Orthogonality and Quantum Geometry

Towards a Relational Reconstruction of Quantum Theory

Shengyang Zhong

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Institute for Logic, Language and Computation
Universiteit van Amsterdam
Science Park 107
1098 XG Amsterdam
phone: +31-20-525 6051
e-mail: illc@uva.nl
homepage: http://www.illc.uva.nl/

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Towards a Relational Reconstruction of Quantum Theory

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Promotor:	Prof. dr. J.F.A.K. van Benthem
Co-promotor:	Dr. A. Baltag
	Dr. S.J.L. Smets
Overige leden:	Dr. N. Bezhanishvili
	Prof. dr. R.I. Goldblatt
	Prof. dr. J.R. Harding
	Prof. dr. Y. Venema
	Prof. dr. R.M. de Wolf
	Prof. dr. M. Ying

Faculteit der Natuurwetenschappen, Wiskunde en Informatica

Contents

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Amsterdam Shengyang Zhong
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Chapter 1

Introduction

1.1 Introduction: Logics of Quantum Theory

Quantum theory, founded at the beginning of the 20th century, has proved successful in describing microscopic phenomena. However, it is very hard to understand conceptually. This can be seen from the debate among the founding fathers of quantum theory, including Bohr and Einstein ([24]). In 1964, during his Messenger Lectures at Cornell University, Feymann claimed 'I can safely say that nobody understands quantum mechanics' (Line 25, Page 129 in [41]). On the one hand, many people think that this is still true today. (Please refer to, for example, Chapter 1 in [30].) On the other hand, since the 1990s the research in quantum computation and quantum information has shown that quantum physics has important applications to high-speed information processing and secure information transmission. However, the number of useful applications of quantum theory to computer science is still rather small, partly due to the novelty of quantum theory ([82]). In a word, quantum theory is novel; this novelty has great potential for applications on the one hand, but makes useful applications hard to find on the other. As a result, the problem of understanding and making good use of quantum theory remains one of the greatest challenges of our time.

Many physicists and philosophers, including Bohr, Born and van Fraassen, try to solve this problem by finding a proper *interpretation* of quantum theory. To be precise, they attempt to reveal the philosophical 'meaning', i.e. the contents in ontology, epistemology, etc., of the formulas and equations in quantum mechanics. However, instead of one interpretation, many competing interpretations were proposed. Famous ones include the Copenhagen interpretation, the statistical interpretation, the de Broglie-Bohm interpretation, the many-worlds interpretation, the modal interpretation. The book [58] is a classic text on this, while [56] provides a nice overview of some of the main important issues.

In the 1960s some physicists and mathematicians started to solve this problem in another way, that is by examining the mathematical theory of quantum mechanics. Quantum mechanics was built by manipulating some complicated mathematical objects such as complex-valued functions in such a way that the outputs of the calculations fit the data that is obtained from the experiments. In [88] von Neumann proposed a formalism of quantum theory within the mathematical theory of Hilbert spaces (over complex numbers).[1] This formalism has been widely accepted and has become standard from then on. Hilbert spaces are mathematical structures with nice algebraic, geometric and topological properties. They enable physicists to do calculations and make precise predications, which can be compared with data obtained from experiments. However, the complicated mathematical structure of Hilbert spaces obscures the physical picture of quantum theory. With this in mind, some physicists and mathematicians started working in another direction. Their aim was to start from physically transparent notions, to model them in mathematical structures much simpler than Hilbert spaces and to characterize their features in quantum theory by simple and natural axioms. The goal of this second line of research is to 'reconstruct' quantum theory. In particular, the focus lies on trying to explain the use of Hilbert spaces in a rigorous way by proving representation theorems for these simpler mathematical structures via Hilbert spaces. Thus the physical significance of the structure of Hilbert spaces can be understood on the basis of the simple and natural axioms. After years of development, research in this line has become an important field in mathematics and physics, and is called *quantum logic*[2] or *the (mathematical) foundations of quantum theory*. Here the name 'quantum logic' is well justified. On the one hand, focusing on the abstract and structural features of quantum theory, this field is in the spirit of logic. On the other hand, this field is inspired by examining quantum theory from a logical perspective, as is explained in the following subsection.

Compared with the interpretations of quantum theory, quantum logic is less conceptual but more technical. Therefore, the results in quantum logic usually can be precisely compared when they are cast in the form of mathematical theorems. Moreover, because of the shared mathematical language, the results in quantum logic can be directly applied to physics and computer science.

1.1.1 A Brief Survey of Quantum Logic

In this subsection, I provide a brief survey of quantum logic. The aim is to sketch the intellectual background of the work in this thesis. Hence it is impossible to cover here all of the main results known in the area of quantum logic. For more detailed surveys on operational quantum logic, a main approach in quantum logic,

[1] Section 1.2 provides a brief review on the Hilbert space formalism of quantum mechanics.

[2] Here the term 'quantum logic' is used in the general sense. In the special sense, 'quantum logic' refers to the lattices of testable properties of quantum systems, which will be introduced in the coming subsection. This term is used in both senses in this thesis, and the context should make it clear in which sense it is used.

I recommend [26] and [70]. For a comprehensive and accessible book on quantum logic, I recommend [17]. For a thorough study of the mathematics of quantum theory from a geometric and logical perspective, I recommend [86]. The two handbooks [36] and [37] reflect the current state of the art.

Traditional Quantum Logic

Quantum logic starts from Birkhoff and von Neumann's seminal paper [22]. Based on the work in [88], Birkhoff and von Neumann investigated the structure formed by the *experimental propositions* of a quantum system. In their words,

> *The object of the present paper is to discover what logical structure one may hope to find in physical theories which, like quantum mechanics, do not conform to classical logic.* (p.823 of [22])

In some of the later literature and in this thesis the word 'testable properties' is used to refer to experimental propositions. In logic it is well known that the testable properties of a classical system form a Boolean algebra. In this paper a main discovery is that the testable properties of a quantum system form an ortho-lattice[3] which is not a Boolean algebra. In particular, the distributivity between join and meet fails.

In [57] Husimi proved that the ortho-lattices of the testable properties of quantum systems satisfy Weak Modularity, a property weaker than distributivity. Since then, the ortho-lattices satisfying Weak Modularity, also called *orthomodular lattices*, has been studied extensively in quantum logic.

Operational Quantum Logic

Inspired by [22], in [64] and [65] Mackey tried to reconstruct quantum theory from the ortho-lattices of testable properties of quantum systems. He defined a state of a quantum system to be a probability measure on the ortho-lattice of testable properties of the quantum system. By imposing eight axioms on the structure of the states and the ortho-lattices of testable properties, he managed to reconstruct quantum theory. Most of these axioms are natural from a physical point of view, except Axiom VII. It requires that the ortho-lattice of testable properties of a quantum system must be isomorphic to that of closed linear subspaces of a (separable) Hilbert space. In his own words,

> *This axiom has rather a different character from Axioms I through VI. These all had some degree of physical naturalness and plausibility. Axiom VII seems entirely ad hoc. … Ideally one would like to have a list of physically plausible assumptions from which one could deduce Axiom VII. … At the moment such lists are not available and we*

[3]For the definition of ortho-lattices and other related notions, please refer to Appendix A.

> *are far from being forced to accept Axiom VII as logically inevitable.* (pp.71-72, [65])

This problem raised by Mackey was addressed by Piron in his PhD thesis [72] and the paper [73][4], who made a significant contribution to the field by providing an almost complete solution. Piron defined an *irreducible propositional system*, which is a special kind of orthomodular lattice called a *Piron lattice* in later literature (e.g. in [14] and [93]) and in this thesis. He proved that every Piron lattice of rank at least 4 is isomorphic to the lattice of closed subspaces of a generalized Hilbert space. On the one hand, Piron's result is a partial solution, because, according to their definitions, Hilbert spaces are not the same as generalized Hilbert spaces. On the other hand, the result is significant, because generalized Hilbert spaces resemble Hilbert spaces very well: Hilbert spaces and generalized Hilbert spaces are both the same kind of mathematical structure, i.e. vector spaces over division rings equipped with orthomodular Hermitian forms; and every Hilbert space is a generalized Hilbert space. Moreover, for a generalized Hilbert space, if its underlying division ring is that of the real numbers, the complex numbers or the quaternions, it will be a Hilbert space. This result is the *Piron-Amemiya-Arakia Theorem*, which is a claim with a wrong proof from Piron in [73] and then correctly proved by Amemiya and Arakia in [9]. Four significant features of Piron's contribution mentioned here are worth pointing out. First, Piron's representation theorem connects ortho-lattices with generalized Hilbert spaces, which are mild generalizations of Hilbert spaces. Second, while Mackey uses probability in his work, Piron's representation theorem does not involve probability and thus is purely qualitative. This is part of Piron's intention:

> *The aim of our work has been to inquire on a more fundamental level about the origin of the superposition principle and thus to justify the use of Hilbert space without appeal at the outset to the notion of probability.* (p.1 of [74])

Considering the definition of Hilbert spaces heavily relies on the real numbers, Piron's theorem is indeed surprising. Third, Piron not only comes up with the mathematical definition of Piron lattices, but also explains the physical intuitions behind it. Moreover, together with Jauch, Randall, Foulis and others, Piron develops a systematic and operational framework of reconstructing quantum theory, which is called *operational quantum logic*[5]. Fourth, to reconstruct quantum theory, the remaining problem is to characterize Hilbert spaces in the class of generalized Hilbert spaces in a way which makes more sense in physics than just

[4]Both [72] and [73] are in French. The results are then presented in English in [74].

[5]This framework, as well as the involved mathematical results, is presented systematically in [59], [74] and [75]. Please note that within operational quantum logic there are clear distinctions to be made between the setting that originated from Piron and Jauch and the one that originated from the work by Randall and Foulis.

stipulating the division ring to be one of the real numbers, the complex numbers and the quaternions.

To characterize Hilbert spaces in the class of generalized Hilbert spaces, the first question is whether every generalized Hilbert space is in fact a Hilbert space. The answer is no. Counterexamples in the finite-dimensional case abound and are easy to find, but the infinite-dimensional case is very hard. The first infinite-dimensional generalized Hilbert space which is not isomorphic to any Hilbert space was found by Keller in [62]. Therefore, characterizing Hilbert spaces in the class of generalized Hilbert spaces is not a trivial problem. In [83] Solèr gave an amazing answer to this question: if it has an infinite orthonormal sequence, a generalized Hilbert space will be a Hilbert space. This result characterizes infinite-dimensional Hilbert spaces in the class of generalized Hilbert spaces. In [55] Holland gave a comprehensive account on the significance of Solèr's theorem in quantum logic. In particular, he showed by known results in projective geometry that the existence of an infinite orthonormal sequence can be stated in the language of lattices. This implies that the Piron lattices which are isomorphic to those of closed linear subspaces of infinite-dimensional Hilbert spaces can be characterized purely in lattice-theoretic terms. At this point, one could say that the problem raised by Mackey had been solved, at least mathematically. Various improvements on Solèr's theorem have been made. For example, in [67] Mayet showed that the existence of an infinite orthonormal sequence is equivalent to a condition on the group of automorphisms on a Piron lattice. Moreover, the three underlying division rings, those of the real numbers, the complex numbers and the quaternions, can be distinguished by the structure of the group of automorphisms. These results make physical sense, because by using Wigner's theorem[6] one can show that the automorphisms on the Piron lattice of closed linear subspaces of a Hilbert space over the complex numbers are all induced by unitary or anti-unitary operators on the Hilbert space. According to quantum theory, these operators model the evolution of a quantum system.

In the above, I outline the main development in quantum logic, which is about recovering Hilbert spaces from the lattices of testable properties. Quantum theory certainly has much more content. Many other important aspects of quantum theory have already been dealt with in quantum logic and in turn this fosters the further development of quantum logic. For example, the evolution of a quantum system is deterministic and described by Schrödinger's equation. This important physical process is studied from the perspective of quantum logic in, for example, Chapter 5 of [74] and [40]. Another example is the description of composite quantum systems. In the Hilbert space formalism of quantum theory, if a quantum system consists of two subsystems described by two Hilbert spaces, respectively, the quantum system can be described by the tensor product of these two Hilbert

[6] Wigner's theorem is proved in [89], of which [90] is the translation into English. In [38] a nice and elementary proof can be found.

spaces, which is also a Hilbert space. In the field of quantum logic, researchers have looked for a construction on ortho-lattices which is the counterpart of tensor products of Hilbert spaces. In [78] Randall and Foulis concluded that tensor products of quantum logics do not exist. They posed several desirable probabilistic properties of the tensor product of two ortho-lattices. Then they found two orthomodular lattices such that any orthomodular lattice can not satisfy those properties and hence can not be the tensor product of the two lattices. In [3] another negative result was obtained. Aerts gave a reasonable way of defining the tensor product of two property lattices[7], and showed that the resulting lattice is an ortho-lattice only if one of the property lattices contains only two elements. These results are frustrating, but they could not stop people from studying tensor products from a logical perspective. For example, in [6] Aerts and Daubechies characterized how the lattices of closed linear subspaces of two Hilbert spaces relate to that of their tensor product in terms of the maps between the lattices. They also revealed the physical intuition behind this characterization. For another example, in [44] Foulis and Bennett considered orthoalgebras, which are more general than ortho-lattices. They defined the tensor product construction on orthoalgebras, and proved that the tensor product of two orthoalgebras exists if they satisfy a minor assumption. Although there still hasn't been a satisfactory solution, the tensor product problem leads researchers to the fruitful study of structures more general than ortho-lattices, such as ortho-posets, orthomodular posets, orthoalgebras and test spaces.

