A Game for the Borel Functions

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A Game for the Borel Functions

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Contents

Chapter 1

Introduction

This thesis is divided into two parts. In the first part, we present a game-theoretic characterization of the Borel functions. We define a Wadge-style game, $G(f)$, and prove the following theorem:

1.0.1. THEOREM. *A function $f : {}^\omega\omega \to {}^\omega\omega$ is Borel $\Leftrightarrow$ Player II has a winning strategy in $G(f)$.*

In the second part of the thesis, we turn our attention to the analysis of low-level Borel functions, summarized by the following diagram:

$$
\begin{tikzcd}
\mathbf{\Lambda}_{1,3} & & \\
& \mathbf{\Lambda}_{2,3} \arrow[ul, hook] & \\
\mathbf{\Lambda}_{1,2} \arrow[ur, hook] & & \mathbf{\Lambda}_{3,3} \arrow[ul, hook] \\
& \mathbf{\Lambda}_{2,2} \arrow[ul, hook] \arrow[ur, hook] & \\
\mathbf{\Lambda}_{1,1} \arrow[ur, hook] & &
\end{tikzcd}
$$

The notation $\mathbf{\Lambda}_{m,n}$ denotes the class of functions $f : A \to {}^\omega\omega$ such that $A \subseteq {}^\omega\omega$ and $f^{-1}[\,Y\,]$ is $\mathbf{\Sigma}^0_n$ in the relative topology of A for any $\mathbf{\Sigma}^0_m$ set Y. The two main results of the second part of the thesis are decomposition theorems for the $\mathbf{\Lambda}_{2,3}$ and $\mathbf{\Lambda}_{3,3}$ functions.

1.0.2. THEOREM. *A function $f : {}^\omega\omega \to {}^\omega\omega$ is $\mathbf{\Lambda}_{2,3}$ $\Leftrightarrow$ there is a $\mathbf{\Pi}^0_2$ partition $\langle A_n : n \in \omega\rangle$ of ${}^\omega\omega$ such that $f \upharpoonright A_n$ is Baire class 1.*

1.0.3. THEOREM. *A function $f : {}^\omega\omega \to {}^\omega\omega$ is $\mathbf{\Lambda}_{3,3}$ $\Leftrightarrow$ there is a $\mathbf{\Pi}^0_2$ partition $\langle A_n : n \in \omega\rangle$ of ${}^\omega\omega$ such that $f \upharpoonright A_n$ is continuous.*

These results extend the decomposition theorem of John E. Jayne and C. Ambrose Rogers for the $\boldsymbol{\Lambda}_{2,2}$ functions.

1.0.4. THEOREM (JAYNE, ROGERS). *A function $f : {}^\omega\omega \to {}^\omega\omega$ is $\boldsymbol{\Lambda}_{2,2} \Leftrightarrow$ there is a closed partition $\langle A_n : n \in \omega \rangle$ of ${}^\omega\omega$ such that $f \restriction A_n$ is continuous.*

It should be noted that Jayne and Rogers proved a more general version of Theorem 1.0.4 [6]. In this thesis, however, we only prove decomposition theorems for total functions on the Baire space.

The author was motivated by two questions of Alessandro Andretta:

(1) Is there a Wadge-style game for the (total) $\boldsymbol{\Lambda}_{3,3}$ functions?
(2) Is Theorem 1.0.3 true?

In the second part of the thesis, we answer both questions affirmatively. The result for the Borel functions was obtained accidentally, while the author was investigating questions (1) and (2).

A brief summary follows. In Chapter 2, we define the *tree game* and show that it characterizes the Borel functions. In Chapter 3, we begin our analysis of low-level Borel functions with the three simplest classes.

$$\begin{array}{ccc} \boldsymbol{\Lambda}_{1,2} & & \\ & \supseteq & \\ & & \boldsymbol{\Lambda}_{2,2} \\ & \subseteq & \\ \boldsymbol{\Lambda}_{1,1} & & \end{array}$$

In preparation for Chapters 4 and 5, we prove the Jayne-Rogers theorem and prove that the above containments are proper. In Chapter 4, we extend the analysis to the $\boldsymbol{\Lambda}_{1,3}$ and $\boldsymbol{\Lambda}_{2,3}$ functions.

$$\begin{array}{ccc} \boldsymbol{\Lambda}_{1,3} & & \\ & \supseteq & \\ & & \boldsymbol{\Lambda}_{2,3} \\ & \subseteq & \\ \boldsymbol{\Lambda}_{1,2} & & \\ & \supseteq & \\ & & \boldsymbol{\Lambda}_{2,2} \\ & \subseteq & \\ \boldsymbol{\Lambda}_{1,1} & & \end{array}$$

We prove the decomposition theorem for $\mathbf{\Lambda}_{2,3}$ and prove that the additional containments are proper. In Chapter 5, we complete the picture with an analysis of the $\mathbf{\Lambda}_{3,3}$ functions.

1.1 Background

Unless otherwise indicated, we use notation that is standard in descriptive set theory. For all undefined terms, we refer the reader to [8].

We use the symbol $\subseteq$ for containment and $\subset$ for proper containment. For sets A and B, we let ${}^{B}A$ denote the set of functions that map B to A. The notation ${}^{<B}A$ denotes

$$\bigcup_{b \in B} {}^{b}A$$

and we define ${}^{\leq B}A := {}^{<B}A \cup {}^{B}A$. In particular, ${}^{<\omega}\omega$ is the set of finite sequences of natural numbers and ${}^{\leq\omega}\omega$ is ${}^{<\omega}\omega \cup {}^{\omega}\omega$.

For a finite sequence $s \in {}^{<\omega}A$, we define $[\,s\,]_A := \{x \in {}^{\omega}A : s \subset x\}$. If the A is understood from the context, we may simply write $[\,s\,]$. We use the symbol $^\frown$ for concatenation of sequences. For $n \in \omega$, let s^n denote the sequence $s^\frown s^\frown \ldots ^\frown s$, with s appearing n times, and let s^* denote the infinite sequence $s^\frown s^\frown s^\frown \ldots$ in ${}^{\omega}A$. If s is a singleton sequence, $\langle a\rangle$, then when concatenating we may write a instead of $\langle a\rangle$ without danger of confusion. Thus, we may write a^n instead of $\langle a\rangle^n$, and the reader will realize that we mean concatenation of sequences and not exponentiation. The notation $\mathrm{lh}(s)$ is used for the length of s, so $\mathrm{lh}(s) := \mathrm{dom}(s)$. If s is non-empty, we define $\mathrm{pred}(s) := s \restriction \mathrm{lh}(s) - 1$ to be the immediate predecessor of s. The set of immedate successors of s is denoted by $\mathrm{succ}_A(s) := \{s^\frown a : a \in A\}$. If the A is understood from the context, we may write $\mathrm{succ}(s)$.

We say that a set $T \subseteq {}^{<\omega}A$ is a **tree** if $s \subset t \in T \Rightarrow s \in T$. For a set $T \subseteq {}^{<\omega}A$, we define $\mathrm{tree}(T) := \{s : \exists t \in T\,(s \subseteq t)\}$. For a tree $T \subseteq {}^{<\omega}A$ and $s \in {}^{<\omega}A$, we define $T[\,s\,] := \{t \in T : t \subseteq s \text{ or } s \subseteq t\}$. The notation $\mathrm{tn}(T)$ is used to denote the terminal nodes of T, so $\mathrm{tn}(T) := \{s \in T : t \supset s \Rightarrow t \notin T\}$. The notation $[\,T\,]$ is used to denote the set of infinite branches of T, so $[\,T\,] := \{x \in {}^{\omega}A : \forall n \in \omega\,(x \restriction n \in T)\}$. The tree T is **linear** if $s \subseteq t$ or $t \subseteq s$ for all $s, t \in T$. The tree T is **finitely branching** if $s \in T \Rightarrow \mathrm{succ}(s) \cap T$ is finite. A function $\phi : T \to {}^{<\omega}B$ is **monotone** if $s \subset t \in T \Rightarrow \phi(s) \subseteq \phi(t)$ and **length-preserving** if $\mathrm{lh}(\phi(s)) = \mathrm{lh}(s)$. A function $\phi : {}^{<\omega}A \to {}^{<\omega}B$ is **infinitary** if

$$\bigcup_{s \subset x} \phi(s)$$

is infinite for every $x \in {}^{\omega}A$.

There is a minor ambiguity regarding the $[\,]$ notation: if $\varnothing$ is considered to be a sequence in ${}^{<\omega}A$, then $[\varnothing] = {}^{\omega}A$. If, however, we view $\varnothing$ as a tree, then $[\varnothing] = \varnothing$. From the context, it will be clear which meaning is intended.

We work in the theory $\mathsf{ZF} + \mathsf{DC}(\mathbb{R})$: that is to say, ZF with dependent choice over the reals. In terms of topological spaces, we will be working exclusively with the Cantor space, the Baire space, and subspaces of the Baire space. If we are considering a subspace $A \subseteq {}^{\omega}\omega$, we will always use the relative topology as the topology of A.

For a metrizable space X, the Borel hierarchy $\mathbf{\Sigma}^0_\alpha(X)$, $\mathbf{\Pi}^0_\alpha(X)$, and $\mathbf{\Delta}^0_\alpha(X) := \mathbf{\Sigma}^0_1(X) \cap \mathbf{\Pi}^0_1(X)$ is defined as usual for $\alpha < \omega_1$. If the space X is understood, then we may write $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, and $\mathbf{\Delta}^0_\alpha$. Above the Borel sets lies the projective hierarchy $\mathbf{\Sigma}^1_n(X)$, $\mathbf{\Pi}^1_n(X)$, and $\mathbf{\Delta}^1_n(X) := \mathbf{\Sigma}^1_1(X) \cap \mathbf{\Pi}^1_1(X)$. In terms of the projective hierarchy, we will only need the classical fact that the Borel sets are equal to $\mathbf{\Delta}^1_1$ for Polish spaces. If X and Y are metrizable spaces, then $f : X \to Y$ is **continuous** if $f^{-1}[U]$ is open for every open set U of Y, and a function $f : X \to Y$ is **Baire class 1** if $f^{-1}[U]$ is $\mathbf{\Sigma}^0_2$ for every open set U of Y. Recursively, for $1 < \xi < \omega_1$, $f : X \to Y$ is **Baire class $\boldsymbol{\xi}$** if it is the pointwise limit of functions $f_n : X \to Y$, where each f_n is Baire class ξ_n with $\xi_n < \xi$. A function $f : X \to Y$ is **Borel** if $f^{-1}[U]$ is Borel for every open (equivalently, Borel) set of Y.

By the classical work of Lebesgue, Hausdorff, and Banach, if Y is also separable, then a function $f : X \to Y$ is Baire class ξ iff $f^{-1}[U]$ is $\mathbf{\Sigma}^0_{\xi+1}$ in X for every open set U of Y. So, in this case, the Borel functions are equal to the union of the Baire class ξ functions. If, in addition, X is separable and zero-dimensional, then f is Baire class 1 iff f is the pointwise limit of continuous functions. We will be working with functions $f : A \to {}^{\omega}\omega$ with $A \subseteq {}^{\omega}\omega$, so the above facts will hold.

We define $\mathbf{\Lambda}_{m,n}$ to be the set of functions $f : A \to {}^{\omega}\omega$ such that $A \subseteq {}^{\omega}\omega$ and $f^{-1}[Y]$ is $\mathbf{\Sigma}^0_n$ for any $\mathbf{\Sigma}^0_m$ set Y. Thus, for example,"$\mathbf{\Lambda}_{1,1}$" is the same as continuous, "$\mathbf{\Lambda}_{1,2}$" is the same as Baire class 1, and "$\mathbf{\Lambda}_{1,3}$" is the same as Baire class 2.

The $\subseteq$ containments for the $\mathbf{\Lambda}_{m,n}$ classes are trivial.

1.1.1. Proposition. *For $m, n \geq 1$, $\mathbf{\Lambda}_{m+1,n} \subseteq \mathbf{\Lambda}_{m,n}$ and $\mathbf{\Lambda}_{m,n} \subseteq \mathbf{\Lambda}_{m+1,n+1}$.*

1.1.2. Proposition. *For $m, n \geq 1$ and $k \geq 0$, $\mathbf{\Lambda}_{m,n} \subseteq \mathbf{\Lambda}_{m+k,n+k}$.*

1.1.3. Proposition. *Let $A \subseteq {}^{\omega}\omega$, $f : A \to {}^{\omega}\omega$, and $m, n \geq 1$. Then $f \in \mathbf{\Lambda}_{m,n} \Leftrightarrow f^{-1}[Y]$ is $\mathbf{\Pi}^0_n$ in the relative topology of A for any $Y \in \mathbf{\Pi}^0_m \Leftrightarrow f^{-1}[Y]$ is $\mathbf{\Delta}^0_n$ in the relative topology of A for any $Y \in \mathbf{\Delta}^0_m$.*

1.1.4. Lemma. *Let $n \geq m \geq 2$, $A \subseteq {}^{\omega}\omega$, $f : A \to {}^{\omega}\omega$, and suppose that there is a partition $\langle A_i : i \in \omega \rangle$ of A such that A_i is $\mathbf{\Pi}^0_{n-1}$ in the relative topology of A and $f \restriction A_i$ is $\mathbf{\Lambda}_{1,n-m+1}$. Then f is $\mathbf{\Lambda}_{m,n}$.*

Proof. Let $Y \in \mathbf{\Sigma}^0_m$ and $Y_j \in \mathbf{\Pi}^0_{m-1}$ such that $Y = \bigcup_j Y_j$. It follows that

$$\begin{aligned} f^{-1}[\,Y\,] &= \bigcup_i (f \restriction A_i)^{-1}[\,Y\,] \\ &= \bigcup_i \bigcup_j (f \restriction A_i)^{-1}[\,Y_j\,] \\ &= \bigcup_i \bigcup_j A \cap X_{i,j}, \text{ where } X_{i,j} \in \mathbf{\Pi}^0_{n-1} \\ &= A \cap X, \text{ where } X \in \mathbf{\Sigma}^0_n. \end{aligned}$$

For the second to last equality, note that $f \restriction A_i \in \mathbf{\Lambda}_{m-1,n-1}$ by Proposition 1.1.2 (take $k = m - 2$). ☐

1.1.5. Lemma. *Let $n \in \omega$ with $n > 0$. Let $A \subseteq {}^\omega\omega$, $h : A \to {}^\omega\omega$, and suppose that $A = B_0 \cup B_1$ such that B_0 and B_1 are $\mathbf{\Sigma}^0_{n+1}$ in A and $B_0 \cap B_1 = \varnothing$. If there is a $\mathbf{\Pi}^0_n$ partition $\langle B_{0,m} : m \in \omega\rangle$ of B_0 and a $\mathbf{\Pi}^0_n$ partition $\langle B_{1,m} : m \in \omega\rangle$ of B_1, then there is a $\mathbf{\Pi}^0_n$ partition $\langle A_m : m \in \omega\rangle$ of A that refines the partitions $B_{0,m}$ and $B_{1,m}$: for every $i \in \omega$, there is a $b < 2$ and a $j \in \omega$ such that $A_i \subseteq B_{b,j}$.*

Proof. We begin by noting that we cannot simply take the sets $B_{b,m}$ to be the partition, since $B_{b,m}$ is not necessarily $\mathbf{\Pi}^0_n$ in A. For $b < 2$ and $m \in \omega$, let $B'_{b,m}$ be $\mathbf{\Pi}^0_n$ in A such that $B_{b,m} = B'_{b,m} \cap B_b$. Let $C_{b,m}$ be $\mathbf{\Pi}^0_n$ in A and pairwise disjoint such that $B_b = \bigcup C_{b,m}$. Note that for any i and j, $C_{b,i} \cap B'_{b,j} = C_{b,i} \cap B_{b,j}$ is $\mathbf{\Pi}^0_n$ in A. The sets $C_{b,i} \cap B_{b,j}$ form the desired partition of A. ☐

We end this section with a brief note about Γ-completeness, following the discussion in [8] on page 169. Suppose Γ is a class of sets in Polish spaces. In other words, for any Polish space X, $\Gamma(X) \subseteq \mathcal{P}(X)$. If Y is a Polish space, then $A \subseteq Y$ is **Γ-complete** if $A \in \Gamma(Y)$ and $B \leq_{\mathrm{W}} A$ for any $B \in \Gamma(X)$, where X is a zero-dimensional Polish space. Note that if A is Γ-complete, $B \in \Gamma$, and $A \leq_{\mathrm{W}} B$, then B is Γ-complete.

1.1.6. Theorem (Wadge). *Let X be a zero-dimensional Polish space. Then $A \subseteq X$ is $\mathbf{\Sigma}^0_\xi$-complete iff $A \in \mathbf{\Sigma}^0_\xi \setminus \mathbf{\Pi}^0_\xi$.*

1.1.7. Fact. The set $\{x \in {}^\omega 2 : \exists i \,\forall j \geq i\,(x(j) = 0)\}$ is $\mathbf{\Sigma}^0_2$-complete.

Let $\ulcorner\cdot,\cdot\urcorner$ be the bijection $\omega \times \omega \to \omega$:

$$\begin{aligned} \ulcorner 0, 0\urcorner &:= 0, \\ \ulcorner 0, j+1\urcorner &:= \ulcorner j, 0\urcorner + 1, \\ \ulcorner i+1, j-1\urcorner &:= \ulcorner i, j\urcorner + 1. \end{aligned}$$

1.1.8. Fact. The set $\{x \in {}^\omega 2 : \exists i\ \exists^\infty j\,(x(\ulcorner i, j\urcorner) = 1)\}$ is $\mathbf{\Sigma}^0_3$-complete.

Chapter 2

A game for the Borel functions

In this chapter, we define the *tree game* and see that it characterizes the Borel functions.

Let $f : {}^{\omega}\omega \to {}^{\omega}\omega$. In the tree game $G(f)$, there are two players who alternate moves for ω rounds. Player I plays elements $x_i \in \omega$ and Player II plays functions $\phi_i : T_i \to {}^{<\omega}\omega$ such that $T_i \subset {}^{<\omega}\omega$ is a finite tree, ϕ_i is monotone and length-preserving, and $i < j \Rightarrow \phi_i \subseteq \phi_j$. After ω rounds, Player I produces $x := \langle x_0, x_1, \ldots \rangle \in {}^{\omega}\omega$ and Player II produces $\phi := \bigcup_i \phi_i$.

$$
\begin{array}{lccccccccc}
\text{I:} & x_0 & & x_1 & & x_2 & & & & x = \langle x_0, x_1, \ldots \rangle \\
 & & & & & & & \cdots & & \\
\text{II:} & & \phi_0 & & \phi_1 & & \phi_2 & & & \phi = \bigcup_i \phi_i
\end{array}
$$

Player II *wins the game* if $\mathrm{dom}(\phi)$ has a unique infinite branch z and

$$\bigcup_{s \subset z} \phi(s) = f(x).$$

Let MOVES be the set of $\psi : T \to {}^{\omega}\omega$ such that $T \subset {}^{<\omega}\omega$ is a finite tree and ψ is monotone and length-preserving. A **strategy** for Player II is a function $\tau : {}^{<\omega}\omega \to \mathsf{MOVES}$ such that $p \subset q \Rightarrow \tau(p) \subseteq \tau(q)$. For $x \in {}^{\omega}\omega$ and a strategy τ for Player II, let

$$\phi_x := \bigcup_{p \subset x} \tau(p)$$

and say that τ is **winning** in $G(f)$ if for all $x \in {}^{\omega}\omega$, $\mathrm{dom}(\phi_x)$ has a unique infinite branch z_x and

$$\bigcup_{s \subset z_x} \phi_x(s) = f(x).$$

2.0.9. Theorem. *A function $f : {}^\omega\omega \to {}^\omega\omega$ is Borel $\Leftrightarrow$ Player II has a winning strategy in the game $G(f)$.*

Proof. Let $\mathcal{F}$ be the set of functions $f : {}^\omega\omega \to {}^\omega\omega$ such that Player II has a winning strategy in $G(f)$. The main part of the proof is to show that $\mathcal{F}$ is closed under countable pointwise limits. Since $\mathcal{F}$ contains the continuous functions, this will show that every Borel function is in $\mathcal{F}$. For the reverse direction, to show that every function in $\mathcal{F}$ is Borel, a simple complexity argument will suffice.

We begin by showing the closure property. Let $f : {}^\omega\omega \to {}^\omega\omega$ and $f_n \in \mathcal{F}$ such that $f(x) = \lim_{n\to\omega} f_n(x)$ for all $x \in {}^\omega\omega$. We want to show that $f \in \mathcal{F}$. Let τ_n be a winning strategy for Player II in $G(f_n)$ and let $z_{n,x}$ be the unique infinite branch produced by τ_n on input $x \in {}^\omega\omega$. The idea is to "squash" the strategies τ_n into a single strategy τ for f. There are two difficulties. Firstly, we do not know ahead of time what the $z_{n,x}$ will be. Secondly, we do not know ahead of time the rate of convergence of the functions f_n. By rate of convergence, we mean the sequence $r_x \in {}^\omega\omega$ where $r_x(m)$ is the least natural number N satisfying $f_n(x) \restriction m = f_N(x) \restriction m$ for all $n \geq N$. The idea is that if we knew the infinite branches $z_{n,x}$ and the rate of convergence r_x, it would be a simple matter to compute $f(x)$. So, we will associate to each finite sequence a finite number of guesses about what will happen with the $z_{n,x}$ and r_x, and from this association we will define the strategy τ.

We define *guessing functions* $\rho_0 : {}^{<\omega}\omega \to \omega$ and $\rho_1 : {}^{<\omega}\omega \to {}^{<\omega}({}^{<\omega}\omega)$. The natural number $\rho_0(s)$ will be a guess for $r_x(\mathrm{lh}(s))$, and for $i < \mathrm{lh}(\rho_1(s))$, the sequence $\rho_1(s)(i)$ will be a guess for $z_{i,x} \restriction \mathrm{lh}(s)$. For technical reasons, the function ρ_1 will satisfy $\mathrm{lh}(\rho_1(s)) = \max(\rho_0(s), \mathrm{lh}(s)) + 1$. This will ensure that $\rho_0(s)$ is in the domain of $\rho_1(s)$ and for any $z \in {}^\omega\omega$,

$$\lim_{s\to z} \mathrm{lh}(\rho_1(s)) = \infty.$$

The definition of the guessing functions is by recursion on s. For the base case, let $\rho_0(\varnothing) := 0$ and $\rho_1(\varnothing) := \langle\varnothing\rangle$. For the recursive case, suppose $\rho_0(s) = N$ and $\rho_1(s) = \langle s_0, \ldots, s_k\rangle$ have been defined with $\mathrm{lh}(s_i) = \mathrm{lh}(s)$ and $k = \max(N, \mathrm{lh}(s))$. Let $\langle \rho_0(s^\frown j), \rho_1(s^\frown j)\rangle$ enumerate all pairs $\langle N', \langle u_0, \ldots, u_{k'}\rangle\rangle$ with $N' \geq N$, $\mathrm{lh}(u_i) = \mathrm{lh}(s) + 1$, $k' = \max(N', \mathrm{lh}(s) + 1)$, and $s_i \subset u_i$ for all i, $0 \leq i \leq k$. This completes the definition of ρ_0 and ρ_1.
For $s, u \in {}^{<\omega}\omega$, the following facts are easy to show:

- $s \subset u \Rightarrow \rho_0(s) \leq \rho_0(u)$,
- $\mathrm{lh}(\rho_1(s)) = \max(\rho_0(s), \mathrm{lh}(s)) + 1$,
- $\forall i < \mathrm{lh}(\rho_1(s))\,(\mathrm{lh}(s) = \mathrm{lh}(\rho_1(s)(i))$, and
- $s \subset u \Rightarrow \forall i < \mathrm{lh}(\rho_1(s))\,(\rho_1(s)(i) \subset \rho_1(u)(i))$.

Moreover, for any non-decreasing $r \in {}^\omega\omega$ and any $z_n \in {}^\omega\omega$, there is a unique $z \in {}^\omega\omega$ that encodes r and z_n via ρ_0 and ρ_1. Conversely, every $z \in {}^\omega\omega$ encodes some r and z_n.

We proceed with the definition of τ. At each round of the game, we consider certain sequences $s \in {}^{<\omega}\omega$ to be *active*. Informally, s is active if it looks like the guesses $\rho_1(s)(i)$ might be correct and are consistent with the guesses we have made along s about the rate of convergence. Let $p \in {}^{<\omega}\omega$ be a finite play of Player I. We say that $s \in {}^{<\omega}\omega$ is *active* if

- $\forall i < \mathrm{lh}(\rho_1(s))\,(\rho_1(s)(i) \in \mathrm{dom}(\tau_i(p)))$,
- $\forall m \leq \mathrm{lh}(s)\,(\rho_0(s \restriction m) > 0 \Rightarrow t_{\rho_0(s\restriction m)} \restriction m \neq t_{\rho_0(s\restriction m)-1} \restriction m)$,
  where $t_i = \tau_i(p)(\rho_1(s)(i))$ for $i < \mathrm{lh}(\rho_1(s))$, and
- $\forall m \leq \mathrm{lh}(s)\,\forall i\,(\rho_0(s \restriction m) < i < \mathrm{lh}(\rho_1(s)) \Rightarrow t_{\rho_0(s\restriction m)} \restriction m = t_i \restriction m)$.

