 \oplus L_2$ plus the two additional principles of commutation $com = \Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p$ and convergence (also known as the Church-Rosser principle) $chr = \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p$. In particular, since **S4** is complete with respect to the universal Horn class of reflexive and transitive frames, the product **S4** $\times$ **S4** is axiomatized as **S4** $\oplus$ **S4** plus *com* and *chr*.

It is known that topological semantics generalizes Kripke semantics for **S4**. In this paper we consider products of topological spaces. We generalize the notions of horizontal and vertical relations to horizontal and vertical topologies and study their relationship with the standard product topology. We show that the modal logic of products of topological spaces with horizontal and vertical topologies is **S4** $\oplus$ **S4**, and the interaction principles *com* and *chr* only become valid when further restrictions are made on the topological spaces under consideration.

Since the topological setting strongly suggests adding the 'true product topology', we also investigate the modal logic of products of topological spaces with all three topologies: horizontal, vertical, and the standard product topology. We show that the modal operator associated with the product topology is not definable in terms of the modal operators associated with the horizontal and vertical topologies, and we axiomatize the modal logic of products of topological spaces with all three topologies.

This chapter is organized as follows. In Section 2.2 we recall some basic facts about topological semantics of **S4** and present a new proof of completeness of **S4** with respect to the rationals. We also review the definitions of the fusion **S4** $\oplus$ **S4** and the product **S4** $\times$ **S4**. In Section 2.3 we introduce the horizontal and vertical topologies, and investigate their relationship with the standard product topology. Section 2.4 is concerned with the commutation and convergence principles in the topological setting, while Sections 2.6 and 2.7 contain completeness results for modal languages with operators corresponding to the horizontal, vertical, and standard product topologies. In the concluding Section 2.8 we point out some of the remaining open questions.

2.2 Preliminaries

2.2.1 Topological completeness of S4

If we interpret the modal operators $\Box$ and $\Diamond$ in topological spaces as the interior and closure operators, then the complete modal logic of all topological spaces is **S4** (McKinsey and Tarski [56]). A much stronger result, also due to McKinsey and Tarski, states that **S4** is in fact the complete modal logic of any metric separable dense-in-itself space. In particular, **S4** is the complete modal logic of the real line $\mathbb{R}$, the rational line $\mathbb{Q}$, or the Cantor space $\mathbf{C}$. An alternative proof of completeness of **S4** with respect to $\mathbf{C}$ can be found in [58], and that with respect to $\mathbb{R}$ in [4]. In the subsequent sections we will need completeness of **S4** with respect to $\mathbb{Q}$. In order to make the paper self-contained, we present here an alternative proof of this fact, which might be of an independent interest.

To this end, recall that a topological space is a structure $\langle X, \tau \rangle$ where $\tau \subseteq \wp(X)$ contains $\emptyset$ and W and is closed under arbitrary unions and finite intersections. The elements of τ are called *open sets*. If, in addition, τ is closed under arbitrary intersections, then $\langle X, \tau \rangle$ is said to be *Alexandroff*. A topological model is a structure $M = \langle X, \tau, \nu \rangle$, where $\langle X, \tau \rangle$ is a topological space and ν is a valuation assigning subsets of X to propositional variables of the modal language. Then for $x \in X$, the modal operators $\Box$ and $\Diamond$ are interpreted as follows.

$$
\begin{array}{lll}
x \models \Box\varphi & \text{iff} & \exists U \in \tau : x \in U \text{ and } \forall y \in U (y \models \varphi) \\
x \models \Diamond\varphi & \text{iff} & \forall U \in \tau : \text{if } x \in U \text{ then } \exists y \in U (y \models \varphi)
\end{array}
$$

A *topo-bisimulation* between two topological models $M = \langle X, \tau, \nu \rangle$ and $M' = \langle X', \tau', \nu' \rangle$ is a non-empty relation $\leftrightharpoons \ \subseteq X \times X'$ such that if $x \leftrightharpoons x'$ then

(i) Base: $x \in \nu(p)$ iff $x' \in \nu'(p)$, for any propositional variable p

(II) FORTH CONDITION: $x \in U \in \tau$ implies that there exists $U' \in \tau'$ such that $x' \in U'$ and for every $y' \in U'$ there is $y \in U$ with $y \leftrightarrows y'$

(III) BACK CONDITION: $x' \in U' \in \tau'$ implies that there exists $U \in \tau$ such that $x \in U$ and for every $y \in U$ there is $y' \in U'$ with $y \leftrightarrows y'$

Points x in M and x' in M' are said to be topo-bisimilar if there is a bisimulation $\leftrightarrows$ in which $x \leftrightarrows x'$.

An important feature of topo-bisimulations that will be used throughout is that they preserve truth of modal formulas [3].

Let T_2 be the infinite binary tree with the (reflexive and transitive) descendant relation. Formally, T_2 can be defined as $\langle W, R \rangle$, where $W = \{0,1\}^*$ is the set of finite strings (including the empty string) over $\{0,1\}$ and sRt iff $\exists u : s \cdot u = t$.

In our proof of completeness we will rely on the following two well-known results.

Theorem 2.2.1 *(van Benthem-Gabbay)* **S4** *is complete with respect to* T_2.

Proof For a proof see, e.g., [42, Theorem 1 and the subsequent discussion]. The proof uses the fact that every finite rooted **S4**-frame is a bounded morphic image of T_2. □

Theorem 2.2.2 *(Cantor) Every countable dense linear ordering without endpoints is isomorphic to* $\mathbb{Q}$.

Proof For a proof see, e.g., [51, Page 217, Theorem 2]. □

Remark 2.2.3 *We recall that if* $\langle X, < \rangle$ *is a linearly ordered set and* $x, y \in X$ *with* $x < y$*, then the* open interval (x, y) *is defined as the set* $\{z \in X : x < z < y\}$*. If we view linearly ordered sets as topological spaces using the set of open intervals as a basis for the topology, then it follows from Cantor's theorem that every countable dense linear ordering without endpoints is (as a topological space) homeomorphic to* $\mathbb{Q}$.

We are now ready to proceed with the proof.

Theorem 2.2.4 **S4** *is complete with respect to* $\mathbb{Q}$.

Proof As we pointed out earlier, this result is a particular case of the McKinsey and Tarski theorem [56]. An alternative proof can be extracted from [4]. Here we give yet another proof of this result, which we will build on in Sections 2.6 and 2.7 to obtain our two main completeness results.

Our strategy is as follows. We use completeness of **S4** with respect to T_2, view T_2 as an Alexandroff space, define a dense subset X of $\mathbb{Q}$ without endpoints, and establish a topo-bisimulation between X and T_2. This will allow us to transfer counterexamples from T_2 to X, which by Cantor's theorem is order-isomorphic, and hence homeomorphic to $\mathbb{Q}$.

Let $X = \bigcup_{n\in\omega} X_n$, where $X_0 = \{0\}$ and

$$X_{n+1} = X_n \cup \{x - \frac{1}{3^n}, x + \frac{1}{3^n} \mid x \in X_n\}$$

Claim 2.2.5 *For $n > 0$ and $x, y \in X_n$, $x \neq y$ implies $|x - y| \geq \frac{1}{3^{n-1}}$.*

Proof By induction on n. If $n = 1$, then $X_1 = \{0, 1, -1\}$, and so $x \neq y$ implies $|x - y| \geq 1$. That the claim holds for $n = k + 1$ is also not hard to see. Note that if $u, v \in X_{n-1}$ with $u \neq v$, then, by induction hypothesis, $|u - v| \geq \frac{1}{3^{n-2}}$ and hence $|(u + \frac{1}{3^{n-1}}) - (v - \frac{1}{3^{n-1}})| \geq \frac{1}{3^{n-1}}$. □

It follows from Claim 2.2.5 that $\langle X, <\rangle$ is a countable dense linear ordering without endpoints, thus order-isomorphic, and hence homeomorphic to $\mathbb{Q}$. It also follows that for each $x \in X$ with $x \neq 0$ there exists n_x with $x \in X_{n_x}$ and $x \notin X_{n_x-1}$, and that there is a unique $y \in X_{n_x-1}$ with $x = y - \frac{1}{3^{n_x-1}}$ or $x = y + \frac{1}{3^{n_x-1}}$. Therefore, the open X-intervals $(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ form a basis for the order-topology on X.

Now we define f from X onto T_2 by recursion (cf. Figure 2.4(a)): If $x = 0$ then we let $f(0)$ be the root r of T_2; if $x \neq 0$ then $x \in X_{n_x} - X_{n_x-1}$ and we let

$$f(x) = \begin{cases} \text{the left successor of } f(y) & \text{if } x = y - \frac{1}{3^{n_x-1}} \\ \text{the right successor of } f(y) & \text{if } x = y + \frac{1}{3^{n_x-1}} \end{cases}$$

Claim 2.2.6 *f is open and continuous.*

Proof We recall that a basis for the Alexandroff topology on T_2 is $\mathcal{B} = \{B_t\}_{t\in T_2}$

where $B_t = \{s \in T_2 : tRs\}$. To show that f is open, for a basic open X-interval $(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$, we show that $f(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}}) = B_{f(x)}$. Indeed, if $y \in (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ then $n_y > n_x$, and so $f(x)Rf(y)$. Conversely, if $f(x)Rt$ then it follows from the definition of f (by induction on the distance between $f(x)$ and t in the tree) that there exists $y \in (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ such that $f(y) = t$. Thus f is open.

To show that f is continuous it suffices to show that for each $t \in T_2$, the f-inverse image of B_t is open. Let $x \in f^{-1}(B_t)$. Then $tRf(x)$. So $f(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}}) = B_{f(x)} \subseteq B_t$. Thus there exists an open interval $I = (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ of x such that $I \subseteq f^{-1}(B_t)$, implying that f is continuous. $\square$

To complete the proof, if $\mathbf{S4} \not\vdash \varphi$, then by Theorem 2.2.1, there is a valuation ν on T_2 such that $\langle T_2, \nu\rangle, r \not\models \varphi$. Define a valuation ξ on X by $\xi(p) = f^{-1}(\nu(p))$. Since f is continuous and open and $f(0) = r$, we have that 0 and r are topo-bisimilar. Therefore, $\langle X, \xi\rangle, 0 \not\models \varphi$. Now since X is homeomorphic to $\mathbb{Q}$, we obtain that φ is also refutable on $\mathbb{Q}$. $\square$

Note that the above completeness proof can also be seen as a representation argument. More precisely, we showed that every finite rooted **S4**-frame is a continuous and open image of $\mathbb{Q}$.

2.2.2 The fusion S4 $\oplus$ S4

Let $\mathcal{L}_{\Box_1\Box_2}$ be a bimodal language with modal operators $\Box_1$ and $\Box_2$. We recall that the *fusion* of **S4** with itself, denoted by $\mathbf{S4} \oplus \mathbf{S4}$, is the least set of formulas containing **S4**-axioms for both $\Box_1$ and $\Box_2$, and closed under modus ponens, substitution, $\Box_1$-necessitation, and $\Box_2$-necessitation.

$\mathbf{S4} \oplus \mathbf{S4}$-frames are triples $\langle W, R_1, R_2\rangle$, where W is a nonempty set and R_1 and R_2 are reflexive and transitive. We call such a frame *rooted* if there is a $w \in W$ such that for all $v \in W$, it holds that $(w, v) \in (R_1 \cup R_2)^*$, where $(R_1 \cup R_2)^*$ is the reflexive transitive closure of $R_1 \cup R_2$.

Theorem 2.2.7 *(Kracht-Wolter and Fine-Schurz)* $\mathbf{S4} \oplus \mathbf{S4}$ *has the finite model property; in fact,* $\mathbf{S4} \oplus \mathbf{S4}$ *is complete with respect to finite rooted* $\mathbf{S4} \oplus \mathbf{S4}$*-frames.*

Proof For a proof see, e.g., [38, Page 196, Theorem 4.2]. $\square$

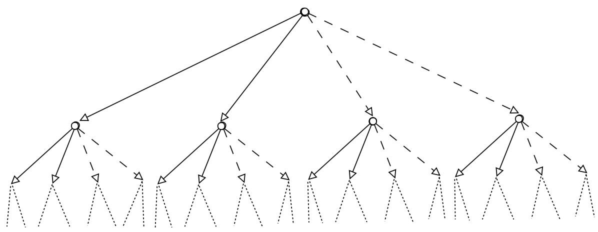

Figure 2.1: $T_{2,2}$. The solid lines represent R_1 and the dashed lines represent R_2. The dotted lines at the final nodes indicate that the pattern repeats on infinitely.

Let $T_{2,2}$ denote the infinite quaternary tree such that each node of $T_{2,2}$ is R_1-related to two of its four immediate successors and R_2-related to the other two; both R_1 and R_2 are taken to be reflexive and transitive. Formally $T_{2,2}$ can be defined as $\langle W, R_1, R_2\rangle$, where $W = \{0,1,2,3\}^*$, sR_1t iff $\exists u \in \{0,1\}^* : s \cdot u = t$, and sR_2t iff $\exists u \in \{2,3\}^* : s \cdot u = t$ (see Figure 2.1). Clearly $T_{2,2}$ is a rooted $\mathbf{S4} \oplus \mathbf{S4}$-frame.

Proposition 2.2.8 $\mathbf{S4} \oplus \mathbf{S4}$ *is complete with respect to* $T_{2,2}$.

Proof See Appendix A. $\square$

2.2.3 The product $\mathbf{S4} \times \mathbf{S4}$

For two $\mathbf{S4}$-frames $\mathcal{F} = \langle W, S\rangle$ and $\mathcal{G} = \langle V, T\rangle$, define the *product frame* $\mathcal{F} \times \mathcal{G}$ to be the frame $\langle W \times V, R_1, R_2\rangle$, where for $w, w' \in W$ and $v, v' \in V$,

$$(w,v)R_1(w',v') \text{ iff } wSw' \text{ and } v = v'$$
$$(w,v)R_2(w',v') \text{ iff } w = w' \text{ and } vTv'$$

The frame $\mathcal{F}\times\mathcal{G}$ can be viewed as an $\mathbf{S4}\oplus\mathbf{S4}$-frame by interpreting the modalities $\Box_1$ and $\Box_2$ of $\mathcal{L}_{\Box_1\Box_2}$ as follows.

$$(w,v) \models \Box_1\varphi \quad \text{iff} \quad \forall(w',v') \text{ if } (w,v)R_1(w',v') \text{ then } (w',v') \models \varphi$$
$$(w,v) \models \Box_2\varphi \quad \text{iff} \quad \forall(w',v') \text{ if } (w,v)R_2(w',v') \text{ then } (w',v') \models \varphi$$

Let $\mathbf{S4} \times \mathbf{S4}$ denote the logic of products of $\mathbf{S4}$-frames. As we pointed out in the introduction, the product logic $\mathbf{S4} \times \mathbf{S4}$ is axiomatized by adding the following two axioms to the fusion $\mathbf{S4} \oplus \mathbf{S4}$:

$$com = \Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p$$

$$chr = \Diamond_1\Box_2 p \rightarrow \Box_2\Diamond_1 p$$

By the Sahlqvist theorem, *com* and *chr* have the following first-order correspondents:

$$\forall x\forall y(\exists z(xR_1z \wedge zR_2y) \leftrightarrow \exists z(xR_2z \wedge zR_1y))$$

$$\forall x\forall y\forall z((xR_1y \wedge xR_2z) \rightarrow \exists w(yR_2w \wedge zR_1w))$$

Besides R_1 and R_2, there is yet another (reflexive and transitive) relation on the product $W \times V$ defined componentwise:

$$(w, v)R(w', v') \text{ iff } wSw' \text{ and } vTv'$$

This allows us to interpret yet another modal operator $\Box$ in $\mathcal{F} \times \mathcal{G}$:

$$(w, v) \models \Box\varphi \quad \text{iff} \quad \forall(w', v') \text{ if } (w, v)R(w', v') \text{ then } (w', v') \models \varphi$$

However, since in product frames we have that $R = R_1 \circ R_2$, $\Box\varphi$ becomes equivalent to $\Box_1\Box_2\varphi$, and so $\Box$ turns out to be definable in terms of $\Box_1$ and $\Box_2$. As we will see shortly, in the subtler setting of topological products, the analogue of $\Box$ is not modally definable in terms of the analogues of $\Box_1$ and $\Box_2$.

2.3 Product spaces and product topo-bisimulations

2.3.1 Horizontal and vertical topologies

Let $\mathcal{X} = \langle X, \eta\rangle$ and $\mathcal{Y} = \langle Y, \theta\rangle$ be two topological spaces. Recall that the *standard product topology* τ on $X \times Y$ is defined by letting the sets $U \times V$ form a basis for τ, where U is open in $\mathcal{X}$ and V is open in $\mathcal{Y}$. Let I denote the interior operator and C denote the closure operator of τ.

We will define two additional one-dimensional topologies on $X \times Y$ by 'lifting' the topologies of the components.

Suppose $A \subseteq X \times Y$. We say that A is *horizontally open (H-open)* if for any $(x, y) \in A$ there exists $U \in \eta$ such that $x \in U$ and $U \times \{y\} \subseteq A$. Similarly, we say that A is *vertically open (V-open)* if for any $(x, y) \in A$ there exists $V \in \theta$ such that $y \in V$ and $\{x\} \times V \subseteq A$. If A is both H- and V-open, then we call it *HV-open.* H-closed, V-closed and HV-closed sets are defined similarly. Let τ_1 denote the set of all H-open subsets of $X \times Y$ and τ_2 denote the set of all V-open subsets of $X \times Y$. It is easy to verify that both τ_1 and τ_2 form topologies on $X \times Y$. We call τ_1 the *horizontal topology* and τ_2 the *vertical topology.* The closure and interior operators C_i and I_i for τ_i can be defined in the usual way $(i = 1, 2)$.

Remark 2.3.1 *It is obvious that a set open in the standard product topology is both horizontally and vertically open. That is $\tau \subseteq \tau_1$ and $\tau \subseteq \tau_2$. However, the converse inclusions don't hold in general. In fact, we will show below that I is not modally definable by means of I_1 and I_2.*

The interpretation of the modal operators $\Box_1$ and $\Box_2$ of $\mathcal{L}_{\Box_1\Box_2}$ in $\langle X \times Y, \tau_1, \tau_2\rangle$ is as expected:

$$(x, y) \models \Box_1\varphi \text{ iff } (\exists U \in \tau_1)((x, y) \in U \text{ and } \forall (x', y') \in U \,.\, (x', y') \models \varphi)$$

$$(x, y) \models \Box_2\varphi \text{ iff } (\exists V \in \tau_2)((x, y) \in V \text{ and } \forall (x', y') \in V \,.\, (x', y') \models \varphi)$$

The modalities $\Diamond_1$ and $\Diamond_2$ are defined dually. Furthermore, all the usual notions, such as satisfiability and validity, generalize naturally to this new language.

The one-dimensional nature of the horizontal and vertical topologies is emphasized by the following proposition.

Proposition 2.3.2

1. *A formula φ constructed from the Booleans and the modal operator $\Box_1$ is valid in $\langle X \times Y, \tau_1, \tau_2\rangle$ iff φ is valid in $\langle X, \eta\rangle$.*

2. *A formula φ constructed from the Booleans and the modal operator $\Box_2$ is valid in $\langle X \times Y, \tau_1, \tau_2\rangle$ iff φ is valid in $\langle Y, \theta\rangle$.*

Proof See Appendix A. $\Box$

The most easily visualized examples of products of topological spaces in our sense are structures like the real plane $\mathbb{R} \times \mathbb{R}$, or its rational variant $\mathbb{Q} \times \mathbb{Q}$ which plays a crucial role in our completeness results below. In these cases, the H- and V-topologies reproduce the topology of the underlying number line, but on the product, our modal language also describes iterated patterns of open sets from both topologies.

More generally, our horizontal and vertical topologies on product spaces generalize a key theme from the usual semantics of modal logic. It is well-known that topological semantics of modal logic generalizes relational semantics for normal extensions of **S4**. Indeed, with every **S4**-frame $\langle W, R\rangle$ there corresponds the topological space $\langle W, \tau_R\rangle$, with τ_R being precisely the R-upward closed subsets of W. Now, for **S4**-frames $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{F}' = \langle W', R'\rangle$, let $\mathfrak{F} \times \mathfrak{F}' = \langle W \times W', R_1, R_2\rangle$ be their product, as defined in the Introduction. Then τ_{R_1} and τ_{R_2} are precisely the horizontal and vertical topologies on the product space $W \times W'$. This shows that our topological product construction is a faithful generalization of the usual product construction for Kripke frames. What our later completeness and representation results show is that this generalization drops the special interaction axioms valid for the binary case. Rather than being general features of a product construction as such, these axioms turn out to be special effects of working with very special topologies, viz. Alexandroff topologies.

This loss of interaction axioms has to do with the 'looseness' of the connection between H- and V-topologies in products. But this looseness also has a positive counterpart, which is well-known from other areas in logical semantics. Moving to a more general class of models usually weakens the logic, but enriches the language. Topological products support many further topologies beyond our two 'conservative copies' for their component topologies. And hence there is scope for a richer family of modalities. One very natural addition from a topological perspective is the 'true product topology', whose modal logic we will axiomatize later on on top of that for the H- and V-topologies. But we think there are many further natural candidates for topologies on product spaces. Cf. [17] for some examples in the case of epistemic logic, where topologies model agents' knowledge and uncertainty, and new topologies on products model various forms of knowledge for groups of such agents. A few more comments on these themes are found in Section 2.8 below.

2.3.2 Failure of *com* and *chr* on $\mathbb{R} \times \mathbb{R}$

We saw in the previous subsection that whenever topological spaces $\mathcal{X}$ and $\mathcal{Y}$ are representable as **S4**-frames (are Alexandroff), then the horizontal and vertical topologies on their product $X \times Y$ can be defined from the horizontal and vertical relations on the product of these frames. In other words, our topological setting generalizes the case for products of Kripke frames. Nevertheless, there are crucial differences between these two settings. In particular, both *com* and *chr*, while valid on products of Kripke frames, can be refuted on topological products. To stimulate intuitions before plunging into general theory, we exhibit their failure on $\mathbb{R} \times \mathbb{R}$.

(a) Failure of *com*: Let

$$\nu(p) = \Big(\bigcup_{x \in (-1,0)} \{x\} \times (x, -x)\Big) \cup (\{0\} \times (-1,1)) \cup \Big(\bigcup_{x \in (0,1)} \{x\} \times (-x, x)\Big)$$

(see Figure 2.2a). Then there is a basic horizontal open $(-1,1) \times \{0\}$ such that $(0,0)$ is in it and every point in $(-1,1) \times \{0\}$ sits in a vertically open subset of p. Thus, $\Box_1\Box_2 p$ is true at $(0,0)$. On the other hand, there is no vertical open containing $(0,0)$ in which every point sits inside a horizontally open subset of p, implying that $\Box_2\Box_1 p$ is false at $(0,0)$.

(b) Failure of *chr*: Let $\nu(p) = \bigcup\{\{\frac{1}{n}\} \times (-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ (see Figure 2.2b). Then in any basic horizontal open around $(0,0)$ there is a point that sits in a basic vertical open in which p is true everywhere. Thus, $\Diamond_1\Box_2 p$ is true at $(0,0)$. On the other hand, since the horizontal closure of $\nu(p)$ is $\nu(p) \cup \{(0,0)\}$ and since the vertical interior of $\nu(p) \cup \{(0,0)\}$ is $\nu(p)$, we have that $(0,0)$ is not in $I_2(C_1(\nu(p)))$, implying that $\Box_2\Diamond_1 p$ is false at $(0,0)$.

As we will see in Section 2.4, the structure of these counterexamples on $\mathbb{R} \times \mathbb{R}$ is not accidental. We will show under which circumstances they can be reproduced in other products of topological spaces.

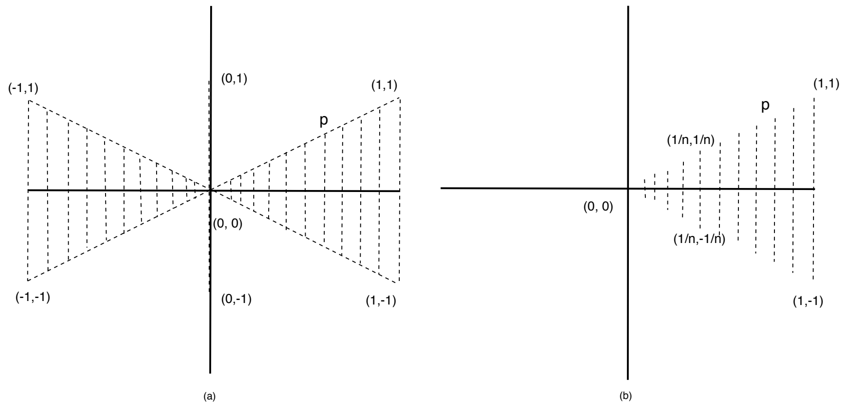

Figure 2.2: Counterexamples for *com* and *chr* on $\mathbb{R} \times \mathbb{R}$.

2.3.3 Product topo-bisimulations

As in Kripke semantics, an appropriate notion of bisimulation plays crucial role in understanding and developing topological semantics. In this subsection we generalize the notion of topo-bisimulation introduced in Section 2.2.1 to topological models equipped with several topologies. We will use it to show that the standard product interior is not definable in terms of the horizontal and vertical interiors. Another important application of multi-dimensional topo-bisimulations will come in the completeness proofs below.

We exhibit the case of two topologies, but the generalization to any number of topologies is straightforward.

Definition 2.3.3 *Let* $M = \langle X, \tau_1, \tau_2, \nu \rangle$ *and* $M' = \langle X', \tau_1', \tau_2', \nu' \rangle$ *be topological models equipped with two topologies each. A* 2-topo-bisimulation *is a nonempty relation* $\rightleftharpoons \; \subseteq X \times X'$ *such that if* $x \rightleftharpoons x'$ *then the following hold for* $i = 1, 2$*:*

(I) Base*:* $x \in \nu(p)$ *iff* $x' \in \nu'(p)$*, for any proposition variable* p

(II) Forth condition*:* $x \in U \in \tau_i$ *implies that there exists* $U' \in \tau_i'$ *such that* $x' \in U'$ *and for all* $z' \in U'$ *there exists* $z \in U$ *with* $z \rightleftharpoons z'$

(III) Back condition*:* $x' \in U' \in \tau_i'$ *implies that there exists* $U \in \tau_i$ *such that* $x \in U$ *and for all* $z \in U$ *there exists* $z' \in U'$ *with* $z \rightleftharpoons z'$

The 2-topo-bisimulation $\rightleftharpoons$ is called *total* if it is defined for all elements of X and X', i.e., $dom(\rightleftharpoons) = X$ and $rng(\rightleftharpoons) = X'$. The fundamental invariance property of 2-topo-bisimulations is given by the following proposition.

Proposition 2.3.4 *Let $M = \langle X, \tau_1, \tau_2, \nu\rangle$ and $M' = \langle X', \tau_1', \tau_2', \nu'\rangle$ be topological models equipped with two topologies each, and let $x \rightleftharpoons x'$ for some 2-topo-bisimulation $\rightleftharpoons \ \subseteq X \times X'$. Then for every modal formula φ in $\mathcal{L}_{\Box_1\Box_2}$ we have that $M, x \models \varphi$ iff $M', x' \models \varphi$.*

Proof The proof is a straightforward generalization of the 1-topo-bisimulation version found in [3] and we omit the details of the induction. $\Box$

Definition 2.3.3 and Proposition 3.2.5 apply to arbitrary topological models M, M' with two (or, via a straightforward generalization, more than two) topologies each. By analogy with Kripke semantics, one can think of such models as fusion models. In the special case when M and M' consist of product spaces with the horizontal and vertical topologies, the 2-topo-bisimulation $\rightleftharpoons$ is called a *product topo-bisimulation.*

Topo-bisimulations are useful for showing that properties are not definable in our language. A nice example of this is given in Proposition 2.3.5 below. For two topological spaces $\mathcal{X}$ and $\mathcal{Y}$, consider the product space $\langle X \times Y, \tau, \tau_1, \tau_2\rangle$, where τ stands for the standard product topology, τ_1 for the horizontal topology, and τ_2 for the vertical topology. We recall that $\Box_1$ and $\Box_2$ are interpreted via the horizontal and vertical topologies, while $\Box$ is interpreted via the standard product topology.