Other Approaches

So far I have only surveyed the results of quantum logic inspired by the paper of Birkhoff and von Neumann. A common feature of them is that they all use the structure of testable properties as primitive and describe it in algebraic structures like posets and lattices. Inspired by these results, researchers have tried to take many other physical notions as primitives and used various mathematical structures to describe them. This has led to a great number of interesting results. For example, although Piron's qualitative representation theorem is very nice, Mackey's probabilistic approach is not forgotten. Many researchers study the structure formed by the probability measures on algebraic structures like orthomodular posets or orthomodular lattices. For another example, many researchers find pure states more intuitive than testable properties, and study various structures on them. Some people prefer the qualitative approach and study the orthogonality relation between states[8], while some study the transition probabilities between states ([68], [94], etc.). In [14] the transitions between states triggered by measurements are the focus. Besides, some people study mixed states of a quantum system, because they think that pure states are not easy to identify in

[7]For the definition of property lattices, please refer to [3].

[8]A detailed and technical survey of the research in this line can be found in Section 2.8.

practice. This results in a deep understanding of the convex sets formed by the mixed states of quantum systems.

Moreover, there is a current trend stimulated by the development of category theory. One example is *categorical quantum mechanics* founded by Abramsky and Coecke ([1]). The paper [2] is an accessible survey of this approach. By using special categories to model quantum entanglement, research in this line has proved to be successful in analysing quantum algorithms and protocols ([1] and [87]). Another example is *topos quantum theory* founded by Isham, Döring, Butterfield, etc. ([32], [33], [34] and [35]), in which quantum systems are modelled by a kind of category-theoretic structure called a topos. The book [43] is a detailed and accessible introduction.

Among the many approaches in quantum logic, I would like to emphasize and discuss in detail *dynamic quantum logic* founded by Baltag and Smets in [14]. The work in this thesis is greatly inspired by this approach.

Dynamic Quantum Logic

According to Baltag and Smets, there are two main motivations for dynamic quantum logic.

One is the extensional and state-based view of systems from computer science and philosophy. In computer science, a (classical) computational system like an automaton is usually modelled by a labelled transition system, i.e. a set equipped with various binary relations. The elements of the set are interpreted as the possible states of the system, and the binary relations as the operations or programs. Moreover, according to the extensional view of properties, a property is nothing more than the set of states possessing the property. This simple and intuitive idea turns out to be very useful in analysing computer programs and protocols. Note that the quantum logic initiated by Birkhoff and von Neumann is not in this spirit, because testable properties are considered as primitive and abstract objects without inner structures.

The other motivation is the dynamic turn in logic. Generally speaking, in this shift of perspective, many non-classical logics are not considered as non-classical laws of reasoning but manifestations of the dynamics of non-classical actions. Intuitionistic logic is a good example. Originally it was studied as a sub-logic of classical logic refuting the law of excluded middle. The Brouwer-Heyting-Kolmogorov interpretation reveals the dynamics underlying it, that is the constructive process of creating mathematics. This interpretation is captured formally in a relational semantics based on partially ordered sets with the orders tracking the progress of the construction. This semantics for intuitionistic logic is very useful and leads to many deep results. In general, researchers have found that the dynamics of actions can be modelled by relational structures and described by formal languages, and research in this line has been very fruitful. Van Benthem's book [19] is a representative in this dynamic turn of logic. Now that quantum logic is shown in [22] to be a non-classical logic with a background in physics, to reveal and study the underlying dynamics of this logic is interesting from both theoretical and practical perspectives. The relation between dynamic quantum logic and the dynamic turn in logic is elaborated in [16].

In [14] Baltag and Smets for the first time deliberately introduced the extensional and state-based view in computer science to study the non-classical dynamics underlying quantum logic. They abstracted a labelled transition system from a quantum system by regarding the pure states of the system as the elements of a set, the transitions between states triggered by measurements as binary relations on the set and the testable properties as special subsets of the set. Based on this idea, they defined *quantum dynamic frames*. These are labelled transition systems which satisfy several axioms reflecting the non-classical behaviour of measurements in quantum theory. Moreover, they hinted that the special subsets in a quantum dynamic frame modelling testable properties form a Piron lattice. In this technical way they explained how a Piron lattice, a kind of abstract and algebraic structure, arises from some concrete and set-theoretic structure. In my opinion, the approach of dynamic quantum logic has a lot of advantages. First, quantum dynamic frames are more intuitive than Piron lattices in modelling quantum systems. Second, because of the shared extensional and stated-based view, dynamic quantum logic has close connections with computer science. For example, in [15] Baltag and Smets devised a logic for quantum dynamic frames and gave formal proofs of the correctness of many important quantum algorithms and protocols.

1.1.2 Motivation of the Thesis

In this thesis I try to contribute to the understanding of quantum theory by following the general idea of quantum logic. For the primitive physical notions, I choose the non-orthogonality relation between (pure) states of a (closed) quantum system. There are several important reasons for this choice.

First, the non-orthogonality relation is intuitive from a physical point of view. There are two ways to interpret two states being orthogonal. One is that there is an observable of the system such that the two states are the final states of the system after measurements yielding two different values of the observable, respectively. The other is that there is no measurement such that one execution of it could trigger the system to change from one state to the other. Both interpretations are intuitive; and, according to standard quantum theory, they are equivalent.

Second, the non-orthogonality relation is simple from a mathematical point of view. In quantum dynamic frames, the transitions between states triggered by measurements are the primitives and are modelled by binary relations. However, by definition, in a quantum dynamic frame the set of binary relations is so big that its cardinality is at least as large as that of the set of states. Moreover, the intricate interaction between the binary relations and subsets of a quantum dynamic frame obscures the essence of the dynamics of quantum logic. Therefore, it is worth finding simpler structures to model quantum systems in the extensional and state-based spirit and to reveal the dynamics of quantum logic. A good candidate to start this investigation is the non-orthogonality relation. First, this relation can be easily defined and plays an important role in quantum dynamic frames. Second, the non-orthogonality relation is just a binary relation on the set of states, hence it can be modelled in Kripke frames. A *Kripke frame* is just a set equipped with a binary relation on it, and hence is probably one of the simplest mathematical structures. Moreover, Kripke frames have been extensively used and studied in logic and computer science. Therefore, on the one hand, the results and techniques of these two fields can be applied to the study of quantum theory. On the other hand, the results about the non-orthogonality relation can be easily applied to logic and computer science. Besides, in its relational semantics, intuitionistic logic is also interpreted on Kripke frames. Hence quantum logic and intuitionistic logic, two important non-classical logics, can be compared from the perspective of relational semantics.

Third, there are results showing that the (non-)orthogonality relation is important to the structure of testable properties in quantum theory. Good examples are [31] and [47]. For a detailed and technical survey of such work, please refer to Section 2.8. However, these results only contain representation theorems via Kripke frames for general kinds of lattices like ortho-lattices and orthomodular lattices, but not for lattices as specific as Piron lattices. Therefore, it is worth having a closer look at the non-orthogonality relation and trying to extend and improve the existing results.

Therefore, based on the existing work, it is attractive and promising to study such a physically intuitive and mathematically simple notion. A relational reconstruction of quantum theory based on the non-orthogonality relation will greatly improve our understanding of quantum phenomena.

1.1.3 Outline of the Thesis

This thesis is mainly an in-depth study of the non-orthogonality relation between (pure) states of a (closed) quantum system. I use Kripke frames to model this relation, and then try to reconstruct quantum theory. To be precise, the following questions are addressed.

The first question is about how crucial the role is of the non-orthogonality relation in quantum theory. As was mentioned before, there have not been any representation theorems via Kripke frames for important quantum structures like Piron lattices. Hence it is reasonable to doubt whether this relation is indeed so important that quantum theory can be reconstructed from it. To be precise:

Research Question 1: Taking the non-orthogonality relation as the only primitive and modelling it in Kripke frames, is it possible to represent Piron lattices and other important quantum structures?

Chapter 2 of this thesis is devoted to this question, and the answer is yes. Based on the properties of the non-orthogonality relation in quantum theory, I define five kinds of Kripke frames, from the general ones to the specific ones: *state spaces*, *geometric frames*, *complete geometric frames*, *quasi-quantum Kripke frames* and *quantum Kripke frames*. These Kripke frames are studied one by one from a geometric perspective. This study leads to a sophisticated understanding of the inner structure of quantum Kripke frames and their correspondence with Piron lattices. As a by-product, I discover that some special kinds of projective geometries are Kripke frames in disguise. Moreover, based on the work of Piron, Solèr and Holland, a special kind of quantum Kripke frame requires a representation involving an infinite-dimensional Hilbert space over the complex numbers. This result, on the one hand, hints at why Hilbert spaces are exclusively used in formalizing quantum theory; and, on the other hand, shows that quantum Kripke frames can be useful in the qualitative modelling of quantum systems.

The second question is about how useful quantum Kripke frames are in modelling quantum systems. The solution to Research Question 1 only involves Hilbert spaces, but to reconstruct quantum theory Hilbert spaces need to be involved in the right way. Hence it is important to answer the following question:

Research Question 2: How much quantum-like structure of Hilbert spaces has a counterpart in quantum Kripke frames?

Chapter 3 is devoted to this question. I define *continuous homomorphisms* from a quantum Kripke frame to another. Using results in projective geometry I show that they require representations via continuous quasi-linear maps, which are mild generalizations of linear maps having adjoints between Hilbert spaces. Moreover, I study three special kinds of continuous homomorphisms defined in parallel to the unitary operators, self-adjoint operators and projectors on Hilbert spaces, respectively. Therefore, according to the Hilbert space formalism of quantum theory, these three kinds of continuous homomorphisms can be used to model the evolution, observables and testable properties of quantum systems, respectively.

Furthermore, in Chapter 3 I give a partial answer to the tensor product problem. According to Hilbert space theory, the tensor product of two Hilbert spaces can be constructed from the set of Hilbert-Schmidt operators from one of them to the other; when both Hilbert spaces are finite-dimensional, this set coincides with the set of linear maps. I show that, given two *finite-dimensional* quantum Kripke frames modelling two quantum systems, a carefully chosen set of continuous homomorphisms from one frame to the other forms a quantum Kripke frame, which can model the compound system consisting of the two systems. This is a solution to a special case of the tensor product problem in the framework of quantum Kripke frames. Moreover, this way of choosing the set of continuous homomorphisms involves a characterization of the quasi-linear maps between two vector spaces which have the same underlying field isomorphism. This leads to a solution to a special case of an open problem in [39].

The third question concerns automated reasoning about quantum phenomena. Results in Chapter 2 and Chapter 3 show that quantum Kripke frames are simple but useful structures in modelling quantum systems. Hence it is natural to investigate the theories of quantum Kripke frames within some formal languages. This investigation will lay the foundation for the automated reasoning of quantum Kripke frames and thus of quantum phenomena.

> **Research Question 3:** Fixing a certain formal language, how complex is the theory of quantum Kripke frames in this language? Is it finitely axiomatizable, decidable?

Chapter 4 is devoted to this question. Since quantum Kripke frames are relational structures, the modal language and the first-order language are two natural candidates of formal languages for describing them. I study the axiomatization problem and the decision problem of the modal and the first-order theories of quantum Kripke frames.

In Chapter 5, the last technical chapter, I attempt to pave the way for future research about quantum probability theory. There are three reasons for this. The first one is the importance of probability in quantum theory. Indeterminacy is intrinsic in the description of measurements in quantum theory, and probability is a powerful tool in describing indeterminate phenomena. Moreover, probabilities in quantum theory are considered to satisfy axioms different from the classical Kolmogorov axioms, so they are a hot research topic ([50]). The second one is the need to extend the purely qualitative framework of quantum Kripke frames to a quantitative one which has more power in modelling. Finally, this change of framework from a qualitative one to a quantitative one is actually very natural and not as huge a step as it appears. The qualitative non-orthogonality relation has a natural quantitative counterpart, that is the transition probabilities between (pure) states of a (closed) quantum system. Two states are non-orthogonal, if and only if the transition probability between them is not zero. As a result, it is natural and it makes sense to ask the following question:

> **Research Question 4:** How do the results on quantum Kripke frames help in the understanding of transition probabilities between states and the study of quantum probability theory?