Note that $\mathrm{lh}(t_i) = \mathrm{lh}(s)$ for all $i < \mathrm{lh}(\rho_1(s))$.

To understand the first condition, recall that $s_i := \rho_1(s)(i)$ is a guess for $z_{i,x} \restriction \mathrm{lh}(s)$. If $s_i \notin \mathrm{dom}(\tau_i(p))$, then we are not yet interested in this guess. For the second condition, recall that $N := \rho_0(s \restriction m)$ is a guess for $r_x(m)$. In words, this is the guess that the sequence of functions converges on the first m digits precisely at the Nth function. If t_N and t_{N-1} agree on the first m digits, then the guess N is too big, given that the other guesses associated to s are correct. Similarly, if t_N and t_i disagree on the first m digits for some i, $N < i < \mathrm{lh}(\rho_1(s))$, then the guess N is too small.

Let

$$S(p) := \{s : s \text{ is active and } \mathrm{lh}(\rho_1(s)) \leq \mathrm{lh}(p)\}.$$

Define $\tau(p)$ to be the function $\phi : S(p) \to {}^{<\omega}\omega$,

$$\phi(s) := t_{\rho_0(s)}.$$

We will show that τ is winning in the game $G(f)$. We begin by checking that τ is indeed a strategy. Firstly, we check that $\mathrm{dom}(\tau(p))$ is a tree. It suffices to show that if $s \subset u$ and u is active, then s is active. To check the first condition of activation, let $i < \mathrm{lh}(\rho_1(s))$. Since $\mathrm{lh}(\rho_1(s)) \leq \mathrm{lh}(\rho_1(u))$ and u is active, it follows that $\rho_1(u)(i) \in \mathrm{dom}(\tau_i(p))$. Since $\rho_1(s)(i) \subset \rho_1(u)(i)$ and $\mathrm{dom}(\tau_i(p))$ is a tree, it follows that $\rho_1(s)(i) \in \mathrm{dom}(\tau_i(p))$ as desired. For the second condition, let $m \leq \mathrm{lh}(s)$, $n = \rho_0(s \restriction m)$, and suppose $n > 0$. For $i < \mathrm{lh}(\rho_1(s))$, let $t_i = \tau_i(p)(\rho_1(s)(i))$ and $v_i = \tau_i(p)(\rho_1(u)(i))$. It follows that $t_i \subset v_i$ for all $i < \mathrm{lh}(\rho_1(s))$. By the second condition of activation of u, $v_n \restriction m \neq v_{n-1} \restriction m$. Therefore, $t_n \restriction m \neq t_{n-1} \restriction m$. For the third condition, let $m \leq \mathrm{lh}(s)$, $n = \rho_0(s \restriction m)$, and t_i and v_i as before. Let i such that $n < i < \mathrm{lh}(\rho_1(s))$. By the third condition of activation of u, it follows that $v_n \restriction m = v_i \restriction m$ and therefore $t_n \restriction m = t_i \restriction m$. This shows that $\mathrm{dom}(\tau(p))$ is a tree.

To show that $\mathrm{dom}(\tau(p))$ is finite, it suffices to show that for any $p \in {}^{<\omega}\omega$ and $k \in \omega$,

$$\{u \in {}^{<\omega}\omega : \mathrm{lh}(\rho_1(u)) = k \text{ and } u \text{ is active}\}$$

is finite. To that end, note that k is an upper bound for $\rho_0(u)$. By the first condition of activation, there are only finitely many possibilities for $\rho_1(u)$ since $\mathrm{dom}(\tau_i(p))$ is finite. For fixed $n \in \omega$ and $\langle s_0, \ldots, s_k \rangle \in {}^{<\omega}({}^{<\omega}\omega)$, there are finitely many u such that $\rho_0(u) = n$ and $\rho_1(u) = \langle s_0, \ldots, s_k \rangle$. It follows that $\mathrm{dom}(\tau(p))$ is finite.

It is immediate that $\tau(p)$ is length-preserving, so let us show that $\tau(p)$ is monotone. Let $s \subset u \in \mathrm{dom}(\tau(p))$, it must be shown that $\tau(p)(s) \subset \tau(p)(u)$. Let t_i and v_i as before: so $t_i = \tau_i(p)(\rho_1(s)(i))$ and $v_i = \tau_i(p)(\rho_1(u)(i))$. It follows that $\tau(p)(s) = t_{\rho_0(s)} = v_{\rho_0(s)} \upharpoonright \mathrm{lh}(s) = v_{\rho_0(u)} \upharpoonright \mathrm{lh}(s) = \tau(p)(u) \upharpoonright \mathrm{lh}(s)$. For the third equality, use that u is active and consider the third condition with $m = \mathrm{lh}(s)$ and $i = \rho_0(u)$. Finally, it must be shown that $p \subset q \Rightarrow \tau(p) \subseteq \tau(q)$, but this can easily be checked using that $p \subset q \Rightarrow \tau_i(p) \subseteq \tau_i(q)$ for all $i \in \omega$. This concludes the proof that τ is a strategy.

It remains to be shown that, on input x, τ produces a unique infinite branch along which the value is $f(x)$. Let r_x be the rate of convergence and let $z_{n,x}$ be the unique infinite branch produced by τ_n on input x. Let $z \in {}^{\omega}\omega$ be unique such that for all $s \subset z$, $\rho_0(s) = r_x(\mathrm{lh}(s))$ and $\rho_1(s) = \langle s_0, \ldots, s_k \rangle$ with $s_i \subset z_i$. In other words, z is the unique infinite sequence along which every guess is correct. Let ϕ_x be the function produced by τ and let $s \subset z$. It follows that $s \in \mathrm{dom}(\phi_x)$, in other words s will become active at some stage, and $\phi_x(s) = f(x) \upharpoonright \mathrm{lh}(s)$.

To show that z is the only infinite branch of $\mathrm{dom}(\phi_x)$, let $z' \in {}^{\omega}\omega$ such that $z' \neq z$. It will be shown that there is an initial segment of z' that is never activated. Let z'_n be the infinite branches encoded by z' via ρ_1, and let $\phi_{n,x}$ be the function produced by τ_n on input x. If $z'_i \neq z_i$ for some i, then there is an $s \subset z'_i$ such that $s \notin \mathrm{dom}(\phi_{i,x})$. Otherwise, τ_i would produce two distinct infinite branches, a contradiction. Let $u \subset z'_i$ such that $s \subseteq \rho_1(u)(i)$. It follows that u is never activated.

If $z'_n = z_n$ for all n, then it must be the case that $\rho_0(s) \neq r_x(\mathrm{lh}(s))$ for some $s \subset z'$. If $\rho_0(s) > r_x(\mathrm{lh}(s))$, then s is never activated. If $\rho_0(s) < r_x(\mathrm{lh}(s))$, then there is an i such that $i > \rho_0(s)$ and $f_{\rho_0(s)}(x) \upharpoonright \mathrm{lh}(s) \neq f_i(x) \upharpoonright \mathrm{lh}(s)$. Let $u \in {}^{<\omega}\omega$ with $s \subseteq u \subset z'$ and $\rho_1(u) = \langle u_0, \ldots, u_k \rangle$ with $i \leq k$. Then u is never activated, as u_i witnesses that the guess $\rho_0(s)$ is too small.

This completes the proof of the closure property.

For the reverse direction, it must be shown that every function in $\mathcal{F}$ is Borel. Let $f \in \mathcal{F}$ and let τ be a winning strategy for Player II in the game $G(f)$. It suffices to show that the preimage of a basic open set $[\,t\,]$ is analytic:

$$\begin{aligned} f^{-1}([\,t\,]) = \{x \in {}^{\omega}\omega :\ & \exists z \in {}^{\omega}\omega\, \exists m \in \omega\, (\tau(x \upharpoonright m)(z \upharpoonright \mathrm{lh}(t)) = t) \text{ and} \\ & \forall n \in \omega\, \exists m \in \omega\, (z \upharpoonright n \in \mathrm{dom}(\tau(x \upharpoonright m)))\} \end{aligned}$$

It follows that $f^{-1}[\,t\,]$ is analytic, since the strategy τ may be encoded as a real parameter. $\square$

Chapter 3

The $\boldsymbol{\Lambda}_{1,1}$, $\boldsymbol{\Lambda}_{2,2}$, and $\boldsymbol{\Lambda}_{1,2}$ functions

In this chapter, which begins our analysis of low-level Borel functions, we prove the Jayne-Rogers theorem. We also show that $\boldsymbol{\Lambda}_{1,1}$ is properly contained in $\boldsymbol{\Lambda}_{1,2}$ and $\boldsymbol{\Lambda}_{2,2}$ is properly contained in $\boldsymbol{\Lambda}_{1,2}$. In preparation, we review the *Wadge*, *backtrack*, and *eraser games*, developed by William W. Wadge, Robert van Wesep, and Jacques Duparc, respectively.

$$\begin{array}{ccc} \boldsymbol{\Lambda}_{1,2} & & \\ & \rotatebox{-45}{$\supsetneq$} & \\ & & \boldsymbol{\Lambda}_{2,2} \\ & \rotatebox{45}{$\subsetneq$} & \\ \boldsymbol{\Lambda}_{1,1} & & \end{array}$$

3.1 The Wadge game

The Wadge game was developed by William W. Wadge in his Ph.D. thesis [15] to characterize the notion of continuous reduction. Given two sets $A, B \subseteq {}^{\omega}\omega$, A is **Wadge reducible** to B ($A \leq_{\mathrm{W}} B$) if there is a continuous function $f : {}^{\omega}\omega \to {}^{\omega}\omega$ such that $f^{-1}[\,B\,] = A$. The Wadge game $G_{\mathrm{W}}(A, B)$ has two Players and is normally defined in such a way that Player II has a winning strategy if and only if $A \leq_{\mathrm{W}} B$. In this thesis, however, it will be convenient to drop the A's and B's and present a version of the Wadge game that characterizes the notion of continuous function instead of continuous reduction. We will also extend the game to handle partial functions on the Baire space.

Let $A \subseteq {}^{\omega}\omega$ and $f : A \to {}^{\omega}\omega$. In the Wadge game $G_{\mathrm{W}}(f)$, Player I plays elements $x_i \in \omega$ and Player II plays sequences $t_i \in {}^{<\omega}\omega$ such that $i < j \Rightarrow t_i \subseteq t_j$. After ω rounds, Player I produces $x := \langle x_0, x_1, \dots \rangle \in {}^{\omega}\omega$ and Player II produces $y := \bigcup_i t_i$.

$$\begin{array}{lccccccl} \text{I:} & x_0 & & x_1 & & x_2 & & x = \langle x_0, x_1, \dots \rangle \\ & & & & & & \dots & \\ \text{II:} & & t_0 & & t_1 & & t_2 & y = \bigcup_i t_i \end{array}$$

Player II wins the game if $x \notin A$ or $x \in A$ and $y = f(x)$.

A **Wadge strategy** for Player II is a function $\tau : {}^{<\omega}\omega \to {}^{<\omega}\omega$ such that $p \subset q \Rightarrow \tau(p) \subseteq \tau(q)$. A Wadge strategy for Player II is **winning** in $G_{\mathrm{W}}(f)$ if for all $x \in A$,

$$\bigcup_{p \subset x} \tau(p) = f(x).$$

3.1.1. THEOREM (WADGE). *A function $f : A \to {}^{\omega}\omega$ is continuous iff Player II has a winning strategy in $G_{\mathrm{W}}(f)$.*

Proof. $\Rightarrow$: Define

$$\tau(p) := (\bigcap f[[p]]) \cap \mathrm{lh}(p),$$

so $\tau : {}^{<\omega}\omega \to {}^{<\omega}\omega$ is a Wadge strategy. Furthermore, τ is winning for Player II in $G_{\mathrm{W}}(f)$. Suppose $x \in A$ and $t \subset f(x)$. Since f is continuous, $f^{-1}[[t]] = X \cap A$ for some open set X. Let $p_i \in {}^{<\omega}\omega$ such that $X = \bigcup_i [p_i]$. Let i such that $x \in [p_i]$ and let $m = \max(\mathrm{lh}(p_i), \mathrm{lh}(t))$. It follows that $\tau(x \upharpoonright m) \supseteq t$.

$\Leftarrow$: Suppose that τ is the winning strategy and let $t \in {}^{<\omega}\omega$. Then

$$f^{-1}[[t]] = (\bigcup \{[p] : \tau(p) \supseteq t\}) \cap A,$$

and therefore f is continuous. $\square$

3.2 The eraser game

Let $A \subseteq {}^{\omega}\omega$ and $f : A \to {}^{\omega}\omega$. We define the eraser game using trees (other definitions are also possible, for example in [11]). In the eraser game $G_{\mathrm{e}}(f)$, Player I plays elements $x_i \in \omega$ and Player II plays finite trees $T_i \subset {}^{<\omega}\omega$ such that $i < j \Rightarrow T_i \subseteq T_j$. After ω rounds, Player I produces $x := \langle x_0, x_1, \dots \rangle \in {}^{\omega}\omega$ and Player II produces $T := \bigcup_i T_i$.

$$\begin{array}{lccccccl} \text{I:} & x_0 & & x_1 & & x_2 & & x = \langle x_0, x_1, \dots \rangle \\ & & & & & & \dots & \\ \text{II:} & & T_0 & & T_1 & & T_2 & T = \bigcup_i T_i \end{array}$$

Player II wins the game if either $x \notin A$ or if T is finitely branching and $f(x)$ is the unique infinite branch of T.

Let MOVES be the set of finite trees $T \subset {}^{<\omega}\omega$. An **eraser strategy** for Player II is a function $\tau : {}^{<\omega}\omega \to \mathsf{MOVES}$ such that $p \subset q \Rightarrow \tau(p) \subseteq \tau(q)$. If $x \in {}^{\omega}\omega$ and τ is an eraser strategy for Player II, let

$$T_x := \bigcup_{p \subset x} \tau(p)$$

and say that τ for Player II is **winning** in $G_{\mathrm{e}}(f)$ if for all $x \in A$, T_x is finitely branching and $f(x)$ is the unique infinite branch of T_x.

3.2.1. THEOREM (DUPARC). *A function $f : A \to {}^{\omega}\omega$ is Baire class 1 iff Player II has a winning strategy in G_{e}.*

Proof. $\Rightarrow$: Let $f = \lim_{n\to\infty} f_n$ with $f_n : A \to {}^{\omega}\omega$ continuous and let τ_n be a winning strategy for Player II in $G_{\mathrm{W}}(f_n)$. Define

$$\tau(p) := \mathrm{tree}(\{\tau_n(p) \upharpoonright n : n \leq \mathrm{lh}(p)\})$$

where $\mathrm{tree}(T) := \{s : \exists t \in T\,(s \subseteq t)\}$. It is easy to check that τ is an eraser strategy. We show that τ is winning in $G_{\mathrm{e}}(f)$. Let $x \in A$ be a play of Player I and let T_x be the function produced by τ on input x. It follows that

$$T_x = \mathrm{tree}(\{f_n(x) \upharpoonright n : n \in \omega\})$$

and T_x is finitely branching since $\{t \upharpoonright m : t \in T_x\} = \{f_n(x) \upharpoonright m : n \geq m\}$ is finite. Furthermore, for any m there is an $n \geq m$ such that $f(x) \upharpoonright m = f_n(x) \upharpoonright m$, so $f(x)$ is an infinite branch of T_x. If $t \not\subset f(x)$ then $T_x \cap \{v : v \supseteq t\}$ is finite, so $f(x)$ is the only infinite branch of T_x.

$\Leftarrow$: Let τ be winning for Player II in $G_{\mathrm{e}}(f)$ and let T_x be the tree produced by τ on input $x \in A$. For $t \in T_x$, let $\mu_x(t)$ be the least n such that $t \in \tau(x \upharpoonright n)$. Let $\prec$ be a well-ordering of ${}^{<\omega}\omega$ and let $\prec_x$ be the well-ordering of T_x given by

$$\begin{aligned} s \prec_x t :\Leftrightarrow\ & \mu_x(s) < \mu_x(t) \text{ or} \\ & \mu_x(s) = \mu_x(t) \text{ and } s \prec t. \end{aligned}$$

Let $f_n(x) : A \to {}^{\omega}\omega$,

$$f_n(x) := t^\frown 0^*,$$

where t is the $\prec_x$-nth element of T_x. The functions f_n are continuous and furthermore, $f = \lim_{n\to\infty} f_n$. Let $x \in A$ and $t \subset f(x)$. Since T_x is finitely branching and $f(x)$ is its unique infinite branch, it follows that $\{s \in T_x : t \not\subseteq s\}$ is finite by König's lemma. Therefore, there are finitely many n such that $t \not\subset f_n(x)$. $\square$

3.3 The backtrack game

Let $A \subseteq {}^\omega\omega$ and $f : A \to {}^\omega\omega$. In the backtrack game $G_{\mathrm{bt}}(f)$, Player I plays elements $x_i \in \omega$ and Player II plays functions $\phi_i : D_i \to {}^{<\omega}\omega$ such that $D_i \subset \omega$ is finite. Player II is subject to the requirements that $i < j \Rightarrow D_i \subseteq D_j$ and $\phi_i(n) \subseteq \phi_j(n)$ for all $n \in \mathrm{dom}(\phi_i)$. After ω rounds, Player I produces $x = \langle x_0, x_1, \dots \rangle \in {}^\omega\omega$ and Player II produces $\phi : D_\omega \to {}^{\leq\omega}\omega$,

$$\phi(n) := \bigcup\{\phi_i(n) : i \in \omega \text{ and } n \in \mathrm{dom}(\phi_i)\},$$

where $D_\omega := \bigcup_i D_i$.

$$\begin{array}{lccccccccl} \text{I:} & x_0 & & x_1 & & x_2 & & & & x = \langle x_0, x_1, \dots \rangle \\ & & & & & & & \cdots & & \\ \text{II:} & & \phi_0 & & \phi_1 & & \phi_2 & & & \phi \text{ as above} \end{array}$$

Player II wins the game if either $x \notin A$ or if D_ω is finite, there is an $n \in D_\omega$ such that $\phi(n) = f(x)$, and $\phi(n')$ is finite for all $n' \neq n$. Informally, we think of the domain of ϕ as consisting of a finite number of *rows*. Player II's task is to produce an infinite sequence, namely $f(x)$, on exactly one of the rows. We refer to this row as the *output row*.

Let MOVES be the set of functions $\psi : D \to {}^{<\omega}\omega$ such that $D \subset \omega$ is finite. A **backtrack strategy** for Player II is a function $\tau : {}^{<\omega}\omega \to \mathsf{MOVES}$ such that $p \subset q \Rightarrow \mathrm{dom}(\tau(p)) \subseteq \mathrm{dom}(\tau(q))$ and $\tau(p)(n) \subseteq \tau(q)(n)$ for all $n \in \mathrm{dom}(\tau(p))$. For an infinite play x of Player I and a backtrack strategy τ for Player II, we let $D_x := \bigcup_{p \subset x} \mathrm{dom}(\tau(p))$ and $\phi_x : D_x \to {}^{\leq\omega}\omega$,

$$\phi_x(n) := \bigcup\{\tau(p)(n) : p \subset x \text{ and } n \in \mathrm{dom}(\tau(p))\}.$$

A backtrack strategy τ for Player II is **winning** in $G_{\mathrm{bt}}(f)$ if for all $x \in A$, D_x is finite, there is an $n \in D_x$ such that $\phi_x(n) = f(x)$, and $\phi_x(n')$ is finite for all $n' \neq n$. We will sometimes denote this n, the output row, by o_x.

The next theorem is due to Alessandro Andretta.

3.3.1. Theorem (Andretta). *A function $f : A \to {}^\omega\omega$ admits a $\mathbf{\Pi}^0_1$ partition $\langle A_n : n \in \omega \rangle$ such that $f \restriction A_n$ is continuous iff Player II has a winning strategy in $G_{\mathrm{bt}}(f)$.*

Proof. $\Rightarrow$: Let $f : A \to {}^\omega\omega$, let A_n be the partition, and let τ_n be a winning strategy for Player II in $G_{\mathrm{W}}(f \restriction A_n)$. Let $T_n \subseteq {}^{<\omega}\omega$ be a tree such that $A_n = [T_n] \cap A$. For $p \in {}^{<\omega}\omega$, let $B(p) := \{\langle n, \tau_n(p) \rangle\}$, where n is least such that $p \in T_n$. Define $\tau(p) : \bigcup\{\mathrm{dom}(B(q)) : q \subseteq p\} \to {}^{<\omega}\omega$,

$$\tau(p)(n) := \bigcup\{B(q)(n) : q \subseteq p \text{ and } n \in \mathrm{dom}(B(q))\}.$$

It is easy to check that τ is a backtrack strategy and winning for Player II in $G_{\mathrm{bt}}(f)$.

$\Leftarrow$: Let τ be the winning strategy for Player II in $G_{\mathrm{bt}}(f)$. For $x \in A$, let D_x, ϕ_x, and o_x as in the previous section. Define

$$A_n := \{x \in A : o_x = n\}.$$

The Wadge strategy τ_n given by

$$\tau_n(p) := \begin{cases} \tau(p)(n) \text{ if } n \in \mathrm{dom}(\tau(p)), \\ \varnothing \text{ otherwise} \end{cases}$$

is winning for Player II in $G_{\mathrm{W}}(f \upharpoonright A_n)$. Furthermore, the sets A_n are $\mathbf{\Sigma}^0_2$. Namely, fix $n \in \omega$ and let T_i be the set of $p \in {}^{<\omega}\omega$ such that

$$\sum_{\substack{m \in \mathrm{dom}(\tau(p)) \\ m \neq n}} \mathrm{lh}(\tau(p)(m)) \leq i.$$

Then $A_n = \bigcup_i [\,T_i\,] \cap A$. Since we are working in the Baire space, $\mathbf{\Sigma}^0_2$ sets are the disjoint union of countably many $\mathbf{\Pi}^0_1$ sets, completing the proof. $\square$

3.4 The Jayne-Rogers theorem

To prove the Jayne-Rogers theorem, we begin with some lemmas.

3.4.1. Lemma. *Let $A \subseteq {}^\omega\omega$, $h : A \to {}^\omega\omega$, and suppose that τ_{e} is a winning strategy for Player II in $G_{\mathrm{e}}(h)$. Let $t_1, t_2 \in {}^{<\omega}\omega$ such that $t_1 \perp t_2$. If Player II has a winning strategy τ_1 in $G_{\mathrm{bt}}(h \upharpoonright h^{-1}[\,[\,t_1\,]^c\,])$ and a winning strategy τ_2 in $G_{\mathrm{bt}}(h \upharpoonright h^{-1}[\,[\,t_2\,]^c\,])$, then Player II has a winning strategy in $G_{\mathrm{bt}}(h)$.*

Proof. For $p \in {}^{<\omega}\omega$, let

$$\begin{aligned} \gamma_1(p) &:= \mathrm{card}(\tau_{\mathrm{e}}(p) \setminus \{v : v \supseteq t_1\}), \text{ and} \\ \gamma_2(p) &:= \mathrm{card}(\tau_{\mathrm{e}}(p)[\,t_1\,]). \end{aligned}$$

Define

$$\begin{aligned} \tau(p) := \{\langle 2n, t\rangle : \langle n, t\rangle \in \tau_1(p \upharpoonright \gamma_1(p))\} \cup \\ \{\langle 2n+1, t\rangle : \langle n, t\rangle \in \tau_2(p \upharpoonright \gamma_2(p))\}. \end{aligned}$$

It is easy to see that the backtrack strategy τ is winning for Player II in $G_{\mathrm{bt}}(h)$. If $x \in [\,t_1\,]$, then as $p \to x$, $\gamma_1(p)$ is bounded by König's lemma and $\gamma_2(p) \to \infty$. It follows that τ will produce the value $h(x)$ on one of its odd rows. Similarly, if $x \notin [\,t_1\,]$, then $\gamma_2(p)$ is bounded and $\gamma_1(p) \to \infty$ as $x \to p$. So, τ will produce the value $h(x)$ on one of its even rows. $\square$

We turn our attention to the eraser game, with another simple lemma.

3.4.2. LEMMA. *Let $A \subseteq {}^\omega\omega$, $h : A \to {}^\omega\omega$, and suppose that τ_e is a winning strategy for Player II in $G_\mathrm{e}(h)$. For $x \in A$, let $T_x \subset {}^{<\omega}\omega$ be the tree produced by τ_e on input x. Let $\langle t_n : n \in \omega \rangle$ be an infinite sequence of pairwise incompatible elements of ${}^{<\omega}\omega$. If $t_n \in T_x$ for infinitely many n, then $h(x) \notin [\, t_n \,]$ for all n.*

Proof. Suppose T_x contains infinitely many t_n. Fix $n \in \omega$. The finitely branching tree $T_x \setminus \{v : v \supseteq t_n\}$ is infinite and thus $h(x) \notin [\, t_n \,]$ by König's lemma. □

The next lemma is the main lemma of the argument. The proof we give here is due to Solecki.

