Characterizing All Models in Infinite Cardinalities

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Chapter 1

Introduction

1.1 Introduction

We shall investigate whether second order equivalence of two models, or equivalence in some stronger logic than second order logic, implies isomorphism of the models in certain cardinalities. We always assume that our vocabulary is finite. The notation which is not yet explained can be found under the heading "Notation" on page 2 or by using the index.

1.1.1. Remark. *We are assuming through this paper that a vocabulary is finite. This is done because if the vocabulary is finite, the isomorphism type of the model is characterizable inside the model in second order logic. In infinitary second order logic $L^2_{\kappa,\omega}$ the isomorphism type of the model is characterizable if the vocabulary is smaller than κ, and our assumption is stronger than what is needed.*

There are some open questions about whether our results can be generalized to bigger vocabularies. An example of such a question is whether it is consistent with ZFC, that in any countable vocabulary any two countable L^2-equivalent models are isomorphic.

Suppose L is a logic [3] (Chapter 2, Definition 1.1.1). The L-theory of a model is the set of L-sentences true in the model. Two models are said to satisfy the same L-theory if they satisfy the same L-sentences.

1.1.2. Definition. *The expression $A(L,\kappa)$ refers to the following condition: For any models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality κ, if $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same L-theory then they are isomorphic.*

We use the expression $A(ZF,\kappa)$ to denote the condition "For all models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality κ in a finite vocabulary, if $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences (with the model as a parameter) in the language of set theory then $\mathfrak{A} \cong \mathfrak{B}$." Note that ZF is not a logic as two isomorphic models can satisfy different sentences in the language of set theory.

1.1.3. Definition. *We call $A(L^2, \omega)$ when restricted to ordinals the* Fraïssé Hypothesis*. This is the Hypothesis: All countable ordinals have different second order theories.*

Ajtai [2] has proved that $A(L^2, \omega)$ is independent of ZFC. We are looking for related results in the cardinality $\aleph_0$ and similar results in bigger infinite cardinalities. The name "Fraïssé Hypothesis" has been used by Wiktor Marek. The Fraïssé Hypothesis has been studied by Fraïssé [6] and Marek [15], [16].

Our results are relative to the consistency of ZFC. If we assume more than the consistency of ZFC it is always explicitly mentioned.

In Chapter 2 we will recall the proof of Ajtai and use his method to prove various results related to $A(L^2, \omega)$ in the countable cardinality.

In Chapter 3 we will develop a forcing technique for coding subsets of ordinals by collapsing certain cardinals. This forcing is used to prove for example the following: If κ is a cardinal in L there is a transitive model of ZFC in which $A(L^4, \lambda)$ holds for exactly those cardinals λ which are smaller or equal to κ.

In Chapter 4 we will show that if κ is a cardinal, there is a language $L^{\kappa *}$ with κ many generalized quantifiers such that $A(L^{\kappa *}, \kappa)$ holds. Given a cardinal κ the language $L^{\kappa *}$ may be different for different models of ZFC containing κ and it is also possible that no such $L^{\kappa *}$ is definable in the language of set theory. This result for $\kappa = \omega$ is due to Scott Weinstein [29] and the generalization for uncountable κ is based on an idea of Per Lindström [14].

In Chapter 5 we will use Ajtai's method to prove that it is independent of ZFC whether $A(L^2_{\kappa,\omega}, \kappa)$ holds for a regular cardinal κ. We will also prove that for different regular cardinals κ and λ, $A(L^2_{\kappa,\omega}, \kappa)$ and $A(L^2_{\lambda,\omega}, \lambda)$ are independent of each other. We will also give an analogous result for singular cardinals.

In Chapter 6 we will investigate the relation between $A(L^2, \omega)$ and various large cardinal axioms. If there are infinitely many Woodin cardinals and a measurable cardinal above them, then $A(L^2, \omega)$ fails. Assuming the consistency of relevant large cardinal axioms, if n is a natural number, there is a model of ZFC in which there are n Woodin cardinals and $A(L^2, \omega)$ holds. As n grows bigger, more complex second order sentences seem to be needed to characterize all countable models up to isomorphism. $A(L^3, \omega)$ is consistent with Martin's Maximum and practically all large cardinal axioms.

Notation

The expression *ZF-formulas* refers to formulas in the language of set theory, i.e., first order language in a vocabulary with one binary relation $\in$. *ZF-equivalence* of two structures, denoted by $\mathfrak{A} \equiv_{ZF} \mathfrak{B}$, refers to the condition that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences of the language of set theory, i.e., for any sentence ϕ in the language of set theory $V \models \phi(\mathfrak{A}) \leftrightarrow \phi(\mathfrak{B})$. If L is a logic $\mathfrak{A} \equiv_L \mathfrak{B}$ refers to the condition that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences of L. $H(\kappa)$ refers to the set of sets hereditarily smaller than κ, i.e., $\{X$: the transitive closure of X has cardinality less than $\kappa\}$. The symbol $\restriction$ means "restricted to". Depending on context this can mean a reduct of a model to a smaller vocabulary or restriction of some operations to some set. The notation $\phi^{\mathfrak{M}}(\cdot)$ refers to the set of tuples which satisfy the formula ϕ in model $\mathfrak{M}$. The forcing name of a given set X is denoted by $\dot{X}$. Interpretation of a set in a given model of ZFC is denoted by the set with the model of ZFC as superscript: for example ω_1^{L} means ω_1 of L. The reals mean the same as the powerset of ω.

Notation which is not explained is standard as used for example in Jech's book [11].

1.2 Preliminaries

1.2.1 The logics L^n

In this section we will present some fundamental definitions and lemmas about higher order logics, forcing and L. This section does not contain any new results. In the rest of the paper we have clearly marked results of other mathematicians that we use. All the results which are not marked for somebody else are, according to our knowledge, new.

1.2.1. Definition. *An n-ary relation $R_i^n \subseteq (\mathrm{dom}(\mathfrak{A}))^n$ is* definable *in a language L in a model $\mathfrak{A}$ if there is an L-formula $\phi(x_1, \ldots, x_n)$ such that $R = \{(a_1, \ldots, a_n) : \mathfrak{A} \models \phi(a_1, \ldots, a_n)\}$.*

A class of structures C is characterizable *in a logic L if there is an L-formula $\phi_C(X_1, \ldots, X_m)$ such that in any model $\mathfrak{A}$ it holds that $\mathfrak{A} \models_s \phi_C(X_1, \ldots, X_m)$ $\Leftrightarrow (A, (X_1)_s^{\mathfrak{A}}, \ldots, (X_m)_s^{\mathfrak{A}}) \in C$. When C is a singleton class $\{\mathfrak{B}\}$ we say that the model $\mathfrak{B}$ is characterizable in L.*

1.2.2. Definition. *Each second order function variable F_i^m has a finite arity m. Each second order relation variable R_j^n has a finite arity n.*

Given a vocabulary τ, the set of $L^2[\tau]$ terms is the smallest set which contains first order variables, contains first order constants in vocabulary τ, is closed under functions in vocabulary τ and is closed under second order function variables.

Given a vocabulary τ, the set of $L^2[\tau]$ atomic formulas is the smallest set which contains equalities of $L^2[\tau]$ terms and contains the formulas $R_i^m(t_1, \ldots, t_m)$, where each t_n is an $L^2[\tau]$ term and R_i^m is either a relation symbol in τ of arity m or a second order relation variable of arity m.

Second order logic, denoted by L^2, is the smallest logic which

1. *Contains atomic formulas,*

2. *Is closed under negation and conjunction,*

3. *Is closed under first order existential and universal quantifiers and*

4. *Is closed under second order existential and universal quantifiers.*

Assume we have defined L^n. An $n+1$st order relation variable R_i^t has a type t which is a finite set of types of nth order relation variables.

The set of $n+1$st order atomic formulas is the smallest set which contains the nth order atomic formulas and the formulas $R_i^t(R_j^{t'})$, where R_i^t is an $n+1$st order variable of type t, $R_j^{t'}$ is an nth order variable of type t' and $t' \in t$.[1] *The $n+1$st order logic L^{n+1} is defined to be the smallest logic which*

1. *Contains atomic formulas*

2. *Is closed under the same operations as L^n and in addition closed under $n+1$st order existential and universal quantifiers.*

If $\mathfrak{A}$ is a model and s is an assignment of variables of L^{n+1} we have $\mathfrak{A} \models_s R_i^t(R_j^{t'})$ iff $(R_j^{t'})_s^{\mathfrak{A}} \in (R_i^t)_s^{\mathfrak{A}}$.

We use L^2 to refer to second order logic. In L^2 we can quantify over all finitary relations over the universe of the model, thus our second order logic means the second order logic with full semantics. There are also other second order logics which do not use full semantics such as *monadic second order logic* where we can quantify over unary relations only, and second order logic with Henkin semantics [10]. More generally L^n refers to nth order logic with full semantics.[2]

1.2.3. Definition. *Let $n \geq 1$ be a natural number. An L^n formula is Σ_0^{n-1} and Π_0^{n-1} if it does not contain any nth order quantifiers. A formula is Σ_{k+1}^{n-1} if it is of the form $\exists \bar{R}\phi$, where $\bar{R}$ is a sequence of nth order variables and ϕ is a Π_k^{n-1} formula. A formula is Π_{k+1}^{n-1} if it is of the form $\forall \bar{R}\phi$, where $\bar{R}$ is a sequence of nth order variables and ϕ is a Σ_k^{n-1} formula. A property is said to be Δ_m^{n-1}, if it is both Σ_m^{n-1} and Π_m^{n-1}.*

1.2.4. Lemma. *The following are characterizable in second order logic in any model $\mathfrak{A}$ of infinite cardinality κ:*

1. *A relation R is a function.*

2. *A relation R is an injection.*

3. *A relation R is a bijection from the set $\{x : \phi(x, \bar{a})\}$ to the set $\{x : \psi(x, \bar{a})\}$.*

4. *The set $\{x : \phi(x, \bar{a})\}$ is infinite.*

[1] To keep things simple we do not allow third order or higher order function variables.

[2] There are several ways to define L^n. By and large they are all equivalent (at least as long as they allow to prove Lemma 1.2.9).

5. *The set $\{x : \phi(x, \bar{a})\}$ is finite.*

6. *A tuple $(X, <_X)$ is a linear order, i.e., $<_X$ is a linear order in the set X .*

7. *A tuple $(X, <_X)$ is isomorphic to $(\omega, <)$.*

8. *A tuple $(X, <_X)$ is well-founded.*

9. *A tuple $(X, <_X)$ is a well-order.*

10. *A tuple $(X, <_X)$ is isomorphic to the model $(\kappa, \in)$.*

11. *A tuple $(X, +', \cdot', 0', 1')$ is isomorphic to the model $(\mathbb{N}, +, \cdot, 0, 1)$.*

12. *A tuple $(X, R_1^{n_1}, \ldots R_m^{n_m})$ is isomorphic to a tuple $(Y, R_{m+1}^{n_1}, \ldots R_{2m}^{n_m})$.*

13. *The canonical bijection π from $\kappa \times \kappa$ to κ and its inverse function: $\pi(\alpha, \beta) = \gamma$, $\pi^{-1}(\gamma) = (\alpha, \beta)$.*
    The canonical bijection π_n from κ^n to κ and its inverse function: $\pi_n(\alpha_1, \ldots \alpha_n) = \gamma$, $\pi_n^{-1}(\gamma) = (\alpha_1, \ldots \alpha_n)$.

14. *A tuple (S, S') is the transitive closure of $a \in R$ with respect to (R, R'), i.e., S is the set of those elements b in the domain of R such that there is a finite sequence $a_1, \ldots a_n$ in the domain of R so that $R'(a_1, a), R'(a_2, a_1), \ldots R'(a_n, a_{n-1})$ and $R'(b, a_n)$ and $S' = R' \cap S \times S$.*

Proof. In each case we give the relevant sentence. The natural numbers in the sentences (for example 1. in sentence number 2.) refer to the sentences in the list with the corresponding number.

1. $\forall x \forall y \forall z((R(x, y) \wedge R(x, z)) \to y = z)$

2. $1. \wedge \forall x \forall y \forall z((R(x, z) \wedge R(y, z)) \to x = y)$

3. $2. \wedge \forall x \forall y(R(x, y) \to (\phi(x, \bar{a}) \wedge \psi(y, \bar{a}))) \wedge \forall x \exists y((\phi(x, \bar{a}) \to R(x, y)) \wedge (\psi(x, \bar{a}) \to R(y, x)))$

4. $\exists R \exists z(\phi(z, \bar{a}) \wedge \theta_{\mathrm{bij}}(R, \bar{a}))$,
   where $\theta_{\mathrm{bij}}(R, \bar{a})$ expresses the condition that R is a bijection from the set $\{x : \phi(x, \bar{a})\}$ to the set $\{x : \phi(x, \bar{a}) \wedge \neg x = z\}$.

5. $\neg 4.$

6. The relevant sentence is the conjunction of the following:

   - $\forall x \forall y(x <_X y \to (X(x) \wedge X(y)))$
   - $\forall x \neg x <_X x$

- $\forall x \forall y \forall z((x <_X y \wedge y <_X z) \to x <_X z)$
- $\forall x \forall y((X(x) \wedge X(y)) \to (x = y \vee x <_X y \vee y <_X x))$

7. The sentence expresses the conjunction of the following:
   - 6.
   - $\{x : X(x)\}$ is infinite
   - $\forall x(X(x) \to (\{y : X(y) \wedge y <_X x\}$ is finite$))$
8. $\neg \exists R \exists y \psi$, where ψ expresses the conjunction of the following:
   - $\{x : R(x)\}$ is infinite
   - $R(y)$
   - $\forall x(R(x) \to (X(x) \wedge \exists z(R(z) \wedge z <_X x)))$
9. 6. $\wedge$ 8.
10. $9 \wedge \forall x(X(x) \to \neg \exists R \psi)$, where ψ expresses the condition that R is a bijection from the set $\{y : X(y)\}$ to the set $\{y : y <_X x\}))$
11. $\exists < \forall H \forall j\ (\phi_{\text{linear}}(<, X)$
    $\wedge \forall x \forall y(x < y \leftrightarrow (x \neq y \wedge \exists z(z +' x = y)))$
    $\wedge \forall x \forall y(x = y +' 1' \to \neg \exists z(y < z \wedge z < x))$
    $\wedge X(0') \wedge X(1') \wedge \neg 0' = 1'$
    $\wedge \phi_f(+', X) \wedge \phi_f(\cdot', X)$
    $\wedge \forall x \forall y(x +' 0 = x \wedge x +' (y' + 1) = (x +' y) +' 1)$
    $\wedge \forall x \forall y(x \cdot' 1' = x \wedge x \cdot' (y +' 1') = (x \cdot' y) +' x)$
    $\wedge \phi_{\text{bij}}(H, j))$

    In the above formula $\phi_{\text{linear}}(<, X)$ says that $<$ is a linear order in X, the formula $\phi_f(+', X)$ says that $+'$ is a function from $X \times X$ to X and the formula $\phi_{\text{bij}}(H, j)$ says that if H is an injection from $\{x : x < j\}$ to $\{x : x < j\}$, then H is a bijection.
12. $\exists P \psi$, where ψ expresses the conjunction of the following:
    - P is a bijection from X to Y
    - $\bigwedge_{1 \leq i \leq m} (\forall x_1, \ldots \forall x_{n_i} R_i^{n_i}(x_1, \ldots x_{n_i}) \leftrightarrow$
      $(\exists y_1, \ldots \exists y_{n_i} (\bigwedge_{1 \leq j \leq n_i} P(x_j, y_j) \wedge R_{m+i}^{n_i}(y_1, \ldots y_{n_i}))))$
13. $\exists P \psi$, where ψ expresses the conjunction of the following:
    - P is a bijection from $\kappa \times \kappa$ to κ
    - $P((\alpha, \beta), \gamma)$

- $\forall\alpha\forall\beta\forall\alpha'\forall\beta'(P(\alpha,\beta) < P(\alpha',\beta') \leftrightarrow (\beta < \beta' \vee (\beta = \beta' \wedge \alpha < \alpha')))$
- $\exists P\exists\alpha'_2,\ldots,\exists\alpha'_{n-1}\theta$, where θ expresses the conjunction of the following:
  a) P is a bijection from $\kappa\times\kappa$ to κ
  b) $P((\alpha_1,\alpha_2),\alpha'_2)\wedge\bigwedge_{2\leq m\leq n-2}P((\alpha'_m,\alpha_m),\alpha'_{m+1})\wedge P((\alpha'_{n-1},\alpha_{n-1}),\alpha'_n)$
  c) $\forall\alpha\forall\beta\forall\alpha'\forall\beta'(P(\alpha,\beta) < P(\alpha',\beta') \leftrightarrow (\beta < \beta' \vee (\beta = \beta' \wedge \alpha < \alpha')))$

14. $\forall x\forall y\psi$, where ψ expresses the conjunction of the following:

    - $S(x) \to R(x)$
    - $(S(x)\wedge S(y)) \to (S'(x,y) \leftrightarrow R'(x,y))$
    - $S(a)$
    - $((S(x)\wedge R(y,x)) \to S(y))$
    - $\forall P\forall P'\forall w((\forall x\forall y(P(x) \to R(x)\wedge((P(x)\wedge P(y)) \to (P'(x,y) \leftrightarrow R'(x,y))\wedge P(a)\wedge((P(x)\wedge R(y,x)) \to P(y))))\wedge S(w)) \to P(w))$

    The first four formulas say that (S,S') is a transitive set (with respect to R') which contains a, and the last formula says that (S,S') is the smallest such set.

$\square$

In the following theorem we say that a relation E is extensional if the extensionality axiom $\forall x\forall y(\forall z(zEx \leftrightarrow zEy) \to x = y)$ holds.

1.2.5. Theorem (Mostowski's Collapsing Theorem). *If E is a well-founded extensional relation on a class P, then there is a transitive class M and an isomorphism π between (P,E) and $(M,\in)$.*

Proof. We define the function π by transfinite induction on the well-founded class P:

$$\pi(x) = \{\pi(y) : E(y,x)\}.$$

It is clear from the definition that $ran(\pi)$ is a transitive class. We will show that $(ran(\pi),\in)$ is isomorphic to (P,E). We will prove that π is one-to-one. Assume not: Then there is an element $z\in P$ of the least possible rank such that $z = \pi(x) = \pi(y)$ for some $x \neq y$. As $x \neq y$ by symmetry and extensionality axiom we can assume there is an element a_0 such that $E(a_0,x)$ and not $E(a_0,y)$. As $\pi(x) = \pi(y)$ there is some element b_0 such that $E(b_0,y)$ and $\pi(a_0) = \pi(b_0)$. But this contradicts the assumption that z was of the least possible rank.

We will now prove that $x\,E\,y \leftrightarrow \pi(x)\in\pi(y)$. Assume $x\,E\,y$, then by definition $\pi(x)\in\pi(y)$. Assume then $\pi(x)\in\pi(y)$. By definition $\pi(x) = \pi(z)$ for an element z such that zEy. Since π is one-to-one we have $x = z$ and thus xEy.

$\square$

1.2.6. Definition. *We say that a second order formula $\phi(X,Y)$ is a second order definable well-order of the reals if in the model $(\mathbb{N},+,\cdot,0,1)$ the formula ϕ defines a well-order of the subsets of the universe of the model.*

We say that a second order formula $\phi(X,Y)$ is a second order definable well-order of the powerset of κ if in the model $(\kappa,\in)$ the formula ϕ defines a well-order of the subsets of the universe of the model.

Let $\tau = \{R_1,\ldots,R_n\}$ be a relational vocabulary and let the arity of R_i be k_i for each i. We will next introduce a way to code a model of infinite cardinality κ in vocabulary τ into a subset of κ^m, where $m = \Sigma_{1\le i\le n} k_i$.

1.2.7. Definition (Coding a model into a subset of κ^m).
Let $\mathfrak{B} = (\kappa, R_1^{\mathfrak{B}},\ldots,R_n^{\mathfrak{B}})$ be a model of cardinality κ in the vocabulary τ. Given an order $<$ of order type κ on B,[3] *the relations of $\mathfrak{B}$ can be coded into an m-ary relation $X_n^m \subseteq \kappa^m$ in the following way: any sequence of ordinals belongs to X_n^m iff for some i it is of the form*

$$(\underbrace{0,0,\ldots}_{\Sigma_{j<i}k_j\ times}\ \alpha_1+1,\alpha_2+1,\ldots\alpha_{n_i}+1,\ \underbrace{0,0,\ldots}_{\Sigma_{i<j\le n}k_j\ times})$$

for some ordinals $\alpha_1,\ldots\alpha_{n_i}$ such that $\mathfrak{B}\models R_i(\alpha_1,\ldots\alpha_{n_i})$. The ordinals α_i, α_i+1 etc. refer to elements of κ which have order type α_i, α_i+1, etc. with respect to $<$.

1.2.8. Lemma. *Let $\mathfrak{A} = (A, R_1^{\mathfrak{A}},\ldots R_n^{\mathfrak{A}})$ be a model of infinite cardinality κ in a finite vocabulary τ. Let $\#R_i = k_i$ for each i and $m = \Sigma_{1\le i\le n}k_i$. Then:*

- *$\mathfrak{A}$ is isomorphic to some models which have κ as universe.*
- *The set $I_{\mathfrak{A}}$ of those subsets of κ^m which are codes of models isomorphic to $\mathfrak{A}$ is L^2-characterizable in $\mathfrak{A}$.*

Proof. Obviously any bijection from A to κ generates a model isomorphic to $\mathfrak{A}$ which has κ as universe.

In $(\mathfrak{B},<)$ the relation X_n^m (introduced in Definition 1.2.7) is L^2-definable and each relation $R_i^{\mathfrak{B}}$ is second order definable from X_n^m. Let $\psi(X_n^m,\mathfrak{B},<)$ be the following second order formula which says that X_n^m is the code of $\mathfrak{B}$ with respect to $<$:

$$\forall x_1,\ldots\forall x_m(X_n^m(x_1,\ldots x_m)\leftrightarrow \bigvee_{1\le i\le m}\phi_i)$$

where ϕ_i is the conjunction of the following formulas[4]:

[3]The reader may wonder where does $<$ come from. If the model does not contain a copy of $(\kappa,\in)$ we can build such a copy by second order quantifiers. We have formulas of the form $\exists K\exists E(\phi_{\kappa,\in}(K,E)\wedge\ldots)$, where $\phi_{\kappa,\in}$ characterizes $(\kappa,\in)$.

[4]To be precise the formulas below, such as $x_j = 0$ are not formulas in our language. However 0, immediate predecessor of an element and immediate successor of an element(all with respect to $<$) are definable so it is possible to write the expressions below formally.

- $\bigwedge_{j\leq \Sigma_{t<i}k_t} x_j = 0$
- $\bigwedge_{j>\Sigma_{t\leq i}k_t} x_j = 0$
- $\bigwedge_{j\in 1+\Sigma_{t\leq i}k_t,\ldots k_i+\Sigma_{t\leq i}k_t} x_j \neq 0 \wedge R(x_{1+\Sigma_{t\leq i}k_t}-1,\ldots x_{k_i+\Sigma_{t\leq i}k_t}-1)$.

Let $C\subseteq \kappa^m$. C is a code of a model isomorphic to $\mathfrak{A}$ (with respect to $<$) iff

$$\exists P_1^{k_1}\exists P_2^{k_2}\ldots,\exists P_n^{k_n}\exists T((\phi_{\mathrm{bij}}(T,A,\mathrm{dom}(<))\wedge \bigwedge_{1\leq i\leq n}\phi_i)\wedge \psi(C,(\kappa,P_1,\ldots P_n),<)),$$

where $\phi_{\mathrm{bij}}(T,A,\mathrm{dom}(<))$ says that T is a bijection from A to $\mathrm{dom}(<)$, ψ is defined above and ϕ_i is the following formula:

$$\forall x_1,\ldots,\forall x_{k_i}(R_i(x_1,\ldots x_{k_i})\leftrightarrow P_i(T(x_1),\ldots T(x_{k_i}))).$$

$\square$

1.2.9. Lemma. *a) Let ϕ be a second order formula in a finite vocabulary τ, $\mathfrak{A}$ a model of cardinality κ with vocabulary τ, and let s be an assignment of the free variables of ϕ in $\mathfrak{A}$. Then ϕ with the assignment s is equivalent to a ZF-formula in $H(\kappa^+)$ with $\mathfrak{A}$ and s as parameters, i.e., there is a formula θ in the language of set theory such that $\mathfrak{A}\models_s \phi \Leftrightarrow H(\kappa^+)\models \theta(\mathfrak{A},s)$. More generally an nth order formula in a model $\mathfrak{A}$ of cardinality κ with assignment s is equivalent to a ZF-formula in $H((\beth_{n-2}(\kappa))^+)$ with $\mathfrak{A}$ and s as parameters.*

b) Let τ be a finite vocabulary and assume we have fixed some second order characterizable way to code models in the vocabulary τ by subsets of κ. There is a translation which translates every ZF(I)-sentence[5] ϕ in $(H(\kappa^+),I_{\mathfrak{A}},\in)$ to a second order sentence ϕ^* in the vocabulary τ in such a way that $(H(\kappa^+),I_{\mathfrak{A}},\in)\models\phi\Leftrightarrow \mathfrak{A}\models\phi^*$ for any model $\mathfrak{A}$ of cardinality κ in vocabulary τ. More generally, there is a translation which translates every $ZF(I)$-sentence ϕ in $H((\beth_{n-2}(\kappa))^+)$ to an nth order sentence ϕ^* in such a way that $(H(\beth_{n-2}(\kappa)^+),I_{\mathfrak{A}},\in)\models\phi\Leftrightarrow\mathfrak{A}\models\phi^*$.