I define *probabilistic quantum Kripke frames* by endowing the pairs of elements in a quantum Kripke frame with transition probabilities, i.e. real numbers between 0 and 1, in a systematic way prescribed by four natural axioms. Quantum probability measures are shown to arise in a probabilistic quantum Kripke frame in a natural way. Finally, by combining the non-orthogonality relations and the transition probabilities in probabilistic quantum Kripke frames, I define *quantum transition probability spaces* in which only the transition probabilities are primitive. They are shown to be in correspondence with probabilistic quantum Kripke frames, and thus have the nice inner structure of quantum Kripke frames revealed in Chapter 2. Therefore, they are promising in the quantitative modelling of quantum systems and in the study of quantum probability theory.

1.1.4 Acknowledgement of Intellectual Contributions

None of the results in this thesis has been published in journals or conference proceedings. Some results from Chapter 2 and Chapter 4 are published informally in the ILLC Technical Notes Series ([95] and [96]).

Moreover, I published three papers with my colleagues during my PhD on reasoning about quantum theory. The knowledge and experience that I gained by writing these papers were indispensable to this thesis. To be precise:

With Dr. Alexandru Baltag, Drs. Jort Bergfeld, Dr. Kohei Kishida, Dr. Joshua Sack and Dr. Sonja Smets, I published two papers, [12] and [13], on logics for formal verification of quantum algorithms and protocols. The discussion and writing of this paper greatly improved my understanding of quantum theory, quantum algorithms and quantum protocols.

With Drs. Jort Bergfeld, Dr. Kohei Kishida and Dr. Joshua Sack, I published the paper [20], which is an extension of [14]. The discussion and writing of this paper helped me to get the key intuition in defining quantum Kripke frames, the protagonists of this thesis. Moreover, many technical tricks in the paper are also used in this thesis.

Finally, some of the topics and results in this thesis were suggested to me by Prof. Johan van Benthem, Dr. Alexandru Baltag and Dr. Alessandra Palmigiano. These are pointed out in the appropriate places in the thesis in the form of footnotes.

1.2 The Hilbert Space Formalism of Quantum Mechanics

In this section, I review some elements of the standard formalism of quantum mechanics in Hilbert spaces. The main purpose is to sketch the physical background of this thesis, and to give an idea of how Hilbert spaces are used in modelling quantum systems and their behaviour. As a result, this account will only contain materials that is relevant to this thesis; and it is by no means complete or self-contained. Moreover, it is presented here in a casual style. For more thorough accounts, I recommend [56] and Chapter 2 in [71]. (The latter is only about special quantum systems described by *finite-dimensional* Hilbert spaces, but this suffices for most purposes.) A more mathematical and detailed exposition can be found in [51]. For a textbook in quantum mechanics, I recommend [80].

1.2.1 Quantum Systems and Hilbert Spaces

The first postulate in quantum mechanics is the following:

> **Postulate 1:** A closed quantum system is described by a Hilbert space over the complex numbers in such a way that the states of the system correspond to the one-dimensional subspaces of the Hilbert space.

The concept of *closed quantum systems*, like those of points of mass and points of electrical charge, is a physical idealization. It means that the quantum system is considered as isolated from its environment. In practise, it is indeed a complicated engineering task to isolate a quantum system, but for some particular quantum systems this can be approximated to a satisfactory extent.

A *Hilbert space* over the complex numbers is a vector space over the field $\mathbb{C}$ of complex numbers equipped with an inner product in such a way that it is complete under the norm topology. For the definition of vector spaces over fields and many other relevant definitions involved in this section, please refer to Appendix B.2. An *inner product* on a vector space V over $\mathbb{C}$ is an anisotropic Hermitian form on V, denoted by $\langle \cdot, \cdot \rangle$, whose accompanying involution is the complex conjugate and which satisfies the additional condition of *positiveness*: for every $\mathbf{v} \in V$, $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$. It is well known that the function $\| \cdot \| : V \to \mathbb{R}^* :: \mathbf{v} \mapsto \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$ is a norm on V. *A Cauchy sequence* $\{\mathbf{v}_n\}_{n=0}^{\infty} \subseteq V$ is one such that for every $\epsilon > 0$ there is an $N \in \mathbb{N}^+$ such that $m, n > N$ implies that $\|\mathbf{v}_n - \mathbf{v}_m\| < \epsilon$. A sequence $\{\mathbf{v}_n\}_{n=0}^{\infty} \subseteq V$ *converges to* $\mathbf{v} \in V$, denoted by $lim_{n\to\infty}\mathbf{v}_n = \mathbf{v}$, if $lim_{n\to\infty}\|\mathbf{v}_n - \mathbf{v}\| = 0$. V is said to be *complete under the norm topology*, if every Cauchy sequence in V converges to an element in V.

A one-dimensional subspace of a Hilbert space $\mathcal{H}$ is a set of the form $\langle \mathbf{v} \rangle \stackrel{\text{def}}{=} \{c\mathbf{v} \mid c \in \mathbb{C}\}$ with a non-zero vector $\mathbf{v} \in \mathcal{H}$. I denote by $\Sigma(\mathcal{H})$ the set of all one-dimensional subspaces of $\mathcal{H}$. A binary relation on $\Sigma(\mathcal{H})$ will play an eminent role in this thesis: for any $s, t \in \Sigma(\mathcal{H})$, s and t are *non-orthogonal*, denoted by $s \to_{\mathcal{H}} t$, if there are $\mathbf{s} \in s \setminus \{\mathbf{0}\}$ and $\mathbf{t} \in t \setminus \{\mathbf{0}\}$ such that $\langle \mathbf{s}, \mathbf{t} \rangle \neq 0$; otherwise, s and t are *orthogonal*, denoted by $s \not\to_{\mathcal{H}} t$. This is equivalent to: for any $\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle \in \Sigma(\mathcal{H})$, $\langle \mathbf{s} \rangle \to_{\mathcal{H}} \langle \mathbf{t} \rangle$, if $\langle \mathbf{s}, \mathbf{t} \rangle \neq 0$; and $\langle \mathbf{s} \rangle \not\to_{\mathcal{H}} \langle \mathbf{t} \rangle$, if $\langle \mathbf{s}, \mathbf{t} \rangle = 0$.

Mathematical as the definition is, the orthogonality relation has a clear physical meaning. Consider any two states represented by orthogonal one-dimensional subspaces. On the one hand, there is an observable of the system such that the two states are the final states of the system after measurements yielding two different values of the observable, respectively. On the other hand, there is no measurement such that one execution of it could trigger the system to change from one state to the other. These two interpretations are equivalent, according to quantum theory. I continue to discuss the formalism of measurements and observables in quantum theory in the next subsection.

1.2.2 Observables and Self-Adjoint Operators

The second postulate in quantum mechanics is the following:

> **Postulate 2:** Every observable A of a (closed) quantum system is described by a self-adjoint operator A on the Hilbert space $\mathcal{H}$ describing the system. The possible values of a measurement of the observable A are those in $\sigma(A)$, the spectrum of A. If the system is in a state described by $\langle \mathbf{v} \rangle$ with $\mathbf{v}$ a unit vector in $\mathcal{H}$, the probability of a measurement yielding a value in a Borel set $E \subseteq \sigma(A)$ is $p(E) = \langle \mathbf{v}, \mu_A(E)(\mathbf{v}) \rangle$, where μ_A is the projector-valued measure on the Borel σ-algebra in $\sigma(A)$; and the state after the measurement is the one described by $\langle \mu_A(E)(\mathbf{v}) \rangle$, if $p(E) \neq 0$. Moreover, the expectation of a measurement of A is $\langle \mathbf{v}, A(\mathbf{v}) \rangle$.

Many notions in this postulate need explanation. Moreover, I confine myself to bounded self-adjoint operators. The general case is structurally similar, but involves many more technical subtleties. A *bounded operator* A on a Hilbert space $\mathcal{H}$ is a linear map on $\mathcal{H}$ such that there is a $b \in \mathbb{R}^+$ satisfying, for every $\mathbf{v} \in \mathcal{H}$, $\|A(\mathbf{v})\| \leq b\|\mathbf{v}\|$. For a bounded operator A, the *spectrum* of it is the set $\sigma(A) \stackrel{\text{def}}{=} \{\lambda \in \mathbb{C} \mid A - \lambda id_{\mathcal{H}} \text{ does not have a bounded inverse}\}$, where $id_{\mathcal{H}}$ is the identity map on $\mathcal{H}$. Every bounded operator A has a unique *adjoint*, which is denoted by $A^\dagger$ and defined to be a bounded operator on $\mathcal{H}$ such that, for any $\mathbf{u}, \mathbf{v} \in \mathcal{H}$, $\langle \mathbf{u}, A(\mathbf{v}) \rangle = \langle A^\dagger(\mathbf{u}), \mathbf{v} \rangle$. A *bounded self-adjoint operator* A is a bounded operator satisfying $A = A^\dagger$. For every bounded self-adjoint operator A, $\sigma(A) \subseteq \mathbb{R}$; and, according to the Spectral Theorem, there is a unique projector-valued measure μ_A on the Borel σ-algebra in $\sigma(A)$ satisfying $A = \int_{\sigma(A)} \lambda d\mu_A(\lambda)$. A *projector* P on $\mathcal{H}$ is a bounded self-adjoint operator which is idempotent, i.e. $\mathsf{P} \circ \mathsf{P} = \mathsf{P}$. Then a *projector-valued measure* μ is a function from a σ-algebra in a set X to the set of projectors on $\mathcal{H}$ satisfying the following:

1. $\mu(\emptyset)$ is the zero map and $\mu(X) = id_{\mathcal{H}}$;
2. if $\{E_i \mid i \in \mathbb{N}\}$ is a sequence of *disjoint* sets in the algebra and $\mathbf{v} \in \mathcal{H}$, $\mu(\bigcup_{i=0}^{\infty} E_i)(\mathbf{v}) = \sum_{i=0}^{\infty} \mu(E_i)(\mathbf{v})$;
3. for any two sets E and F in the algebra, $\mu(E \cap F) = \mu(E) \circ \mu(F)$.

When $\sigma(A)$ is a finite set, say $\{\lambda_0, ..., \lambda_n\}$, the Spectral Theorem boils down to asserting that A determines a set of projectors $\{P_0, ..., P_n\}$ such that $P_i \circ P_j$ is the zero map whenever $i \neq j$, $\sum_{i=0}^{n} P_i = id_{\mathcal{H}}$ and $A = \sum_{i=0}^{n} \lambda_i P_i$.

The projectors on $\mathcal{H}$ are in correspondence[9]with the special subsets of $\mathcal{H}$ called closed linear subspaces. A *closed linear subspace* P *of* $\mathcal{H}$ is a subspace such that every Cauchy sequence in P converges in $\mathcal{H}$ to a vector in P. According to the Orthogonal Decomposition Theorem, for every closed linear subspace P, $P \oplus P^{\perp} = \mathcal{H}$, where $P^{\perp} \stackrel{\text{def}}{=} \{\mathbf{v} \in \mathcal{H} \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for every } \mathbf{w} \in P\}$ is called the *orthocomplement of* P. Then the projectors on $\mathcal{H}$ are in correspondence with the closed linear subspaces of $\mathcal{H}$: $\mathsf{P}[\mathcal{H}]$ is a closed linear subspace for every projector P, and every closed linear subspace P determines a projector P satisfying $P = \mathsf{P}[\mathcal{H}]$.

This correspondence makes it possible to talk about properties of quantum systems. In this thesis I adopt the extensional view of properties, i.e. properties are sets of states of a system. Then a *testable property* is a set of states that are included in the same closed linear subspace, i.e. $\{\langle \mathbf{v} \rangle \in \wp(P) \mid \mathbf{v} \neq \mathbf{0}\}$ for some closed linear subspace P of $\mathcal{H}$. It is called testable, because the closed linear subspace corresponds to a projector on $\mathcal{H}$, which is a self-adjoint operator and thus describes the measurement of an observable of the system.

In summary, the picture of measurements described in Postulate 2 is as follows: Let A be an observable described by an operator A, E an element in the Borel σ-algebra in $\sigma(A)$, V_E the closed linear subspace corresponding to the projector $\mu_A(E)$ and $\langle \mathbf{v} \rangle$ with $\mathbf{v}$ a unit vector describe a state of the system. Moreover, the concern is whether the result will be a value in E, if the system in this state undergoes a measurement of A. Consider three cases. If $\langle \mathbf{v} \rangle \subseteq V_E$, $\mu_A(E)(\mathbf{v}) = \mathbf{v}$ and thus $p(E) = \langle \mathbf{v}, \mathbf{v} \rangle = 1$. This means that the measurement will certainly yield a value in E and the state of the system is not changed. If $\langle \mathbf{v} \rangle \subseteq V_E^{\perp}$, $\mu_A(E)(\mathbf{v}) = \mathbf{0}$ and thus $p(E) = \langle \mathbf{v}, \mathbf{0} \rangle = 0$. This means that the measurement will not yield a value in E. If $\langle \mathbf{v} \rangle \not\subseteq V_E \cup V_E^{\perp}$, $\mu_A(E)(\mathbf{v}) \notin \langle \mathbf{v} \rangle$ and thus $p(E) \in (0, 1)$. This means that the measurement will yield a value in E with a probability strictly between 0 and 1; and, if this happens, the state of the system will be changed to $\langle \mu_A(E)(\mathbf{v}) \rangle \subseteq V_E$ which is different from $\langle \mathbf{v} \rangle$.