3.4.3. LEMMA. *Let $A \subseteq {}^\omega\omega$, $h : A \to {}^\omega\omega$, and suppose that τ_e is a winning strategy for Player II in $G_\mathrm{e}(h)$. If Player II does not have a winning strategy in $G_\mathrm{bt}(h)$, then there is an $x \in A$ and a $t \in {}^{<\omega}\omega$ such that $t \subset h(x)$ and for all $p \subset x$, Player II does not have a winning strategy in*

$$G_\mathrm{bt}(h \restriction (h^{-1}[\,[\,t\,]^c\,] \cap [\,p\,])).$$

Proof. By contradiction. Suppose for every $x \in A$ and $t \subset h(x)$, there is a $p \subset x$ such that Player II has a winning strategy in

$$G_\mathrm{bt}(h \restriction (h^{-1}[\,[\,t\,]^c\,] \cap [\,p\,])).$$

Let P be the set of $p \in {}^{<\omega}\omega$ such that Player II has a winning strategy in $G_\mathrm{bt}(h \restriction [\,p\,])$ and let $U := \bigcup\{[\,p\,] : p \in P\}$. By assumption, Player II does not have a winning strategy in $G_\mathrm{bt}(h)$. It follows that Player II does not have a winning strategy in $G_\mathrm{bt}(h \restriction (A \setminus U))$, and therefore $h \restriction (A \setminus U)$ is not continuous. Let $x \in A \setminus U$ be a discontinuity point, so there is a $t_0 \subset h(x)$ such that for any $p \subset x$, there exists $y \supset p$ with $y \in A \setminus U$ and $t_0 \not\subset h(y)$. By the failure of the conclusion, there is a $p_0 \subset x$ such that Player II has a winning strategy in

$$G_\mathrm{bt}(h \restriction (h^{-1}[\,[\,t_0\,]^c\,] \cap [\,p_0\,])).$$

Let $y \supset p_0$ such that $y \in A \setminus U$ and $t_0 \not\subset h(y)$. Let $t_1 \subset h(y)$ such that $t_0 \perp t_1$. Again by the failure of the conclusion, there is a $p_1 \subset y$, of which we can assume $p_0 \subseteq p_1$, such that Player II has a winning strategy in

$$G_\mathrm{bt}(h \restriction (h^{-1}[\,[\,t_1\,]^c\,] \cap [\,p_1\,])).$$

By Lemma 3.4.1, Player II has a winning strategy in $G_\mathrm{bt}(h \restriction [\,p_1\,])$, contradicting $y \notin U$. □

Before proving the Jayne-Rogers theorem, we want to generalize the idea of Lemma 3.4.3. Fix $f : {}^\omega\omega \to {}^\omega\omega$ and suppose that τ_e is a winning strategy for Player II in $G_\mathrm{e}(f)$. For $x \in {}^\omega\omega$ and $\sigma \subseteq {}^\omega\omega$, say that x is ***σ*-good** if for every $p \subset x$, Player II does not have a winning strategy in

$$G_\mathrm{bt}(f \restriction (f^{-1}[\,\sigma\,] \cap [\,p\,])).$$

3.4.4. Lemma. *Let $x \in {}^\omega\omega$, $\sigma \subseteq {}^\omega\omega$, and let $\langle t_0, \dots, t_m \rangle$ be a sequence of pairwise incompatible elements of ${}^{<\omega}\omega$. If x is σ-good, then*

$$\{i \leq m : x \textit{ is not } (\sigma \setminus [\, t_i \,])\textit{-good}\}$$

has at most one element.

Proof. Suppose there are $i \neq j \leq m$ such that x is not $(\sigma \setminus [\, t_i \,])$-good and x is not $(\sigma \setminus [\, t_j \,])$-good. Then it follows easily from Lemma 3.4.1 that x is not σ-good, a contradiction. □

3.4.5. Theorem. *(Jayne, Rogers) A function $f : {}^\omega\omega \to {}^\omega\omega$ is $\mathbf{\Lambda}_{2,2} \Leftrightarrow$ there is a $\mathbf{\Pi}^0_1$ partition $\langle A_n : n \in \omega \rangle$ of ${}^\omega\omega$ such that $f \restriction A_n$ is continuous.*

To prove the Jayne-Rogers theorem, we will assume that we are given $f : {}^\omega\omega \to {}^\omega\omega$, $f \in \mathbf{\Lambda}_{1,2}$ such that there is no closed partition A_n of ${}^\omega\omega$ with $f \restriction A_n$ continuous. We will then define an open set Y and a continuous reduction from a $\mathbf{\Sigma}^0_2$-complete set X to $f^{-1}[\,Y\,]$. This will show that $f \notin \mathbf{\Lambda}_{2,2}$, as desired. The reduction will be constructed in stages, using the notion of a *snake*. Say that a sequence $\psi_n : T_n \to {}^{<\omega}\omega$ is a **snake** if

- $T_n \subset {}^{<\omega}2$ is a finite tree,
- ψ_n is monotone,
- $i < j \Rightarrow T_i \subseteq T_j$,
- $i < j$ and $p \in \mathrm{tn}(T_i) \Rightarrow \psi_i(p) \subseteq \psi_j(p)$,
- $i < j$ and $p \in T_i \setminus \mathrm{tn}(T_i) \Rightarrow \psi_i(p) = \psi_j(p)$,
- $\displaystyle\bigcup_n T_n = {}^{<\omega}2$, and
- the function $\psi : {}^{<\omega}2 \to {}^{<\omega}\omega$,
  $\psi(p) := \bigcup\{\psi_n(p) : n \in \omega \text{ and } p \in \mathrm{dom}(\psi_n)\}$ is infinitary.

If ψ_n is a snake and $\psi = \bigcup_n \psi_n$, then $\hat{\psi} : {}^\omega 2 \to {}^\omega\omega$,

$$\hat{\psi}(x) := \bigcup_{p \subset x} \psi(p)$$

is continuous and we refer to $\hat{\psi}$ as the **lifting** of ψ_n.

Proof of Theorem 3.4.5. By Lemma 1.1.4, it suffices to prove $\Rightarrow$. Suppose that there is no such partition A_n, we will show that $f \notin \mathbf{\Lambda}_{2,2}$. If $f \notin \mathbf{\Lambda}_{1,2}$, then we are done, so we may let τ_{e} be a winning strategy for Player II in $G_{\mathrm{e}}(f)$.

We will define an open set Y and a snake ψ_n such that the lifting $\hat{\psi}$ of ψ_n is a reduction from

$$X := \{z \in {}^{\omega}2 : \exists i\, \forall j \geq i\ (z(j) = 0)\}$$

to $f^{-1}[Y]$. Let $\beta : \omega \to {}^{<\omega}2$ be the enumeration given by $\beta(0) := \varnothing$, $\beta(2n+1) := \beta(n)^\frown 0$, and $\beta(2n+2) := \beta(n)^\frown 1$. We will define by recursion:

$$\begin{aligned}
\psi_n &: \beta[n+1] \to {}^{<\omega}\omega,\\
\xi_n &: \beta[n+1] \to {}^{\omega}\omega, \text{ and}\\
\eta_n &: \beta[n+1] \to {}^{<\omega}\omega
\end{aligned}$$

such that $i < j \Rightarrow \xi_i \subset \xi_j$, $i < j \Rightarrow \eta_i \subset \eta_j$, and for all n and all $p \in \beta[n+1]$,

- $\psi_n(p) \subset \xi_n(p)$,
- $\eta_n(p) \subset f(\xi_n(p))$,
- $\mathrm{ran}(\eta_n)$ is an antichain,
- $\xi_n(p)$ is σ_n-good, where $\sigma_n := \bigcap_{t \in \mathrm{ran}(\eta_n)} [\,t\,]^c$, and
- $(*)$ $\mathrm{ran}(\eta_n) \cap \tau_{\mathrm{e}}(\psi_n(p)) > \mathrm{card}(\{k : p(k) = 1\})$.

Let x and t be given by Lemma 3.4.3 applied to f, so $t \subset f(x)$ and x is $[\,t\,]^c$-good. Let $q \subset x$ such that $t \in \tau_{\mathrm{e}}(q)$. Define

$$\begin{aligned}
\psi_0 &:= \{\langle \varnothing, q\rangle\},\\
\xi_0 &:= \{\langle \varnothing, x\rangle\}, \text{ and}\\
\eta_0 &:= \{\langle \varnothing, t\rangle\}.
\end{aligned}$$

The reader should check that ψ_0, ξ_0, and η_0 satisfy the desired requirements. For the recursive case, suppose that ψ_n, ξ_n, and η_n have been defined.

Case A: n is even. Let p such that $\beta(n+1) = p^\frown 0$. Define

$$\begin{aligned}
\psi_{n+1} &:= \psi_n \cup \{\langle p^\frown 0, \xi_n(p) \restriction \mathrm{lh}(\psi_n(p)) + 1\rangle\},\\
\xi_{n+1} &:= \xi_n \cup \{\langle p^\frown 0, \xi_n(p)\rangle\}, \text{ and}\\
\eta_{n+1} &:= \eta_n \cup \{\langle p^\frown 0, \eta_n(p)\rangle\}.
\end{aligned}$$

Case B: n is odd. Let p such that $\beta(n+1) = p^\frown 1$. We want to find x and t such that $\psi_n(p) \subset x$, $t \subset f(x)$, t and elements of $\mathrm{ran}(\eta_n)$ are pairwise incompatible, and every element of $\mathrm{ran}(\xi_n) \cup \{x\}$ is $(\sigma_n \setminus [\,t\,])$-good, with

$$\sigma_n := \bigcap_{v \in \mathrm{ran}(\eta_n)} [\,v\,]^c.$$

We will define sequences $\langle x_0, x_1, \ldots\rangle$ and $\langle t_0, t_1, \ldots\rangle$ such that x_l and t_l will be the desired values of x and t for some l. Let

$$h := f \restriction (f^{-1}[\,\sigma_n\,] \cap [\,\psi_n(p)\,]).$$

By the induction hypothesis, $\psi_n(p) \subset \xi_n(p)$ and $\xi_n(p)$ is σ_n-good. Therefore, Player II does not have a winning strategy in $G_{\mathrm{bt}}(h)$. Let x_0 and t_0 be given by Lemma 3.4.3 applied to h, so $\psi_n(p) \subset x_0$, $t_0 \subset f(x_0)$, $v \not\subseteq t_0$ for all $v \in \mathrm{ran}(\eta_n)$, and x_0 is $(\sigma \setminus [\,t_0\,])$-good.

Now, suppose $\langle x_0, \ldots, x_j\rangle$ and $\langle t_0, \ldots t_j\rangle$ have been defined such that for all $i \leq j$, $\psi_n(p) \subset x_i$, $t_i \subset f(x_i)$, $v \not\subseteq t_i$ for all $v \in \mathrm{ran}(\eta_n) \cup \{t_0, \ldots, t_{i-1}\}$, and x_i is

$$(\sigma_n \cap [\,t_0\,]^c \cap \cdots \cap [\,t_i\,]^c)\text{-good.}$$

Let

$$h := f \restriction (f^{-1}[\,\sigma_n \cap [\,t_0\,]^c \cap \cdots \cap [\,t_j\,]^c\,] \cap [\,\psi_n(p)\,])$$

and let x_{j+1} and t_{j+1} be given by Lemma 3.4.3 applied to h. It follows that $\psi_n(p) \subset x_{j+1}$, $t_{j+1} \subset f(x_{j+1})$, $v \not\subseteq t_{j+1}$ for all $v \in \mathrm{ran}(\eta_n) \cup \{t_0, \ldots, t_j\}$, and x_{j+1} is

$$(\sigma_n \cap [\,t_0\,]^c \cap \cdots \cap [\,t_{j+1}\,]^c)\text{-good.}$$

We claim that there is an l such that t_l and elements of $\mathrm{ran}(\eta_n)$ are pairwise incompatible and every element of $\mathrm{ran}(\xi_n)$ is $(\sigma \setminus [\,t_l\,])$-good. Namely, we may consider an arbitrarily long subsequence of $\langle t_0, t_1, \ldots\rangle$ such that the elements of the subsequence are pairwise incompatible with themselves and elements of $\mathrm{ran}(\eta_n)$. By Lemma 3.4.4, the claim follows. Let $x := x_l$ and $t := t_l$. Let $q \supset \psi_n(p)$ such that $q \subset x$ and $t \in \tau_{\mathrm{e}}(q)$. Define

$$\begin{aligned}\psi_{n+1} &:= \psi_n \cup \{\langle p^\frown 1, q\rangle\},\\ \xi_{n+1} &:= \xi_n \cup \{\langle p^\frown 1, x\rangle\}, \text{ and}\\ \eta_{n+1} &:= \eta_n \cup \{\langle p^\frown 1, t\rangle\}.\end{aligned}$$

This completes the definition of ψ_n, ξ_n, and η_n.

Now, let $\xi := \bigcup \xi_n$, $\eta := \bigcup \eta_n$, and $\hat{\psi}$ be the lifting of ψ_n. Let

$$Y := \bigcup_{t \,\in\, \mathrm{ran}(\eta)} [\,t\,].$$

The continuous function $\hat{\psi}$ is a reduction from X to $f^{-1}[\,Y\,]$. If $x \in X$, then let $p \subset x$ such that $x = p^\frown 0^*$. It follows that $\hat{\psi}(x) = \xi(p)$ and thus $f(\hat{\psi}(x)) \in Y$. If $x \notin X$, then let T be the tree produced by the eraser strategy τ_{e} on input $\hat{\psi}(x)$. By $(*)$, it follows that T contains infinitely many elements of $\mathrm{ran}(\eta)$ and thus $f(\hat{\psi}(x)) \notin Y$ by Lemma 3.4.2. □

3.5 $\mathbf{\Lambda}_{2,2} \not\subseteq \mathbf{\Lambda}_{1,1}$ and $\mathbf{\Lambda}_{1,2} \not\subseteq \mathbf{\Lambda}_{2,2}$

In this section, we show that the containments between these classes are proper. These results are already known and are not difficult to prove. However, we will use a game-theoretic diagonalization method that will be useful in Chapters 4 and 5. The method is similar to the diagonalization methods used in computability theory.

3.5.1. FACT. $\mathbf{\Lambda}_{2,2} \not\subseteq \mathbf{\Lambda}_{1,1}$

Proof. Let $\beta : {}^{<\omega}\omega \to \omega$ be a bijection. If $\tau : {}^{<\omega}\omega \to {}^{<\omega}\omega$ is a Wadge strategy, then say that $x \in {}^{\omega}\omega$ is a **code** for τ if $\tau(p) = \beta^{-1}(x(\beta(p)))$ for all $p \in {}^{<\omega}\omega$. Note that for every Wadge strategy τ, there is a unique x that encodes it. For $T \subset {}^{<\omega}\omega$, say that $\tau : T \to {}^{<\omega}\omega$ is a **partial Wadge strategy** if $s, t \in T$ and $s \subset t \Rightarrow \tau(s) \subseteq \tau(t)$.

It suffices to define a backtrack strategy τ_{bt} that is winning for Player II in $G_{\mathrm{bt}}(f)$ for some $f : {}^{\omega}\omega \to {}^{\omega}\omega$ that is not continuous. On input x, the strategy τ_{bt} will attempt to decode x into a Wadge strategy τ_x and diagonalize against the first digit of the output of τ_x on input x.

Fix $p \in {}^{<\omega}\omega$. Let

$$T := \{\beta^{-1}(n) : n < \mathrm{lh}(p)\}.$$

Let $\tau : T \to {}^{<\omega}\omega$, $\tau(s) := \beta^{-1}(p(\beta(s)))$. If τ is a partial Wadge strategy and there is a $q \subseteq p$ such that $q \in \mathrm{dom}(\tau)$, $\mathrm{lh}(\tau(q)) > 0$, and $\tau(q)(0) = 0$, then let $B(p) := \{\langle 1, 1^{\mathrm{lh}(p)}\rangle\}$. Otherwise, let $B(p) := \{\langle 0, 0^{\mathrm{lh}(p)}\rangle\}$. Define $\tau_{\mathrm{bt}}(p) : \{0,1\} \to {}^{<\omega}\omega$,

$$\tau_{\mathrm{bt}}(p)(n) := \bigcup\{B(q)(n) : q \subseteq p \text{ and } n \in \mathrm{dom}(B(q))\}.$$

Let $f : {}^{\omega}\omega \to \{0^*, 1^*\}$ such that τ_{bt} is winning for Player II in $G_{\mathrm{bt}}(f)$. Suppose for contradiction that f is continuous. Let τ be a Wadge strategy that is winning for Player II in $G_{\mathrm{W}}(f)$. Let $x \in {}^{\omega}\omega$ be the code of τ and consider $f(x)$. If $f(x) = 0^*$ then it follows that $f(x) = 1^*$, and if $f(x) = 1^*$ then it follows that $f(x) = 0^*$. Therefore, f is not continuous. $\square$

Fact 3.5.1 can easily be seen without the use of games. Fix $y \in {}^{\omega}\omega$, and let $h : {}^{\omega}\omega \to {}^{\omega}\omega$,

$$h(x) := \begin{cases} 0^* \text{ if } x = y \\ 1^* \text{ if } x \neq y. \end{cases}$$

It follows that $h \in \mathbf{\Lambda}_{2,2} \setminus \mathbf{\Lambda}_{1,1}$.

3.5.2. FACT. $\mathbf{\Lambda}_{1,2} \not\subseteq \mathbf{\Lambda}_{2,2}$

Proof. As in Section 3.2, let MOVES be the set of functions $\psi : D \to {}^{<\omega}\omega$ such that $D \subset \omega$ is finite. Let $\beta : {}^{<\omega}\omega \to \omega$ and $\gamma : \omega \to \mathsf{MOVES}$ be bijections. If $\tau : {}^{<\omega}\omega \to \mathsf{MOVES}$ is a backtrack strategy, then $x \in {}^{\omega}\omega$ is a **code** for τ if $\tau(p) = \gamma(x(\beta(p)))$ for all $p \in {}^{<\omega}\omega$. Note that for every backtrack strategy τ, there is a unique x that encodes it. For $T \subseteq {}^{<\omega}\omega$, say that $\tau : T \to \mathsf{MOVES}$ is a **partial backtrack strategy** if $s, t \in T$ and $s \subset t \Rightarrow \mathrm{dom}(\tau(s)) \subseteq \mathrm{dom}(\tau(t))$ and $\tau(s)(n) \subseteq \tau(t)(n)$ for all $n \in \mathrm{dom}(\tau(s))$.

It suffices to define an eraser strategy τ_{e} and $f : {}^{\omega}\omega \to {}^{\omega}\omega$ such that τ_{e} is winning in $G_{\mathrm{e}}(f)$ and $f \notin \mathbf{\Lambda}_{2,2}$. On input x, the strategy τ_{e} will attempt to decode x into a backtrack strategy τ_x and diagonalize against the output of τ_x on input x.

Fix $p \in {}^{<\omega}\omega$. Let

$$T := \{\beta^{-1}(n) : n < \mathrm{lh}(p)\}.$$

Let $\tau : T \to \mathsf{MOVES}$, $\tau(s) := \gamma(p(\beta(s)))$. If τ is a partial backtrack strategy, then let $r := \bigcup\{q : q \subseteq p \text{ and } q \in \mathrm{dom}(\tau)\}$ and $\psi := \tau(r)$. Let $E(p) \in {}^{\mathrm{lh}(p)}\omega$,

$$E(p)(m) := \begin{cases} 1 \text{ if } m \in \mathrm{dom}(\psi),\ m \in \mathrm{dom}(\psi(m)),\ \text{and } \psi(m)(m) = 0, \\ 0 \text{ otherwise.} \end{cases}$$

If τ is not a partial backtrack strategy, then let

$$E(p) := 0^{\mathrm{lh}(p)}.$$

Define

$$\tau_{\mathrm{e}}(p) := \mathrm{tree}(\{E(p) : q \subseteq p\}).$$

It is easy to check that τ_{e} is an eraser strategy and that there is an $f : {}^{\omega}\omega \to {}^{2}\omega$ such that τ_{e} is winning in $G_{\mathrm{e}}(f)$. Suppose for contradiction that $f \in \mathbf{\Lambda}_{2,2}$. By Theorems 3.3.1 and 3.4.5, there is a backtrack strategy τ that is winning for Player II in $G_{\mathrm{bt}}(f)$. Let $x \in {}^{\omega}\omega$ be the code of τ, let m be the output row of τ on input x, and consider $f(x)$. If $f(x)(m) = 0$ then it follows that $f(x)(m) = 1$, and if $f(x)(m) = 1$ it follows that $f(x)(m) = 0$. Therefore, $f \notin \mathbf{\Lambda}_{2,2}$. $\square$

It is also easy to show Fact 3.5.2 without using games. Let

$$A := \{x \in {}^{\omega}\omega : \exists N\, \forall n > N\, x(n) = 0\}$$

and let $\beta : A \to \omega$ be a bijection. Let $h : {}^{\omega}\omega \to {}^{\omega}\omega$,

$$h(x) := \begin{cases} 0^{\beta(x)}{}^\frown 1^\frown 0^* \text{ if } x \in A, \\ 0^* \text{ if } x \notin A. \end{cases}$$

It is clear that $h \notin \mathbf{\Lambda}_{2,2}$. Namely, let $Y = \bigcup\{[\,t\,] : t = (0^n)^\frown 1 \text{ for some } n\}$, then $h^{-1}[\,Y\,]$ is $\mathbf{\Sigma}^0_2$-complete. To see that $h \in \mathbf{\Lambda}_{1,2}$, it suffices to show that the preimage of a basic open set $[\,t\,]$ is $\mathbf{\Sigma}^0_2$. If $t = (0^n)^\frown 1^\frown 0^m$, then $h^{-1}[\,[\,t\,]\,]$ is a singleton and thus closed. If $t = 0^n$, then $h^{-1}[\,[\,t\,]\,]$ is cofinite and thus open. Otherwise, $h^{-1}[\,[\,t\,]\,]$ is empty.

Chapter 4

The $\mathbf{\Lambda}_{2,3}$ and $\mathbf{\Lambda}_{1,3}$ functions

In this chapter, we extend the methods from Chapter 3 to analyze the $\mathbf{\Lambda}_{1,3}$ and $\mathbf{\Lambda}_{2,3}$ functions.

$$\begin{array}{ccc}
\mathbf{\Lambda}_{1,3} & & \\
 & \rotatebox{45}{$\supset$} & \\
 & & \mathbf{\Lambda}_{2,3} \\
 & \rotatebox{-45}{$\subset$} & \\
\mathbf{\Lambda}_{1,2} & & \\
 & \rotatebox{45}{$\supset$} & \\
 & & \mathbf{\Lambda}_{2,2} \\
 & \rotatebox{-45}{$\subset$} & \\
\mathbf{\Lambda}_{1,1} & &
\end{array}$$

4.1 The game $G_{1,3}(f)$

Let $A \subseteq {}^{\omega}\omega$ and $f : A \to {}^{\omega}\omega$. As in the tree game from Chapter 2, Player I plays elements $x_i \in \omega$ and Player II plays functions $\phi_i : T_i \to {}^{<\omega}\omega$ such that $T_i \subset {}^{<\omega}\omega$ is a finite tree, ϕ_i is monotone and length-preserving, and $i < j \Rightarrow \phi_i \subseteq \phi_j$. After ω rounds, Player I produces $x = \langle x_0, x_1, \dots \rangle \in {}^{\omega}\omega$ and Player II produces $\phi = \bigcup_i \phi_i$.

$$\begin{array}{lccccccl}
\text{I:} & x_0 & & x_1 & & x_2 & & x = \langle x_0, x_1, \dots \rangle \\
 & & & & & & \cdots & \\
\text{II:} & & \phi_0 & & \phi_1 & & \phi_2 & \phi = \bigcup_i \phi_i
\end{array}$$

Player II wins the game if either $x \notin A$ or if $\operatorname{dom}(\phi)$ has a unique infinite branch z, $\operatorname{dom}(\phi)[\, s\,]$ is infinite $\Rightarrow s \subset z$, and

$$\bigcup_{s \subset z} \phi(s) = f(x).$$

This game is exactly the same as the tree game except for the extra requirement that $\mathrm{dom}(\phi)[\,s\,]$ is infinite $\Rightarrow s \subset z$. Alternatively, this requirement may be stated as follows: in the tree $\mathrm{dom}(\phi)$, any node that is not an initial segment of the infinite branch may only be extended finitely many times. Equivalently, for $s \subset z$, there may be infinitely many k such that $s^\frown k \in \mathrm{dom}(\phi)$, but $\mathrm{dom}(\phi)[\,s^\frown k\,]$ is finite for every $k \neq z(\mathrm{lh}(s))$.