Proof. a) Assignments of finitely many first order and second order variables in the model $\mathfrak{A}$ belong to $H(\kappa^+)$. To formalize truth definition of ϕ in $\mathfrak{A}$ with an assignment s we need only quantify over those assignments which are in $H(\kappa^+)$. Generally third order variables are sets of second order variables and have cardinality at most 2^κ, fourth order variables have cardinality at most $2^{(2^\kappa)}$ and so on. It follows that interpretations of finitely many nth order variables belong to $H((\beth_{n-2}(\kappa))^+)$ and the truth definition of an nth order formula ϕ with an

[5]This means we have added an extra unary predicate I to $H(\kappa^+)$ and interpreted it as the set of those subsets of κ^m which are codes of models isomorphic to $\mathfrak{A}$. A $ZF(I)$-sentence is a first order sentence in vocabulary $\{\in,I\}$ where $\in$ is a binary relation symbol and I is a unary relation symbol.

assignment s in a model $\mathfrak{A}$ can be formalized in $H((\beth_{n-2}(\kappa))^+)$ with $\mathfrak{A}$ and s as parameters.

b) In second order logic we can quantify over transitive closures of the sets in $H(\kappa^+)$ in the following way. If R is a well-founded binary relation which satisfies the extensionality axiom $\forall x \forall y(\forall z(Rzx \leftrightarrow Rzy) \to x = y)$, then $(\mathrm{dom}(R), R)$ is by Mostowski's Collapsing Theorem 1.2.5 isomorphic to a transitive set. If R is also either empty or has a maximal element (i.e., $\exists y \in \mathrm{dom}(R) \neg \exists x Ryx$), then $(\mathrm{dom}(R), R)$ is isomorphic to $(TC(a), \epsilon)$ for some $a \in H(\kappa^+)$. On the other hand, if $a \in H(\kappa^+)$ then $|TC(a)| \leq \kappa$ and there is a well-founded and extensional relation $R_a \subset A \times A$ such that $(\mathrm{dom}(R), R)$ is isomorphic to $(TC(a), \epsilon)$. Thus in second order logic we can in a sense quantify over transitive closures of sets in $H(\kappa^+)$.

Let $\psi(R)$ be a second order formula which says that R is a well-founded binary relation which satisfies the extensionality axiom and is either empty or has a greatest element.

We can define two sets R and R' to be equal if and only if there is an isomorphism from $(\mathrm{dom}(R), R)$ to $(\mathrm{dom}(R'), R')$. Now we can define $x =^* y$ to be $\psi(R_x) \wedge \psi(R_y) \wedge (\mathrm{dom}(R_x), R_x) \cong (\mathrm{dom}(R_y), R_y)$.

We can define $\in$ as follows: $R_x \in^* R_y$ iff $\exists v \exists w \exists Q \exists T \psi$, where θ expresses the conjunction of the following:

1. x is maximal element in R_y
2. $R'(w, v)$
3. $Q = (P, R_y \restriction P)$ is the transitive closure of w with respect to R_y (See Lemma 1.2.4(14))
4. T is an isomorphism from $(\mathrm{dom}(R_x), R_x)$ to Q

Let then $x \in^* y = \psi(R_x) \wedge \psi(R_y) \wedge \exists x \exists y \exists Q \exists T \theta$. Let then $(\neg\phi)^* = \neg(\phi^*)$, $(\phi \wedge \theta)^* = \phi^* \wedge \theta^*$ and $(\exists x \phi)^* = \exists R_x \phi^*$.

We have shown that any ZF-sentence in $H(\kappa^+, \in)$ is equivalent to an L^2-sentence in the model $\mathfrak{A}$.

By Lemma 1.2.8 the relation $I_{\mathfrak{A}}$ is also second order definable so the claim follows.

Next we will generalize the above result for $n > 2$. First we will define the concept of a *hereditarily monadic variable*. For a start we say that a monadic second order variable is hereditarily monadic. If we have defined what it means for an nth order variable to be hereditarily monadic, we define an $n+1$st order variable to be hereditarily monadic iff it has arity 1 and its only argument is a relation of one type: hereditarily monadic nth order variable. It is easy to prove by induction that in a model of cardinality κ there are $\beth_{n-1}(\kappa)$ hereditarily monadic nth order relations, i.e., interpretations of hereditarily monadic n:th order variables.

Now the above proof works for L^{n+1} when we replace first order variables by hereditarily monadic nth order variables and second order variables of arity m by $n+1$st order variables which have as arguments only hereditarily monadic nth order variables.

We denote $A_n = \{A : A$ is a hereditarily monadic nth order relation over $\mathfrak{A}\}$. Thus $|A_n| = \beth_{2^{n-1}}(\kappa)$ and in L^{n+1} we can quantify over subsets of $(A_n \times A_n)$. Certain subsets B of $A_n \times A_n$ correspond to transitive closures of sets in $H((\beth_{2^{n-1}}(\kappa))^+)$, namely those sets B such that $(\mathrm{dom}(B), B)$ satisfies the axiom of extensionality, B has a largest element and B is well-founded. As before, we can define two sets of the above form to be the same if they are isomorphic and a set a belongs to another set b if and only if there is an element b_0 in the domain of b which belongs to the greatest element in b, and the transitive closure of b_0 with respect to b is isomorphic to a. Also, we can characterize the set of those sets of the form κ^n which are isomorphic to $\mathfrak{A}$, so $I_{\mathfrak{A}}$ is characterizable as well. Thus there is a translation of $ZF(I)$-sentences of $(H((\beth_{2^{n-1}}(\kappa))^+), I, \in)$ to L^{n+1}-sentences in the model $\mathfrak{A}$, likewise there is a translation of $ZF(I)$-sentences of $(H((\beth_{2^{n-2}}(\kappa))^+), I, \in)$ to L^n-sentences in model $\mathfrak{A}$.

□

1.2.2 Infinitary second order languages

We will next define second order infinitary language $L^2_{\kappa,\omega}$. The nth order infinitary languages $L^n_{\kappa,\omega}$ can be defined in an analogous way.

1.2.10. Definition. *Let $n \in \omega$ and let κ be a regular cardinal. The logic $L^2_{\kappa,\omega}$ is the smallest logic which*

1. *Contains all atomic formulas,*
2. *Is closed under negation, conjunctions of size less than κ, disjunctions of size less than κ, first order existential and universal quantifiers and second order existential and universal quantifiers*[6].

1.2.11. Lemma. *Let κ be a regular cardinal. In the logic $L_{\kappa,\omega}$ all ordinals $(\alpha, <)$ smaller than κ are characterizable.*

Proof. This is done by induction on the ordinal $\alpha < \kappa$. Assume the Induction Hypothesis holds for all $\beta < \alpha$, i.e., there are formulas $\theta_\beta(y)$ which characterize $(\beta, <)$ for ordinals $\beta < \alpha$. Now the formula which characterizes $(\alpha, <)$ is

$$\bigwedge_{\beta<\alpha} \exists y(y < x \wedge \theta_\beta(y)) \wedge \forall y(y < x \rightarrow \bigvee_{\beta<\alpha} \theta_\beta(y))$$

[6] We allow here both second order relation variables and second order function variables

□

We will now present a lemma which is needed to show the indenpendence of $A(L^2_{\kappa,\omega},\kappa)$ at a regular cardinal κ. In Definition 1.2.13 we will give an exact coding of $L^2_{\kappa,\omega}$-formulas as set theoretic objects and prove the lemma.

1.2.12. LEMMA. *Let $n \in \omega$. Every formula of $L^2_{\kappa^+,\omega}$ can be defined in $(V,\in)$ (or in $(H((\kappa^+)^+),\in)$) by a ZF-formula using a subset of κ as a parameter.*

If κ is an inaccessible cardinal, every formula of $L^2_{\kappa,\omega}$ can be defined in $(V,\in)$ (or in $(H(\kappa^+),\in)$) by a ZF-formula using a subset of some $\lambda < \kappa$ as a parameter.

The next definition proves Lemma 1.2.12.

1.2.13. DEFINITION. *We will introduce a coding where all $L^2_{\kappa,\omega}$ formulas are coded by subsets of κ, or in fact by subsets of ordinals smaller than κ. First the atomic formulas:*

1. *A symbol in the vocabulary of the model which has been assigned a prime number code n by a chosen Gödel numbering (as described in Lemma 6.1.2) is $\langle 1, n\rangle$.*
2. $x_\alpha = \langle 2, \alpha\rangle$.
3. $c_\alpha = \langle 3, \alpha\rangle$.
4. $R^n_\alpha = \langle 4, n, \alpha\rangle$, *these are the codes for relation variables.*
5. $F^n_\alpha = \langle 5, n, \alpha\rangle$, *these are the codes for function variables.*
6. $F^n_i(t_1,\ldots,t_n) = \langle 6, F^n_i, t_1,\ldots,t_n\rangle$.
7. $t_i \equiv t_j = \langle 7, t_i, t_j\rangle$.
8. $R^n_i(t_1,\ldots,t_n) = \langle 8, R^n_i, t_1,\ldots,t_n\rangle$.

We describe now how to code objects of the above form by subsets of κ in a systematic way. There is a second order definable bijection from κ to $\kappa\times\kappa$ (see Lemma 1.2.4(13)). The objects are coded in such a way that the nth κ codes the nth coordinate in the tuple. For example $c_{\omega+1}$ has in the beginning of the first κ three ones and the rest are zeros. In the beginning of the second κ it has $\omega+1$ ones and the rest are zeros, and all the other κ:s have just zeros. The code of $F^1_1(c_{\omega+1})$ has 6 ones in the beginning of the first κ, code of F^1_1 in the second κ and in the third κ the subset of κ coding $c_{\omega+1}$ we just described.

By this coding the predicates "X is a $L^2_{\kappa,\omega}(\tau)$ term" and "X is a $L^2_{\kappa,\omega}(\tau)$ atomic formula" are characterizable in second order logic.

The non-atomic formulas are coded as follows:

1. $\neg\phi = \langle 9, \{\phi\}\rangle$.
2. $\bigwedge X = \langle 10, X\rangle$.
3. $\bigvee X = \langle 11, X\rangle$.
4. $\exists x_\alpha \phi = \langle 12, x_\alpha, \{\phi\}\rangle$.
5. $\forall x_\alpha \phi = \langle 13, x_\alpha, \{\phi\}\rangle$.
6. $\exists R^n_\alpha \phi = \langle 14, R^n_\alpha, \{\phi\}\rangle$.
7. $\forall R^n_\alpha \phi = \langle 15, R^n_\alpha, \{\phi\}\rangle$.
8. $\exists F^n_\alpha \phi = \langle 16, F^n_\alpha, \{\phi\}\rangle$.
9. $\forall F^n_\alpha \phi = \langle 17, F^n_\alpha, \{\phi\}\rangle$.

These are coded by subsets of κ in a similar way as atomic formulas except that there are also sets of formulas. For example objects of type 2 are coded by 10 ones in the first κ and in the second κ a code for the set X. X is a set of formulas of size less than κ and we use again the second order definable bijection from κ to $\kappa \times \kappa$. The first κ codes some formula in X, the second κ codes some other formula in X, if there is any, and so on until after some α many κ:s there are no more formulas in X and the rest are just zeros.

We have defined $L^2_{\kappa,\omega}$-formulas as subsets of κ, but in fact in all subsets coding a $L^2_{\kappa,\omega}$-formula the set of ones is not cofinal in κ. Thus we can as well think them as subsets of some $\alpha < \kappa$ when we have cut away the zeros from the end. This proves Lemma 1.2.12.

We will next present the definition of generalized quantifiers:

1.2.14. Definition. *Suppose τ is a finite relational vocabulary $\{R_1, ..., R_k\}$, where R_i is r_i-ary. Suppose K is a class of τ-structures, closed under isomorphisms. We get an extension of first order logic by adding to the syntax a new quantifier symbol Q_K, a new formula generation rule*

- *If $\phi_1, ..., \phi_k$ are formulas, then so is $Q_K x^1_1 ... x^1_{r_1}, ..., x^k_1 ... x^k_{r_k}; \phi_1, ..., \phi_k$.*

The semantics is defined as follows:

- *$M \models_s Q_K x^1_1 ... x^1_{r_1}, ..., x^k_1 ... x^k_{r_k}; \phi_1, ..., \phi_k$ if and only if*[7]
$$(\mathrm{dom}(M), P_1, ..., P_k) \in K,$$
where
$$P_i = \{(s(x^i_i), ..., s(x^i_{r_i})) : M \models_s \phi_i\}.$$

[7] Here $\mathrm{dom}(M)$ means the domain of the model M.

The extension of first order logic is denoted $L_{\omega\omega}(Q_K)$.

Note that K is always definable in $L_{\omega\omega}(Q_K)$ by the sentence

$$Q_K x_1^1 ... x_{r_1}^1, ..., x_1^k ... x_{r_k}^k; R_1(x_1^1, ..., x_{r_1}^1), ..., R_k(x_1^k, ..., x_{r_k}^k).$$

We will next give some examples of generalized quantifiers:

1. The cardinality quantifier Q_α, "There are $\aleph_\alpha$-many":

$$\mathfrak{A} \models Q_\alpha x \phi(x, \bar{y}) \Leftrightarrow |\{x : \mathfrak{A} \models \phi(x, \bar{y})\}| \geq \aleph_\alpha.$$

2. Härtig quantifier I, the "equicardinality quantifier":

$$\mathfrak{A} \models Ix, y(\phi(x, \bar{z}), \psi(y, \bar{z})) \Leftrightarrow |\{x : \mathfrak{A} \models \phi(x, \bar{z})\}| = |\{y : \mathfrak{A} \models \psi(y, \bar{z})\}|.$$

Lots of cardinality quantifiers Q_α of the form 1. are definable in L^2, see Remark 4.2.3. From Lemma 1.2.4 (3.) it follows that Härtig quantifier is also definable in L^2.

1.2.3 The constructible universe L

In 1938 Kurt Gödel introduced L, the class of constructible sets, see for example [7]. Recall the V-hierarchy of sets:

- $V_0 = \emptyset$.
- $V_{\alpha+1} = \mathcal{P}(V_\alpha)$.
- $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$ for α limit ordinal.
- $V = \bigcup_{\alpha \in On} V_\alpha$.

The L-hierarchy is similar to the V-hierarchy, except that in the successor steps we take only the definable powerset $def(\mathrm{L}_\alpha)$, i.e., the set of those subsets of L_α which are definable in L_α using elements of L_α as parameters.

- $\mathrm{L}_0 = \emptyset$.
- $\mathrm{L}_{\alpha+1} = def(\mathrm{L}_\alpha)$.
- $\mathrm{L}_\alpha = \bigcup_{\beta<\alpha} \mathrm{L}_\beta$ for α limit ordinal.
- $\mathrm{L} = \bigcup_{\alpha \in On} \mathrm{L}_\alpha$.

A few notes about L: GCH holds in L. The operation $def(\mathrm{L}_\alpha)$ can be defined as the closure of $\mathrm{L}_\alpha \cup \{\mathrm{L}_\alpha\}$ under certain finitely many "Gödel functions":

1. $G_1(Y,Z) = \{Y,Z\}$,
2. $G_2(Y,Z) = Y \times Z$,
3. $G_3(Y,Z) = \{(u,v) : u \in Y \wedge v \in Z \wedge u \in v\}$,
4. $G_4(Y,Z) = Y - Z$,
5. $G_5(Y,Z) = Y \cap Z$,
6. $G_6(Y) = \bigcup Y$,
7. $G_7(Y) = \mathrm{dom}(Y)$,
8. $G_8(Y) = \{(u,v) : (v,u) \in Y\}$,
9. $G_9(Y) = \{(u,v,w) : (u,w,v) \in Y\}$,
10. $G_{10}(Y) = \{(u,v,w) : (v,w,u) \in Y\}$.

By induction it can be proved that $|\mathrm{L}_\alpha| = |\alpha|$ for all infinite α. All Gödel functions are second order characterizable so the operation $def(\mathrm{L}_\alpha)$ is second order characterizable.

For each L_α there is a canonical well-order of L_α denoted by $<_{\mathrm{L}_\alpha}$. If $\alpha < \beta$ then $<_{\mathrm{L}_\beta}$ is an end extension of $<_{\mathrm{L}_\alpha}$. We will next describe how the canonical well-order is defined.

The well-order is defined by induction on α. Assume we have defined $<_{\mathrm{L}_\alpha}$. We will define next $<_{\mathrm{L}_{\alpha+1}}$. The idea is that in the beginning is $<_{\mathrm{L}_\alpha}$, then L_α and then the rest of $\mathrm{L}_{\alpha+1}$ in the order

i) How many times Gödel functions need to be iterated starting from the elements of L_α in order to reach the elements in question.

ii) Which Gödel functions need to be used.

iii) To which sets in L_α the Gödel functions need to be applied (here we can use the already defined canonical well-order of L_α and define the element which can be reached from smaller elements to be smaller).

We will next present a technical definition for the idea described above:

1.2.15. Definition. *We define* $W_0^\alpha = \mathrm{L}_\alpha \cup \{\mathrm{L}_\alpha\}$,
$W_{n+1}^\alpha = \{G_i(Y,Z) : Y,Z \in W_n^\alpha, i \in \{1,\ldots,10\}\}$.

1. $<_{\alpha+1}^0$ *is the well-ordering of* $\mathrm{L}_\alpha \cup \{\mathrm{L}_\alpha\}$ *that extends* $<_{\mathrm{L}_\alpha}$ *such that* L_α *is the greatest element.*

2. $<^{n+1}_{\alpha+1}$ *is the following well-order of* W^α_{n+1}*:*

$x <^{n+1}_{\alpha+1} y$ *iff one of the conditions below hold:*

a) $x <^n_{\alpha+1} y$

b) $x \in W^\alpha_n$ *and* $y \notin W^\alpha_n$

c) $x \notin W^\alpha_n$ *and* $y \notin W^\alpha_n$ *and one of the following holds:*

- *"The least* i *such that* $\exists u, v \in W^\alpha_n(x = G_i(u, v))$*"* $<$ *"the least* j *such that* $\exists s, t \in W^\alpha_n(y = G_j(s, t))$*"*
- *"The least* i *such that* $\exists u, v \in W^\alpha_n(x = G_i(u, v))$*"* $=$ *"the least* j *such that* $\exists s, t \in W^\alpha_n(y = G_j(s, t))$*" and*

  "the $<^n_{\alpha+1}$*-least* $u \in W^\alpha_n$ *such that* $\exists v \in W^\alpha_n(x = G_i(u, v))$*"* $<^n_{\alpha+1}$ *"the* $<^n_{\alpha+1}$*-least* $s \in W^\alpha_n$ *such that* $\exists t \in W^\alpha_n(y = G_i(s, t))$*"*
- *"The least* i *such that* $\exists u, v \in W^\alpha_n(x = G_i(u, v))$*"* $=$ *"the least* j *such that* $\exists s, t \in W^\alpha_n(y = G_j(s, t))$*" and "the* $<^n_{\alpha+1}$*-least* $u \in W^\alpha_n$ *such that* $\exists v \in W^\alpha_n(x = G_i(u, v))$*"* $=$ *"the* $<^n_{\alpha+1}$*-least* $s \in W^\alpha_n$ *such that* $\exists t \in W^\alpha_n(y = G_i(s, t))$*" and "the* $<^n_{\alpha+1}$*-least* $v \in W^\alpha_n$ *such that* $x = G_i(u, v)$*"* $<^n_{\alpha+1}$ *"the* $<^n_{\alpha+1}$*-least* $t \in W^\alpha_n$ *such that* $y = G_i(u, t)$*".*

Now we let $<_{\alpha+1}= \bigcup_{n\in\omega} <^n_{\alpha+1} \cap (P(\mathrm{L}_\alpha) \times P(\mathrm{L}_\alpha))$ *(where* $P(\mathrm{L}_\alpha)$ *refers to the powerset of* L_α*).*

For limit ordinals γ *we define* $<_{\mathrm{L}_\gamma} = \bigcup_{\alpha<\gamma} <_{\mathrm{L}_\alpha}$*.*

The inductive definition of the class function $\alpha \mapsto <_{\mathrm{L}_\alpha}$ is clearly definable in the language of set theory. Looking at the definition, in order to define $<_{\mathrm{L}_\alpha}$ from α there is no need to quantify over sets outside $H(|\alpha|^+)$. Now it follows from Lemma 1.2.9 that the function $f : |\alpha|^+ \to H(|\alpha|^+), f(\beta) = <_{\mathrm{L}_\beta}$ is second order characterizable in any model of cardinality $|\alpha|$. It follows that in L in a model of cardinality κ there is a second order characterizable well-order of the powerset of κ: $X < Y \Leftrightarrow \exists\alpha(X <_{\mathrm{L}_\alpha} Y)$.

If X is a subset of an ordinal we can form the class $\mathrm{L}[X]$, the least transitive model of ZFC containing all ordinals and X.[8] The construction of $\mathrm{L}[X]$ is similar to the construction of L except that we are allowed to intersect any set with X (this can be done by adding 11th Gödel function $G_{11}(Y) = Y \cap X$). By a reasoning similar to what was presented above, if X is second order characterizable in a model then the functions $\alpha \mapsto \mathrm{L}_\alpha[X]$ and $\alpha \mapsto <_{\mathrm{L}_\alpha[X]}$ are second order characterizable in the model.

[8] The structure of any set can be coded by a subset of some ordinal so we can make this assumption w.l.o.g. The assumption "X is a subset of some ordinal" turns out to be very useful as in infinitary second order languages $L^2_{\kappa,\omega}$ we can characterize all subsets of ordinals smaller than κ.

The class $\mathrm{L}[X]$, the least transitive model of ZFC containing all ordinals and X, should not be confused with $\mathrm{L}(\mathbb{R})$, the least model of ZF containing all the reals.

1.2.4 Forcing

Forcing is a method invented by Paul Cohen in 1963. He used forcing to prove independence of the Continuum Hypothesis from ZFC [4], [5]. Forcing is a very general method for proving independence results and constructing different models of ZFC. The invention of forcing has had huge impact to the development of set theory.

The idea of forcing is briefly as follows: We have a transitive model M of ZFC. Inside M we can form a forcing language which describes the model $M[G]$ which is the smallest transitive model of ZFC extending M and containing G. What kind of generic set G a forcing adds depends on the type of the forcing. The notation $p \Vdash \phi$ means that p forces ϕ i.e. whenever $p \in G$ then $M[G] \models \phi$. We don't give a detailed introduction to forcing here as it is a broad and complicated topic. For a reader who wants to study forcing we recommend Jech [11].

We present without proofs the following fundamental theorems about forcing. The proofs can be found for example from [11].

1.2.16. Theorem (The Generic Model Theorem). *Let M be a transitive model of ZFC and let $(P,<)$ be a notion of forcing in M. If $G \subset P$ is generic over P, then there exists a transitive model $M[G]$ such that the following hold:*

1. *$M[G]$ is a model of ZFC.*
2. *$M \subset M[G]$ and $G \in M[G]$.*
3. *$M[G]$ and M have the same ordinals.*
4. *If N is a transitive model of ZF such that $M \subset N$ and $G \in N$, then $M[G] \subset N$.*

1.2.17. Theorem (The Forcing Theorem). *Let $(P,<)$ be a notion of forcing in the ground model M. If σ is a sentence of the forcing language, then for every $G \subset P$ generic over M,*

$$M[G] \models \sigma \Leftrightarrow (\exists p \in G) p \Vdash \sigma.$$

In the left-hand-side σ one interprets the constants of the forcing language according to G.