Proposition 2.3.5 *$\Box$ is not definable in the language $\mathcal{L}_{\Box_1\Box_2}$.*

Proof It is sufficient to find two product models $M = \langle X \times Y, \tau_1, \tau_2, \nu\rangle$ and $M' = \langle X' \times Y', \tau_1', \tau_2', \nu'\rangle$ with $(x, y) \in X \times Y$ and $(x', y') \in X' \times Y'$, and a product topo-bisimulation $\leftrightharpoons \subseteq (X \times Y) \times (X' \times Y')$ such that $(x, y) \leftrightharpoons (x', y')$, that $M, (x, y) \models \Diamond p$, and that $M', (x, y) \not\models \Diamond p$. Since all formulae in the language $\mathcal{L}_{\Box_1\Box_2}$ are preserved by product topo-bisimulations and $\Diamond p$ is not, we conclude that $\Diamond p$ is not equivalent to any formula of $\mathcal{L}_{\Box_1\Box_2}$ (or to any infinite conjunction of such formulae for that matter). It follows that neither is $\Box p$.

For the product space we take $\mathbb{Q} \times \mathbb{Q}$. Let $\nu(p) = \{(\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ and $\nu'(p) = \emptyset$. Let also $\leftrightharpoons$ be the identity relation on $(\mathbb{Q} \times \mathbb{Q}) \backslash \{(\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$. It is not hard to see that $\leftrightharpoons$ is a product topo-bisimulation between the models $\langle \mathbb{Q} \times \mathbb{Q}, \nu \rangle$ and $\langle \mathbb{Q} \times \mathbb{Q}, \nu' \rangle$ that connects $(0,0)$ to $(0,0)$. Since $(0,0)$ is in the closure of $\nu(p)$, we have that $\langle \mathbb{Q}\times\mathbb{Q}, \nu \rangle, (0,0) \models \Diamond p$. On the other hand, it is obvious that $\langle \mathbb{Q}\times\mathbb{Q}, \nu' \rangle, (0,0) \models \Box\neg p$. $\Box$

Another example for the use of topo-bisimulations is Proposition 2.3.6. Given topological product spaces $\langle X \times Y, \tau_1, \tau_2 \rangle$ and $\langle X' \times Y', \tau_1', \tau_2' \rangle$, we say that a map $f : X \times Y \to X' \times Y'$ is *HV-continuous* if it is continuous with respect to both horizontal and vertical topologies, and that f is *HV-open* if it is open withe respect to both topologies. HV-open HV-continuous bijections are called *HV-homeomorphisms*. Note that if X is homeomorphic to X' and Y is homeomorphic to Y', then $X \times Y$ is HV-homeomorphic to $X' \times Y'$. For $U \subseteq X \times Y$, we say that U is a *HV-open* subset of $X \times Y$ if U is both H- and V-open in $X \times Y$.

Proposition 2.3.6

1. *Surjective HV-continuous HV-open maps preserve validity of formulas of* $\mathcal{L}_{\Box_1\Box_2}$.

2. *HV-open subsets preserve validity of formulas of* $\mathcal{L}_{\Box_1\Box_2}$.

Proof (1) Let $\langle X \times Y, \tau_1, \tau_2 \rangle$ and $\langle X' \times Y', \tau_1', \tau_2' \rangle$ be given, and let $f : X \times Y \to X' \times Y'$ be surjective, HV-continuous, and HV-open. For a valuation ν' on $X' \times Y'$ we can define a valuation ν on $X \times Y$ by putting $\nu(p) = f^{-1}(\nu'(p))$. Then it is easy to verify that f is a total 2-topo-bisimulation between the models $M = \langle X \times Y, \tau_1, \tau_2, \nu \rangle$ and $M' = \langle X' \times Y', \tau_1', \tau_2', \nu' \rangle$. It follows that whenever a formula of $\mathcal{L}_{\Box_1\Box_2}$ is refuted on the latter model, it can also be refuted on the former one.

(2) is proved similar to (1). $\Box$

2.4 Correspondence for *com* and *chr*

As we have seen above, unlike products of Kripke frames, products of topological spaces do not always validate *com* and *chr*. In this section we specify those classes of products of topological spaces in which *com* and *chr* hold. We start by investigating the validity of *com*. It is useful to split *com* into $com_{\rightarrow} = \Box_1\Box_2 p \rightarrow \Box_2\Box_1 p$ and $com_{\leftarrow} = \Box_2\Box_1 p \rightarrow \Box_1\Box_2 p$.

Let $\mathcal{X} = \langle X, \eta\rangle$ be a topological space. We recall that $\mathcal{X}$ is Alexandroff if the intersection of any family of open sets is again open. We call $\mathcal{X}$ *κ-Alexandroff* if the intersection of any family of open sets of cardinality κ is again open; that is, $\eta' \subseteq \eta$ and $|\eta'| \leq \kappa$ imply $\bigcap \eta' \in \eta$.

Proposition 2.4.1 *If $\mathcal{X} = \langle X, \eta\rangle$ is κ-Alexandroff and $|Y| \leq \kappa$, then $\mathcal{X} \times \mathcal{Y} \models com_{\leftarrow}$ and $\mathcal{Y} \times \mathcal{X} \models com_{\rightarrow}$.*

Proof We show that $\mathcal{X} \times \mathcal{Y} \models com_{\leftarrow}$. That $\mathcal{Y} \times \mathcal{X} \models com_{\rightarrow}$ is proved symmetrically. Suppose for a point $(x, y) \in X \times Y$ and a valuation ν on $\mathcal{X} \times \mathcal{Y}$ we have that $(x, y) \models \Box_2\Box_1 p$. Then there exists a neighborhood V of y such that for each $z \in V$ there is a neighborhood U_z of z with $U_z \times \{z\} \subseteq \nu(p)$. Since $|V| \leq \kappa$ and $\mathcal{X}$ is κ-Alexandroff, we have that $U = \bigcap\{U_z : z \in V\} \in \eta$. But then $U \times V \subseteq \nu(p)$, implying that $(x, y) \models \Box_1\Box_2 p$. $\Box$

Corollary 2.4.2 *If $\mathcal{X}$ is Alexandroff, then $\mathcal{X} \times \mathcal{Y} \models com_{\leftarrow}$ and $\mathcal{Y} \times \mathcal{X} \models com_{\rightarrow}$ for any topological space $\mathcal{Y}$.*

Proof It is sufficient to observe that every Alexandroff space is κ-Alexandroff for every cardinal κ, and apply Proposition 2.4.1. $\Box$

It follows that if both $\mathcal{X}$ and $\mathcal{Y}$ are Alexandroff, then $\mathcal{X} \times \mathcal{Y} \models com$. Given the well-known correspondence between Kripke frames for **S4** and Alexandroff topologies, the above corollary sheds some topological light on the validity of *com* on products of Kripke frames.

The converse of Corollary 2.4.2 does not hold. For instance, every topology commutes with the discrete topology of any cardinality. Thus, it can happen that $\mathcal{X}$ or $\mathcal{Y}$ are not Alexandroff and yet $\mathcal{X} \times \mathcal{Y} \models com$. However, if $\mathcal{X}$ and $\mathcal{Y}$ coincide, then the converse of Corollary 2.4.2 holds. To see this, for $x \in X$, let η_x denote the set of all neighborhoods of x.

Lemma 2.4.3 *If $\mathcal{X}$ is not Alexandroff, then there is a point $x \in X$ such that $\bigcap \eta_x \notin \eta$.*

Proof Since $\mathcal{X}$ is not Alexandroff, there exists a set B of opens such that $\bigcap B \notin \eta$. Let $x \in \bigcap B$. Obviously $\bigcap \eta_x \subseteq \bigcap B$ and $\bigcap B = \bigcup\{\bigcap \eta_x : x \in \bigcap B\}$. If $\bigcap \eta_x$ were open for every $x \in \bigcap B$, then $\bigcap B$ would be open. Therefore, there exists $x \in \bigcap B$ such that $\bigcap \eta_x$ is not open. $\square$

Proposition 2.4.4 *If $\mathcal{X}$ is not Alexandroff, then $\mathcal{X} \times \mathcal{X} \not\models com_{\leftarrow}$ and $\mathcal{X} \times \mathcal{X} \not\models com_{\rightarrow}$.*

Proof We show that $\mathcal{X} \times \mathcal{X} \not\models com_{\leftarrow}$. The case for $\mathcal{X} \times \mathcal{X} \not\models com_{\rightarrow}$ is symmetric. Since $com_{\leftarrow}$ is equivalent to $\Diamond_1\Diamond_2 p \rightarrow \Diamond_2\Diamond_1 p$, it is enough to show that $\mathcal{X} \times \mathcal{X} \not\models \Diamond_1\Diamond_2 p \rightarrow \Diamond_2\Diamond_1 p$. As $\mathcal{X}$ is not Alexandroff, by Lemma 2.4.3 there exists $x \in X$ such that $\bigcap \eta_x \notin \eta$. Let $\eta_x = \{U_i\}_{i \in I}$. We order I by putting $i \leq j$ iff $U_i \supseteq U_j$. Since $U_i, U_j \in \eta_x$ implies $U_i \cap U_j \in \eta_x$, it follows that $(I, \leq)$ is a directed partial order. Let $J = \{i \in I : \exists j \geq i \text{ with } U_i - U_j \neq \emptyset\}$. We show that J is cofinal in I. If not, then there exists $i \in I$ such that for any $j \geq i$ we have $U_i - U_j = \emptyset$. Therefore, $U_i = U_j$ for any $j \geq i$. Thus, $\bigcap \eta_x = \bigcap_{i \in I} U_i = \bigcap_{j \geq i} U_i = U_i \in \eta$, a contradiction. For $i \in J$ let $j \geq i$ be such that $U_i - U_j \neq \emptyset$ and pick $x_i \in U_i - U_j$. Then $\{x_i\}_{i \in J}$ is a net converging to x. Let ν be a valuation on $\mathcal{X} \times \mathcal{X}$ such that $\nu(p) = \{(x_i, x_j) : i, j \in J \text{ and } i \leq j\}$. For $U \in \eta_x$ and $i \in J$, let $U_j = U \cap U_i$. Then $i \leq j$. Since J is cofinal in I we can assume that $j \in J$. Therefore, $(x_i, x_j) \in \nu(p)$. It follows that $(x_i, x) \models \Diamond_2 p$. Thus, $(x, x) \models \Diamond_1\Diamond_2 p$. On the other hand, for any $U \in \eta_x$ and for any $x_j \in U$ we have $(U_i \times \{x_j\}) \cap \nu(p) = \emptyset$ for any $i \in J$ with $i > j$. Therefore, $(x, x) \not\models \Diamond_2\Diamond_1 p$. $\square$

From Corollary 2.4.2 and Proposition 2.4.4 we obtain the following characterization of Alexandroff spaces.

Corollary 2.4.5 *The following conditions are equivalent:*

1. *$\mathcal{X}$ is Alexandroff.*
2. *$\mathcal{X} \times \mathcal{X} \models com$.*
3. *$\mathcal{X} \times \mathcal{Y} \models com_{\leftarrow}$ for every topological space $\mathcal{Y}$.*
4. *$\mathcal{Y} \times \mathcal{X} \models com_{\rightarrow}$ for every topological space $\mathcal{Y}$.*

We end this section by investigating validity of *chr* in the products of topological spaces.

Proposition 2.4.6 *If either $\mathcal{X}$ or $\mathcal{Y}$ is Alexandroff, then $\mathcal{X} \times \mathcal{Y} \models chr$.*

Proof Let $\mathcal{X} = \langle X, \eta \rangle$ and $\mathcal{Y} = \langle Y, \theta \rangle$. First suppose that $\mathcal{X}$ is Alexandroff. So every $x \in X$ has a least neighborhood U_x. If for a valuation ν on $\mathcal{X} \times \mathcal{Y}$ and a point $(x, y) \in X \times Y$ we have that $(x, y) \models \Diamond_1 \Box_2 p$, then there exists $z \in U_x$ such that $(z, y) \models \Box_2 p$. Therefore, there exists $V \in \theta_y$ such that $\{z\} \times V \subseteq \nu(p)$. But then for every $u \in V$ we have $(x, u) \models \Diamond_1 p$, implying that $(x, y) \models \Box_2 \Diamond_1 p$.

Now suppose that $\mathcal{Y}$ is Alexandroff. So every $y \in Y$ has a least neighborhood V_y. If for a valuation ν on $\mathcal{X} \times \mathcal{Y}$ and a point $(x, y) \in X \times Y$ we have that $(x, y) \models \Diamond_1 \Box_2 p$, then for every $U \in \eta_x$ there exists $z \in U$ such that $\{z\} \times V_y \subseteq \nu(p)$. But then for every $u \in V_y$ and for every $U \in \eta_x$ there exists $z \in U$ such that $(z, u) \in \nu(p)$. Thus, $(x, y) \models \Box_2 \Diamond_1 p$. $\Box$

Since Kripke frames for **S4** correspond to Alexandroff topologies, the above proposition gives a topological insight into the soundness of *chr* with respect to products of Kripke frames. Even though the converse of Proposition 2.4.6 is not in general true, similar to the case with *com*, we have that if $\mathcal{X}$ and $\mathcal{Y}$ coincide, then the converse does indeed hold.

Proposition 2.4.7 *If $\mathcal{X}$ is not Alexandroff, then $\mathcal{X} \times \mathcal{X} \not\models chr$.*

Proof Let $x \in X$, $\eta_x = \{U_i\}_{i \in I}$, $J \subseteq I$, and the net $\{x_i\}_{i \in J}$ be chosen as in the proof of Proposition 2.4.4. We define a valuation ν on $\mathcal{X} \times \mathcal{X}$ by putting $\nu(p) = \bigcup_{i \in J}(\{x_i\} \times U_i)$. Then it is easy to verify that $(x, x) \models \Diamond_1 \Box_2 p$ but $(x, x) \not\models \Box_2 \Diamond_1 p$. $\Box$

Propositions 2.4.6 and 2.4.7 lead to yet another characterization of Alexandroff spaces.

Corollary 2.4.8 *The four equivalent conditions in Corollary 2.4.5 are equivalent to the following one:*

(5) $\mathcal{X} \times \mathcal{X} \models chr$.

2.5 Cardinal Spaces and *chr* and *com*

In this section, (Corollaries 2.5.3 & 2.5.5), we give an example for non-Alexandroff spaces $\mathbf{X}$ and $\mathbf{Y}$ such that

$$\mathbf{X}, \mathbf{Y} \models \mathrm{com}_{\rightarrow} \,\&\, \mathrm{chr}.$$

For this, we define the **cardinal space** of cardinality κ, denoted by $\mathbf{card}_\kappa$ as follows: The underlying set of the space is $\kappa \cup \{\infty\}$ where $\infty \notin \kappa$. The open neighbourhood base for each $\alpha \in \kappa$ is $\{\{\alpha\}\}$, the open neighbourhood base for ∞ is

$$\{\{\xi\,;\,\alpha < \xi < \kappa\} \cup \{\infty\}\,;\,\alpha \in \kappa\}.$$

The topology of $\mathbf{card}_\kappa$ is the discrete topology on κ and a point at infinity that is infinitely far away (can be reached only by sequences cofinal in κ). An alternative way of viewing these spaces is as the ordinal topology on the ordinal $\kappa + 1$ with all limit points below κ removed.

Note that for infinite cardinals κ, the cardinal space $\mathbf{card}_\kappa$ is not Alexandroff, and for uncountable cardinals κ, it is not first-countable.

2.5.1 Bimodal formulae in products of cardinal spaces

If $\mu \leq \nu$ are ordinals, and $\gamma \in \nu$, we can form the Cantor Normal Form of γ to the base μ:

$$\gamma = \mu^{\alpha_n} \cdot \gamma_n + \mu^{\alpha_{n-1}} \cdot \gamma_{n-1} + \ldots + \mu^{\alpha_1} \cdot \gamma_1 + \gamma_0.$$

We write $\mathrm{S}_\mu(\gamma) := \gamma_0$ and call it the **scalar term** of γ to the base μ.

Lemma 2.5.1 *If $\mu \leq \nu$ are cardinals, $\gamma < \mu$ and $\beta < \nu$, then there is some $\beta < \eta < \nu$ such that $\mathrm{S}_\mu(\eta) \geq \gamma$.*

Proof Let $\xi := \mathrm{S}_\mu(\beta)$. If $\xi \geq \gamma$, then $\eta := \beta + 1$ does the job. Otherwise, there is a unique $0 < \sigma < \mu$ such that $\gamma = \xi + \sigma$. Let $\eta := \beta + \sigma$. □

Theorem 2.5.2 *Let κ and λ be cardinals. Then the following are equivalent:*

1. $\mathbf{card}_\kappa, \mathbf{card}_\lambda \models \Diamond_1 \Diamond_2\, p \rightarrow \Diamond_2 \Diamond_1\, p$, *and*

2. $\lambda < \operatorname{cf} \kappa$.

Proof "*(i)⇒(ii)*". Suppose $\operatorname{cf} \kappa \leq \lambda$. We'll construct a subset of $\mathbf{card}_\kappa \times \mathbf{card}_\lambda$ that constitutes a counterexample to $\mathrm{com}_\rightarrow$. Let $A = \{\alpha_\gamma\,; \gamma < \operatorname{cf} \kappa\} \subseteq \kappa$ be an increasing enumeration of a cofinal subset of κ. We define a subset of $\kappa \times \lambda$ as follows:

$$\langle \alpha_\gamma, \beta \rangle \in X :\Longleftrightarrow \gamma \leq \mathrm{S}_{\operatorname{cf} \kappa}(\beta).$$

Note that $\mathrm{S}_{\operatorname{cf} \kappa}(\beta) < \operatorname{cf} \kappa$, so if you fix an element $\beta \in \lambda$ and look at the horizontal section $X_\beta = \{\alpha\,; \langle \alpha, \beta \rangle \in X\}$, then each of these sets has cardinality less than $\operatorname{cf} \kappa$. In particular, none of these can be cofinal in κ ($\star$).

Moreover, if you fix $\alpha_\gamma \in A$ and look at the vertical section

$$X^{\alpha_\gamma} = \{\beta\,; \langle \alpha_\gamma, \beta \rangle \in X\},$$

then this set is cofinal in λ ($\star\star$) by the following argument: Take an arbitrary $\beta < \lambda$. By Lemma 2.5.1 applied to $\operatorname{cf} \kappa \leq \lambda$, we find $\beta < \eta < \lambda$ such that $\mathrm{S}_{\operatorname{cf} \kappa}(\eta) \geq \gamma$. But that means that $\langle \alpha_\gamma, \eta \rangle \in X$, so $\beta < \eta \in X^{\alpha_\gamma}$.

By ($\star$), the horizontal closure of X is X itself: none of the elements of the form $\langle \infty, \beta \rangle$ are reached by horizontal sections of X. By ($\star\star$), the vertical closure of X is $X \cup A \times \{\infty\}$. Of course, since A is cofinal in κ, the horizontal closure of $A \times \{\infty\}$ includes the point $\langle \infty, \infty \rangle$.

But then

$$\mathrm{vcl}(\mathrm{hcl}(X)) = X \cup A \times \{\infty\}, \text{ yet}$$

$$\mathrm{hcl}(\mathrm{vcl}(X)) = X \cup A \times \{\infty\} \cup \{\langle \infty, \infty \rangle\}.$$

But this means that

$$\mathbf{card}_\kappa, \mathbf{card}_\lambda \not\models \Diamond_1 \Diamond_2 p \rightarrow \Diamond_2 \Diamond_1 p.$$

"*(ii)⇒(i)*". Assume that $\lambda < \operatorname{cf} \kappa$. We have to show that $\mathbf{card}_\kappa, \mathbf{card}_\lambda \models \Diamond_1 \Diamond_2 p \rightarrow \Diamond_2 \Diamond_1 p$, so we have to show for every subset X of the product that $\mathrm{hcl}(\mathrm{vcl}(X)) \subseteq \mathrm{vcl}(\mathrm{hcl}(X))$. Note that the only point for which the order of horizontal and vertical closures matters is the point $\langle \infty, \infty \rangle$, so the we are done if we can show that

$$\langle\infty,\infty\rangle \in \mathrm{hcl}(\mathrm{vcl}(X)) \text{ implies } \langle\infty,\infty\rangle \in \mathrm{vcl}(\mathrm{hcl}(X)).$$

Without loss of generality, $X \subseteq \kappa \times \lambda$.

If $\langle\infty,\infty\rangle \in \mathrm{hcl}(\mathrm{vcl}(X))$, there is a cofinal set $C \subseteq \kappa$ of cardinality $\mathrm{cf}\,\kappa$ such that for all $\gamma \in C$, we have

$$\langle\gamma,\infty\rangle \in \mathrm{vcl}(X).$$

This in turn means that for each such γ, the vertical section $X^\gamma = \{\beta\,;\,\langle\gamma,\beta\rangle \in X\}$ must be cofinal in λ. In other words, if you fix $\eta \in \lambda$, then

$$X^*_{>\eta} := \{\langle\alpha,\beta\rangle \in X\,;\,\beta > \eta\} \cap (\kappa \times C)$$

must have cardinality at least $\mathrm{cf}\,\kappa$.

For each $\beta \in \lambda$, let

$$P_\beta := \{\langle\alpha,\beta\rangle \in X\,;\,\alpha \in C\}.$$

The family $\{P_\beta\,;\,\eta < \beta < \lambda\}$ is a partition of $X^*_{>\eta}$ into at most λ many pieces. Consequently, by the pigeon hole principle, there must be a $\beta^* > \eta$ such that P_{β^*} has $\mathrm{cf}\,\kappa$ many elements. But since $P_{\beta^*} \subseteq X_{\beta^*}$ and C was cofinal in κ, this means that $\langle\infty,\beta^*\rangle \in \mathrm{hcl}(X)$.

Since η was arbitrary, we just showed that the set of such β^* is cofinal in λ, and thus $\langle\infty,\infty\rangle \in \mathrm{vcl}(\mathrm{hcl}(X))$. This was the claim. $\square$

Corollary 2.5.3 *For $\aleph_0 \leq \lambda < \mathrm{cf}\,\kappa$, $\mathbf{card}_\kappa$ and $\mathbf{card}_\lambda$ are non-Alexandroff spaces such that* $\mathrm{com}_\leftarrow$ *holds in $\mathbf{card}_\kappa \times \mathbf{card}_\lambda$. In particular, this is true in $\mathbf{card}_{\aleph_1} \times \mathbf{card}_{\aleph_0}$. Also,* $\mathrm{com}_\rightarrow$ *holds in $\mathbf{card}_{\aleph_0} \times \mathbf{card}_{\aleph_1}$.*

Theorem 2.5.4 *Let κ and λ be cardinals. Then the following are equivalent:*

1. $\mathbf{card}_\kappa, \mathbf{card}_\lambda \models \Diamond_1\Box_2\, p \to \Box_2\Diamond_1\, p$, *and*

2. $\mathrm{cf}\,\lambda \neq \mathrm{cf}\,\kappa$.

Proof "*(i)⇒(ii)*". Suppose that $\vartheta := \operatorname{cf}\kappa = \operatorname{cf}\lambda$. Let $A = \{\alpha_\gamma\,;\,\gamma < \vartheta\} \subseteq \kappa$ and $B = \{\beta_\gamma\,;\,\gamma < \vartheta\} \subseteq \lambda$ be increasing enumerations of cofinal subsets. Define

$$X := \{\langle\alpha_\gamma, \beta\rangle\,;\,\beta \geq \beta_\gamma, \gamma < \vartheta\} \cup \{\langle\alpha_\gamma, \infty\rangle\,;\,\gamma < \vartheta\}.$$

Then for each $\gamma < \vartheta$, $\{\infty\} \cup \{\beta\,;\,\beta_\gamma \leq \beta < \lambda\} \subseteq X^{\alpha_\gamma}$ which is an open neighbourhood of ∞ in $\mathbf{card}_\lambda$. Consequently, $\langle\alpha_\gamma, \infty\rangle \in \operatorname{vint}(X)$. Since A was cofinal in κ, this means that $\langle\infty, \infty\rangle \in \operatorname{hcl}(\operatorname{vint}(X))$.

Yet, for each $\beta \in \lambda$ there is an upper bound for X_β: if $\beta_\gamma \leq \beta < \beta_{\gamma+1}$, then

$$X_\beta \subseteq \{\alpha \in \kappa\,;\, 0 \leq \alpha < \alpha_{\gamma+1}\}.$$

That means that $\operatorname{hcl}(X)$ doesn't contain any element of the form $\langle\infty, \beta\rangle$, and so $\langle\infty, \infty\rangle \notin \operatorname{vint}(\operatorname{hcl}(X))$.

"*(ii)⇒(i)*". The symmetry of chr makes sure that we only have to check the case $\operatorname{cf}\kappa < \operatorname{cf}\lambda$.

To start, let us notice that for subsets X of $\mathbf{card}_\kappa \times \mathbf{card}_\lambda$, we always have that

$$\operatorname{hcl}(\operatorname{vint}(X)) \backslash \{\langle\infty, \infty\rangle\} \subseteq \operatorname{vint}(\operatorname{hcl}(X)),$$

since the elements of $\kappa \times \lambda$ are not affected by any of the interior and closure operations. Thus, we only have to show

$$\langle\infty, \infty\rangle \in \operatorname{hcl}(\operatorname{vint}(X)) \text{ implies } \langle\infty, \infty\rangle \in \operatorname{vint}(\operatorname{hcl}(X)).$$

Fix X such that $\langle\infty, \infty\rangle \in \operatorname{hcl}(\operatorname{vint}(X))$. This means that there is some cofinal set $A = \{\alpha_\gamma\,;\,\gamma < \operatorname{cf}\kappa\} \subseteq \kappa$ such that $A \times \{\infty\} \subseteq \operatorname{vint}(X)$, so for each γ, there is some $\beta_\gamma < \lambda$ such that

$$\{\infty\} \cup \{\beta\,;\,\beta_\gamma \leq \beta < \lambda\} \subseteq X^{\alpha_\gamma}.$$

The set $\{\beta_\gamma\,;\,\gamma < \operatorname{cf}\kappa\}$ has cardinality $\operatorname{cf}\kappa < \operatorname{cf}\lambda$, so $\beta^* := \sup\{\beta_\gamma\,;\,\gamma < \operatorname{cf}\kappa\} < \lambda$. But then for every $\beta > \beta^*$, we have that $A \subseteq X_\beta$, and so $\langle\infty, \beta\rangle \in \operatorname{hcl}(X)$. This means that $\{\infty\} \cup \{\beta\,;\, \beta^* < \beta\}$ is an $\mathbf{card}_\lambda$-open neighbourhood contained in $(\mathrm{hcl}(X))^\infty$, so $\langle\infty, \infty\rangle \in \mathrm{vint}(\mathrm{hcl}(X))$. $\square$

Corollary 2.5.5 *For $\aleph_0 \leq \mathrm{cf}\,\kappa < \mathrm{cf}\,\lambda$, $\mathbf{card}_\kappa$ and $\mathbf{card}_\lambda$ are non-Alexandroff spaces such that* chr *holds in $\mathbf{card}_\kappa \times \mathbf{card}_\lambda$. In particular, this is true in $\mathbf{card}_{\aleph_0} \times \mathbf{card}_{\aleph_1}$.*

Corollaries 2.5.3 and 2.5.5 together answer Question **??** negatively:

$$\mathbf{card}_{\aleph_0}, \mathbf{card}_{\aleph_1} \models \mathrm{chr} \,\&\, \mathrm{com}_\rightarrow \,\&\, \neg\mathrm{com}_\leftarrow, \text{ and}$$

$$\mathbf{card}_{\aleph_1}, \mathbf{card}_{\aleph_0} \models \mathrm{chr} \,\&\, \mathrm{com}_\leftarrow \,\&\, \neg\mathrm{com}_\rightarrow.$$

2.6 The logic of product spaces

As we saw in the previous section, both *com* and *chr* can be refuted on products of topological spaces. This suggests that the complete logic of all products of topological spaces is weaker than $\mathbf{S4} \times \mathbf{S4}$. The main goal of this section is to show that this logic is $\mathbf{S4} \oplus \mathbf{S4}$. In fact, we will show that $\mathbf{S4} \oplus \mathbf{S4}$ is complete with respect to $\mathbb{Q} \times \mathbb{Q}$.

Theorem 2.6.1 $\mathbf{S4} \oplus \mathbf{S4}$ *is complete with respect to $\mathbb{Q} \times \mathbb{Q}$.*

Proof By Proposition A.0.1, $\mathbf{S4} \oplus \mathbf{S4}$ is complete with respect to the infinite quaternary tree $T_{2,2} = \langle W, R_1, R_2\rangle$. We view $T_{2,2}$ as equipped with two Alexandroff topologies defined from R_1 and R_2. To prove completeness of $\mathbf{S4} \oplus \mathbf{S4}$ with respect to $\mathbb{Q} \times \mathbb{Q}$ we take the X constructed in the proof of Theorem 2.2.4, define recursively a HV-open subspace Y of $X \times X$ and a continuous open map g from Y onto $T_{2,2}$ with respect to both topologies: this will allow us to transfer counterexamples from $T_{2,2}$ to Y, then from Y to $X \times X$, and finally from $X \times X$ to $\mathbb{Q} \times \mathbb{Q}$.