We define the set MOVES, the notion of a strategy, z_x and ϕ_x as in the definition of the tree game. In the game $G_{1,3}(f)$, a strategy τ is **winning** for Player II if for all $x \in A$, $\mathrm{dom}(\phi_x)$ has a unique infinite branch z_x, $\mathrm{dom}(\phi_x)[\,s\,]$ is infinite $\Rightarrow s \subset z_x$, and

$$\bigcup_{s \subset z_x} \phi_x(s) = f(x).$$

4.1.1. Theorem. *A function $f : A \to {}^\omega\omega$ is Baire class 2 iff Player II has a winning strategy in $G_{1,3}(f)$.*

Proof. $\Rightarrow$: As in the proof of Theorem 2.0.9, we will define a winning strategy for Player II by defining *guessing functions*. Let $f_n : A \to {}^\omega\omega$ such that $f = \lim_{n\to\infty} f_n$ and f_n is Baire class 1. By Theorem 3.2.1, there is a winning strategy τ_n for Player II in $G_{\mathrm{e}}(f_n)$. Let $T_{n,x}$ be the tree produced by τ_n on input x:

$$T_{n,x} := \bigcup_{p \subset x} \tau_n(p).$$

Note that $T_{n,x} \subset {}^{<\omega}\omega$ is a finitely branching tree whose unique infinite branch is $f_n(x)$, by the definition of the game G_{e}.

We proceed by defining guessing functions $\rho_0 : {}^{<\omega}\omega \to {}^{<\omega}\omega$, $\rho_1 : {}^{<\omega}\omega \to \omega$, and $\rho_2 : {}^{<\omega}\omega \to \omega$ satisfying:

- $\mathrm{lh}(\rho_0(s)) = \mathrm{lh}(s)$,
- $s \subset u \Rightarrow \rho_0(s) \subset \rho_0(u)$, and
- $s \subset u \Rightarrow \rho_1(s) \leq \rho_1(u)$.

Let $x \in A$ be an infinite play of Player I. The sequence $\rho_0(s)$ will be a guess for $f(x) \upharpoonright \mathrm{lh}(s)$, the natural number $\rho_1(s)$ will be a guess for the least N such that $f_n(x) \upharpoonright \mathrm{lh}(s) = f_N(x) \upharpoonright \mathrm{lh}(s)$ for all $n \geq N$, and the natural number $\rho_2(s)$ will be a guess for $\mathrm{card}(T_{\rho_1(s)-1,x}[\,\rho_0(s)\,])$. (If $\rho_1(s) = 0$, then we let $\rho_2(s) := 0$.)

We define the guessing functions as follows. Let $\rho_0(\varnothing) = \varnothing$ and $\rho_1(\varnothing) = \rho_2(\varnothing) = 0$. If ρ_0, ρ_1 and ρ_2 are defined at $s \in {}^{<\omega}\omega$, let $\langle \rho_0(s^\frown k), \rho_1(s^\frown k), \rho_2(s^\frown k)\rangle$ enumerate all triples $\langle t, r, m\rangle \in {}^{<\omega}\omega \times \omega \times \omega$ with $\rho_0(s) \subset t$, $\mathrm{lh}(t) = \mathrm{lh}(\rho_0(s)) + 1$, $r \geq \rho_1(s)$, and $m = 0$ if $r = 0$.

For $p \in {}^{<\omega}\omega$, let $S(p)$ be the set of $s \in {}^{\leq \mathrm{lh}(p)}\mathrm{lh}(p)$ such that for all $u \subseteq s$,

$$\rho_1(u) > 0 \Rightarrow \mathrm{card}(\tau_{\rho_1(u)-1}(p)[\,\rho_0(u)\,]) = \rho_2(u)$$

and for all n such that $\rho_1(u) \leq n \leq \max(\mathrm{lh}(s), \mathrm{ran}(s))$,

$$\mathrm{card}(\tau_n(p)[\,\rho_0(u)\,]) \geq \max(\mathrm{lh}(s), \mathrm{ran}(s)).$$

Define

$$\tau(p) : \bigcup_{q \subseteq p} S(q) \to {}^{<\omega}\omega,\ \tau(p)(s) := \rho_0(s).$$

It is not difficult to check that τ is a strategy. It remains to be shown that τ is winning for Player II in $G_{1,3}(f)$. Fix $x \in A$, let

$$\phi_x := \bigcup_{p \subset x} \tau(p),$$

and let $z_x \in {}^{\omega}\omega$ be the unique infinite sequence whose encoded guesses are all correct. This means that the following holds for every $s \subset z_x$: $\rho_0(s) = f(x) \restriction \mathrm{lh}(s)$, $\rho_1(s)$ is the least N such that $f_n(x) \restriction \mathrm{lh}(s) = f_N(x) \restriction \mathrm{lh}(s)$ for all $n \geq N$, and $\rho_2(s)$ is the cardinality of $T_{\rho_1(s)-1,x}[\,\rho_0(s)\,]$ if $\rho_1(s) > 0$ and 0 otherwise.

Note that for every $s \subset z_x$, there exists a $p \subset x$ such that $s \in S(p)$. Namely, let $s \subset z_x$ and choose $p_0 \subset x$ such that for all $u \subseteq s$, $\rho_1(u) > 0 \Rightarrow \mathrm{card}(\tau_{\rho_1(u)-1}(p_0)[\,\rho_0(u)\,]) = \mathrm{card}(T_{\rho_1(u)-1,x}[\,\rho_0(u)\,])$. Such p_0 exists since $T_{\rho_1(u)-1,x}[\,\rho_0(u)\,]$ is finite for all such u. Now, for all $u \subseteq s$ and $n \geq \rho_1(u)$, $\mathrm{card}(\tau_n(p)[\,\rho_0(u)\,]) \to \infty$ as $p \to x$. Choose $p_1 \subset x$ such that $\mathrm{card}(\tau_n(p_1)[\,\rho_0(u)\,]) \geq \max(\mathrm{lh}(s), \mathrm{ran}(s))$ for all $u \subseteq s$ and n such that $\rho_1(u) \leq n \leq \max(\mathrm{lh}(s), \mathrm{ran}(s))$. Let $p_2 = x \restriction \max(\mathrm{lh}(s), \mathrm{ran}(s))$ and let $p = p_0 \cup p_1 \cup p_2$. It follows that $s \in S(p)$. This shows that z_x is an infinite branch of $\mathrm{dom}(\phi_x)$ and

$$\bigcup_{s \subset z_x} \phi_x(s) = f(x).$$

To finish the proof, we show that $u \not\subset z_x \Rightarrow \mathrm{dom}(\phi_x)[\,u\,]$ is finite. Note that if $u \not\subset z_x$ then there is a $v \subseteq u$ and an $i \in \{0, 1, 2\}$ such that the guess $\rho_i(v)$ is incorrect. In the case of $i = 0$, this implies that the guess $\rho_0(u)$ is incorrect since $\rho_0(v) \subseteq \rho_0(u)$.

Case A. The guess $\rho_0(u)$ is incorrect. Let $N \geq \rho_1(u)$ such that $T_{N,x}[\,\rho_0(u)\,]$ is finite. Such N exists since otherwise for all $n \geq \rho_1(u)$, we would have that $T_{n,x}[\,\rho_0(u)\,]$ is infinite and thus $f_n(x) \restriction \mathrm{lh}(u) = \rho_0(u)$. Let $m = \mathrm{card}(T_{N,x}[\,\rho_0(u)\,])$ and let $k = \max(m, N)$. Suppose $s \supseteq u$ such that $\max(\mathrm{lh}(s), \mathrm{ran}(s)) > k$. It follows that $s \notin S(p)$ for all $p \subset x$. Namely, for any $p \subset x$, we have that $\rho_1(u) \leq N \leq \max(\mathrm{lh}(s), \mathrm{ran}(s))$ but $\mathrm{card}(\tau_N(p)[\,\rho_0(u)\,]) \leq \mathrm{card}(T_{N,x}[\,\rho_0(u)\,]) < \max(\mathrm{lh}(s), \mathrm{ran}(s))$. It follows that $\mathrm{dom}(\phi_x)[\,u\,]$ is finite.

Case B. The guess $\rho_0(u)$ is correct, but there is a $v \subseteq u$ such that the guess $\rho_1(v)$ is incorrect. If $\rho_1(v)$ is too small, then let $N \geq \rho_1(v)$ such that $T_{N,x}[\,\rho_0(v)\,]$ is finite and argue as in Case A. If $\rho_1(v)$ is too large, then $T_{\rho_1(v)-1,x}[\,\rho_0(v)\,]$ is infinite. Let $p \subset x$ such that

$$\mathrm{card}(\tau_{\rho_1(v)-1}(p)[\,\rho_0(v)\,]) > \rho_2(v).$$

It follows that $s \in S[q]$ implies $v \not\subseteq s$ for all $q \supseteq p$ with $q \subset x$, and thus $\mathrm{dom}(\phi_x)[u]$ is finite.

Case C. Case A and Case B do not hold, but there is a $v \subseteq u$ such that the guess $\rho_2(v)$ is incorrect. If $\rho_2(v)$ is too small then let $p \subset x$ such that

$$\mathrm{card}(\tau_{\rho_1(v)-1}(p)[\rho_0(v)]) > \rho_2(v).$$

It follows that $s \in S[q] \Rightarrow v \not\subseteq s$ for all $q \supseteq p$ with $q \subset x$, and thus $\mathrm{dom}(\phi_x)[u]$ is finite. If $\rho_2(v)$ is too large then $u \not\subseteq s$ for all $s \in \mathrm{dom}(\phi_x)$.

$\Leftarrow$: Let τ be winning for Player II in $G_{1,3}(f)$ and define ϕ_x and z_x for $x \in A$ as earlier. For $x \in A$ and $n \in \omega$, let s^n_x be the least sequence s of length n in the lexicographic ordering $<_{\mathrm{lex}}$ of ${}^{<\omega}\omega$ such that

$$\mathrm{card}(\{u \in \mathrm{dom}(\phi_x) : u \supseteq s\}) \geq n.$$

Define $f_n(x) = \phi_x(s^n_x)^\frown 0^*$. We claim that the functions f_n are Baire class 1 (in fact, $\mathbf{\Lambda}_{2,2}$) and $f = \lim_{n\to\infty} f_n$. Note that $f_0 \in \mathbf{\Lambda}_{2,2}$ trivially.

Fix $n > 0$. We will define a backtrack strategy τ_{bt} that is winning for Player II in $G_{\mathrm{bt}}(f_n)$. We will use a guessing function $\rho : \omega \to {}^n\omega$, where the sequence $\rho(m)$ is a guess for s^n_x. To define ρ, take any bijection $\omega \to {}^n\omega$.

Let $p \in {}^{<\omega}\omega$ and let s be the least sequence of length n in the lexicographic ordering such that

$$\mathrm{card}(\{u \in \mathrm{dom}(\tau(p)) : u \supseteq s\}) \geq n$$

if such a sequence exists and $\varnothing$ otherwise. If s is non-empty then let $m = \rho^{-1}(s)$. Let

$$B(p) := \begin{cases} \{\langle m, \tau(p)(s)^\frown 0^{\mathrm{lh}(p)}\rangle\} \text{ if } s \neq \varnothing, \\ \varnothing \text{ otherwise.} \end{cases}$$

Define $\tau_{\mathrm{bt}}(p) : \bigcup\{\mathrm{dom}(B(q)) : q \subseteq p\} \to {}^{<\omega}\omega$,

$$\tau_{\mathrm{bt}}(p)(n) := \bigcup\{B(q)(n) : q \subseteq p \text{ and } n \in \mathrm{dom}(B(q))\}.$$

It is easy to check that the backtrack strategy τ_{bt} is winning for Player II in $G_{\mathrm{bt}}(f_n)$.

It remains to be shown that $f = \lim_{n\to\infty} f_n$. Suppose $t \subset f(x)$ and let $s = z_x \restriction \mathrm{lh}(t)$. It suffices to show that there is an N such that $s^n_x \supseteq s$ for all $n \geq N$. We may assume that s is non-empty as otherwise the statement is trivial. For $i < \mathrm{lh}(s)$, let $L_i = \{(s \restriction i)^\frown k : k < s(i)\}$ and let $N_i \in \omega$ such that for all $u \in L_i$,

$$\mathrm{card}(\{v \in \mathrm{dom}(\phi_x) : v \supseteq u\}) \leq N_i.$$

Note that such N_i exists because τ is winning for Player II in $G_{1,3}(f)$ and every $u \in L_i$ is not an initial segment of the infinite branch z_x. Also note that any $u <_{\mathrm{lex}} s$ must have some element of one of the L_i's as an initial segment. Let

$$N = \sup\left(\{N_i + 1 : i < \mathrm{lh}(s)\} \cup \{\mathrm{lh}(s)\}\right).$$

It follows that $s^n_x \supseteq s$ for all $n \geq N$. Namely, let $n \geq N$ and consider $z_x \upharpoonright n$. Since the cardinality of $\{v \in \mathrm{dom}(\phi_x) : v \supseteq z_x \upharpoonright n\}$ is infinite, it follows that $s^n_x \leq_{\mathrm{lex}} z_x \upharpoonright n$. By choice of N, $s^n_x \upharpoonright \mathrm{lh}(s)$ cannot extend any element of any of the L_i's, so $s^n_x \upharpoonright \mathrm{lh}(s) \geq_{\mathrm{lex}} s$. But if $s^n_x \upharpoonright \mathrm{lh}(s) >_{\mathrm{lex}} s$ then we would have that $s^n_x >_{\mathrm{lex}} z_x \upharpoonright n$, a contradiction. It follows that $s^n_x \supseteq s$ and thus $f = \lim_{n\to\infty} f_n$.

□

4.2 The game $G_{2,3}(f)$

Let $A \subseteq {}^\omega\omega$ and $f : A \to {}^\omega\omega$. In the game $G_{2,3}(f)$, Player I plays elements $x_i \in \omega$ and Player II plays functions $\phi_i : D_i \to \mathcal{P}({}^{<\omega}\omega)$ such that $D_i \subset \omega$ is finite and $\phi_i(n)$ is a finite tree. Player II is subject to the requirements that $i < j \Rightarrow D_i \subseteq D_j$ and $\phi_i(n) \subseteq \phi_j(n)$ for all $n \in \mathrm{dom}(\phi_i)$. After ω rounds, Player I produces $x = \langle x_0, x_1, \ldots\rangle \in {}^\omega\omega$ and Player II produces $\phi : D_\omega \to \mathcal{P}({}^{<\omega}\omega)$,

$$\phi(n) := \bigcup\{\phi_i(n) : i \in \omega \text{ and } n \in \mathrm{dom}(\phi_i)\},$$

where $D_\omega := \bigcup_i D_i$.

I:	x_0		x_1		x_2		$x = \langle x_0, x_1, \ldots\rangle$
						$\ldots$	
II:		ϕ_0		ϕ_1		ϕ_2	ϕ as above

Player II wins the game if either $x \notin A$ or if there is a unique $n \in D_\omega$ such that $\phi(n)$ is infinite (so $\phi(n')$ is finite for all $n' \in D_\omega$ such that $n' \neq n$), $\phi(n)$ is finitely branching, and $f(x)$ is the unique infinite branch of $\phi(n)$. Informally, we think of the domain of ϕ as consisting of countably many *rows*. As the game progresses, Player II builds trees on finitely many of these rows. In the limit, Player II may use infinitely many rows but may only play an infinite tree on one of them. If Player I plays $x \in A$, then Player II wins if and only if this tree is finitely branching and $f(x)$ is its unique infinite branch.

Let MOVES be the set of functions $\psi : D \to \mathcal{P}({}^{<\omega}\omega)$ such that $D \subset \omega$ is finite and $\psi(n)$ is a finite tree. A $\mathbf{\Lambda_{2,3}}$ **strategy** for Player II is a function $\tau : {}^{<\omega}\omega \to \mathsf{MOVES}$ such that $p \subset q \Rightarrow \mathrm{dom}(\tau(p)) \subseteq \mathrm{dom}(\tau(q))$ and $\tau(p)(n) \subseteq \tau(q)(n)$ for all $n \in \mathrm{dom}(\tau(p))$. If $x \in A$ and τ is a $\mathbf{\Lambda}_{2,3}$ strategy for Player II, let D_x be the set of $n \in \omega$ such that $n \in \mathrm{dom}(\tau(p))$ for some $p \subset x$ and let $\phi_x : D_x \to \mathcal{P}({}^{<\omega}\omega)$,

$$\phi_x(n) = \bigcup\{\tau(p)(n) : p \subset x \text{ and } n \in \mathrm{dom}(\tau(p))\}.$$

A $\mathbf{\Lambda}_{2,3}$ strategy τ is **winning** for Player II in $G_{2,3}(f)$ if for all $x \in A$, there is a unique $n \in D_x$ such that $\phi_x(n)$ is infinite (so $\phi_x(n')$ is finite for all $n' \in D_x$ such that $n' \neq n$), $\phi_x(n)$ is finitely branching, and $f(x)$ is the unique infinite branch of $\phi_x(n)$. We will sometimes denote the output row n by o_x.

4.2.1. THEOREM. *A function $f : A \to {}^\omega\omega$ admits a $\mathbf{\Pi}^0_2$ partition $\langle A_n : n \in \omega\rangle$ such that $f \restriction A_n$ is Baire class 1 iff Player II has a winning strategy in $G_{2,3}(f)$.*

Proof. $\Rightarrow$: Let A_n be the partition and τ_n be a winning strategy for Player II in $G_\mathrm{e}(f \restriction A_n)$. Let $B_{n,m} \subseteq A$ be open in A such that $A_n = \bigcap_m B_{n,m}$. For $p \in {}^{<\omega}\omega$, let

$$\gamma_n(p) = \sup\{m : [\,p\,] \cap A \subseteq B_{n,i} \text{ for all } i \leq m\}.$$

Note that $\gamma_n(p)$ may be a natural number or may be ω. Also note that $p \subset q \Rightarrow \gamma_n(p) \leq \gamma_n(q)$ and that for any $x \in A$, there is a unique $n \in \omega$ such that $\lim_{p\to x} \gamma_n(p) = \infty$. Define $\tau(p) : \mathrm{lh}(p) \to \mathsf{MOVES}$,

$$\tau(p)(n) = \tau_n(p \restriction \gamma_n(p)).$$

It is easy to check that τ is a $\mathbf{\Lambda}_{2,3}$ strategy. We claim that τ is winning in $G_{2,3}(f)$. Let $x \in A$, n such that $x \in A_n$, and let ϕ_x be defined as in the previous section. It follows that n is unique such that $\phi_x(n)$ is infinite. Moreover, it is easy to see that

$$\phi_x(n) = \bigcup_{p \subset x} \tau_n(p).$$

It follows that $\phi_n(x)$ is finitely branching and $f(x)$ is the unique infinite branch of $\phi_x(n)$, since τ_n is winning in $G_\mathrm{e}(f \restriction A_n)$.

$\Leftarrow$: Let τ be the winning strategy for Player II in $G_{2,3}(f)$. For $x \in A$, let ϕ_x and D_x be defined as in the previous section, and let o_x denote the output row of τ on input x. Define

$$A_n := \{x \in A : o_x = n\}.$$

The eraser strategy τ_n defined by

$$\tau_n(p) = \begin{cases} \tau(p)(n) \text{ if } n \in \mathrm{dom}(\tau(p)), \\ \varnothing \text{ otherwise} \end{cases}$$

is winning for Player II in $G_\mathrm{e}(f \restriction A_n)$. Furthermore, it is easy to check that the sets A_n are $\mathbf{\Pi}^0_2$ in A, completing the proof. $\square$

4.3 Decomposing $\mathbf{\Lambda}_{2,3}$

In this section, we proceed with the main goal of this chapter, to prove Theorem 4.3.7.

4.3.1. LEMMA. *Suppose $A \subseteq {}^\omega\omega$, $h : A \to {}^\omega\omega$, and that h is Baire class 2. Let $t_1, t_2 \in {}^{<\omega}\omega$ such that $t_1 \perp t_2$. If Player II has a winning strategy in $G_{2,3}(h \restriction h^{-1}[\,[\,t_1\,]^c])$ and a winning strategy in $G_{2,3}(h \restriction h^{-1}[\,[\,t_2\,]^c])$ then Player II has a winning strategy in $G_{2,3}(h)$.*

Proof. Since $[\,t_1\,] \subset [\,t_2\,]^c$, it follows that Player II has a winning strategy in $G_{2,3}(h \restriction h^{-1}[\,[\,t_1\,]\,])$. Let $B = h^{-1}[\,[\,t_1\,]\,]$ and $C = h^{-1}[\,[\,t_1\,]^c\,]$. It follows that $A = B \cup C$ and that B and C are $\mathbf{\Sigma}^0_3$ in A. The lemma follows from Theorem 4.2.1 and Lemma 1.1.5. A game-theoretic proof in the style of Lemma 3.4.1 is also possible, but we leave this to the reader. $\square$

4.3.2. LEMMA. *Suppose $f : {}^\omega\omega \to {}^\omega\omega$ and that $\tau_{1,3}$ is a winning strategy for Player II in $G_{1,3}(f)$. Let $s_1, s_2, t_1, t_2 \in {}^{<\omega}\omega$ such that* $\mathrm{lh}(s_1) = \mathrm{lh}(t_1)$, $\mathrm{lh}(s_2) = \mathrm{lh}(t_2)$, *and $t_1 \perp t_2$. On input $x \in {}^\omega\omega$, let ϕ_x be the function produced by $\tau_{1,3}$ and let z_x be the unique infinite branch of* $\mathrm{dom}(\phi_x)$. *Suppose $T \subseteq {}^{<\omega}\omega$ is a non-empty tree, $p \in T$, and for all $q \supseteq p$ such that $q \in T$,*

$$\{x : s_1 \subset z_x \text{ and } t_1 \subset f(x)\} \cap [\,T[\,q\,]\,] \neq \varnothing.$$

Then there is a $q \supseteq p$ such that $q \in T$ and

$$\{x : s_2 \subset z_x \text{ and } t_2 \subset f(x)\} \cap [\,T[\,q\,]\,] = \varnothing.$$

Proof. If s_1 is compatible with s_2, then let $x \in [\,T[\,p\,]\,]$ such that $s_1 \subset z_x$ and $t_1 \subset f(x)$. Let $q \supseteq p$ such that $q \subset x$ and $\langle s_1, t_1\rangle \in \tau_{1,3}(q)$. Such q exists since $\tau_{1,3}$ is winning for Player II in $G_{,1,3}(f)$. It follows that $\langle s_2, t_2\rangle \notin \tau_{1,3}(r)$ for all $r \supseteq q$ since $t_1 \perp t_2$.

If $s_1 \perp s_2$ then suppose for contradiction that the conclusion of the lemma does not hold. Let $p_0 = p$ and suppose $p_n \in T$ has been defined. If n is even, let $p_{n+1} \supset p_n$ such that $p_{n+1} \in T$ and

$$\mathrm{card}(\mathrm{dom}(\tau_{1,3}(p_{n+1}))[\,s_1\,]) > \mathrm{card}(\mathrm{dom}(\tau_{1,3}(p_n))[\,s_1\,]).$$

If n is odd, let $p_{n+1} \supset p_n$ such that $p_{n+1} \in T$ and

$$\mathrm{card}(\mathrm{dom}(\tau_{1,3}(p_{n+1}))[\,s_2\,]) > \mathrm{card}(\mathrm{dom}(\tau_{1,3}(p_n))[\,s_2\,]).$$

Let $x = \bigcup p_n$. It follows that both $\mathrm{dom}(\phi_x)[\,s_1\,]$ and $\mathrm{dom}(\phi_x)[\,s_2\,]$ are infinite. Since $\tau_{1,3}$ is winning for Player II in $G_{1,3}(f)$, it follows $s_1 \subset z_x$ and $s_2 \subset z_x$. This is a contradiction since s_1 and s_2 are incompatible. $\square$

The following lemma is an analogue of Lemma 3.4.3.

4.3.3. LEMMA. *Suppose $A \subseteq {}^\omega\omega$, $h : A \to {}^\omega\omega$, and that $\tau_{1,3}$ is a winning strategy for Player II in $G_{1,3}(h)$. On input $x \in A$, let ϕ_x be the function produced by $\tau_{1,3}$ and let z_x be the unique infinite branch of* $\mathrm{dom}(\phi_x)$. *If Player II does not have a winning strategy in $G_{2,3}(h)$, then there is a non-empty tree $T \subseteq {}^{<\omega}\omega$ and $s, t \in {}^{<\omega}\omega$ such that* $\mathrm{lh}(s) = \mathrm{lh}(t)$ *and for every $p \in T$, Player II does not have a winning strategy in*

$$G_{2,3}(h \restriction (h^{-1}[\,[\,t\,]^c\,] \cap [\,T[\,p\,]\,]))$$

and

$$\{x \in A : s \subset z_x \textit{ and } t \subset h(x)\} \cap [T[p]] \neq \varnothing.$$

Proof. By contradiction. We assume that the conclusion of the Lemma does not hold and give a winning strategy for Player II in $G_{2,3}(h)$. For each $s, t \in {}^{<\omega}\omega$ with $\mathrm{lh}(s) = \mathrm{lh}(t)$, we will define by transfinite recursion a $\subseteq$-decreasing sequence of trees $\langle T_\alpha : \alpha \leq \gamma \rangle$ for some $\gamma < \omega_1$. We will think of this sequence as an attempt to find the T in the conclusion of the lemma. By assumption, all such attempts will fail, and we will use this fact to define a winning strategy τ for Player II in $G_{2,3}(h)$.