The most important forcings used in this paper are the following:

- Cohen forcings: If κ is a regular cardinal in M there is a Cohen forcing (which is a different forcing for different cardinals κ) which adds a new subset to κ.
- Cardinal collapses: If κ and λ are infinite cardinals in M such that $\kappa < \lambda$ there is a forcing which adds a bijection from κ to λ, i.e., collapses λ to κ. After the forcing λ and all cardinals strictly between κ and λ are not cardinals anymore.
- Iterated forcing: Iterated forcing was developed by Solovay and Tennenbaum [26] in a paper where they proved the independence of Souslin Hypothesis from ZFC. Iterated forcing is a technically complicated topic and instead of defining iterated forcing here we refer to Jech [11]. The idea is that by iterated forcing we can do α many successive forcings, where α is the length of the iterated forcing.

  We say that at a limit ordinal γ the support of a forcing condition p of length γ is the set of those ordinals $\alpha < \gamma$ where $p(\alpha)$ is non-zero. A forcing can have at a limit ordinal (for example) the following:

  Finite support: Any forcing condition p has finite support.

  Countable support: Any forcing condition p has countable support.

  Full support: A forcing condition may have whole λ as support.

  Direct limit: $p \in P_\alpha$ if and only if $\exists \beta < \alpha\, (p \restriction \beta \in P_\beta$ and $\forall \xi \geq \beta\; p(\xi) = 1)$.

  Inverse limit: $p \in P_\alpha$ if and only if $\forall \beta < \alpha\; p \restriction \beta \in P_\beta$.

1.2.18. Lemma (The Factor Lemma). *Let $P_{\alpha+\beta}$ be a forcing iteration of $\langle \dot{Q}_\xi : \xi < \alpha + \beta\rangle$, where each P_ξ, $\xi \leq \alpha + \beta$ is either a direct limit or an inverse limit. In V^{P_α}, let $\dot{P}_\beta^{(\alpha)}$ be the forcing iteration of $\langle \dot{Q}_{\alpha+\xi} : \xi < \beta\rangle$ such that for every limit ordinal $\xi < \beta$, $\dot{P}_\xi^{(\alpha)}$ is either a direct limit or an inverse limit, according to whether $P_{\alpha+\beta}$ is a direct limit or an inverse limit. If $P_{\alpha+\beta}$ is an inverse limit for every limit ordinal $\xi \leq \beta$ such that $cf\xi \leq |P_\alpha|$, then $P_{\alpha+\beta}$ is isomorphic to $P_\alpha * \dot{P}_\beta^{(\alpha)}$.*

1.2.19. Definition. *A forcing is* κ-closed *if for any increasing sequence of conditions of length less than κ there is a condition which is stronger than all the conditions in the sequence.*

A forcing satisfies the κ-chain-condition *if any antichain of forcing conditions (i.e., a set of pairwise incompatible conditions) has cardinality less than κ.*

A κ-closed forcing does not add any new subsets to cardinals smaller than κ. A κ-chain-condition forcing does not collapse cardinals greater or equal to κ.

Chapter 2

Ajtai's result, the countable case

2.1 $A(L^2, \omega)$ and L^2-definable well-order of the reals

In this chapter we will present two theorems by Ajtai which show that $A(L^2, \omega)$ is independent of ZFC. After that we will discuss some related topics concerning countable models.

We recall that Ajtai proved the independence of $A(L^2, \omega)$ from ZFC. We will now present the first part of the proof of Ajtai:

2.1.1. Theorem (Ajtai [2]). *If there is a second order definable well-order of the powerset of ω, then $A(L^2, \omega)$ holds. If the well-order is Σ^1_n for $n \geq 2$, then $A(\Sigma^1_{n+1}, \omega)$ holds.*

Proof. We will show that if there is a second order definable well-order of the reals, $A(\Sigma^1_k, \omega)$ holds for certain k. Let us assume our second order definable well-order of the reals is Δ^1_n for some $n \geq 2$. We make the assumption $n \geq 2$ to make complexity calculations simpler; in all our applications $n \geq 2$ so it does not do any harm. Note that if a well-order is Σ^1_n then it is Π^1_n because $x < y \Leftrightarrow x \neq y \wedge \neg y < x$. Similarly every Π^1_n well-order is Σ^1_n. Thus a well-order is Σ^1_n iff it is Π^1_n iff it is Δ^1_n. Also two models are Σ^1_n-equivalent iff they are Π^1_n-equivalent as we will show. Assume not: there are Σ^1_n-equivalent models $\mathfrak{A}$ and $\mathfrak{B}$ which are not Π^1_n-equivalent. Assume ϕ is such a Π^1_n formula that $\mathfrak{A} \models \phi$ and $\mathfrak{B} \not\models \phi$. Now $\neg\phi$ is such a Σ^1_n formula that $\mathfrak{A} \not\models \neg\phi$ and $\mathfrak{B} \models \neg\phi$, so the models are not Σ^1_n-equivalent, which is a contradiction. The proof that Π^1_n-equivalence implies Σ^1_n-equivalence is the same.

As we have shown in Lemma 1.2.8, a model of cardinality $\aleph_0$ in a finite vocabulary is isomorphic to some models which have ω as universe. These models can be coded into n-ary relations on ω in a second order definable way, and the set I of codes of models which have ω as their universe and are isomorphic to the model in question is second order characterizable in the model in question. As there is a second order definable well-order of the reals and a second order characterizable bijection from ω^n to ω, we can talk in second order logic about the least subset A_0 of ω which is mapped to a set in I by the bijection. For each natural number n we can say in second order logic that n belongs to A_0, and also that n does not belong to A_0. If two countable models in a finite vocabulary have the same second order theory then they have the same set A_0. Consequently they have the same isomorphism type and they are isomorphic.

We will next present the definition of these sentences mentioned above and calculate the complexity of them. Let Φ be the second order sentence:

$$\begin{aligned}&\exists N \exists 0' \exists 1' \exists +' \exists \cdot' \exists < \exists \pi_n \exists \bar{A}_0 \exists A'_0 \exists A^*_0\\ &\quad (\mathrm{def}(N, 0', 1', +', \cdot') \ \wedge \ \mathrm{def}(\pi_n) \ \wedge \ \theta_{\bar{A},\tau}\\ &\quad \wedge \psi_{\cong}(\bar{A}_0) \ \wedge \ \phi_{code}(\bar{A}_0, A'_0) \ \wedge \ \eta_n(A'_0, A^*_0) \ \wedge\\ &\quad \forall \bar{A}_1 \forall A'_1 \forall A^*_1((\theta_{\bar{A}_1,\tau} \ \wedge \ \psi_{\cong}(\bar{A}_1) \ \wedge \ \phi_{code}(\bar{A}_1, A'_1) \ \wedge \ \eta_n(A'_1, A^*_1)\\ &\quad\quad \to (\phi'(A^*_0, A^*_1) \vee \forall x(A^*_0(x) \leftrightarrow A^*_1(x)))) \wedge A^*_0(47)).\end{aligned} \tag{2.1}$$

Here is an explanations of the different components of the sentence:

- $\mathrm{def}(N, 0', 1', +', \cdot')$ is the Π^1_1-formula which defines the structure $(\mathbb{N}, 0, 1, +, \cdot)$,
- $\mathrm{def}(\Pi_n)$ is the first order formula which defines a bijection from N^n to N, see Lemma 1.2.4.
- $\theta_{\bar{A},\tau}$ is a first order formula which says that $\bar{A}$ is a sequence of relations on N such that the arities correspond to arities of relations in τ.
- $\psi_{\cong}(\bar{A}_0)$ is a Σ^1_1 formula which says that $\mathfrak{A}$ (i.e., the model itself) is isomorphic to $\bar{A}_0$.
- $\phi_{code}(\bar{A}_0, A'_0)$ is the first order formula which says that (A_0) is the subset of N^n which codes $\bar{A}_0$, see Lemma 1.2.8.
- $\eta_n(A'_0, A^*_0)$ is the first order formula which say that A^*_0 is the image of A'_0 under π_n, see Lemma 1.2.4.
- $\phi'(A^*_0, A^*_1)$ is the Δ^1_n-formula which says that A^*_0 is strictly smaller than A^*_1 in the well-order of the powerset of N defined by ϕ'. The formula ϕ' is formed from ϕ by replacing 0 by 0', 1 by 1', + by +', $\cdot$ by $\cdot'$ and by relativising all the first order and second order quantifiers to N.
- $A^*_0(47)$ is the first order formula which says that the natural number 47 (in the sense of N) belongs to A^*_0. Similarly we could say by a first order formula that n belongs to (or does not belong to) A^*_0 for any chosen n.

The formula

$$((\theta_{\bar{A}_1,\tau} \wedge \psi_{\cong}(\bar{A}_1) \wedge \phi_{code}(\bar{A}_1, A'_1) \wedge \eta_n(A'_1, A^*_1))$$
$$\rightarrow (\phi'(A^*_0, A^*_1) \vee \forall x(A^*_0(x) \leftrightarrow A^*_1(x))))$$

has the same complexity as $\neg\phi'(A^*_0, A^*_1)$, which is Δ^1_n, as ϕ' is Δ^1_n. Then the formula

$$\forall \bar{A}_1 \forall A'_1 \forall A^*_1((\theta_{\bar{A}_1,\tau} \wedge \psi_{\cong}(\bar{A}_1) \wedge \phi_{code}(\bar{A}_1, A'_1) \wedge \eta_n(A'_1, A^*_1))$$
$$\rightarrow (\phi'(A^*_0, A^*_1) \vee \forall x(A^*_0(x) \leftrightarrow A^*_1(x))))$$

has complexity Π^1_n. Now the formula

$$\begin{aligned}&\mathrm{def}(N, 0', 1', +', \cdot') \ \wedge \ \mathrm{def}(\pi_n) \ \wedge \ \theta_{\bar{A},\tau}\\ &\wedge \psi_{\cong}(\bar{A}_0) \ \wedge \ \phi_{code}(\bar{A}_0, A'_0) \ \wedge \ \eta_n(A'_0, A^*_0) \ \wedge\\ &\forall \bar{A}_1 \forall A'_1 \forall A^*_1((\theta_{\bar{A}_1,\tau} \ \wedge \ \psi_{\cong}(\bar{A}_1) \ \wedge \ \phi_{code}(\bar{A}_1, A'_1) \ \wedge \ \eta_n(A'_1, A^*_1))\\ &\rightarrow (\phi'(A^*_0, A^*_1) \vee \forall x(A^*_0(x) \leftrightarrow A^*_1(x)))) \wedge A^*_0(47)).\end{aligned}$$

has complexity Π^1_n and the formula (2.1) has complexity Σ^1_{n+1}. The sentence Φ is true in $\mathfrak{A}$, hence true in $\mathfrak{B}$. So $\mathfrak{A} \cong \mathfrak{B}$. Thus $A(\Sigma^1_{n+1}, \omega)$ has been proved.

□

2.1.2. Corollary (Ajtai [2]). *If $V = \mathrm{L}$ then $A(L^2, \omega)$ holds.*

Proof. In L there is a second order definable well-order of the powerset of ω (See the notes about L on page 14 in the preliminaries).

□

The well-order of the reals in L is Δ^1_2, thus if $V = \mathrm{L}$ then Σ^1_3-equivalence implies isomorphism for countable models. More generally, if there is a Σ^1_n well-order of the reals, any two countable Σ_{n+1}-equivalent models are isomorphic. Hence they are second order equivalent and the full second order theory of a countable model is determined by its Σ_{n+1}-theory.

However, it does not follow that every second order sentence is equivalent to a Σ^1_{n+1} sentence for countable models [22] (Corollary 14.5 VIII(b)).

2.1.3. Corollary (Ajtai [2], Harrington [8]). *$A(L^2, \omega)$ is consistent with $V \neq \mathrm{L}$.*

Proof. By a result of Harrington [8] it is consistent with ZFC that the continuum is as big as desired but has a Δ^1_3-definable well-order.

□

If we have a second order definable well-order of the reals with a parameter[1] r then any two countable models which satisfy the same second order theory with parameter r are isomorphic. This can be seen by just adding a parameter to the proof of Theorem 2.1.1. However, in this article we do not give much attention to the case where we allow parameters: We are generally interested in possibility to determine isomorphism types of models by their theories in languages having sentences smaller than the cardinality of the model. Thus using a real parameter in a language to determine isomorphism type of a countable model (a real) is a bit disappointing.

However, we note the following result of Harrington [8]: It is consistent with ZFC that Martin's Axiom holds, the continuum is as big as wanted and there is a second order definable well-order of the reals using a real parameter. It follows that there is a model of ZFC in which the following hold:

1. Martin's Axiom
2. For some real parameter r, second order equivalence with the real parameter r implies isomorphism for countable models.

2.1.4. QUESTION. *Is Martin's Axiom consistent with $A(L^2, \omega)$?*

A second order definable well-order of the reals is also consistent with measurable and Woodin cardinals, which cannot exist in L. We will return to these large cardinals in Chapter 6.

By Theorem 2.1.1 $A(L^2, \omega)$ is consistent. In all our examples where $A(L^2, \omega)$ holds this is based on a second order definable well-order of the reals.

2.1.5. QUESTION. *Is it consistent with ZFC that $A(L^2, \omega)$ holds, but there is no second order definable well-order of the reals?*

2.2 Optimality

We proved before that $A(\Sigma^1_3, \omega)$ is consistent with ZFC. We will show next that $A(\Sigma^1_1, \omega)$ is not consistent with ZFC.

2.2.1. THEOREM. *For any infinite cardinal κ there are two non-isomorphic Σ^1_1-equivalent models of Peano Axioms of cardinality κ. In particular there are two Σ^1_1-equivalent countable models of Peano Axioms which are not isomorphic.*

[1]The logic for second order logic with a real parameter is $L^2(Q_r)$, the second order logic with a generalized quantifier Q_r. The quantifier Q_r is defined as $\mathfrak{A} \models Q_r(x)\phi(x) \Leftrightarrow |\{x : \mathfrak{A} \models \phi(x)\}| \in r$. Note that if we have $(\omega, <)$ in the model (either in the vocabulary of the model or as interpretation of second order variables) then the formula $\psi(X) = \forall x \in \omega(X(x) \leftrightarrow Q_r y(y < x))$ defines the real r as a subset of ω.

Proof. We start by proving the claim for $\kappa = \omega$. This proof works equally well for all $\kappa < 2^{\aleph_0}$. For $\kappa \geq 2^{\aleph_0}$ the claim follows from a simple cardinality argument.

We construct an elementary chain of length ω_1 of countable models of Peano Axioms. Let $\mathfrak{A}_0$ be the standard model of arithmetic. We recall that there are $2^{\aleph_0}$ different types in arithmetic. If $A \subseteq \omega$, then by the Compactness Theorem $\Sigma_A = \{\theta_n(x) : n \in A\} \cup \{\neg\theta_n(x) : n \notin A\}$, where $\theta_n(x)$ says that the n:th prime number divides x, is a consistent set of formulas. Thus if $A \subseteq \omega$, $B \subseteq \omega$ and $A \neq B$, Σ_A and Σ_B can be completed to types and these types are different. In any countable model only countably many types Σ_A are satisfied, so by the Compactness Theorem there is always a countable elementary extension which realizes some new type Σ_A. It is thus easy to get an elementary chain of length ω_1 of countable non-isomorphic models of Peano Axioms. However, we want some of the models in the chain to be Σ^1_1-equivalent. In order to do that, we make sure that the Σ^1_1 sentences true in the models of the chain are increasing. Thus for each Σ^1_1 formula $\exists R\phi$ which is true in the standard model of arithmetic we put a new relation to the vocabulary of $\mathfrak{A}_0$ and interpret it in such a way that the formula ϕ is satisfied. If $\mathfrak{A}_{\alpha+1}$ satisfies some Σ^1_1 sentences (in the original vocabulary) which are not true in $\mathfrak{A}_\alpha$ then we add new relations to the model so that every Σ^1_1 sentence is satisfied by a relation in the model. We are making the vocabulary bigger and bigger, but it does not matter. If σ is the vocabulary of $\mathfrak{A}_\alpha$ and τ is the vocabulary of $\mathfrak{A}_\beta$, $\alpha < \beta$, then $\mathfrak{A}_\alpha \preccurlyeq \mathfrak{A}_\beta \restriction \sigma$. Since there are only countably many Σ^1_1 sentences, there is such an $\alpha < \omega_1$ that from α forward all models in the chain are Σ^1_1-equivalent. Thus from some α forward, all models in the chain are Σ^1_1-equivalent but not isomorphic.

The above proof works equally well for all cardinalities $\aleph_\alpha < 2^{\aleph_0}$. In any cardinality κ there are 2^κ nonisomorphic models of arithmetic. Therefore Σ^1_1-equivalence does not imply isomorphism in cardinalities $\kappa \geq 2^{\aleph_0}$ and thus Σ^1_1-equivalence does not imply isomorphism in any infinite cardinality.

□

Theorem 2.2.1 is formulated for Peano Axioms, but the proof works equally well for any theory which has $2^{\aleph_0}$ many types and more than continuum many non-isomorphic models in all cardinalities greater than or equal to the continuum.

We showed above that $A(\Sigma^1_1, \omega)$ does not hold. We proved earlier that $A(\Sigma^1_3, \omega)$ holds in L. However we don't know yet whether $A(\Sigma^1_2, \omega)$ is consistent.

2.2.2. QUESTION. *Is it consistent with ZFC that $A(\Sigma^1_2, \omega)$ holds?*

2.2.3. QUESTION. *If $V = \mathrm{L}$, are there two countable non-isomorphic models which have the same monadic second order theory?*

2.3 Failure of $A(L^2, \omega)$

We will now recall the second part of the independence proof of Ajtai [2].

2.3.1. Theorem (Ajtai). *It is consistent with ZFC, that there are two countable non-isomorphic models which satisfy the same sentences of the language of set theory. In particular the models are second order equivalent and L^n-equivalent for all n.*

Proof. We add a Cohen-generic real to the set theoretic universe. Recall that the forcing conditions are functions from finite subsets of ω to $\{0,1\}$. A forcing condition p is stronger than a forcing condition q iff p extends q. If G is a subset of ω, we denote by F^G the set of all subsets of ω which differ from G only in finitely many points. Let now G be a generic real and $-G$ the complement of G. We are discussing the models $(F^G \cup \omega, <_\omega, P_G)$, where $<_\omega$ is the natural order of ω and P_G is the relation which tells which natural numbers n belong to which sets in F^G, and the corresponding model to $-G$.[2] We denote these models M^G and M^{-G}. We claim that these two models satisfy the same sentences of the language of set theory, but are not isomorphic. If some formula $\phi(x)$ of the language of set theory is satisfied by M^G, then by the Forcing Theorem 1.2.17 it is forced by some forcing condition p. But p is finite and does not determine M^G at all. Assume $p \Vdash \phi(\dot{M}^G) \wedge \neg\phi(\dot{M}^{-G})$. So there is a generic filter G containing p such that $V^G \models \phi(M^G) \wedge \neg\phi(M^{-G})$. Now consider another generic filter G' which agrees with G on the domain of p but is the complement of G outside the domain of p. Now $V^G = V^{G'}$, but the models M^G and M^{-G} swap places: $(\dot{M}^G)^{V^G} = (\dot{M}^{-G})^{V^{G'}}$ and $(\dot{M}^G)^{V^{G'}} = (\dot{M}^{-G})^{V^G}$. Thus the forcing condition p can not force any formula of the language of set theory with parameters from the ground model to be satisfied in M^G and false in M^{-G}.

But $(F^G \cup \omega, <_\omega, P_G)$ and $(F^{-G} \cup \omega, <_\omega, P_{-G})$ are non-isomorphic: Since ω is a rigid structure, in an isomorphism every set in F^G should be mapped to exactly the same set in F^{-G}. But this is impossible because $G \notin F^{-G}$.

□

Note that in the proof we do not assume anything about the ground model. Consequently if we add a Cohen real to any model of ZFC, as is done in the proof, $A(L^2, \omega)$ fails in the generic extension.

2.3.2. Remark. *If two countable models are not isomorphic to each other then they can be separated by some $L_{\omega_1,\omega}$-sentence. The logic $L_{\omega_1,\omega}$ is related to Dynamic Ehrenfeucht-Fraïssé games, see for example [28] for the definitions. For*

[2] In fact the union of the relations $<_\omega$ and P_G is $\in$, so we could also form the model in vocabulary $\{\in\}$ instead of $\{<_\omega, P_G\}$. We follow here Ajtai, whose vocabulary is maybe more intuitive than the alternative vocabulary.

any non-$L_{\omega_1,\omega}$-equivalent countable models $\mathfrak{A}$ and $\mathfrak{B}$ there is an $\alpha < \omega_1$ such that I has a winning strategy in Dynamic Ehrenfeucht-Fraïssé game $EFD_\alpha(\mathfrak{A},\mathfrak{B})$. The least such α is called the Scott Watershed for $\mathfrak{A}$ and $\mathfrak{B}$. The bigger the Scott Watershed is, the harder the models are to distinguish by an $L_{\omega_1,\omega}$-sentence. The models M^G and M^{-G} satisfy the same sentences of the language of set theory, so they are in a way hard to distinguish from each other. However, the Scott Watershed of the pair $(M^G\ , M^{-G})$ is a very small ordinal: $\omega+1$. Thus the difference between M^G and M^{-G} is not of the kind that is well reflected in the approach of Ehrenfeucht-Fraïssé games.

In the proof of Theorem 2.3.1 we added one generic real to the set theoretic universe and got two second order equivalent non-isomorphic models. But actually by a little modification of the proof, we can add many generic reals to the universe and get many countable second order equivalent non-isomorphic models:

2.3.3. THEOREM. *Let κ^+ be an infinite cardinal. There is a cardinals preserving notion of forcing P that forces that there are κ^+ countable ZF-equivalent non-isomorphic models.*

Proof. We add κ^+ generic reals to L. Forcing conditions are finite functions from $\kappa^+ \times \omega$ to $\{0,1\}$. A forcing condition p is stronger than another forcing condition q iff p extends q. If G is a generic set for this notion of forcing, for all $\alpha < \kappa^+$, $f_\alpha = \{n : G(\alpha, n) = 1\}$ is a generic real. Note that for all $\alpha < \beta < \kappa^+$, f_α and f_β differ in infinitely many points. Thus if we construct models around f_α and f_β as in Theorem 2.3.1, we get countable non-isomorphic models. We denote these models by M^{f_α} and M^{f_β}. We will show that the models are ZF-equivalent. Assume not: then by the Forcing Theorem 1.2.17 there is a forcing condition p and a ZF-sentence ϕ with possibly parameters from the ground model such that $p \Vdash \phi(\dot{M}^{f_\alpha}) \wedge \neg\phi(\dot{M}^{f_\beta})$. So there is a generic filter G containing p such that $V^G \models \phi(M^{f_\alpha}) \wedge \neg\phi(M^{f_\beta})$. But there is another generic filter G' which agrees with G in all ordinals different from α and β, agrees with G in α and β in the domain of p and chances digits of α to digits of β and vice versa outside the domain of p. Now $V^G = V^{G'}$, $p \in G'$ and the interpretations of $\dot{M}^{f_\alpha}$ and $\dot{M}^{f_\beta}$ swap places in the two generic extensions. Thus it is impossible that $p \Vdash \phi(\dot{M}^{f_\alpha}) \wedge \neg\phi(\dot{M}^{f_\beta})$.

□

2.4 The Fraïssé Hypothesis

Given a language L, a cardinal κ and a model class C we can ask whether $A(L,\kappa)$ restricted to C is true, i.e., whether any two L-equivalent models of cardinality κ which belong to C are isomorphic. In this section we will discuss the following model classes: the ordinals, the linear orders and the models of arithmetic.

Recall the definition of the *Fraïssé Hypothesis*, Definition 1.1.3.

2.4.1. THEOREM. *The Fraïssé Hypothesis implies that there is a third order definable well-order of length ω_1 of a subset of the reals.*

Proof. The ordinal ω_1 is characterizable in third order logic in any countable model as a third order predicate (a set of sets). In third order logic we can also characterize a truth definition for all countable ordinals, i.e., a mapping from ω_1 to the second order theories of the ordinals in ω_1. For details see Definition 6.1.8 and Lemma 6.1.9. We fix some Gödel-numbering of second order sentences and consider second order theories as real numbers. From the Fraïssé Hypothesis it follows that countable ordinals have different second order theories and thus our mapping maps them to different reals. Thus we have a third order characterizable injective mapping from ω_1 to the reals. So we have a third order definable well-order of length ω_1 of a subset of the reals.

$\square$

2.4.2. THEOREM. *If there is a second order definable well-order of length ω_1 of a subset of the reals then the Fraïssé Hypothesis holds.*

Proof. Let X be the subset of the reals in the assumption and let α be a countable ordinal. In the second order definable well-order of X there is the α:th real in the well-order of X. In second order logic we can talk about this real by sentences of the following form:

"There is an initial segment of the well-order of X which has the same order type with this model and the supremum of this initial segment contains (or does not contain) n."