Let $Y = \bigcup_{n\in\omega} Y_n$, where $Y_0 = \{(0, 0)\}$ and

$$Y_{n+1} \;=\; Y_n \cup \{(x - \frac{1}{3^n}, y), (x + \frac{1}{3^n}, y), (x, y - \frac{1}{3^n}), (x, y + \frac{1}{3^n}) \mid (x, y) \in Y_n\}$$

Claim 2.6.2 *Y is a HV-open subspace of $X \times X$.*

Proof Let $(x,y) \in Y$. Then $x \in (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}}) \subseteq X$. Therefore, $(x,y) \in (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}}) \times \{y\} \subseteq Y$. Thus, Y is a H-open subspace of $X \times X$. That Y is a V-open subspace of $X \times X$ is proved symmetrically. □

A similar argument as before shows that for each $(x,y) \in Y$ such that $(x,y) \neq (0,0)$ there exists $n_{(x,y)}$ with $(x,y) \in Y_{n_{(x,y)}}$ and $(x,y) \notin Y_{n_{(x,y)}-1}$, and that there is a unique $(u,v) \in Y_{n_{(x,y)}-1}$ such that $(x,y) = (u \pm \frac{1}{3^{n(x,y)-1}}, v)$ or $(x,y) = (u, v \pm \frac{1}{3^{n(x,y)-1}})$.

We define g from Y onto $T_{2,2}$ by recursion (cf. Figure 2.4(b)): If $(x,y) = (0,0)$ then we let $g(0,0)$ be the root r of $T_{2,2}$; if $(x,y) \neq (0,0)$ then $(x,y) = (u \pm \frac{1}{3^{n(x,y)-1}}, v)$ or $(x,y) = (u, v \pm \frac{1}{3^{n(x,y)-1}})$ for a unique $(u,v) \in Y_{n_{(x,y)}-1}$, and we let

$$g(x,y) = \begin{cases} \text{the left } R_1\text{-successor of } g(u,v) & \text{if } (x,y) = (u - \frac{1}{3^{n(x,y)-1}}, v) \\ \text{the right } R_1\text{-successor of } g(u,v) & \text{if } (x,y) = (u + \frac{1}{3^{n(x,y)-1}}, v) \\ \text{the left } R_2\text{-successor of } g(u,v) & \text{if } (x,y) = (u, v - \frac{1}{3^{n(x,y)-1}}) \\ \text{the right } R_2\text{-successor of } g(u,v) & \text{if } (x,y) = (u, v + \frac{1}{3^{n(x,y)-1}}) \end{cases}$$

Claim 2.6.3 *g is open and continuous with respect to both topologies.*

Proof Let τ_1 and τ_2 denote the restrictions of the horizontal and vertical topologies of $X \times X$ to Y, respectively. We prove that g is open and continuous with respect to τ_1. That it is open and continuous with respect to τ_2 is proved symmetrically. We observe that

$$\{(x - \frac{1}{3^{n_{(x,y)}}}, x + \frac{1}{3^{n_{(x,y)}}}) \times \{y\} | (x,y) \in Y\}$$

forms a basis for τ_1. We also recall that a basis for the Alexandroff topology on $T_{2,2}$ defined from R_1 is $\mathcal{B}_1 = \{B^1_t\}_{t \in T_{2,2}}$ where $B^1_t = \{s \in T_{2,2} : tR_1s\}$.

To see that g is open, let $(x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\}$ be a basic open for τ_1. Then the same argument as in Claim 2.2.6 guarantees that $g((x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\}) = B^1_{g(x,y)}$. Thus g is open. To see that g is continuous it suffices to show that for each $t \in T_{2,2}$, the g-inverse image of B^1_t belongs to τ_1. Let $(x,y) \in g^{-1}(B^1_t)$. Then $tR_1g(x,y)$. So $g((x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\}) = B^1_{g(x,y)} \subseteq B^1_t$. Thus there exists an open neighborhood $U = (x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\}$ of (x,y) such that $U \subseteq g^{-1}(B^1_t)$, implying that g is continuous. □

To complete the proof, if $\mathbf{S4} \oplus \mathbf{S4} \nvdash \varphi$, then by Proposition A.0.1, there is a valuation ν on $T_{2,2}$ such that $\langle T_{2,2}, \nu \rangle, r \not\models \varphi$. Define a valuation ξ on Y by $\xi(p) = g^{-1}(\nu(p))$. Since g is continuous and open with respect to both topologies and $g(0,0) = r$, we have that $(0,0)$ and r are 2-topo-bisimilar. Therefore, $\langle Y, \xi \rangle, (0,0) \not\models \varphi$. Now since Y is a HV-open subset of $X \times X$, we obtain that φ is refutable on $X \times X$. Finally, Theorem 2.2.4 implies that X is homeomorphic to $\mathbb{Q}$. Therefore, $X \times X$ is HV-homeomorphic to $\mathbb{Q} \times \mathbb{Q}$, and hence φ is also refutable on $\mathbb{Q} \times \mathbb{Q}$. $\square$

Corollary 2.6.4 $\mathbf{S4} \oplus \mathbf{S4}$ *is the logic of products of arbitrary topologies.*

It follows that the logic of products of arbitrary topologies is decidable and has a $PSPACE$-complete satisfiability problem [65]. This stands in contrast with the satisfiability problem for $\mathbf{S4} \times \mathbf{S4}$, which turned out to be undecidable [40].

Let us say that a logic L in the language $\mathcal{L}_{\Box_1 \Box_2}$ has the *finite topo-product model property* if any non-theorem of L is refuted on a finite product space. Then the logic of products of arbitrary topologies does not have the finite topo-product model property as finite spaces are Alexandroff, and hence validate *com* and *chr*.[1] This remark is not to be confused with the non existence of finite Kripke models: it follows from Theorem 2.2.7 that every non-theorem of $\mathbf{S4} \oplus \mathbf{S4}$ does indeed fail on a finite model.

2.7 Adding the true product interior

So far, we have only focused on the horizontal and vertical topologies on the product space, by analogy to products of relational structures. However, the topological semantics suggests a further addition to the language. In this section we investigate the modal logic of products of topological spaces with all three horizontal, vertical, and standard product topologies. We add to the language $\mathcal{L}_{\Box_1 \Box_2}$ an extra modal operator $\Box$ with the intended interpretation as the interior operator of the standard product topology.

[1] In fact, the same argument implies that no logic in the interval $[\mathbf{S4} \oplus \mathbf{S4}, \mathbf{S4} \times \mathbf{S4}[$ has the finite topo-product model property.

For two topological spaces $\mathcal{X} = \langle X, \eta\rangle$ and $\mathcal{Y} = \langle Y, \theta\rangle$, we will consider the product $\langle X \times Y, \tau, \tau_1, \tau_2\rangle$ with three topologies, where τ is the standard product topology, τ_1 is the horizontal topology, and τ_2 is the vertical topology. Then $\Box$ is interpreted as follows.

$$(x, y) \models \Box\varphi \quad \text{iff} \quad \exists U \in \eta \text{ and } \exists V \in \theta : U \times V \models \varphi$$

Since $\tau \subseteq \tau_1 \cap \tau_2$, we obtain that the modal principle

$$\Box p \to \Box_1 p \wedge \Box_2 p$$

is valid in product spaces. Our main goal in this section is to show that adding this principle to the fusion of three copies of **S4** axiomatizes the logic of products of topological spaces (with three topologies).

Definition 2.7.1 *Let $\mathcal{L}_{\Box,\Box_1,\Box_2}$ be a modal language with three modal operators $\Box$, $\Box_1$, and $\Box_2$. We define the* topological product logic **TPL** *as the least set of formulas in $\mathcal{L}_{\Box,\Box_1,\Box_2}$ containing all axioms of* **S4** $\oplus$ **S4** $\oplus$ **S4** *plus the axiom $\Box p \to \Box_1 p \wedge \Box_2 p$, and closed under modus ponens, substitution, and $\Box$-, $\Box_1$-, and $\Box_2$-necessitation.*

Let $T_{6,2,2}$ denote the infinite six branching tree such that each node of $T_{6,2,2}$ is R-related to all six of its immediate successors, R_1-related to the first two, and R_2-related to the last two; R, R_1, and R_2 are taken to be reflexive and transitive. Formally $T_{6,2,2}$ can be defined as $\langle W, R, R_1, R_2\rangle$, where $W = \{0, 1, 2, 3, 4, 5\}^*$,

sRt iff $\exists u \in \{0, 1, 2, 3, 4, 5\}^* : s \cdot u = t$

sR_1t iff $\exists u \in \{0, 1\}^* : s \cdot u = t$

sR_2t iff $\exists u \in \{4, 5\}^* : s \cdot u = t$ (see Figure 2.3)

Theorem 2.7.2 **TPL** *is complete with respect to $T_{6,2,2}$.*

Proof See Appendix A. $\Box$

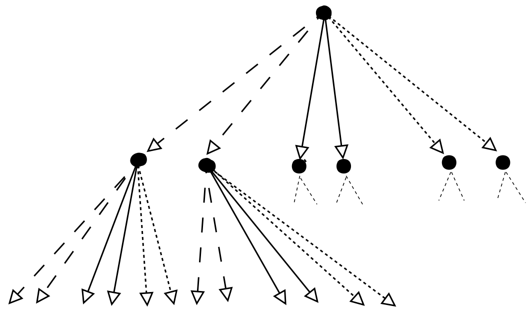

Figure 2.3: $T_{6,2,2}$. The solid lines represent R, the dashed lines represent R_1, and the dotted lines represent R_2. We assume that all dashed and dotted lines are also solid.

(a)

(b)

(c)

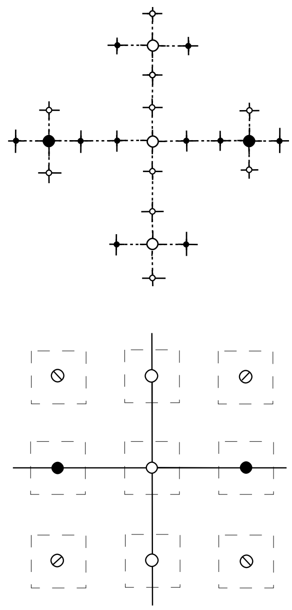

Figure 2.4: The first stages of the labelling in the completeness proofs for (a) **S4**, (b) **S4** $\oplus$ **S4**, and (c) **TPL**.

Theorem 2.7.3 **TPL** *is complete with respect to* $\mathbb{Q} \times \mathbb{Q}$.

Proof Our strategy is similar to that of the proof of Theorem 2.6.1. By Theorem A.0.8 **TPL** is complete with respect to $T_{6,2,2} = \langle W, R, R_1, R_2 \rangle$. We view $T_{6,2,2}$ as equipped with three Alexandroff topologies defined from R, R_1, and R_2. So for completeness of **TPL** with respect to $\mathbb{Q} \times \mathbb{Q}$ it is sufficient to show that there exists a total 3-topo-bisimulation between the $X \times X$ defined in the proof of Theorem 2.6.1 and $T_{6,2,2}$.

We define h from $X \times X$ onto $T_{6,2,2}$ by recursion following the inductive definition of X (cf. Figure 2.4(c)): If $(x, y) = (0, 0)$ then we let $h(0, 0)$ be the root r of $T_{6,2,2}$; if $(x, y) \neq (0, 0)$ then there is a unique (u, v) that is labelled before (x, y) such that $(x, y) = (u \pm \frac{1}{3^{n(x,y)-1}}, v)$ or $(x, y) = (u, v \pm \frac{1}{3^{n(x,y)-1}})$ or $(x, y) = (u \pm \frac{1}{3^{n(x,y)-1}}, v \pm \frac{1}{3^{n(x,y)-1}})$. Then we let

$$h(x,y) = \begin{cases} \text{the left } R_1\text{-successor of } h(u,v) & \text{if } (x,y) = (u - \frac{1}{3^{n(x,y)-1}}, v) \\ \text{the right } R_1\text{-successor of } h(u,v) & \text{if } (x,y) = (u + \frac{1}{3^{n(x,y)-1}}, v) \\ \text{the left } R_2\text{-successor of } h(u,v) & \text{if } (x,y) = (u, v - \frac{1}{3^{n(x,y)-1}}) \\ \text{the right } R_2\text{-successor of } h(u,v) & \text{if } (x,y) = (u, v + \frac{1}{3^{n(x,y)-1}}) \\ \text{the first remaining } R\text{-successor} & \text{if } (x,y) = (u + \frac{1}{3^{n(x,y)-1}}, v + \frac{1}{3^{n(x,y)-1}}) \\ & \text{or } (x,y) = (u - \frac{1}{3^{n(x,y)-1}}, v - \frac{1}{3^{n(x,y)-1}}) \\ \text{the last remaining } R\text{-successor} & \text{if } (x,y) = (u + \frac{1}{3^{n(x,y)-1}}, v - \frac{1}{3^{n(x,y)-1}}) \\ & \text{or } (x,y) = (u - \frac{1}{3^{n(x,y)-1}}, v + \frac{1}{3^{n(x,y)-1}}) \end{cases}$$

Claim 2.7.4 *h is open and continuous with respect to all three topologies.*

Proof The argument that h is open and continuous with respect to τ_1 and τ_2 carries over directly from Claim 2.6.3. The same technique can be used to show that h is open and continuous with respect to τ. To see this, we observe that

$$\{(x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times (y - \frac{1}{3^{n(x,y)}}, y + \frac{1}{3^{n(x,y)}}) \; : \; (x,y) \in X \times X\}$$

form a basis for τ on $X \times X$. We also observe that a basis for the topology on $T_{6,2,2}$

defined from R is $\mathcal{B} = \{B_t\}_{t \in T_{6,2,2}}$ where $B_t = \{s \in T_{6,2,2} : tRs\}$.

To see that h is open, let $(x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times (y - \frac{1}{3^{n(x,y)}}, y + \frac{1}{3^{n(x,y)}})$ be a basic open for τ. Then the same argument as in Claim 2.6.3 guarantees that

$$h((x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times (y - \frac{1}{3^{n(x,y)}}, y + \frac{1}{3^{n(x,y)}})) = B_{h(x,y)}.$$

Thus h is open. To see that h is continuous it suffices to show that for each $t \in T_{6,2,2}$, the h-inverse image of B_t belongs to τ. Let $(x, y) \in h^{-1}(B_t)$. Then $tRh(x, y)$. So

$$h((x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times (y - \frac{1}{3^{n(x,y)}}, y + \frac{1}{3^{n(x,y)}})) = B_{h(x,y)} \subseteq B_t.$$

Thus there exists an open neighborhood $U = (x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times (y - \frac{1}{3^{n(x,y)}}, y + \frac{1}{3^{n(x,y)}})$ of (x, y) such that $U \subseteq h^{-1}(B_t)$, implying that h is continuous. □

To complete the proof, if **TPL** $\nvdash \varphi$, then by Theorem A.0.8, there is a valuation ν on $T_{6,2,2}$ such that $\langle T_{6,2,2}, \nu\rangle, r \not\models \varphi$. Define a valuation ξ on $X \times X$ by $\xi(p) = h^{-1}(\nu(p))$. Since h is continuous and open with respect to all three topologies and $h(0, 0) = r$, we have that $(0, 0)$ and r are 3-topo-bisimilar. Therefore, $\langle X \times X, \xi\rangle, (0, 0) \not\models \varphi$. Now since $X \times X$ is homeomorphic to $\mathbb{Q} \times \mathbb{Q}$ with respect to all three topologies, it follows that φ is also refutable on $\mathbb{Q} \times \mathbb{Q}$. □

Corollary 2.7.5 *In the language $\mathcal{L}_{\Box,\Box_1,\Box_2}$,* **TPL** *is the logic of products of arbitrary topologies.*

Incidentally, (using Kripke semantics) it is easy to show that **TPL** is a conservative extension of **S4** $\oplus$ **S4**, and that **S4** $\oplus$ **S4** is a conservative extension of **S4**. Therefore, Theorem 2.2.4 becomes a corollary of Theorem 2.6.1, while Theorem 2.6.1 becomes a corollary of Theorem 2.7.3.

2.8 Conclusions and further directions

We introduced the horizontal and vertical topologies on the product of two topological spaces and we showed that the modal logic of products of topological spaces with two horizontal and vertical topologies is the fusion $\mathbf{S4} \oplus \mathbf{S4}$. In addition, we axiomatized the modal logic of products of topological spaces with three horizontal, vertical, and standard product topologies. We conclude by mentioning several open questions that arise naturally from this study.

2.8.1 Special spaces

Although we showed that $\mathbf{S4} \oplus \mathbf{S4}$ is complete with respect to $\langle \mathbb{Q} \times \mathbb{Q}, \tau_1, \tau_2 \rangle$, and that **TPL** is complete with respect to $\langle \mathbb{Q} \times \mathbb{Q}, \tau, \tau_1, \tau_2 \rangle$, it is still an open question what the logics of $\langle \mathbb{R} \times \mathbb{R}, \tau_1, \tau_2 \rangle$ and $\langle \mathbb{R} \times \mathbb{R}, \tau, \tau_1, \tau_2 \rangle$ are.

Since Alexandroff spaces can be represented as **S4**-frames, it follows from Gabbay at al. [38] that the modal logic of the products of Alexandroff spaces (with horizontal and vertical topologies) is $\mathbf{S4} \times \mathbf{S4}$. On the other hand, it is still unknown what the modal logic is of the products of Alexandroff spaces with arbitrary topological spaces. We conjecture that this logic is $\mathbf{S4} \oplus \mathbf{S4} + com_{\leftarrow} + chr$.

2.8.2 Enriching the language

From a topological perspective, our topological completeness result for $\mathbf{S4} \oplus \mathbf{S4}$ seems to suggest that the basic modal language is not expressive enough to model interesting interactions between horizontal and vertical topologies. This suggests to consider richer languages. In adding $\Box$ we have made the first step in this direction, but there are several others that can be taken. For instance, adding the universal modality or nominals.

A very natural extension of the language would be with the common knowledge operator. In the standard Kripke setting, there are several ways of defining common knowledge, but they all turn out to be equivalent (see [10]). In [17] we examine two most prominent such ways and show that in the topological setting the two are in fact distinct. The first defines the common knowledge as an infinite conjunction of claims in the original language, and the second takes common knowledge to be the greatest fixed point of an operator. Thus in our setting the two are:

1. $C_{1,2}\varphi :=$ an infinite conjunction of all finite nestings of $\Box_1$ and $\Box_2$:

$$\varphi \wedge \Box_1\varphi \wedge \Box_2\varphi \wedge \Box_1\Box_2\varphi \wedge \ldots$$

2. $K_{1,2}\varphi :=$ the greatest fixed point of the operator $\lambda X.([|\phi|] \cap I_1X \cap I_2X)$, as in the following formula of the modal μ-calculus:

$$\nu p.(\varphi \wedge \Box_1 p \wedge \Box_2 p)$$

We argue in [17] that the common knowledge as the greatest fixed point is most interesting from the topological perspective.

2.8.3 Further exploration of the connection with Kripke semantics

We have shown that the topological setting has greater power of discrimination than the relational setting. In particular, topological products validate less principles than products of Kripke frames, and the true product interior modality is not definable in terms of the horizontal and vertical modalities. Several further lines of study might help us understand better the difference between topological products and relational products. Here, we will name two.

Given topological spaces $\langle X, \eta\rangle$ and $\langle Y, \theta\rangle$, call a subset of $X \times Y$ a *block* if it is of the form $A \times B$ for some $A \subseteq X$ and $B \subseteq Y$. Next, call a valuation ν for the product $\langle X \times Y, \tau_1, \tau_2\rangle$ *admissible* if it assigns to each propositional letter a finite union of blocks. Interestingly, when attention is restricted to admissible valuations only, the interaction principles *com* and *chr* become valid again. In fact, we conjecture that the logic obtained in this way (when no restrictions are made on the topological spaces themselves) is precisely **S4** $\times$ **S4**.

The second line of study concerns generalizations of the product construction on Kripke frames. One such generalization is obtained by restricting the universe of admissible product subsets (see, e.g., [14]). The latter is a well-known strategy in relational algebra and arrow logic (see Chapter 7 of [15]). In particular, over such generalized relational products we have that *com* and *chr* are no longer valid, and that the product $\Box$ is no longer definable as $\Box_1\Box_2$. This similarity suggests a connection between topological products and generalized relational products.

Incidentally, we believe that the product construction discussed in this paper is of independent interest. Like the fusion and the Kripke product operation, it induces an operation on modal logics. Given normal modal logics L_1, L_2 above **S4**, we can define $L_1 \times_t L_2$ as the bi-modal logic of the class of products of topological spaces of L_1 and L_2 (with horizontal and vertical topologies). One of our main results, then, tells us that $\mathbf{S4} \times_t \mathbf{S4} = \mathbf{S4} \oplus \mathbf{S4}$. More generally, it is not hard to see that

$$(L_1 \oplus L_2) \quad \subseteq \quad (L_1 \times_t L_2) \quad \subseteq \quad (L_1 \times L_2)$$

Many questions arise from this perspective. For instance, does decidability transfer under $\times_t$?

Chapter 3

Combining Order and Topology: Topo-Directional and Topo-Compass Logics

3.1 Introduction

In this chapter we investigate topological multi-modal logics which recognize directionality. Adding direction, for instance left-right, or compass directions, seems a natural step in the programme of adding extra expressive power to languages for reasoning about space. In particular in the product spaces of chapter 2, adding compass directions seems very natural. In this chapter, however, we start with a one dimensional case, to get a better sense of what is involved. In this way we obtain a class of what can best be viewed as interval temporal logics.

We will enable the modalities to recognize directionality by incorporating a linear ordering into the topological interpretation. This places this chapter in between two significant current trends in modal logic. On the one hand, there is the topological interpretation, described in more detail in Chapter 2, and on the other the topic of modal logic over linear orders. The latter has been introduced by Segerberg in [62] and studied a great deal since in temporal logic. [For more recent work, see for instance [44].]

More recently, within the general trend of combining modal logics, [c.f. [38]] the combination of topological logic with various logics of linear orders have been studied. A key example is Shehtman: [64]. In the most basic case, one combines our topological modality $\Box$ with the temporal modalities F, P for the linear order. The appropriate frames are then triples $(X, \tau, <)$ consisting of a set X, a topology τ on X, and a linear ordering $<$ on X. A model M adds a valuation function and the key clauses of the truth definition read as follows:

$M, x \models \Box\phi$ iff $\exists U \in \tau, \forall y \in U, M, y \models \phi$
$M, x \models F\phi$ iff $\exists y, x < y$, and $M, y \models \phi$
$M, x \models P\phi$ iff $\exists y, y < x$, and $M, y \models \phi$

In the simplest case, there is no interaction between τ and $<$, therefore the minimal logic of the models is likely to be the fusion of **S4** and the minimal logic for linear orders, although we are not aware of a published proof of this.

In more interesting cases, however, the topology and the order are related. In particular, consider the order topology whose base is defined the collection of open intervals of the form $\{z \in X \mid x < z < y\}$. The general logic of order topologies has not been studied as far as we know. But there are completeness results for order topologies on specific structures. Shehtman axiomatized the logic for the case of the rational numbers $\mathbb{Q}$, and his proof was considerably simplified by Gerhardt in his Master Thesis [41].

The most challenging open problem in this area appears to be an axiomatization of order topology on $\mathbb{R}$. This has proved surprisingly difficult, but an interesting new approach is found in some recent research on this question by G. Bezhanishvili, N. Bezhanishvili, and C. Kupke [21].

It is worth noting that the logic of $\Box, F, P$ over order topology on linear orders can be embedded into the the stronger temporal logic of Until/Since via the equivalences:

$$\Box p \equiv S(\top, p) \wedge p \wedge U(\top, p)$$

$$Fp \equiv U(p, \top)$$

$$Pp \equiv S(p, \top).$$

The Since/Until logics for the general linear orders, $\mathbb{Q}$, and $\mathbb{R}$ are all known and well studied [44]. So in this specific case we are really asking for perspicuous axiomatization of fragments.

This is not true, of course, for general order topology. Cf. the order topologies on Minkowski space in [73], [74], or [42].

In this chapter, however, we want to introduce a new perspective and integrate topology and order in a more intimate fashion.

3.2 Topo-Directional Logic

It seems that what is interesting about order topologies is that they integrate an order and a topology. In this chapter we follow that line of thought, and instead of having a separate modality for the topology and a pair of modalities for the order, we introduce a pair of directional modalities that in their semantics combine the topology and the order, rather than treating them as separate notions.

To begin with, our language will be a propositional bimodal language **TDL** with the modalities $\Box_L$ and $\Box_R$ (to be read 'box left' and 'box right' respectively). Later on in a product setting we will have four such modalities for compass directions.

3.2.1 Semantics

Definition 3.2.1 *A* topo-directional structure *$\mathcal{X}$ is a triple $(X, \tau, <)$, where X is a set, τ a topology on X, and $<$ a linear ordering of X.*

A model M is defined with as a topo-directional structure plus a valuation function $\nu : At \to \wp(X)$. The interesting cases of the truth definition are:

$\mathcal{M}, x \models \Box_R\phi$ iff $\exists U \in \tau$, $x \in U$ and $\forall z \in U$, if $x < z$, then $\mathcal{M}, z \models \phi$,
$\mathcal{M}, x \models \Box_L\phi$ iff $\exists U \in \tau$, $x \in U$ and $\forall z \in U$, if $z < x$, then $\mathcal{M}, z \models \phi$.

Corresponding to each box we define a diamond dual in the usual way:

$$\Diamond_R\phi =_{def.} \neg\Box_R\neg\phi.$$

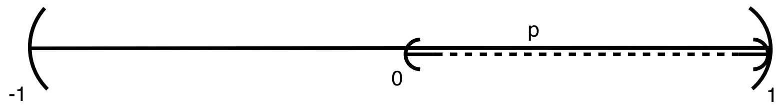

Figure 3.1: Truth definition for $\Box_R p$.

$$\Diamond_L \phi =_{def.} \neg \Box_L \neg \phi.$$

Semantically this amounts to

$$\mathcal{M}, x \models \Diamond_R \phi \text{ iff } \forall U \in \tau \text{, if } x \in U \text{ then } \exists z \in U, x < z, \text{ and } \mathcal{M}, z \models \phi$$

Or in words, $\Diamond_R \phi$ is true if ϕ points are strictly approaching the current point from the right, and,

$$\mathcal{M}, x \models \Diamond_L \phi \text{ iff } \forall U \in \tau \text{, if } x \in U \text{ then } \exists z \in U, z < x, \text{ and } \mathcal{M}, z \models \phi$$

Or in words, $\Diamond_L \phi$ is true if ϕ points are strictly approaching the current point from the left.

To visualize the truth definitions, we think of $\Box_R$ modality as saying that ϕ holds in some neighborhood of current point along a direction, i.e. in this case to the right.

In Figure 3.1, we assume we are working with standard metric topology on $\mathbb{R}$. Since p is true in the interval $(0, 1)$, it is true everywhere to the right of 0 in the open $(-1, 1)$ say, and hence $\Box_R p$ is true at 0.

In general, however, there is no immediate connection between the ordering $<$ and the topology τ, i.e., the topology is not necessarily the order topology defined by means of $<$. For instance, we could take the topology to be the standard metric topology on $\mathbb{R}^2$ and the ordering defined as follows,

$$\{(x, y) <' (x', y') \text{ iff } x < x' \text{ or } x = x' \text{ and } y < y'$$

where $<$ is the standard ordering of the reals.[1] Here we have an ordering and a topology that cannot be defined in terms of this ordering in any obvious way.

The semantics is of independent interest on a temporal interpretation. If we are working in some order topology, we can think of $\Box_R p$ as expressing the fact that p will be the case for a while in the future, and, similarly, of $\Box_L p$ as expressing the fact that p has been true for a while. Referring to the earlier since until temporal language, our two modalities can again be defined:

$U(p, \top)$ (p is true until the top becomes true, i.e., p will be true for a while) and

$S(\top, p)$ p has been true for a while.

3.2.2 Expressive Power

Since it is a kind of temporal language, we want to situate the langauge of Topo-Directional Logic within the class of related languages over topo-directional structures. As we will see, the logic is sufficiently independent from its immediate logical relatives to be studied on its own as a temporal logic. Its closest relatives are the modal logic $S4$ with the topological interpretation, the basic temporal logic of F, P interpreted over a linear order $<$ and their combined language $\Box, F, P$ over a topology equipped with an order. As we will see although as one should expect our modalities $\Box_R$ and $\Box_L$ are definable in the language of until and since, none of the above weaker relatives define $\Box_R$ and $\Box_L$.