1.2.3 Evolution and Unitary Operators

The third postulate in quantum mechanics is the following:

> **Postulate 3:** The state of a (closed) quantum system may change with time purely depending on the mechanism of the system itself. This process is called *evolution*, and is described by the *Schrödinger equation:*
>
> $$i\hbar \frac{\partial \mathbf{v}(t)}{\partial t} = H(t)(\mathbf{v}(t))$$
>
> where $\langle \mathbf{v}(t) \rangle$ is the state of the system at time t, $\hbar$ is *Planck's constant* and, for every t, $H(t)$ is a self-adjoint operator on the Hilbert space $\mathcal{H}$ describing the system, called the *Hamiltonian*.

[9]For a technical definition of correspondences, please refer to Footnote 6 in Chapter 2.

In physics, one of the main concerns is to find the appropriate Hamiltonian for a quantum system and to solve this partial differential equation. Here I only care about the general and structural picture given by this postulate. Moreover, the case when $H(t)$ changes with time is complicated, hence here I only discuss the case when $H(t)$ is a constant function on t and write H for $H(t)$.

According to Stone's theorem, the (not necessarily bounded) self-adjoint operators on a Hilbert space $\mathcal{H}$ are in correspondence with the strongly continuous one-parameter unitary groups on $\mathcal{H}$. A *unitary operator U on $\mathcal{H}$* is a bounded operator on $\mathcal{H}$ such that $U \circ U^\dagger = U^\dagger \circ U = id_\mathcal{H}$. A *strongly continuous one-parameter unitary group on $\mathcal{H}$* is a set of unitary operators on $\mathcal{H}$ indexed by the real numbers, $\{U_t\}_{t\in\mathbb{R}}$, satisfying both of the following:

1. for every $\mathbf{v} \in \mathcal{H}$ and $t_0 \in \mathbb{R}$, $lim_{t\to t_0} U_t(\mathbf{v}) = U_{t_0}(\mathbf{v})$;

2. for every $s, t \in \mathbb{R}$, $U_{s+t} = U_s \circ U_t$.

Then the self-adjoint operators on $\mathcal{H}$ and the strongly continuous one-parameter unitary groups on $\mathcal{H}$ are in correspondence: for every self-adjoint operator A, $\{e^{itA}\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group, and every strongly continuous one-parameter unitary group $\{U_t\}_{t\in\mathbb{R}}$ determines a self-adjoint operator A such that $U_t = e^{itA}$ for each $t \in \mathbb{R}$.

The picture of the evolution of a quantum system given by the postulate and Stone's theorem is as follows: Let H be the Hamiltonian and $\{U_t\}_{t\in\mathbb{R}}$ the strongly continuous one-parameter unitary group corresponding to it. If the state at time t_0 is described by $\langle \mathbf{v}(t_0)\rangle$ with $\mathbf{v}(t_0)$ a unit vector, then the state at t is described by the one-dimensional subspace generated by $\mathbf{v}(t) = U_{t-t_0}(\mathbf{v}(t_0))$, which can be proved to be a unit vector by the definition of unitary operators. Note that by definition U_{t-t_0} only depends on H and $t - t_0$, and not on t_0, t, $\mathbf{v}_{t_0}$ or $\mathbf{v}_t$.

Finally, the unitary operators are automorphisms of a Hilbert space, in that they preserve not only the vector space structure but also the inner product. To be precise, it is easy to show from the definition that $\langle U(\mathbf{u}), U(\mathbf{v})\rangle = \langle \mathbf{u}, \mathbf{v}\rangle$, for any vectors $\mathbf{u}, \mathbf{v}$ in a Hilbert space $\mathcal{H}$ and any unitary operator U. This implies that, if neither of $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{0}$, $\langle U(\mathbf{u})\rangle \to_\mathcal{H} \langle U(\mathbf{v})\rangle$ if and only if $\langle \mathbf{u}\rangle \to_\mathcal{H} \langle \mathbf{v}\rangle$.

1.2.4 Composite Systems and Tensor Products

The previous postulates only describe quantum systems as a whole and disregard their composition. The next postulate is about describing a composite quantum system given the descriptions of its subsystems.

> **Postulate 4:** If two quantum systems are described by Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, the system consisting of them is described by the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of $\mathcal{H}_1$ and $\mathcal{H}_2$.

Tensor products of Hilbert spaces are technical and complicated, especially when infinite-dimensional Hilbert spaces are involved. A thorough discussion can be found in Section 2.6 in [61]. Here I confine myself to tensor products of finite-dimensional Hilbert spaces.

Given two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ over $\mathbb{C}$ of finite dimensions n and m, respectively, if $\mathcal{B}_1 = \{\mathbf{u}_1, ..., \mathbf{u}_n\}$ and $\mathcal{B}_2 = \{\mathbf{v}_1, ..., \mathbf{v}_m\}$ are orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, the *tensor product of* $\mathcal{H}_1$ *and* $\mathcal{H}_2$, denoted by $\mathcal{H}_1 \otimes \mathcal{H}_2$, is the Hilbert space over $\mathbb{C}$ with $\mathcal{B}_1 \times \mathcal{B}_2$ as an orthonormal basis and the inner product defined as follows:

$$\left\langle \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}(\mathbf{u}_i, \mathbf{v}_j), \sum_{k=1}^{n}\sum_{l=1}^{m} c_{kl}(\mathbf{u}_k, \mathbf{v}_l) \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{l=1}^{m} c_{ij}^{*} c_{kl} \langle \mathbf{u}_i, \mathbf{u}_k \rangle \langle \mathbf{v}_j, \mathbf{v}_l \rangle,$$

where $(\cdot)^*$ is the complex conjugate. Usually a basis vector $(\mathbf{u}_i, \mathbf{v}_j)$ is written as $\mathbf{u}_i \otimes \mathbf{v}_j$, and one can write $(c\mathbf{u}) \otimes \mathbf{v}$ or $\mathbf{u} \otimes (c\mathbf{v})$ for $c(\mathbf{u} \otimes \mathbf{v})$, $(c\mathbf{u} + c'\mathbf{u}) \otimes \mathbf{v}$ for $c\mathbf{u} \otimes \mathbf{v} + c'\mathbf{u} \otimes \mathbf{v}$ and $\mathbf{u} \otimes (c\mathbf{v} + c'\mathbf{v})$ for $c\mathbf{u} \otimes \mathbf{v} + c'\mathbf{u} \otimes \mathbf{v}'$.

Note that, for some vector $\mathbf{w} \in \mathcal{H}_1 \otimes \mathcal{H}_2$, there are $\mathbf{w}_1 \in \mathcal{H}_1$ and $\mathbf{w}_2 \in \mathcal{H}_2$ such that $\mathbf{w} = \mathbf{w}_1 \otimes \mathbf{w}_2$. Typical examples are the basis vectors. The states described by the one-dimensional subspaces generated by them are called *separated*. However, there are vectors in $\mathcal{H}_1 \otimes \mathcal{H}_2$ for which this can not be done; for example, $\mathbf{u}_1 \otimes \mathbf{v}_1 + \mathbf{u}_2 \otimes \mathbf{v}_2$. The states described by the one-dimensional subspaces generated by such vectors are called *entangled*.

Finally, the set of linear maps from $\mathcal{H}_1$ to $\mathcal{H}_2$ forms a vector space $Hom(\mathcal{H}_1, \mathcal{H}_2)$ over $\mathbb{C}$. The function $tr(\cdot^\dagger \circ \cdot)$, with $tr(\cdot)$ being the trace function on linear maps, is proved to be an inner product on it, and thus turns the space into a Hilbert space. This Hilbert space is isomorphic to $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Chapter 2

Quantum Kripke Frames

In this chapter, I investigate the non-orthogonality relation between (pure) states of a (closed) quantum system. This relation is modelled in the mathematical structures called Kripke frames. I introduce five kinds of special Kripke frames, from the general ones to the specific ones: state spaces, geometric frames, complete geometric frames, quasi-quantum Kripke frames and quantum Kripke frames. The states of a quantum system equipped with the non-orthogonality relation will be shown to form a quantum Kripke frame. Hence quantum Kripke frames are the protagonists of this chapter and this thesis. The other kinds of Kripke frames are introduced not just for pedagogical reasons. As I will show, some of them are useful from other perspectives. For example, geometric frames did appear in the literature before in the disguise of projective geometries equipped with pure polarities or pure orthogeometries. This discovery will turn out to be useful in the study of projective geometry.

This chapter is organized as follows: In Section 2.1 I define the notion of Kripke frames and the five kinds of Kripke frames mentioned above. I also give examples of them. Sections 2.2 to 2.5 are devoted to the investigations of the inner structure of each kind of Kripke frame. In Section 2.6 I introduce a useful construction on Kripke frames called a subframe. Based on the previous sections, I discuss in Section 2.7 the significance of quantum Kripke frames for quantum theory, and the relation between quantum Kripke frames and some other important quantum structures such as Piron lattices and quantum dynamic frames. Finally, in Section 2.8 I survey the research on the (non-)orthogonality relation in the literature, which has inspired the work in this thesis.

2.1 Definitions and Examples

In this section, I define quantum Kripke frames and some related mathematical structures, namely, state spaces, geometric frames, complete geometric frames and quasi-quantum Kripke frames. I also give examples of these structures.

2.1.1 Definitions

In this subsection, I define quantum Kripke frames and some related mathematical structures. All of these structures are special cases of the simple structures called Kripke frames defined as follows:

2.1.1. Definition. A *Kripke frame* is a tuple $\mathfrak{F} = (\Sigma, \rightarrow)$ in which Σ is a non-empty set and $\rightarrow \subseteq \Sigma \times \Sigma$.

Before defining other structures, I introduce some important notions and notations in a Kripke frame $\mathfrak{F} = (\Sigma, \rightarrow)$:

- If $(s, t) \in \rightarrow$, call that s and t are *non-orthogonal* and write $s \rightarrow t$.
- If $(s, t) \notin \rightarrow$, call that s and t are *orthogonal* and write $s \not\rightarrow t$.
- $P \subseteq \Sigma$ is *orthogonal*, if, for any $s, t \in P$, $s \neq t$ implies that $s \not\rightarrow t$.
- The *orthocomplement of* $P \subseteq \Sigma$ *(with respect to* $\rightarrow$*)*, denoted by $\sim P$, is the set $\{s \in \Sigma \mid s \not\rightarrow t, \text{ for every } t \in P\}$.
- $P \subseteq \Sigma$ is *bi-orthogonally closed*, if $\sim\sim P = P$. $\mathcal{L}_{\mathfrak{F}}$ is used to denote the set $\{P \subseteq \Sigma \mid \sim\sim P = P\}$.
- $s, t \in \Sigma$ are *indistinguishable with respect to* $P \subseteq \Sigma$, denoted by $s \approx_P t$, if $s \rightarrow x \Leftrightarrow t \rightarrow x$ for every $x \in P$.
- $t \in \Sigma$ is *an approximation of* $s \in \Sigma$ *in* $P \subseteq \Sigma$, if $t \in P$ and $s \approx_P t$.

2.1.2. Remark. As it turns out, the notion of indistinguishability is important. I list some basic properties of this relation, which are all easy to verify:

- for every $P \subseteq \Sigma$, $\approx_P$ is an equivalence relation on Σ;
- $\approx_\emptyset = \Sigma \times \Sigma$ and $Id_\Sigma \stackrel{\text{def}}{=} \{(s, s) \mid s \in \Sigma\} \subseteq \approx_\Sigma$[1];
- $\approx_Q \subseteq \approx_P$, if $P \subseteq Q \subseteq \Sigma$.

Next, I define several conditions on a Kripke frame, some of which are well known in the literature:

2.1.3. Definition. The following is a list of conditions which a Kripke frame $\mathfrak{F} = (\Sigma, \rightarrow)$ may satisfy:

- **Reflexivity:** $s \rightarrow s$, for every $s \in \Sigma$.

[1] Under Separation (introduced in Definition 2.1.3), which is assumed for most of the time in this thesis, this inclusion becomes the identity.

- **Symmetry:** $s \to t$ implies that $t \to s$, for any $s, t \in \Sigma$.
- **Separation:** For any $s, t \in \Sigma$, if $s \neq t$, there is a $w \in \Sigma$ such that $w \to s$ and $w \not\to t$.
- **Existence of Approximation for Lines (AL):**

  For any $s, t \in \Sigma$, if $w \in \Sigma \setminus {\sim}\{s, t\}$, there is a $w' \in \Sigma$ which is an approximation of w in ${\sim}{\sim}\{s, t\}$, i.e. $w' \in {\sim}{\sim}\{s, t\}$ and $w \approx_{\sim\sim\{s,t\}} w'$.
- **Existence of Approximation for Hyperplanes (AH):**

  For each $s \in \Sigma$, if $w \in \Sigma \setminus {\sim}{\sim}\{s\}$, there is a $w' \in \Sigma$ which is an approximation of w in ${\sim}\{s\}$, i.e. $w' \in {\sim}\{s\}$ and $w \approx_{\sim\{s\}} w'$.
- **Existence of Approximation (A):**

  For each $P \subseteq \Sigma$ with ${\sim}{\sim}P = P$, if $s \in \Sigma \setminus {\sim}P$, there is an $s' \in \Sigma$ which is an approximation of s in P, i.e. $s' \in P$ and $s \approx_P s'$.
- **Superposition:** For any $s, t \in \Sigma$, there is a $w \in \Sigma$ such that $w \to s$ and $w \to t$.