If α and β are different countable ordinals, then X has an α:th real a and a β:th real b and $a \neq b$. Thus there is some $n \in \omega$ where a and b disagree and for this n the ordinals α and β disagree about a second order sentence of the above form.

$\square$

2.4.3. THEOREM. *Consider the following conditions:*

1. *There is a second order definable well-order of the reals.*
2. $A(L^2, \omega)$.
3. *The Fraïssé Hypothesis.*
4. *There is a third order definable well-order of a subset of the reals which has length ω_1.*
5. *There is a second order definable well-order of length ω_1 of a subset of the reals.*

The following implications hold:
1. $\Rightarrow$ 2. $\Rightarrow$ 3. $\Rightarrow$ 4.
5. $\Rightarrow$ 3.

Proof. 1. $\Rightarrow$ 2. Theorem 2.1.1. 2. $\Rightarrow$ 3. is trivial. 3. $\Rightarrow$ 4. Theorem 2.4.1. 5. $\Rightarrow$ 3. Theorem 2.4.2.

$\square$

Recall that the implication 2. $\Rightarrow$ 1. in Theorem 2.4.3 above is an open question. From the next Theorem it follows that the negation of the Fraïssé Hypothesis implies $\omega_1 \neq \omega_1^L$. As a consequense implication 2. $\Rightarrow$ 3. is proper, as after adding a Cohen real to L countable ordinals still satisfy different second order theories.

2.4.4. Theorem. *Assume $\mathfrak{A}$ is a countable model in the universe of constructible sets, and M is a transitive model of ZFC containing all ordinals. Then for any model $\mathfrak{B}$ in M, if $M \models \mathfrak{A} \equiv_{L^2} \mathfrak{B}$ then $\mathfrak{A} \cong \mathfrak{B}$.*

Proof. Given a countable model in L (say $\mathfrak{A}$), we can say in second order logic which second order sentences it satisfies in L. This is because without too much trouble we can relativize all second order quantifiers to L. Also there is a second order sentence ϕ which says that the model in question is countable in L. Assume now $M \models \mathfrak{A} \equiv_{L^2} \mathfrak{B}$. As $\mathfrak{B} \models \phi$ the model $\mathfrak{B}$ is countable in L. As $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same second order sentences in L, it follows from Theorem 2.1.1 that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.

$\square$

At this point we note that $A(L^2, \omega)$ and the Continuum Hypothesis do not decide each other in any way. We give the following examples:

1. $A(L^2, \omega)$ and the Continuum Hypothesis both hold in L, see Theorem 2.1.1.
2. If we add $\aleph_2$ Cohen generic reals to L (see Theorem 2.3.3), then $A(L^2, \omega)$ and the Continuum Hypothesis both fail.
3. Harrington[8] gives a model of ZFC in which the continuum is large but has a Δ^1_3 well-order. From Theorem 2.1.1 it follows that in Harrington's model the Continuum Hypothesis fails but $A(L^2, \omega)$ holds.
4. If we add one Cohen generic real to L, as is done in Theorem 2.3.1, then $A(L^2, \omega)$ fails but the Continuum Hypothesis holds.

We will next define the diamond principle $\diamondsuit$ and show that $\diamondsuit$ does not decide $A(L^2, \omega)$ either.

2.4.5. DEFINITION. *The diamond principle $\diamondsuit$ is the following condition:*
There exists a sequence of sets $\langle S_\alpha : \alpha < \omega_1 \rangle$ with $S_\alpha \subset \alpha$, such that for every $X \subset \omega_1$, the set $\{\alpha < \omega_1 : X \cap \alpha = S_\alpha\}$ is a stationary subset of ω_1.

We will now introduce a forcing which makes $\diamondsuit$ true. We use the forcing from Jech [11], exercise 15.23.

2.4.6. LEMMA. *(Folklore) Let $Q = \{\langle S_\beta : \beta < \alpha \rangle, \alpha < \omega_1\}$, where $S_\beta \subseteq \beta$ for all $\beta < \alpha$. Let p be stronger than q if and only if p extends q. Let G be Q-generic. Then $V[G] \models \diamondsuit$.*

Proof. We will show that $\bigcup G$ is a $\diamondsuit$-sequence. Thus we need to show that for any forcing names $\dot{C}$ and $\dot{X}$, if $p \Vdash (\dot{C}$ is closed unbounded subset of ω_1 and $\dot{X} \subseteq \omega_1)$ then there is a q stronger than p such that $q = \langle S_\beta : \beta \leq \alpha \rangle$ and $q \Vdash (\alpha \in \dot{C}$ and $\dot{X} \cap \alpha = S_\alpha)$.

So assume $p \Vdash (\dot{C}$ is a closed unbounded subset of ω_1 and $\dot{X} \subseteq \omega_1)$. We will define inductively an ω-sequence of forcing conditions in such a way, that the upper limit of this sequence will do the job. We use $\mathrm{len}(p)$ to denote the length of the forcing condition p.

1. $p_0 = p$

2. p_1 is a forcing condition strictly stronger than p_0 such that $p_1 \Vdash \alpha_1 \in \dot{C}$ for some $\alpha_1 > \mathrm{len}(p_0)$. This is possible because p_0 proves that $\dot{C}$ is unbounded subset of ω_1.

3. p_2 is a forcing condition strictly stronger than p_1 such that it decides $\dot{X} \cap \alpha_1$, and $\mathrm{len}(p_2) > \alpha_1$. This is possible because our forcing is $\aleph_0$-closed and it does not add any new subsets to countable sets. Thus $\dot{X} \cap \alpha_1$ is some set from the ground model and there is some forcing condition which decides which set from the ground model it is.

4. p_3 is a forcing condition strictly stronger than p_2 such that $p_3 \Vdash \alpha_2 \in \dot{C}$ for some $\alpha_2 > \mathrm{len}(p_2)$.

5. p_4 is a forcing condition strictly stronger than p_3 such that it decides $\dot{X} \cap \alpha_2$, and $\mathrm{len}(p_4) > \alpha_2$.

   $\vdots$

Let α be the supremum of the ordinals $\mathrm{len}(p_n)$, $n \in \omega$. Since the sequence $\alpha_1, \alpha_2, \ldots$ converges to α and $\dot{C}$ is closed, $q \Vdash \alpha \in \dot{C}$ for any q which is stronger than all p_n:s. Also for any $\beta < \alpha$ there is some forcing condition p_n which decides whether $\beta \in \dot{X}$. Now we can define q to be as p_n:s for $\beta < \alpha$ and at α we can define it to be $\dot{X} \cap \alpha$.

□

In L the $\diamondsuit$ principle holds and $A(L^2, \omega)$ holds. We just showed that $\diamondsuit$ can be forced to be true by a small forcing which does not destroy large cardinals. Thus we can have a model with $\diamondsuit$ and infinitely many Woodin cardinals with a measurable cardinal above them (assuming the consistency of the large cardinal axiom above). Then, looking ahead, from Theorem 6.1.6 it follows that $\diamondsuit$ is consistent with the negation of $A(L^2, \omega)$.

Ajtai [2] has proved that it is consistent with ZFC that there are two different countable ordinals which satisfy the same standard ZF-formulas. However, the model of ZFC in the proof is not necessarily transitive, so there might be some non-standard ZF or second order formulas which do not agree about those ordinals.

Marek [16] notes without a proof that in the Levy model, where all cardinals below the first inaccessible cardinal are collapsed to countable ordinals, the Fraïssé Hypothesis fails. He also notes a result of G. Sacks that if ω_1^{L} is collapsed to ω, then the Fraïssé Hypothesis fails. We will next present a proof for this. Note that the failure of the Fraïssé Hypothesis is consistent relative to the consistency of ZFC.

2.4.7. THEOREM (SACKS). *It is consistent with ZFC that the Fraïssé Hypothesis fails.*

Proof. Let L be the ground model. We make a forcing which collapses ω_1 to ω. The forcing conditions are injective functions from finite subsets of ω to ω_1. A condition p is stronger than a condition q iff p extends q.

We make the following remark: The forcing is *homogeneous* (see [23] for the definition), and consequently if a is an element of the ground model, ϕ is a second order sentence and $p \Vdash \phi(a)$ then $1 \Vdash \phi(a)$. This is because in this forcing any forcing condition can not determine the generic extension in any way. If G is a generic filter for this forcing and p is a forcing condition then there is another generic filter G' containing p such that $V^G = V^{G'}$.

We claim that after the forcing there are two different ordinals smaller than ω_2^{L} which have the same second order theory. Assume not. Then after the forcing all ordinals smaller than ω_2^L have different second order theories. For each ordinal $\alpha < \omega_2^L$, the relation $1 \Vdash \phi(\alpha)$ is definable in the ground model and the real $r_\alpha = \{n : n \text{ is a Gödel number of such a second order sentence } \phi \text{ that } 1 \Vdash \phi(\alpha)\}$ is definable in the ground model and belongs to the ground model. Now the mapping $\alpha \mapsto r_\alpha$ is an injective mapping from ω_2 to the reals and it exists in L which is a contradiction.

□

We will give another proof for the consistency of the existence of two non-isomorphic second order equivalent countable linear orders. In the proof we construct two linear orders, which "look like" the two models in the proof of Theorem 2.3.1.

2.4.8. THEOREM. *It is consistent with ZFC that there are two (or κ^+) countable non-isomorphic second order equivalent linear orders.*

Proof. Recall the models $(F^G \cup \omega, <_\omega, P_G)$ and $(F^{-G} \cup \omega, <_\omega, P_{-G})$ from the proof of Theorem 2.3.1. We expand these models by adding linear orders ("lexicographic orders") to the sets F^G and F^{-G}. In "lexicographic order" $X < Y$ iff there is an $n \in \omega$ such that below n the sets X and Y have the same elements, but $n \notin X$ and $n \in Y$. Note that these lexicographic orders are characterizable in second order logic in the models in question, so the expanded models are second order equivalent.

Now we want to modify these lexicographic orders in such a way that they reflect the structure of the sets in F^G and F^{-G}. For each subset X of ω we construct the following linear order denoted by $<_X$:

We denote by $<_X^1$ the following linear order: In the beginning there are four points. After the four starting points there is a $\mathbb{Q}$-component. Then if X has the first digit zero there are two points in the linear order. If X has the first digit one there are three points in the linear order. If $<_X^n$ has been defined, we denote by $<_X^{n+1}$ the linear order which has $<_X^n$ in the beginning, then a $\mathbb{Q}$-component and then two points (if the $n+1$st digit of X is 0) or three points (if the $n+1$st digit of X is 1). Finally we define $<_X = \bigcup_{n\in\mathbb{N}} <_X^n$.

The construction is characterizable in second order logic, so the mapping $X \mapsto <_X$ with domain F^G is characterizable by a L^2 formula in $(F^G \cup \omega, <_\omega, P_G)$. Similarly the mapping $X \mapsto <_X$ with domain F^{-G} is characterizable by a L^2 formula in $(F^{-G} \cup \omega, <_\omega, P_{-G})$.

Now we can define the linear order $<_G$ as follows:

$$dom <_G = \bigcup_{X \in F^G} dom <_X$$

where $dom <_X \cap dom <_Y = \emptyset$ for all different X and Y. If x and y are in $dom <_G$ then $x <_G y$ iff one of the following holds:

1. There are X and Y such that $x \in dom <_X$ and $y \in dom <_Y$ and X is smaller than Y in the lexicographic order of F^G.

2. There is X such that $x \in dom <_X$ and $y \in dom <_X$ and $x <_X y$.

The construction of $<_G$ is second order characterizable in $(F^G \cup \omega, <_\omega)$. In a similar way we can characterize another linear order $<_{-G}$ in $(F^{-G} \cup \omega, <_\omega, P_{-G})$.

Because $(F^G \cup \omega, <_\omega, P_G)$ and $(F^{-G} \cup \omega, <_\omega, P_{-G})$ are second order equivalent, also the linear orders $<_G$ and $<_{-G}$ are second order equivalent.

But the models are not isomorphic. The model constructed from $-G$ does not have an interval which starts with four points, then has ω copies of Q-components and some points between the Q-components as we will describe below: For each $k \in \omega$ between the $k+1$st and $k+2$nd Q-components there are 2 points when the kth digit of G is 0 and there are 3 points when the kth digit of G is 1.

If we add κ^+ generic reals as in Theorem 2.3.3 then we get κ^+ non-isomorphic second order equivalent linear orders.

□

2.4.9. Theorem. *It is consistent with ZFC that there are two countable second order equivalent non-isomorphic models of arithmetic.*

Proof. Let α and β be second order equivalent countable non-isomorphic ordinals, which consistently exist by Theorem 2.4.7. Let σ be a minimal type [12]. We extend the prime model of arithmetic by taking α-canonical and β-canonical extensions over the type σ. That is: we take the Ehrenfeucht-Mostowski models which are generated by the sequences of elements of the minimal type σ, and we let the generating sequences have order types α and β. The models are second order equivalent, but they are not isomorphic as there is no order preserving mapping of the generators of the first model to the generators of the second model. It is also impossible to have an isomorphism from one model to the other which would map the set of generators to a set other than the generators in the other, because both structures are rigid [12] (p.70).

□

2.5 Submodels

In this section we discuss elementary submodels. As we will see, the concept of second order elementary submodel is too strong to be useful. However, using Ajtai's technique we will prove a theorem which demonstrates the possibility of having a stronger version of first order elementary submodel.

2.5.1. Definition. $\mathfrak{A} \preceq^*_{L^2} \mathfrak{B}$ *means* $\mathfrak{A}$ *is a second order elementary submodel of* $\mathfrak{B}$*. This means:* $\mathfrak{A}$ *is a submodel of* $\mathfrak{B}$ *and for any second order formula* $\phi(X_1, \dots X_n, x_1, \dots, x_m)$ *and relations* $A_1, \dots A_n \in A$ *and elements* $a_1, \dots, a_m \in A$*,*

if $\mathfrak{A} \models \phi(A_1, \dots A_n, a_1, \dots, a_m)$ *then* $\mathfrak{B} \models \phi(A_1, \dots A_n, a_1, \dots , a_m)$*.*

Clearly it is impossible to have $\mathfrak{A} \preceq^*_{L^2} \mathfrak{B}$ when $A \neq B$, as A satisfies the formula saying that every element belongs to it in $\mathfrak{A}$ but that is not the case in $\mathfrak{B}$.

We need a weaker formulation for second order elementary submodel:

2.5.2. DEFINITION. *$\mathfrak{A} \preceq_{L^2} \mathfrak{B}$ if $\mathfrak{A} \subseteq \mathfrak{B}$ and for any finite sequence of parameters $\bar{a} \in A$ $(\mathfrak{A}, \bar{a}) \equiv_{L^2} (\mathfrak{B}, \bar{a})$.*

Is it possible to find non-isomorphic models $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A} \preceq_{L^2} \mathfrak{B}$? In fact we will prove a stronger result: we will give such models $\mathfrak{A}$ and $\mathfrak{B}$ that $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{A} \ncong \mathfrak{B}$ and $(\mathfrak{A}, a_1, \ldots, a_n)$ and $(\mathfrak{B}, a_1, \ldots, a_n)$ satisfy the same formulas of the language of set theory for all first order parameters $a_1, \ldots, a_n \in A$. This result is easy to get if one thinks models of empty vocabulary in different cardinalities, but we give an example were both models $\mathfrak{A}$ and $\mathfrak{B}$ have the same cardinality $\aleph_0$.

2.5.3. THEOREM. *It is consistent with ZFC that there exist two models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality $\aleph_0$ satisfying the following: $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{A} \ncong \mathfrak{B}$ and $(\mathfrak{A}, a_1, \ldots, a_n) \equiv_{ZF} (\mathfrak{B}, a_1, \ldots, a_n)$ for all elements $a_1, \ldots a_n \in A$.*

Proof. We force ω generic reals to the set theoretic universe. Forcing conditions are finite functions $f : \omega \times \omega \to \{0, 1\}$, and a forcing condition p is stronger than a forcing condition q iff p extends q.[3] If G is a generic set of conditions and $i \in \omega$ we say that $G_i = \{n : G(i, n) = 1\}$ is the ith generic real. Define $\mathrm{dom}\mathfrak{A} = \bigcup_{i \in 6\mathbb{N}} \mathrm{dom}\mathfrak{A}_i$, where $\mathfrak{A}_i$ is the Ajtai model constructed from the ith generic real. Define $<^{\mathfrak{A}}=$ the natural order of ω. Define $P^{\mathfrak{A}} = \bigcup_{i \in 6\mathbb{N}} P^{\mathfrak{A}_i}$. Define similar way $\mathrm{dom}\mathfrak{B} = \bigcup_{i \in 2\mathbb{N}} \mathrm{dom}\mathfrak{A}_i$, $<^{\mathfrak{B}}=$ the natural order of ω and $P^{\mathfrak{B}} = \bigcup_{i \in 2\mathbb{N}} P^{\mathfrak{A}_i}$.

The models are not isomorphic because $\mathfrak{B}$ contains some subsets of ω which $\mathfrak{A}$ does not contain, and in an isomorphism every subset of ω is mapped to itself.

We claim that $(\mathfrak{A}, a_1, \ldots, a_n) \equiv_{ZF} (\mathfrak{B}, a_1, \ldots, a_n)$ for arbitrary $a_1, \ldots a_n \in \mathfrak{A}$. Suppose not: there is a forcing condition p and a formula ϕ such that $p \Vdash \phi(\dot{\mathfrak{A}}, \dot{a_1}, \ldots, \dot{a_n}) \wedge \neg\phi(\dot{\mathfrak{B}}, \dot{a_1}, \ldots, \dot{a_n})$. Let G be a generic filter which contains p. It is possible to construct another generic filter G' such that $V^G = V^{G'}$, $\dot{a_1}^{V^G} = \dot{a_1}^{V^{G'}}, \ldots, \dot{a_n}^{V^G} = \dot{a_n}^{V^{G'}}$ and $\dot{\mathfrak{B}}^{V^G} = \dot{\mathfrak{A}}^{V^{G'}}$. This is possible because the forcing condition p is finite. For those i which determine the interpretations of the forcing names $\dot{a_1}, \ldots, \dot{a_n}$ we let G and G' agree about everything[4]. In the domain of p we let G and G' agree about everything. Otherwise we let G' produce in the indexes $6\mathbb{N}$ those generic reals which G produces in the indexes

[3]In fact this forcing is the same as the usual Cohen real forcing where the forcing conditions are finite functions from ω to $\{0, 1\}$, but we feel that this formulation is more intuitive here.

[4]$\dot{a_1}$ is either a natural number or a subset of ω. If it is a natural number then the interpretation of (for example) $\dot{a_1}$ is determined by the trivial forcing condition. If $\dot{a_1}$ is a subset of ω then $\dot{a_1}$ is the same as one of G_i:s (where $i \in 2\mathbb{N}$), except for finitely many digits. Thus it is determined by $G_i = \{n : G(i, n) = 1\}$.

$2\mathbb{N}$, and in the indexes $\mathbb{N} \setminus 6\mathbb{N}$ those generic reals which G produces in the indexes $2\mathbb{N}+1$. Because of p it may be impossible to produce exactly the same generic reals, but it is possible to produce reals which are the same except in finitely many digits. However, finitely many digits do not make any difference to the model $\dot{\mathfrak{A}}^{VG'}$ and we get $\dot{\mathfrak{B}}^{VG} = \dot{\mathfrak{A}}^{VG'}$. But now it can not be so that $p \Vdash \phi((\dot{\mathfrak{A}}, \dot{a}_1, \ldots, \dot{a}_n)) \wedge \neg\phi((\dot{\mathfrak{B}}, \dot{a}_1, \ldots, \dot{a}_n))$.

$\square$

Chapter 3

Fourth order logic

3.1 Coding subsets by collapsing cardinals

In Chapter 2 we showed that it is independent of ZFC whether $A(L^2, \omega)$ holds. A natural question is whether analogous results can be proved for other higher order logics L^n or various uncountable cardinals κ. Our results in this chapter were inspired by the following theorem of Ajtai [2].

3.1.1. Theorem (Ajtai). *There is a model of ZFC in which $A(L^n, \omega)$ fails for every $n \in \omega$ but $A(ZF, \omega)$ holds.*

Proof. [sketch] We add first a Cohen real G to L as in Theorem 2.3.1. After this forcing there are two ZF-equivalent non-isomorphic models of cardinality ω and GCH holds.

Next we will make G definable by a ZF sentence. For all natural numbers n we add by Easton forcing $\aleph_{\omega+n+2}$ Cohen subsets to $\aleph_{\omega+n}$ for those n where the nth digit of G is 1. After the forcing G is definable by a ZF-formula as the function $f : \omega \to \{0, 1\}$,

- $f(n) = 0$, if GCH holds at $\aleph_{\omega+n}$
- $f(n) = 1$ if GCH does not hold at $\aleph_{\omega+n}$

is definable by a ZF-formula. It follows that after the forcing the canonical well-order of $L[G]$ is definable by a ZF-formula and $A(ZF, \omega)$ holds after the forcing. By Lemma 1.2.9 the truth of L^n sentences in a model of cardinality ω are determined by sets in $H((\beth_{n-2}(\omega))^+)$, thus the two models remain non-isomorphic and L^n-equivalent for all n.

□

Next we will give some motivation to our definition of a forcing $P_{X',\kappa}$, which is used a lot in this chapter. The forcing uses some ideas of Kenneth McAloon [18].

Assume $M = \mathrm{L}[X]$, λ is a cardinal and $X \subseteq \lambda$. Assume also that M and L have the same cardinals, $\kappa = \aleph_\alpha^M$ is a cardinal in M and GCH holds above κ in M. We will next introduce the forcing $P_{X,\kappa}$, which makes X definable from κ, but does not add any new subsets to κ. Let X' be a subset of $\lambda \setminus \{\beta : \beta$ is a limit ordinal$\}$ such that X' and X contain the same information[1]. The forcing is an iterated forcing of length λ with full support at all limit stages. The idea is that $P_{X,\kappa}$ collapses $\aleph_{\alpha+\omega\cdot\beta+2}$ to $\aleph_{\alpha+\omega\cdot\beta+1}$ for $\beta \in X'$, and does not collapse any other cardinals. After the forcing X' (and hence X) is definable from α as $X' = \{\beta < \lambda : \aleph^{\mathrm{L}}_{\alpha+\omega\cdot\beta+2}$ is not a cardinal$\}$. Next we will give an exact definition of the forcing conditions:

3.1.2. Definition ($P_{X',\kappa}$). *The forcing conditions are sequences $(p_\beta)_{\beta<\lambda}$ such that the following hold:*

1. *If $0 \in X'$, then P_0 is the set of partial functions from $\aleph_{\alpha+1}$ to $\aleph_{\alpha+2}$ of cardinality smaller than $\aleph_{\alpha+1}$. A forcing condition p is stronger than a forcing condition q if and only if p extends q. If $0 \notin X'$, then P_0 is the trivial forcing.*

2. *Assume $\beta = \gamma + 1$ and $P_{\gamma'}$ has been defined for all $\gamma' \leq \gamma$.*

   If $\beta \in X'$, we define P_β to be the set of sequences $p_\gamma, \gamma \leq \beta$ where the γth coordinate belongs to P_γ for each $\gamma < \beta$ and the βth coordinate is a forcing name $\dot{Y}$ such that $p \restriction \beta \Vdash \dot{Y}$ is a partial function from $\aleph_{\alpha+\omega\cdot\beta+1}$ to $\aleph_{\alpha+\omega\cdot\beta+2}$ of cardinality smaller than $\aleph_{\alpha+\omega\cdot\beta+1}$. If p and q are two conditions of length β then p is stronger than q if and only if $p \restriction \gamma$ is stronger than $q \restriction \gamma$ and $p \restriction \beta \Vdash$ ($p(\beta)$ and $q(\beta)$ are partial functions from $\aleph_{\alpha+\omega\cdot\beta+1}$ to $\aleph_{\alpha+\omega\cdot\beta+2}$ of cardinality smaller than $\aleph_{\alpha+\omega\cdot\beta+1}$ and $p(\alpha) \supseteq q(\alpha)$).

   If $\beta \notin X'$ then P_β is the trivial forcing.