Fact 3.2.2 *The standard topological modality $\Box$ is definable in the language of* **TDL**.

$\Box p$ can be defined as $\Box_L p \wedge p \wedge \Box_R p$. This should be no surprise, since we intended the language as a more expressive addition to the standard topo-semantics.

To measure the expressive power of our language, we need an appropriate notion of bisimulation. For a start, here is a useful perspective. Using the given topology τ, we can in addition define two topologies τ_R, τ_L via the following bases:

[1] The ordering is known in the literature as the *alphabetic ordering.*

$$B_{\tau_R} = \{U' \subset X \mid \exists U \in \tau, \exists x \in U, \text{ and } U' = \{y \in U \mid x < y \text{ or } x = y\}\}$$

and

$$B_{\tau_L} = \{U' \subset X \mid \exists U \in \tau, \exists x \in U, \text{ and } U' = \{y \in U \mid y < x \text{ or } x = y\}\}$$

It is easily seen that τ_R, τ_L are topologies.

Fact 3.2.3 *The standard topological bisimulation [see Chapter 2] with respect to topologies τ_R and τ_L preserves formulae of the language of* **TDL** *in the sense that a bimodal formula ϕ is true in the* **TDL** *model at a point x iff ϕ interpreted topologically is true in the bisimilar point x' in the bimodal topological model.*

Thus we can look for models of modal formula in the **TDL** langauge in the standard topological semantics. In particular, we can use Alexandroff topologies, or Kripke frames to find appropriate models.

There is of course no guarantee in general that for a model bisimilar in the standard sense there will be an ordering $<$, and a topology τ such that at bisimilar points formulae are all true under the semantics of **TDL**. For that we need a stronger notion of bisimulation.

Definition 3.2.4 *Let $M = (X, \tau, <, \nu)$ and $M' = X', \tau', <', \nu')$ be Topo-Directional models. A* topo-directional bisimulation *is a nonempty relation $\leftrightharpoons \ \subseteq X \times X'$ such that if $x \leftrightharpoons x'$ then the following hold:*

(I) Base*: $x \in \nu(p)$ iff $x' \in \nu'(p)$, for any proposition variable p*

(II) Forth condition R*: $x \in U \in \tau$ implies that there exists $U' \in \tau'$ such that $x' \in U'$ and for all $z' \in U'$, if $x' <' z'$ then there exists $z \in U, x < z$ and $z \rightleftharpoons z'$*

(III) Forth condition L*: $x \in U \in \tau$ implies that there exists $U' \in \tau'$ such that $x' \in U'$ and for all $z' \in U'$, if $z' <' x'$ then there exists $z \in U, z < x$ and $z \rightleftharpoons z'$*

(IV) Back condition R*: $x' \in U' \in \tau'$ implies that there exists $U \in \tau$ such that $x \in U$ and for all $z \in U$, if $x < z$, then there exists $z' \in U', x' <' z'$ and $z \rightleftharpoons z'$*

(IV) BACK CONDITION L*: $x' \in U' \in \tau'$ implies that there exists $U \in \tau$ such that $x \in U$ and for all $z \in U$, if $x < z$, then there exists $z' \in U', x' <' z'$ and $z \rightleftharpoons z'$*

Proposition 3.2.5 *Let $M = (X, \tau, <, \nu)$ and $M' = (X', \tau', <', \nu')$ be topo-directional models, and let $x \rightleftharpoons x'$ for some topo-directional bisimulation $\rightleftharpoons \ \subseteq X \times X'$. Then for every modal formula φ in the language of* **TDL** *we have that $M, x \models \varphi$ iff $M', x' \models \varphi$.*

Proof The proof is a straightforward induction on depth of formulae. □

As usual, the notion of bisimulation can be applied in undefinability results. For instance,

Fact 3.2.6 *F, P are not definable in the language of* **TDL***.*

Proof We let X be $\mathbb{R}$ and X' be $(0,1) \subset \mathbb{R}$. Further, we let $\nu(p) = \{-2, 2\}$, and $\nu'(p) = \emptyset$. Then obviously $M, 0 \models Fp \wedge Pp$ and $M', 0 \not\models Fp \vee Pp$. It can be checked that the identity relation on $(0,1)$ is a topo-directional bisimulation on which $0 \rightleftharpoons 0$. Since Fp, Pp are true in in one of the bisimilar points and false in the other, the formulae cannot be definable in the language of **TDL**. □

Corollary 3.2.7 *Until and Since are not definable in the language of* **TDL***.*

We now turn to results showing that **TDL** itself is not definable in in the modal logics of its components.

Fact 3.2.8 *$\Box_L, \Box_R$ are not definable in the language of basic temporal logic.*

Proof If $\Box_L$, $\Box_R$ were definable with F, P, then we could define $\Box$ in the language of F and P alone, which is known to be impossible. □

Fact 3.2.9 *$\Box_L, \Box_R$ are not definable in the language of the interior operator $\Box$ alone.*

Proof Let X, X' be $\mathbb{R}$, $\nu(p) = \{r \in R \,|\, r < 0\}$, $\nu'(p) = \{r \in R \,|\, r > 0\}$. Then the following function is a topo-bisimulation

$$f(r) = -r$$

But note that this bisimulation does not preserve the formulas of **TDL**. For instance $\Box_R p$ is true at 0 in X' but it is false at 0 in X. □

Using a similar argument we can prove

Fact 3.2.10 *$\Box_L, \Box_R$ are not definable in the language of basic temporal logic together with the interior operator $\Box$.*

Proof □

3.3 Minimal Logic of TDL

In this section, we examine the logic of the language of **TDL** over its most general class of models. The axioms that we find, seem of independent interest and increase our understanding of what **TDL** is about.

3.3.1 Axioms

We need the following four axioms:

$\mathbf{4_R}$ $\Box_R p \to \Box_R \Box_R p$
$\mathbf{4_L}$ $\Box_L p \to \Box_L \Box_L p$
$\mathbf{LR_1}$ $(p \wedge \Box_R p \wedge \Box_L p) \to \Box_R \Box_L p$
$\mathbf{LR_2}$ $(p \wedge \Box_R p \wedge \Box_L p) \to \Box_L \Box_R p$

Here the first two formulae are the usual **.4** axioms. But the latter two are more interesting.

$\mathbf{LR_i}$ stand not only for 'left-right', but also for 'linearity' since, as we will see in the soundness proof below, they reflect the linearity of the underlying ordering.

These axioms have Sahlqvist form[2], and their respective first-order correspondents are as follows:

Fact 3.3.1 $\mathbf{LR_1}$ *corresponds to*

$\mathbf{FO-LR_1}$ $\forall x \forall y((R_R xy \wedge R_L yz) \to (x = z \vee R_R xz \vee R_L xz))$

$\mathbf{LR_2}$ *corresponds to*

$\mathbf{FO-LR_1}$ $\forall x \forall y((R_L xy \wedge R_R yz) \to (x = z \vee R_R xz \vee R_L xz))$

[2]For more on Sahlqvist correspondence see, for instance, [25].

With some abuse of notation, we baptize our logic.

Definition 3.3.2 *Let* $\mathbf{TDL} = \mathbf{K4} \oplus \mathbf{K4} \cup \{\mathbf{LR_1}, \mathbf{LR_2}\}$.

Recall that $\mathbf{K4} \oplus \mathbf{K4}$ stands for the fusion of **K4** with itself.

Definition 3.3.3 *A two-relation Kripke frame* $\mathcal{F} = (W, R_L, R_R)$ *is a* **TDL**-frame*, if* R_L, R_R *are transitive, and they jointly satisfy* $\mathbf{FO} - \mathbf{LR_1}$ *and* $\mathbf{FO} - \mathbf{LR_2}$.

The following results follows from general modal completeness theory:

Lemma 3.3.4 **TDL** *is sound and complete with respect to the class* $\mathfrak{F}$ *of* **TDL**-*frames.*

Proof We simply note once again that all axioms have Sahlqvist form. $\Box$

The preceding result is not yet what we want, however, because it refers to relational Kripke frames instead of the topo-directional frames which are our real concern.

3.3.2 Soundness

Let us first check that **TDL** is indeed sound for the intended semantics.

Proposition 3.3.5 *The axioms and rules of* **TDL** *are valid in topo-directional models.*

Proof The proofs that Necessitation and Modus Ponens preserve truth are standard. We show only that axioms $\mathbf{4_R}$ and $\mathbf{LR_1}$ are valid. Proofs that $\mathbf{4_L}$ and $\mathbf{LR_2}$ are valid are symmetric.

For $\mathbf{4_R}$, assume that $M, x \models \Box_R p$. Then, there is an open $U \in \tau$, and for all $y \in \tau$, if $x < y$, then $M, y \models p$. It would suffice to show that for any such y, $M, y \models \Box_R p$. But, we can take the open set U, and indeed, for any z in U, if $(x <) y < z$ then $M, z \models p$.

For $\mathbf{LR_1}$, assume that $M, x \models (p \wedge \Box_R p \wedge \Box_L p)$. We wish to prove that $M, x \models \Box_R \Box_L p$. The assumption implies that x is in an open U, and for all $y \in U$, $M, y \models p$ (c.f. the proof that $\Box p$ is definable in **TDL**). It now suffices to show than at every point y in U to the right of x, $\Box_L p$ holds. But once again this is the case, since for all $z \in U$, if $z < y$, $M, z \models p$.

$\Box$

Note that the verification of soundness uses both transitivity and linearity of the given ordering.

The next step is to prove that the axioms of **TDL** as complete for the actual topo-directional semantics over the class of general topo-directional frames. The proof, which we leave for a different occasion, proceeds by unravelling of the **TDL**-frames into a linear ordering.

Conjecture: **TDL** is complete for its intended semantics.

Instead, in this chapter we concentrate on several more restricted classes of topologies and prove decidability and completeness for those.

3.4 The logic TDL$\mathbb{GO}$ over the class of generalized order topologies

A natural question to ask is what the logic of topo-directional frames for which the topology τ is the order topology for $<$? In this section we explore this question.

To begin with at least, we will use a slightly generalized definition of order topology.

Definition 3.4.1 *Suppose that $(X, <)$ is a linearly ordered set. The class $\mathcal{G}(X, <)$ is the class of topologies τ on X for which there is a base B such that*

1. *Every open interval of $(X, <)$ is in B*
2. *If Y is in B, then Y is an interval of $(X, <)$ (that is, either Y is an open, (left or right) half-closed, or closed interval of $(X, <)$).*

Thus $\mathcal{G}$ associates with every linearly ordered set $(X, <)$ a set of topologies τ on X. Condition 1 ensures that all open intervals belong to τ. Condition 2 allows for more than just the open intervals of $(X, <)$ to be open in τ. Note that the standard order topology on $(X, <)$, for which a base consists of just the open intervals of $(X, <)$, belongs to $\mathcal{G}(X, <)$, and so $\mathcal{G}(X, <)$ represents a generalization of the notion of order topology.

Definition 3.4.2 *A structure* $(X, <, \tau)$ *for which* τ *is in* $\mathcal{G}(X, <)$ *is called a* generalized order topology*.*

We begin with some preliminary observations. We shall see that the axioms we had for TDL simplify dramatically when we give ourselves the extra information that the ambient topology is a generalized order topology.

Fact 3.4.3 *The following principles are valid in every (generalized) order topology:*

$\mathbf{LR}\mathbb{GO}_\mathbf{1}$ $\Box_R p \to \Box_R \Box_L p$

$\mathbf{LR}\mathbb{GO}_\mathbf{2}$ $\Box_L p \to \Box_L \Box_R p$

Proof $\mathbf{LR}\mathbb{GO}_\mathbf{1}$: Let M be some model over some order [generalized] topology $\mathcal{O}$ and $M, x \models \Box_R p$. Then there is an open interval (y, z) [or $[y, z), (y, z], [y, z]$] for $y < x < z$, and by the truth definition every point in the interval (x, z) makes p true. Hence at every point in the interval (x, z), $\Box_L p$ is true, and thus $\Box_R \Box_L p$ is true at x as desired. $\mathbf{LR}\mathbb{GO}_\mathbf{2}$ is shown to be valid in a similar way.

□

The two principles have Sahlqvist form, and their first-order correspondents are:

$\mathbf{FO} - \mathbf{LR}\mathbb{GO}_\mathbf{1}$ $\forall x \forall y \forall z((R_R xy \wedge R_L yz) \to R_R xz)$

$\mathbf{FO} - \mathbf{LR}\mathbb{GO}_\mathbf{2}$ $\forall x \forall y \forall z((R_L xy \wedge R_R yz) \to R_L xz)$

Remark 3.4.4 $\mathbf{LR}\mathbb{GO}_\mathbf{i}$ *implies* $\mathbf{LR_i}$*, for* $i \in \{1, 2\}$*.*

Remark 3.4.5 *The transitivity of* R_R *and* $\mathbf{FO} - \mathbf{LR}\eth\mathbb{O}_\mathbf{1}$ *are equivalent to the following principle:*

$\forall x \forall y \forall z((R_R xy \wedge (R_R yz \vee R_L yz)) \to R_R xz)$

The modal formula that corresponds to this principle is:

$\mathbf{RL}\mathbb{GO}$ $\Box_R p \to \Box_R(\Box_R p \wedge \Box_L p)$

Similarly, the transitivity of R_L *and* $\mathbf{FO} - \mathbf{LR}\mathbb{GO}_\mathbf{2}$ *are equivalent to:*

$\forall x \forall y \forall z((R_L xy \wedge (R_L yz \vee R_R yz)) \to R_L xz)$

The modal formula that corresponds to this principle is:

LR$\mathbb{GO}$ $\Box_L p \to \Box_L(\Box_L p \wedge \Box_R p)$

We opted for the original version in order to keep the relation with the general topological case more transparent and the axioms more basic and simpler. We will however freely use the fact that the two are equivalent.

Definition 3.4.6 *The logic* **TDL$\mathbb{GO}$** *is* $\mathbf{K4} \oplus \mathbf{K4} \cup \{\mathbf{LR}\mathbb{GO}_\mathbf{1}, \mathbf{LR}\mathbb{GO}_\mathbf{2}\}$

Definition 3.4.7 *Call a two relation Kripke frame* $\mathcal{F} = (W, R_L, R_R)$*, a* **TDL$\mathbb{GO}$***-frame, if it it is transitive in both* R_R *and* R_L *and it satisfies* $\mathbf{FO} - \mathbf{LR}\mathbb{GO}_\mathbf{1}$ *and* $\mathbf{FO} - \mathbf{LR}\mathbb{GO}_\mathbf{2}$.

Fact 3.4.8 **TDL$\mathbb{GO}$** *is sound and complete with respect to the class of* **TDL$\mathbb{GO}$***-frames.*

Proof We simply observe that all axioms are Sahlqvist. □

3.5 Decidability of TDL$\mathbb{GO}$

We will show that **TDL$\mathbb{GO}$** has the strong finite model property, that is, given a formula ϕ, if ϕ is satisfiable, it is satisfied on a model with at most $2^{|\phi|}$ points. Since we can effectively list all models of some finite size, the decidability follows. We will show not that the logics have finite topological model property, but rather that with respect to standard Kripke frames they have the desired property.

Recall that **TDL$\mathbb{GO}$** is complete with respect to the class of frames F, for $\mathcal{F} \in F$, $\mathcal{F} = (W, R_N, R_S)$, where R_N, R_S are transitive and they jointly satisfy $(FO - NS\mathbb{GO}1 - 2)$. Given a formula ϕ satisfied on such a frame $\mathcal{F}$, we now wish to construct a finite frame $\mathcal{F}'$ that also satisfies the formula. We will use the standard technique of filtrating a model on the frame $\mathcal{F}$ through the subformulae of ϕ. Neither the transitivity nor $(FO - NS\mathbb{O}1 - 2)$ are preserved by the standard filtration, so we will need to adjust the definition to ensure that the resulting model has the required relational properties.

Definition 3.5.1 *A set of formulae Σ is closed under subformulae if for all formulae ϕ, ψ, if $\phi \vee \psi \in \Sigma$ then $\phi, \psi \in \Sigma$; if $\neg\phi \in \Sigma$ then, $\phi \in \Sigma$; and finally if $\Diamond_* \phi \in \Sigma$ then $\phi \in \Sigma$, for $* \in \{N, S\}$.*

Definition 3.5.2 *Let $M = (\mathcal{F}, \nu)$ be a model as above satisfying ϕ, and let Σ_ϕ be the set of subformulae of ϕ. We define $\leftrightsquigarrow_\phi$ on the states of M by*

$$x \leftrightsquigarrow_\phi y \text{ iff for all } \psi \text{ in } \Sigma_\phi : M, x \models \psi \text{ iff } M, y \models \psi.$$

Since $\leftrightsquigarrow_\phi$ is an equivalence relation, as usual, we denote the equivalence class of a point with respect to $\leftrightsquigarrow_\phi$ by $|x|_\phi$. Now we say that a model $M_f^\phi = (W^f, R_N^f, R_S^f, \nu^f)$ is a filtration if it satisfies:

1. $W^f = \{|x|_\phi \mid x \in W\}$,
2. *If $R_N xy$ then $R_N^f |x|_\phi |y|_\phi$,*
3. *If $R_S xy$ then $R_S^f |x|_\phi |y|_\phi$,*
4. *If $R_N^f |x|_\phi |y|_\phi$ then for all $\Diamond_N \psi \in \Sigma_\phi$, if $M, y \models \phi$ then $M, x \models \Diamond_N \phi$,*
5. *If $R_S^f |x|_\phi |y|_\phi$ then for all $\Diamond_S \psi \in \Sigma_\phi$, if $M, y \models \phi$ then $M, x \models \Diamond_S \phi$,*
6. *$\nu^f(p) = \{|x|_\phi \mid M, x \models p\}$ for all $p \in \Sigma_\phi$.*

Theorem 3.5.3 *If ϕ is satisfied in M then it is satisfied in the filtration model M_f^ϕ, if one exists.*

Proof See, for instance, [25], pp. 78–79. □

Proposition 3.5.4 *There is a filtration that preserves transitivity for R_N, R_S, as well as ($FO - NS\mathbb{GO}1 - 2$), if the original model M satisfies these properties.*

Proof We define the relations R_N^f and R_N^f as follows:

$R_N^f |x|_\phi |y|_\phi$ iff

(i) for all $\psi \in \Sigma_\phi$, if $\Diamond_N \psi \in \Sigma_\phi$ and $M, y \models \psi \vee \Diamond_N \psi$ then $M, x \models \Diamond_N \psi$; and

(ii) for all $\psi \in \Sigma_\phi$, if $M, y \models \Diamond_S \psi$ then $M, x \models \Diamond_N \psi$.

Similarly,

$R^f_S|x|_\phi|y|_\phi$ iff

(i) for all $\psi \in \Sigma_\phi$, if $\Diamond_S\psi \in \Sigma_\phi$ and $M, y \models \psi \vee \Diamond_S\psi$ then $M, x \models \Diamond_S\psi$; and

(ii) for all $\psi \in \Sigma_\phi$, if $M, y \models \Diamond_N\psi$ then $M, x \models \Diamond_S\psi$.

We show that the model M'^f_ϕ based on the frame with relations R^f_N and R^f_S defined above is a filtration, and that underlying frame satisfies the appropriate relations.

We first show that the model M'^f_ϕ is a filtration. The clauses 1 and 6 are satisfied by definition. Furthermore, 4 and 5 follow from the clause (i) in the definition of R^f_N and R^f_S. The interesting clauses are 2 and 3. We prove 2 only. Assume that R_Nxy. Then, for any ψ whatsoever, if ψ is true at y then $\Diamond_N\psi$ is true at x, and by transitivity of R_N, if $\Diamond_N\psi$ is true at y, then the same formula is true at x. Thus clause (i) holds. For (ii) suppose that $\Diamond_S\psi$ is true at y, where ψ is any formula. We need to show that $\Diamond_N\psi$ is true at x. If $\Diamond_S\psi$ is true at y, then there is a z such that R_Syz and ψ is true at z. By $(FO - NS\mathbb{GO}1)$, we know that R_Nxz, that is, $\Diamond_N\psi$ is true at x, as desired. Thus, M'^f_ϕ is indeed a filtration, but to show that it is an appropriate finite model, we need to show that transitivity holds for both relations, and that $(FO - NS\mathbb{GO}1 - 2)$ are satisfied.

(Transitivity)

Assume that $R^f_N|x|_\phi|y|_\phi$ and $R^f_N|y|_\phi|z|_\phi$. We show that $R^f_N|x|_\phi|z|_\phi$ by showing that clauses (i) and (ii) hold for x and z. For (i), assume that $M, z \models \psi \vee \Diamond_N\psi$. Then, by (i) for y and z, $M, y \models \Diamond_N\psi$, and again by (i) for x and y $M, x \models \Diamond_N\psi$. For (ii), assume that $M, z \models \Diamond_S\psi$. Then by (ii) for y and z, $M, y \models \Diamond_N\psi$, and thus by (i) for x and y, $M, x \models \Diamond_N\psi$.

$(FO - NS\mathbb{GO}1 - 2)$

We prove only that (FO-NS$\mathbb{GO}$ 1) is preserved. The proof that (FO-NS$\mathbb{GO}$ 2) is preserved is symmetric. Assume that $R^f_N|x|_\phi|y|_\phi$ and $R^f_S|y|_\phi|z|_\phi$. We show that $R^f_N|x|_\phi|z|_\phi$ by showing that clauses (i) and (ii) hold for x and z. For (i), assume that $M, z \models \psi \vee \Diamond_N\psi$. Then either $M, z \models \psi$ or $M, z \models \Diamond_N\psi$. If the former, then $M, y \models \Diamond_S\psi$ and hence by clause (ii) for $R^f_N|x|_\phi|y|_\phi$, $M, x \models \Diamond_N\psi$ as desired. If the latter, then then $M, y \models \Diamond_S\psi$ by clause (i) for $R^f_N|y|_\phi|z|_\phi$, and again $M, x \models \Diamond_N\psi$.

For (ii) assume that $M, z \models \Diamond_S \psi$. Then by clause (i) for $R^f_S|y|_\phi|z|_\phi$, $M, y \models \Diamond_S \psi$, and by clause (ii) for $R^f_N|x|_\phi|y|_\phi$, $M, x \models \Diamond_N \psi$ as desired. □

Corollary 3.5.5 **TDL**$\mathbb{GO}$ *has a finite model property, that is, if a formula has a model on a* **TDL**$\mathbb{GO}$*-frame, then it has a model on a finite* **TDL**$\mathbb{GO}$ *frame.*

We can also observe that the complexity of the logic is between PSPACE since the logic contains S4 and EXPTIME, as this is the upper bound set by filtration.

We will need the finite model property in the completeness proof, which shows that

Theorem 3.5.6 **TDL**$\mathbb{GO}$ *is a complete logic of generalized order topology.*

Our strategy for the completeness is to use the fact that the logic is complete for finite **TDL**$\mathbb{GO}$-frames. We will provide a procedure for unravelling such frames into linear orders that are modally equivalent to the original frames, but under the topo-directional semantics rather than original Kripke semantics. This procedure is akin to various embedding procedures we use in Chapter 2.

Let $\mathcal{F} = (W, R_L, R_R)$ be a finite rooted **TDL**$\mathbb{GO}$-frame satisfying some formula ϕ at the root x_0. We will build a linear order O and a labelling function f from O onto $\mathcal{F}$ such that if $a \in O, x \in \mathcal{F}$ and $f(a) = x$, then a and x are modally equivalent (albeit under different semantics).

Definition 3.5.7 *We define* cycling functions $r : W \times \mathbb{N} \to W$ *and* $l : W \times \mathbb{N} \to W$ *as follows.*

Let $y_1, ..., y_n$ be the finite (and possibly empty) set of R_R successors of x in $\mathcal{F}$. If the set is nonempty, then we let $r(x, 1) = y_1, ..., r(x, n) = y_n, r(x, n+1) = y_1, ..., r(x, n+n) = y_n, ...$ In other words, $r(x, i)$ infinitely cycles through all finitely many R_R successors of x. If the set is empty, then for all i we let $r(x, i) = x$. Similarly, let $z_1, ..., z_m$ be the finite (and possibly empty) set of R_L successors of x in $\mathcal{F}$. If the set is nonempty, then we let $l(x, 1) = z_1, ..., l(x, m) = z_m, l(x, m+1) = z_1, ..., l(x, m+m) = z_m, ...$ Else, we let $l(x, i) = x$.

Definition 3.5.8 *We will have the ordering,* $O = \bigcup_{n\in\omega} O_n$, *and the labelling function,* $f : X \subseteq O \to W$ *is* $\bigcup_{n\in\omega} f_n$, *where* O_n, f_n, *are defined inductively as follows:*

$O_0 = \{0\}$ *and* $f_0(0) = x_0$, *the root of* W.

Let O_n, f_n *be defined. We simultaneously define* O_{n+1}, f_{n+1} *as extensions of* O_n, f_n.

(R) For any $o \in O_n$, $y \in W$, *if* $f_n(o) = y$, *then:*

If the set of R_R *successors of* $y \in \mathcal{F}$ *is nonempty, then add* $o + \frac{1}{3^n}$ *to* O_{n+1} *and let* $f_{n+1}(o + \frac{1}{3^n}) = r(y, n+1)$.

(L) Similarly, for any $o \in O_n$, *and* $y \in W$, *if* $f_n(o) = y$ *then:*

If the set of R_L *successors of* $y \in \mathcal{F}$ *is nonempty, then add* $o - \frac{1}{3^n}$ *to* O_{n+1} *and let* $f_{n+1}(o - \frac{1}{3^n}) = r(y, n+1)$.

Nothing is in O_{n+1}, f_{n_1} *except in virtue of the clauses (R) and (L), and membership in* O_n, f_n.

Next we define an ordering and a generalized order topology on O.

Definition 3.5.9 *(i) For* $o, o' \in O$, *we say that* $o < o'$ *iff in* $\mathbb{Q}$ *with the standard ordering,* $o < o'$.

(ii) We define topology τ_O *via the following base. For all* $o, o' \in O, \{o'' \mid o < o'' < o'\}$ *is in the base. In addition,*

(iii) if $f(o) = y$ *and in* $\mathcal{F}$ y *has no* R_L *successors, then all sets of the form* $\{o' \mid \exists o'' o \leq o'' < o'\}$. *Symmetrically,*

(iv) if $f(o) = y$ *and in* $\mathcal{F}$ y *has no* R_R *successors, then all sets of the form* $\{o' \mid \exists o'' o' < o'' \leq o\}$.

Fact 3.5.10 *In the structure* $(O, <, \tau_O)$, $<$ *is a linear ordering, and* τ_O *is a generalized order topology.*

Definition 3.5.11 *We say that a sequence* s *converges to a point* $o \in O$ *from the right iff*

(i) for all o' *the interval* $(o', o]$ *is* not *open.*

(ii) for all $o_i \in s$, $o_i < o$, and

(iii) for any $p < o$ there is a $o_i \in s$, and $p < o_i < o$.

Similarly, a sequence s converges to a point $o \in O$ from the left iff

(i) for all o' the interval $[o, o')$ is not *open.*

(ii) for all $o_i \in s$, $o < o_i$, and

(iii) for any p, if $o < p$ then there is a $o_i \in s$, and $o < o_i < p$.

Fact 3.5.12 *Given O and some valuation $V : At \to O$, for any formula ϕ*

(i) $(O, <, \tau_O, V), o \models \Diamond_R \phi$ iff there is a sequence s converging to o from the right and every $o_i \in s$, $(O, V), o_i \models \phi$, and

$(O, <, \tau_O, V), o \models \Diamond_L \phi$ iff there is a sequence s converging to o from the left and every $o_i \in s$, $(O, V), o_i \models \phi$.

Lemma 3.5.13 *(i) $R_R xy \in \mathcal{F}$ iff for every $o \in O$, if $f(o) = x$ then there is a sequence s converging to o from the right, and for every $o_i \in s$, $f(o_i) = y$.*

(ii) $R_L xy \in \mathcal{F}$ iff for every $o \in O$, if $f(o) = x$ then there is a sequence s converging to o from the left, and for every o_i in the sequence, $f(o_i) = y$.

Proof (i), ($\Rightarrow$) Suppose that $R_R xy$ and that for some o, $f(o) = x$. Let n be the least such that $o \in O_n$. Then since the set of R_R-successors of x is nonempty, $o + \frac{1}{3^n}$ is added to O_{n+1} and $f_{n+1}(o + \frac{1}{3^n}) = r(x, n+1)$. By the same reasoning, $o + \frac{1}{3^{n+1}}$ is added to O_{n+2} and $f_{n+2}(o + \frac{1}{3^{n+1}}) = r(x, n+2)$, and in general, for every $m \in \mathbb{N}$ O contains the sequence $o + \frac{1}{3^{n+m}}$ which approaches o. Since the sequence is labelled by the cycling function $r(x, n+1+m)$ which cycles through R_R-successors of x, a countable subsequence of $o + \frac{1}{3^{n+m}}$ that approaches o from the right will be labelled by y, as desired. Furthermore, since x has an actual R_R successor, for no o' is the interval $[o, o')$ in the base of τ_O, and hence not in τ_O.