In the above, I refer to a set of the form ${\sim}{\sim}\{s, t\}$ as a 'line' and one of the form ${\sim}\{s\}$ as a 'hyperplane'. The reason for my use of these terms will be made clear below (Remark 2.3.8 and Proposition 2.3.9).

I now define the following structures:

2.1.4. DEFINITION. A *state space*[2] is a Kripke frame satisfying Reflexivity, Symmetry and Separation.

The following structures are all special kinds of state spaces:

- A *geometric frame* is a state space having Property AL and Property AH.
- A *complete geometric frame* is a state space satisfying Property A.
- A *quasi-quantum Kripke frame* is a state space satisfying Property AL, Property AH and Superposition.
- A *quantum Kripke frame* is a state space satisfying Property A and Superposition.

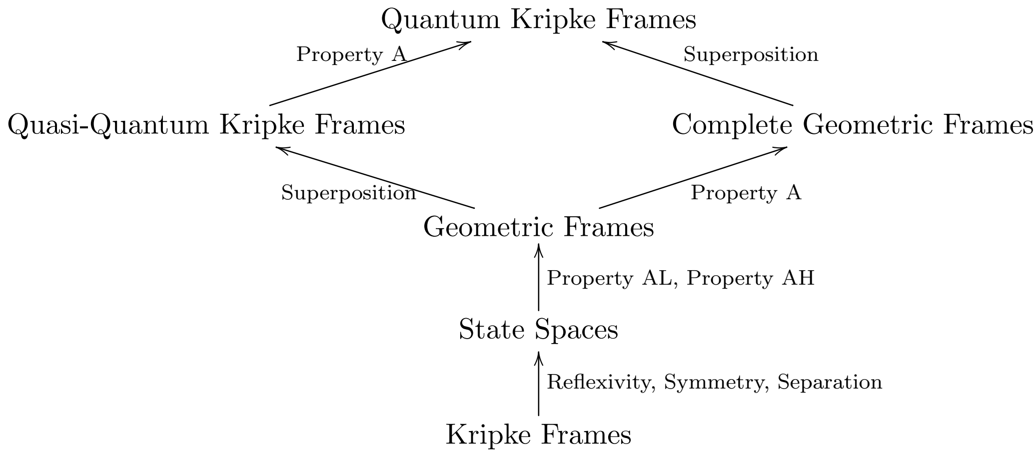
I emphasize that, by definition a quantum Kripke frame is a Kripke frame with five properties: Reflexivity, Symmetry, Separation, Property A and Superposition[3].

---

[2] I adopt this name from [69], and the reasons are explained in Section 2.8.

[3] Both Symmetry and Separation turn out to follow from a slight variant of Separation: for any $s, t \in \Sigma$, $s \neq t$, if and only if there is a $w \in \Sigma$ such that $s \to w$ and $w \not\to t$. This was pointed out to me by Drs. Jort Bergfeld.

Below I will prove that Property A implies both Property AL and Property AH in state spaces (Proposition 2.2.4). It follows that a complete geometric frame is a geometric frame, and a quantum Kripke frame is a quasi-quantum Kripke frame (Corollary 2.2.5). Therefore, my way of naming these structures is consistent. Given this, the relations among these structures can be summarized in the following picture:



In this picture, for example, the arrow from the node 'Quasi-Quantum Kripke Frames' to the node 'Quantum Kripke Frames' labelled by 'Property A' should be read as that quantum Kripke frames are quasi-quantum Kripke frames satisfying Property A.

The main motivation for these structures comes from quantum theory. Given a quantum system, the set of all states of it, equipped with the non-orthogonality relation, can be shown to form a quantum Kripke frame. It follows that every condition listed in Definition 2.1.3 is satisfied in a structure of this kind. (Please refer to Example 2.1.5 and Remark 2.7.2 below for a more detailed explanation.)

Finally, I would like to point out that some of these structures are not new but have been proposed and studied in the literature. Please refer to Section 2.8 for detailed historical notes.

2.1.2 Examples

In this subsection, I give examples of the structures defined in the previous subsection. There are two reasons. The first reason is that these structures are abstract, and concrete examples help to develop intuitions about them. The second reason is that some of these examples show that the inclusions among the classes of Kripke frames, state spaces, geometric frames, complete geometric frames, quasi-quantum Kripke frames and quantum Kripke frames mentioned in the previous subsection are all proper. In these examples, the direct verifications may not be hard but could be tedious, hence some of them are sketched and the others are left to the readers. In some of these examples, the verifications follow in a straightforward way from some sophisticated results proved later, and I will point them out in such situations.

I start with three examples of quantum Kripke frames.

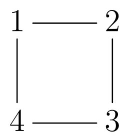
2.1.5. Example. $\mathfrak{F} = (\Sigma(\mathcal{H}), \to_{\mathcal{H}})$ is a quantum Kripke frame, for every Hilbert space $\mathcal{H}$ over the complex numbers $\mathbb{C}$. (For the notations involved, please refer to Subsection 1.2.1, or Corollary 2.5.5 and Theorem B.3.1 in a more general setting.) The direct verification is not hard, but this follows from some general results (Theorem 2.7.1).

According to Subsection 1.2.1, the states of a quantum system are modelled by the elements in $\Sigma(\mathcal{H})$ and the non-orthogonality relation between states are modelled by $\to_{\mathcal{H}}$, for some Hilbert space $\mathcal{H}$ over $\mathbb{C}$. Therefore, the set of all states of a quantum system, equipped with the non-orthogonality relation, forms a quantum Kripke frame.

2.1.6. Example. $\mathfrak{F} = (\Sigma, \to)$ is a finite quantum Kripke frame, when $\Sigma = \{1, 2, 3, 4\}$ and

$$\begin{aligned} \to = \{ \quad & (1,1), (2,2), (3,3), (4,4), \\ & (1,2), (2,3), (3,4), (4,1), \\ & (2,1), (3,2), (4,3), (1,4) \quad \} \end{aligned}$$

It can be depicted as follows, where the self-loops are omitted for simplicity and the arrows are omitted due to Symmetry:



Reflexivity, Symmetry, Separation and Superposition are all easy to verify. To verify Property A, it is crucial to note that there are only 6 bi-orthogonally closed sets in $\mathfrak{F}$: $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and Σ.

It is not hard to see that there is no quantum Kripke frame whose underlying set is of cardinality 2 or 3. Hence the finite quantum Kripke frame in this example is of the simplest kind except for those of the form $(\{s\}, \{(s, s)\})$.

Moreover, this quantum Kripke frame only has orthogonal sets of cardinality at most 2. This is not coincidental. It follows from Theorem 5 in [10] that there is no finite quantum Kripke frame having an orthogonal set of cardinality 3.

The following example is nice, because it gives a quantum Kripke frame which has an orthogonal set of cardinality 3 and can be easily visualized, although it is an infinite one.

2.1.7. EXAMPLE. A hemisphere of radius 1 in the three-dimensional Euclidean space E^3 forms a quantum Kripke frame.

To describe it more precisely, I use a spherical coordinate system with the origin set on the center O of the hemisphere: in a spherical coordinate (r, θ, φ), $r \in [0, +\infty)$ is the radial distance, $\theta \in [0, 2\pi)$ is the azimuthal angle and $\varphi \in [0, \pi]$ is the polar angle. Then $\mathfrak{F} = (\Sigma, \rightarrow)$ is a quantum Kripke frame, when

1. Σ consists of the points in E^3 with spherical coordinates in the set $\{(r, \theta, \varphi) \mid r = 1, \theta \in [0, \pi), \varphi \in [0, \frac{\pi}{2}]\}$;
2. for any $A, B \in \Sigma$, $A \rightarrow B$, if and only if the lines OA and OB are *not* perpendicular in E^3.

It is not hard to verify from the definition that $\mathfrak{F}$ is a quantum Kripke frame. Another way to show this, using some results developed below, is to prove that it is isomorphic[4] to $(\Sigma(\mathbb{R}^3), \rightarrow_{\mathbb{R}^3})$, which is a quantum Kripke frame.

Since $\mathbb{R}^3$ is a vector space over the real numbers $\mathbb{R}$ with the inner product as the orthomodular Hermitian form (Lemma B.2.15), by Corollary 2.5.6 $(\Sigma(\mathbb{R}^3), \rightarrow_{\mathbb{R}^3})$ is a quantum Kripke frame. Then consider $\mathbb{R}^3$ as the Cartesian coordinates of points in E^3 with respect to some Cartesian coordinate system with O as the origin. Then $\Sigma(\mathbb{R}^3)$ can be visualized as the set of lines passing through O, and it is not hard to see that each of these lines intersects with Σ at exactly one point. With this in mind, an isomorphism between $\mathfrak{F}$ and $(\Sigma(\mathbb{R}^3), \rightarrow_{\mathbb{R}^3})$ is not hard to find.

Now I present examples that separate the classes of structures defined above.

2.1.8. EXAMPLE. $\mathfrak{F} = (\Sigma, \rightarrow)$ is a quasi-quantum Kripke frame but not a complete geometric frame, and not a quantum Kripke frame, when $\mathfrak{F} = (\Sigma(l_2^{(0)}), \rightarrow_{l_2^{(0)}})$. Here $l_2^{(0)}$ is the pre-Hilbert space of finitely non-zero sequences of complex numbers, which is not a Hilbert space. (Please refer to Example 2.1.13 in [61].)

Since $l_2^{(0)}$ is a vector space over $\mathbb{C}$ with the inner product as the anisotropic Hermitian form, by Corollary 2.5.5 $\mathfrak{F}$ is a quasi-quantum Kripke frame.

However, since $l_2^{(0)}$ is not complete, by the Piron-Amemiya-Arakia Theorem (Theorem 2.7.3) the inner product is not an orthomodular Hermitian form, so it is not hard to show that $\mathfrak{F}$ does not satisfy Property A. Hence $\mathfrak{F}$ is not a complete geometric frame, and thus is not a quantum Kripke frame.

2.1.9. EXAMPLE. $\mathfrak{F} = (\Sigma, \rightarrow)$ is a complete geometric frame, not a quasi-quantum Kripke frame, and thus not a quantum Kripke frame, when $\Sigma = \{1, 2, 3\}$ and $\rightarrow = \{(1, 1), (2, 2), (3, 3)\}$.

[4]In this thesis, isomorphisms are in the sense of universal algebra, not in the sense of category theory.

According to Lemma 2.7.27, $\mathfrak{F}$ is a classical frame, and thus is a complete geometric frame. However, $\mathfrak{F}$ does not satisfy Superposition by Proposition 2.7.29. Hence $\mathfrak{F}$ is neither a quasi-quantum Kripke frame nor a quantum Kripke frame.

2.1.10. Example. $\mathfrak{F} = (\Sigma, \to)$ is a geometric frame but neither a complete geometric frame nor a quasi-quantum Kripke frame, when

1. $\Sigma = \Sigma(l_2^{(0)}) \cup \{o\}$, where $o \notin \Sigma(l_2^{(0)})$;
2. $\to \,=\, \to_{l_2^{(0)}} \cup \{(o, o)\}$.

According to Example 2.1.8, $(\Sigma(l_2^{(0)}), \to_{l_2^{(0)}})$ is a quasi-quantum Kripke frame where Property A does not hold.

Note that, for every $P \subseteq \Sigma$, the orthocomplement of P in $\mathfrak{F}$ equals to the orthocomplement of $P \setminus \{o\}$ in $(\Sigma(l_2^{(0)}), \to_{l_2^{(0)}})$, if $o \in P$; and it equals to the union of the orthocomplement of P in $(\Sigma(l_2^{(0)}), \to_{l_2^{(0)}})$ and $\{o\}$, if $o \notin P$. It follows that adding o hardly changes the structure of $(\Sigma(l_2^{(0)}), \to_{l_2^{(0)}})$. With this in mind, it is not hard to verify that $\mathfrak{F}$ is a geometric frame where Property A does not hold.

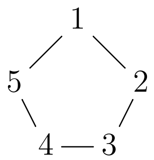
Moreover, $\mathfrak{F}$ does not satisfy Superposition. Take an $s \in \Sigma(l_2^{(0)})$. Then it is easy to see from the definition that there is no $t \in \Sigma$ such that $t \to s$ and $t \to o$.

As a result, $\mathfrak{F}$ is a geometric frame but neither a complete geometric frame nor a quasi-quantum Kripke frame.