3. *If β is a limit ordinal, the forcing conditions in P_β are the tuples p of length β such that for each $\gamma < \beta \quad p \restriction \gamma \Vdash p(\gamma) \in P_\gamma$. This forcing has full support in all limit stages, which means that in limit stages all coordinates of a forcing condition may be non zero. A forcing condition p is stronger than q if and only if $p \restriction \gamma$ is stronger than $q \restriction \gamma$ for each $\gamma < \beta$.*

3.1.3. Lemma. *Assume $M = \mathrm{L}[X]$, λ is a cardinal and $X \subseteq \lambda$. Assume also that M and L have the same cardinals and GCH holds above κ in M. Let G be a $P_{X,\kappa}$-generic set over M. $M[G] \models X' = \{\beta < \lambda : \aleph^{\mathrm{L}}_{\alpha+\omega\cdot\beta+2}$ is not a cardinal$\}$.*

[1] For example for all $\alpha < \lambda : \alpha \in X \leftrightarrow \alpha + 1 \in X'$.

Proof. We prove by induction on β that after P_β the claim holds for all $\gamma \leq \beta$, i.e., for all $\gamma \leq \beta$, $\aleph^{\mathrm{L}}_{\alpha+\omega\cdot\gamma+2}$ is a cardinal iff $\gamma \in X'$. The rest of the iterated forcing is $\aleph_{\alpha+\omega\cdot(\beta+1)}$-closed and does not add subsets to $\aleph_{\alpha+\omega\cdot\beta+1}$ so the claim follows.

1. Let $\beta = 0$. If $0 \notin X'$ then P_0 is the trivial forcing and the claim holds. If $\beta \in X'$ then P_β collapses $\aleph_{\alpha+\omega\cdot\beta+2}$ to $\aleph_{\alpha+\omega\cdot\beta+1}$. The forcing P_β is $< \aleph_{\alpha+\omega\cdot\beta+1}$-closed and has cardinality $\aleph_{\alpha+\omega\cdot\beta+2}$ (because GCH holds above $\kappa = \aleph_\alpha$), so other cardinals and GCH above κ are preserved.

2. Let $\beta = \gamma + 1$ and assume Induction Hypothesis holds for γ. If $\beta \notin X'$ then P_β is the trivial forcing and the claim holds. If $\beta \in X'$ then P_β collapses $\aleph_{\alpha+\omega\cdot\beta+2}$ to $\aleph_{\alpha+\omega\cdot\beta+1}$. Note that P_β is $< \aleph_{\alpha+\omega\cdot\beta+1}$-closed and has cardinality $\aleph_{\alpha+\omega\cdot\beta+2}$, because GCH above κ holds. It follows that P_β preserves other cardinals. Also GCH above κ is preserved so the claim holds.

3. Assume β is a limit ordinal and the Induction Hypothesis holds for all smaller ordinals. The forcing P_β has cardinality at most $\aleph_{\alpha+\omega\cdot\beta}$ so it does not collapse any cardinals greater than $\aleph_{\alpha+\omega\cdot\beta}$. Also $\aleph_{\alpha+\omega\cdot\beta}$ is not collapsed because there are cofinally many cardinals below which are not collapsed. GCH above κ is also preserved.

4. The whole forcing $P_{X,\kappa}$ has cardinality at most $\aleph_{\alpha+\omega\cdot\lambda}$ so cardinals greater than $\aleph_{\alpha+\omega\cdot\lambda}$ are preserved. The cardinal $\aleph_{\alpha+\omega\cdot\lambda}$ itself is preserved, as cofinally many cardinals below it are preserved.

□

3.1.4. THEOREM. *Let κ be a cardinal in* L. *There is a model of ZFC in which $2^\kappa = 2^{\aleph_0}$, $A(L^4, \kappa)$ holds, $A(L^2, \kappa)$ fails and all cardinals $\leq \kappa$ of* L *are preserved.*

Proof. Let L be the ground model. We make an iterated forcing which has three parts and length $\kappa^+ + 1$. After the forcing fourth order equivalence implies isomorphism in cardinality κ but second order equivalence does not.

1. First we add 2^κ Cohen-subsets to ω. This forcing does not collapse any cardinals and after the forcing $2^{\aleph_0} = 2^\kappa$.

2. Now let G be the generic set we added in step 1. and let Π be a bijection from $2^{\aleph_0}$ to 2^κ in $V[G]$. We want to make G and Π definable from κ in the language of set theory, but not to make them second order characterizable in cardinality κ.

   As G and Π are of cardinality smaller or equal to $2^\kappa = \kappa^+$, there is a subset X of 2^κ which codes them both. Let X' be a subset of $2^\kappa \setminus \{\gamma : \gamma \text{ is } 0 \text{ or a limit ordinal }\}$ such that X and X' are definable from each other. We will now make one such X' definable from κ in the language of set theory.

Let $\dot{X}'$ be a canonical name for X'. After the forcing P_0 the GCH holds above κ, $\dot{X}'$ has cardinality 2^κ, and the cardinals are the same as in L, so by Lemma 3.1.3 $P_0 * P_{\dot{X}',\kappa}$ does not add any new subsets to κ and makes $\dot{X}'$ definable from κ in the language set theory.

3. In the last step we add $\aleph_{\alpha+\omega\cdot\kappa^++1}$ Cohen subsets for κ^+. This does not collapse cardinals or add new subsets to κ. Now $\dot{X}'$ is definable in $H((\beth_2(\kappa))^+)$ so by Lemma 1.2.9 there is a fourth order definable well-order of the powerset of κ and a fourth order definable bijection from the powerset of κ to the reals. Now as in Theorem 2.1.1 we can have fourth order sentences which say "There are $R_0 \subseteq \kappa$ and $R_0' \subseteq \omega$ such that R_0 is the least subset in the well-order isomorphic to the model in question and Π maps R_0 to R_0' and $R_0'(8743)$". Sentences of this form determine the isomorphism type of the model so $A(L^4,\kappa)$ holds after the forcing. $A(L^2,\kappa)$ fails after the forcing as it fails after the first Cohen forcing[2] and we did not add any subsets to κ after that.

□

3.1.5. Theorem. *Let κ be a cardinal in* L. *There is a model of ZFC in which $2^\kappa = 2^{\aleph_0}$, and $A(L^4,\lambda)$ holds and $A(L^2,\lambda)$ fails in any cardinality $\lambda \leq \kappa$.*

Proof. Let L be the ground model. We use an iterated forcing which has the following steps:

1. We add $2^\kappa = \kappa^+$ Cohen subsets to ω. Cardinals are preserved in this forcing and after this forcing $2^\lambda = 2^\kappa = \kappa^+$ for any $\lambda \leq \kappa$. Also $A(L^2,\lambda)$ fails for all $\lambda \leq \kappa$, see Theorem 5.2.7 below.

2. Now let G be the generic set we added in step 1. and let $\{\Pi_\lambda : \lambda \leq \kappa\}$ be a set such that each Π_λ is a bijection from $2^{\aleph_0}$ to 2^λ in $V[G]$. Let $\dot{X}'$ be a subset of $2^\kappa \setminus \{\gamma : \gamma$ is 0 or a limit ordinal$\}$ which codes G and all the bijections Π_λ.

   As in the previous theorem $P_0 * P_{\dot{X}',\kappa}$ makes X' definable from κ and adds the same subsets of κ as P_0 alone.

3. In the last step we add $\aleph_{\alpha+\omega\cdot\kappa^++1}$ Cohen subsets to 2^κ.

   After the forcing $A(L^2,\lambda)$ fails for every $\lambda \leq \kappa$ as we did not add any new subsets to λ after step 1. After the forcing $A(L^4,\lambda)$ holds for all $\lambda \leq \kappa$ as in $H(\beth_2(\lambda)^+)$ there is a definable well-order of the powerset of λ and a definable bijection from 2^λ to 2^ω.

[2]Theorem 2.3.1 proves this for $\kappa = \omega$ and Theorem 5.2.7 below proves the uncountable case.

□

3.1.6. THEOREM. *Let κ be a cardinal in* L *and let n be a natural number greater or equal to 2. There is a model of ZFC in which $A(L^n, \kappa)$ fails but $A(L^{n+2}, \kappa)$ holds.*

Proof. Let L be the ground model. Our iterated forcing has the following steps:

1. We add 2^κ Cohen subsets to ω. After this step $A(L^n, \kappa)$ fails for every n, $2^\kappa = 2^\omega$, GCH holds above κ and all cardinals of L remain cardinals.
2. Let $\dot{X}'$ be a canonical name for a subset of $\kappa^+ \setminus \{\gamma : \gamma$ is a limit ordinal or $0\}$ which codes the generic set added in step 1. and a bijection Π from 2^ω to 2^κ. The second step is $P_{\dot{X}', \beth_{n-2}(\kappa)}$ This step does not add any subsets to $\beth_{n-2}(\kappa)$.
3. Cohen forcing which adds $\aleph_{\alpha+\omega\cdot 2^\kappa+1}$ subsets to $\beth_{n-1}(\kappa)$.

   After the forcing $A(L^n, \kappa)$ fails as it fails after the first Cohen forcing and no subsets are added to $\beth_{n-2}(\kappa)$ after that. After the forcing X' is definable in $H((\beth_n(\kappa))^+)$ and thus there is an L^{n+2}-characterizable well-order of the powerset of κ and an L^{n+2}-characterizable bijection from 2^κ to 2^ω. It follows that $A(L^{n+2}, \kappa)$ holds.

□

Note that there are several open questions left, for example the following:

3.1.7. QUESTION. *Does $A(L^{n+1}, \kappa)$ hold after the above forcing? Or does it depend on κ and n whether $A(L^{n+1}, \kappa)$ holds after the above forcing?*

3.1.8. THEOREM. *Let κ be a cardinal definable in* L. *There is a model of ZFC in which $A(L^n, \kappa)$ fails for any n but $A(ZF, \kappa)$ holds and all cardinals $\leq \kappa$ of* L *are preserved.*

Proof. This is just an obvious generalization of Ajtai's theorem 3.1.1. Note that the theorem could be also proved by using the forcing $P_{X,\kappa}$. Let L be the ground model. We do an iterated forcing with two steps:

1. Let P_0 be a forcing which adds $2^\kappa = \kappa^+$ Cohen subsets for ω. After this forcing there are two ZF-equivalent non-isomorphic models of cardinality κ in a finite vocabulary. The models are also L^n-equivalent for any natural number n. This forcing does not collapse any cardinals and also GCH above κ is preserved. After this forcing $2^\kappa = 2^\omega$.

2. In the second step we make the Cohen subset G which we added in step 1 and a bijection Π from 2^κ to 2^ω definable in the language of set theory. We make this in such a way, that the truth of all L^n sentences in models of cardinality κ is preserved, and after the forcing the powerset of κ has a ZF-definable well-order and there is a ZF-definable bijection from 2^κ to 2^ω. Consequently $A(ZF, \kappa)$ holds after the forcing.

   As κ is a definable cardinal in L, also $\aleph_{\kappa+\omega}$ is a definable cardinal in L. As GCH holds above κ in $L[G]$, the truth of L^n sentences in models of cardinality κ in $L[G]$ is determined by sets which are hereditarily smaller than $\aleph_{\kappa+\omega}$. We will introduce a forcing which makes G and Π definable in the language of set theory but does not add any sets which are hereditarily smaller than $\aleph_{\kappa+\omega}$.

   Let $X \subseteq \kappa^+$ be a set which codes G and Π. Let P_1 be a forcing which adds $\aleph_{\kappa+\omega+\alpha+2}$ Cohen subsets to $\aleph_{\kappa+\omega+\alpha}$ for those α for which $\alpha \in X$. After the forcing we can read X as the function from κ^+ to $\{0, 1\}$ which maps α to 0 if GCH holds at $\aleph_{\kappa+\omega+\alpha}$ and to 1 otherwise. Now as X is definable by a ZF-formula we have a ZF-definable well-order of the powerset of κ and a ZF-definable bijection from 2^κ to 2^ω. It follows that $A(ZF, \kappa)$ holds.

$\square$

3.2 Solovay's result on complete second order sentences

In this section we will present a Solovay's result about complete second order sentences, Theorem 3.2.3, and prove some related results.

3.2.1. Definition. *A L-sentence ϕ is a* complete L-sentence*, if all such models $\mathfrak{A}$ and $\mathfrak{B}$ that $\mathfrak{A} \models \phi$ and $\mathfrak{B} \models \phi$ are L-equivalent.*

Note that an equivalent definition would be that for all L-sentences ψ, $\phi \models \psi$ or $\phi \models \neg\psi$.

3.2.2. Definition. *We use $S(L)$ to denote the hypothesis that any complete L-sentence ϕ has at most one model up to isomorphism.*

We use $S(L, \kappa)$ to denote the hypothesis that any complete L-sentence ϕ has at most one model of cardinality κ up to isomorphism.

The following is an unpublished result of Solovay [25].

3.2.3. Theorem (Solovay). *It is independent of ZFC whether $S(L^2)$ holds. However, it is provable in ZFC that models which satisfy the same complete second order sentence have the same cardinality.*

Proof. Let $V = \mathrm{L}$ and let ϕ be a complete second order sentence. If there were more than one non-isomorphic models of ϕ then there would be some model $\mathfrak{A}$ which is the $<_{\mathrm{L}}$-least model of ϕ and some model $\mathfrak{B}$ of ϕ which is not isomorphic to the $<_{\mathrm{L}}$-least model of ϕ. But now ϕ can not be complete because $\mathfrak{A}$ satisfies second order sentence "is isomorphic to the $<_{\mathrm{L}}$-least model of ϕ" and $\mathfrak{B}$ does not.

We have proved earlier that if we add a Cohen-generic real G to L, we get L^2-equivalent non-isomorphic models $(F^G \cup \omega, <_\omega, P_G)$ and $(F^{-G} \cup \omega, <_\omega, P_{-G})$. In fact the models satisfy the same complete second order sentence. This sentence says: The universe of the model is $\omega \cup \{X \subseteq \omega : |X \cap -G| < \aleph_0\}$ where G is some Cohen-generic[3] subset of ω over L such that all reals are constructible from G, and there is also the natural order of ω and a relation which tells which elements of ω belong to which subsets of ω.

We will now show that models which satisfy the same complete second order sentence have the same cardinality. Assume not. Then there are models of different cardinalities which satisfy a complete second order sentence ϕ. Some of these models is of the smallest cardinality where there is a model of ϕ and some others are not. Assume $\mathfrak{A}$ is a model of ϕ of the least possible cardinality and $\mathfrak{B}$ is a model of ϕ of some bigger cardinality. Now in $\mathfrak{B}$ the second order sentence "there is a model of ϕ which has cardinality less than cardinality of this model" is true and in $\mathfrak{A}$ it is false. Thus ϕ is not a complete second order sentence.

$\square$

3.2.4. Lemma. *Assume L^* is a logic extending L^2 such that L^* has* relativization property *(see [3] for the definition) and κ is a cardinal characterizable by an L^*-formula. If there is a well-order of the powerset of κ characterizable by an L^*-formula in models of cardinality κ, then $S(L^*, \kappa)$ holds.*

Proof. Let ϕ be a complete L^*-sentence and let $\mathfrak{A}$ and $\mathfrak{B}$ be models of ϕ of cardinality κ. Let $<$ be the well-order of the powerset of κ characterizable by an L^*-formula. As there is a model of ϕ of cardinality κ there is a model of ϕ of cardinality κ which is up to isomorphism the $<$-least model of ϕ. Because ϕ is a complete L^*-sentence and the property "is isomorphic to the $<$-least model of ϕ" is expressible by an L^*-sentence, every model of ϕ is isomorphic to the $<$-least model of ϕ. It follows that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.

$\square$

[3] We can say in second order logic that G meets all dense subsets of the Cohen real forcing over L. This is because in the Cohen real forcing over L the set of forcing conditions is countable and hence characterizable in second order logic in any transitive model of ZFC. Consequently we can quantify over dense subsets in second order logic and "G is a Cohen generic subset of ω over L" is expressible in second order logic. All "Ajtai models" over Cohen generic subsets of ω over L are ZF-equivalent (follows from the proof of Theorem 2.3.1) thus the sentence is a complete second order sentence.

The following theorem is essentially Theorem 4 in Ajtai [2].

3.2.5. THEOREM (AJTAI). *It is consistent with ZFC that $S(L^2, \omega)$ fails but $S(L^3, \omega)$ holds.*

Proof. [sketch] We describe the forcing of Ajtai which adds a Cohen real to L and makes it third order characterizable. The idea is to add first a Cohen real to L and then make that Cohen real third order characterizable by adding uncountable branches to suitably chosen Suslin trees.

After we have added a Cohen real there are two non-isomorphic countable models which satisfy the same complete second order sentence, i.e., $S(L^2, \omega)$ fails. Adding new subsets to ω_1 does not make those models isomorphic or chance truth of second order sentences, thus in the end $S(L^2, \omega)$ fails. On the other hand, there is a third order definable well-order of the reals, so $S(L^3, \omega)$ holds by Lemma 3.2.4.

□

3.2.6. DEFINITION. *Let L^* be a logic. A* categorical *L^*-theory is an L^*-theory which has exactly one model up to isomorphism. A* categorical *L^*-sentence is an L^*-sentence which has exactly one model up to isomorphism.*

3.2.7. THEOREM. *Assume the Fraïssé Hypothesis holds. There is a cardinal κ and a model $\mathfrak{A}$ of cardinality κ such that $\mathfrak{A}$ satisfies a categorical L^2-theory but there is no model of cardinality κ satisfying a categorical second order sentence.*

In case $V = \mathrm{L}$ we can replace "categorical second order sentence" by "complete second order sentence" above.

Proof. There is a second order sentence ϕ which says that the model has cardinality $\aleph_\alpha$ where α is the order type of a well-ordered predicate U, see Remark 4.2.3 below. Assume the Fraïssé Hypothesis holds and α is a countable ordinal. Consider the theory $T_\alpha = \{\phi\} \cup \{\psi^U : \alpha \models \psi\} \cup U_\omega$ where ψ^U is the relativization of a second order sentence ψ to U and U_ω is a sentence which says that U is countable. Any model of T_α has cardinality $\aleph_\alpha$, U has order type α and the whole model has order type $\aleph_\alpha$. Consequently every model of T_α is categorical. As there are uncountably many countable ordinals α but only countably many categorical second order sentences, in some cardinality $\aleph_\alpha$ no model satisfies a categorical second order sentence.

In case $V = \mathrm{L}$ every complete second order sentence is categorical so the claim follows.

□

3.2.8. THEOREM. *Let κ be a cardinal in* L *such that κ and all smaller cardinals are definable in* L*, i.e., in the model $(L, \in)$. There is a model of ZFC in which $S(L^2, \lambda)$ holds for all $\lambda < \kappa$ but $S(L^2, \kappa)$ fails and all cardinals $\leq \kappa$ of* L *are preserved.*

Proof. Let L be the ground model. We add a Cohen subset G to κ. This does not add any new subsets to cardinals smaller than κ so $S(L^2, \lambda)$ is preserved for all $\lambda < \kappa$. In cardinality κ there are now two non-isomorphic models which satisfy the same complete second order sentence ϕ. The sentence ϕ says that the model has as universe the ordinal κ and a set of subsets of κ which differ from a Cohen generic subset of κ over L in less than κ digits, and there is epsilon relation in the model. This can be said in second order logic as the notion of forcing which adds a Cohen subset for κ to L has κ forcing conditions, and we can quantify over dense subsets of this notion of forcing in a model of cardinality κ.

$\square$

3.2.9. THEOREM. *Let $\kappa = \aleph_\alpha$ be a cardinal such that κ and all smaller cardinals are definable in* L *and let $n \geq 3$ be a natural number. There is a model of ZFC in which $S(L^n, \lambda)$ fails for all $\lambda \leq \kappa$ but $S(L^{n+2}, \lambda)$ holds for all $\lambda \leq \kappa$ and all cardinals $\leq \kappa$ of* L *are preserved.*

Proof. We use the iterated forcing developed earlier in this chapter. Let L be the ground model.

1. We add 2^κ Cohen subsets to ω. After this forcing the cardinals are preserved, i.e., every cardinal of L remains a cardinal. Also for any infinite cardinal $\lambda \leq \kappa$ it holds that $2^\lambda = 2^\kappa$ and GCH holds at and above κ.

   As this notion of forcing has cardinality κ^+, after the forcing we can quantify over dense subsets of this forcing in L^3 in any model of infinite cardinality. From a generic set over this forcing we can construct countable ZF-equivalent non-isomorphic "Ajtai models" (see Theorem 2.3.3) and we can expand these models to have cardinality $\lambda \leq \kappa$ as is described in Theorem 5.2.7. Consequently $A(L^n, \lambda)$ fails for any $\lambda \leq \kappa$ after this forcing.

2. We add a definable well-order for the powersets of all cardinals $\lambda \leq \kappa$ as follows. Let $\dot{X}$ be a canonical name for a sequence $(\dot{X}_\beta : \beta \leq \alpha)$ such that each $\dot{X}_\beta$ is a canonical name for a well-order of the powerset of $\aleph_\beta$. Let $\dot{X}'$ be a canonical name for a subset of $\kappa^+ \setminus \{\gamma : \gamma \text{ is 0 or a limit ordinal}\}$ which codes $\dot{X}$. The next step is the forcing $P_{\dot{X}', \beth_{n-2}(\kappa)}$.

   This step does not add any such subsets to $\beth_{n-2}(\kappa)$ which were not added in step 1. Thus $S(L^n, \lambda)$ remains false.

3. The last step adds $\aleph_{\alpha+\omega\cdot\kappa^++1}$ Cohen subsets to $\beth_{n-1}(\kappa)$. After this step $S(L^n, \lambda)$ remains false for all $\lambda \leq \kappa$. After the last step there will be a definable well-order of the powerset of λ in $H(\beth_n(\lambda)^+)$ and from Lemma 3.2.4 it follows that $S(L^{n+2}, \lambda)$ holds.

$\square$

Chapter 4

Generalized quantifiers

4.1 The countable case

In this chapter we ask whether higher order logics can be replaced in the above results by a logic with generalized quantifiers. A clear limitation is provided by the following result [9]:

4.1.1. Theorem (Hella). *Let n be a natural number. Let $\{Q_i : i \in I\}$ be a set of generalized quantifiers of arity $\leq n$ and let κ be any infinite cardinal. Then there are two models of cardinality κ which are $L(\{Q_i : i \in I\})$-equivalent but not isomorphic.*

In view of the above theorem, in order to characterize all models of an infinite cardinality by their theories in a logic $L(\{Q_i : i \in I\})$, the arity of the generalized quantifiers of the logic has to increase beyond any finite bound. On the other hand, if we let the arity grow we can find a generalized quantifier logic L such that $A(L, \kappa)$ holds provably in ZFC. We will next give the definition of the above mentioned language in case $\kappa = \omega$.

4.1.2. Definition. *Let $(\mathfrak{A}_r)_{r\in\mathbb{R}}$ be an indexing of all countable models in finite vocabularies by real numbers, i.e., for any countable model $\mathfrak{A}$ in a finite vocabulary there is exactly one $r \in \mathbb{R}$ such that $\mathfrak{A}$ is isomorphic to $\mathfrak{A}_r$.*

The language $L^* = L(Q_{r,s} : r, s \in \mathbb{Q})$ contains atomic formulas, is closed under negation, conjunction and first order existential and universal quantifiers. L^* is also closed under the quantifiers

$$(*) \qquad Q_{r,s}\bar{x}^1, \ldots, \bar{x}^n(\phi_1(\bar{x}^1), \ldots \phi_n(\bar{x}^n))$$

for all $r, s \in \mathbb{Q}$. The notation $\bar{x}^k$ is a shorthand for $x_1^k, \ldots, x_{N_k}^k$, where $x_m^n \neq x_o^p$ whenever $m \neq o$ or $n \neq p$.

The formula $(*)$ is defined to be true in a model $\mathfrak{M}$ if and only if $|\mathfrak{M}| = \aleph_0$ and $(M, \phi_1^{\mathfrak{M}}(\cdot), \ldots, \phi_n^{\mathfrak{M}}(\cdot))$ is isomorphic to a structure $\mathfrak{A}_t$ such that $r < t < s$.

We can't prove in ZFC that there is any such indexing $(\mathfrak{A}_r)_{r\in\mathbb{R}}$ of the countable models, which is definable in the language of set theory. But we fix one such indexing no matter whether it is definable or not.

4.1.3. THEOREM (WEINSTEIN (UNPUBLISHED)). [1]
In any model of ZFC there is a countable language L^* such that $A(L^*, \omega)$ holds.