($\Leftarrow$) Let s be a sequence approaching o from the right, and let o appear for the first time in O_n. First we consider the possibility that x has no R_R successors. Then for some o', $(o', o]$ is in τ_O, and thus no sequence approaches o from the right, and certainly not s. So we can assume that x has at least one R_R successor. Now we consider what happens in the interval $(o, o + \frac{1}{n^3})$.

Claim: for every $p \in (o, o+\frac{1}{n^3}) \subset O, R_R x f(p)$. We know by assumption that there is some z, $R_R xz$ and $f(o+\frac{1}{n^3}) = z$. Furthermore, in O_{n+1} there is no points between o and $o+\frac{1}{n^3}$ (by construction of O_{n+1} from O_n). Now by inspecting the definition, we can see that in O_{n+m} for $m > 1$, the points that procedure adds to this interval are all labelled by right successors of x, or left and right successors of those, or the left and right successors of those, etc., or left successors of z and left and right successors of those, etc.

In other words, all labels in the interval are accessible along $R_R \cup R_L$ from x in finitely many steps starting with a R_R step. But then all those points are accessible from x via R_R in one step by axiom **LR**$\mathbb{GO}$.

Finally, if s approaches o then for any $o_i \in s$, if o_i is in $(o, o+\frac{1}{n^3})$, then $R_R x f(o_i)$, and thus $R_R xy$ as desired.

(ii) is proved symmetrically.

$\square$

Theorem 3.5.14 *For any formula ϕ of our language, and any valuation V, on $\mathcal{F}$, $(\mathcal{F}, V), x \models \phi$ iff for every $o \in O$, if $f(o) = x$, then $((O, <, \tau_O), f^{-1} \circ V), o \models \phi$.*

Proof The base case for propositional variables is given to us by the definition of the valuation on O. For inductive hypothesis (skipping the obvious boolean cases) we assume that the property holds for ψ to prove that it holds for $\Diamond_R\psi, \Diamond_L\psi$. We consider one of these cases only.

($\Rightarrow$) Assume that $(\mathcal{F}, \nu), x \models \Diamond_R\psi$. Then there is a point y, $R_R xy$ and $(\mathcal{F}, \nu), y \models \psi$. Let o be an arbitrary point such that $f(o) = x$. Then by Lemma 3.11, there is a sequence of y points approaching o from the left, and then again by Fact 3.5.12 and inductive hypothesis, $((O, <, \tau_O), f^{-1} \circ \nu), o \models \Diamond_R\psi$.

($\Rightarrow$) Pick an arbitrary o such that $f(o) = x$ and assume that $((O, <, \tau_O), f^{-1}), o \models \Diamond_R\psi$. Then by 3.5.12, there is a sequence s converging to o from the right and every $o_i \in s$, $(O, \nu), o_i \models \psi$. Since $\mathcal{F}$ is finite, there is a subsequence s' approaching o such that for every $o'_i \in s'$, $f(o'_i) = y$ for some $y \in \mathcal{F}$, and by inductive hypothesis $(\mathcal{F}, \nu), y \models \psi$. But by 3.11, $R_R xy$, and hence $(\mathcal{F}, \nu), x \models \Diamond_R\psi$, as desired. $\square$

This finalizes the proof the completeness proof for $\mathbf{TDL}\mathbb{GO}$ for the class of generalized order topologies. We have shown that if a formula ϕ has a counter model on a $\mathbf{TDL}\mathbb{GO}$-frame, then a linearly ordered set O can be constructed and equipped with a generalized order topology that invalidates the same formula on topo-directional semantics.

As interesting as generalized order topologies are, their main interest still lies in the fact that they provide a convenient base system and a set of methodologies for proving interesting facts about specific order topologies. In the following sections we look at few examples of specific order topologies.

3.6 The logic TDL$\mathbb{Q}$ over rational numbers

In this section we explore the logic of $\Box_R$ and $\Box_L$ on the standard order topology on $\mathbb{Q}$. This logic is slightly stronger than $\mathbf{TDL}\mathbb{GO}$, stemming from the fact that $\mathbb{Q}$ is not only equipped with an order topology, but the order is also dense.

Thus, since for every point $q \in \mathbb{Q}$ there is a sequence of points strictly approaching it, additional consistency principles are valid:

Fact 3.6.1

$$\mathbf{D_R} : \Diamond_\mathbf{R} \top$$

$$\mathbf{D_L} : \Diamond_\mathbf{L} \top$$

are valid on $\mathbb{Q}$.

Let **D4** be logic that to basic normal modal logics **K** adds principles **D** and **4**.

Definition 3.6.2 *We call the logic* $\mathbf{D4} \oplus \mathbf{D4} \cup \{\mathbf{LR}\mathbb{O}_\mathbf{1}, \mathbf{LR}\mathbb{O}_\mathbf{2}\}$, $\mathbf{TDL}\mathbb{Q}$

Definition 3.6.3 *Call a two relation Kripke frame* $\mathcal{F} = (W, R_L, R_R)$, *a* $\mathbf{TDL}\mathbb{Q}$-*frame, if it it is transitive and serial*[3] *in both* R_R *and* R_L *and it satisfies* $\mathbf{FO} - \mathbf{LR}\mathbb{O}_\mathbf{1}$ *and* $\mathbf{FO} - \mathbf{LR}\mathbb{O}_\mathbf{2}$.

[3] I.e. every point has at least one successor.

Fact 3.6.4 $\mathbf{TDL}\mathbb{Q}$ *is sound and complete with respect to the class of* $\mathbf{TDL}\mathbb{Q}$ *frames.*

Proof We simply observe that all axioms are Sahlqvist. $\square$

We are of course interested in a completeness for topo-directional semantics on the actual $\mathbb{Q}$.

Theorem 3.6.5 *On the topo-directional semantics, the logic* $\mathbf{TDL}\mathbb{Q}$ *is sound and complete with respect to the singleton class* $\{\mathbb{Q}\}$ *with the standard ordering and the standard order topology.*

Given the proof of completeness of $\mathbf{TDL}\mathbb{GO}$ for generalized order topologies, the proof for $\mathbb{Q}$ is rather straightforward. Since every $\mathbf{TDL}\mathbb{Q}$-frame is also a $\mathbf{TDL}\mathbb{GO}$-frame, we can use the same unravelling procedure given a $\mathbf{TDL}\mathbb{Q}$-frame $\mathcal{F}$ to obtain a linear ordering and a topology $(O, <, \tau_O)$. The main difference is in the following lemma:

Lemma 3.6.6 *If* $\mathcal{F}$ *is a* $\mathbf{TDL}\mathbb{Q}$*-frame then*

(i) $(O, <)$ *is isomorphic to* $\mathbb{Q}$*,*

(ii) τ_O *is the usual order topology on* $\mathbb{Q}$*.*

Proof (ii) is obvious since every point always has both R_R and R_L successors so the clauses (iii) and (iv) of definition of the base of τ_O never apply.

For (i) we note that since, once again, every point has both a R_R and R_L successor, for any point $o \in O_n$, both $o + \frac{1}{n^3}$ and $o - \frac{1}{n^3}$ are always in $n+1$. Thus $O = X$ where $X =$

$\bigcup_{n \in \omega} X_n$, where $X_0 = \{0\}$ and

$$X_{n+1} = X_n \cup \{x - \frac{1}{3^n}, x + \frac{1}{3^n} \mid x \in X_n\}$$

And we proved in Chapter 2 (Claim 2.2.5 and immediate vicinity) that X is isomorphic to $\mathbb{Q}$. $\square$

Thus for any formula ϕ that has a counterexample on a **TDL**$\mathbb{Q}$-frame, a counterexample can be constructed on $(\mathbb{Q}, <, \tau_O))$ which proves the completeness.

Lemma 3.6.7 *The logic a* **TDL**$\mathbb{Q}$ *has a finite model property.*

Proof This follows from the fact that the finite model property of **TDL**$\mathbb{GO}$ was proved via a filtration together with the fact that filtrations preserve consistency principles $\Diamond_R\top, \Diamond_L\top$ [see for instance [25]]. $\square$

3.7 The logic TDL$\mathbb{N}$ over natural numbers

Topo-directional logic of the natural numbers with the standard ordering and the standard order topology $(\mathbb{N}, <, O)$ turns out to be rather trivial. This as we will see stems from the fact that in this topology all singletons are open and thus $\Box_L\bot$ and $\Box_R\bot$ are valid.

3.7.1 Axioms

The logic **TDL**$\mathbb{N}$ consists of **TDL**$\mathbb{GO} \cup \{\Box_{\mathbf{L}}\bot, \Box_{\mathbf{R}}\bot\}$.

It is a well known fact that the axiom $\Box_L\bot$ corresponds to $\forall x\forall y\neg R_L xy$ and similarly $\Box_R\bot$ corresponds to $\forall x\forall y\neg R_L xy$.

Definition 3.7.1 *We call a frame on which both R_L and R_R are empty a* **TDL**$\mathbb{N}$-*frame.*

The following is only a slight extension of the well-known result:

Fact 3.7.2 **TDL**$\mathbb{N}$ *is complete with respect a singleton frame* $SINGL = (\{x\}, R_L, R_R)$, *where* $R_L = R_R = \emptyset$

Once again we are seeking a way of transferring this completeness result onto $\mathbb{N}$ with the topo-directional semantics.

3.7.2 Soundness

Let us first check that **TDL**$\mathbb{N}$ is indeed sound and complete for $\mathbb{N}$ on the intended topo-directional semantics.

Proposition 3.7.3 *The axioms and rules of* **TDL**$\mathbb{N}$ *are valid in topo-directional models.*

Proof The only axiom worth checking is $\Box_L \bot$ (or symmetrically $\Box_R \bot$). As we remarked earlier, every singleton n is open in $(\mathbb{N}, <, O)$. For instance, 5 can be defined as $(4, 6)$. Thus at 5, to continue with the example, there is an open, viz. $(4, 6)$, such that, for all $n \in (4, 6)$, if $n < 5$ then n makes $\bot$ true (since there is no such n). Thus $\Box_L \bot$ is true at 5. $\Box$

Let $\mathbb{N}_\emptyset = (\mathbb{N}, R_L, R_R)$ where $R_L = R_R = \emptyset$.

Fact 3.7.4 *The function f from $\mathbb{N}_\emptyset$ onto $SINGL$ such that $\forall n, f(n) = x$ is a p-morphism.*

For completeness for topo-directional semantics we need the following two observations:

If there is a smallest open U around a point x in a topo-directional structure, then for any formula ϕ, $\Box_L \phi$ is true at x iff ϕ is true at every $y \in U$ such that $y < x$ and symmetrically, $\Box_R \phi$ is true at x iff ϕ is true at every $y \in U$ such that $x < y$.

Thus for any point n in $(\mathbb{N}, <, O)$, $\Box_L \phi$ is true at n iff for all $m \in \{n\}$ such that $m < n$, that is, all m in the empty set, m makes ϕ true. Similarly, for an n' in $\mathbb{N}_\emptyset = (\mathbb{N}, R_L, R_R)$, $\Box_L \phi$ is true in n' iff for all m such that $R_L nm$, or once again, all m in the empty set, m makes ϕ true. Thus, assuming that the valuation ν is the same for $\mathbb{N}_\emptyset$ and $(\mathbb{N}, <, O)$, n makes ϕ true in $\mathbb{N}_\emptyset$ iff n makes ϕ true in $\mathbb{N}_\emptyset$, that is the completeness follows.

Corollary 3.7.5 **TDL**$\mathbb{N}$ *is complete for any discrete linear order, finite or infinite, and thus for all finite linear orders.*

3.8 The logic TDL$\mathbb{O}$ over the class of (standard) order topologies

The topo-directional logic of the class of standard order topologies defined in the usual way via the base for all $x, y \in O$,

$$\{z \,|\, x < z < y\}$$

turns out to be stronger than **TDL**$\mathbb{GO}$. We end the discussion of **TDL** language with some observations about the logic of standard order topologies. The following additional principles are valid in every order topology:

LR$\mathbb{SO}_1$ $\Diamond_L \Box_R \bot \leftrightarrow \Diamond_L \Box_L \bot$

LR$\mathbb{SO}_2$ $\Diamond_R \Box_L \bot \leftrightarrow \Diamond_R \Box_R \bot$

Fact 3.8.1 *(i) $\Box_R \bot$ is true at a point o in an order topology $(O, <)$ iff (a) o has an immediate $<$ successor on the right, i.e., there is o', $o < o'$ and $\forall o'' \in O\ \neg(o < o'' < o')$, or else (b) o is the right endpoint, i.e., $\neg \exists o' o < o'$.*

(ii) $\Box_L \bot$ is true at a point o in an order topology $(O, <)$ iff o (a) has an immediate $<$ successor on the left, or else (b) o is the left endpoint, i.e., $\neg \exists o' o' < o$.

Proof (i) Suppose that for some point o, neither a nor b hold, and o makes $\Box_R \bot$ true. That is, for any basic open set U, of the form $o_1 < o < o_2$ there is a point o_3, $o_1 < o < o_3 < o_2$, and since $\top$ holds at o_3, and U was arbitrary basic set, it follows that for every open that o is in, there is a point on the right of that open that makes $\top$ true, in that is o makes $\Diamond_R \top$ (i.e. $\neg \Box_R \bot$) true, contrary to supposition.

(ii) is proved symmetrically.

$\Box$

Fact 3.8.2 **LR**$\mathbb{SO}_1$ *and* **LR**$\mathbb{SO}_2$ *are valid in every order topology.*

Proof (**LR**$\mathbb{SO}_1$) For some order topology $(O, <)$, some valuation ν, and some point $o \in O$ let $((O, <), \nu), o \models \Diamond_L \Box_R \bot$. What that means by Fact 3.8.1 is that there is a sequence of pairs (o_1, o_2) approaching o form the right, such that, o_2 is the immediate right successor of o_1. But once again by Fact 3.8.1, at every such o_2 makes $\Box_L$ true, and hence $\Diamond_L \Box_L \bot$ is true at o as desired. The argument is similar if we assume that $\Diamond_L \Box_L \bot$ is true at o.

The argument for $\mathbf{LR\mathbb{SO}_2}$ is analogous.

$\Box$

Fact 3.8.3 *In standard Kripke semantics,*

(i) the first-order correspondent of $\mathbf{LR\mathbb{SO}_1}$ *is:*

$\mathbf{FO_L LR\mathbb{SO}_1}$ $(\exists y R_L xy \wedge \forall z \neg R_R yz) \leftrightarrow (\exists y R_L xy \wedge \forall z \neg L_L yz)$ *and*

(ii) the first-order correspondent of $\mathbf{LR\mathbb{SO}_2}$ *is:*

$\mathbf{FO - LR\mathbb{SO}_2}$ $(\exists y R_R xy \wedge \forall z \neg R_L yz) \leftrightarrow (\exists y R_R xy \wedge \forall z \neg L_R yz)$.

Definition 3.8.4 *Let* $\mathbf{TDL\mathbb{O}}$ *be the logic* $\mathbf{TDL\mathbb{GO}} \cup \{\mathbf{LR\mathbb{SO}_1}, \mathbf{LR\mathbb{SO}_2}\}$.

Definition 3.8.5 *Let* $\mathbf{TDL\mathbb{O}}$*-frame be a* $\mathbf{TDL\mathbb{GO}}$*-frame that in addition satisfies* $\mathbf{FO - LR\mathbb{SO}_1}$ *and* $\mathbf{FO - LR\mathbb{SO}_2}$.

Lemma 3.8.6 *The logic* $\mathbf{TDL\mathbb{O}}$ *is complete with respect to the class of all* $\mathbf{TDL\mathbb{O}}$*-frames.*

Proof Once again by observing that all axioms have Sahlqvist form. $\Box$

We leave the reader with a couple of questions about this system.

Question 3.8.7 *If* $\mathbf{FO_L LR\mathbb{SO}_1}$ *and* $\mathbf{FO_L LR\mathbb{SO}_2}$ *are true in a model, then are they true in a filtration of that model?*

Question 3.8.8 *Is* $\mathbf{TDL\mathbb{O}}$ *is complete with respect to the class of topo-directional frames based on order topologies?*

Our hypothesis in fact is that additional axioms are needed for the completeness over standard order topologies but for now we leave this as an open question.

3.9 Topological Compass Logic

So far we have looked at logics on a single topology and and a linear ordering in the language of **TDL**. In line with the main topic of this dissertation, we now consider a product space with topo-directional modalities on two dimensions. As we will see, the two dimensional variant of **TDL** is best seen as a topological variant of Venema's Compass Logic from [71]. As in Chapter 2, we will generalize from the products of Kripke frames to topological product spaces.

Our goal in this section is to define a topological version of the basic compass logic introduced in [71]. Recall that in its simplest incarnation, the language of compass logic contains four interrelated boxes $\Box_N, \Box_S, \Box_E$ and $\Box_W$ (to be read 'box north', 'south', 'east', and 'west' respectively). Structures for this language contain two linear orders, $(T_1, <_1)$ and $(T_2, <_2)$. The valuation function assigns sets of ordered pairs from $T_1 \times T_2$ to propositional variables. $(T_1, <_1)$ is the horizontal and $(T_2, <_2)$ the vertical dimension on the cartesian grid. Formulas are built from a countable set of propositional variables At, using boolean connectives $\neg, \wedge, \vee, \rightarrow$, and modal operators above. Models are then quintuples $\mathcal{M} = (T_1, <_1, T_2, <_2, \nu)$, where $\nu : At \rightarrow \wp(T_1 \times T_2)$. Truth is defined recursively (skipping the obvious boolean cases):

$$\mathcal{M}, (x, y) \models p \quad \text{iff} \quad (x, y) \in \nu(p)$$

$$\mathcal{M}, (x, y) \models \Box_N \phi \quad \text{iff} \quad \forall z \in T_2, \text{ if } y <_2 z, \text{ then } \mathcal{M}, (x, z) \models \phi$$

$$\mathcal{M}, (x, y) \models \Box_S \phi \quad \text{iff} \quad \forall z \in T_2, \text{ if } z <_2 y, \text{ then } \mathcal{M}, (x, z) \models \phi$$

$$\mathcal{M}, (x, y) \models \Box_W \phi \quad \text{iff} \quad \forall z \in T_1, \text{ if } x <_1 z, \text{ then } \mathcal{M}, (z, y) \models \phi$$

$$\mathcal{M}, (x, y) \models \Box_E \phi \quad \text{iff} \quad \forall z \in T_1, \text{ if } z <_1 x, \text{ then } \mathcal{M}, (z, y) \models \phi$$

This logic has found many applications and a variety of extensions have been considered. For more information about variants and axioms, we refer the reader to [2].

The main drawback of this family of logics is that it describes the two dimensional grid so closely, that it facilitates encoding of the grid and thus in most cases the logics end up undecidable. The undecidability of this logic is closely connected to the undecidability of the product logic $\mathbf{K4} \times \mathbf{K4}$ on standard Kripke frames via the connection of this logic with two orthogonal direction fragment of compass logic such as for instance *north-west* [For more on undecidability of $\mathbf{K4} \times \mathbf{K4}$, see [40]]. This suggests that the topological approach may give us an interesting decidable compass logic. In this chapter we show two examples of decidable topo-compass Logic. But before any task of measuring the complexity of this logic can be undertaken, we first have to see what the logic is. Axiomatizing the topo-variant of Venema's compass logic over the products $\mathbb{Q} \times \mathbb{Q}$ and $\mathbb{N} \times \mathbb{N}$ and proving that logic decidable is the task of the remainder of this chapter.

We will work with the language of the basic compass logic.
Let $\mathcal{X}_1 = (X_1, \tau_1, <_1)$ and $\mathcal{X}_2 = (X_2, \tau_2, <_2)$ be topo-directional structures.

Definition 3.9.1 Topo-compass structures *are tuples* $\mathcal{X} = (X, \tau_V, \tau_H, <_V, <_H)$*, where* $X = X_1 \times X_2$*,* τ_H *and* τ_V *are horizontal and vertical topologies derived from* τ_1, τ_2 *respectively, and where the order satisfies*

$$(x, y) <_H (x', y') \text{ iff } x <_1 x' \text{ and } y = y',$$

and

$$(x, y) <_V (x', y') \text{ iff } x = x' \text{ and } y <_2 y'.$$

Thus, a topo-compass structure is a topological product of two topo-directional structures.

Next, a valuation function ν assigns propositional variables to subsets of $X_1 \times X_2$. A *topo-compass model* $\mathcal{M}$ is a pair $(\mathcal{X}, \nu)$, where X is a topo-compass structure and ν a valuation. It is often convenient to spell out $\mathcal{M}$ as a quintuple $(X, \tau_H, \tau_V, <_H, <_V\ \nu)$. The truth clauses for the booleans are defined in the standard way and are the same as in the case of the standard modal semantics. The interesting difference lies in the truth definitions for the modal operators. The novelty is guided by the topo-directional semantics:

$$\mathcal{M},(x,y) \models p \quad \text{iff } (x,y) \in \nu(p)$$

$$\mathcal{M},(x,y) \models \Box_N\phi \quad \text{iff } \exists U \in \tau_V,\ (x,y) \in U \text{ and } \forall (x,z) \in U, \text{ if } (x,y) <_V (x,z), \text{ then } \mathcal{M},(x,z) \models \phi$$

$$\mathcal{M},(x,y) \models \Box_S\phi \quad \text{iff } \exists U \in \tau_V,\ (x,y) \in U \text{ and } \forall (x,z) \in U, \text{ if } (x,z) <_V (x,y), \text{ then } \mathcal{M},(x,z) \models \phi$$

$$\mathcal{M},(x,y) \models \Box_W\phi \quad \text{iff } \exists U' \in \tau_H,\ (x,y) \in U' \text{ and } \forall (z,y) \in U', \text{ if } (x,y) <_H (z,y), \text{ then } \mathcal{M},(z,y) \models \phi$$

$$\mathcal{M},(x,y) \models \Box_E\phi \quad \text{iff } \exists U' \in \tau_H,\ (x,y) \in U' \text{ and } \forall (z,y) \in U', \text{ if } (z,y) <_H (x,y) \text{ then } \mathcal{M},(z,y) \models \phi$$

Put in words, $\Box_N\phi$, for instance, is true at a pair (x,y) if and only if there is a vertical open, (x,y) is in that open, and all of the northern portion of that open makes ϕ true.

Corresponding to each box, a diamond dual is defined in the usual way.

Example 3.9.2 *Take both topologies to be $\mathbb{R}$ with the standard order topology. Then the topologies induced by the boxes are all variously directed versions of Sorgenfrey topology. Thus the topology induced by $\Box_N$ has as its base the set $B_{\Box_N} = \{[(x_1,x_2),(x_1,y_2))\,|\,x_1,x_2 \in \mathbb{R}, y_2 \in \mathbb{Q}\}$. The topology induced by $\Box_S$ on the other hand is $B_{\Box_S} = \{((x_1,x_2),(x_1,y_2)]\,|\,x_1,x_2 \in \mathbb{R}, y_2 \in \mathbb{Q}\}$.*

3.10 Axioms

Definition 3.10.1 *Let* **TC** *be the following axiom set. We take the* **K4** *axioms and rules for all of $\Box_N, \Box_S, \Box_E$ and $\Box_W$, and we add the following two pairs of interaction* *principles:*

NS$_1$ $(p \wedge \Box_N p \wedge \Box_S p) \rightarrow \Box_N \Box_S p$

NS$_2$ $(p \wedge \Box_N p \wedge \Box_S p) \rightarrow \Box_S \Box_N p$

EW$_1$ $(p \wedge \Box_W p \wedge \Box_E p) \rightarrow \Box_W \Box_E p$

EW$_2$ $(p \wedge \Box_W p \wedge \Box_E p) \rightarrow \Box_E \Box_W p$

That is, **TC** is the fusion **TDL** $\oplus$ **TDL**.

3.10.1 Completeness on abstract frames

Each axiom in **TC** has Sahlqvist form, and thus we have

Fact 3.10.2 *The standard Kripke first-order conditions corresponding to* **NS$_1$**, **NS$_2$** *and* **EW$_1$**, **EW$_2$** *are:*

FO − NS$_1$ $\forall x \forall y((R_N xy \wedge R_S yz) \rightarrow (x = z \vee R_N xz \vee R_S xz))$

FO − NS$_1$ $\forall x \forall y((R_S xy \wedge R_N yz) \rightarrow (x = z \vee R_N xz \vee R_S xz))$

FO − EW$_1$ $\forall x \forall y((R_W xy \wedge R_E yz) \rightarrow (x = z \vee R_W xz \vee R_E xz))$

FO − NS$_1$ $\forall x \forall y((R_E xy \wedge R_W yz) \rightarrow (x = z \vee R_W xz \vee R_E xz))$

By the Sahlqvist theorem, once again, we have

Lemma 3.10.3 **TC** *is sound and complete with respect to the class of Kripke frames* $\mathfrak{F}$, *for* $\mathcal{F} \in \mathfrak{F}$, $\mathcal{F} = (W, R_N, R_S, R_W, R_E)$ *where all four relations are transitive, and they jointly satisfy* **FO − NS$_1$**, **FO − NS$_2$** *and* **FO − EW$_1$**, **FO − EW$_2$**.

While of interest, this is not the main result we would like to have. What we would like to answer is

Question 3.10.4 *Is* **TC** *complete for the class of arbitrary topo-compass structures?*

and perhaps more generally,

Question 3.10.5 *Is the logic of the class of arbitrary topo-compass structures decidable?*

As in the case of **TDL** we will dwell on these issues on a different occasion. For the purposes of this thesis, and as an example, we show that the logics over $\mathbb{Q} \times \mathbb{Q}$ and $\mathbb{N} \times \mathbb{N}$ are finitely axiomatized and decidable.

3.11 TCℚ, the complete logic for $\mathbb{Q} \times \mathbb{Q}$

Definition 3.11.1 *The logic* **TC**$\mathbb{Q}$ *is the fusion* **TDL**$\mathbb{Q} \oplus$ **TDL**$\mathbb{Q}$.

Definition 3.11.2 *We call a frame* $\mathcal{F} = (W, R_N, R_S, R_W, R_E)$, *a* **TC**$\mathbb{Q}$-frame *if* R_N, R_S, R_W, R_E *are transitive and serial, and the following hold:*

FO − NS$\mathbb{Q}_1$ $\forall x \forall y \forall z((R_N xy \wedge R_S yz) \to R_N xz)$

FO − NS$\mathbb{Q}_2$ $\forall x \forall y \forall z((R_S xy \wedge R_N yz) \to R_S xz)$

FO − EW$\mathbb{Q}_1$ $\forall x \forall y \forall z((R_W xy \wedge R_E yz) \to R_W xz)$

FO − EW$\mathbb{Q}_2$ $\forall x \forall y \forall z((R_E xy \wedge R_W yz) \to R_E xz)$.

Proposition 3.11.3 **TC**$\mathbb{Q}$ *is sound and complete for the class of* **TC**$\mathbb{Q}$*-frames.*

Proof Once again, all axioms in **TC**$\mathbb{Q}$ have Sahlqvist form. □

Proposition 3.11.4 **TC**$\mathbb{Q}$ *is complete for the class of pointed finite* **TC**$\mathbb{Q}$*-frames.*

Proof We observe that a fusion logic of a logic with a finite model property has finite model property. □

To prove completeness for $\mathbb{Q} \times \mathbb{Q}$ we are going to unravel an arbitrary pointed finite **TC**$\mathbb{Q}$-frame $\mathcal{F}$ into a subset of $\mathbb{Q} \times \mathbb{Q}$. The procedure combines our strategy for proving that **S4** $\oplus$ **S4** is complete for $\mathbb{Q} \times \mathbb{Q}$ of 2and the unravelling procedure that we used in Definition 3.5.8 above.

From Chapter 2 we borrow the set,

$Y = \bigcup_{n \in \omega} Y_n$, where $Y_0 = \{(0, 0)\}$ and

$$Y_{n+1} = Y_n \cup \{(x - \frac{1}{3^n}, y), (x + \frac{1}{3^n}, y), (x, y - \frac{1}{3^n}), (x, y + \frac{1}{3^n}) \mid (x, y) \in Y_n\}$$

The results of 2 already tell us that Y can be viewed as a subset of $X \times X$, where X is isomorphic to $\mathbb{Q}$. Thus to show completeness for $\mathbb{Q} \times \mathbb{Q}$ it is sufficient to define a map f from Y onto W in $\mathcal{F}$ which has the following property:

Let ν be some valuation on $\mathcal{F}$.