2.1.11. Example. $\mathfrak{F} = (\Sigma, \to)$ is a state space but not a geometric frame, when $\Sigma = \{1, 2, 3, 4, 5\}$ and

$$\begin{aligned} \to \,=\, \{ \quad & (1,1), (2,2), (3,3), (4,4), (5,5), \\ & (1,2), (2,3), (3,4), (4,5), (5,1), \\ & (2,1), (3,2), (4,3), (5,4), (1,5) \quad \} \end{aligned}$$

It can be depicted as follows, where the self-loops are omitted for simplicity and the arrows are omitted due to Symmetry:



Reflexivity and Symmetry can be easily verified. For Separation, if $s \not\to t$, then it holds trivially. When $s \to t$, e.g. $s = 3$ and $t = 4$, then $2 \to 3$, $2 \not\to 4$, $5 \to 4$ and $5 \not\to 3$; other cases can be verified similarly.

It is not a geometric frame because Property AH does not hold. On the one hand, $1 \in \Sigma \setminus {\sim}{\sim}\{5\} = \{1, 2, 3, 4\}$. On the other hand, neither 2 nor 3 in ${\sim}\{5\} = \{2, 3\}$ is an approximation of 1 in ${\sim}\{5\}$.

It may be interesting to compare this with Example 2.1.6.

2.1.12. Example. $\mathfrak{F} = (\Sigma, \to)$ is a Kripke frame but not a state space, when $\Sigma = \{1, 2\}$ and $\to = \{(1, 2)\}$, because neither Reflexivity nor Symmetry holds.

2.2 State Spaces

In this section, I investigate the structure inside a state space. I start from some basic properties of orthocomplements, and then proceed to study the structure formed by the bi-orthogonally closed subsets. This section end with the relation among Property A, Property AH and Property AL.

First, I present some elementary properties of orthocomplements.

2.2.1. Proposition. *In every state space* $\mathfrak{F} = (\Sigma, \to)$,

1. $\sim\Sigma = \emptyset$ *and* $\sim\emptyset = \Sigma$, *so both* Σ *and* $\emptyset$ *are bi-orthogonally closed;*
2. $P \subseteq Q$ *implies that* $\sim Q \subseteq \sim P$, *for any* $P, Q \subseteq \Sigma$;
3. $P \subseteq \sim\sim P$, *for every* $P \subseteq \Sigma$;
4. $\sim P$ *is bi-orthogonally closed, for every* $P \subseteq \Sigma$;
5. $P \cap \sim P = \emptyset$, *for every* $P \subseteq \Sigma$.

Proof. *1*, *2*, *3* and *5* follow easily from the definition of state spaces.

For *4*, $P \subseteq \sim\sim P$ by *3*, so $\sim\sim\sim P \subseteq \sim P$ by *2*. Using *3* again, $\sim P \subseteq \sim\sim\sim P$. Hence $\sim P = \sim\sim\sim P$ and $\sim P$ is bi-orthogonally closed. ⊣

2.2.2. Remark. In a state space $\mathfrak{F} = (\Sigma, \to)$, from *2*, *3* and *4* of the above proposition, one can easily deduce that $\sim\sim(\cdot)$ is a *closure operator on* Σ in the sense that, for any $P, Q \subseteq \Sigma$,

(1) $P \subseteq \sim\sim P$;

(2) $P \subseteq Q$ implies that $\sim\sim P \subseteq \sim\sim Q$;

(3) $\sim\sim\sim\sim P = \sim\sim P$.

In the following, I call $\sim\sim P$ *the bi-orthogonal closure of* $P \subseteq \Sigma$.

Second, I study the structure of the set of all bi-orthogonally closed subsets of a state space in some detail.

2.2.3. Proposition. *For a state space* $\mathfrak{F} = (\Sigma, \to)$, *the set* $\mathcal{L}_{\mathfrak{F}}$ *of bi-orthogonally closed subsets of* $\mathfrak{F}$ *forms a complete atomistic ortho-lattice*[5] *with* $\subseteq$ *as the partial order and* $\sim(\cdot)$ *(restricted to* $\mathcal{L}_{\mathfrak{F}}$*) as the orthocomplementation. In particular,*

[5] Relevant notions in lattice theory are reviewed in Appendix A.

1. *for every* $\{P_i \mid i \in I\} \subseteq \mathcal{L}_{\mathfrak{F}}$*,* $\bigcap_{i\in I} P_i$ *is bi-orthogonally closed and is the greatest lower bound, or the meet, of* $\{P_i \mid i \in I\}$*;*
2. *for each* $s \in \Sigma$*,* $\{s\}$ *is bi-orthogonally closed, and is the atom of this lattice;*
3. *for* $\{P_i \mid i \in I\} \subseteq \mathcal{L}_{\mathfrak{F}}$*,* $\bigvee_{i\in I} P_i \stackrel{\text{def}}{=} \bigcap\{Q \in \mathcal{L}_{\mathfrak{F}} \mid P_i \subseteq Q$ *for each* $i \in I\}$ *is bi-orthogonally closed and is the least upper bound, or the join, of* $\{P_i \mid i \in I\}$*;*
4. $P = \bigvee\{\{s\} \in \mathcal{L}_{\mathfrak{F}} \mid \{s\} \subseteq P\}$*, for every* $P \in \mathcal{L}_{\mathfrak{F}}$*.*
5. $\sim\sim P = P$*, for each* $P \in \mathcal{L}_{\mathfrak{F}}$*;*
6. $P \subseteq Q$ *implies that* $\sim Q \subseteq \sim P$*, for any* $P, Q \in \mathcal{L}_{\mathfrak{F}}$*;*
7. $P \cap \sim P = \emptyset$ *and* $P \vee \sim P = \Sigma$*, for each* $P \in \mathcal{L}_{\mathfrak{F}}$*.*
8. *De Morgan's laws hold, i.e.* $\bigcap_{i\in I} \sim P_i = \sim \bigvee_{i\in I} P_i$ *and* $\bigvee_{i\in I} \sim P_i = \sim \bigcap_{i\in I} P_i$*, for every* $\{P_i \mid i \in I\} \subseteq \mathcal{L}_{\mathfrak{F}}$*.*

The lattice is complete in the sense of 1, and it is atomistic in the sense of 4. $\sim(\cdot)$ *is an orthocomplementation in the sense of 5, 6 and 7.*

Proof. *1* to *7* follow from Lemma 5.5 in [69] and *8* follows from *1* to *7* together with Lemma 4.1 in the same paper, although there the complement of $\rightarrow$ in $\Sigma \times \Sigma$, which is irreflexive, is taken as primitive. Direct proofs are not very hard. ⊣

Birkhoff in [21] essentially showed that the bi-orthogonally closed subsets of a Kripke frame satisfying Reflexivity and Symmetry form a complete ortho-lattice. Moore in [69] showed that by adding Separation the lattice will become atomistic. Moreover, he showed that the above proposition can be strengthened to a duality between a category with complete atomistic ortho-lattices as objects and one with state spaces as objects. For more details, please refer to Section 2.8.

Finally, I use the above results to establish the relation among Property A, Property AL and Property AH.

2.2.4. Proposition. *In every state space* $\mathfrak{F} = (\Sigma, \rightarrow)$*, Property A implies both Property AL and Property AH.*

Proof. Assume that Property A holds.

For Property AL, assume that $w \in \Sigma \setminus \sim\{s,t\}$. According to *4* of Proposition 2.2.1, $\sim\sim\{s,t\}$ is bi-orthogonally closed, and $\sim\{s,t\} = \sim\sim\sim\{s,t\}$. Hence $w \notin \sim\sim\sim\{s,t\}$. By Property A there is an approximation of w in $\sim\sim\{s,t\}$.

For Property AH, assume that $w \in \Sigma \setminus \sim\sim\{s\}$. By *4* of Proposition 2.2.1 $\sim\{s\}$ is bi-orthogonally closed. Since $w \notin \sim\sim\{s\}$, by Property A there is an approximation of w in $\sim\{s\}$. ⊣

2.2.5. Corollary. *Every complete geometric frame is a geometric frame, and every quantum Kripke frame is a quasi-quantum Kripke frame.*

Proof. Straightforward from the definitions and the above proposition. ⊣

This corollary justifies my way of naming the structures.

I end this section with a remark on a strengthened version of Property AH.

2.2.6. Remark. Observe that, in a Kripke frame $\mathfrak{F} = (\Sigma, \to)$ satisfying Reflexivity and Symmetry, Property AH and Separation are equivalent to the following:

> **Strong Existence of Approximation for Hyperplanes (AH'):** For each $s \in \Sigma$, if $w \in \Sigma \setminus \{s\}$, then there is a $w' \in \Sigma$ which is an approximation of w in $\sim\{s\}$, i.e. $w' \in \sim\{s\}$ and $w \approx_{\sim\{s\}} w'$.

For one direction, assume that Property AH and Separation hold. Then $\mathfrak{F} = (\Sigma, \to)$ is a state space. According to *2* of Proposition 2.2.3, $w \in \Sigma \setminus \{s\}$ implies that $w \in \Sigma \setminus \sim\sim\{s\}$. Hence Property AH implies that there is an approximation of w in $\sim\{s\}$.

For the other direction, assume that Property AH' holds. First I show that Property AH holds. Since $\{s\} \subseteq \sim\sim\{s\}$ holds by Reflexivity and Symmetry, $w \in \Sigma \setminus \sim\sim\{s\}$ implies that $w \in \Sigma \setminus \{s\}$. Hence Property AH' implies that there is an approximation of w in $\sim\{s\}$. Second I show that Separation holds. Suppose that $s, t \in \Sigma$ are such that $s \neq t$. According to Property AH', there is an $s' \in \Sigma$ which is an approximation of s in $\sim\{t\}$, i.e. $s' \in \sim\{t\}$ and $s \approx_{\sim\{t\}} s'$. Then $s' \not\to t$ follows from $s' \in \sim\{t\}$. Moreover, $s' \to s$ follows from $s \approx_{\sim\{t\}} s'$, $s' \in \sim\{t\}$ and $s' \to s'$. Therefore, s' has the required property.

In the following, I mainly discuss the Kripke frames that are state spaces, and hence I may use Property AH' more often than Property AH, for the antecedent of Property AH' is simpler.

2.3 Geometric Frames

In this section, I investigate the structure inside a geometric frame. The main result is a correspondence[6] between geometric frames and projective geometries with pure polarities. Based on this correspondence, notions and results in projective geometry can be imported into the study of geometric frames and yield useful results. The definitions and results in projective geometry used in this thesis are reviewed in Appendix B.1.

[6] In this thesis, a correspondence between two classes C and D means a pair of class functions $\mathbf{S} : C \to D$ and $\mathbf{T} : D \to C$ such that $\mathbf{T} \circ \mathbf{S}(c) \cong c$ and $\mathbf{S} \circ \mathbf{T}(d) \cong d$, for any $c \in C$ and $d \in D$.

2.3.1 From Geometric Frames to Projective Geometries

In this subsection, I show that every geometric frame can be organised as a projective geometry with a pure polarity. For convenience, I fix an arbitrary geometric frame $\mathfrak{F} = (\Sigma, \to)$ throughout this subsection.

Define a tuple $\mathbf{G}'(\mathfrak{F}) = (\Sigma, \star, p)$ as follows: First, $\star : \Sigma \times \Sigma \to \wp(\Sigma)$ is a function such that $s \star t \stackrel{\text{def}}{=} {\sim}{\sim}\{s, t\}$ for any $s, t \in \Sigma$. Intuitively, one can think of $s \star t$ as the unique line passing through two points s and t, and, in the degenerate case when $s = t$, $s \star t$ is the point itself: this intuition will be justified formally below (Remark 2.3.8). Note that by definition $s \star t = t \star s$ for any $s, t \in \Sigma$. Second, $p : \Sigma \to \wp(\Sigma)$ is a function such that $p(s) \stackrel{\text{def}}{=} {\sim}\{s\}$ for every $s \in \Sigma$.

I will show that $\mathbf{G}'(\mathfrak{F})$ is a projective geometry with a pure polarity by verifying the conditions in the definition one by one, and this will be presented as propositions. I will also prove some useful lemmas in the meantime.

As a start, note that Σ is a non-empty set by definition.

2.3.1. Proposition. $\mathbf{G}'(\mathfrak{F})$ *satisfies (P1), i.e.* $s \star s = \{s\}$, *for every* $s \in \Sigma$.

Proof. By definition and Proposition 2.2.3 $s \star s = {\sim}{\sim}\{s, s\} = {\sim}{\sim}\{s\} = \{s\}$. ⊣

2.3.2. Proposition. $\mathbf{G}'(\mathfrak{F})$ *satisfies (P2), i.e.* $s \in t \star s$, *for every* $s, t \in \Sigma$.

Proof. For each $w \in {\sim}\{t, s\}$, $s \not\to w$ by definition. Hence $s \in {\sim}{\sim}\{t, s\} = t \star s$. ⊣

Before continuing to (P3), I prove some useful lemmas. The following lemma intuitively says that, for every $u \in \Sigma$, ${\sim}\{u\}$ intersects with every line.