Proof. Let $\mathfrak{A}$ be a countable model in a finite relational vocabulary $(R_1, \ldots, R_n)$. Note that constants can be coded into unary relations and n-ary functions can be coded into $n+1$-ary relations so restriction to relational vocabularies does not make the result less general. The sentence $Q_{r_0,s_0}\bar{x}^1, \ldots, \bar{x}^n(R_1(\bar{x}^1), \ldots, R_n(\bar{x}^n))$ is true in a model $\mathfrak{A}$ if and only if the r such that $\mathfrak{A}_r$ which is isomorphic to $(A, R_1^{\mathfrak{A}}, \ldots, R_n^{\mathfrak{A}})$ (and thus isomorphic to $\mathfrak{A}$ itself) is between r_0 and s_0. Let now $\mathfrak{A}$ and $\mathfrak{B}$ be two countable non-isomorphic models in vocabulary τ. Now $\mathfrak{A}$ is isomorphic to some $\mathfrak{A}_p$ and $\mathfrak{B}$ is isomorphic to some $\mathfrak{A}_q$ for different p and q. Let r_0 and s_0 be such that $r_0 < p < s_0$ and either $q < r_0$ or $s_0 < q$. Then

$$\mathfrak{A} \models Q_{r_0,s_0}\bar{x}^1, \ldots, \bar{x}^n(R_1(\bar{x}^1), \ldots, R_n(\bar{x}^n))$$

but

$$\mathfrak{B} \models \neg Q_{r_0,s_0}\bar{x}^1, \ldots, \bar{x}^n(R_1(\bar{x}^1), \ldots, R_n(\bar{x}^n)).$$

□

4.2 The uncountable case

Theorem 4.1.3 can be generalized to any infinite cardinality as we will do next.

The proof is based on an idea of Per Lindström [14]. First we will give the definition of the relevant logic:

4.2.1. DEFINITION. *Let $(\mathfrak{A}_f)_{f:\kappa\to\{0,1\}}$ be an indexing of all models of cardinality κ in finite vocabularies.*

Define the sets X_α and X'_α as follows: $X_\alpha = \{f : \kappa \to \{0,1\} : f(\alpha) = 0\}$, $X'_\alpha = \{f : \kappa \to \{0,1\} : f(\alpha) = 1\}$.

Let $L^{\kappa *} = L(Q^S_\alpha : \alpha < \kappa$, S a finite set of variables$)$ contain atomic formulas, be closed under negation, conjunction and first order existential and universal quantifiers. Let S be any finite sequence of finite sequences of distinct variables $(S = (\bar{x}^1, \ldots, \bar{x}^k))$. Let L^κ be also closed under the following quantifiers Q^S_α and R^S_α:

$$Q^S_\alpha \bar{x}^1, \ldots, \bar{x}^k(\phi_1(\bar{x}^1), \ldots \phi_k(\bar{x}^k))$$

[1] This result and its proof is presented here with the permission of Professor Scott Weinstein.

$$R^S_\alpha \bar{x}^1, \ldots, \bar{x}^k(\phi_1(\bar{x}^1), \ldots \phi_k(\bar{x}^k)).$$

The formula $Q^S_\alpha \bar{x}^1, \ldots, \bar{x}^k(\phi_1(\bar{x}^1), \ldots \phi_k(\bar{x}^k))$ *is true in a model* $\mathfrak{M}$ *iff* $|\mathfrak{M}| = \kappa$ *and* $(M, \phi_1^{\mathfrak{M}}(\cdot), \ldots, \phi_k^{\mathfrak{M}}(\cdot))$ *is isomorphic to an* $\mathfrak{A}_f$ *such that* $f \in X_\alpha$.
The formula $R^S_\alpha \bar{x}^1, \ldots, \bar{x}^k(\phi_1(\bar{x}^1), \ldots \phi_k(\bar{x}^k))$ *is true in a model* $\mathfrak{M}$ *iff* $|\mathfrak{M}| = \kappa$ *and* $(M, \phi_1^{\mathfrak{M}}(\cdot), \ldots, \phi_k^{\mathfrak{M}}(\cdot))$ *is isomorphic to an* $\mathfrak{A}_f$ *such that* $f \in X'_\alpha$.

Note that there are countably many finite vocabularies, and for any finite vocabulary there are at most 2^κ pairwise non-isomorphic models of cardinality κ with the vocabulary. Thus an indexing $(\mathfrak{A}_f)_{f:\kappa\to\{0,1\}}$ of all models of cardinality κ in finite vocabularies always exists though may be impossible to define in the language of set theory.

4.2.2. THEOREM. *Let* κ *be an infinite cardinal. There is a language* $L^{\kappa *}$ *of cardinality* κ *such that* $A(L^{\kappa *}, \kappa)$ *holds.*

Proof. Any $f : \kappa \to \{0, 1\}$ can be expressed as an intersection of κ many sets of the form X_α and X'_α, namely $\{f\} = \bigcap\{X_\alpha : f(\alpha) = 0\} \cap \bigcap\{X'_\alpha : f(\alpha) = 1\}$. On the other hand, if f and g are two different functions from κ to $\{0, 1\}$, there is an X_α such that one of f and g belongs to X_α and the other does not.

As in the previous theorem, assume without loss of generality that a model $\mathfrak{A}$ of cardinality κ has a relational vocabulary $R_1, \ldots R_n$.

The sentence $Q_\alpha \bar{x}^1, \ldots, \bar{x}^n(R_1(\bar{x}^1), \ldots, R_n(\bar{x}^n))$ is true in a model $\mathfrak{A}$ if and only if the f such that $\mathfrak{A}_f$ which is isomorphic to $(A, R_1^{\mathfrak{A}}, \ldots, R_n^{\mathfrak{A}})$ (and thus isomorphic to $\mathfrak{A}$ itself) belongs to X_α. The sentence $R_\alpha \bar{x}^1, \ldots, \bar{x}^n(R_1(\bar{x}^1), \ldots, R_n(\bar{x}^n))$ is true in a model $\mathfrak{A}$ if and only if the f such that $\mathfrak{A}_f$ which is isomorphic to $(A, R_1^{\mathfrak{A}}, \ldots, R_n^{\mathfrak{A}})$ (and thus isomorphic to $\mathfrak{A}$ itself) belongs to X'_α.

Let τ be a finite relational vocabulary $(R_1^{m_1}, \ldots, R_n^{m_n})$ where the superscripts denote the arities of the relation symbols. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two non-isomorphic models of cardinality κ with vocabulary τ. Let P be a sequence of variables which corresponds to arities of the relation symbols i.e. $P = \bar{x}^1, \ldots, \bar{x}^n$ such that each $\bar{x}^p$ contains m_p variables. Now $(A, R_1(\cdot)^{\mathfrak{A}}, \ldots, R_n(\cdot)^{\mathfrak{A}})$ (which is isomorphic to $\mathfrak{A}$) is isomorphic to some $\mathfrak{A}_f$ and $(B, R_1(\cdot)^{\mathfrak{B}}, \ldots, R_n(\cdot)^{\mathfrak{B}})$ (which is isomorphic to $\mathfrak{B}$) is isomorphic to some $\mathfrak{A}_g$ and $f \neq g$. So there is an β such that one of f and g (say f) gets value 0 at β and the other gets value 1. Now $\mathfrak{A} \models Q^P_\beta \bar{x}^1, \ldots, \bar{x}^n(R_1(\bar{x}^1), \ldots R_n(\bar{x}^k))$ but $\mathfrak{B} \models \neg Q^P_\beta \bar{x}^1, \ldots, \bar{x}^n(R_1(\bar{x}^1), \ldots R_n(\bar{x}^k))$.

$\square$

Hella's result 4.1.1 showed that in order to characterize all models of cardinality κ the arity of the quantifiers in the language must be unbounded. However, if we look at the proof above we see that if we want to characterize all models of cardinality κ in a fixed finite vocabulary, the arity of quantifiers in the language can be bounded.

4.2.3. Remark. *Many cardinality quantifiers are expressible in second order logic. In the logics $L^2_{\kappa^+,\omega}$ even more cardinality quantifiers are expressible. If an ordinal α is characterizable in second order logic then, as we show below, the quantifiers $\exists^{\geq \aleph_\alpha}$ and $\exists^{\aleph_\alpha}$ are definable in second order logic. This is also true for infinitary second order languages. Consequently if κ is a regular cardinal and $\alpha < \kappa$ is an ordinal, the quantifiers $\exists^{\geq \aleph_\alpha}$ and $\exists^{\aleph_\alpha}$ are definable in the logic $L^2_{\kappa,\omega}$.*

Assume now α is characterizable in second order logic (or in infinitary second order logic). We will introduce a sentence which defines the quantifier $\exists^{\aleph_\alpha}$:

The sentence says that there are U, V, c and $<$ such that the following hold:

1. *The relation $<$ defines a well-order in the unary predicate U,*
2. *Unary predicate V contains those elements x satisfying the following:*
   - $x \in U$
   - *x has infinitely many predecessors*
   - *For all $y < x$: $|\{z : z < y\}| < |\{z : z < x\}|$,*
3. *c is the greatest element of V and $(V \setminus \{c\}, <\restriction V \setminus \{c\}) \cong (\alpha, \epsilon)$,*
4. *$\exists \pi$ (π is a bijection from $\{y : y < c\}$ to $\{y : \phi(y)\}$).*

When an ordinal α is given, 1.-3. characterize the cardinal $\aleph_\alpha$. Finally 4. says that there is a bijection from this cardinal to those elements which satisfy the formula ϕ. Thus this sentence is equivalent to $\exists^{\aleph_\alpha} x\phi(x)$. By replacing bijection by injection in 4. we get a sentence equivalent to $\exists^{\geq \aleph_\alpha} x\phi(x)$.

4.2.4. Definition. *We define the ordinal ϵ_0 as follows: $\alpha_0 = \omega$, $\alpha_{n+1} = \alpha_n^\omega$, $\epsilon_0 = sup\{\alpha_n : n \in \omega\}$.*

4.2.5. Lemma. *The ordinal ϵ_0 and all ordinals $\alpha < \epsilon_0$ are characterizable by second order formulas.*

Proof. We prove the claim by induction on ordinal α. Let $\alpha_n, n \in \omega$, be as in Definition 4.2.4. First of all, it is clear that ϵ_0 and all the ordinals α_n are second order characterizable.

All ordinals smaller or equal to α_0 are finite or ω and are thus second order characterizable.

Assume all ordinals smaller or equal to α_n are second order characterizable. The ordinal α_{n+1} is second order characterizable and if $\alpha_n < \alpha < \alpha_{n+1}$ then α is of the form $\alpha_n^m + \alpha_n \cdot m' + \beta$ for some (unique) natural numbers m and m' and ordinal $\beta < \alpha_n$. Now α is second order characterizable as α_n, m, m', β, ordinal addition, ordinal multiplication and ordinal exponentiation are second order characterizable.

□

We say that a model $\mathfrak{A}$ is *monadic* if the vocabulary of $\mathfrak{A}$ contains only unary relation symbols, and no constants or function symbols.

4.2.6. THEOREM. *For any monadic model $\mathfrak{A}$ of cardinality smaller or equal to $\aleph_{\epsilon_0}$ there is a second order sentence $\phi_{\mathfrak{A}}$ which characterizes the model $\mathfrak{A}$ up to isomorphism.*

Let κ be a regular cardinal. For any monadic model $\mathfrak{A}$ of cardinality smaller than $\aleph_\kappa$ there is an $L^2_{\kappa,\omega}$-sentence $\phi_{\mathfrak{A}}$ which characterizes the model $\mathfrak{A}$ up to isomorphism.

Proof. Let the vocabulary of $\mathfrak{A}$ be $P_1, \ldots, P_n$ where each predicate has arity 1. Consider formulas $\phi(x)$ of the form $\bigwedge_{1\leq m\leq n} \psi_m$, where each ψ_m is either $P_m(x)$ or $\neg P_m(x)$. It is clear that if $\mathfrak{B}$ is a model in the same vocabulary as $\mathfrak{A}$ and for each formula of the form above the cardinality of elements satisfying the formula is the same in $\mathfrak{A}$ and $\mathfrak{B}$, then the models $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. Thus if we have in our language cardinality quantifiers which correspond to $|\phi(\cdot)^{\mathfrak{A}}|$ for the ϕ's of the form above, then we can write a sentence which tells cardinalities of these sets and this sentence characterizes $\mathfrak{A}$ up to isomorphism.

Now the first part of the theorem follows from Remark 4.2.3 and Lemma 4.2.5. The second part follows from Remark 4.2.3 and the fact that every ordinal smaller than κ is characterizable in $L_{\kappa,\omega}$ (see Lemma 1.2.11).

□

Note that $\aleph_{\epsilon_0}$ is not an upper limit of those cardinalities where all monadic structures can be characterized by a second order sentence. We can iterate the idea of the proof of Lemma 4.2.5 and get bigger countable ordinals α such that all ordinals up to α are second order characterizable. However, as there are uncountably many countable ordinals and countably many second order sentences, there is some countable ordinal α and a monadic stucture $\mathfrak{A}$ of cardinality $\aleph_\alpha$ such that $\mathfrak{A}$ can not be characterized by a second order sentence. But the infinitary second order language $L^2_{\omega_1,\omega}$ is strong enough to characterize all monadic structures of cardinality $< \aleph_{\omega_1}$.

Chapter 5

Infinitary second order languages

5.1 Discussion

We have shown in Theorem 2.1.1 that $A(L^2,\omega)$ is consistent with ZFC. But is $A(L^2,\aleph_1)$ consistent with ZFC? It is easy to show by a simple cardinality argument that $A(L^2,\aleph_1)$ does not necessarily hold:

In any finite vocabulary with a binary predicate there are $2^{\aleph_0}$ many L^2-theories. In a finite vocabulary with a binary predicate there are $2^{\aleph_1}$ models of cardinality $\aleph_1$ which are pairwise non-isomorphic. It is clear that if $2^{\aleph_0} < 2^{\aleph_1}$, then there are two second order equivalent non-isomorphic models of cardinality $\aleph_1$. However, if $2^{\aleph_0} = 2^{\aleph_1}$ we don't know what happens:

5.1.1. Question. *Is it consistent that $2^{\aleph_0} = 2^{\aleph_1}$ and $A(L^2,\aleph_1)$ holds?*

In Chapter 2 we saw the result of Ajtai that it is independent of ZFC whether all countable models in any finite vocabulary can be characterized up to isomorphism by their second order theories. By appropriate coding sentences of second order logic are natural numbers and second order theories are real numbers. Via coding, countable models are also real numbers, so the question whether any two different reals of the latter type correspond to two different reals of the former type is meaningful. We note that first order theories also correspond to real numbers but all countable models can not be characterized up to isomorphism by their first order theories.

All models of cardinality κ can be characterized up to isomorphism by a L_{κ^+,κ^+} sentence, as we will show. Let $\mathfrak{A}$ be a model of cardinality κ. Let $(a_\alpha : \alpha < \kappa)$ be a well-ordering of the domain of $\mathfrak{A}$. Let $\phi_{\mathfrak{A}}$ be the sentence

$$\exists (x_\alpha)_{\alpha<\kappa}(\bigwedge_{\phi\in L_{\omega,\omega}\text{atomic}}\{\phi(x_{\alpha_1},\ldots,x_{\alpha_m}) : \mathfrak{A}\models\phi(a_{\alpha_1},\ldots,a_{\alpha_m})\}$$
$$\wedge\bigwedge_{\phi\in L_{\omega,\omega}\text{atomic}}\{\neg\phi(x_{\alpha_1},\ldots,x_{\alpha_m}) : \mathfrak{A}\models\neg\phi(a_{\alpha_1},\ldots,a_{\alpha_m})\}$$
$$\wedge\forall y\bigvee_{\alpha<\kappa} y = x_\alpha)$$

The sentences of the form $\phi_{\mathfrak{A}}$ say that there is a sequence of elements which satisfies exactly the same atomic formulas as the elements $(a_\alpha : \alpha < \kappa)$ satisfy in $\mathfrak{A}$ and there are no other elements. It is clear that for each sentence of the form $\phi_{\mathfrak{A}}$ there can be only one model up to isomorphism which satisfies $\phi_{\mathfrak{A}}$. However, these sentences have the same cardinality as the model in question. In this paper we are interested in the possibility of characterizing models up to isomorphism by theories, where the sentences have cardinality smaller than the model.

We make the following observations about the possibility to characterize models up to isomorphism by infinitary languages. In the countable cardinality of the models $L_{\omega_1,\omega}$-equivalence implies isomorphism. Generally $L_{\infty,\omega}$ equivalence is equivalent to the existence of a back-and-forth set. Back-and-forth-equivalence implies isomorphism only in the countable cardinality so $L_{\infty,\omega}$ is not good in characterizing uncountable models up to isomorphism. Nadel and Stavi [21] have investigated logics $L_{\infty,\lambda}$ and showed that these are not successful in characterizing all models up to isomorphism in cardinality λ, where λ is an uncountable successor cardinal.

Thus infinitary languages are not sufficient for characterizing all models up to isomorphism in an uncountable cardinality λ, if we don't allow the infinitary language to have sentences of cardinality λ. Higher order languages are also not very successful. As they have only continuum many theories they cannot characterize all models up to isomorphism in a cardinality which has more than continuum many models.

5.2 Regular cardinals

We have introduced the infinitary second order language $L^2_{\kappa,\omega}$ for a regular cardinal κ in the preliminaries. We will now prove that it is independent of ZFC whether all models of cardinality κ in any finite vocabulary can be characterized up to isomorphism by their $L^2_{\kappa,\omega}$-theories. Sentences of $L^2_{\kappa,\omega}$ correspond to subsets of cardinals $\lambda < \kappa$ so this logic is not "too strong".

5.2.1. Remark. We decided to formulate our theorems for $L^2_{\kappa,\omega}$ because it is a natural logic. However, most of our results hold equally well for a fragment of $L^2_{\kappa,\omega}$ which contains atomic formulas, in which ordinals smaller than κ are characterizable and which is closed under second order quantifiers, first order quantifiers and finite connectives. We don't know whether this fragment is a proper fragment of $L^2_{\kappa,\omega}$.

The logic $L^2_{\kappa,\omega}$ has the following properties in models of cardinality $\geq \kappa$:

1. All sets in $H(\kappa)$ are characterizable.
2. The isomorphism type (in the sense of Lemma 1.2.8) of the model in question is characterizable.

3. The logic is closed under quantifying over sets in $H(\lambda^+)$, where λ is the cardinality of the model in question.

4. The logic is closed under negation and finite conjunction.

It might be the case that $L^2_{\kappa,\omega}$ is the least logic satisfying the conditions 1.-4. above, but we have not found a proof for that.

5.2.2. THEOREM. *If κ is a regular cardinal and there is a second order definable well-order of the powerset of κ, then $A(L^2_{\kappa,\omega}, \kappa)$ holds. In particular $A(L^2_{\kappa,\omega}, \kappa)$ holds if $V = \mathrm{L}$.*

Proof. We omit the details as the proof is entirely similar to the proof of Theorem 2.1.1. See also the proof of Theorem 5.2.3 below.

As in Theorem 2.1.1, a model can be coded into an n-ary relation $R \subseteq \kappa^n$. By Lemma 1.2.11 all ordinals smaller than κ are characterizable. For all n-tuples of ordinals smaller than κ we can say whether the tuple belongs to or does not belong to the least subset of κ^n in the well-order which is isomorphic with the model. The canonical well-order of L up to sets of cardinality κ is second order characterizable in any cardinality κ.

□

In Theorem 5.2.2 we saw that $A(L^2_{\kappa,\omega}, \kappa)$ holds in L at any regular cardinal κ as there is a second order definable well-order of the powerset of κ. In fact we will get a better result:

5.2.3. THEOREM. *Let κ be a regular cardinal and let $H(\kappa^+) \subset \mathrm{L}[X]$ for some set X with $X \subseteq \lambda < \kappa$. Then $A(L^2_{\kappa,\omega}, \kappa)$ holds.*

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two models of cardinality κ. By assumption $\mathfrak{A}$ and $\mathfrak{B}$ belong up to isomorphism to $\mathrm{L}[X]$ and hence are isomorphic to some sets in $\mathrm{L}[X]$. In the infinitary second order language $L^2_{\kappa,\omega}$ we can talk about the least subset of κ^n in the canonical well-order of $\mathrm{L}[X]$ which is isomorphic to $\mathfrak{A}$. We will now describe how this is done.

In the logic $L^2_{\kappa,\omega}$ all ordinals $\alpha < \kappa$ are characterizable by certain formulas θ_α(see Lemma 1.2.11). Now the set X is characterizable in a model of cardinality κ by the formula

$$\exists <^* (\phi_{(\kappa,\epsilon)}(A, <^*) \wedge \forall x(P(x) \leftrightarrow \bigvee_{\alpha \in X} \theta_\alpha(x)))$$

In the above formula $\phi_{(\kappa,\epsilon)}$ denotes the formula which characterizes $(\kappa, <)$ and A denotes the domain of the model in question. We denote this formula which characterizes X by ϕ_X.

If the set X and an ordinal $\alpha < \kappa^+$ are given, the αth level of the sets constructible from X is second order characterizable from these parameters. Also the canonical well-order of $\mathrm{L}_\alpha[X]$ is second order definable on $L_\alpha[X]$ from X and α. Let $\phi_{\mathrm{L}_\alpha[X]}(Y, X, \alpha)$ be a second order formula which says that Y is the αth level of the sets constructible from X(up to isomorphism) and let $\phi_{<_{\mathrm{L}_\alpha[X]}}(Z, X, \alpha)$ be a second order formula which says that Z is the canonical well-order of the αth level of the sets constructible from X(up to isomorphism).

As usual, we assume that the model in question has been coded into an n-ary relation R. We are interested in sentences of the following form:

There are X, a, M, $<$ and R_0 such that the following hold:

1. $\phi_X(X)$
2. a is an ordinal
3. $\phi_{\mathrm{L}_\alpha[X]}(M, X, a)$
4. $\phi_{<_{\mathrm{L}_\alpha[X]}}(<, X, a)$
5. $R_0 \in M \wedge R_0 \cong R \wedge \forall R_1((R_1 \in M \wedge R_1 \cong R) \rightarrow (R_0 < R_1 \vee R_0 = R_1))$
6. $(\alpha_1, \ldots, \alpha_n) \in R_0$

The first four formulas say that a is an ordinal, X is what it is supposed to be (up to isomorphism), M is $\mathrm{L}_a[X]$ (up to isomorphism) and $<$ is $<_{\mathrm{L}_a[X]}$ (up to isomorphism). The fifth formula says that R_0 belongs to $\mathrm{L}_a[X]$ and it is the least model in the canonical well-order of $\mathrm{L}_a[X]$ which is isomorphic to the model in question. The sixth formula says that a tuple $(\alpha_1, \ldots, \alpha_n)$ belongs to R_0. Similarly we can say that a tuple does not belong to R_0.

If two models of cardinality κ are now $L^2_{\kappa,\omega}$-equivalent, then they satisfy all the same sentences of the form above. Thus they have the same set R_0 and consequently they are isomorphic.

□

5.2.4. COROLLARY. *It is consistent that there is a measurable cardinal κ and $A(L^2_{\lambda,\omega}, \lambda)$ holds for any $\lambda > 2^\kappa$.*

Proof. There is a model of ZFC [24] such that there is a measurable cardinal κ and every set is constructible from a certain subset of the powerset of κ.

□

5.2.5. QUESTION. *Are the following conditions equivalent?*

1. *There is an $L^2_{\kappa,\omega}$-definable well-order of the powerset of κ.*

2. $A(L^2_{\kappa,\omega}, \kappa)$.

Ajtai proved the following theorem in case $\kappa = \omega$, see Theorem 2.3.1.

5.2.6. THEOREM. *Let κ be a regular cardinal. It is consistent with ZFC that there are two ZF-equivalent non-isomorphic models of cardinality κ. The models are also $L^n_{\kappa,\omega}$-equivalent for all n.*

Proof. We add a Cohen-generic subset G to κ. The forcing conditions are mappings of cardinality smaller than κ from κ to $\{0,1\}$. We define the model $(F^G \cup \kappa, <_\kappa, R_G)$. Here F^G is the set of all subsets of κ which agree with G everywhere except in a set of cardinality smaller than κ, $<_\kappa$ is the natural order of κ and R_G is a relation which tells which elements of κ belong to which sets in F^G. The model $(F^{-G} \cup \kappa, <_\kappa, R_{-G})$ is defined in an analogous way.

We note that this forcing is $< \kappa$-closed so it does not add any new subsets to cardinals smaller than κ. If κ is inaccessible, all cardinals below κ are preserved and κ remains inaccessible.