Lemma 3.11.5 *Then for all* **TC** *formulae ϕ, and any $y \in Y$,*

$(\mathcal{F}, \nu), f(y) \models \phi$ *on the relational semantics* $\iff$ $(Y, f^{-1} \circ \nu), y \models \phi$ *on the topo-compass semantics.*

To prove this we use a slight modification of Definition 3.5.8.

Definition 3.11.6 *For each stage n we take set Y_n. We will define f_n with the help of cycling functions. This time around, we will have four cycling functions, one for each relation in $\mathcal{F}$, but things are otherwise unchanged.*

We define cycling functions $e : W \times \mathbb{N} \to W$, $w : W \times \mathbb{N} \to W$, $n : W \times \mathbb{N} \to W$, *and* $s : W \times \mathbb{N} \to W$ *as follows.*

Let $y_1, ..., y_n$ be the finite and nonempty set of R_E successors of x in $\mathcal{F}$. Let $e(x, 1) = y_1, ..., e(x, n) = y_n, e(x, n + 1) = y_1, ..., e(x, n + n) = y_n, ...$ *In other words, $e(x, i)$ infinitely cycles through all finitely many R_E successors of x.*

Next, let $z_1, ..., z_m$ be the finite and nonempty set of R_W successors of x in $\mathcal{F}$. We let $w(x, 1) = z_1, ..., w(x, m) = z_m, w(x, m + 1) = z_1, ..., w(x, m + m) = z_m, ...$

Further, let $y_1, ..., y_n$ be the finite and nonempty set of R_N successors of x in $\mathcal{F}$. Let $n(x, 1) = y_1, ..., n(x, n) = y_n, n(x, n + 1) = y_1, ..., n(x, n + n) = y_n, ...$

Finally, let $z_1, ..., z_m$ be the finite and nonempty set of R_S successors of x in $\mathcal{F}$. We let $s(x, 1) = z_1, ..., s(x, m) = z_m, s(x, m + 1) = z_1, ..., s(x, m + m) = z_m, ...$

We are ready now to define f_n for every n. The function f will then be defined as $\bigcup_{n\in\omega} f_n$

f_n is defined inductively as follows:

$f_0((0, 0)) = x_0$, the root of W.

Let f_n be defined. We define $f_{n+1} : Y_{n+1} \to W$ as an extension of f_n.

For any $(p, q) \in Y_n$, $x \in W$, if $f_n((p, q)) = x$, then:

(E) Let $f_{n+1}((q, p - \frac{1}{3^n}) = e(x, n + 1)$.

(W) Let $f_{n+1}((q, p + \frac{1}{3^n}) = w(x, n + 1)$.

(N) Let $f_{n+1}((q + \frac{1}{3^n}, p) = n(x, n + 1)$.

(S) Let $f_{n+1}((q - \frac{1}{3^n}, p) = r(x, n + 1)$.

In the following two results, $(i, j) \in \{(E, \text{east}), (W, \text{west}), (N, \text{north}), (S,, \text{south})\}$,

Lemma 3.11.7 *$R_i xy \in \mathcal{F}$ iff for every $(q,p) \in Y$, if $f((q,p)) = x$ then there is a sequence s converging to (q,p) from the direction j, and for every $(q',p') \in s$, $f((q',p')) = y$.*

Proof The proof of this lemma is identical to that of with the two extra cases. □

Fact 3.11.8 *At a pair (p,q) in a model $M = (Y,\nu)$ based on any valuation ν, $M,(p,q) \models \Diamond_i \phi$ iff there is a sequence of points s approaching (p,q) from the direction j and every $(p',q') \in s$ makes ϕ true.*

Proof By inspection of truth conditions for $\Diamond_i$. □

We can now prove Lemma 3.11.5 by induction on the complexity of formulae.

The base case where ϕ is a propositional variable is given definition of valuation on Y, and boolean cases are sufficiently simple.

The interesting cases are the four cases of modal formulae. We prove one such case. Let ϕ be of the form $\Diamond_E \psi$. Then we reason as follows:

$(\mathcal{F},\nu), f((q,p)) \models \Diamond_E \psi$ iff there is a $z \in W, R_E f((q,p))z$, and $(\mathcal{F},\nu), z \models \psi$.

By Lemma 3.11.7 and induction hypothesis,

there is a $z \in W, R_E f((q,p))z$, and $(\mathcal{F},\nu), z \models \psi$ iff there is a sequence s converging to (q,p) from the east, and for every $(q',p') \in s$, $(Y, f^{-1} \circ \nu), (q',p') \models \psi$.

But by 3.11.8,

there is a sequence s converging to (q,p) from the east, and for every $(q',p') \in s$, $(Y, f^{-1} \circ \nu), (q',p') \models \psi$ iff $(Y, f^{-1} \circ \nu), (q,p) \models \Diamond_E \psi$. □

This completes the main steps of the completeness proof as we can now argue in the following way. Suppose that there is model based on **TC**$\mathbb{Q}$-frame that satisfies some formula ϕ. Then we can find a point in Y that also satisfies ϕ by the reasoning above, and since Y is a subset of $X \times X$ for X isomorphic to $\mathbb{Q}$, we can satisfy ϕ on $\mathbb{Q} \times \mathbb{Q}$, as desired.

Fact 3.11.9 **TC**$\mathbb{Q}$ *is decidable.*

Proof $\mathbf{TC}\mathbb{Q}$ is a fusion $\mathbf{TDL}\mathbb{Q} \oplus \mathbf{TDL}\mathbb{Q}$, $\mathbf{TDL}\mathbb{Q}$ has finite model property, $\mathbf{TC}\mathbb{Q}$ also has a finite model property as fusion preserves complexity for complexity classes PSPACE and above. □

Question 3.11.10 *What is the complexity of satisfiablity checking for* $\mathbf{TC}\mathbb{Q}$*?*

Thus $\mathbf{TC}\mathbb{Q}$ is our first example of an interesting decidable compass logic based on the topological semantics. The next example is also informative, even though perhaps less intricate.

3.12 $\mathbf{TC}\mathbb{N}$, the complete logic for $\mathbb{N} \times \mathbb{N}$

Let $\mathbf{TC}\mathbb{N}$ be the fusion $\mathbf{TDL}\mathbb{N} \oplus \mathbf{TDL}\mathbb{N}$.

Theorem 3.12.1 $\mathbf{TC}\mathbb{N}$ *is the complete logic of* $\mathbb{N} \times \mathbb{N}$ *on topo-compass semantics.*

Fact 3.12.2 *Once again, we make the observation that* $\mathbf{TC}\mathbb{N}$ *on relational semantics is complete for a frame consisting of a single point with all four relations empty.*

Thus a map that maps every point in $\mathbb{N} \times \mathbb{N}$ onto a single point with empty relations can be easily proved to preserve modal formulae from the topo-compass semantics to the relational semantics on the single point, and completeness follows.

Corollary 3.12.3 $\mathbf{TC}\mathbb{N}$ *in decidable, and its complexity is NPTIME.*

Question 3.12.4 *What is the logic of products topo-directional frames based on generalized order topologies? Is it* $\mathbf{TDL}\mathbb{GO} \oplus \mathbf{TDL}\mathbb{GO}$*? What is its complexity?*

And finally,

Question 3.12.5 *What is the logic of products topo-directional frames based on standard order topologies? Is it* $\mathbf{TDL}\mathbb{O} \oplus \mathbf{TDL}\mathbb{O}$*? What is its complexity?*

3.13 Conclusion

This chapter has shown that there are interesting languages that mix topology and ordering that are different form just adding interior box to temporal operators. We have found some new complete logics, and we have also shown how the style of product analysis developed in 2, extends to this new setting. Of course, new open questions now arise. One example is finding the complete logic of order topologies; we stand by our conjecture but we haven't proved it yet. Another interesting challenge is axiomatizing the complete logic in our mixed language for the product $\mathbb{R} \times \mathbb{R}$. Finally, we mention that there are also complexity and decidability issues here, for a brief discussion see Chapter 5.

Chapter 4

The Geometry of Knowledge

The most widely used attractive logical account of knowledge uses standard epistemic models, i.e., *graphs* whose edges are indistinguishability relations for agents. In this chapter, we discuss more general *topological* models for a multi-agent epistemic language, whose main uses so far have been in reasoning about space. We show that this more geometrical perspective affords greater powers of distinction in the study of common knowledge, defining new collective agents, and merging information for groups of agents.

4.1 Epistemic logic in its standard guise

4.1.1 Basic epistemic logic

Epistemic logic is in wide use today as a description of knowledge and ignorance for agents in philosophy [43], computer science [35], [77], game theory [23], and other areas. In this chpter, we assume familiarity with the basic language of propositional epistemic logic, interpreted over multi-agent **S4** models whose accessibility relations are reflexive and transitive. Alternative model classes occur, too, such as equivalence relations for each agent in multi-agent **S5**–but our discussion is largely independent from such choices. The key semantic clause about an agent's knowledge of a proposition says that $K_i\phi$ holds at a world x if and only if ϕ is true in all worlds y accessible

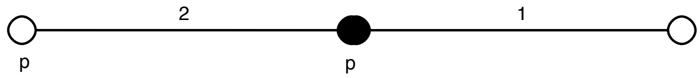

Figure 4.1: In the black central world, 1 does not know if p, while 2 does know that p. In the world to the left, 1 does know that p, so in the central world, 2 does not know if 1 knows that p.

for i from x. That is, the epistemic knowledge modality is really a modal box $\Box_i\phi$. For technical convenience, we will use the latter notation for knowledge in the rest of this paper. The main modern interest in epistemic logic has to do with analyzing iterated knowledge of agents about themselves and what others know, for purposes of communication and interaction. Cf. [9], [13] on systems that combine epistemic logic and dynamic logic to describe information update in groups of agents. A simple example of how the basic logic works is the model in Figure 4.1.

The universally valid principles in our models are those of multi-agent **S4**. In an epistemic setting, the usual modal axioms get a special flavor. E.g., the iteration axiom $\Box_1\phi \to \Box_1\Box_1\phi$ now expresses 'positive introspection': agents who know something know that they know it. More precisely, we have **S4**-axioms for each separate agent, but no valid further 'mixing axioms' for iterated knowledge of agents, such as $\Box_1\Box_2\phi \to \Box_2\Box_1\phi$. Indeed, the latter implication fails in the above example. For instance, in the world on the left, 1 has no uncertainties, and so 1 knows that 2 knows that p. But 2 does not know there that 1 knows that p, because the latter assertion is false in the central world. Another way of describing the set of valid principles is as a *fusion* $\mathbf{S4} \oplus \mathbf{S4}$ of separate logics **S4** for each agent, a perspective of 'merging logics' to which we will return below. In what follows, we shall mostly work with two-agent groups, $G = \{1, 2\}$, since most phenomena of interest can be studied there. Generalizations to finite k-agent cases are straightforward.

4.1.2 Group knowledge

Perhaps the most interesting topic in an interactive epistemic setting has been the discovery of various notions of what may be called *group knowledge*. Two well-known examples are as follows:

1. $E_G\phi$: every agent in group G knows that ϕ,
2. $C_G\phi$: ϕ is *common knowledge* in the group G.

The latter notion of group knowledge is much stronger than the former. It has been proposed in the philosophical, economic and linguistic literature as a necessary precondition for coordinated behavior between agents, cf. [47]. The usual semantic definition of common knowledge runs as follows:

$$M, x \models C_{1,2}\phi \text{ iff for all } y \text{ with } x\ (R_1 \cup R_2)^* y,\ M, y \models \phi$$

where $x(R_1 \cup R_2)^* y$ if there is a finite sequence of successive steps from either of the two accessibility relations connecting x to y. This relation is the reflexive transitive closure of the union of the relations for both agents. The key valid principles for common knowledge are the following additional axiom and rule:

Equilibrium Axiom: $C_{1,2}\phi \leftrightarrow (\phi \wedge (\Box_1 C_{1,2}\phi \wedge \Box_2 C_{1,2}\phi))$
Induction Rule: $\frac{\vdash p \rightarrow (\Box_1(q \wedge p) \wedge \Box_2(q \wedge p))}{\vdash p \rightarrow C_{1,2}q}$

This logic is known as $\mathbf{S4_2^C}$. It has been shown to be complete and decidable in [35] via a simple variation on similar proofs for propositional dynamic logic.

But there are still further interesting notions of knowledge for a group of agents. A prominent one is so-called *implicit knowledge*, $D_G\phi$, which describes what a group would know if its members decided to merge their information:

$$M, x \models D_{1,2}\phi \text{ iff for all } y \text{ with } xR_1 \cap R_2 y,\ M, y \models \phi$$

where $R_1 \cap R_2$ is the intersection of the accessibility relations for the separate agents. This new notion is technically somewhat different from the earlier two in that, unlike universal and common knowledge, it is not invariant under modal *bisimulations* of epistemic models. It also involves a new phenomenon of independent epistemic interest: viz. merging the information possessed by different agents. The latter topic will return throughout this paper.

4.1.3 Agents as epistemic accessibility relations

We can also think of new notions of group knowledge as introducing *new agents*. E.g., C_G defines a new kind of **S4**-agent, since $R_{(1\cup 2)^*}$ was again a pre-order. Note that $R_1 \cup R_2$ by itself is not a pre-order, so the new 'agent' corresponding to the fact that 'everybody knows' would have different epistemic properties. In particular, it would lack positive introspection as to what it knows. In contrast, the relation $R_1 \cap R_2$ for D_G is again an **S4**-agent as it stands, since Horn conditions like transitivity and reflexivity are preserved under intersections of relations. So, given a group of individual agents, our logical models suggest new agents. In particular, with two **S4**-agents $1, 2$, two additional ones supervene on these, one weaker, one stronger:

$$\begin{tikzcd}
 & R_1 \cap R_2 & \\
R_1 \arrow[ur] & & R_2 \arrow[ul] \\
 & (R_1 \cup R_2)^* \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

All this seems quite rich as an account for epistemic agents. And yet, there are indications that this framework is not yet flexible enough for its tasks.

4.1.4 Alternative views of common knowledge

Despite the success of the standard epistemic logic framework, there are still doubts about its expressive power and sensitivity. Some recurrent complaints seem endemic to logical approaches as such, like the vexing problem of logical omniscience: agents automatically know all laws of the system. But a more serious concern is the lack of epistemic distinctions in the standard modal setting. Notably, in his well-known critical paper [10], Barwise claimed that a proper analysis of common knowledge must distinguish three different approaches, that we may label

1. countably *infinite iteration* of individual knowledge modalities,
2. the *fixed-point view* of common knowledge as 'equilibrium',
3. agents' having a *shared epistemic situation.*

He then showed how to distinguish all three in a special situation-theoretic framework. As we will see below, however, Barwise's distinctions make sense in mainstream logic too–provided that we move to a broader topological semantics for the epistemic language involving products of models for individual agents. But before we do that, let us first analyze the reason why standard epistemic logic fails to distinguish the first two options. The third notion of 'shared understanding' is somewhat more mysterious, and harder to grasp in a standard relational modal setting. We will have a stab at it in the richer topological models of Section 2.

4.1.5 Computing epistemic fixed-points

The above Equilibrium Axiom for the common knowledge operator $C_G\phi$ shows how it may be viewed as defining a fixed-point of an epistemic operator $\lambda X.\phi \wedge \Box_1 X \wedge \Box_2 X$.

In conjunction with the Induction Rule, it may even be seen to be a *greatest fixed-point* definable in the standard modal *μ-calculus* as:

$$C_G\phi := \nu p.\phi \wedge \Box_1 p \wedge \Box_2 p.$$

With a perhaps more familiar modal μ-operator, its existential variant would be defined as a smallest fixed-point

$$\Diamond_G^C\phi := \mu p.\phi \vee \Diamond_1 p \vee \Diamond_2 p.$$

As usual, a greatest fixed-point is defined as the fixed-point of a descending approximation sequence defined over the set of ordinals. We write $[|\phi|]$ for the truth set of ϕ in the relevant model where evaluation takes place:

$C^0_{1,2}\phi := [|\phi|]$,
$C^{\kappa+1}_{1,2}\phi := [|\phi \wedge \Box_1(C^\kappa_{1,2}\phi) \wedge \Box_2(C^\kappa_{1,2}\phi)|]$,
$C^\lambda_{1,2}\phi := [|\bigwedge_{\kappa<\lambda} C^\kappa_{1,2}\phi|]$, for λ a limit ordinal.

Finally, we let $C_{1,2}\phi := C^\kappa_{1,2}\phi$ where κ is the least ordinal for which the approximation procedure halts: i.e., $C^{\kappa+1}_{1,2}\phi = C^\kappa_{1,2}\phi$. This approximation procedure must stop at some ordinal because the operator F applied is *monotonic*, a fact which is guaranteed by the positive occurrence of the propositional variable p in the body of F's definition. As a result, the approximation sequence for a greatest fixed-point operator always descends to subsets, and hence it must stop eventually. In general μ-calculus, reaching this stopping point may take any number of ordinal stages. A standard example is the least-fixed-point formula $\mu p.\Box p$ which computes the so-called 'well-founded part' of the binary accessibility relation for the modality. But in certain cases, stabilization is guaranteed to occur by the first infinite stage.

Fact 4.1.1 *In every relational epistemic model, the approximation procedure for the common knowledge modality stabilizes at $\kappa \leq \omega$.*

This simple behavior is most easily understood by observing that knowledge modalities $\Box_i$ distribute over any infinite conjunction. Thus, $\Box_i(\bigwedge_{n<\omega} C^n_{1,2}\phi)$ is simply $\bigwedge_{n<\omega} \Box_i C^n_{1,2}\phi$ which is equivalent to $\bigwedge_{n<\omega} C^n_{1,2}\phi$. More generally, stabilization for a formula $\nu p.\phi(p)$ is guaranteed by stage ω in any model just in case the syntax defining the monotone approximation operator is constrained as follows [15]. The formula $\phi(p)$ must be a disjunction whose members are constructed using only

1. arbitrary literals $(\neg)q$,
2. any epistemic formulas that do not contain q at all,
3. conjunctions and universal modalities.

The preceding Fact says that the fixed-point approach to common knowledge and that with countably infinite conjunctions of repeated knowledge modalities are equivalent in the standard setting, as $\nu p.\phi \wedge \Box_1 p \wedge \Box_2 p$ is equivalent to

$$K_{1,2}p := \phi \wedge \Box_1\phi \wedge \Box_2\phi \wedge \Box_1\Box_2\phi ...$$

This equivalence is often considered a technical convenience. But it may also indicate that our standard models are too weak to make a relevant distinction, and that more general models are needed. As we shall see, these two definitions of common knowledge are different in a *topological* modelling for epistemic logic— and even stronger ones can then be modelled, resembling Barwise's use of 'shared situations'.

4.1.6 Merging Information

Many further interesting issues are raised by a multi-agent epistemic setting. In particular, multi-agent models will often arise by *merging* models for separate agents,

or groups of agents, so that common knowledge for the whole group becomes possible at all. One natural way of combining models for two or more agents emphasized in the recent literature on combining modal logics employs *products* of their underlying frames.
Sometimes one also adds the direct product relation $R_{1,2}$ which requires successor steps in both components. But in the present setting, this is definable as the relational composition of R_1 and R_2 in any order.

This way of combining modal logics is explored in detail in [38]. As we mention in Chapter 2, the separate logics of the component frames are preserved in the product. But the really interesting question is what happens in the joint language containing both modalities $\Box_1$ and $\Box_2$, which can express interaction between epistemic agents. As noted, product frames automatically validate *com*, $\Box_1\Box_2 p \equiv \Box_2\Box_1 p$, and *chr*, $\Diamond_1\Box_2 p \rightarrow \Box_2\Diamond_1 p$. But note that these two principles were not valid in the general fusion logic $\mathbf{S4} \oplus \mathbf{S4}$ of epistemic agents, as we saw earlier. Figure 2.2 in Chapter 2 provided a formal counterexample to *com*. To put such a scenario in words: a student may know that the teacher knows the answer to questions on the test, while the teacher does not know if the student knows the answer. Moreover, if *com* does become valid, common knowledge trivializes, since any finite sequence of knowledge modalities will be equivalent to one of $\Box_1, \Box_2$ or $\Box_1\Box_2$.

Now there are other notions of merge for epistemic models, and the preceding collapse of common knowledge need not occur with other operations. Often, merging information for single agents or groups of agents is more naturally viewed as an operation on *models*, rather than frames. And in that case, the necessity of obtaining a consistent atomic valuation on pairs of worlds may complicate the above product construction, and thereby block *com* and *chr*. We discuss this issue briefly in Section 4.2.7 But for our purposes later on with analyzing common knowledge, frame products are important, provided we generalize them, again, to a wider topological setting. In that case, the two undesirable epistemic interaction laws no longer hold, and the above trivialization of common knowledge goes away.

We have now accumulated enough motivation for looking into broader alternative semantics for a multi-agent language, which should be fine-grained enough to distinguish different notions of common knowledge, while being sufficiently robust to still provide a plausible version of epistemic logic. We find this in the following mathematical generalization of relational models.

4.2 Epistemic Models in Topological Semantics

4.2.1 From graphs to topological spaces.

The major alternatives to relational semantics for modal logics that we emphasize in this thesis, and historically even the earlier approach, employs *topological* models. Topology is an abstract mathematical theory of space, emphasizing qualitative notions of open environment, closure, boundary, or connectedness. All topological modalities in this chapter satisfy the axioms of the modal logic **S4**, which reflect key properties of the topological interior operation. The interesting epistemic details then lie in the interaction among such modalities.

There is a way to view topological semantics as a generalization of standard modal model theory. The general connection between the two classes of models for modal or epistemic languages is best seen in the fact that standard relational models can be viewed as a special kind of topological spaces via the following notion.

Definition 4.2.1 *A topological space $\mathcal{X}$ is* Alexandroff *if every intersection of open sets of $\mathcal{X}$ is again open.*

Any Alexandroff topology $\mathcal{X} = \langle X, \tau \rangle$ induces a standard relational frame $\langle X, R \rangle$ with a reflexive transitive relation Rxy iff $y \in \bigcap\{U \in \tau \mid x \in U\}$. Conversely, any reflexive transitive relational frame $\langle X, R \rangle$ induces an Alexandroff topology by taking the sets $U_x = \{y \mid Rxy\}$ for each $x \in X$ as a basis for τ. It is easily shown that topological interpretation of modal formulas in a relational model yields the same results as in their associated Alexandroff spaces, and vice versa. In this way, modal logics of relational models describe special sets of topological models. But in general, topological models include settings without a clear relational counterpart. E.g., the standard topologies on $\mathbb{Q}$ and $\mathbb{R}$ are clearly not Alexandroff: any singleton set (a non-open) is the intersection of the open intervals containing it.

There is a recent revival of interest in modal **S4** interpreted over topological spaces, because of its applications to spatial reasoning. [1] and [4] survey the expressive power of **S4** and its extensions for this purpose. We will use a few results from this spatial line later on. But before we cite them, let us make a connection with our major concern of what agents know.

4.2.2 Topology and information

Dating back to the 1930s, there has also been a more epistemic use of topological models, viz. for *intuitionistic* logic, cf. [70]. In that case, open sets are rather interpreted as 'pieces of evidence', e.g., about the location of a point, reflecting the intuitionistic idea of truth-as-provability. We can generalize this idea to epistemic logic, reading the above truth condition for a knowledge modality $\Box_i p$ as saying that there exists a piece of evidence for agent i (viz. an open set in i's topology) which validates the proposition p. Alternatively, we could also think of the topology as a collection of theories or data bases that an agent has at its disposal. [72] contains more abstract versions of this idea. As we will see, one of the side benefits of this information-based interpretation of the epistemic language is that common knowledge arises in a group of agents precisely when they share the same piece of information. But first, we explore the new handle that we get on the issue of merging information structures for different agents.

From the results of Chapter 2 we know that we can use products of topological models to combine agents, and further it follows that the fusion $\mathbf{S4} \oplus \mathbf{S4}$ is the logic of two epistemic agents combined into one framework using topological products, without any dramatic interaction enforced as in the case of products of relational frames. This result gives us the technical means to analyze different versions of common knowledge in a concrete setting of merged multi-agent models.

4.2.3 Common knowledge in product spaces

The earlier definitions of common knowledge still make sense in topological models. For instance, countably infinite iteration of all finite sequences of alternating knowledge modalities for the individual agents $1, 2$ is as before:

$K_{1,2}p := \bigwedge_n^\omega K_{1,2}^n p,$

with $K_{1,2}^n p$ defined inductively as follows:

$$K_{1,2}^0 p := p$$
$$K_{1,2}^{n+1} p := \Box_1(K_{1,2}^n p) \wedge \Box_2(K_{1,2}^n p)$$

And the same is true for the fixed-point definition

$$C_{1,2}\phi := \nu p.\phi \wedge \Box_1 p \wedge \Box_2 p,$$

provided we make the appropriate adjustments in computing fixed points. In particular, the monotone operations generated by formulas positive in p now work a bit differently from before. In relational models, the operator $\Box_i$ applied to a set X yielded $\Box_i(X) = \{y \mid \forall x(R_i yx \to x \in X)\}$, making the modality a bounded universal quantifier. In topological semantics, however, the relevant operator is

$$\Box_i(X) = \{y \mid \exists U \in \tau_i \,\&\, \forall x(x \in U \to x \in X)\}$$

This reads a modality as an existential quantifier over open sets followed by a universal quantifier over elements of those sets. This two-quantifier combination complicates matters when approximating greatest or smallest fixed-points. Indeed, the definitions of common knowledge by fixed-points and by countably infinite iteration will now diverge. Here is a first indication why this may happen. The topological semantics validates the finitary logic **S4**, but it diverges from the relational validities in its infinitary behavior.

Fact 4.2.2 *Topological interior does not distribute over infinite conjunctions:*

$$\Box_i \bigwedge_n p_n \quad \textit{is not always equivalent to} \quad \bigwedge_n \Box_i p_n$$

Proof Take the standard topology on $\mathbb{Q}$. Define a valuation ν with, for all n, $\nu(p_n) = (-\frac{1}{n}, \frac{1}{n})$. Note that the intersection of these open sets is the singleton 0. Then $\bigwedge_n \Box_i p_n$ is true at 0, whereas $\Box_i \bigwedge_n p_n$ is not true anywhere. $\Box$

This result, though suggestive, is not yet a proof that the two definitions of common knowledge diverge. To do that, we will show that given a set p, the operator $K_{1,2}p$ does not always define a horizontally and vertically open set. Since the fixed-point version of $C_{1,2}p$ is always open in both these senses, the two cannot be the same.

We construct the relevant example by choosing a countable sequence of points in the rational plane $\mathbb{Q} \times \mathbb{Q}$ horizontally converging to the origin $(0, 0)$. The first point in the sequence makes $\Box_1 p$ true but not $\Box_2\Box_1 p$, the second $\Box_1\Box_2 p, \Box_2\Box_1 p$ but not $\Box_2\Box_1\Box_2 p$, etc. This is possible by Theorem 2.6.1 for the logic of $\Box_1, \Box_2$: no finite iteration level of knowledge implies the next in the fusion logic $\mathbf{S4} \oplus \mathbf{S4}$, and hence situations as described must exist in suitable models over $\mathbb{Q} \times \mathbb{Q}$. In particular, at each point of the sequence, $K_{1,2}$ will be false, and hence $\Box_1 K_{1,2}p$ is false at the origin $(0, 0)$. It then remains to show that $K_{1,2}p$ itself does hold at $(0, 0)$, but this will happen because of a well-chosen total valuation $\nu(p)$ for p on $\mathbb{Q} \times \mathbb{Q}$. To make this work, we make a number of more precise observations– while also slightly changing the formulas involved:

Theorem 4.2.3 $K_{1,2}p \to \Box_1 K_{1,2}p$ *is not valid on topological product spaces.*

Let ψ_n be the formula $\Box_1(K^n_{1,2}p) \to \Box_2(K^n_{1,2}p)$.

Fact 4.2.4 *(a) For all n, ψ_n is not a theorem of the fusion logic* $\mathbf{S4} \oplus \mathbf{S4}$.

(b) There is a model M_n on $\mathbb{Q} \times \mathbb{Q}$ such that $M_n, (0,0) \not\models \Box_2(K^n_{1,2}p)$, and for all $q \in \mathbb{Q}$, $M_n, (q,0) \models K^n_{1,2}p$.

Proof As for (a), one can easily construct finite fusion frames invalidating any given principle ψ_n.