2.3.3. Lemma. *For any* $s, t, u \in \Sigma$, *if* $s \neq t$, *there is a* $v \in s \star t$ *such that* $u \not\to v$.

Proof. If $u \in {\sim}\{s, t\}$, taking v to be t will work, since $u \not\to t$ and $t \in s \star t$ by Proposition 2.3.2. In the following, I focus on the case when $u \notin {\sim}\{s, t\}$.

By Property AL there is a $u' \in {\sim}{\sim}\{s, t\}$ such that $u \approx_{{\sim}{\sim}\{s,t\}} u'$. Since $s \neq t$, $u' \neq s$ or $u' \neq t$. Without loss of generality, assume that $u' \neq s$. Then by Separation there is a $w \in \Sigma$ such that $w \to s$ and $w \not\to u'$. Since $w \to s$, $w \notin {\sim}\{s, t\}$. Hence by Property AL there is a $v \in {\sim}{\sim}\{s, t\}$ such that $w \approx_{{\sim}{\sim}\{s,t\}} v$.

I claim that this v has the required property. In fact, by the construction $v \in {\sim}{\sim}\{s, t\} = s \star t$. Moreover, $u' \in {\sim}{\sim}\{s, t\}$ and $v \approx_{{\sim}{\sim}\{s,t\}} w$ together with $w \not\to u'$ imply that $v \not\to u'$. Then, combining with $v \in {\sim}{\sim}\{s, t\}$ and $u \approx_{{\sim}{\sim}\{s,t\}} u'$, $v \not\to u$ is implied. Therefore, v has the required property. ⊣

The following lemma and its corollaries justify the intuition that $s \star t$ is a line when $s \neq t$ in the sense that $s \star t$ is determined by two distinct elements in Σ.

2.3.4. Lemma. *For any* $s, t, w \in \Sigma$, *if* $w \neq t$ *and* $w \in s \star t$, *then* $s \star t = w \star t$.

Proof. First, observe that $s \neq t$; otherwise, from $w \in s \star t = \{t\}$, $w = t$ can be derived, which contradicts $w \neq t$.

Second, I prove from $s \neq t$ that there is an $s' \in {\sim}\{t\}$ such that ${\sim}\{s,t\} = {\sim}\{s',t\}$. Since $s \neq t$, by Property AH' there is an $s' \in {\sim}\{t\}$ such that $s \approx_{\sim\{t\}} s'$. Then, for every $u \in \Sigma$,

$$\begin{aligned} u \in {\sim}\{s,t\} &\Leftrightarrow u \not\to s \text{ and } u \not\to t \\ &\Leftrightarrow u \not\to s' \text{ and } u \not\to t \\ &\Leftrightarrow u \in {\sim}\{s',t\} \end{aligned}$$

Hence ${\sim}\{s,t\} = {\sim}\{s',t\}$.

Similarly, from $w \neq t$ one can find a $w' \in {\sim}\{t\}$ such that $w \approx_{\sim\{t\}} w'$, and thus ${\sim}\{w,t\} = {\sim}\{w',t\}$.

I claim that $w' = s'$. Suppose (towards a contradiction) that $w' \neq s'$. By Separation there is a $v \in \Sigma$ such that $v \to w'$ and $v \not\to s'$. Since $v \to w'$ and $w' \not\to t$, $v \neq t$. By Property AH' there is a $v' \in {\sim}\{t\}$ such that $v \approx_{\sim\{t\}} v'$. Now, on the one hand, since $w', s' \in {\sim}\{t\}$ and $v \approx_{\sim\{t\}} v'$, one can deduce that $v' \to w'$ and $v' \not\to s'$. On the other hand, since $v' \to w'$ and $v' \in {\sim}\{t\}$, by $w \approx_{\sim\{t\}} w'$ I get $v' \to w$. Since $w \in s \star t = {\sim}{\sim}\{s,t\}$, $v' \notin {\sim}\{s,t\}$, i.e. $v' \to s$ or $v' \to t$. Then $v' \to s$ follows, for $v' \not\to t$. Since again $v' \in {\sim}\{t\}$, by $s \approx_{\sim\{t\}} s'$ I get $v' \to s'$, contradicting that $v' \not\to s'$ which is proved just before. Therefore, $s' = w'$.

As a result, $s \star t = {\sim}{\sim}\{s,t\} = {\sim}{\sim}\{s',t\} = {\sim}{\sim}\{w',t\} = {\sim}{\sim}\{w,t\} = w \star t$. ⊣

2.3.5. Corollary. *For any $s, t, w \in \Sigma$, if $w \neq t$ and $w \in s \star t$, then $s \in w \star t$.*

Proof. $s \in t \star s = s \star t = w \star t$ by Proposition 2.3.2 and the previous lemma. ⊣

2.3.6. Corollary. *For any $s, t, u, v \in \Sigma$, if $u \neq v$ and $u, v \in s \star t$, then $u \star v = s \star t$.*

Proof. By the assumption $s \star t$ is not a singleton, and thus $s \neq t$ by Proposition 2.3.1. Then $u \neq s$ or $u \neq t$. Without loss of generality, assume that $u \neq s$. Then it follows from $u \neq s$ and $u \in s \star t$ that $u \star s = s \star t$, according to Lemma 2.3.4. Again by Lemma 2.3.4 it follows from $u \neq v$ and $v \in s \star t = u \star s$ that $u \star v = u \star s$. Therefore, $u \star v = u \star s = s \star t$. ⊣

2.3.7. Proposition. $\mathbf{G}'(\mathfrak{F})$ *satisfies (P3), i.e. $s \in t \star r$, $r \in x \star y$ and $s \neq x$ imply that $(s \star x) \cap (t \star y) \neq \emptyset$, for all $s, t, x, y, r \in \Sigma$.*

Proof.[7] Three cases need to be considered.

Case 1: $r = y$. In this case, $s \in t \star r = t \star y$. Hence $s \in (s \star x) \cap (t \star y)$, so $(s \star x) \cap (t \star y) \neq \emptyset$.

[7] This proof is inspired by the proof of the Theorem of Buekenhout and Parmentier in [25]. In principle, I can prove this proposition by introducing their terminologies and applying their theorem. However, since it is not long, a direct proof may be more helpful in developing intuitions.

Case 2: $s \star x \subseteq t \star y$. In this case, $x \in s \star x = (s \star x) \cap (t \star y)$ and thus $(s \star x) \cap (t \star y) \neq \emptyset$.

Case 3: $r \neq y$ and $s \star x \not\subseteq t \star y$. In this case $\sim\{t, y\} \not\subseteq \sim\{s, x\}$ by Proposition 2.2.1. Then there is a $u \in \sim\{t, y\}$ such that $u \notin \sim\{s, x\}$. Since $s \neq x$, by Lemma 2.3.3 there is a $v \in s \star x$ such that $v \not\to u$. I claim that $v \in t \star y$. Under this claim, $v \in (s \star x) \cap (t \star y)$, and thus $(s \star x) \cap (t \star y) \neq \emptyset$.

It remains to prove the claim that $v \in t \star y$, i.e. $v \in \sim\sim\{t, y\}$.

Let $w \in \sim\{t, y\}$ be arbitrary. To establish the claim, it suffices to show that $v \not\to w$. If $w = u$, then one can deduce that $v \not\to w$ by the construction of v. Now it remains to deal with the case when $w \neq u$.

First observe that there is a $z \in w \star u$ such that $z \in \sim\{t, y, r\}$. Since $w \neq u$, by Lemma 2.3.3 there is a $z \in w \star u$ such that $z \not\to r$. Since $w, u \in \sim\{t, y\}$, it is not hard to show that $w \star u \subseteq \sim\{t, y\}$. Hence $z \in w \star u \subseteq \sim\{t, y\}$. Combining this with $z \not\to r$, $z \in \sim\{t, y\} \cap \sim\{r\} = \sim\{t, y, r\}$.

Second observe that $v \not\to z$. Since $s \in t \star r$, $z \not\to s$. Since $r \in x \star y$ and $r \neq y$, $x \in r \star y$ by Corollary 2.3.5. Hence $z \not\to x$. I have shown that $z \in \sim\{s, x\}$. Since $v \in s \star x$, $v \not\to z$.

Now I am ready to show that $v \not\to w$. Since $z \in \sim\{s, x\}$ and $u \notin \sim\{s, x\}$, $z \neq u$. Since $z \in w \star u$, by Corollary 2.3.5 $w \in z \star u$. Remembering that $v \in \sim\{z, u\}$, $v \not\to w$. ⊣

2.3.8. Remark. By now I have proved that $(\Sigma, \star)$ satisfies (P1), (P2) and (P3), so it is a projective geometry. Hence it is justified to think of $s \star t = \sim\sim\{s, t\}$ as the line passing through s and t whenever $s \neq t$.

Now I continue to show that the function p is a pure polarity on $(\Sigma, \star)$.

2.3.9. Proposition. *For every $s \in \Sigma$, $p(s) = \sim\{s\}$ is a hyperplane of $(\Sigma, \star)$.*

Proof. First show that $\sim\{s\}$ is a subspace. Assume that $u, v \in \sim\{s\}$. Then it is not hard to derive from Proposition 2.2.1 that $u \star v = \sim\sim\{u, v\} \subseteq \sim\{s\}$.

Second, to see that $\sim\{s\} \neq \Sigma$, note that $s \notin \sim\{s\}$ by Reflexivity.

Finally, for maximality, assume that P is a subspace of $(\Sigma, \star)$ such that $\sim\{s\} \subseteq P$ and there is an $x \in P \backslash \sim\{s\}$. To prove that $\sim\{s\}$ is a hyperplane, by definition it suffices to show that $\Sigma \subseteq P$. Let $u \in \Sigma$ be arbitrary. If $u = x$, then $u \in P$ because $x \in P$. In the following, I focus on the case when $u \neq x$. By Lemma 2.3.3 there is a $v \in u \star x$ such that $v \not\to s$. Since $x \notin \sim\{s\}$, $x \neq v$. By Corollary 2.3.5 $u \in x \star v$. From $v \in \sim\{s\} \subseteq P$ and $x \in P$, one can deduce that $x \star v \subseteq P$ for P is a subspace. Therefore, $u \in P$. For u is arbitrary, $\Sigma \subseteq P$. ⊣

2.3.10. Proposition.

1. *$s \in p(t) \Leftrightarrow t \in p(s)$, for any $s, t \in \Sigma$.*
2. *$s \notin p(s)$, for every $s \in \Sigma$.*

Proof. These follow from the definition of p, Reflexivity and Symmetry. ⊣

Finally, I arrive at the conclusion of this subsection:

2.3.11. THEOREM. *For every geometric frame $\mathfrak{F} = (\Sigma, \to)$, $\mathbf{G}'(\mathfrak{F}) = (\Sigma, \star, p)$ defined as above is a projective geometry with a pure polarity.*

Moreover, $\mathfrak{F} \mapsto \mathbf{G}'(\mathfrak{F})$ is a class function, denoted by $\mathbf{G}'$, from the class of geometric frames to the class of projective geometries with pure polarities.

Proof. This follows immediately from the definition and all propositions proved in this subsection. ⊣

2.3.2 From Projective Geometries to Geometric Frames

In this subsection, I show that every projective geometry with a pure polarity can be organised as a geometric frame. For convenience, I fix an arbitrary projective geometry with a pure polarity $\mathcal{G} = (G, \star, p)$ throughout this subsection.

Define a tuple $\mathbf{F}'(\mathcal{G}) = (G, \to)$, where $\to \subseteq G \times G$ is such that, for any $a, b \in G$, $a \to b \Leftrightarrow a \notin p(b)$. Denote the orthocomplement operator with respect to $\to$ by $\sim(\cdot)$. I will show that $\mathbf{F}'(\mathcal{G})$ is a geometric frame by verifying the conditions in the definition one by one. (Given Remark 2.2.6, I will deal with Property AH' instead of Separation and Property AH.)

As a start, note that G is non-empty by definition.

2.3.12. PROPOSITION. *$\mathbf{F}'(\mathcal{G})$ satisfies Reflexivity and Symmetry.*

Proof. Reflexivity holds because $a \notin p(a)$ for every $a \in G$. Symmetry holds because $a \notin p(b) \Leftrightarrow b \notin p(a)$ for any $a, b \in G$. ⊣

I continue with a characterization of $\star$ and p in terms of $\sim(\cdot)$.

2.3.13. LEMMA. *For any $a, b \in G$, $p(a) = \sim\{a\}$ and $a \star b = \sim\sim\{a, b\}$.*

Proof. $p(a) = \sim\{a\}$ is obvious from the definition of $\sim(\cdot)$.