No forcing condition can determine the model $(F^G \cup \kappa, <_\kappa, R_G)$ in any way, as a forcing condition defines the value of G only in a subset of κ which has cardinality less than κ. For any forcing condition p there are two generic filters G and G' containing p such that

$$V^G = V^{G'}, \quad (\dot{F}^G \cup \kappa, <_\kappa, R_G)^{V^G} = (\dot{F}^{-G} \cup \kappa, <_\kappa, R_{-G})^{V^{-G}}$$

and

$$(\dot{F}^G \cup \kappa, <_\kappa, R_G)^{V^{-G}} = (\dot{F}^{-G} \cup \kappa, <_\kappa, R_{-G})^{V^G}.$$

Thus the models $(F^G \cup \kappa, <_\kappa, R_G)$ and $(F^{-G} \cup \kappa, <_\kappa, R_{-G})$ are ZF-equivalent with parameters from the ground model. As the forcing does not add any new subsets to any cardinals smaller than κ, by Lemma 1.2.12 the models are $L^2_{\kappa,\omega}$-equivalent. But they are not isomorphic: the well-ordered structure $(\kappa, <_\kappa)$ is rigid, so every subset of κ would be mapped in an isomorphism to itself. However $G \in (F^G \cup \kappa, <_\kappa, R_G)$ and $G \notin (F^{-G} \cup \kappa, <_\kappa, R_{-G})$, so there is no isomorphism.

□

5.2.7. THEOREM. *Let M be a transitive model of ZFC and let κ be a regular cardinal in M. If we force a Cohen subset for κ in M, in the generic extension there are two ZF-equivalent non-isomorphic models of cardinality λ in all cardinalities $\lambda \geq \kappa$.*

Proof. We have proved that adding a Cohen subset to a regular cardinal κ produces two models of cardinality κ which are non-isomorphic but satisfy the same formulas of the language of set theory with parameters from the ground model. In fact Cohen subsets produce such models in all cardinalities $\lambda \geq \kappa$.

This is because we can extend the universes of the models defined in Theorem 5.2.6 by adding λ new elements and putting them to some new unary relation. These new models can be constructed from the models introduced in Theorem 5.2.6 and the term λ, and thus they are ZF-equivalent.

□

5.3 Independence

We have proved that it is independent of ZFC whether $A(L^2_{\kappa,\omega},\kappa)$ holds at a regular cardinal κ. It happens that these are also relatively independent of each other, as the following theorem demonstrates.

5.3.1. THEOREM. *Let J be a finite set of regular cardinals. It is consistent that $A(L^2_{\kappa,\omega},\kappa)$ fails for all cardinals κ in J and holds at every regular cardinal κ not in J.*

Proof. We start from L and use iterated forcing to add Cohen subsets to all cardinals in J, adding a Cohen subset first to the smallest cardinal in J and proceeding this way from down to up. We note that GCH holds in L and adding a single Cohen subset preserves GCH so GCH is preserved all the way through our forcing. Also cardinals are preserved. Let κ be a cardinal in J. It follows from the Factor Lemma that the iterated forcing can be decomposed into $P_{<\kappa} * P_\kappa * P_{>\kappa}$ as follows. The forcing $P_{<\kappa}$ preserves GCH and cardinals and P_κ adds a Cohen subset to κ. Thus after $P_{<\kappa} * P_\kappa$ we have GCH, cardinals are preserved and $A(L^2_{\kappa,\omega},\kappa)$ fails because of the proof of Theorem 5.2.6 applied after $P_{<\kappa}$. The forcing $P_{>\kappa}$ is κ^+ closed and thus does not add any subsets to cardinals smaller than or equal to κ. Consequently, $P_{>\kappa}$ does not chance the truth value of $A(L^2_{\kappa,\omega},\kappa)$, which is false after the forcing $P_{<\kappa} * P_\kappa$.

Let now $\kappa \notin J$. Our forcing can be decomposed to $P_{<\kappa} * P_{>\kappa}$. The forcing $P_{<\kappa}$ adds some Cohen subsets below κ and $P_{>\kappa}$ adds subsets only to cardinals greater than κ. Thus after the forcing $H(\kappa^+) \subseteq L[X]$ for some $X \subseteq \lambda < \kappa$ and from Theorem 5.2.3 it follows that $A(L^2_{\kappa,\omega},\kappa)$ holds.

□

5.3.2. THEOREM. *Let J be a set which contains some successor cardinals and possibly ω. It is consistent that $A(L^2_{\kappa,\omega},\kappa)$ fails for all $\kappa \in J$, and holds for all successor cardinals outside J and for all inaccessible cardinals which do not have a cofinal subset in J.*

Proof. Let L be the ground model. We make an iterated forcing with full support in all limit stages, which proceeds from down up and adds Cohen subsets to all cardinals in J. Menas calls it *backward Easton forcing* [19].

The forcing conditions are as follows:

1. If $\omega \in J$, then P_0 is the set of finite partial functions from ω to $\{0,1\}$. A forcing condition p is stronger than forcing condition q if and only if p extends q. If $\omega \notin J$, then P_0 is the trivial forcing.

2. Assume $\alpha = \beta^+$ and P_γ has been defined for all $\gamma \leq \beta$.

   If $\aleph_\alpha \in J$, we define P_α to be the set of sequences $p_\gamma, \gamma \leq \alpha$ where the γth coordinate belongs to P_γ for each $\gamma < \alpha$, and the αth coordinate is a forcing name $\dot{X}$ such that $p \restriction \alpha \Vdash \dot{X}$ is a partial function from $\aleph_\alpha$ to $\{0,1\}$ and $|\dot{X}| < \aleph_\alpha$. If p and q are two conditions of length α, then p is stronger than q if and only if $p \restriction \alpha$ is stronger than $q \restriction \alpha$ and $p \restriction \alpha \Vdash$ "$p(\alpha)$ and $q(\alpha)$ are partial functions from $\aleph_\alpha$ to $\{0,1\}$ which have cardinality smaller than $\aleph_\alpha$ and $p(\alpha) \supseteq q(\alpha)$".

   If $\aleph_\alpha \notin J$ then P_α is the trivial forcing.

3. If α is a limit ordinal, forcing conditions in P_α are tuples p of length α such that for each $\beta < \alpha, \quad p \restriction \beta \Vdash p(\beta) \in P_\beta$. This forcing has full support in all limit stages, which means that in limit stages all coordinates of a forcing condition may be non zero. A forcing condition p is stronger than a forcing condition q if and only if $p \restriction \beta$ is stronger than $q \restriction \beta$ for each $\beta < \alpha$.

We will inductively show that for all cardinals κ the following conditions will hold after the forcing:

1. κ remains a cardinal.

2. If κ is ω or a successor cardinal, $A(L^2_{\kappa,\omega}, \kappa)$ fails iff $\kappa \in J$. If κ is inaccessible cardinal and there is no cofinal subset of κ in J then $A(L^2_{\kappa,\omega}, \kappa)$ holds.

3. Generalized Continuum Hypothesis holds up to cardinal κ.

Let us assume the claim holds for all cardinals below κ. By the Factor Lemma the forcing can be decomposed into parts:

$$P_{<\kappa} * P_\kappa * P_{>\kappa}$$

in such a way that after the forcing $P_{<\kappa}$ Induction Hypothesis holds below κ and if $\kappa \in J$, then P_κ adds a Cohen subset to κ, and if $\kappa \notin J$, then P_κ is the trivial forcing. The forcing $P_{>\kappa}$ is κ^+-closed, so it does not make any chance to Induction Hypothesis in cardinals less or equal to κ.

If $\kappa \in J$, then the Cohen forcing makes $A(L^2_{\kappa,\omega}, \kappa)$ false, and adding a single Cohen subset does not make GCH false at κ.

If $\kappa \notin J$, the trivial forcing does not make GCH false at κ. Also $H(\kappa^+) \subseteq \mathrm{L}(X)$ for $X \subseteq \lambda < \kappa$ which codes all the previously added generic subsets, so from Theorem 5.2.3 it follows that $A(L^2_{\kappa,\omega}, \kappa)$ holds.

We still need to show that GCH is preserved at limit cardinals.

1. Assume λ is a singular limit cardinal. From the Induction Hypothesis we know that GCH holds below λ. Because our ground model was L and the failure of the $SCH(\lambda)$ implies $0^\sharp$ exists, after our forcing it can't be that $\neg SCH(\lambda)$. Thus SCH (λ). Now λ is a strong limit cardinal so $2^\lambda = \lambda^{cf(\lambda)} = \lambda^+$ by $SCH(\lambda)$.

2. Let κ be an inaccessible cardinal. All subsets of κ in V^G are constructible from a single set of cardinality κ which codes all the generic sets added below κ. Thus the powerset of κ has cardinality κ^+.

□

5.3.3. Remark. *If we allow J to be a proper class in the assumption of Theorem 5.3.2, the theorem seems still to be valid. Then we just need to use a proper class of forcing conditions and the length of the iteration is a proper class.*

Ajtai's original proof (see Theorem 2.3.1) did not only show the independence of $A(L^2, \omega)$, but it showed the independence of whether n:th order equivalence implies isomorphism for countable models for arbitrary $n \geq 2$. This is also true for the generalization of Ajtai's result to arbitrary regular cardinals, Theorem 5.2.6, which we presented earlier in this chapter. When we use iterated forcing and add Cohen subsets first to smaller cardinals and then to bigger cardinals, adding Cohen subsets to bigger cardinals does not change (infinitary) second order equivalence of models at smaller cardinals. However, it might change (infinitary) higher order equivalence of models for some stronger higher order logics. The following question is an example about the problem:

5.3.4. Question. *Let P be an iterated forcing which adds first a Cohen subset to $\aleph_0$ and then a Cohen subset to $\aleph_1$. Let M_0^G and M_0^{-G} be the usual models constructed from the generic set and its complement in cardinality $\aleph_0$. Are the models M_0^G and M_0^{-G} third order equivalent after the forcing?*

5.4 Singular cardinals

In this chapter we have already given a generalization of Ajtai's result to regular cardinals. Next we will turn our attention to the case of singular cardinals. For the case of regular cardinals the languages $L^2_{\kappa,\omega}$ had an important role. For the singular cardinals κ we introduce a language which has the same role as the languages $L^2_{\kappa,\omega}$ had for regular cardinals κ:

5.4.1. Definition. *Let κ be a singular cardinal. We define $L^2_\kappa = \bigcup_{\lambda<\kappa} L^2_{\lambda^+,\omega}$.*

Note that the set of L^2_κ-formulas is closed under finitary first order connectives and quantifiers, but not under conjunctions or disjunctions of length $cf(\kappa)$.

Two important facts about the languages L^2_κ are the following:

1. Every ordinal $\alpha < \kappa$ is characterizable in L^2_κ.

2. Every formula of L^2_κ can be expressed as a formula of the language of set theory using a subset of some $\lambda < \kappa$ as a parameter.

As the formulas of L^2_κ are the formulas of $L^2_{\lambda,\omega}$ for regular cardinals $\lambda < \kappa$, the above facts follow from Lemma 1.2.12 and Lemma 1.2.11.

5.4.2. THEOREM. *If $V = \mathrm{L}$ then $A(L^2_\kappa, \kappa)$ holds for any singular cardinal κ.*

Proof. We showed before in Theorem 5.2.2 that if $V = \mathrm{L}$ then all $L^2_{\kappa,\omega}$-equivalent models of cardinality κ are isomorphic for any regular cardinal κ. Because all ordinals less than κ are characterizable in L^2_κ, the proof we used there works without any changes for L^2_κ.

□

5.4.3. THEOREM. *Let $\kappa = \aleph_\alpha$ be a singular cardinal. There is a forcing extension of* L *in which $A(L^2_\kappa, \kappa)$ fails and all cardinals are preserved.*

Proof. Let L be the ground model. As in Theorem 5.3.2, we use the full support iterated Cohen forcing. This time we add generic subsets to all regular cardinals smaller than κ.

Recall that for each regular $\aleph_\beta < \kappa$ our forcing creates two models M^G_β and M^{-G}_β of cardinality $\aleph_\beta$ which are $L^2_{\aleph_\beta,\omega}$-equivalent and non-isomorphic. We define the models M^G_κ and M^{-G}_κ as follows:

M^G_κ contains the α-sequences which satisfy the following conditions:

1. If $\beta < \alpha$ and $\aleph_\beta$ is regular, the βth coordinate is either M^G_β or M^{-G}_β,

2. If $\beta < \alpha$ and $\aleph_\beta$ is singular, the βth coordinate is $\emptyset$,

3. The set of indexes β where the βth coordinate is M^{-G}_β is not cofinal in α.

Similarly we define M^{-G}_κ to contain those α-sequences which satisfy the following conditions:

1. If $\beta < \alpha$ and $\aleph_\beta$ is regular, the βth coordinate is either M^G_β or M^{-G}_β,

2. If $\beta < \alpha$ and $\aleph_\beta$ is singular, the βth coordinate is $\emptyset$,

3. The set of indexes β where the βth coordinate is M^G_β is not cofinal in α.

Clearly the models are non-isomorphic as there is no sequence in M_κ^{-G} which could be mapped to the sequence in M_κ^G which contains only the models M_β^G.

We will now prove that the models are L_κ^2-equivalent. Assume not. Then there is a forcing condition p such that $p \Vdash \dot\phi \in L_\kappa^2 \wedge \dot\phi(\dot{M_\kappa^G}) \wedge \neg\dot\phi(\dot{M_\kappa^{-G}})$ for some forcing name $\dot\phi$. Thus there is some generic filter G such that $p \in G$ and $V^G \models \phi(M^G) \wedge \neg\phi(M^{-G})$. The sentence ϕ is a sentence in the language of set theory with a subset of some $\aleph_{\gamma^+} < \kappa$ as a parameter.

We will now construct another generic filter G' which contains p such that $\dot\phi^{V^G} = \dot\phi^{V^{G'}}$. The elements of G' are made from elements of G by the following modification:

1. Up to stage γ^+ (where the formula ϕ appears) no modification is done.
2. In the the domain of p no modification is done.
3. Above stage γ^+ outside the domain of p the forcing condition is chanced to its mirror image, i.e., the domain remains the same but zeros and ones chance places.

Clearly $p \in G'$. Also up to stage γ^+ the generic sets G' and G agree about everything, so $\dot\phi^{V^G} = \dot\phi^{V^{G'}}$. After stage γ^+ the generic set G' adds essentially complements of those sets which G adds to all regular cardinals. There is a difference only in the domain of p which is always of a smaller cardinality. In particular $M_\beta^G = M_\beta^{-G'}$ and $M_\beta^{-G} = M_\beta^{G'}$ for all $\gamma^+ < \beta < \alpha$. Also $V^G = V^{G'}$. Now $\dot{M^G}^{V^{G'}} = M^{-G}$ and $\dot{M^{-G}}^{V^{G'}} = M^G$, i.e., the models chance places in the generic extensions. However, the formula ϕ is the same and $V^G = V^{G'}$ so ϕ can not be true in one model and false in the other.

$\square$

We will next present a model of ZFC in which the infinitary second order languages cannot characterize all models in any cardinality.

5.4.4. COROLLARY. *Assuming the consistency of an inaccessible cardinal, there is a model of ZFC in which $A(L_\kappa^2, \kappa)$ fails for all singular cardinals κ and $A(L_{\kappa,\omega}^2, \kappa)$ fails for all regular cardinals κ.*

Proof. We start from a model of ZFC which satisfies $V = \mathrm{L}$ and there is an inaccessible cardinal. Let λ be the least inaccessible cardinal in that model. We proceed from down to up and add by iterated Cohen forcing generic subsets to all regular cardinals smaller than λ. At limit stages we take full support. After the forcing $A(L_\kappa^2, \kappa)$ fails for all singular cardinals $\kappa < \lambda$ and $A(L_{\kappa,\omega}^2, \kappa)$ fails for all regular cardinals $\kappa < \lambda$ and λ remains inaccessible. Thus $V_\kappa^{(V^G)}$ satisfies ZFC and $A(L_\kappa^2, \kappa)$ fails for all singular cardinals κ and $A(L_{\kappa,\omega}^2, \kappa)$ fails for all regular cardinals κ.

□

5.4.5. QUESTION. Is it consistent with ZFC that there is a singular cardinal κ such that $A(L^2_{\kappa,\kappa})$ fails but $A(L^2_{\lambda,\omega},\lambda)$ does not fail in cofinally many regular cardinals λ below κ?

5.5 $A(L^2_{\kappa,\omega},\kappa)$ at a measurable cardinal

In this section we prove two theorems of the form "If $A(L^2_{\lambda,\omega},\lambda)$ holds for all regular cardinals λ below a suitable large cardinal κ then $A(L^2_{\kappa,\omega},\kappa)$ holds. The large cardinals we are dealing with are the measurable cardinals and the Σ^m_n indescribable cardinals, which we will define next.

5.5.1. DEFINITION. *A cardinal κ is Σ^m_n indescribable if for all $U\subseteq V_\kappa$ and for all Σ^m_n sentences ϕ if $(V_\kappa,\epsilon,U)\models\phi$ then there is an $\alpha<\kappa$ such that $(V_\alpha,\epsilon,U\cap V_\alpha)\models\phi$.*

5.5.2. THEOREM. *If $A(L^2_{\lambda,\omega},\lambda)$ holds for every regular cardinal λ below a Σ^2_1 indescribable cardinal κ then $A(L^2_{\kappa,\omega},\kappa)$ holds..*

Proof. Assume towards contradiction that $A(L^2_{\kappa,\omega},\kappa)$ fails. As we will see in Lemma 6.1.10 below, the failure of $A(L^2_{\kappa,\omega},\kappa)$ is Σ^2_1 in models of cardinality κ. Then by Σ^2_1 indescribability there is an $\alpha<\kappa$ such that $(V_\alpha,\epsilon)\models\phi$, where ϕ expresses the negation of $A(L^2_{\kappa,\omega},\kappa)$ at the cardinality of the model in question. But then $A(L^2_{\kappa,\omega},\kappa)$ fails at the cardinality of V_α, so κ is not the first cardinal where $A(L^2_{\kappa,\omega},\kappa)$ fails, contradiction. In fact we need here only an apparently weaker version of Σ^2_1 indescribability: we don't need to use any subset of V_κ as a parameter.

□

5.5.3. THEOREM. *If $A(L^2_{\lambda,\omega},\lambda)$ holds for every regular cardinal λ below a measurable cardinal κ then $A(L^2_{\kappa,\omega},\kappa)$ holds.*

Proof. Assume that is not the case. Then $A(L^2_{\lambda,\omega},\lambda)$ holds for every regular λ below a measurable cardinal κ, but there are two models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality κ which are $L^2_{\kappa,\omega}$-equivalent but not isomorphic. Let j be an elementary embedding from V into a transitive class M with critical point κ. Since j is an elementary embedding, $j(\kappa)$ is the least cardinal κ' such that $M\models\neg A(L^2_{\kappa',\omega},\kappa')$. We will show that $A(L^2_{\kappa,\omega},\kappa)$ fails in M, which will be a contradiction. We assume that $\mathfrak{A}$ and $\mathfrak{B}$ are subsets of κ^n. Then $j(\mathfrak{A})$ and $j(\mathfrak{B})$ are subsets of $j(\kappa)^n$. $\mathfrak{A}=j(\mathfrak{A})\cap\kappa^n$ and $\mathfrak{B}=j(\mathfrak{B})\cap\kappa^n$, thus the models $\mathfrak{A}$ and $\mathfrak{B}$ belong to M. Similarly any subset of κ in V belongs to M. Thus the models $\mathfrak{A}$ and $\mathfrak{B}$ are $L^2_{\kappa,\omega}$-equivalent but not isomorphic in M.

□

Chapter 6

$A(L^2, \omega)$ and large cardinal axioms

6.1 Large cardinals

In this chapter we will discuss how some large cardinal axioms are related to $A(L^2, \omega)$. First we will discuss consistency of some large cardinal axioms with second order definable well-orders of the reals. Then we will show that if there are enough large cardinals then $A(L^2, \omega)$ is false. We will also show that the sentence "There are two $L^2_{\kappa,\omega}$-equivalent non-isomorphic models of cardinality κ" is Σ^2_1, i.e., third order Σ_1 in models of cardinality κ. In the end we will discuss third order definable well-orders of the reals and forcing axioms.

From the proof of Theorem 2.1.1 and some well-known facts about the consistency of well-orders of the reals with large cardinals we get the following results:

6.1.1. Theorem (Ajtai [2], Silver [24], Martin and Steel [17]). *It is consistent that there is a measurable cardinal and $A(\Sigma^1_4, \omega)$ holds. It is consistent that there are n Woodin cardinals and $A(\Sigma^1_{n+3}, \omega)$ holds. The above results are relative to consistency of the relevant large cardinal axioms.*

Proof. The existence of a measurable cardinal with a Δ^1_3 well-order of the reals is consistent [24], so by Theorem 2.1.1 it is consistent that there is a measurable cardinal and $A(\Sigma^1_4, \omega)$ holds. Also for each natural number n it is consistent to have n Woodin cardinals and a Σ^1_{n+2} well-order of the reals [17]. From Theorem 2.1.1 it follows that it is consistent that there are n Woodin cardinals and $A(\Sigma^1_{n+3}, \omega)$ holds.

□

We will next prove several lemmas which are needed to prove Theorem 6.1.6: "If there are enough large cardinals then $A(L^2, \omega)$ fails."

6.1.2. Lemma. *It is possible to code all finite vocabularies as natural numbers by some Gödel numbering.*

Proof. Divide the set of prime numbers to infinitely many infinite parts P_n in some second order characterizable way. Then take a countably infinite set of constants and a countably infinite set of relation and function symbols of each arity, and assign in some second order characterizable way a different prime number code for any symbol. Now a finite vocabulary can be coded as the number which we get if we multiply all codes of the symbols in the vocabulary with each other.

$\square$

6.1.3. LEMMA. *Given a finite vocabulary σ, the set of L^2-terms in vocabulary σ is second order definable on $(\omega,<)$. Also the set of free variables in a L^2-term is second order definable on $(\omega,<)$. Given an infinite model $\mathfrak{A}$ in vocabulary σ, a L^2-term t and an assignment of L^2-variables s which contains the free variables of t in its domain, the interpretation of term $t_s^{\mathfrak{A}}$ is second order characterizable.*

Proof. We define the rank for $L^2(\sigma)$-terms as follows:

1. Constants and variables have rank 0.
2. If rank of terms $t_1,\dots,t_n$ have been defined and F is an n-ary function symbol in σ or n-ary second order function variable then rank $F(t_1,\dots t_n)$ is $\sup\{\text{rank } (t_i)+1\colon 1\leq i\leq n\}$.

A set t is an $L^2(\sigma)$-term iff the following hold:

a) There is a set X such that $t\in X$ and every set in X is either an $L^2(\sigma)$-term of rank 0 or is a result of applying a function in σ or a second order function variable to sets in X.

The condition a) can be formalized in second order logic.

For an $L^2(\sigma)$-term t define X' to be the smallest set which satisfies the condition a) above. X' is the set of subterms of t and it is characterizable in second order logic. Once we have X' characterized, we can characterize the rank for terms in X' and by induction on rank characterize the free variables of all subterms and interpretations of subterms with a given assignment.

$\square$

6.1.4. LEMMA. *$A(L^2,\omega)$ is true in V if and only if it is true in* L($\mathbb{R}$).

Proof. We define inductively the rank for $L^2_{\kappa,\omega}$ formulas ϕ as follows:

1. rank(ϕ)=0 for atomic ϕ.
2. rank($\bigwedge\Psi$)= rank($\bigvee\Psi$) $=\sup\{\text{ rank }(\phi)+1 : \phi\in\Psi\}$.
3. If $\phi=\neg\psi, \phi=\exists x_n\psi, \phi=\forall x_n\psi, \phi=\exists X_i^n\psi, \phi=\forall X_i^n\psi, \phi=\exists F_i^n\psi$ or $\phi=\forall F_i^n\psi$ then rank(ϕ)=rank(ψ) +1.

In case of second order formulas, conjunctions and disjunctions are of length 2 and rank is always finite. More generally $L^2_{\kappa,\omega}$ formulas have rank less than κ.

Given a finite vocabulary τ, the second order formulas in vocabulary τ are inductively characterizable in a similar way as terms in Lemma 6.1.3. Also for any $L^2(\tau)$-formula the set of its subformulas is characterizable. In this set we can define the rank for all subformulas, and by induction on the rank the set of free variables in a given subformula. An interpretation for finitely many first order and second order variables in a countable model can be coded into a real number. Consequently every interpretation which exists in V exists in $\mathrm{L}(\mathbb{R})$. The truth predicate for a countable model $\mathfrak{A}$, i.e., the set of ordered tuples $\langle \phi, s \rangle$ such that $\mathfrak{A} \models_s \phi$ is inductively characterizable. This means that the truth predicate for formulas of rank 0 is characterizable, and if the truth predicate for formulas of rank $< n$ is characterizable, then it is characterizable also for formulas of rank n. Finally the truth predicate is characterizable as the union of these "partial truth predicates". The truth predicate is definable in $\mathrm{L}(\mathbb{R})$ because it is an inductive definition and its existence is provable from ZF. Axiom of Choice may be false in $\mathrm{L}(\mathbb{R})$ but it is not needed. Also the truth predicate of V equals truth predicate of $\mathrm{L}(\mathbb{R})$ because they are determined by the reals and V and $\mathrm{L}(\mathbb{R})$ have the same reals.