Fix $s, t \in {}^{<\omega}\omega$ with $\mathrm{lh}(s) = \mathrm{lh}(t)$. To define the transfinite sequence of trees we will use two operations, Ξ and Ω. For a tree $T \subseteq {}^{<\omega}\omega$, let $\Xi(T)$ be the set of $p \in T$ such that Player II does not have a winning strategy in

$$G_{2,3}(h \upharpoonright (h^{-1}[[t]^c] \cap [T[p]])),$$

and let $\Omega(T)$ be the set of $p \in T$ such that

$$\{x \in A : s \subset z_x \text{ and } t \subset h(x)\} \cap [T[p]] \neq \varnothing.$$

It is immediate that $\Xi(T)$ and $\Omega(T)$ are trees, $\Xi(\Xi(T)) = \Xi(T)$, and $\Omega(\Omega(T)) = \Omega(T)$. Define

$$\begin{aligned} T^0 &:= \Omega({}^{<\omega}\omega), \\ T^{\alpha+1} &:= \Xi(T^\alpha) \ (\alpha \text{ even}), \\ T^{\alpha+1} &:= \Omega(T^\alpha) \ (\alpha \text{ odd}), \\ T^\lambda &:= \Omega(\bigcap_{\alpha<\lambda} T^\alpha) \ (\lambda \text{ limit}). \end{aligned}$$

Since the T^α's are $\subseteq$-decreasing subsets of ${}^{<\omega}\omega$, we may let $\gamma < \omega_1$ be the least ordinal such that $T^\gamma = T^{\gamma+1}$. If γ is odd, then $T^\gamma = \Xi(T)$ for some T and $T^{\gamma+1} = \Omega(T^\gamma) = T^\gamma$. Since $\Xi(\Xi(T)) = \Xi(T)$, it follows that $\Xi(T^\gamma) = T^\gamma$. If $T^\gamma \neq \varnothing$, then it would satisfy the requirements for T in the conclusion of the lemma, so $T^\gamma = \varnothing$. Similarly, if γ is odd, then $\Omega(T^\gamma) = \Xi(T^\gamma) = T^\gamma$ and $T^\gamma = \varnothing$. We may carry out this procedure for any s and t with $\mathrm{lh}(s) = \mathrm{lh}(t)$. For this, we use the notation $\langle T^\alpha_{s,t} : \alpha \leq \gamma_{s,t} \rangle$, $\Xi_{s,t}$, and $\Omega_{s,t}$.

For $p \in {}^{<\omega}\omega$, define $\iota_{s,t}(p)$ to be the least α such that $p \notin T^\alpha_{s,t}$. It is immediate that $s \in \mathrm{dom}(\tau_{1,3}(p))$ and $\tau_{1,3}(p)(s) \neq t$ implies $\iota_{s,t}(p) = 0$. To simplify the notation, for $s \in \mathrm{dom}(\tau_{1,3}(p))$ and $t = \tau_{1,3}(p)(s)$, let $\iota_s(p) := \iota_{s,t}(p)$. Note that $s \in \mathrm{dom}(\tau_{1,3}(p))$ and $p \subseteq q \Rightarrow \iota_s(p) \geq \iota_s(q)$. It follows that for any $s \in \mathrm{dom}(\phi_x)$, $\iota_s(p)$ must converge to some ordinal as $p \to x$, since otherwise there would be an infinite descending sequence of ordinals. So, for any infinite play x of Player I, there is an N such that for all $n \geq N$, $\iota_s(x \upharpoonright n) = \iota_s(x \upharpoonright N)$. Extending the ι_s notation to infinite sequences, let $\iota_s(x) := \iota_s(x \upharpoonright N)$.

In general, we are interested in whether $\iota_s(x)$ is even or odd. Suppose, for example, that $\iota_s(x)$ is an even successor ordinal $\alpha + 1$. This means that $x \in [\,T_\alpha\,] \setminus [\,\Omega(T_\alpha)\,]$. In this run of the game, s may be pruned from the domain of the function produced by $\tau_{1,3}$, since the infinite branch will not extend s by the definition of Ω. Similarly, if $\iota_s(x)$ is an odd ordinal $\alpha+1$, then $x \in [\,T_\alpha\,]\setminus[\,\Xi(T_\alpha)\,]$. In this case, we may use the fact that Player II has a winning strategy in

$$G_{2,3}(h \upharpoonright (h^{-1}[\,[\,t\,]^c\,] \cap [\,T_\alpha\,] \setminus [\,\Xi(T_\alpha)\,])).$$

We proceed by defining a winning strategy for Player II in $G_{2,3}(h)$. For each $s \in \mathrm{dom}(\phi_x)$, say that s is *green* if $\iota^s(x)$ is odd and *red* if $\iota^s(x)$ is even. Recall that limit ordinals are considered to be even. Note that every $s \subset z_x$ must be green, since by definition $s \not\subset z_x$ if s is red. For $x \in A$, there are two cases to consider:

Case A: $\phi_x(s) \subset h(x)$ for all green $s \in \mathrm{dom}(\phi_x)$,
Case B: there are green $s_1, s_2 \in \mathrm{dom}(\phi_x)$ such that
$\phi_x(s_1) \perp \phi_x(s_2)$.

To handle Case A, let $\prec$ be a well-ordering of ${}^{<\omega}\omega$ and fix $p \in {}^{<\omega}\omega$. Let $S(p)$ be the set of $s \in \mathrm{dom}(\tau_{1,3}(p))$ such that

- $\iota_s(p)$ is odd, and
- for all $u \prec s$, $u \in \mathrm{dom}(\tau_{1,3}(p))$ and $\iota_u(p)$ is odd $\Rightarrow$ $\tau_{1,3}(p)(u)$ is compatible with $\tau_{1,3}(p)(s)$.

Let

$$E(p) := \bigcup_{s\,\in\, S(p)} \tau_{1,3}(p)(s).$$

It is easy to check that $E(p) \in {}^{<\omega}\omega$. Let

$$\tau_{\mathrm{A}}(p) := \mathrm{tree}(\{E(q) : q \subseteq p\}).$$

If Case A holds, then $h(x)$ is the unique infinite branch of the finitely branching tree

$$T_x := \bigcup_{p\,\subset\, x} \tau_{\mathrm{A}}(p).$$

Namely, let $t \subset h(x)$ and let $s = z_x \upharpoonright \mathrm{lh}(t)$. Let

$$U := \{u \prec s : u \in \mathrm{dom}(\phi_x) \text{ and } \phi_x(u) \perp \phi_x(s)\}.$$

It follows that U is finite and every $u \in U$ is red. Let $V = U \cup \{s\}$ and let $p \subset x$ such that $V \subseteq \mathrm{dom}(\tau_{1,3}(p))$ and $\iota_v(q) = \iota_s(p)$ for every $v \in V$ and every q,

$p \subseteq q \subset x$. It follows that $E(q) \supseteq t$ for all $q \supseteq p$. If Case A does not hold, then it is easy to check that T_x is finite.

To handle Case B, let $\gamma := \sup\{\gamma_{s,t} : s, t \in {}^{<\omega}\omega \text{ and } \mathrm{lh}(s) = \mathrm{lh}(t)\}$. Note that γ is a countable ordinal by the regularity of ω_1. We proceed by defining guessing functions

$$\begin{aligned}
&\rho_0 : \omega \to {}^{<\omega}\omega,\\
&\rho_1 : \omega \to {}^{<\omega}\omega,\\
&\rho_2 : \omega \to \gamma,\\
&\rho_3 : \omega \to {}^{<\omega}\omega,\\
&\rho_4 : \omega \to {}^{<\omega}\omega, \text{ and}\\
&\rho_5 : \omega \to \gamma.
\end{aligned}$$

Let $\langle \rho_i(m) : i < 6\rangle$ enumerate all sextuples $\langle s_1, t_1, \alpha_1, s_2, t_2, \alpha_2\rangle$ such that $\mathrm{lh}(s_1) = \mathrm{lh}(t_1)$, $\mathrm{lh}(s_2) = \mathrm{lh}(t_2)$, $s_1 \perp s_2$, $t_1 \perp t_2$, $\alpha_1 < \gamma^{s_1,t_1}$, $\alpha_2 < \gamma^{s_2,t_2}$, and α_1 and α_2 are both even. For each $m \in \omega$, the sextuple $\langle \rho_i(m) : i < 6\rangle = \langle s_1, t_1, \alpha_1, s_2, t_2, \alpha_2\rangle$ encodes guesses that $s_1, s_2 \in \mathrm{dom}(\phi_x)$, $\phi_x(s_1) = t_1$, $\phi_x(s_2) = t_2$, $\iota_{s_1}(x) = \alpha_1 + 1$, and $\iota_{s_2}(x) = \alpha_2 + 1$. Since we are in Case B, there is an m whose encoded guesses are correct. The $\mathbf{\Lambda}_{2,3}$ strategy we define will use the least such m to compute $h(x)$.

Fix $m \in \omega$ and suppose $\langle \rho_i(m) : i < 6\rangle = \langle s_1, t_1, \alpha_1, s_2, t_2, \alpha_2\rangle$. For $j \in \{1, 2\}$, let

$$A_j := [\,T^{\alpha_j}_{s_j,t_j}\,] \setminus [\,T^{\alpha_j+1}_{s_j,t_j}\,].$$

It follows that Player II has a winning strategy in

$$G_{2,3}(h \restriction (h^{-1}[\,[\,t_j\,]^c\,] \cap A_j))$$

for both j. Letting $g := h \restriction (A_1 \cap A_2)$, it follows that Player II has a winning strategy in

$$G_{2,3}(g \restriction (g^{-1}[\,[\,t_j\,]^c\,])$$

for both j. By Lemma 4.3.1 applied to g, let π_m be a winning strategy for Player II in $G_{2,3}(g)$.

Now, fix $p \in {}^{<\omega}\omega$. Let $m \in \omega$ be least, if it exists, such that $\tau_{1,3}(p)(s_j) = t_j$ and

$$p \in T^{\alpha_j}_{s_j,t_j} \setminus T^{\alpha_j+1}_{s_j,t_j},$$

where $\langle \rho_i(m) : i < 6\rangle = \langle s_1, t_1, \alpha_1, s_2, t_2, \alpha_2\rangle$ and $j \in \{1, 2\}$. Let $\ulcorner\cdot,\cdot\urcorner : \omega \times \omega \to \omega$ be a bijection and let

$$M(p) := \{\langle \ulcorner m, n\urcorner, \pi_m(p)(n)\rangle : n \in \mathrm{dom}(\pi_m(p))\}$$

if such m exists and $\varnothing$ otherwise.

Define

$$\tau_{\mathrm{B}}(p) := \bigcup\{M(q)(k) : q \subseteq p \text{ and } k \in \mathrm{dom}(M(q))\}.$$

It is easy to check that τ_{B} is a $\mathbf{\Lambda}_{2,3}$ strategy. For $x \in A$, let D_x be the set of $k \in \omega$ such that $k \in \mathrm{dom}(\tau_{\mathrm{B}}(p))$ for some $p \subset x$ and let $\psi_x : D_x \to \mathcal{P}({}^{<\omega}\omega)$,

$$\psi_x(k) := \bigcup\{\tau_{\mathrm{B}}(p)(k) : p \subset x \text{ and } k \in \mathrm{dom}(\tau_{\mathrm{B}}(p))\}.$$

Suppose that Case B holds. Let m be least such that the guesses encoded by m are correct and let n be the output row of π_m on input x. It follows that $\psi_x(\ulcorner m, n \urcorner)$ is a finitely branching tree whose unique infinite branch is $h(x)$, and $\psi_x(k')$ is finite for all $k' \neq \ulcorner m, n \urcorner$. If Case B does not hold, then $\psi_x(k)$ is finite for all $k \in D_x$. This completes the setup to handle Case B.

To complete the proof, define

$$\begin{aligned}\tau(p) := \{&\langle 0, \tau_{\mathrm{A}}(p)\rangle\} \cup \\ &\{\langle n+1, T\rangle : \langle n, T\rangle \in \tau_{\mathrm{B}}(p)\}.\end{aligned}$$

The strategy τ is winning for Player II in $G_{2,3}(h)$. □

In the following, we fix $f : {}^{\omega}\omega \to {}^{\omega}\omega$ and suppose that Player II has a winning strategy in $G_{1,3}(f)$. Let δ be a (possibly empty) finite sequence of trees $\langle T_0, \dots, T_k\rangle$ with $T_i \subseteq {}^{<\omega}\omega$ and $T_0 \supseteq \cdots \supseteq T_k$. Let $\sigma \subseteq {}^{\omega}\omega$. If $\delta = \varnothing$, then say that $p \in {}^{<\omega}\omega$ is **$\boldsymbol{\delta}$-$\boldsymbol{\sigma}$-good**. If $\delta = \langle T_0, \dots, T_k\rangle$ and $p \in T_k$, then p is **$\boldsymbol{\delta}$-$\boldsymbol{\sigma}$-good** if for all $q \supseteq p$ with $q \in T_k$, Player II does not have a winning strategy in

$$G_{2,3}(f \restriction (f^{-1}[\,\sigma\,] \cap [\,T_k[\,q\,]\,]))$$

and there is an $r \supseteq q$ such that r is pred(δ)-σ-good. (Recall that $\mathrm{pred}(s) := s \restriction \mathrm{lh}(s) - 1$ for non-empty finite sequences s.) Note that if p is δ-σ-good and $\delta = \langle T_0, \dots, T_k\rangle$, the definition requires that $p \in T_k$.

4.3.4. Proposition. *Suppose $\delta = \langle T_0, \dots, T_k\rangle$ and p is δ-σ-good. Then q is δ-σ-good for all $q \supseteq p$ with $q \in T_k$.*

4.3.5. Proposition. *Suppose $\delta = \langle T_0, \dots, T_k\rangle$, $\sigma \subseteq {}^{\omega}\omega$, and $p \in T_k$ is δ-σ-good. Then for any $i < k+1$, there exists $q \supseteq p$ such that q is $(\delta \restriction i)$-σ-good.*

4.3.6. Lemma. *Let $\delta = \langle T_0, \dots, T_k\rangle$, $\sigma \subseteq {}^{\omega}\omega$, and let $\langle t_0, \dots, t_m\rangle$ be a sequence of pairwise incompatible elements of ${}^{<\omega}\omega$. If p is δ-σ-good, then*

$$\{i \leq m : \textit{no } q \supseteq p \textit{ is } \delta\text{-}(\sigma \setminus [\,t_i\,])\text{-}\textit{good}\}$$

has at most $k+1$ elements.

Proof. Proof by induction on k. For the base case $k = 0$, suppose $\delta = \langle T_0 \rangle$ and p is δ-σ-good. If p is δ-$(\sigma \setminus [\,t_i\,])$-good for each $i \leq m$, then there is nothing to prove by Proposition 4.3.4. Otherwise, there is an $i \leq m$ such that p is not δ-$(\sigma \setminus [\,t_i\,])$-good. Let $q \supseteq p$ with $q \in T_0$ such that Player II has a winning strategy in

$$G_{2,3}(f \restriction (f^{-1}[\,\sigma \setminus [\,t_i\,]\,] \cap [\,T_0[\,q\,]\,])).$$

Since q is δ-σ-good, for any $r \supseteq q$ with $r \in T_0$, Player II does not have a winning strategy in

$$G_{2,3}(f \restriction (f^{-1}[\,\sigma\,] \cap [\,T_0[\,r\,]\,])).$$

Let $j \leq m$ with $j \neq i$ and let $r \supseteq q$ with $r \in T_0$. By Lemma 4.3.1, Player II does not have a winning strategy in

$$G_{2,3}(f \restriction (f^{-1}[\,\sigma \setminus [\,t_j\,]\,] \cap [\,T_0[\,r\,]\,])).$$

Therefore, q is δ-$(\sigma \setminus [\,u_j\,])$-good.

For the inductive step, let $\delta = \langle T_0, \ldots, T_{k+1} \rangle$ and suppose p is δ-σ-good. Suppose w.l.o.g. that there is an $i \leq m$ and a $q \supseteq p$ with $q \in T_{k+1}$ such that Player II has a winning strategy in

$$G_{2,3}(f \restriction (f^{-1}[\,\sigma \setminus [\,t_i\,]\,] \cap [\,T_{k+1}[\,q\,]\,])).$$

As before, Player II does not have a winning strategy in

$$G_{2,3}(f \restriction (f^{-1}[\,\sigma \setminus [\,t_j\,]\,] \cap [\,T_{k+1}[\,r\,]\,]))$$

for any $j \leq m$ with $j \neq i$ and $r \supseteq q$ with $r \in T_{k+1}$. Suppose there are distinct $j_0, \ldots, j_k \leq m$ with $j_0, \ldots, j_k \neq i$ such that for any $j \in \{j_0, \ldots, j_k\}$, no $r \supseteq q$ is δ-$(\sigma \setminus [\,t_j\,])$-good. Let $l \leq m$ with $l \notin \{j_0, \ldots, j_k, i\}$. It will be shown that q is δ-$(\sigma \setminus [\,t_l\,])$-good, completing the proof. It suffices to show that for any $r \supseteq q$ with $r \in T_{k+1}$, there is an $s \supseteq r$ such that s is $\mathrm{pred}(\delta)$-$(\sigma \setminus [\,t_l\,])$-good. Let $r \supseteq q$ with $r \in T_{k+1}$. By choice of j_0, there is an $r_0 \supseteq r$ with $r_0 \in T_{k+1}$ such that no $s \supseteq r_0$ is $\mathrm{pred}(\delta)$-$(\sigma \setminus [\,u_{j_0}\,])$-good. Find $r_1 \supseteq r_0$, $r_2 \supseteq r_1$, $\ldots$, up to $r_k \supseteq r_{k-1}$ such that $r_i \in T_{k+1}$ and for any $j \in \{j_0, \ldots, j_k\}$, no $s \supseteq r_k$ is $\mathrm{pred}(\delta)$-$(\sigma \setminus [\,t_j\,])$-good. Since r_k is δ-σ-good, there is a $\mathrm{pred}(\delta)$-σ-good $s \supseteq r_k$. By the induction hypothesis, there is a $t \supseteq s$ such that t is $\mathrm{pred}(\delta)$-$(\sigma \setminus [\,u_l\,])$-good. $\square$

4.3.7. Theorem. *A function $f : {}^\omega\omega \to {}^\omega\omega$ is $\mathbf{\Lambda}_{2,3}$ $\Leftrightarrow$ there is a $\mathbf{\Pi}^0_2$ partition $\langle A_n : n \in \omega \rangle$ of ${}^\omega\omega$ such that $f \restriction A_n$ is Baire class 1.*

Proof. The $\Leftarrow$ direction is immediate by Proposition 1.1.4. For the $\Rightarrow$ direction, we assume for contradiction that there is no such partition A_n and show that $f \notin \mathbf{\Lambda}_{2,3}$. By Theorem 4.2.1, Player II does not have a winning strategy in $G_{2,3}(f)$. Since we wish to show that $f \notin \mathbf{\Lambda}_{2,3}$, we may assume that $f \in \mathbf{\Lambda}_{1,3}$, so there is a winning strategy $\tau_{1,3}$ for Player II in $G_{1,3}(f)$ by Theorem 4.1.1. For $x \in {}^\omega\omega$, let ϕ_x be the function produced by $\tau_{1,3}$ and let z_x be the unique infinite branch of $\mathrm{dom}(\phi_x)$. Let $\ulcorner\cdot,\cdot\urcorner$ be the bijection $\omega\times\omega\to\omega$:

$$\begin{aligned}\ulcorner 0,0\urcorner &:= 0,\\ \ulcorner 0,j+1\urcorner &:= \ulcorner j,0\urcorner + 1,\\ \ulcorner i+1,j-1\urcorner &:= \ulcorner i,j\urcorner + 1.\end{aligned}$$

Let

$$X := \{x \in {}^\omega 2 : \exists i\ \exists^\infty j\ (x(\ulcorner i,j\urcorner) = 1)\},$$

so X is $\mathbf{\Sigma}^0_3$-complete.

We will define an open set Y and a snake ψ_n such that the lifting of ψ_n is a reduction from X to $f^{-1}[Y]$. Define $\mathrm{row}(\ulcorner i,j\urcorner) := i$, so if $\mathrm{row}(k) = i$ then $\mathrm{row}(k+1) = i+1$ or $\mathrm{row}(k+1) = 0$. Let $\beta : \omega \to {}^{<\omega}2$ be the enumeration given by $\beta(0) := \varnothing$, $\beta(2n+1) := \beta(n)^\frown 0$, and $\beta(2n+2) := \beta(n)^\frown 1$. Let $\mathcal{D}$ be the set of sequences $\langle T_0,\ldots,T_k\rangle$ such that $T_i \subseteq {}^{<\omega}\omega$ is a tree and $T_0 \supseteq \cdots \supseteq T_k$. We will define by recursion

$$\begin{aligned}\psi_n &: \beta[2n+1] \to {}^{<\omega}\omega,\\ \delta_n &: \beta[2n+1] \to \mathcal{D},\\ \zeta_n &: \mathcal{T}_n \to {}^{<\omega}\omega, \text{ and}\\ \eta_n &: \mathcal{T}_n \to {}^{<\omega}\omega\end{aligned}$$

where $\mathcal{T}_n := \{\delta_n(p)(k) : p \in \beta[2n+1] \text{ and } k < \mathrm{lh}(\delta_n(p))\}$. So, $\mathcal{T}_n$ is the set of trees that occur in the sequences $\delta_n(p)$. The construction will satisfy $i < j \Rightarrow \delta_i \subseteq \delta_j \wedge \zeta_i \subseteq \zeta_j \wedge \eta_i \subseteq \eta_j$, and for all n and all $p \in \mathrm{tn}(\beta[2n+1])$,

- $\delta_n(p) \neq \varnothing$,
- $\mathrm{ran}(\eta_n)$ is an antichain,
- $\mathrm{lh}(\zeta_n(T)) = \mathrm{lh}(\eta_n(T))$ for all $T \in \mathcal{T}_n$,
- $\mathrm{row}(\mathrm{lh}(p)) \leq \mathrm{lh}(\delta_n(p))$,
- $\psi_n(p)$ is $\delta_n(p)$-σ_n-good, where $\sigma_n := \bigcap_{t\,\in\,\mathrm{ran}(\eta_n)} [\,t\,]^c$, and
- $(*)$ for all $T \in \mathcal{T}_n$ and all $q \in T$, $\{x : \zeta_n(T) \subset z_x \text{ and } \eta_n(T) \subset f(x)\} \cap [\,T[\,q\,]\,] \neq \varnothing$.

The following properties will hold for n, p, q and i such that $q \in \mathrm{dom}(\psi_{n+1}) \setminus \mathrm{dom}(\psi_n) = \{p^\frown 0, p^\frown 1\}$ and $\mathrm{row}(\mathrm{lh}(p)) = \mathrm{lh}(\delta_n(p)) = i$:

- $\mathrm{lh}(\delta_{n+1}(q)) = i+1$,
- $\mathcal{T}_{n+1} \setminus \mathcal{T}_n = \{T\}$, where $T := \delta_{n+1}(q)(i)$, and
- $(**)$ for all $v \in \mathrm{ran}(\eta_n)$ and $u \in {}^{<\omega}\omega$, $\langle u,v\rangle \in \tau_{1,3}(\psi_n(p)) \Rightarrow \{x : u \subset z_x\} \cap [\,T[\,\psi_{n+1}(q)\,]\,] = \varnothing$.

Let T, s, and t be given by Lemma 4.3.3 applied to f, so $\varnothing$ is $\langle T\rangle$-$[\,t\,]^c$-good and for all $p \in T$, $\{x : s \subset z_x \text{ and } t \subset f(x)\} \cap [\,T[\,p\,]\,] \neq \varnothing$. Define

$$\begin{aligned}\psi_0 &:= \{\langle \varnothing, \varnothing\rangle\},\\ \delta_0 &:= \{\langle \varnothing, \langle T\rangle\rangle\},\\ \zeta_0 &:= \{\langle T, s\rangle\}, \text{ and}\\ \eta_0 &:= \{\langle T, t\rangle\}.\end{aligned}$$

The reader should check that ψ_0, δ_0, ζ_0, and η_0 satisfy the desired properties.