(b) Since $\mathbf{S4} \oplus \mathbf{S4}$ is complete for $\mathbb{Q} \times \mathbb{Q}$, by (a) there is a model M'_n such that $M'_n, (0,0) \not\models \psi_n$, that is,

$$M'_n, (0,0) \models \Box_1(K^n_{1,2}p)$$

as well as

$$M'_n, (0,0) \not\models \Box_2(K^n_{1,2}p).$$

It follows that there is an open interval $((-q,0),(q,0))$ and every $(q',0)$ in this interval satisfies $K^n_{1,2}p$. By Locality (Proposition 2.3.6 in Chapter 2), in $(-q,q) \times \mathbb{Q}$ with the valuation from M'_n restricted to this space it is still true that $\Box_2(K^n_{1,2}p)$ fails at $(0,0)$ and that $K^n_{1,2}p$ holds at each point $(q',0)$. But $(-q,q) \times \mathbb{Q}$ is homeomorphic to $\mathbb{Q} \times \mathbb{Q}$ itself, and hence the valuation of M'_n transfers to $\mathbb{Q} \times \mathbb{Q}$ via the homeomorphism.

Fact 4.2.5 *There is a sequence of positive irrational numbers converging to 0 such that for any two adjacent numbers r, r' in the sequence, the distance $r - r'$ is a rational number.*

Take for instance $\sqrt{2}$, $\sqrt{2}-1$, $\sqrt{2}-1.4$, $\sqrt{2}-1.41$, etc. Next, for each rational interval, we form squares $S_1, S_2, ...$ of decreasing sizes over these intervals bounded by the separating irrationals [see Figure 4.2]. In the above example, the first square would be $(\sqrt{2}, \sqrt{2}-1) \times (-\frac{1}{2}, \frac{1}{2})$, the second $(\sqrt{2}-1, \sqrt{2}-1.4) \times (-0.2, 0.2)$, etc. Each of these squares is still homeomorphic to the rational plane $\mathbb{Q} \times \mathbb{Q}$ with some valuation for the proposition letter p.

Now, we create a new big model M over $\mathbb{Q} \times \mathbb{Q}$ as follows. In the sequence of squares S_n, we embed the earlier counter-examples M_n into S_n in such a way that its horizontal axis becomes the horizontal axis of the square S_n. This ensures that $K^n_{1,2}p$ holds everywhere on S_n's $X-$axis while $\Box_2(K^n_{1,2}p)$ fails somewhere on it. Outside of

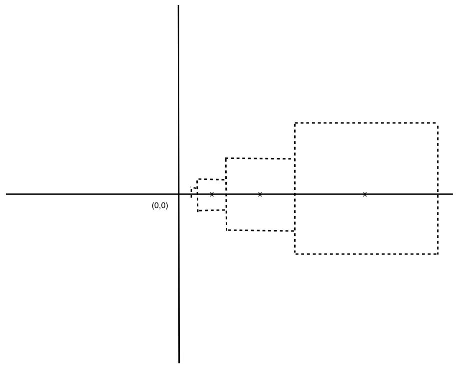

Figure 4.2:

the squares, we put every point of the total rational plane in $V(p)$. Now we can prove the earlier informal assertion.

Claim 4.2.6 *(a)* $M, (0,0) \models K_{1,2}p$
(b) $M, (0,0) \not\models \Box_1 K_{1,2}p$.

Proof (a) We will prove that for all n, $K^n_{1,2}p$ holds at $(0,0)$. The proof is by induction. First note that any point on the y axis or to the left of it (except $(0,0)$) sits in an open circle interior in which p is true everywhere. Inside such a circle, these points evidently satisfy all formulas $K^n_{1,2}p$, and hence by Locality again, they also satisfy all these formulas in the whole model M.

Now we consider the origin $(0,0)$. The base step is simple: $K^0_{1,2}p$ is true by the definition of $\nu(p)$. Next consider the inductive step $K^n_{1,2}p \Rightarrow K^{n+1}_{1,2}p$, where $K^{n+1}_{1,2}p$ is $\Box_1(K^n_{1,2}p) \wedge \Box_2(K^n_{1,2}p)$. We show that the two conjuncts hold separately. To see that $\Box_2(K^n_{1,2}p)$ holds at $(0,0)$ we need an open set $((0,y),(0,-y))$ with $K^n_{1,2}p$ true at each point in this set. Evidently, this formula holds at $(0,0)$ itself by the inductive hypothesis. And it holds at any other point on the Y axis by the preceding observation about open p-circles.

Next we show that $\Box_1(K^n_{1,2}p)$ holds at $(0,0)$. This time we need an interval of the form $((-y,0),(x,0))$ with $K^n_{1,2}p$ true at every point in the interval. Here, points in $((y,0),(0,0))$ are covered by the observation about open p-circles again, and the origin itself by the inductive hypothesis. Then, looking toward the right, by the construction of the squares S_n, we know that $K^n_{1,2}p$ holds everywhere at the horizontal axis of S_n, and the same obviously remains true for S_m with $m > n$. Thus, for the desired right end-point $(x, 0)$ we can take any point on the horizontal axis of the square S_n. Since every point in $((0, 0), (x, 0))$ is in some S_m for $m \geq n$, we have the desired interval, and hence $\Box_1(K^n_{1,2}p)$ is true at the origin. In this connection, the idea behind our 'gluing' the squares at irrationals was that inside $\mathbb{Q} \times \mathbb{Q}$, there are then no boundary points to consider.

(b) To see that $\Box_1 K_{1,2}p$ fails at $(0, 0)$, we observe that in any horizontal open interval I around $(0, 0)$ there is a point where $K_{1,2}p$ fails. Note that for some n, the horizontal axis of S_n is a subset of I, by our construction of ever smaller squares S_n, and hence there is a point inside our interval where $\Box_2(K^n_{1,2}p)$ fails, and hence also $K_{1,2}p$, as desired. $\Box$

Corollary 4.2.7 *$K_{1,2}p$ is not equivalent to $C_{1,2}p$ in topological models.*

Corollary 4.2.8 *Stabilization of the fixed-point version of $C_{1,2}X$ may occur later than ordinal stage ω.*

Thus, the topological setting achieves a natural separation between the first two definitions of common knowledge that Barwise distinguished. Moreover, our method raises further issues. First, it is rather 'logicky', and one might want a concrete independently motivated set of points in the rational plane for which the separation occurs. Also, it would be of interest to determine the exact (countable) ordinals at which epistemic fixed-point definitions do stabilize in this model.

This still leaves Barwise's third account of common knowledge in terms of 'shared situations'. We shall return to this matter in Section 4.2.6.

4.2.4 Complete logic of common knowledge on topo-products

Now what is the basic logic of the greatest fixed-point common knowledge modality $C_{1,2}$ on topological models? Perhaps surprisingly, the general answer is: 'the same as that for relational **S4**-models'. The reason is that the usual system $\mathbf{S4_2^C}$ already has principles for common knowledge that are satisfied by the fixed-point definition. Moreover, that system is complete w.r.t. relational models [35], and the latter are Alexandroff topological models at the same time. More interesting is what happens in our topological product models. In fact, the logic does not change here either, but this time, the argument takes a little more thought.

Theorem 4.2.9 *$S4_2^C$ is complete for products of arbitrary topologies. In fact it is even the complete logic of $\mathbb{Q} \times \mathbb{Q}$.*

The completeness argument runs along the lines of the proof of Theorem 2.6.1 in Chapter 2. By the usual completeness proof with respect to relational models, any non-theorem of $\mathbf{S4_2^C}$ fails on some finite rooted modal model. Next, such a model can be unravelled via a bisimulation into the double-binary branching tree $T_{2,2}$ with an appropriate valuation. Now we do the labelling construction described in the proof of Theorem 2.6.1. In the end, this procedure produced a topo-bisimulation between the given model on $T_{2,2}$ and some model on the rational plane $\mathbb{Q} \times \mathbb{Q}$. Now the only thing we need to observe is that topo-bisimulations do not just preserve truth values of ordinary modal formulas. They also evidently preserve truth values of formulas in any modal language allowing *infinite* conjunctions and disjunctions of formulas. And, the latter observation gives us exactly what we need to transfer counterexamples to formulas in the epistemic language with common knowledge viewed as a fixed-point operator.

Fact 4.2.10 *Topological bisimulations preserve arbitrary fixed-point formulas.*

Proof In any given model M, any modal fixed-point formula ϕ is equivalent to some modal formula $\phi(\alpha)$ which has no fixed-point operators any more, but which uses infinite conjunctions and disjunctions up to a size determined by the ordinal α to 'unwind' approximation sequences. What this α is depends on the size of the model M. Moreover, it does not matter if we unwind up to any higher ordinal. Now, suppose that some fixed-point formula ϕ is true at M, s, and E is a bisimulation connecting s to t in a model N, t. Let α^* be the maximum of the unwinding ordinals for ϕ in the two models M, N. Then $\phi(\alpha^*)$ is true at s in M, and therefore also true at t in N. It follows that the original fixed-point formula ϕ is true in N, t. $\Box$

Even so, given the difference between $C_{1,2}\phi$ and $K_{1,2}\phi$ that we have now found, a new completeness question arises, yet to be solved:

Question:

What is the complete logic of $K_{1,2}\phi$?

Given all this emphasis on geometrical models like the rational plane, can we really claim that they are also epistemically relevant? Our discussion only shows their use as visualizations of abstract distinctions. Whether there is any deeper *informational* meaning to $\mathbb{Q} \times \mathbb{Q}$ still remains to be seen.

In the remainder of this paper, we discuss some further aspects of the topological semantics for knowledge, analogous to those raised in Section 4.1.

4.2.5 More on epistemic agents as topologies

In relational semantics, agents were really just accessibility relations. Likewise, in our topological models, agents are topologies! As was explained in Section 4.2.2, what the agent knows in a world of some model is what holds there according to the box modality of its topology. Let us now draw some comparisons with the situation in Section 4.1.3, where two agents $1, 2$ generated at least two further 'introspective collective agents', one being their supremum $R_{(1\cup 2)^*}$ leading to common knowledge, and the other their infimum $R_1 \cap R_2$ leading to 'implicit knowledge' for the group. The topological semantics gives us interesting counterparts to these operations.

Remark.

Introspection principles If we are less strict in our logic, without requiring positive introspection, then many further options arise, just as with relational models. If we are more strict, as in relational **S5**-models with negative introspection, then we must only use topologies that do satisfy the axiom $\phi \to \Box\Diamond\phi$. It is easy to see that, on T_0 spaces in which all singletons are closed, imposing this principle makes the topology discrete, trivializing the epistemic logic. But then, even a weak separation axiom like T_0 is not plausible epistemically. On general spaces, $\phi \to \Box\Diamond\phi$ corresponds to the property that every set is a subset of the interior of its closure. Unpacked further this says that:

$$\forall x, \exists U \in \tau : x \in U \,\&\, \forall y \in U, y \in V \in \tau : x \in V$$

This means the space is a union of open sets whose points have the same open neighbourhoods – which is a topological counterpart of relational **S5** models.

Our favorite setting for studying new collective agents are the product models that we used so far. We start with a simple but perhaps surprising observation. Common knowledge as a greatest fixed-point corresponds to taking the following very natural operation on the given topologies for the individual agents. Consider the *intersection* $\tau_{1\cap 2}$ of the earlier topologies τ_1 and τ_2 on a product space. It is easy to see that this is again a topology: all closure conditions are satisfied. Now we observe the following connection:

Fact 4.2.11 $\forall M \forall x, M, x \models C_{1,2}\phi$ *iff* $M, x \models [1 \cap 2]\phi$

Proof We will show that the truth sets $[|C_{1,2}\phi|]$ and $[|[1\cap 2]\phi|]$ are identical in all models. First, $[|C_{1,2}\phi|] \in \tau_i$ for $i \in \{1, 2\}$ since the truth set is a fixed-point of $\nu p.\phi \wedge \Box_1 p \wedge \Box_2 p$. But then $[|C_{1,2}\phi|] \in \tau_{1\cap 2}$ by the definition, and so $[|C_{1,2}\phi|] \subseteq [|[1\cap 2]\phi|]$. Next, $[|[1\cap 2]\phi|]$ satisfies $[|\Box_i[1\cap 2]\phi|] = [|[1\cap 2]\phi|]$ for $i \in \{1, 2\}$. Hence $[|[1\cap 2]\phi|]$ is a fixed-point. Since $[|C_{1,2}\phi|]$ is the greatest fixed-point, $[|[1\cap 2]\phi|] \subseteq [|C_{1,2}\phi|]$. $\Box$

It is worth observing that this argument holds in general, for any two given topologies on some space, not just the vertical and horizontal ones in products. In fact, intersection of topologies is the counterpart, under the model-to topology transformation sketched earlier, of taking the reflexive transitive closure of given accessibility relations.

Thus, we also expect a topological counterpart for the earlier operation of *relational intersection*, which modelled implicit group knowledge D_G. This should be the union of two topologies, and then closing off in the minimal way that produces a topology again. The result is the *sum topology* $\tau_1 + \tau_2$ which takes all pairwise intersections of opens of the two topologies as a basis. The latter topology need not always be of great interest. E.g., on our recurrent topo-product $\mathbb{Q} \times \mathbb{Q}$, it will just be the discrete topology, making every point an open. From an informational perspective, this means that merging the information that we get about points in the horizontal and vertical directions fixes their position uniquely.

The result of all this is again an inclusion diagram:

$$
\begin{tikzcd}
& \tau_1 + \tau_2 & \\
\tau_1 \arrow[ur] & & \tau_2 \arrow[ul] \\
& \tau_{1\cap 2} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Let us now return to the three distinctions made in [10]. So far, we have separated the countably infinite conjunction view from the greatest fixed-point view of common knowledge. What about the third view of having a 'shared situation'? In some ways, using the intersection topology seems to model this. Its opens are precisely those information pieces that are accepted by both agents. But if that is the case, then we have not separated the second and third notions. Fact 4.2.11 tells us precisely that the two amount to the same thing. But topological product models have further resources! In particular, so far, we have not discussed what topologists would call the real *product topology* τ on spaces $X \times Y$. This topology is defined by letting the sets $U \times V$ form a basis, where U is open in $\mathcal{X}$ and V is open in $\mathcal{Y}$. An example is the natural metric topology on the plane $\mathbb{Q} \times \mathbb{Q}$, used briefly in the argument for Claim 4.2.6, with open circles around points as neighbourhoods. The agent corresponding to this new group concept τ only accepts very strong collective evidence for any proposition. Here are two relevant results from Chapter 2:

Theorem 4.2.12 *The epistemic box modality for the true product topology is not definable in the language of the separate modalities $\Box_1, \Box_2$, even when we add fixed-point operators.*

Theorem 4.2.13 *The complete logic including the true product topology is the smallest normal modal logic in the language of three modalities $\Box, \Box_1, \Box_2$ that contains (i) the* **S4** *axioms for $\Box_1, \Box_2$ and $\Box$, (ii) $\Box p \to \Box_1 p$ and $\Box p \to \Box_2 p$.*

Thus, we have found an even stronger notion of common knowledge that might be said to model Barwise's third stage. Nevertheless, there are some difficulties with this identification. For instance, unlike the preceding two operations of intersection and union closure, true product topology has no general definition on arbitrary models for our language, as it exploits the product structure essentially. This makes it rather specialized, and this same fact is also reflected in the poverty of the complete logic given above. Nevertheless, there are also interesting logical aspects to this situation. In contrast with the *sequential quantification* embodied in the greatest fixed-point reading of common knowledge, the true product modality reads more like a *branching quantifier* as defined in [11]. We do not know what to make epistemically of this tantalizing analogy at this stage.

4.2.6 Operations that are safe for topo-bisimulation

To illustrate the preceding notions of knowledge and agency a bit further, we add a brief digression on simulations between topological models.

In relational semantics for modal languages, most natural operations $f(R_1, R_2)$ have the property of being *safe for bisimulation*, that is,

- any given bisimulation between two models w.r.t. the relations R_1, R_2 is also a bisimulation for the relation $f(R_1, R_2)$.

This says that the new operation stays at the same level of model structure as the old. The regular operations of composition, union, and iteration on binary relations are all safe in this sense, while a typical non-safe operation is *intersection*. Safety is a natural extension of invariance for static formulas to dynamic transition relations ([15] has a complete characterization of all first-order definable safe operations). Safety constrains the repertoire of definable transition relations within one given model. In general process theories, new relations can also be constructed out of old while forming a new model at the same time, as happens with products for concurrent processes in Process Algebra. In that setting, safety for operations generalizes to *respect for bisimulation*, e.g., if we let $\cong$ signify bisimulation:

- if $M \cong M'$ and $N \cong N'$, then $f(M, N) \cong f(M', N')$.

Most natural product operations show respect for bisimulation. As a check on our new notions, we can also look at operations on topologies in the same way, substituting the above topological bisimulations for the usual relational ones.

Of the repertoire of regular operations, only a small part matters in our perspective. When working only with reflexive transitive relations, composition and union by themselves do not qualify as operations, and we need to take *-closures. And for reflexive-transitive R_1, R_2, $(R_1 \cup R_2)^*$ and $(R_1; R_2)^*$ yield even the same relation. The topological counterpart for the latter operation was *intersection of topologies* $\tau_1 \cap \tau_2$, as noted above. Fact 2.23 expressed the observation that the modality for this is the same as the common knowledge fixed-point modality for the modal operators $[\tau_1]$, $[\tau_2]$. The latter is invariant for topological bisimulations by earlier observations. Indeed we have the following

Fact 4.2.14 *Intersection of topologies is safe for topological bisimulation.*

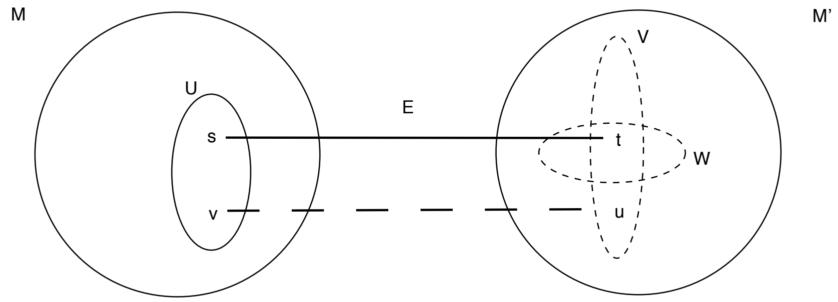

Figure 4.3:

Proof Let E be a relation between topological models M, N which is a topological bisimulation for their two separate topologies, as in Figure 4.3.

For a start, let sEt, and $s \in U$ with U in $\tau_1 \cap \tau_2$. Since E is a bisimulation w.r.t. τ_1, there is a τ_1-open set V in M' such that every point $v \in V$ is E-related to some point u in U. Likewise, there is an τ_2-open set W in M' such that every point $v \in W$ is E-related to some point u in U. Now, it may be tempting to take the intersection of V and W at t for the required matching neighbourhood of U, but this need not be open in either topology. Instead, we consider every E-link between points u in U and points v in the union $V \cup W$. Using the bisimulation properties again, there are again both τ_1 and τ_2-open neighbourhoods for all such points u, which satisfy the backward zigzag condition toward U. Continuing this procedure countably many times, the union of all these successively produced subsets of M' is both τ_1- and τ_2 open, and moreover, it still satisfies the correct backward zigzag condition w.r.t. the original open neighbourhood U of s in M. The argument in the opposite direction is similar. $\square$

This result may sound strange because intersection of binary relations led to non-invariance for bisimulation. But the topological counterpart of this operation was the sum topology $\tau_1 + \tau_2$ defined above, and its behaviour is indeed unsafe.

Fact 4.2.15 *Taking the sum of topologies is not safe for topological bisimulation.*

The counterexample is the same as for the relational case. Consider the two three-point models of Figure 5, with their topologies plus a binary relation E between their

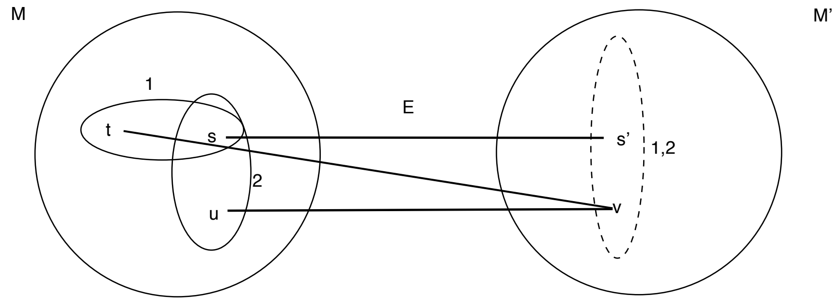

Figure 4.4:

points as indicated.

Note that the sum topology on the left-hand side has the singleton set $\{s\}$ as an open, whereas the sum topology on the right has only the whole two-element space for a non-empty open. Also, the relation E is a bisimulation for both topologies τ_1 and τ_2. Next, consider the link sEt, with the open subset $\{s\}$ on the left. The only matching open set on the right can be $\{s', v\}$, but this fails to satisfy the backward zigzag condition, as sEv does not hold. □

Finally, more general operations may produce new topologies over combined spaces. Our characteristic example was topological product.

Fact 4.2.16 *Topological products $\tau_1 \times \tau_2$ respect topological bisimulation.*

Proof Let E_1 be a bisimulation w.r.t. τ_1 between models M, M', and likewise E_2 a bisimulation w.r.t. τ_2 between models N, N'. Now define a bisimulation E between $M \times N$, $M' \times N'$ by setting:

$$(s,t)E(s',t') \text{ iff } sE_1s' \text{ and } tE_2t'.$$

Given Definition 2.3.3, it is completely straightforward to check that E is a bisimulation w.r.t both topologies on the product. □

In contrast to this, taking a product of two topological spaces with the true product topology τ introduced a little while ago does not respect topological bisimulation.

The reason is the earlier fact that the true product modality $\Box$ is not invariant for topological bisimulations w.r.t. the two component topologies.

4.2.7 Merging information revisited

Finally, we make a few comments on the issue of merging epistemic situations. We have shown that products of topological spaces are a natural setting for combining knowledge by different agents, and for distinguishing various forms of knowledge in the group of all agents. But as in Section 4.1, there is a broader question behind this. Our topological products are just one way of merging information models. The general subject of merging epistemic models goes far beyond the scope of this paper (cf. [?] for more on this topic). We only make one general point here which seems relevant to our move from relational semantics to topological models.

In general, we need to *specify* what we want to happen with existing knowledge and ignorance of agents when merging their information. Suppose we are given two epistemic models M for group G_1 and N for G_2, where G_1, G_2 overlap. In that case, we may want to require that the intersection group does not learn anything new in the 'merge model' $M * N$, at least w.r.t. formulas in its old language. This situation is reminiscent of the process of *amalgamation* of relational models in semantic proofs of the interpolation theorem for the basic modal language (cf. [?] for an elementary exposition). Such proofs often start with a $G_1 \cap G_2$ *bisimulation* between models M, s and N, t, which serves as an initial connection between the two different settings. The relevant merge $M * N$ then turns out to be a *submodel of the full product* $M \times N$, viz. just those pairs which stand in that bisimulation. One then shows that the projections from pairs to the original models M, N are bisimulations for the separate languages. Hence, formulas in the intersection of the two languages retain one unambiguous truth value: the one they had before under the bisimulation. In the case of interpolation theorems for shared modalities, this amalgamation construction has to be complicated, but the point remains the same. General merging of models for groups of agents may presuppose some initial connection, and its effects on modal formulas can be prescribed to some extent. In particular, we need not accept all pairs in a product as members of a merge model. Once we do this, the connection between topological models and relational models becomes more complicated, as we could also try to get the results of this paper with sub-product constructions on relational models.

4.3 Conclusion

Topological semantics for epistemic logic is a natural extension of the usual relational modelling. It provides distinctions that can be used to differentiate between various notions of common knowledge, and define various sorts of collective agents. Also, using product spaces, topological semantics suggests 'low-interaction' merges for epistemic models for separate groups of agents. Thus, we believe that there are good reasons for further development of this currently still marginal perspective.

Chapter 5

Conclusions and Further Directions

Exploring spatial structure by means of modal languages and their logics is still in an experimental stage. We know that there are some elegant and tractable modal fragments of full topology or geometry, which motivate us to look for more. But once we venture into new languages beyond the safe territory explored by Tarski and other pioneers, things can get hard. A typical example is the surprising difficulty in obtaining completeness when adding natural further structure to the topological interior modality. For instance, despite many interesting results of [48], the complete dynamic topological logic of the reals with an added continuous function has yet to be axiomatized. And the same is true for the ongoing efforts by Shehtman [64] and the recent work of G. and N. Bezhanishvili, and C. Koepke from [21] mentioned earlier of axiomatizing the complete topological order logic of the reals. Part of this may be inevitable combinatorial complexity, part also the issue of choosing a most appropriate modal language–not too weak, and not too strong–describing the relevant spatial structures in a most 'transparent' format. Against this background, we see the main results of this thesis as follows.

In Chapter 2, we have introduced the notion of topological product models as a natural way of obtaining further spatial structure. We have shown how to axiomatize these, bringing out the true features of combination in a topological sense–while avoiding the technical peculiarities of products of relational Kripke frames. Of course, much further structure could be studied on product models, but our results on true product topology give some hope that more can be done along the lines of this chapter. Next, in Chapter 3, we have provided a new take on combining topology and ordering, avoiding earlier complexity by mixing the two in the definition of a new directional neighborhood modality. Again, we found that this leads to complete logics, and interesting follow-up issues, also in product spaces. Finally, Chapter 4 provides additional evidence that all this is not aimless generalization. We show how our product models make sense in epistemic logic, with information of agents now viewed as topological structure, in line with intuitionistic semantics [72] and Scott's information systems [61]. Here, too, we feel that many new patterns and complete logics, can be found, either on geometrical analogies or equivalently, on an epistemic interpretation of the models.

Nevertheless, what we have so far is obviously just a sequence of first steps. In the remainder of this chapter, we briefly discuss a few directions where we foresee further results, partly based on results that we have already obtained - but which in their present state did not fit yet into the general line of this thesis.

5.1 Products in richer modal languages

The most obvious first-order extension of modal languages these days uses hybrid languages (cf. [5] [27]). In particular, hybrid modal languages have been introduced into topology in [39], and [67]. While we have not looked yet at what these expressive extensions would do for our product languages, we have undertaken one preliminary study. The following result is from [28]. Consider the product of relational frames defined in Chapter 4 in the style of [37]. We show how enriching the language with quite modest modal operations will allow us to capture the ordered pair construction in a direct and perspicuous manner. For this purpose we use a *hybrid language*, adding nominals naming specific worlds. The following sequence of results shows that this is a fruitful perspective. In particular, we get a completeness transfer result.

Definition 5.1.1 Products of frames. *For all frames* $\mathfrak{F} = (W, R)$ *and* $\mathfrak{G} = (W', R')$, *let* $\mathfrak{F} \times \mathfrak{G} = (W \times W', R_H, R_V)$, *where* $(x, y)R_H(x', y')$ *iff* xRx' *and* $y = y'$; *and* $(x, y)R_V(x', y')$ *iff* $x = x'$ *and* $yR'y'$.

Definition 5.1.2 Products of hybrid logics. *For any hybrid logics L_1, L_2, we define $PROD(L_1, L_2)$ to be the smallest hybrid logic that contains the fusion of the L_1 and L_2 and the following axioms, and that is closed under the usual inference rules for hybrid logic (including the* Name *and* Paste *rules, c.f. [27]).*

1. $com : \Diamond_H \Diamond_V i \leftrightarrow \Diamond_V \Diamond_H i$
2. $chr : \Diamond_H \Box_V p \rightarrow \Box_V \Diamond_H p$
3. $uc : \Diamond_H(\Diamond_V i \wedge \Diamond_V j) \rightarrow \Box_V(\Diamond_H i \rightarrow \Box_H(j \rightarrow i))$
4. $disjV : \Diamond_V(i \wedge \Diamond_H j) \rightarrow \Box_V(\Diamond_H j \rightarrow i)$
5. $disjH : \Diamond_H(i \wedge \Diamond_V j) \rightarrow \Box_H(\Diamond_V j \rightarrow i)$

Proposition 5.1.3 *Let L_1 and L_2 be hybrid logics preserved under disjoint union. For all $\mathfrak{F}, \mathfrak{G}$, if $\mathfrak{F} \models L_1$ and $\mathfrak{G} \models L_2$ then $\mathfrak{F} \times \mathfrak{G} \models PROD(L_1, L_2)$.*

Proof Suppose $\mathfrak{F} = (W, R) \models L_1$ and $\mathfrak{G} = (W', R') \models L_2$. By closure under disjoint union, it follows that $(W \times W', \{((x, y), (x', y')) \mid xRx' \text{ \& } y = y'\}) \models L_1$ and $(W \times W', \{((x, y), (x', y')) \mid x = x' \text{ \& } yR'y'\}) \models L_2$. From this, it follows that $\mathfrak{F} \times \mathfrak{G}$ satisfies the fusion of L_1 and L_2. That the other five axioms are satisfied can be checked by immediate inspection. □

Proposition 5.1.4 *Let $\mathfrak{F}$ be a point-generated 2-frame, and let L_1, L_2 be hybrid logics extending K4. If $\mathfrak{F} \models PROD(L_1, L_2)$, then $\mathfrak{F}$ is isomorphic to the product of two (point-generated) frames satisfying L_1 and L_2 respectively.*

Proof Let $\mathfrak{F} = (W, R_H, R_V)$ be generated by w, and let $\mathfrak{F}^H_w$ and $\mathfrak{F}^V_w$ be the subframes of (W, R_H) and (W, R_V) respectively, generated by w. Notice that since L_1 and L_2 extend K4, both R_H and R_V are transitive. By *chr* and *uc*, it follows that for all $x \in \mathfrak{F}^H_w$ and $y \in \mathfrak{F}^V_w$, there is a unique $z \in \mathfrak{F}$ such that xR_Hz and yR_Hz. Let $f : \mathfrak{F}^H_w \times \mathfrak{F}^V_w \rightarrow \mathfrak{F}$ be the function that maps every pair (x, y) to this unique convergence point z. We will show that f is an isomorphism.