Let us now look at the sentence which says that $A(L^2, \omega)$ fails:

The sentence starts with $\exists\tau\exists\mathfrak{A}\exists\mathfrak{B}\exists\Pi_{\mathfrak{A}}\exists\Pi_{\mathfrak{B}}$ and then the conjunction of the following:

- $\mathfrak{A}$ and $\mathfrak{B}$ have vocabulary τ.
- $|\mathfrak{A}| = |\mathfrak{B}| = \omega$.
- $\neg\mathfrak{A} \cong \mathfrak{B}$.
- $\Pi_{\mathfrak{A}}$ is a truth predicate of second order formulas for $\mathfrak{A}$ and $\Pi_{\mathfrak{B}}$ is a truth predicate of second order formulas for $\mathfrak{B}$.
- $\Pi_{\mathfrak{A}}$ and $\Pi_{\mathfrak{B}}$ contain exactly the same sentences.

If the sentence is true in one of V and $\mathrm{L}(\mathbb{R})$ then all the sets witnessing the truth of the sentence exist also in the other. Thus the sentence is also true in the other and the claim follows.

□

The proof of the next theorem of Woodin can be found in Woodin's book [31].

6.1.5. THEOREM. *If δ is a limit of Woodin cardinals and there exists a measurable cardinal above δ, then no forcing construction in V_δ can change the theory of $\mathrm{L}(\mathbb{R})$.*

6.1.6. THEOREM. *If there is a measurable cardinal above a limit of Woodin cardinals then $A(L^2,\omega)$ fails.*

Proof. Assume there is a measurable cardinal above a limit of Woodin cardinals. We add a Cohen generic real G to V as in Theorem 2.3.1. Now $A(L^2,\omega)$ is false in $V[G]$. By Lemma 6.1.4 $A(L^2,\omega)$ is false in $\mathrm{L}(\mathbb{R})^{V[G]}$. By assumption and Theorem 6.1.5 $A(L^2,\omega)$ is false in $\mathrm{L}(\mathbb{R})^V$ and by Lemma 6.1.4 $A(L^2,\omega)$ is false in V.

□

We note that the proof of Lemma 6.1.3 works also for $L^2_{\kappa,\omega}(\sigma)$-terms in cardinality κ.

6.1.7. LEMMA. *Given a finite vocabulary σ the relation "X is an $L^2_{\kappa,\omega}(\sigma)$-sentence" is second order characterizable in a model of cardinality κ.*

Proof. The second order sentence says that there is a set $Y = Y_1 \cup Y_2$ containing X such that every set in Y_1 is either $L^2_{\kappa,\omega}(\sigma)$-term of rank 0 or is a result of applying functions in σ or second order function variables to elements in Y_1. Also every element in Y_2 is either $L^2_{\kappa,\omega}$ atomic formula or is formed from other sets in Y by operations described in Definition 1.2.13 and Y is the smallest set satisfying this definition. By this definition Y is the set of subformulas and subterms of X. The sentence says further that there is a function F which maps all the elements of Y to the set of their free variables and F maps X to $\emptyset$.

□

6.1.8. DEFINITION. *Let $\mathfrak{A}$ be a model and τ be a finite vocabulary. The truth predicate T for the logic $L^2_{\kappa,\omega}(\tau)$ in the model $\mathfrak{A}$ is a binary relation. As elements it has ordered pairs of $L^2_{\kappa,\omega}(\tau)$-formulas and interpretations of less than κ many variables of $L_{\kappa,\omega}(\tau)$ in the model $\mathfrak{A}$ satisfying the following conditions:*

1. *If t_i and t_j are $L^2_{\kappa,\omega}(\tau)$ terms and variables of t_i and t_j belong to the domain of an interpretation s, then $\langle t_i = t_j, s\rangle \in T$ if and only if $(t_i)^{\mathfrak{A}}_s = (t_j)^{\mathfrak{A}}_s$.*
2. *If R is an n-ary relation symbol in τ and $t_1,\dots t_n$ are $L^2_{\kappa,\omega}(\tau)$-terms such that their variables belong to the domain of s, then $\langle R(t_1,\dots t_n), s\rangle \in T$ if and only if $\langle (t_1)^{\mathfrak{A}}_s,\dots (t_n)^{\mathfrak{A}}_s\rangle \in R^{\mathfrak{A}}$.*
3. *If X is an n-ary relation variable and $t_1,\dots t_n$ are $L^2_{\kappa,\omega}$-terms such that their variables belong to the domain of s, then $\langle X(t_1,\dots t_n), s\rangle \in T$ if and only if $\langle (t_1)^{\mathfrak{A}}_s,\dots (t_n)^{\mathfrak{A}}_s\rangle \in X^{\mathfrak{A}}_s$.*
4. *$\langle \neg\phi, s\rangle \in T$ if and only if $\langle \phi, s\rangle \notin T$.*

5. *If Ψ is a set of $L^2_{\kappa,\omega}(\tau)$-formulas and for all $\phi \in \Psi$ it is defined whether $\langle \phi, s \rangle \in T$ or not, then $\langle \bigwedge \Psi, s \rangle \in T$ if and only if $\langle \phi, s \rangle \in T$ for all $\phi \in \Psi$.*

6. *If Ψ is a set of $L^2_{\kappa,\omega}(\tau)$-formulas and for all $\phi \in \Psi$ it is defined whether $\langle \phi, s \rangle \in T$ or not, then $\langle \bigvee \Psi, s \rangle \in T$ if and only if $\langle \phi, s \rangle \in T$ for some $\phi \in \Psi$.*

7. *$\langle \exists x_\alpha \phi, s \rangle \in T$ if and only if $\langle \phi, s' \rangle \in T$ for some interpretation s' such that s and s' are the same except possibly in x_α.*

8. *$\langle \forall x_\alpha \phi, s \rangle \in T$ if and only if $\langle \phi, s' \rangle \in T$ for all interpretations s' such that s and s' are the same except possibly in x_α.*

9. *$\langle \exists X_\alpha \phi, s \rangle \in T$ if and only if $\langle \phi, s' \rangle \in T$ for some interpretation s' such that s and s' are the same except possibly in X_α.*

10. *$\langle \forall X_\alpha \phi, s \rangle \in T$ if and only if $\langle \phi, s' \rangle \in T$ for all interpretations s' such that s and s' are the same except possibly in X_α.*

11. *$\langle \exists F_\alpha \phi, s \rangle \in T$ if and only if $\langle \phi, s' \rangle \in T$ for some interpretation s' such that s and s' are the same except possibly in F_α.*

12. *$\langle \forall F_\alpha \phi, s \rangle \in T$ if and only if $\langle \phi, s' \rangle \in T$ for all interpretations s' such that s and s' are the same except possibly in F_α.*

6.1.9. LEMMA. *If Π is a set of ordered pairs of $L^2_{\kappa,\omega}(\tau)$-sentences and assignments for less than κ variables in a model $\mathfrak{A}$ then there is a second order sentence with a third order parameter Π which is true if and only if Π is the truth predicate of $\mathfrak{A}$.*

Proof. This is just formalizing Definition 6.1.8 in second order logic. This is possible because given a model $\mathfrak{A}$ of cardinality κ in a vocabulary τ, the set of $L^2_{\kappa,\omega}$-terms, formulas, free variables in formulas, assignments for less than κ variables and interpretations of terms with given assignments including the free variables of the term are second order characterizable. From these it follows that the case of atomic formulas is definable in second order logic. The other cases are definable as well, because we need to quantify only over sets of cardinality κ in the truth definition. Note that we cannot quantify over the truth predicate in second order logic because it is too big and we need a third order quantifier for that. But given a model and a predicate, checking whether the predicate is the truth predicate for the model is possible in second order logic.

□

6.1.10. LEMMA. *"There are two $L^2_{\kappa,\omega}$-equivalent non-isomorphic models of cardinality κ" is a Σ^2_1 property in cardinality κ.*

Proof. In the following Σ^2_1-sentence $\phi_\omega(Y, Z)$ says that (Y, Z) has ordertype of $(\omega, <)$, $\phi_{\text{voc}}(x, X, Y, Z)$ says that X has vocabulary x, where x is a natural number with respect to (Y, Z), $\phi_\kappa(X)$ says that X has cardinality κ(where κ is the cardinality of the model in question). The formula $\phi_{\text{truth}}(\Pi, X, x, Y, Z)$ says that Π is a truth definition of $L^2_{\kappa,\omega}(x)$ in X. The formula $\phi_{\text{sentence}}(A, x, Y, Z)$ says that A is a $L^2_{\kappa,\omega}$ sentence in vocabulary x. The formula $\phi_{\neg\cong}$ says that two models are not isomorphic to each other.

The sentence starts with $\exists Y \exists Z \exists \sigma \exists \mathfrak{A} \exists \mathfrak{B} \exists \Pi_1 \exists \Pi_2$ and then the conjunction of the following:

- $\phi_\omega(Y, Z)$
- $\phi_{\text{voc}}(\sigma, \mathfrak{A}, Y, Z) \wedge \phi_{\text{voc}}(\sigma, \mathfrak{B}, Y, Z)$
- $\phi_\kappa(\mathfrak{A}) \wedge \phi_\kappa(\mathfrak{B})$
- $\phi_{\text{truth}}(\Pi_1, \mathfrak{A}, \sigma, Y, Z) \wedge \phi_{\text{truth}}(\Pi_2, \mathfrak{B}, \sigma, Y, Z)$
- $\forall A(\phi_{\text{sentence}}(A, \sigma, Y, Z) \rightarrow (\Pi_1(A, \emptyset) \leftrightarrow \Pi_2(A, \emptyset)))$
- $\phi_{\neg\cong}(\mathfrak{A}, \mathfrak{B})$

□

Some large cardinal axioms imply that there is no second order definable well-order of the reals. In particular this holds for large cardinal axioms that imply the Projective Determinacy, as we will show in the next section. These axioms possibly imply that $A(L^2, \omega)$ fails. If that is the case, we can ask the question: does $A(L^3, \omega)$ hold? By the following theorem most large cardinal axioms are consistent with $A(L^3, \omega)$ (relative to the consistency of the large cardinal axiom in question).

6.1.11. THEOREM. *$A(L^3, \omega)$ is consistent with practically all known consistent large cardinal axioms.*

Proof. Let the ground model be a model of ZFC which satisfies your favorite large cardinal axiom. By a result of Abraham and Shelah [1] it is possible to force a third order definable well-order of the reals with a small forcing[1]. In the generic extension $A(L^3, \omega)$ holds because of the same reasoning as in Theorem 2.1.1. If the large cardinal axiom was preserved in the forcing, then the generic extension satisfies the large cardinal axiom and $A(L^3, \omega)$.

□

[1]If κ is a large cardinal we say that a notion of forcing P is small (relative to κ) if $|P| < \kappa$. Practically all large cardinals are preserved in small forcings [11] (Theorem 21.2).

6.2 Forcing axioms

As we already noted in Chapter 2, it is an open question whether Martin's axiom is consistent with $A(L^2, \omega)$. Unlike the consistency of the Proper Forcing Axiom and Martin's Maximum, the consistency of Martin's axiom $+2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ can be proved from the consistency of ZFC.

6.2.1. Lemma (Vitali [30], Mycielski and Steinhaus [20]). *If there is a second order definable well-order of the reals, then there is a second order definable non-measurable set of reals. If Projective Determinacy holds, then all second order definable sets of reals are Lebesgue measurable. Consequently if Projective Determinacy holds, there is no second order definable well-order of the reals.*

Proof. Recall the construction of a non-measurable set of reals by Vitali. We define an equivalence relation in the interval $[0, 1]$: $x \sim y \Leftrightarrow \quad x - y$ is a rational number. By the Axiom of Choice there is a set which contains exactly one member from each equivalence class. Such a set turns out, as is well-known, to be non-measurable. If $<$ is a second order definable well-order of the reals then there is a second order definable non-measurable set of reals. We can define this set to contain the $<$-least element from each equivalence class.

By a result of Mycielski and Steinhaus [20], every second order definable set of reals is measurable assuming Projective Determinacy.

□

Next we will note that if the Proper Forcing Axiom holds, then there is no second order definable well order of the reals. Consequently one cannot use Ajtai's proof to show the consistency of $A(L^2, \omega)$ with the Proper Forcing Axiom. If $A(L^2, \omega)$ is consistent with the Proper Forcing Axiom, then $A(L^2, \omega)$ can hold without a second order definable well-order of the reals.

6.2.2. Theorem (Steel). *The Proper Forcing Axiom implies that there is no second order definable well-order of the reals.*

Proof. The Proper Forcing Axiom implies that Axiom of Determinacy holds in $L(\mathbb{R})$, which in turn implies Projective Determinacy [27].

□

6.2.3. Question. *Is the Proper Forcing Axiom consistent with $A(L^2, \omega)$?*

6.2.4. Theorem. *Assuming the consistency of the relevant large cardinal axioms it is consistent that Martin's Maximum holds with $A(L^3, \omega)$.*

Proof. By Paul Larson's result [13] Martin's Maximum is consistent with the existence of a well-order of the reals definable in $H(\aleph_2)$ without parameters.

By Lemma 1.2.9 we can quantify over elements of $H(\aleph_2)$ in third order logic thus Martin's Maximum is consistent with a third order definable well-order of the reals. Consequently it is consistent that Martin's Maximum holds and $A(L^3, \omega)$ holds.

$\square$

Chapter 7

Summary and future work

7.1 Summary

If κ is an infinite cardinal we can ask the question what is the least logic L such that every L-theory is κ categorical. If κ is a regular cardinal, adding a Cohen subset for κ makes sure that no such small definable logic L exists. If κ is a singular cardinal, adding Cohen subsets for cofinally many $\lambda < \kappa$ by an iterated forcing, taking full support at all limits, does essentially the same. However, there is always a small logic L with generalized quantifiers such that all L-theories are κ-categorical but L may be not definable in the language of set theory.

In the countable cardinality the "small" logic can be second order logic. If V=L even Σ^1_3 is enough. With n Woodin cardinals Σ^1_{n+3} can be enough. But if there are infinitely many Woodin cardinals and a measurable cardinal above them then $A(L^2, \omega)$ fails. However $A(L^3, \omega)$ is consistent with practically all large cardinal axioms. $A(L^3, \omega)$ is also consistent with Martin's Maximum.

In an uncountable cardinality the small logic can be $L^2_{\kappa,\omega}$ or L^n where $n \geq 4$. Whether $A(L^2_{\kappa,\omega}, \kappa)$ holds for different cardinals κ is very much independent of each other.

The following table contains information about whether $A(L, \kappa)$ holds for certain language L and cardinal κ. In the intersection of an L-row and a κ-column we have described in the left-hand-side a model of ZFC where $A(L, \kappa)$ holds and on the right-hand-side a model of ZFC where $A(L, \kappa)$ fails (if they exist). The question mark means an open question. Cohen, iter., and $P_{X,\kappa}$ refer to suitable Cohen forcing, iterated Cohen forcing with full support in all limit stages and the forcing $P_{X,\kappa}$ defined in Chapter 3, respectively. Regular column refers to arbitrary uncountable regular cardinals and singular column refers to arbitrary uncountable singular cardinals. The ground model is L in all the forcings.

$A(L,\kappa)$	$\aleph_0$	*regular*	*singular*
FO	$-/always$	$-/always$	$-/always$
$L_{\kappa^+,\omega}$	$always/-$	$\kappa=\aleph_0/\kappa\neq\aleph_0$	$-/always$
L^2	$V=L/Cohen$	$?/V=L$	$?/V=L$
$L^2_{\kappa,\omega}/L^2_\kappa$	$V=L/Cohen$	$V=L/Cohen$	$V=L/iter.$
L^3	$V=L/Cohen$	$?/V=L$	$?/V=L$
L^4	$V=L/Cohen$	$P_{X,\kappa}/V=L$	$P_{X,\kappa}/V=L$
L^n	$V=L/Cohen$	$P_{X,\kappa}/V=L$	$P_{X,\kappa}/V=L$
ZF	$V=L/Cohen$	$P_{X,\kappa}/V=L$	$P_{X,\kappa}/V=L$

7.2 Future work

In this section we list the most important open questions and possible directions of future research.

Recall Question 2.1.5:

7.2.1. Question. *Is it consistent that $A(L^2,\omega)$ holds, but there is no second order definable well-order of the reals?*

If that is not consistent, then $A(L^2,\omega)$ is equivalent to the existence of a second order definable well-order of the reals. We have an idea how it might be possible to prove that these conditions are not equivalent.

Suppose there is a model of ZFC with the following properties (We do not know yet if such a model exists) :

1. There is no second order definable well-order of the reals.

2. There are second order definable sets $X_i \subset \mathbb{R} : i \in \omega$ such that each X_i has a second order definable well-order and $\mathbb{R} = \bigcup_{i\in\omega} X_i$.

Suppose now $\mathfrak{A}$ and $\mathfrak{B}$ are two second order equivalent countable models. Now $\mathfrak{A}$ is isomorphic to some real a and $\mathfrak{B}$ is isomorphic to some real b. Assume i and j are such indexes that $a \in X_i$ and $b \in X_j$. Let $X = X_i \cup X_j$. Now X is second order definable and there is a second order definable well-order of X. We assumed $\mathfrak{A}$ and $\mathfrak{B}$ are second order equivalent, so for all $n \in \omega$ the natural number n belongs to the the least real in X isomorphic to $\mathfrak{A}$ if and only if n belongs to the the least real in X isomorphic to $\mathfrak{B}$. Now $\mathfrak{A}$ and $\mathfrak{B}$ have the same isomorphism type and they are isomorphic.

We have another idea, suggested by Saharon Shelah, how it might be possible to have $A(L^2,\omega)$ without second order definable well-order of the reals. Assume there is a second order definable set of reals which contains exactly one real of each isomorphism type. Then we can use the idea of Ajtai's proof to show that $A(L^2,\omega)$ holds. The problem is to find a model of ZFC in which there is a second order definable set of reals which contains exactly one real of each isomorphism type but there is no second order definable well-order of the reals. We are working on finding such a model, using a construction suggested by Shelah.

Adding a Cohen subset to a regular cardinal produces two ZF-equivalent non-isomorphic models of cardinality κ. When we do iterated Cohen forcings we have not been able to prove that the models remain ZF-equivalent. The following question is an example of that: Let L be the ground model and $P = P_0 * P_1$ be an iterated forcing which adds first a Cohen subset to ω and then a Cohen subset to $\aleph_1$. Let G be a P-generic set over L and G_0 the P_0-generic set over L determined by G and M^{G_0} and M^{-G_0} the "Ajtai models" constructed from G_0 and $-G_0$ (see Theorem 2.3.1). Are M^{G_0} and M^{-G_0} third order equivalent in L$[G]$?

7.2.2. QUESTION. *Is it consistent with ZFC that $A(L^2, \kappa)$ holds for an uncountable cardinal κ? If not, is it consistent that $A(L^3, \kappa)$ holds for an uncountable cardinal κ?*

7.2.3. QUESTION. *Is it consistent with ZFC that Martin's axiom + $2^{\aleph_0} = \aleph_2$ holds with $A(L^2, \omega)$.*

Possible directions for future research:

1. Our results are often related to models which resemble L a lot (Theorem 5.2.3 is used in many results). An interesting question is whether our results could be generalized to inner models of some large cardinals.

2. The question about whether every L-theory is κ-categorical in a model class C. We have here only discussed briefly the Fraïssé Hypothesis, i.e., the above question in case $L = L^2$, $\kappa = \omega$ and C is the class of ordinals.

3. Ehrenfeucht-Mostowski models. Adding a Cohen real introduces two countable non-isomorphic ZF-equivalent linear orders. Suitable cardinal collapse makes the Fraïssé Hypothesis fail. For which theories T we can construct non-isomorphic ZF-equivalent Ehrenfeucht-Mostowski models over these linear orders (or ordinals)? Is this possible for all unstable theories? Hyttinen, Kangas and Väänänen are working on this question.

4. Bigger vocabularies. Here is an example of an open question: Is it consistent with ZFC that in any countable vocabulary second order equivalence implies isomorphism for countable models? One can ask the same question for any L^n.

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Samenvatting

$A(L,\kappa)$ betekent dat voor alle modellen $\mathfrak{A}$ en $\mathfrak{B}$ van kardinaliteit κ, als $\mathfrak{A}$ en $\mathfrak{B}$ dezelfde L-theorie vervullen, ze isomorphism zijn.

Als κ een oneindig kardinaalgetal is, kunnen we de vraag stellen wat de minste logica L is zodanig dat elke L-theorie κ categorisch is. Als κ een regulier kardinaalgetal is, maakt het toevoegen van een Cohen deelverzameling aan κ zeker dat er geen kleine definieerbare logica L bestaat. Hetzelfde geldt voor het toevoegen van Cohen deelverzamelingen voor een co-eindig aantal $\lambda < \kappa$ door een geitereerde forcing die de volledige ondersteuning bij alle limieten geeft. Er is echter altijd een kleine logica L met gegeneraliseerde kwantoren zodanig dat alle L-theorieën κ-categorisch zijn maar niet definieerbaar hoeven te zijn in de taal van de verzamelingentheorie.

In de telbare kardinaliteit kan de "kleine" logica een tweede order logica zijn. Als V =L dan is zelfs Σ^1_3 voldoende. Met een n aantal Woodin kardinaalgetallen is Σ^1_{n+3} genoeg. Echter, als er oneindig veel Woodin kardinaalgetallen zijn en er is een meetbaar kardinaalgetal boven ze, dan faalt $A(L^2,\omega)$. $A(L^3,\omega)$ is echter consistent met alle grote kardinaal axioma's. $A(L^3,\omega)$ is ook consistent met Martin's Maximum.

In een ontelbare kardinaliteit κ kan de kleine logica $L^2_{\kappa,\omega}$ zijn of L^n waar $n \geq 4$. De vraag of $A(L^2_{\kappa,\omega},\kappa)$ geldt voor verschillende kardinalen κ is hiervan onafhankelijk.

De volgende tabel laat zien of $A(L,\kappa)$ geldt voor een taal L en een kardinaalgetal κ

In de doorsnede van een L-regel; en een κ-kolom hebben we aan de linkerkant een model van ZFC beschreven waar $A(L,\kappa)$ geldt en aan de rechterkant een model van ZFC waar $A(L,\kappa)$ niet geldt (als het bestaat). Met het vraagteken beduigen wij een open vraag. Cohen, iter., en $P_{X,\kappa}$ betekenen Cohen forcing, geïtereerde Cohen forcing met volledige ondersteuning in alle limieten en de forcing $P_{X,\kappa}$, die in hoofdstuk 3 is gedefinieerd. Reguliere en respectievelijk singuliere kolommen verwijden naar arbitraire ontelbare reguliere/singuliere kardinaalgetallen. Het basismodel is L in alle forcings.

$A(L,\kappa)$	$\aleph_0$	*regulier*	*singulier*
FO	$-/altijd$	$-/altijd$	$-/altijd$
$L_{\kappa^+,\omega}$	$altijd/-$	$\kappa=\aleph_0/\kappa\neq\aleph_0$	$-/altijd$
L^2	V=$L/Cohen$	?/V=L	?/V=L
$L^2_{\kappa,\omega}/L^2_\kappa$	V=$L/Cohen$	V=$L/Cohen$	V=$L/iter.$
L^3	V=$L/Cohen$	?/V=L	?/V=L
L^4	V=$L/Cohen$	$P_{X,\kappa}$/V=L	$P_{X,\kappa}$/V=L
L^n	V=$L/Cohen$	$P_{X,\kappa}$/V=L	$P_{X,\kappa}$/V=L
ZF	V=$L/Cohen$	$P_{X,\kappa}$/V=L	$P_{X,\kappa}$/V=L

Abstract

Fix a cardinal κ. We can ask the question what kind of a logic L is needed to characterize all models of cardinality κ (in a finite vocabulary) up to isomorphism by their L-theories. In other words: for which logics L it is true that if any models $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same L-theory then they are isomorphic.

It is always possible to characterize models of cardinality κ by their L_{κ^+,κ^+}-theories, but we are interested in finding a "small" logic L, i.e. the sentences of L are hereditarily smaller than κ. For any cardinal κ it is independent of ZFC whether any such small definable logic L exists. If it exists it can be second order logic for $\kappa = \omega$ and fourth order logic or certain infinitary second order logic $L^2_{\kappa,\omega}$ for uncountable κ. All models of cardinality κ can always be characterized by their theories in a small logic with generalized quantifiers, but the logic may be not definable in the language of set theory.

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