Now, suppose ψ_n, δ_n, ζ_n, and η_n have been defined. Let p such that $\beta(2n+1) = p^\frown 0$ and let $i = \mathrm{row}(\mathrm{lh}(p))$, so $i \leq \mathrm{lh}(\delta_n(p))$. Let

$$\sigma_n := \bigcap_{t \in \mathrm{ran}(\eta_n)} [\,t\,]^c.$$

Case A: $i < \mathrm{lh}(\delta_n(p))$. Since $\psi_n(p)$ is $\delta_n(p)$-σ_n-good, we may find $q \supseteq \psi_n(p)$ such that q is $(\delta_n(p) \restriction i+1)$-$\sigma_n$-good, by Proposition 4.3.5. Let $T := \delta_n(p)(i)$. By $(*)$, we may find $r \supset q$ with $r \in T$ such that

$$\mathrm{card}(\mathrm{dom}(\tau_{1,3}(r)) \cap \{u : u \supseteq \zeta_n(T)\})$$

is strictly greater than

$$\mathrm{card}(\mathrm{dom}(\tau_{1,3}(\psi_n(p))) \cap \{u : u \supseteq \zeta_n(T)\}).$$

Define

$$\begin{aligned}\psi_{n+1} &:= \psi_n \cup \{\langle p^\frown 0, \psi_n(p)\rangle\} \cup \{\langle p^\frown 1, r\rangle\},\\ \delta_{n+1} &:= \delta_n \cup \{\langle p^\frown 0, \delta_n(p)\rangle\} \cup \{\langle p^\frown 1, \delta_n(p) \restriction i+1\rangle\},\\ \zeta_{n+1} &:= \zeta_n, \text{ and}\\ \eta_{n+1} &:= \eta_n.\end{aligned}$$

Case B: $i = \mathrm{lh}(\delta_n(p))$. In this case, we want to find a tree $T \subset {}^{<\omega}\omega$, s, t, $q \in {}^{<\omega}\omega$, and $\chi : \beta[2n+1] \to {}^{<\omega}\omega$ such that $T \notin \mathcal{T}_n$, $\mathrm{lh}(s) = \mathrm{lh}(t)$,

- $\mathrm{ran}(\eta_n) \cup \{t\}$ is an antichain,
- $\chi(r) \supseteq \psi_n(r)$ and $\chi(r)$ is $\delta_n(r)$-$(\sigma_n \setminus [\,t\,])$-good for all $r \in \mathrm{tn}(\beta[2n+1]) \setminus \{p\}$,
- $\chi(r) = \psi_n(r)$ for all $r \in (\beta[2n+1] \setminus \mathrm{tn}(\beta[2n+1])) \cup \{p\}$,
- $q \supset \psi_n(p)$,
- q is $(\delta_n(p)^\frown T)$-$(\sigma_n \setminus [\,t\,])$-good,
- for all $r \in T[\,q\,]$, $\{x : s \subset z_x \text{ and } t \subset f(x)\} \cap [\,T[\,r\,]\,] \neq \varnothing$, and
- for all $v \in \mathrm{ran}(\eta_n)$ and $u \in {}^{<\omega}\omega$, $\langle u, v\rangle \in \tau_{1,3}(\psi_n(p)) \Rightarrow \{x : u \subset z_x\} \cap [\,T[\,q\,]\,] = \varnothing$.

We will define sequences $\langle T_0, T_1, \ldots\rangle$, $\langle s_0, s_1, \ldots\rangle$, $\langle t_0, t_1, \ldots\rangle$, $\langle q_0, q_1, \ldots\rangle$ such that T_l, s_l, t_l, and an extension of q_l will be the desired values of T, s, t, and q for some l. By the induction hypothesis, $\psi_n(p)$ is $\delta_n(p)$-σ_n-good. Let S be the last element of the sequence $\delta_n(p)$ and let

$$h := f \restriction (f^{-1}[\,\sigma\,] \cap [\,S[\,\psi_n(p)\,]\,]),$$

so Player II does not have a winning strategy in $G_{2,3}(h)$. Let T, s, and t be given by Lemma 4.3.3 applied to h and let $T_0 := T$, $s_0 := s$, and $t_0 := t$. Note that $T_0 \subseteq S[\,\psi_n(p)\,]$ and $v \not\subseteq t_0$ for all $v \in \mathrm{ran}(\eta_n)$. Also note that $\psi_n(p)$ satisfies the first condition of being $(\delta_n(p)^\frown T_0)$-$(\sigma_n \setminus [\,t_0\,])$-good. Suppose that for every $r \supseteq \psi_n(p)$ with $r \in T_0$, there is an $r' \supseteq r$ such that r' is $\delta_n(p)$-$(\sigma_n \setminus [\,t_0\,])$-good. Let $q_0 := \psi_n(p)$. Otherwise, there is an $r \supseteq \psi_n(p)$ with $r \in T_0$ such that no $r' \supseteq r$ is $\delta_n(p)$-$(\sigma_n \setminus [\,t_0\,])$-good. Let $q_0 := r$.

Suppose $\langle T_0, \ldots, T_j\rangle$, $\langle s_0, \ldots s_j\rangle$, $\langle t_0, \ldots t_j\rangle$, and $\langle q_0, \ldots, q_j\rangle$ have been defined such that $v \not\subseteq t_j$ for all $v \in \mathrm{ran}(\eta_n) \cup \{t_0, \ldots, t_{j-1}\}$, $T_0 \supseteq \cdots \supseteq T_j$, $q_0 \subseteq \cdots \subseteq q_j$, $q_i \in T_i$, q_j satisfies the first condition of being

$$(\delta_n(p)^\frown T_j)\text{-}(\sigma_n \cap [\,t_0\,]^c \cap \cdots \cap [\,t_j\,]^c)\text{-good},$$

and either q_j is $(\delta_n(p)^\frown T_j)$-$(\sigma_n \setminus [\,t_j\,])$-good or no $r \supseteq q_j$ is $\delta_n(p)$-$(\sigma_n \setminus [\,t_j\,])$-good. Let

$$h := f \restriction (f^{-1}[\,\sigma_n \cap [\,t_0\,]^c \cap \cdots \cap [\,t_j\,]^c\,] \cap [\,T_j[\,q_j\,]\,]).$$

Let T, s, and t be given by Lemma 4.3.3 applied to h and let $T_{j+1} := T$, $s_{j+1} := s$, and $t_{j+1} := t$. Suppose for every $r \supseteq q_j$ with $r \in T_{j+1}$, there is an $r' \supseteq r$ such that r' is $\delta_n(p)$-$(\sigma_n \setminus [\,t_{j+1}\,])$-good. Let $q_{j+1} := q_j$. Otherwise, there is an $r \supseteq q_j$ with $r \in T_{j+1}$ such that no $r' \supseteq r$ is $\delta_n(p)$-$(\sigma_n \setminus [\,t_{j+1}\,])$-good. Let $q_{j+1} := r$.

We claim that there is an l such that t_l and elements of $\mathrm{ran}(\eta_n)$ are pairwise incompatible, q_l is $(\delta_n(p)^\frown T_l)$-$(\sigma_n \setminus [\,t_l\,])$-good, and for every $p' \in \mathrm{tn}(\beta[2n{+}1]) \setminus \{p\}$ there is a $\delta_n(p')$-$(\sigma_n \setminus [\,t_l\,])$-good extension of $\psi_n(p')$. Namely, we may consider an arbitrarily long subsequence of $\langle t_0, t_1, \ldots\rangle$ such that the elements of the subsequence are pairwise incompatible with themselves and elements of $\mathrm{ran}(\eta_n)$. Using Lemma 4.3.4, the claim follows. Let χ be as desired and let $T := T_l$, $s := s_l$, and $t := t_l$.

As the final step, let

$$U := \{u \in \mathrm{dom}(\tau_{1,3}(\psi_n(p))) : \tau_{1,3}(\psi_n(p))(u) \in \mathrm{ran}(\eta_n)\}.$$

By Proposition 4.3.2, let $q \supset q_l$ such that $q \in T$ and

$$\{x : u \subset z_x\} \cap [\,T[\,q\,]\,] = \varnothing$$

for all $u \in U$. Define

$$\begin{aligned}
\psi_{n+1} &:= \chi \cup \{\langle p^\frown 0, q\rangle\} \cup \{\langle p^\frown 1, q\rangle\},\\
\delta_{n+1} &:= \delta_n \cup \{\langle p^\frown 0, \delta_n(p)^\frown T\rangle\} \cup \{\langle p^\frown 1, \delta_n(p)^\frown T\rangle\},\\
\zeta_{n+1} &:= \zeta_n \cup \{\langle T, s\rangle\}, \text{ and}\\
\eta_{n+1} &:= \eta_n \cup \{\langle T, t\rangle\}.
\end{aligned}$$

This completes the construction of ψ_n, δ_n, ζ_n, and η_n. Let $\mathcal{T} := \bigcup_n \mathcal{T}_n$, $\delta := \bigcup_n \delta_n$, $\zeta := \bigcup_n \zeta_n$, $\eta := \bigcup_n \eta_n$, and let $\hat{\psi}$ be the lifting of the ψ_n. Let

$$Y := \bigcup_{t \in \operatorname{ran}(\eta)} [\, t\,].$$

The function $\hat{\psi}$ is a reduction from X to $f^{-1}[\,Y\,]$. If $x \in X$, then let i be least such that $x(\ulcorner i, j\urcorner) = 1$ for infinitely many j. Let N such that $x(n) = 1 \Rightarrow \operatorname{row}(n) \geq i$ for all $n \geq N$. Let $p \in {}^{<\omega}\omega$, $x \restriction N \subset p \subset x$ such that $\operatorname{lh}(\delta(p)) \geq i+1$. It follows that $\operatorname{lh}(\delta(q)) \geq i+1$ and $\delta(q)(i) = \delta(p)(i)$ for all q, $p \subseteq q \subset x$. Let $T := \delta(p)(i)$. It follows that $\operatorname{card}(\operatorname{dom}(\tau_{1,3}(r)) \cap \{u : u \supseteq \zeta(T)\}) \to \infty$ as $r \to \hat{\psi}(x)$, so $f(\hat{\psi}(x)) \supset \eta(T)$. Thus $\hat{\psi}(x) \in f^{-1}[\,Y\,]$.

If $x \notin X$, then for any i, there is an N such that $x(n) = 1 \Rightarrow \operatorname{row}(n) \geq i$ for all $n \geq N$. As before, there is a $p \subset x$ such that $\operatorname{lh}(\delta(q)) \geq i+1$ and $\delta(q)(i) = \delta(p)(i)$ for all q, $p \subseteq q \subset x$. So, there is a $\delta_x \in {}^{\omega}(\mathcal{P}({}^{<\omega}\omega))$ such that $\delta(p) \to \delta_x$ as $p \to x$ and $\hat{\psi}(x) \in \bigcap_i [\,\delta_x(i)\,]$. Now, suppose $\langle s, t\rangle \in \phi_{\hat{\psi}(x)}$ and $t \in \operatorname{ran}(\eta)$. Let $p \subset x$ and m such that $p \in \operatorname{dom}(\psi_m)$ and $\langle s, t\rangle \in \tau_{1,3}(\psi_m(p))$. Let q, $p \subseteq q \subset x$ and $n \geq m$ such that $\operatorname{dom}(\psi_{n+1}) \setminus \operatorname{dom}(\psi_n) = \{q^\frown 0, q^\frown 1\}$ and $\mathcal{T}_{n+1} \setminus \mathcal{T}_n = \{T\}$ for some $T \in \operatorname{ran}(\delta_x)$. By $(**)$, it follows that

$$\{y : s \subset z_y\} \cap [\, T[\,\psi_{n+1}(r)\,]\,] = \varnothing$$

for $r \in \{q^\frown 0, q^\frown 1\}$. Therefore, $\hat{\psi}(x) \notin \{y : s \subset z_y\}$ and thus $t \not\subset f(\hat{\psi}(x))$ for any $t \in \operatorname{ran}(\eta)$. This shows that $\hat{\psi}(x) \notin f^{-1}[\,Y\,]$, as desired. $\square$

4.4 $\mathbf{\Lambda}_{2,3} \not\subseteq \mathbf{\Lambda}_{1,2}$ and $\mathbf{\Lambda}_{1,3} \not\subseteq \mathbf{\Lambda}_{2,3}$

In this section, we show that the containments between $\mathbf{\Lambda}_{1,2}$ and $\mathbf{\Lambda}_{2,3}$ and between $\mathbf{\Lambda}_{2,3}$ and $\mathbf{\Lambda}_{1,3}$ are proper.

4.4.1. Theorem. $\mathbf{\Lambda}_{2,3} \not\subseteq \mathbf{\Lambda}_{1,2}$

Proof. As in Section 3.2, let MOVES be the set of finite trees $T \subset {}^{<\omega}\omega$. Let $\beta : {}^{<\omega}\omega \to \omega$ and $\gamma : \omega \to \mathsf{MOVES}$ be bijections. If $\tau : {}^{<\omega}\omega \to \mathsf{MOVES}$ is an eraser strategy, then $x \in {}^{\omega}\omega$ is a **code** for τ if $\tau(p) = \gamma(x(\beta(p)))$ for all $p \in {}^{<\omega}\omega$.

Note that for every eraser strategy τ, there is a unique x that encodes it. For $S \subset {}^{<\omega}\omega$, say that $\tau : S \to \mathsf{MOVES}$ is a **partial eraser strategy** if $s, t \in S$ and $s \subset t \Rightarrow \tau(s) \subseteq \tau(t)$.

It suffices to define a strategy $\tau_{2,3}$ and $f : {}^{\omega}\omega \to {}^{\omega}\omega$ such that $\tau_{2,3}$ is winning for Player II in $G_{2,3}(f)$ and $f \notin \mathbf{\Lambda}_{1,2}$. On input x, the strategy $\tau_{2,3}$ will attempt to decode x into an eraser strategy τ_x and diagonalize against the output of τ_x on input x. If x is the code of a valid eraser strategy τ_x, then let T_x be the tree produced by τ_x on input x. The strategy $\tau_{2,3}$ will use the following guessing function: row 0 will correspond to the guess that x does not encode a valid eraser strategy, row 1 will correspond to the guess that $T_x[\,0\,]$ is infinite, and row $k+2$ will correspond to the guess that $\mathrm{card}(T_x[\,0\,]) = k$.

Fix $p \in {}^{<\omega}\omega$. Let

$$S := \{\beta^{-1}(n) : n < \mathrm{lh}(p)\}.$$

Let $\tau : S \to {}^{<\omega}\omega$, $\tau(s) := \gamma(p(\beta(s)))$. If τ is a partial eraser strategy, then let $r := \bigcup\{q : q \subseteq p \text{ and } q \in \mathrm{dom}(\tau)\}$. Let $T := \tau(r)$ and $k := \mathrm{card}(T[\,0\,])$. Let $M(p) := \{\langle 1, 1^k\rangle\} \cup \{\langle k+2, 0^{\mathrm{lh}(p)}\rangle\}$. If τ is not a partial eraser strategy, then let $M(p) := \{\langle 0, 0^{\mathrm{lh}(p)}\rangle\}$.

Define $\tau_{2,3}(p) : \bigcup\{\mathrm{dom}(M(q)) : q \subseteq p\} \to \mathcal{P}({}^{<\omega}\omega)$,

$$\tau_{2,3}(p)(n) := \mathrm{tree}(\{M(q)(n) : q \subseteq p \text{ and } n \in \mathrm{dom}(M(q))\}).$$

Let $f : {}^{\omega}\omega \to \{0^*, 1^*\}$ such that $\tau_{2,3}$ is winning for Player II in $G_{2,3}(f)$. Suppose for contradiction that $f \in \mathbf{\Lambda}_{1,2}$. Let τ be the eraser strategy that is winning for Player II in $G_{1,2}(f)$. Let $x \in {}^{\omega}\omega$ be the code of τ and consider $f(x)$. If $f(x) = 0^*$ then it follows that $f(x) = 1^*$, and if $f(x) = 1^*$ then it follows that $f(x) = 0^*$. Therefore, $f \notin \mathbf{\Lambda}_{1,2}$. □

4.4.2. Theorem. $\mathbf{\Lambda}_{1,3} \not\subseteq \mathbf{\Lambda}_{2,3}$

Proof. As in Section 4.2, let MOVES be the set of functions $\psi : D \to \mathcal{P}({}^{<\omega}\omega)$ such that $D \subset \omega$ is finite and $\psi(n)$ is a finite tree for all $n \in \mathrm{dom}(\psi)$. Let $\beta : {}^{<\omega}\omega \to \omega$ and $\gamma : \omega \to \mathsf{MOVES}$ be bijections. If $\tau : {}^{<\omega}\omega \to \mathsf{MOVES}$ is a strategy for Player II in the game $G_{2,3}$, then $x \in {}^{\omega}\omega$ is a **code** for τ if $\tau(p) = \gamma(x(\beta(p)))$ for all $p \in {}^{<\omega}\omega$. For $S \subseteq {}^{<\omega}\omega$, say that $\tau : S \to \mathsf{MOVES}$ is a **partial $\mathbf{\Lambda}_{2,3}$ strategy** if $s, t \in S$ and $s \subset t \Rightarrow \mathrm{dom}(\tau(s)) \subseteq \mathrm{dom}(\tau(t))$ and $\tau(s)(n) \subseteq \tau(t)(n)$ for all $n \in \mathrm{dom}(\tau(s))$.

It suffices to define a strategy $\tau_{1,3}$ and $f : {}^{\omega}\omega \to {}^{\omega}\omega$ such that $\tau_{1,3}$ is winning for Player II in $G_{1,3}(f)$ and $f \notin \mathbf{\Lambda}_{2,3}$. On input x, the strategy $\tau_{1,3}$ will attempt to decode x into a $\mathbf{\Lambda}_{2,3}$ strategy τ_x and diagonalize against the output of τ_x on input x. If x is the code of a $\mathbf{\Lambda}_{2,3}$ strategy, let ϕ_x be the function produced by τ_x on input x and let $T_{n,x} := \phi_x(n)$.

The strategy $\tau_{1,3}$ considers three cases:

Case A: The input x does not encode a valid $\mathbf{\Lambda}_{2,3}$ strategy.
Case B: The input x encodes a valid $\mathbf{\Lambda}_{2,3}$ strategy τ_x and $\{t(n) : t \in T_{n,x} \cap {}^{n+1}\omega\}$ is infinite for some n.
Case C: The input x encodes a valid $\mathbf{\Lambda}_{2,3}$ strategy τ_x and $\{t(n) : t \in T_{n,x} \cap {}^{n+1}\omega\}$ is finite for all n.

Note that if Case A holds, then $\tau_{1,3}$ just needs to produce a valid output. Similarly, if Case B holds, then $T_{n,x}$ is not finitely branching so $\tau_{1,3}$ just needs to produce a valid output. If Case C holds, then $\tau_{1,3}$ will output $y \in {}^{\omega}\omega$ such that $y(n) > \sup\{t(n) : t \in T_{n,x} \cap {}^{n+1}\omega\}$ for all n. This will ensure that y cannot be an infinite branch of any of the $T_{n,x}$.

Fix $p \in {}^{<\omega}\omega$ Let

$$S := \{\beta^{-1}(n) : n < \mathrm{lh}(p)\}.$$

Let $\tau : S \to \mathsf{MOVES}$, $\tau(s) := \gamma(p(\beta(s)))$. If τ is a partial $\mathbf{\Lambda}_{2,3}$ strategy, then let $r := \bigcup\{q : q \subseteq p \text{ and } q \in \mathrm{dom}(\tau)\}$ and $\psi := \tau(r)$. Let $U(p) \in {}^{\mathrm{lh}(p)}(\omega \setminus \{0\})$,

$$U(p)(n) := \sup\{t(n) : t \in \psi(n) \cap {}^{n+1}\omega\} + 1$$

for all $n < \mathrm{lh}(p)$. Define

$$Z(p) := \{s^\frown 0^k : s^\frown k \subseteq U(p)\}.$$

The above definition of $Z(p)$ is under the assumption that τ is a partial $\mathbf{\Lambda}_{2,3}$ strategy. If τ is not a partial strategy, then let $Z(p) := \{0^n\}$.

Define

$$\tau_{1,3}(p) := \bigcup_{q \subseteq p}\{\langle s, s\rangle : s \in \mathrm{tree}(Z(q))\}.$$

It is easy to check that $\tau_{1,3}$ is a strategy. Note the following fact: $(*)$ if p encodes a partial $\mathbf{\Lambda}_{2,3}$ strategy, $s \in {}^{<\omega}(\omega \setminus \{0\})$, and $s^\frown 0^k \in Z(p)$, then $s^\frown k \subseteq U(p)$. For an infinite play x of Player I, let χ_x be the function produced by $\tau_{1,3}$ on input x. Note a second fact: $(**)$ every $u \in \mathrm{dom}(\chi_x)$ is of the form $s^\frown 0^k$ for some $s \in {}^{<\omega}(\omega \setminus \{0\})$ and $k \geq 0$.

We will show that there is an $f : {}^{\omega}\omega \to {}^{\omega}\omega$ such that $\tau_{1,3}$ is winning in $G_{1,3}(f)$. Let x be an infinite play of Player I and suppose that Case A holds. It follows that 0^* is an infinite branch of $\mathrm{dom}(\chi_x)$ and $\mathrm{dom}(\chi_x)[\,u\,]$ is finite for every $u \not\subset 0^*$. If Case B holds, then let n be least such that $\{t(n) : t \in T_{n,x} \cap {}^{n+1}\omega\}$ is infinite. Let $s \in {}^{n}\omega$,

$$s(m) := \sup\{t(m) : t \in T_{m,x} \cap {}^{m+1}\omega\} + 1.$$

It follows that $U(p) \restriction n$ converges to s and $U(p)(n) \to \infty$ as $p \to x$. Therefore, $s^\frown 0^*$ is an infinite branch of $\mathrm{dom}(\chi_x)$. Suppose $u \in \mathrm{dom}(\chi_x)$ and $u \not\subset s^\frown 0^*$.

By $(**)$, let $u = v^\frown 0^k$ with $v \in {}^{<\omega}(\omega \setminus \{0\})$ and $k \geq 0$. If $v \subset s$, then it must be the case that $k > 0$. Again by $(**)$, it follows that $u' \in \mathrm{dom}(\chi_x)$ and $u' \supseteq u \Rightarrow u' = v^\frown 0^{k'}$ for some $k' \geq k$. Thus $\mathrm{dom}(\chi_x)[\,u\,]$ is finite as k' is bounded by $s(\mathrm{lh}(v))$, by $(*)$. If $v \not\subset s$, then it must be the case that either $v \perp s$ or $v \supset s$. In either case, $v \subset U(p)$ for finitely many $p \subset x$. By $(*)$, it follows that $v \in \mathrm{tree}(Z(p)) \Rightarrow v \subset U(p)$ and thus $\mathrm{dom}(\chi_x[\,u\,])$ is finite. If Case C holds, then let $y \in {}^{\omega}(\omega \setminus \{0\})$,

$$y(n) := \sup\{t(n) : t \in T_{n,x} \cap {}^{n+1}\omega\} + 1.$$

It follows that $U(p) \to y$ as $p \to x$ and that y is an infinite branch of $\mathrm{dom}(\chi_x)$. Suppose $u \in \mathrm{dom}(\chi_x)$ and $u \not\subset y$. Let $u = v^\frown 0^k$ with $v \in {}^{<\omega}(\omega \setminus \{0\})$ and $k \geq 0$. If $k = 0$, then $v \not\subset y$ and thus $v \subset U(p)$ for finitely many $p \subset x$. As in Case B, it follows that $\mathrm{dom}(\chi_x)[\,u\,]$ is finite. If $k > 0$, then $u' \in \mathrm{dom}(\chi_x)$ and $u' \supseteq u \Rightarrow u' = v^\frown 0^{k'}$ for some $k' \geq k$. As in Case B, $\mathrm{dom}(\chi_x)[\,u\,]$ is finite as k' is bounded, this time by $y(\mathrm{lh}(v))$.

Now, suppose for contradiction that $f \in \mathbf{\Lambda}_{2,3}$. By Theorems 4.2.1 and 4.3.7, there is a strategy τ that is winning for Player II in $G_{2,3}(f)$. Let $x \in {}^{\omega}\omega$ be the code of τ, let ϕ_x be the function produced by τ on input x, and let m be the output row of ϕ_x. Consider the behavior of $\tau_{1,3}$ on input x. Since τ is winning for Player II in $G_{2,3}(f)$, it follows that Case C holds. Let $y \in {}^{\omega}\omega$ be unique such that

$$y(n) = \sup(\{t(n) : t \in \phi_x(n) \cap {}^{n+1}\omega\}) + 1$$

for all n. It follows that y is the output of $\tau_{1,3}$ on input x and $y(m) = f(x)(m) > f(x)(m)$, a contradiction. Therefore, $f \notin \mathbf{\Lambda}_{2,3}$. $\square$

Chapter 5

The $\mathbf{\Lambda}_{3,3}$ functions

In this chapter, we finish up our analysis of low-level Borel functions with the $\mathbf{\Lambda}_{3,3}$ class. We begin with the definition of the *multitape* game and show that it characterizes the class of functions f admitting a $\mathbf{\Pi}^0_2$ partition $\langle A_n : n < \omega \rangle$ such that $f \restriction A_n$ is continuous. It is immediate that this class is contained in $\mathbf{\Lambda}_{3,3}$ by Lemma 1.1.4; the main point of this chapter is to show that the reverse inclusion holds for total functions $f : {}^\omega\omega \to {}^\omega\omega$. This is done in Section 5.2. In Section 5.3, we see that neither $\mathbf{\Lambda}_{3,3}$ nor $\mathbf{\Lambda}_{1,2}$ is contained in the other.