- f is surjective. Consider any point $z \in \mathfrak{F}$. By point-generatedness, z is reachable from w in a number of steps by the relation $R_H \cup R_V$. By repeated application of commutativity and transitivity, it follows that there are points x and y such that wR_Hx, xR_Vz, wR_Vy and yR_Hz. Notice that $x \in \mathfrak{F}_w^H$ and $y \in \mathfrak{F}_w^V$. By the definition of f, $f(x,y) = z$ and therefore, z is in the range of f.

- f is injective. Suppose $f(x,y) = f(x',y')$. then by definition of f, we have the following situation:

  By $DisjH$ and $DisjV$ it follows immediately that $x = x'$ and $y = y'$.

- $f(x,y)R_Hf(x',y')$ iff xR_Hx' and $y = y'$. First, suppose $f(x,y)R_Hf(x',y')$. Then by the definition of f, we have the following situation: By *comm*, there must be a point x'' such that xR_Hx'' and $x''R_Vf(x',y')$. By *uc*, it follows that $x'' = x'$ and therefore, xR_Hx'. That $y = y'$ follows directly from the $disjV$ axiom.

  Next, suppose xR_Hx' and $y = y'$. Then we have the following situation: By *chr*, there must be a z such that $x'R_Vz$ and $f(x,y)R_Hz$. By *uc*, it follows that $z = f(x',y')$, and therefore $f(x,y)R_Hf(x',y')$.

- $f(x,y)R_Vf(x',y')$ iff $x = x'$ and yR_Vy'. Analogous to the previous claim.

Finally, since $\mathfrak{F} \models PROD(L_1, L_2)$, it follows that $(W, R_H) \models L_1$, and therefore by closure under generated subframes, $\mathfrak{F}_w^H \models L_1$. Similarly, $\mathfrak{F}_w^V \models L_2$. □

Corollary 5.1.5 *Let L_1 and L_2 be hybrid logics preserved under disjoint unions of frames. A point generated 2-frame satisfies $PROD(L_1, L_2)$ iff it is isomorphic to a product of (point-generated) frames satisfying L_1 and L_2 respectively.*

Corollary 5.1.6 *If $PROD(L_1, L_2)$ is frame complete, then it is complete for the products of L_1- and L_2-frames.*

For further details on this and related results, we refer the reader to the forthcoming [28], which is the more extensive study of product frames in hybrid logic.

5.2 First-order extensions

In the limit, the obvious language to consider when studying frame products is first-order logic. Indeed, there are quite a few results from classical model theory that apply here. E.g., we know that the first-order formulas preserved under the formation of products are the generalized Horn sentences [29]. Such results can interact with modal model theory. E.g., [53] contains an extensive study of interpolation theorems for modal logics whose frame classes are closed under direct products. The corresponding algebras have amalgamation properties which underlie interpolation properties by a Robinson joint-consistency argument. (C.f. also the discussion of merging epistemic models at the end of Chapter 4.) Now, indeed, the Gabbay-Shehtman construction does not form direct products in the classical sense. It rather forms 'interleaved products' keeping their operations separate as follows:

$(x, y)R_1(z, u)$ iff xR_1z and $y = u$,
$(x, y)R_2(z, u)$ iff $x = z$ and yR_2u.

Such products are well-known from the theory of concurrency, and logical studies of 'Process Algebra' ([18], [45]). In this case, it is not preservation that we want, but rather some systematic information on how truth of first-order statement about R_1 in $\mathcal{F}$ and about R_2 in $\mathcal{G}$ influence truth of statement sin the combined language in the product model $\mathcal{F} \times \mathcal{G}$. We give one sample result, concerning the information models of [12], where again this type of product construction occurs. In the latter setting, the component models also have proposition letters p, q, which get lifted to componentwise unary predicates in the product:

$(x, y) \models P_1$ iff $x \models p$
$(x, y) \models P_2$ iff $x \models p$,

Now it is easy to prove the following decomposition result.

Theorem 5.2.1 *There is an algorithm taking first-order formulas ϕ in the language of R_1, R_2, P_1, P_2 into Boolean combinations BC of first-order formulas in the separate* *first-order languages of R_1, P and R_2, P such that for any product $\mathcal{F} \times \mathcal{G}$, $\mathcal{F} \times \mathcal{G} \models \phi$ iff BC holds when its component formulas are interpreted in 'their' models $\mathcal{F}, \mathcal{G}$.*

The proof of the theorem replaces quantifier $\exists x$ in the language of the product by pairs of quantifiers $\exists x_1 \exists x_2$, and in atoms of the form, say, $R_1 xy$ it rewrites to Rx_1y_1 and $x_2 = y_2$. The resulting first-order formula can be rewritten to a boolean combination of the required kind.

This sort of result suggests that we could also aim for 'decomposition results' on our product models, provided that we do not interpret proposition letters as arbitrary sets of ordered pairs, but only as 'blocks' of the forms $Y \times \mathcal{G}$, $\mathcal{F} \times Y$.

5.3 Constraints on product operations

We have not studied our product operations as a mathematical construction *per se*. In the literature on product models, however, there are some general constraints that such constructions should satisfy. Of greatest relevance to us seems the following. In Process Algebra, operations O forming new transition systems should 'respect bisimulation' (cf.[19], [57], [15]), that is,

If M is bisimilar to N, and M' is bisimilar to N', ..., then $O(M, N, ...)$ is bisimilar to $O(M', N', ...)$.

This stipulation constrains possible operations on models, though not very deeply. Van Benthem in [15] studied some constraints on defining formats for model operations satisfying a sort of 'constructive' version of *respect*, viz. that the value of bisimulation be definable in some sense out of the given ones for the arguments. [45] is an extensive study of classification results for first-order definable operations that respect bisimulation, even though the notion also makes sense for arbitrary logical languages defining products. In our setting of Chapter 2, it is easy to prove an analogue of respect by routine inspection of the clauses for topo-bisimulation:

Proposition 5.3.1 *If M is topo-bisimilar to M', and N is topo-bisimilar to N', then $M \times N$ is topo-bisimilar to $M' \times N'$.*

But there is also another way of formulating constraints here, viz. in terms of equivalence of theories in the given language. For example, in [36], we have that,

If $\mathcal{F}$ is elementarily equivalent to $\mathcal{F}'$, $\mathcal{G}$ is elementarily equivalent to $\mathcal{G}'$, then the direct products $\mathcal{F} \times \mathcal{G}$ and $\mathcal{F}' \times \mathcal{G}'$ are elementarily equivalent.

Such results are not always obvious. For instance, it is still an open question whether elementary equivalence of argument models implies elementary equivalence for the binary linear product of models in the sense of Chu-spaces [60]. More generally, connections between the Feferman-Vaught analysis of products and our topological setting would be an interesting subject to explore.

Likewise, we have not been able to determine whether our topological product construction $M \times N$ of Chapter 2 preserves modal equivalence of its arguments: i.e, having the same modal theory in the distinguished world. We conjecture that the answer is positive, since the definition of the product seems so simple–but a direct reduction as in the above result for Barwise-Seligman products seems much harder, because of the non-componentwise treatment of proposition letters in our products.

5.4 Decidability and complexity

All our earlier topics in this thesis have complexity aspects. For instance, in Chapter 2, we were able to show that the complexity of modal product logic does not go up from the component logics, since both **S4** and **S4** $\oplus$ **S4** are PSPACE-complete. More generally, however, we know that forming product model can be dangerous, as products form a natural grid pattern that can be used in principle to encode undecidable Tiling Problems (for basic introduction to tiling, see for instance [25]). Indeed, there exist very simple examples of undecidable modal logics that are obtained in this way: **S4** $\times$ **S4** is a relevant example (cf. [40]). Likewise, it is known that Venema's compass logics in a two-dimensional plane are undecidable [55].

Nevertheless, as we have seen some of our languages, especially the mixed topological order modalities of Chapter 3, tread a fine line between expressive power and undecidability. In particular, we have the following result:

Proposition 5.4.1 **TC**$\mathbb{Q}$, *the logic of the topo-compass language over* $\mathbb{Q} \times \mathbb{Q}$ *is decidable. Further, its complexity is between PSPACE and EXPTIME.*

Conjecture 5.4.2 *The complexity of* **TC** *is between PSPACE and EXPTIME.*

The proof of this claim is in currently in the works, but if the conjecture holds, topo-compass logic would retain some of the advantages of Venema's approach while avoiding the undecidability.

Nevertheless, decidability and undecidability of modal product logics is a very tricky area (cf. [38], [40], [54]). Here is a result from the earlier-mentioned working paper [28] showing this for the case of, again, products of relational models. We start with a known decidability result.

Theorem 5.4.3 *The fusion of the minimal temporal hybrid logic* $\mathbf{K_t}$ *with itself* $\mathbf{K_t} \oplus \mathbf{K_t}$ *is decidable.*

However, grid encoding and tiling problems strike in the following, apparently not much stronger combination:

Theorem 5.4.4 *The product logic of* $K_t \otimes K_t$ *is undecidable.*

Let K_t be the basic temporal hybrid logic. That is, the basic hybrid logic in the language with a countable set NOM of nominals $i, j, ...$, a countable set PROP of propositional variables, $p, q, ...$ The formulae are defined recursively:

$$\phi := p \,|\, i \,|\, \neg\phi \,|\, \phi \wedge \psi \,|\, @_i\phi \,|\, F\phi \,|\, Pp \,|\, Gp \,|\, Hp$$

where $i \in NOM$, $p \in PROP$, F and P are existential modalities and G and H are the corresponding universal modalities. Let $K_t \otimes K_t$ be the hybrid product of this logic with itself. That is,

$$K_t \otimes K_t = Log\{\mathcal{F}_1 \otimes \mathcal{F}_2 \,|\, \mathcal{F}_1, \mathcal{F}_2 \in Fr(K_t)\}$$

where the product $\mathcal{F}_1 \otimes \mathcal{F}_2$ of Kripke frames $\mathcal{F}_1 = \langle W_1, R_1 \rangle$ and $\mathcal{F}_2 = \langle W_2, R_2 \rangle$ is the frame $\langle W_1 \times W_2, R_h, R_v \rangle$ in which for all $x, x' \in W_1, y, y' \in W_2$,

$$\langle x, y \rangle R_h \langle x', y' \rangle \Leftrightarrow x R_1 x' \,\&\, y = y'$$

and

$$\langle x, y \rangle R_v \langle x', y' \rangle \Leftrightarrow x = x' \,\&\, y R_2 y'.$$

A model $\mathcal{M}$ over such a frame $\mathcal{F}_1 \otimes \mathcal{F}_2$ is an ordered pair $(\mathcal{F}_1 \otimes \mathcal{F}_2, V), V : PROP \cup NOM \rightarrow \wp(W_1 \times W_2)$ where for any propositional variable $p \in PROP$, $V(p) \subseteq W_1 \times W_2$ and for any nominal i, $V(i)$ is a singleton subset of $W_1 \times W_2$.

To prove theorem 5.4.4, we construct a temporal formula ψ from a given modal formula ϕ such that ψ is satisfiable in a temporal product model iff ϕ is globally satisfiable in a modal product model.

Obtaining ψ from ϕ.

First we translate ϕ which is in the standard modal language of $\Diamond_v$ and $\Diamond_h$ into the temporal product language F_v, F_h and P_v, P_h by simply translating $\Diamond_v$ to F_v and $\Diamond_h$ into F_h. We call the resulting formula $Tr(\phi)$.

ψ is then the conjunction of the following formulae (in the brackets, for the later reference, we include a semantic role the formulae will play in the construction):

(i) $i \wedge @iF_v\top \wedge @iF_h\top$ (Ensures that we are evaluating at i, both vertical frame and the horizontal frame have at least 2 points.)

(ii) $@_i\neg F_h i, @_i \neg F_v i$ (Ensures that the point i does not see itself either along R_h or along R_v.)

(iii) $@iG_vG_vP_vi$ (Says that every point vertically reachable is reachable in one vertical step.)

(iii') $@iG_hG_hP_hi$ (The horizontal version of (ii).)

(iv) $@iG_v\neg F_v i$ (No vertical successor of i has i as vertical successor.)

(iv') $@iG_h\neg F_h i$ (The horizontal version of (iii).)

(v) $@iG_vG_hTr(\phi)$ where $Tr(\phi)$ is the above translation of ϕ into the temporal language. (Simulates the universal modality U.)

Definition 5.4.5 *A formula ϕ is globally satisfiable in $K \times K$ model iff there is product frame $\mathcal{F} \times \mathcal{G}$ and a modal valuation V, that is, a valuation $V : PROP \rightarrow \wp(\mathcal{F} \times \mathcal{G})$, such that for all pairs $(x, y) \in \mathcal{F} \times \mathcal{G}$, $(\mathcal{F} \times \mathcal{G}, V), (x, y) \models \phi$.*

Proposition 5.4.6 *ϕ is globally satisfiable in a $K \times K$ model iff ψ is satisfiable in a hybrid $K_t \otimes K_t$ model.*

Proof $(\Longrightarrow)$ Let $\mathcal{F} \times \mathcal{G}$ with some *modal* valuation V be the $K \times K$ model globally satisfying ϕ. We build a rooted temporal model $\mathcal{F}' \otimes \mathcal{G}'$ that satisfies ψ at the root (x, y) as follows. Let $\mathcal{F} = (W_1, R_1)$, and $\mathcal{G} = (W_2, R_2)$. Then in $\mathcal{F}' = (W_1', R_1')$, $W_1' = W_1 \cup \{x\}$, and $R_1' = R_1 \cup \{(x, w) \,|\, w \in W_1\}$. Similarly, in $\mathcal{G}' = (W_2', R_2')$, $W_2' = W_2 \cup \{y\}$, and $R_2' = R_2 \cup \{(y, u) \,|\, u \in W_2\}$. We define a *hybrid* valuation V' over $\mathcal{F}' \otimes \mathcal{G}'$ from V by $(w, u) \in V'(p)$ if $w \in W_1$, $u \in W_2$ and $(w, u) \in V(p)$, for all $p \in PROP$. We let $V'(j) = \{(x, y)\}$ for all $j \in NOM$.

We then want to show the following.

Claim 5.4.7 *The identity is a standard modal bisimulation between $(\mathcal{F} \times \mathcal{G}, V)(w, u)$ and $(\mathcal{F}' \times \mathcal{G}', V')(w, u)$ for $w \neq x$, $u \neq y$, in other words, for any formula χ built from the booleans, F_v and F_h, $(\mathcal{F} \times \mathcal{G}, V)(w, u) \models \chi$ iff $(\mathcal{F}' \times \mathcal{G}', V')(w, u) \models \chi$.*

But $Tr(\phi)$ by construction only involves F_v and F_h, that is, no P_v or P_h. Since ϕ was true everywhere in $(\mathcal{F} \times \mathcal{G}, V)$, given the valuation V', for all $w \neq x, u \neq y, (\mathcal{F}' \times \mathcal{G}', V')(w, u) \models Tr(\phi)$.

Claim 5.4.8 *i-v' are all true at (x, y) in $(\mathcal{F}' \otimes \mathcal{G}', V')$.*

$(\Longleftarrow)$ Let $(F_1 \otimes F_2, V), (x, y) \models \psi$, with V a hybrid valuation, $\mathcal{F}_1 = (W_1, R_1)$ and $\mathcal{F}_2 = (W_2, R_2)$. We want to extract a modal product model $(F_1' \times F_2', V')$ such that

$$\text{for each pair } (w, v) \in F_1' \times F_2',\ (F_1' \times F_2', V'), (w, v) \models \phi.$$

We define $(F_1' \times F_2', V')$ as follows. $F_1' = (W_1', R_1')$ where $W_1' = W_1 - \{x\}$, $R_1' = R_1 \restriction W_1'$, that is, the restriction of R_1 to W_1'. Similarly, $F_2' = (W_2', R_2')$ where $W_2' = W_2 - \{y\}$, and $R_2' = R_2 \restriction W_2'$. For V' we stipulate that $V'(p) = (V(p) \cap W_1' \times W_2')$.

We want to show that

Fact 5.4.9 *(1) $F_1 \times F_2$ is well defined. This amounts to showing that $W_1' \times W_2'$ is nonempty.*

(2) The identity is a standard modal bisimulation between $(F_1' \times F_2', V')(w, v)$ and $(F_1 \otimes F_2, V)(w, v)$.

Proof For (1), by (i) both W_1 and W_2 have at least two points, and thus $W_1 - \{x\} \times W_2 - \{y\}$ contains at least one pair. To prove (2) we need to show that:

(a) $(w, v) \in V(p)$ iff $(w, v) \in V'(p)$,

(b) for $(w, v) \in W_1' \times W_2'$, $(w, v)R_1(w', v)$ iff $(w, v)R_1'(w', v)$,

(c) for $(w, v) \in W_1' \times W_2'$, $(w, v)R_2(w, v')$ iff $(w, v)R_2'(w, v')$.

(a) is immediate from the definition of V'.

(b) Since $R_1' \subset R_1$, the 'if' direction is immediate. To show that $(w, v)R_1(w', v)$ only if $(w, v)R_1'(w', v)$, it is sufficient to show that if $w \neq x$ and $(w, v)R_1(w', v)$, then $w' \neq x$. But this follows by (iv), since by (iii) we know that for all u, $(x, y)R_1(u, y)$. Thus if for some u, $(u, y)R_1(x, y)$ we would have $(F_1 \otimes F_2, V), (x, y) \models @_i F_v F_v i$, and thus the negation of (iv) would hold at (x, y) as well. Since we are working in a product frame, it follows that for no $u \in W_1'$ and no $w' \in W_2'$, $(u, w')R_1'(x, w')$.

(c) We prove (c) symmetrically using (iii') and (iv'). □

Since the identity is a bisimulation, for any modal formula χ and its temporal translation $Tr(\chi)$,

$$(F_1' \times F_2', V'), (w, v) \models \chi \text{ iff } (F_1 \otimes F_2, V), (w, v) \models \chi.$$

In particular, if we can show that for all pairs $(w, v) \in W_1' \times W_2'$, $(F_1 \otimes F_2, V), (w, v) \models Tr(\phi)$, we would have shown that ϕ is globally satisfied in $(F_1' \times F_2', V')$ as desired.

Fact 5.4.10 *ϕ is globally satisfied in $(F_1' \times F_2', V')$.*

Proof Let (w, v) be an arbitrary pair in $W_1' \times W_2'$. It is sufficient to show that $(x, y)R_2(x, v)R_1(w, v)$. Then by (v), $Tr(\phi)$ is true at (w, v). To see that $(x, y)R_2(x, v)R_1(w, v)$ observe that by (iii) and (iii') respectively, xR_1w and yR_2v.

□

This completes the reduction of global satisfiability in $K \times K$ to satisfiability in $K_t \otimes K_t$ and thus the proof of undecidability of $K_t \otimes K_t$.

Again, the general issue would be whether this sort of undecidability argument also crops up in topological product models, where the grid directions are not directly accessed by relational modalities.

5.5 Conclusion once more

In this thesis, we have given new languages for spatial structures, with an emphasis on modal description of product models introducing more than one spatial 'dimension'. We have developed some basic model theory, and completeness theory. But, as our discussion in this final chapter clearly shows: the area is open for many further rounds of questions, and hence new research!

Appendix A

Plug-And-Play Unravelling and other issues

In this appendix, we clear some of the remaining issues from Chapter 2. We start with an unravelling procedure that enables us to prove completeness of transitive reflexive multi-modal logics with respect to certain kinds of trees.

We will first prove that,

Proposition A.0.1 $\mathbf{S4} \oplus \mathbf{S4}$ *is complete with respect to* $T_{2,2}$.

Proof

Let $\mathbb{F}$ be a finite $\mathbf{S4} \oplus \mathbf{S4}$ frame, and let $w_0, w_1, ...$ be an enumeration of its nodes starting with the root. We show how to label $T_{2,2}$ with the nodes of $\mathbb{F}$ in such a way that the labelling is a bounded morphism from $T_{2,2}$ onto $\mathbb{F}$.

But the unravelling procedure is more general and we will describe it in most general terms.

Let $\mathbb{G}$ be a finite or countable frame on the set W' and R_i for $i \in \{1, ...n\}$ be finitely many transitive reflexive relations on that frame, and let $l : W' \to \mathbb{N}$ be an enumeration of W'.

Definition A.0.2 *A finite* R'_i-plug *for a node* x *is the binary branching tree described as follows: The root of the tree is the node* x*. The root is then* R'_i *related to two distinct points, on the left to a copy of* x*, and on the right to* y *such that* $l(y)$

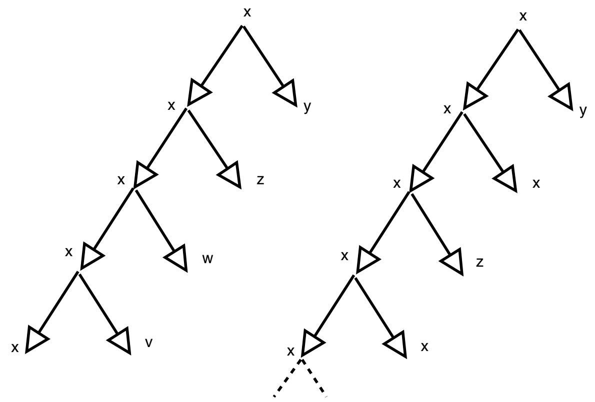

Figure A.1: A finite and a countable plug.

is the the least number such that R_ixy, i.e., the first point in the enumeration of x's successors. The right successor is then a leaf in the tree and the left successor is further related to two points, the left a copy of x and the right the second successor of x on the enumeration of W', z. And so on: the right is a leaf, and the left has two successors, x itself and another successor of x until all successors are exhausted. See Figure A.1. The finite plug is pictured on the left. Obviously, if x has finitely many successors, then the R_i' plug for x is finite.

*A countable R_i'-*plug *for a node x is needed when the logic in question lacks finite model property. A slight modification of the definition of the finite plug is required to ensure that every copy of x indeed sees all of the successors of x from the original frame $\mathcal{F}$. Let x be a node and $l(y) < l(z) < l(w), \ldots$. Then as before, we start with the root x. The root is again related to two distinct points, a copy of x on the left and a copy of y, the first successor of x in the enumeration, on the right. The right successor is then a leaf in the tree and the left successor is further related to two points, but this time both points are copies of x. In the next round the left is related to two points, the left one a copy of x and the right one a copy of z, the second successor in the enumeration. The right successor is always a leaf, and the left is related to two copies of x. And so on. Once again, Figure A.1 is telling.*

In the case of $\mathbf{S4} \oplus \mathbf{S4}$, $\mathbb{F}$ can always be made finite so both R_1 and R_2 plugs for any given point will be finite. Note that since every point in a reflexive frame has a successor, plugs will always consist of at least the root and two successors. Note also that every point in a plug is either a leaf or it has exactly two successors.

We define,

Definition A.0.3 Plug-and-play tree *for* $\mathbb{G}$ *as the infinite* $2n$*-ary branching tree constructed as follows. The tree starts with* R'_1 *plug for the root of* $\mathbb{G}$*. Then in the next round every* $R'_1, R'_2..., R'_n$ *leaf of the finite tree is replaced by its respective* $R'_1, R'_2, ..., R'_n$ *plugs until the tree has no more leafs.*

Fact A.0.4 *In the plug and play tree for* $\mathbb{G}$*, every point has exactly two successors along each of the relations* R'_i*.*

Fact A.0.5 *For any point* y *in a* R'_i *plug for* x*,* y *is a leaf if* $y \neq x$*, and* y *sees itself as a leaf in a finitely many steps that go through other copies of* y *alone, if* $y = x$*.*

The transitive closure of plug-and-play tree for $\mathbb{G}$ is the $R'_1, R'_2, ...R'_n$ transitive (and reflexive if needed) closure of that tree. We will call the resulting infinite tree, $T_{n\times 2}$.

Fact A.0.6 *By transitivity and reflexivity of* $R_i \in \mathcal{G}$*, every point* y *such that* $R'_i xy$ *in* $T_{n\times 2}$ *is a copy of a point* z *such that* $R_i xz$ *in* $\mathcal{G}$*.*

Fact A.0.7 *The map* $f : T_{n\times 2} \to W'$ *that sends every copy of* x *in* $T_{n\times 2}$ *to* x *is a p-morphism with respect to each of* $R_1, R_2, ...R_n$*.*

Proof We show (i) if $R'_i x'y'$, then $R_i f(x')f(y')$, and (ii) if $R_i xy$, then for each x' such that $f(x') = x$ there is a y' such that $R'_i x'y'$.

(i) Let $R'_i x'y'$. Then $R_i f(x)f(y)$ by Fact A.0.6.

(ii) Since by A.0.5, every point in a plug either is a leaf or it sees itself as a leaf and every leaf is replaced with a plug for that leaf that contains all of its successors, every point sees all of its original successors.

$\square$

Thus applying the plug and play method to $T_{2,2}$, for any frame $\mathcal{F}$ we can obtain an unravelling of $\mathcal{F}$ to $T_{2,2}$ with a p-morphism $f : T_{2,2} \to W$, that is, the singleton $T_{2,2}$ is complete for $\mathbf{S4} \oplus \mathbf{S4}$.

The following is ever so slightly more complex to demonstrate:

Theorem A.0.8 **TPL** *is complete with respect to* $T_{6,2,2}$.

Proof We will show how to unravel an arbitrary countable rooted **TPL** frame $\mathcal{F}$ to $T_{6,2,2}$ in such a way that there is a natural p-morphism from $T_{6,2,2}$ onto $\mathcal{F}$. For this we use plug-and-play technique above, except that we need to make a slight adjustment. The tree $T_{3\times 2}$ that we obtain via the plug and play from a **TPL**-frame $\mathcal{F}$ is not quite a **TPL** frame since it does not satisfy the requirement that $R_1, R_2 \subseteq R$. We ensure that the tree we get satisfy this property by simply stipulating that if R'_1xy then $R'xy$ and if R'_2xy then $R'xy$. Because this was true of the points in the original frame, it can be shown that the p-morphism f still has the required properties. $\Box$

Proposition A.0.9 1. *A formula φ constructed from the Booleans and the modal operator $\Box_1$ is valid in $\langle X \times Y, \tau_1, \tau_2\rangle$ iff φ is valid in $\langle X, \eta\rangle$.*

2. *A formula φ constructed from the Booleans and the modal operator $\Box_2$ is valid in $\langle X \times Y, \tau_1, \tau_2\rangle$ iff φ is valid in $\langle Y, \theta\rangle$.*

Proof (1) ($\Rightarrow$)Let there be a model $\mathcal{M}$ over X with a valuation ν such that for some x, $\mathcal{M}, x \not\models \phi$. Define a valuation ν' over $X \times Y$ as follows: $\langle x_1, x_2\rangle \in \nu'(p)$ iff $x_1 \in \nu(p)$. For a formula ψ built from propositional variables, $\wedge, \neg$, and $\Box_1$, it is shown by induction that $\mathcal{M}, x \models \psi$ iff $\langle X \times Y, \nu'\rangle, \langle x, x_2\rangle \models \psi$.

($\Leftarrow$) Let $\langle X \times Y, \nu'\rangle, \langle x, x_2\rangle \not\models \psi$ and let $y \in \nu(p)$ iff $\langle y, x_2\rangle \in \nu'(p)$. For relevant ψ, it is proved by induction that $\mathcal{M}, x \models \psi$ iff $\langle X \times Y, \nu'\rangle, \langle x, x_2\rangle \models \psi$.

The proof of (2) is symmetric. $\Box$

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