The multitape game was first presented in [11] by the author of this thesis, although in a different form. The name "multitape" derives from its usage in conjunction with Turing machines where it signifies that more than one tape may be used.

$$
\begin{array}{ccccc}
\mathbf{\Lambda}_{1,3} & & & & \\
 & \rotatebox{45}{$\supset$} & & & \\
 & & \mathbf{\Lambda}_{2,3} & & \\
 & \rotatebox{-45}{$\subset$} & & \rotatebox{45}{$\supset$} & \\
\mathbf{\Lambda}_{1,2} & & & & \mathbf{\Lambda}_{3,3} \\
 & \rotatebox{45}{$\supset$} & & \rotatebox{-45}{$\subset$} & \\
 & & \mathbf{\Lambda}_{2,2} & & \\
 & \rotatebox{-45}{$\subset$} & & & \\
\mathbf{\Lambda}_{1,1} & & & &
\end{array}
$$

5.1 The multitape game

The multitape game is the same as the backtrack game except that the domain of the function produced by Player II is allowed to be infinite. Let $A \subseteq {}^\omega\omega$ and $f : A \to {}^\omega\omega$. In the multitape game $G_{\mathrm{mt}}(f)$, Player I plays elements $x_i \in \omega$ and Player II plays functions $\phi_i : D_i \to {}^{<\omega}\omega$ such that $D_i \subset \omega$ is finite. Player II is subject to the requirements that $i < j \Rightarrow D_i \subseteq D_j$ and $\phi_i(n) \subseteq \phi_j(n)$ for all $n \in \mathrm{dom}(\phi_i)$. After ω rounds, Player I produces $x = \langle x_0, x_1, \dots \rangle$ and Player II produces $\phi : D_\omega \to {}^{\leq\omega}\omega$,

$$\phi(n) := \bigcup\{\phi_i(n) : i \in \omega \text{ and } n \in \mathrm{dom}(\phi_i)\},$$

where $D_\omega := \bigcup_i D_i$.

$$\begin{array}{lccccccc}
\text{I:} & x_0 & & x_1 & & x_2 & & x = \langle x_0, x_1, \dots \rangle \\
 & & & & & & \dots & \\
\text{II:} & & \phi_0 & & \phi_1 & & \phi_2 & \phi \text{ as above}
\end{array}$$

Player II wins the game if either $x \notin A$ or if there is an $n \in D_\omega$ such that $\phi(n) = f(x)$ and $\phi(n')$ is finite for all $n' \neq n$. Informally, we think of Player II as playing finite sequences on a certain number of *rows*. As the game progresses, Player II may extend these finite sequences and may increase the number of rows she is using. In the limit, Player II's task is to produce an infinite sequence, namely $f(x)$, on exactly one of the rows. We refer to this row n as the *output row*.

Let MOVES be the set of functions $\psi : D \to {}^{<\omega}\omega$ such that $D \subset \omega$ is finite. A **multitape strategy** for Player II is a function $\tau : {}^{<\omega}\omega \to \mathsf{MOVES}$ such that $p \subset q \Rightarrow \mathrm{dom}(\tau(p)) \subseteq \mathrm{dom}(\tau(q))$ and $\tau(p)(n) \subseteq \tau(q)(n)$ for all $n \in \mathrm{dom}(\tau(p))$. For an infinite play x of Player I and a multitape strategy τ for Player II, we let $D_x := \bigcup_{p \subset x} \mathrm{dom}(\tau(p))$ and $\phi_x : D_x \to {}^{\leq\omega}\omega$,

$$\phi_x(n) := \bigcup\{\tau(p)(n) : p \subset x \text{ and } n \in \mathrm{dom}(\tau(p))\}.$$

A multitape strategy τ for Player II is **winning** in $G_{\mathrm{mt}}(f)$ if for all $x \in A$, there is an $n \in D_x$ such that $\phi_x(n) = f(x)$ and $\phi(n')$ is finite for all $n' \neq n$. We will sometimes denote this n, the output row, by o_x.

5.1.1. Theorem (Andretta, S.). *Suppose $A \subseteq {}^\omega\omega$ and $f : A \to {}^\omega\omega$. Then there is a $\mathbf{\Pi}^0_2$ partition $\langle A_n : n \in \omega\rangle$ of A such that $f \restriction A_n$ is continuous iff Player II has a winning strategy in $G_{\mathrm{mt}}(f)$.*

Proof. The proof is essentially the same as the proof of Theorem 4.2.1.

$\Rightarrow$: Let A_n be the partition and τ_n be a winning strategy for Player II in $G_{\mathrm{W}}(f \restriction A_n)$. Let $B_{n,m}$ be open in A such that $A_n = \bigcap_m B_{n,m}$. For $p \in {}^{<\omega}\omega$, let

$$\gamma_n(p) = \sup\{m : [\,p\,] \cap A \subseteq B_{n,i} \text{ for all } i \leq m\}.$$

Note that $\gamma_n(p)$ may be a natural number or may be ω. Also note that $p \subset q \Rightarrow \gamma_n(p) \leq \gamma_n(q)$ and that for any $x \in A$, there is a unique $n \in \omega$ such that $\lim_{p\to x} \gamma_n(p) = \infty$. Define $\tau(p) : \mathrm{lh}(p) \to \mathsf{MOVES}$,

$$\tau(p)(n) := \tau_n(p \restriction \gamma_n(p)).$$

It is easy to check that τ is a multitape strategy. We claim that τ is winning in $G_{\mathrm{mt}}(f)$. Let $x \in A$, n such that $x \in A_n$, and let ϕ_x be defined as in the previous section. It follows that n is unique such that $\phi_x(n)$ is infinite. Moreover, it is easy to see that

$$\phi_x(n) = \bigcup_{p \subset x} \tau_n(p).$$

It follows that $\phi_n(x) = f(x)$ since τ_n is winning in $G_{\mathrm{W}}(f \upharpoonright A_n)$.

$\Leftarrow$: Let τ be the winning strategy for Player II in $G_{\mathrm{mt}}(f)$. For $x \in A$, let ϕ_x and D_x be defined as in the previous section, and let o_x denote the output row of τ on input x. Define

$$A_n := \{x \in A : o_x = n\}.$$

The Wadge strategy τ_n defined by

$$\tau_n(p) := \begin{cases} \tau(p)(n) \text{ if } n \in \mathrm{dom}(\tau(p)), \\ \varnothing \text{ otherwise} \end{cases}$$

is winning for Player II in $G_{\mathrm{W}}(f \upharpoonright A_n)$. Furthermore, the sets A_n are $\mathbf{\Pi}^0_2$ in A. Fix $n \in \omega$. Let $B_m := \bigcup \{[p] : p \in {}^{<\omega}\omega,\ n \in \mathrm{dom}(\tau(p)) \text{ and } \mathrm{lh}(\tau(p)(n)) \geq m\}$. Then $A_n = \bigcap_m B_m \cap A$. □

5.2 Decomposing $\mathbf{\Lambda}_{3,3}$

We proceed with the main goal of this chapter, which is to prove Theorem 5.2.8.

5.2.1. Lemma. *Let $f : {}^{\omega}\omega \to {}^{\omega}\omega$. Suppose that Player II has a winning strategy in $G_{2,3}(f)$ but not in $G_{\mathrm{mt}}(f)$. Then there is a $\mathbf{\Pi}^0_2$ set $A \subseteq {}^{\omega}\omega$ such that Player II has a winning strategy in $G_{\mathrm{e}}(f \upharpoonright A)$ but not in $G_{\mathrm{mt}}(f \upharpoonright A)$.*

Proof. Let τ be the winning strategy for Player II in $G_{2,3}(f)$ and let ϕ_x, D_x, and o_x be defined as in Section 4.2. Let

$$A_n := \{x \in {}^{\omega}\omega : o_x = n\},$$

so the sets A_n are $\mathbf{\Pi}^0_2$. It is clear that Player II has a winning strategy in $G_{\mathrm{e}}(f \upharpoonright A_n)$ for each n, namely:

$$\tau_n(p) := \begin{cases} \tau(p)(n) \text{ if } n \in \mathrm{dom}(\tau(p)), \\ \varnothing \text{ otherwise.} \end{cases}$$

Suppose for contradiction that for each n, there is a winning strategy π_n for Player II in $G_{\mathrm{mt}}(f \upharpoonright A_n)$. For each n and $x \in A_n$, let $\phi_{n,x}$, $D_{n,x}$, and $o_{n,x}$ be the ϕ_x, D_x, and o_x as defined in Section 5.1 for π_n. We proceed by giving a winning strategy for Player II in $G_{\mathrm{mt}}(f)$, by defining guessing functions $\rho_0 : \omega \to \omega$ and $\rho_1 : \omega \to \omega$. For an infinite play x of Player I, the natural numbers $\rho_0(k)$ and $\rho_1(k)$ are guesses that

$$x \in A_{\rho_0(k)} \text{ and } o_{\rho_0(k),x} = \rho_1(k).$$

To define the guessing functions, let $\langle \rho_0(k), \rho_1(k) \rangle$ enumerate all pairs $\langle i, j \rangle \in \omega \times \omega$. For $p \in {}^{<\omega}\omega$, let

$$\gamma_n(p) := \begin{cases} \mathrm{card}(\tau(p)(n)) \text{ if } n \in \mathrm{dom}(\tau(p)), \\ 0 \text{ otherwise.} \end{cases}$$

Define $\pi(p) : \mathrm{lh}(p) \to {}^{<\omega}\omega$,

$$\pi(p)(k) := \pi_{\rho_0(k)}(p \restriction \gamma_{\rho_0(k)}(p))(\rho_1(k)).$$

It is easy to check that π is a multitape strategy. It remains to be shown that π is winning for Player II in $G_{\mathrm{mt}}(f)$. Let x and n such that $x \in A_n$, and let k be unique such that $\rho_0(k) = n$ and $\rho_1(k) = o_{n,x}$. It follows that $\gamma_n(p) \to \infty$ as $p \to x$. Therefore, on input x, π will produce the sequence $\phi_{n,x}(o) = f(x)$ on row k.

It remains to be shown that on input x, π produces a finite sequence on every row $k' \neq k$. If the guess $\rho_0(k')$ is incorrect, then $\gamma_{\rho_0(k')}(p)$ converges to some natural number as $p \to x$. If $\rho_0(k')$ is correct but $\rho_1(k')$ is incorrect, then π produces the sequence $\phi_{\rho_0(k'),x}(\rho_1(k'))$ on row k'. In either case, the sequence produced by π on row k' is finite. □

5.2.2. Lemma. *Let $A \subseteq {}^{\omega}\omega$, $h : A \to {}^{\omega}\omega$, and suppose that Player II does not have a winning strategy in $G_{\mathrm{mt}}(h)$. Then there is a non-empty tree $T \subseteq {}^{<\omega}\omega$ such that for any $p \in T$, Player II does not have a winning strategy in $G_{\mathrm{mt}}(h \restriction [T[p]])$.*

Proof. Let T be the set of $p \in {}^{<\omega}\omega$ such that Player II does not have a winning strategy in $G_{\mathrm{mt}}(h \restriction [p])$. Then T is a non-empty tree, as $\varnothing \in T$ by assumption and T is closed under predecessors. Let $p \in T$. If there were a winning strategy for Player II in $G_{\mathrm{mt}}(h \restriction [T[p]])$, then there would be a winning strategy for Player II in $G_{\mathrm{mt}}(h \restriction [p])$. □

5.2.3. Lemma. *Suppose $A \subseteq {}^{\omega}\omega$ and $h : A \to {}^{\omega}\omega$ is Baire class 2. Let $t_1, t_2 \in {}^{<\omega}\omega$ such that $t_1 \perp t_2$. If Player II has winning strategy τ_1 in $G_{\mathrm{mt}}(h \restriction h^{-1}[[t_1]^c])$ and a winning strategy τ_2 in $G_{\mathrm{mt}}(h \restriction h^{-1}[[t_2]^c])$ then Player II has a winning strategy in $G_{\mathrm{mt}}(h)$.*

Proof. Similar to the proof of Lemma 4.3.1. Since $[t_1] \subset [t_2]^c$, it follows that Player II has a winning strategy in $G_{\mathrm{mt}}(h \restriction h^{-1}[[t_1]])$. Let $B = h^{-1}[[t_1]]$ and $C = h^{-1}[[t_1]^c]$. It follows that $A = B \cup C$ and that B and C are $\mathbf{\Sigma}^0_3$ in A. The lemma follows from Theorem 5.1.1 and Lemma 1.1.5. □

The next lemma is the main lemma of the argument. It is analogous to Lemmas 3.4.3 and 4.3.3.

5.2.4. Lemma. *Let $A \subseteq {}^\omega\omega$, $h : A \to {}^\omega\omega$, and suppose that τ_{e} is a winning strategy for Player II in $G_{\mathrm{e}}(h)$. If Player II does not have a winning strategy in $G_{\mathrm{mt}}(h)$ then there is a non-empty tree $T \subseteq {}^{<\omega}\omega$ and $t \in {}^{<\omega}\omega$ such that Player II does not have a winning strategy in*

$$G_{\mathrm{mt}}(h \restriction (h^{-1}[\,[\,t\,]\,] \cap [\,T\,]))$$

and for every $p \in T$, Player II does not have a winning strategy in

$$G_{\mathrm{mt}}(h \restriction (h^{-1}[\,[\,t\,]^c\,] \cap [\,T[\,p\,]\,])).$$

Proof. By contradiction. We assume that the conclusion of the lemma does not hold and define a winning strategy for Player II in $G_{\mathrm{mt}}(h)$.

Fix $t \in {}^{<\omega}\omega$. If Player II does not have a winning strategy in $G_{\mathrm{mt}}(h \restriction (h^{-1}[\,[\,t\,]^c\,]))$, let T^t be given by the proof of Lemma 5.2.2 applied to $h \restriction (h^{-1}[\,[\,t\,]^c\,])$. So, T^t is the set of $p \in {}^{<\omega}\omega$ such that Player II does not have a winning strategy in

$$G_{\mathrm{mt}}(h \restriction (h^{-1}[\,[\,t\,]^c\,] \cap [\,p\,])),$$

and Player II does not have a winning strategy in

$$G_{\mathrm{mt}}(h \restriction (h^{-1}[\,[\,t\,]^c\,] \cap T^t[\,p\,]))$$

for any $p \in T^t$. Since we have assumed that the conclusion of the lemma does not hold, it follows that Player II has a winning strategy in

$$G_{\mathrm{mt}}(h \restriction (h^{-1}[\,[\,t\,]\,] \cap [\,T^t\,])).$$

If Player II does have a winning strategy in $G_{\mathrm{mt}}(h \restriction (h^{-1}[\,[\,t\,]^c\,]))$, let $T^t := \varnothing$. Again, it follows that Player II has a winning strategy in

$$G_{\mathrm{mt}}(h \restriction (h^{-1}[\,[\,t\,]\,] \cap [\,T^t\,])),$$

namely since $[\,T^t\,] = \varnothing$. Thus, for every $t \in {}^{<\omega}\omega$, we define T^t as indicated. Note that $t \subseteq v \Rightarrow T^t \subseteq T^v$.

For $x \in A$, let T_x be the tree produced by τ_{e} on input x as in Section 3.2. For each $t \in {}^{<\omega}\omega$, say that t is *blue* if $x \in [\,T^t\,]$. Otherwise, namely if there is a $p \subset x$ such that $p \notin T^t$, say that t is *red*. There are three cases to consider:

Case A: there is a blue $t \subset h(x)$,
Case B: there is a $p \subset x$ such that Player II has a
winning strategy in $G_{\mathrm{mt}}(h \restriction [\,p\,])$,
Case C: neither Case A nor Case B holds.

It is immediate that Cases A, B, and C are mutually exclusive. By Lemma 5.2.3, if Case C holds, then all $t \subset h(x)$ are red and all $t \not\subset h(x)$ are blue.

To handle Case A, we define a multitape strategy τ_{A} via guessing functions $\rho_0 : \omega \to {}^{<\omega}\omega$ and $\rho_1 : \omega \to \omega$. For $t \in {}^{<\omega}\omega$, let π_t be a winning strategy for Player II in

$$G_{\mathrm{mt}}(h \restriction (h^{-1}[\,[\,t\,]\,] \cap [\,T^t\,])).$$

The finite sequence $\rho_0(n)$ is a guess for the $\subseteq$-least blue initial segment of $h(x)$, and the natural number $\rho_1(n)$ is a guess for the output row of the strategy $\pi_{\rho_0(n)}$ on input $x \in h^{-1}[\,[\,\rho_0(n)\,]\,] \cap [\,T^{\rho_0(n)}\,]$. To define the guessing functions, let $\langle \rho_0(n), \rho_1(n)\rangle$ enumerate all pairs $\langle t, k\rangle \in {}^{<\omega}\omega \times \omega$.

For $p \in {}^{<\omega}\omega$ and $t \in {}^{<\omega}\omega$, let

$$\gamma_t(p) := \mathrm{card}(\{v \in \tau_{\mathrm{e}}(p) : v \supseteq t\}),$$

let

$$\begin{aligned} D(p) := \{n < \mathrm{lh}(p) :\ & p \in T^{\rho_0(n)}, \\ & p \notin T^v \text{ for all } v \subset \rho_0(n), \text{ and} \\ & \rho_1(n) \in \mathrm{dom}(\pi_{\rho_0(n)}(p \restriction \gamma_{\rho_0(n)}(p)))\}, \end{aligned}$$

and let $M(p) : D(p) \to {}^{<\omega}\omega$,

$$M(p)(n) := \pi_{\rho_0(n)}(p \restriction \gamma_{\rho_0(n)}(p))(\rho_1(n)).$$

Define $\tau_{\mathrm{A}}(p) : \bigcup\{D(q) : q \subset p\} \to {}^{<\omega}\omega$,

$$\tau_{\mathrm{A}}(p)(n) := \bigcup\{M(q)(n) : q \subseteq p \text{ and } n \in D(q)\}.$$

It is easy to check that τ_{A} is a multitape strategy.

We will show, if Case A holds, that τ_{A} computes $h(x)$. In other words, we will show that τ_{A} is winning for Player II in the game

$$G_{\mathrm{mt}}(h \restriction \{x : \text{there is a blue } t \subset h(x)\}).$$

Suppose that there is a blue $t \subset h(x)$. Let $\phi_{\mathrm{A},x}$ be the ϕ_x defined in Section 5.1 for τ_{A} and let n such that the guesses $\rho_0(n)$ and $\rho_1(n)$ are correct. It follows that $\phi_{\mathrm{A},x}(n) = h(x)$. To humor the reader, we provide a proof here. Let $q \subset x$ such that $n \in D(q)$, so $n \in D(p)$ for all p, $q \subseteq p \subset x$. Then

$$\begin{aligned} \phi_{\mathrm{A},x}(n) &= \bigcup\{\tau_{\mathrm{A}}(p)(n) : p \subset x \text{ and } n \in \mathrm{dom}(\tau_{\mathrm{A}}(p))\} \\ &= \bigcup\{\tau_{\mathrm{A}}(p)(n) : q \subseteq p \subset x\} \\ &= \bigcup\{\pi_{\rho_0(n)}(p \restriction \gamma_{\rho_0(n)}(p))(\rho_1(n)) : q \subseteq p \subset x\} \\ &= \bigcup\{\pi_{\rho_0(n)}(p)(\rho_1(n)) : q \subseteq p \subset x\} \\ &\quad \text{(since } \gamma_{\rho_0(n)}(p) \to \infty \text{ as } p \to x) \\ &= h(x). \end{aligned}$$

If $n' \neq n$ then at least one of the guesses $\rho_0(n')$ or $\rho_1(n')$ is incorrect. We want to show that $\phi_{\mathrm{A},x}(n')$ is finite. Suppose the guess $\rho_0(n')$ is incorrect, so $\rho_0(n')$ is not the $\subseteq$-least blue initial segment of $h(x)$. If $\rho_0(n')$ is not blue, then there is a $p \subset x$ such that $n' \notin D(q)$ for all q, $p \subseteq q \subset x$. If $\rho_0(n')$ is not an initial segment of $h(x)$, then $\gamma_{\rho_0(n')}(p)$ converges to some natural number as $p \to x$. If $\rho_0(n')$ is a blue initial segment of $h(x)$, but not the $\subseteq$-least such, then $n' \notin D(p)$ for all $p \subset x$. It follows from these observations that $\phi_{\mathrm{A},x}(n')$ is finite if the guess $\rho_0(n')$ is incorrect. If the guess $\rho_0(n')$ is correct but the guess $\rho_1(n')$ is incorrect, then $\phi_{\mathrm{A},x}(n')$ is is the finite sequence produced by $\pi_{\rho_0(n')}$ on row $\rho_1(n')$, on input x. We have shown that τ_{A} is a multitape strategy that computes $h(x)$ if Case A holds. If Case A does not hold then $\phi_{\mathrm{A},x}(n)$ is finite for every $n \in \mathrm{dom}(\phi_{\mathrm{A},x})$.

For Case B, let P be the set of $p \in {}^{<\omega}\omega$ such that Player II has a winning strategy in $G_{\mathrm{mt}}(h \restriction [\,p\,])$, and let Q be the maximal antichain of P such that $p \subset q \in Q \Rightarrow p \notin P$. For $q \in Q$, let τ_q be winning for Player II in $G_{\mathrm{mt}}(h \restriction [\,q\,])$. Define

$$\tau_{\mathrm{B}}(p) := \begin{cases} \tau_q(p) & \text{if } p \supseteq q \text{ for some } q \in Q, \\ \varnothing & \text{otherwise.} \end{cases}$$

It is easy to check that τ_{B} is a multitape strategy and winning for Player II in

$$G_{\mathrm{mt}}(h \restriction \{x : \text{Case B holds}\}).$$

If Case B does not hold then the function produced by τ_{B} is empty.

For Case C, let P as in Case B, let $R(p)$ be the set of $t \in \tau_{\mathrm{e}}(p)$ such that Player II has a winning strategy in

$$G_{\mathrm{mt}}(h \restriction (h^{-1}[\,[\,t\,]^c\,] \cap [\,p\,])),$$

and let

$$\mu(p) := \bigcup\{q \subseteq p : q \notin P\}.$$

Define $\tau_{\mathrm{C}}(p) : 1 \to {}^{<\omega}\omega$,

$$\tau_{\mathrm{C}}(p)(0) := \bigcup R(\mu(p)).$$

Note that $\mu(p) \notin P$, so $\bigcup R(\mu(p)) \in {}^{<\omega}\omega$ by Lemma 5.2.3. If Case C holds, then every $t \subset h(x)$ is red. Since Case B does not hold, $\mu(p) \to x$ as $p \to x$ and $\bigcup\{\tau_{\mathrm{C}}(p)(0) : p \subset x\} = h(x)$. If Case C does not hold, then it must be the case that either Case A or Case B holds. In either case, it is easy to check that $\bigcup\{\tau_{\mathrm{C}}(p)(0) : p \subset x\}$ is finite.

To complete the proof, define

$$\begin{aligned} \tau(p) := {} & \tau_{\mathrm{C}}(p) \cup \\ & \{\langle 2n+1, t\rangle : \langle n, t\rangle \in \tau_{\mathrm{A}}(p)\} \cup \\ & \{\langle 2n+2, t\rangle : \langle n, t\rangle \in \tau_{\mathrm{B}}(p)\}. \end{aligned}$$

The multitape strategy τ is winning for Player II in $G_{\mathrm{mt}}(h)$, a contradiction. $\square$

In the following, we fix $A \subseteq {}^{\omega}\omega$, $g : A \to {}^{\omega}\omega$, and suppose that Player II has a winning strategy in $G_{\mathrm{e}}(g)$. Let δ be a (possibly empty) finite sequence of trees $\langle T_0, \ldots, T_k\rangle$ with $T_i \subseteq {}^{<\omega}\omega$ and $T_0 \supseteq \cdots \supseteq T_k$. Let σ be a finite sequence $\langle X_0, \ldots, X_k\rangle$ of pairwise disjoint subsets of ${}^{\omega}\omega$ such that $\mathrm{lh}(\delta) = \mathrm{lh}(\sigma)$. If $\delta = \sigma = \varnothing$ then say that every $p \in {}^{<\omega}\omega$ is **$\boldsymbol{\delta}$-$\boldsymbol{\sigma}$-good**. If the length of δ and σ is $k+1$, then say that $p \in T_k$ is **$\boldsymbol{\delta}$-$\boldsymbol{\sigma}$-good** if for all $q \supseteq p$ with $q \in T_k$, Player II does not have a winning strategy in

$$G_{\mathrm{mt}}(g \restriction (g^{-1}[\,X_k\,] \cap [\,T_k[\,q\,]\,]))$$

and there is an $r \supseteq q$ such that r is $\mathrm{pred}(\delta)$-$\mathrm{pred}(\sigma)$-good. Note that if p is δ-σ-good and $\delta = \langle T_0, \ldots, T_k\rangle$, the definition requires that $p \in T_k$. The following propositions are immediate.

5.2.5. Proposition. *Suppose $\delta = \langle T_0, \ldots, T_k\rangle$, $\sigma = \langle X_0, \ldots, X_k\rangle$, and $p \in T_k$ is δ-σ-good. Then q is δ-σ-good for all $q \supseteq p$ with $q \in T_k$.*

5.2.6. Proposition. *Suppose $\delta = \langle T_0, \ldots, T_k\rangle$, $\sigma = \langle X_0, \ldots, X_k\rangle$, and $p \in T_k$ is δ-σ-good. Then for any $i < k+1$, there exists $q \supseteq p$ such that q is $(\delta \restriction i)$-$(\sigma \restriction i)$-good.*

For $\sigma = \langle X_0, \ldots, X_k\rangle$ and $t \in {}^{<\omega}\omega$, we abuse notation and define $\sigma \setminus t := \langle X_0 \setminus [\,t\,], \ldots, X_k \setminus [\,t\,]\rangle$.

5.2.7. Lemma. *Let $\delta = \langle T_0, \ldots, T_k\rangle$, $\sigma = \langle X_0, \ldots, X_k\rangle$, suppose $\langle t_0, \ldots, t_m\rangle$ is a sequence of pairwise incompatible elements of ${}^{<\omega}\omega$, and $\bigcup_i [\,t_i\,]$ is contained in some X_j. If p is δ-σ-good, then*

$$\{i \le m : \textit{no } q \supseteq p \textit{ is } \delta\text{-}(\sigma \setminus t_i)\textit{-good}\}$$

has at most one element.

Proof. Proof by induction on k. For the base case $k = 0$, let $\delta = \langle T\rangle$, $\sigma = \langle X\rangle$, and $\langle t_0, \ldots, t_m\rangle$ be given. By assumption, the t_i are pairwise incompatible, $[\,t_i\,] \subseteq X$ for each $i \le m$, and $p \in T$ is δ-σ-good. If p is δ-$(\sigma \setminus t_i)$-good for each $i \le m$, then we are done. Otherwise, there is an $i \le m$ such that p is not δ-$(\sigma \setminus t_i)$-good. Let $q \supseteq p$ such that $q \in T$ and Player II has a winning strategy in

$$G_{\mathrm{mt}}(g \restriction (g^{-1}[\,X \setminus [\,t_i\,]\,] \cap [\,T[\,q\,]\,])).$$

Since q is δ-σ-good, Player II does not have a winning strategy in

$$G_{\mathrm{mt}}(g \restriction (g^{-1}[\,X\,] \cap [\,T[\,r\,]\,]))$$

for any $r \supseteq q$ with $r \in T$. Let $l \le m$ with $l \ne i$ and $r \supseteq q$ with $r \in T$. By Lemma 5.2.3, Player II does not have a winning strategy in

$$G_{\mathrm{mt}}(g \restriction (g^{-1}[\,X \setminus [\,t_l\,]\,] \cap [\,T[\,r\,]\,])).$$

It follows that q is δ-$(\sigma \setminus t_l)$-good.

For the inductive step, let $\delta = \langle T_0, \ldots, T_{k+1}\rangle$, $\sigma = \langle X_0, \ldots, X_{k+1}\rangle$ and suppose $p \in T_{k+1}$ is δ-σ-good. Let $j \leq k+1$ such that $[\,t_i\,] \subseteq X_j$ for each $i \leq m$. If $j = k+1$ then suppose that there is an $i \leq m$ and a $q \supseteq p$ with $q \in T_{k+1}$ such that Player II has a winning strategy in

$$G_{\mathrm{mt}}(g \restriction (g^{-1}[\,X_{k+1} \setminus [\,t_i\,]\,] \cap [\,T_{k+1}[\,q\,]\,])).$$

Otherwise, if there is no such i and q, then p is δ-$(\sigma \setminus t_i)$-good for all $i \leq m$ and we are done. Let $l \leq m$ with $l \neq i$ and $r \supseteq q$ with $r \in T_{k+1}$. As before, Player II does not have a winning strategy in

$$G_{\mathrm{mt}}(g \restriction (g^{-1}[\,X_{k+1} \setminus [\,t_l\,]\,] \cap [\,T_{k+1}[\,r\,]\,])).$$

It follows that q is δ-$(\sigma \setminus t_l)$-good.

If $j < k+1$ then suppose there is an $i \leq m$ such that no $q \supseteq p$ is δ-$(\sigma \setminus t_i)$-good. Let $l \leq m$ such that $l \neq i$. It will be shown that p is δ-$(\sigma \setminus t_l)$-good, completing the proof. It suffices to show that the pred(δ)-pred$(\sigma \setminus t_l)$-good nodes are dense in $T_{k+1}[\,p\,]$. Let $q \supseteq p$ with $q \in T_{k+1}$. By choice of i, q is not δ-$(\sigma \setminus t_i)$-good. Since q is δ-σ-good, it must be the second part of the definition of δ-$(\sigma \setminus t_i)$-goodness that fails for q. Let $r \supseteq q$ with $r \in T_{k+1}$ such that no $s \supseteq r$ is pred(δ)-pred$(\sigma \setminus t_i)$-good. Since r is δ-σ-good, there is a pred(δ)-pred(σ)-good $u \supseteq r$ with $u \in T_k$. By the induction hypothesis, there is a pred(δ)-pred$(\sigma \setminus t_l)$-good extension of u. $\square$

5.2.8. THEOREM. *A function* $f : {}^\omega\omega \to {}^\omega\omega$ *is* $\mathbf{\Lambda}_{3,3} \Leftrightarrow$ *there is a* $\mathbf{\Pi}^0_2$ *partition* $\langle A_n : n \in \omega\rangle$ *of* ${}^\omega\omega$ *such that* $f \restriction A_n$ *is continuous.*

Proof. The direction $\Leftarrow$ is immediate, so it suffices to prove $\Rightarrow$. Suppose for contradiction that there is no winning strategy for Player II in $G_{\mathrm{mt}}(f)$, we will show that $f \notin \mathbf{\Lambda}_{3,3}$. By Theorems 4.2.1 and 4.3.7, we may assume that Player II has a winning strategy in $G_{2,3}(f)$. Let A and τ_{e} be given by the proof of Lemma 5.2.1, so τ_{e} is winning for Player II in $G_{\mathrm{e}}(f \restriction A)$ and Player II does not have a winning strategy in $G_{\mathrm{mt}}(f \restriction A)$. For $x \in A$, let T_x be the tree produced by τ_{e} on input x as in Section 3.2. Let $\ulcorner \cdot, \cdot \urcorner$, X, row, β, and $\mathcal{D}$ as in the proof of Theorem 4.3.7.

We will define a $\mathbf{\Sigma}^0_2$ set Y and a snake ψ_n such that the lifting $\hat{\psi}$ of ψ_n is a reduction from X to $f^{-1}[\,Y\,]$. The $\mathbf{\Sigma}^0_2$ set Y will be defined using a Lusin scheme $\eta : {}^{<\omega}\omega \to {}^{<\omega}\omega$ satisfying

- $\eta(\varnothing) = \varnothing$,
- $\eta(s^\frown k) \supset \eta(s)$, and
- $\{\eta(s^\frown k) : k \in \omega\}$ is an antichain.

Note that proper containment is required for the second condition. Recursively, we will define a sequence of functions $\eta_n : D_n \to {}^{<\omega}\omega$ such that $D_n \subset {}^{<\omega}\omega$ is a finite tree and $i < j \Rightarrow \eta_i \subseteq \eta_j$. We will then let

$$\eta := \bigcup_n \eta_n.$$

To define Y, we will let

$$Y_m := \bigcup_{s \in {}^{m+1}\omega} [\,\eta(s)\,],$$

and

$$Y := \bigcup_m {Y_m}^c.$$

The behavior of the strategy τ_{e} will ensure that $\hat{\psi}$ is a reduction from X to $f^{-1}[\,Y\,]$. Recall that we view each $x \in {}^{\omega}2$ as a two-dimensional matrix of 0's and 1's via the mapping $\ulcorner\cdot,\cdot\urcorner$. If we encounter infinitely many 1's on a row of x, then we want $\hat{\psi}$ to map x inside of $f^{-1}[\,Y\,]$. This will be accomplished as follows: if m is such a row of x, then on input $\hat{\psi}(x)$, the eraser strategy τ_{e} will extend $\eta(s)$ for infinitely many $s \in {}^{m+1}\omega$. By Lemma 3.4.2, we will have that $f(\hat{\psi}(x)) \notin Y_m$ and thus $\hat{\psi}(x) \in f^{-1}[\,Y\,]$.

If, on the other hand, we encounter only finitely many 1's on each row of x, then we want $\hat{\psi}$ to map x outside of $f^{-1}[\,Y\,]$. In this case, for every row m, there will be an $s \in {}^{m+1}\omega$ such that τ_{e} extends $\eta(s)$ infinitely many times on input $\hat{\psi}(x)$. This will imply that $f(\hat{\psi}(x)) \in Y_m$ for all m, so $\hat{\psi}(x) \notin f^{-1}[\,Y\,]$.

We define by recursion

$$\begin{aligned}
\psi_n &: \beta[2n+1] \to {}^{<\omega}\omega,\\
\delta_n &: \beta[2n+1] \to \mathcal{D},\\
\iota_n &: \beta[2n+1] \to D_n, \text{ and}\\
\eta_n &: D_n \to {}^{<\omega}\omega,
\end{aligned}$$

such that $D_n \subset {}^{<\omega}\omega$ is a finite tree, $i < j \Rightarrow \delta_i \subseteq \delta_j \,\wedge\, \iota_i \subseteq \iota_j \,\wedge\, \eta_i \subseteq \eta_j$, and for all $p \in \mathrm{tn}(\beta[2n+1])$,

- $\mathrm{row}(\mathrm{lh}(p)) < \mathrm{lh}(\iota_n(p)) + 1 = \mathrm{lh}(\delta_n(p))$,
- $\psi_n(p)$ is $\delta_n(p)$-$\sigma_n(p)$-good, where $\sigma_n(p) := \langle X_\varnothing, X_{\iota_n(p)\restriction 1}, \ldots, X_{\iota_n(p)}\rangle$, with $X_u := [\,\eta_n(u)\,] \setminus \bigcup\{[\,\eta_n(v)\,] : t \in \mathrm{succ}(u) \cap D_n\}$, and
- the eraser strategy τ_{e} extends $\eta_n(\iota_n(p))$ at least once on input $\psi_n(p)$.

We must also ensure that the sequence $\langle \psi_n : n \in \omega\rangle$ is a snake and that the union of the η_n is a Lusin scheme as described above.

Let T and t be given by Lemma 5.2.4 applied to $g := f \restriction A$. Let $h := g \restriction (g^{-1}[\,[\,t\,]\,] \cap [\,T\,])$ and let $U \subseteq T$ be the tree given by Proposition 5.2.2 applied to h. It follows that $\varnothing$ is $\langle T, U\rangle$-$\langle [\,t\,]^c, [\,t\,]\rangle$-good. Let $r \in U$ such that τ_{e} extends t at least once on input r. Define

$$\begin{aligned}\psi_0 &:= \{\langle \varnothing, r\rangle\},\\ \delta_0 &:= \{\langle \varnothing, \langle T, U\rangle\rangle\},\\ \iota_0 &:= \{\langle \varnothing, \langle 0\rangle\rangle\}, \text{ and}\\ \eta_0 &:= \{\langle \varnothing, \varnothing\rangle\} \cup \{\langle\langle 0\rangle, t\rangle\}.\end{aligned}$$

The reader should check that ψ_0, δ_0, ι_0, and η_0 satisfy the desired requirements.

Now, suppose ψ_n, δ_n, ι_n, and η_n have been defined. Let p such that $\beta(2n+1) = p^\frown 0$ and $i := \mathrm{row}(\mathrm{lh}(p))$. For each $q \in \mathrm{tn}(\beta[2n+1])$, let

$$\sigma_n(q) := \langle X_\varnothing, X_{\iota_n(q)\restriction 1}, \dots, X_{\iota_n(q)}\rangle,$$

where

$$X_u := [\,\eta_n(u)\,] \setminus \bigcup \{[\,\eta_n(v)\,] : v \in \mathrm{succ}(u) \cap \mathrm{dom}(\eta_n)\}.$$

Now, let $u := \iota_n(p) \restriction i$. We want to find T, U, t, r, and $\chi : \beta[2n+1] \to {}^{<\omega}\omega$ such that

- $t \supset \eta_n(u)$,
- $\{\eta_n(v) : v \in \mathrm{succ}(u) \cap D_n\} \cup \{t\}$ is an antichain,
- $\chi(q) \supseteq \psi_n(q)$ and $\chi(q)$ is $\delta_n(q)$-$(\sigma_n(q) \setminus t)$-good for all $q \in \mathrm{tn}(\beta[2n+1]) \setminus \{p\}$,
- $\chi(q) = \psi_n(q)$ for all $q \in (\beta[2n+1] \setminus \mathrm{tn}(\beta[2n+1])) \cup \{p\}$,
- $r \supset \psi_n(p)$, and
- r is $(\delta_n(p) \restriction i)^\frown T^\frown U$-$(\sigma_n(p) \restriction i)^\frown(\sigma_n(p)(i) \setminus [\,t\,])^\frown[\,t\,]$-good.

By Proposition 5.2.6, we may find $q \supseteq \psi_n(p)$ such that q is

$$(\delta_n(p) \restriction i+1)\text{-}(\sigma_n(p) \restriction i+1)\text{-good.}$$

Let $S = \delta_n(p)(i)$, $Z := \sigma_n(p)(i)$, and

$$h := g \restriction (g^{-1}[\,Z\,] \cap [\,S[\,q\,]\,]),$$

so Player II does not have a winning strategy in $G_{\mathrm{mt}}(h)$. We will define sequences $\langle T_0, T_1, \dots\rangle$ and $\langle t_0, t_1, \dots\rangle$ such that T_l and t_l will be the desired values of T and t for some l. Let T_0 and t_0 be given by Lemma 5.2.4 applied to h. Note that $T_0 \subseteq S[\,q\,]$, $\eta_n(u) \subset t_0$, $\eta_n(v) \not\subseteq t_0$ for all $v \in \mathrm{succ}(u) \cap D_n$, and q is

$$(\delta_n(p) \restriction i)^\frown T_0\text{-}(\sigma_n(p) \restriction i)^\frown(Z \setminus [\,t_0\,])\text{-good.}$$

Suppose $\langle T_0, \dots, T_j\rangle$ and $\langle t_0, \dots t_j\rangle$ have been defined such that $\eta_n(u) \subset t_j$, $\eta_n(v) \not\subseteq t_j$ for all $v \in \operatorname{succ}(u) \cap D_n$, $t_i \not\subseteq t_j$ for all $i < j$, $T_0 \supseteq \cdots \supseteq T_j$, and q is

$$(\delta_n(p) \restriction i)^\frown T_j\text{-}(\sigma_n(p) \restriction i)^\frown(Z \cap [\,t_0\,]^c \cap \cdots \cap [\,t_j\,]^c)\text{-good.}$$

Let

$$h := g \restriction (g^{-1}[\,Z \cap [\,t_0\,]^c \cap \cdots \cap [\,t_j\,]^c\,] \cap [\,T_j\,])$$

and let T_{j+1} and t_{j+1} be given by Lemma 5.2.4.

We claim that there is an l such that $\{\eta_n(v) : v \in \operatorname{succ}(u) \cap D_n\} \cup \{t_l\}$ is an antichain and for every $r \in \operatorname{tn}(\beta[2n+1]) \setminus \{p\}$, there is an $\delta_n(r)$-$(\sigma_n(r) \setminus t_l)$-good extension of $\psi_n(r)$. Namely, we may consider an arbitrarily long subsequence of $\langle t_0, t_1, \dots\rangle$ such that the elements of the subsequence are pairwise incompatible with themselves and elements of $\{\eta_n(v) : v \in \operatorname{succ}(u) \cap D_n\}$. Using Lemma 5.2.5, the claim follows. Let χ be as desired, $T := T_l$, and $t := t_l$.

As the final step, since Player II does not have a winning strategy in

$$h := g \restriction (g^{-1}[\,[\,t\,]\,] \cap [\,T\,]),$$

let $U \subseteq T$ be given by Proposition 5.2.2 applied to h. Let $r \supset q$ such that $r \in U$ and τ_{e} has extended t at least once on input r. Let $k := \sup\{j + 1 : u^\frown j \in \operatorname{dom}(\eta_n)\}$.

Case A: $i = \operatorname{lh}(\iota_n(p))$. Note in this case that $u = \iota_n(p)$. Define

$$\begin{aligned}
\psi_{n+1} &:= \chi \cup \{\langle p^\frown 0, r\rangle\} \cup \{\langle p^\frown 1, r\rangle\},\\
\delta_{n+1} &:= \delta_n \cup \{\langle p^\frown 0, (\delta_n(p) \restriction i)^\frown T^\frown U\rangle\} \cup \{\langle p^\frown 1, (\delta_n(p) \restriction i)^\frown T^\frown U\rangle\},\\
\iota_{n+1} &:= \iota_n \cup \{\langle p^\frown 0, u^\frown k\rangle\} \cup \{\langle p^\frown 1, u^\frown k\rangle\},\\
\eta_{n+1} &:= \eta_n \cup \{\langle u^\frown k, t\rangle\}.
\end{aligned}$$

Case B: $i < \operatorname{lh}(\iota_n(p))$. Define

$$\begin{aligned}
\psi_{n+1} &:= \chi \cup \{\langle p^\frown 0, \psi_n(p)\rangle\} \cup \{\langle p^\frown 1, r\rangle\},\\
\delta_{n+1} &:= \delta_n \cup \{\langle p^\frown 0, \delta_n(p)\} \cup \{\langle p^\frown 1, (\delta_n(p) \restriction i)^\frown T^\frown U\rangle\},\\
\iota_{n+1} &:= \iota_n \cup \{\langle p^\frown 0, \iota_n(p)\rangle\} \cup \{\langle p^\frown 1, u^\frown k\rangle\},\\
\eta_{n+1} &:= \eta_n \cup \{\langle u^\frown k, t\rangle\}.
\end{aligned}$$

This completes the construction of ψ_n, δ_n, ι_n, and η_n. Let Y_m and Y be defined as indicated earlier, let $\iota = \bigcup_n \iota_n$, $\eta = \bigcup_n \eta_n$ and let $\hat{\psi}$ be the lifting of ψ_n.

The function $\hat{\psi}$ is a reduction from X to $f^{-1}[\,Y\,]$. If $x \in X$, then let i be least such that $x(\ulcorner i, j\urcorner) = 1$ for infinitely many j. It follows that the strategy τ_{e} extends infinitely many $t \in \eta[\,{}^{i+1}\omega\,]$ on input $\hat{\psi}(x)$. Since elements of $\eta[\,{}^{i+1}\omega\,]$ are pairwise disjoint and $Y_i = \bigcup\{[\,t\,] : t \in \eta[\,{}^{i+1}\omega\,]\}$, it follows that $f(\hat{\psi}(x)) \notin Y_i$ by Lemma 3.4.2. Therefore, $f(x) \in Y$.

Suppose $x \notin X$. Fix $i \in \omega$ and let N such that $x(n) = 1 \Rightarrow \mathrm{row}(n) > i$ for all $n \geq N$. Let $p \in {}^{<\omega}\omega$, $x \restriction N \subseteq p \subset x$ such that $\mathrm{lh}(\iota(p)) \geq i+1$. It follows that $\iota(q) \restriction i+1 = \iota(p) \restriction i+1$ for all q, $p \subseteq q \subset x$. Since τ_{e} extends $\eta(\iota(p) \restriction i+1)$ infinitely many times on input $\hat{\psi}(x)$, it follows that $f(\hat{\psi}(x)) \in Y_i$. As $i \in \omega$ was arbitrary, $f(\hat{\psi}(x)) \notin Y$. $\square$

5.3 $\mathbf{\Lambda}_{3,3} \not\subseteq \mathbf{\Lambda}_{1,2}$ and $\mathbf{\Lambda}_{1,2} \not\subseteq \mathbf{\Lambda}_{3,3}$

These facts follow immediately from earlier proofs. To see that $\mathbf{\Lambda}_{3,3} \not\subseteq \mathbf{\Lambda}_{1,2}$, consider the $\mathbf{\Lambda}_{2,3}$ strategy $\tau_{2,3}$ and $f : {}^{\omega}\omega \to {}^{\omega}\omega$ given in the proof of Theorem 4.4.1. The strategy $\tau_{2,3}$, winning for Player II in $G_{2,3}(f)$, can trivially be converted into a multitape strategy that is winning for Player II in $G_{\mathrm{mt}}(f)$. The fact that $\mathbf{\Lambda}_{1,2} \not\subseteq \mathbf{\Lambda}_{3,3}$ can be shown with the same eraser strategy used in the proof of Theorem 3.5.2, using Theorems 5.1.1 and 5.2.8.

Chapter 6

Conclusion

We have seen a number of games in this thesis. In Chapter 2, we saw the tree game and its characterization of the Borel functions. In the second part of the thesis, we saw more games for certain subclasses of Borel functions, and we saw that they can be used to prove decomposition theorems.

The question nautrally arises: can we obtain a result for more general subclasses of Borel functions? It is hoped that the game-theoretic tools we have developed in this thesis can be generalized to obtain a more elegant characterization theorem. In particular, all of the games we have looked at can be viewed as restricted tree games. The Wadge game can be viewed as the restricted tree game in which Player II is required to produce ϕ such that $\mathrm{dom}(\phi)$ is linear; for the eraser game, we require that $\mathrm{dom}(\phi)$ is finitely branching; for the backtrack game, we require that $\mathrm{dom}(\phi)$ branches finitely at the root and is linear therafter; for the game $G_{2,3}$, we require that $\mathrm{dom}(\phi)$ may branch infinitely at the root but is finitely branching thereafter; and for the multitape game, we require that $\mathrm{dom}(\phi)$ may branch infinitely at the root but is linear thereafter.

Thus, it would seem natural to come up with more general restrictions on $\mathrm{dom}(\phi)$, and work with m's and n's or α's and β's instead of numbers between 1 and 3. (The author refuses to prove any decomposition theorems with 4's in them.)

The tree game itself is simple and characterizes a class of functions widely considered in descriptive set theory. Going beyond the Borel functions, one might try to generalize the tree game to characterize classes of projective functions, possibly by allowing Player II to produce multiple infinite branches.

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Samenvatting

Dit proefschrift gaat over klassen van functies gedefinieerd op de Baire-ruimte. Voor een aantal belangrijke klassen van functies zijn representaties door middel van spelen ontwikkeld, die zeer nuttig blijken te zijn. Het meest in het oog springende voorbeeld hiervan is Wadge's karakterisering van de continue functies, die heeft geleid tot de theorie van de Wadge-hiërarchie. Zich baserend op een resultaat van Jayne en Rogers heeft Andretta In 2006 een representatie door middel van spelen gegeven van de $\mathbf{\Delta}^1_2$ functies (in de taal van dit proefschrift is dit de klasse $\mathbf{\Lambda}_{2,2}$). Karakteriseringen met behulp van spelen zijn belangrijk omdat ze zogenaamde "Wadge-style" bewijstechnieken mogelijk maken. Andretta en Martin klagen in hun paper over Borel-functies dat

> "there is no analogue of the Wadge/Lipschitz games for Borel functions, [and] hence many of the standard proofs for the Wadge hierarchy do not generalize in a straightforward way to the Borel set-up."

Dit gaf aanleiding tot twee belangrijke vragen:

1. Kunnen vergelijkbare karakteriseringen worden gegeven van andere klassen van functies, in het bijzonder van de klasse van alle Borel-functies en de klasse $\mathbf{\Lambda}_{3,3}$?

2. Bestaat er een parallel van de Jayne-Rogers stelling op het derde niveau van de Borel-hiërarchie?

Dit proefschrift bevestigt deze vragen (stellingen 2.0.9, 5.1.1 en 5.2.8).

Abstract

In this thesis, we deal with classes of functions on Baire space. For some important function classes, game representations are known and proved to be very useful. The most prominent example is Wadge's characterization of the continuous functions that allowed the development of the theory of the Wadge hierarchy; in 2006, based on a result of Jayne and Rogers, Andretta gave a game representation for the $\mathbf{\Delta}^1_2$ functions (in the language of this thesis, this is the class $\mathbf{\Lambda}_{2,2}$). Game characterizations are important as they allow for "Wadge-style proof techniques". In their paper on Borel functions, Andretta and Martin lament that

> "there is no analogue of the Wadge/Lipschitz games for Borel functions, [and] hence many of the standard proofs for the Wadge hierarchy do not generalize in a straightforward way to the Borel set-up."

This suggested two important questions:

1. Can similar characterizations be given for other function classes, most notably for the class of all Borel functions and the class $\mathbf{\Lambda}_{3,3}$?

2. Is there an analogue of the Jayne-Rogers theorem at the third level of the Borel hierarchy?

In this thesis, we give positive answers to these questions (Theorems 2.0.9, 5.1.1, and 5.2.8).

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