For $a \star b = \sim\sim\{a, b\}$, observe that by Proposition B.1.16, for every $c \in G$,

$$\begin{aligned} c \in a \star b &\Leftrightarrow p(a) \cap p(b) \subseteq p(c) \\ &\Leftrightarrow x \in p(a) \text{ and } x \in p(b) \text{ imply that } x \in p(c), \text{ for every } x \in G \\ &\Leftrightarrow x \not\to a \text{ and } x \not\to b \text{ imply that } x \not\to c, \text{ for every } x \in G \\ &\Leftrightarrow x \in \sim\{a, b\} \text{ implies that } x \not\to c, \text{ for every } x \in G \\ &\Leftrightarrow c \in \sim\sim\{a, b\} \end{aligned}$$

Therefore, $a \star b = \sim\sim\{a, b\}$. ⊣

2.3.14. Proposition. *$\mathbf{F}'(\mathcal{G})$ satisfies Property AH'.*

Proof. Assume that $a, b \in G$ are such that $a \neq b$. It is required to show that there is a $c \in {\sim}\{b\}$ such that $a \approx_{\sim\{b\}} c$. By the previous lemma it suffices to find a $c \in p(b)$ such that $x \in p(a) \Leftrightarrow x \in p(c)$ for every $x \in p(b)$.

Observe that $(a \star b) \cap p(b)$ is a singleton. Since $a \neq b$, $(a \star b) \cap p(b)$ is either $a \star b$ or a singleton, according to Theorem B.1.4. Since p is pure, $b \notin p(b)$, so $(a \star b) \cap p(b) \neq a \star b$. Therefore, $(a \star b) \cap p(b)$ is a singleton. Denote by c the unique element in this singleton.

I claim that this c has the required property. On the one hand, by construction $c \in a \star b$, so by *1* of Proposition B.1.16

$$p(a) \cap p(b) \subseteq p(c) \tag{2.1}$$

On the other hand, since $c \in p(b)$ and $b \notin p(b)$, $b \neq c$. Since $c \in a \star b$, by (P4) of Lemma B.1.2 $a \in b \star c$. Then by *1* of Proposition B.1.16

$$p(c) \cap p(b) \subseteq p(a) \tag{2.2}$$

The required property of c follows easily from (2.1) and (2.2). ⊣

2.3.15. Proposition. *$\mathbf{F}'(\mathcal{G})$ satisfies (AL).*

Proof. Assume that $c, a, b \in G$ are such that $c \notin {\sim}\{a, b\}$. It is required to show that there is a $d \in {\sim\sim}\{a, b\}$ such that $c \approx_{\sim\sim\{a,b\}} d$. By Lemma 2.3.13 it suffices to find a $d \in a \star b$ such that $c \in p(x) \Leftrightarrow d \in p(x)$ for every $x \in a \star b$.

As a start, note that, when $a = b$, $a \in a \star b$ has the required property. In the following, I focus on the case when $a \neq b$.

First, observe that $(a \star b) \cap p(c)$ is a singleton. Since $a \neq b$, $(a \star b) \cap p(c)$ is either $a \star b$ or a singleton, according to Theorem B.1.4. Since $c \notin {\sim}\{a, b\}$, $c \notin p(a)$ or $c \notin p(b)$, and thus $a \notin p(c)$ or $b \notin p(c)$. Hence $(a \star b) \cap p(c) \neq a \star b$. Therefore, $(a \star b) \cap p(c)$ is a singleton. Denote by s the unique element in this set.

Second, observe that $(a \star b) \cap p(s)$ is a singleton. Since $a \neq b$, $(a \star b) \cap p(s)$ is either $a \star b$ or a singleton, according to Theorem B.1.4. Since by construction $s \in a \star b$, $s \in {\sim\sim}\{a, b\}$ by Lemma 2.3.13. Since p is pure, it is not hard to see that $s \notin {\sim}\{a, b\}$. Hence $s \notin p(a)$ or $s \notin p(b)$, and thus $a \notin p(s)$ or $b \notin p(s)$. Hence $(a \star b) \cap p(s) \neq a \star b$. Therefore, $(a \star b) \cap p(s)$ is a singleton. Denote by d the unique element in this set.

Now I show that this d has the required property. Let $x \in a \star b$ be arbitrary. First assume that $c \in p(x)$. Then $x \in p(c)$, and thus $x \in (a \star b) \cap p(c)$. According to the construction of s, $x = s$. Hence $d \in p(s) = p(x)$. Second assume that $d \in p(x)$. I claim that $x = s$. Suppose (towards a contradiction) that $s \neq x$. Since $s, x \in a \star b$, (P8) in Lemma B.1.2 implies that $a \star b = s \star x$. Since $p(d)$ is a hyperplane and $s, x \in p(d)$, $a \star b = s \star x \subseteq p(d)$. Then $d \in a \star b \subseteq p(d)$, contradicting that p is pure. Therefore, $x = s$. Since $s \in p(c)$, $c \in p(s) = p(x)$. As a result, d has the required property. ⊣

Finally, I come to the main conclusion of this subsection:

2.3.16. THEOREM. *For a projective geometry with a pure polarity $\mathcal{G} = (G, \star, p)$, $\mathbf{F}'(\mathcal{G}) = (G, \rightarrow)$ defined as above is a geometric frame.*

Moreover, $\mathcal{G} \mapsto \mathbf{F}'(\mathcal{G})$ is a class function, denoted by $\mathbf{F}'$, from the class of projective geometries with pure polarities to the class of geometric frames.

Proof. This follows immediately from the definition and all propositions proved in this subsection. ⊣

2.3.3 Correspondence

In this subsection, I strengthen the results in the previous two subsections to a correspondence between geometric frames and projective geometries with pure polarities. Based on this, I import some notions and techniques from projective geometry and obtain some useful results about geometric frames.

The correspondence between geometric frames and projective geometries with pure polarities goes as follows:

2.3.17. THEOREM.

1. *For every geometric frame $\mathfrak{F}$, $\mathbf{F}' \circ \mathbf{G}'(\mathfrak{F}) = \mathfrak{F}$.*

2. *For every projective geometry with a pure polarity $\mathcal{G}$, $\mathbf{G}' \circ \mathbf{F}'(\mathcal{G}) = \mathcal{G}$.*

Proof. **For *1*:** Let $\mathfrak{F} = (\Sigma, \rightarrow)$ be an arbitrary geometric frame. Let $\mathbf{G}'(\mathfrak{F}) = (\Sigma, \star, p)$, as is defined in Theorem 2.3.11, and $\mathbf{F}' \circ \mathbf{G}'(\mathfrak{F}) = (\Sigma, \rightarrowtail)$, as is defined in Theorem 2.3.16. Then by the relevant definitions, for any $s, t \in \Sigma$,

$$s \rightarrowtail t \Leftrightarrow s \notin p(t) \Leftrightarrow s \notin {\sim}\{t\} \Leftrightarrow s \rightarrow t.$$

Therefore, $\mathbf{F}' \circ \mathbf{G}'(\mathfrak{F}) = (\Sigma, \rightarrowtail) = (\Sigma, \rightarrow) = \mathfrak{F}$.

For *2*: Let $\mathcal{G} = (G, \star, p)$ be an arbitrary projective geometry with a pure polarity. Let $\mathbf{F}'(\mathcal{G}) = (G, \rightarrow)$, as is defined in Theorem 2.3.16, and $\mathbf{G}' \circ \mathbf{F}'(\mathcal{G}) = (G, \circledast, p')$, as is defined in Theorem 2.3.11. It is easy to see from relevant definitions and Lemma 2.3.13 that, for any $a, b \in G$,

$$a \circledast b = {\sim}{\sim}\{a, b\} = a \star b \qquad a \in p'(b) \Leftrightarrow a \not\rightarrow b \Leftrightarrow a \in p(b).$$

Therefore, $\mathbf{G}' \circ \mathbf{F}'(\mathcal{G}) = (G, \circledast, p') = (G, \star, p) = \mathcal{G}$. ⊣

Another correspondence can be drawn from this theorem.

2.3.18. Corollary.

1. *For every geometric frame $\mathfrak{F} = (\Sigma, \to)$, define a function $\star : \Sigma \times \Sigma \to \Sigma$ such that $s \star t \stackrel{\text{def}}{=} \sim\sim\{s,t\}$ for any $s,t \in \Sigma$. Then the tuple $\mathbf{G}(\mathfrak{F}) = (\Sigma, \star, \not\to)$ is a pure orthogeometry.*

2. *For every pure orthogeometry $\mathcal{G} = (G, \star, \perp)$, the tuple $\mathbf{F}(\mathcal{G}) = (G, \not\perp)$ is a geometric frame.*

3. *$\mathfrak{F} \mapsto \mathbf{G}(\mathfrak{F})$ is a class function, denoted by $\mathbf{G}$, from the class of geometric frames to the class of pure orthogeometries.*

   $\mathcal{G} \mapsto \mathbf{F}(\mathcal{G})$ is a class function, denoted by $\mathbf{F}$, from the class of pure orthogeometries to the class of geometric frames.

   Moreover, for any geometric frame $\mathfrak{F}$ and pure orthogeometry $\mathcal{G}$,

$$\mathbf{F} \circ \mathbf{G}(\mathfrak{F}) = \mathfrak{F} \qquad\qquad \mathbf{G} \circ \mathbf{F}(\mathcal{G}) = \mathcal{G}$$

Proof. This follows from Theorem 2.3.17 and Theorem B.1.19. ⊣

The theorem and its corollary are significant in two ways. On the one hand, they mean that projective geometries with pure polarities and pure orthogeometries are Kripke frames in disguise. This may not be a surprise, given Lemma 2.3.13; but the definition of geometric frames is still much simpler than the result of just replacing the occurrences of $\star$ and p by $\sim(\cdot)$ in the definition of projective geometries with pure polarities. On the other hand, the theorem and its corollary mean that geometric frames have nice geometric structures. Therefore, notions and results in projective geometry can be introduced into the study of geometric frames. The most useful and relevant ones are reviewed in Appendix B.1. Among them, I emphasize the following:

2.3.19. Definition. Let $\mathfrak{F} = (\Sigma, \to)$ be a geometric frame.

- For any $s,t \in \Sigma$, the *line generated by s and t*, denoted by $s \star t$, is $\sim\sim\{s,t\}$.
- $P \subseteq \Sigma$ is a *subspace of* $\mathfrak{F}$, if $s \star t \subseteq P$ for any $s,t \in P$.
- A *hyperplane* H of $\mathfrak{F}$ is a subspace of $\mathfrak{F}$ satisfying both of the following:
  1. H is proper, i.e. $H \neq \Sigma$;
  2. H is maximal, i.e. $H \subseteq P$ implies that $P = H$ or $P = \Sigma$ for every subspace P of $\mathfrak{F}$.
- $\mathcal{C}(P) \stackrel{\text{def}}{=} \bigcap\{Q \subseteq \Sigma \mid Q \text{ is a subspace of } \mathfrak{F} \text{ and } P \subseteq Q\}$ is called the *linear closure of* $P \subseteq \Sigma$.

According to Remark 2.2.2 and Lemma B.1.6, both the bi-orthogonal closure and the linear closure are closure operators on a geometric frame. In the remaining part of this subsection, I collect some useful facts relating them.

2.3.20. LEMMA. *In a geometric frame $\mathfrak{F} = (\Sigma, \rightarrow)$, if $P \subseteq \Sigma$ is bi-orthogonally closed, it is a subspace of $\mathfrak{F}$.*

Proof. It suffices to show that, if $s, t \in P$, $\sim\sim\{s, t\} \subseteq P$. Assume that $s, t \in P$. Then $\{s, t\} \subseteq P$. Applying *2* of Proposition 2.2.1 twice, one can obtain $\sim\sim\{s, t\} \subseteq \sim\sim P$. Since P is bi-orthogonally closed, $\sim\sim\{s, t\} \subseteq \sim\sim P = P$. ⊣

2.3.21. LEMMA. *In a geometric frame $\mathfrak{F} = (\Sigma, \rightarrow)$, $\sim Q = \sim\mathcal{C}(Q)$, for every $Q \subseteq \Sigma$.*

Proof. By definition $Q \subseteq \mathcal{C}(Q)$, so $\sim\mathcal{C}(Q) \subseteq \sim Q$ by *2* of Proposition 2.2.1. It remains to show that $\sim Q \subseteq \sim\mathcal{C}(Q)$.

I define a sequence of sets $\{Q_i\}_{i \in \mathbb{N}}$ in the same way as in Proposition B.1.7. Then by the proposition $\mathcal{C}(Q) = \bigcup_{i \in \mathbb{N}} Q_i$. It is easy to see from the definition that $\sim\mathcal{C}(Q) = \sim\bigcup_{i \in \mathbb{N}} Q_i = \bigcap_{i \in \mathbb{N}} \sim Q_i$. I prove $\sim Q \subseteq \bigcap_{i \in \mathbb{N}} \sim Q_i = \sim\mathcal{C}(Q)$ by showing that $\sim Q \subseteq \sim Q_i$, for every $i \in \mathbb{N}$. Use induction on i.

Base Step: $i = 0$. $\sim Q \subseteq \sim Q = \sim Q_0$ obviously holds.

Induction Step: $i = n + 1$. Let $s \in \sim Q$ and $t \in Q_{n+1}$ be arbitrary. By definition there are $u, v \in Q_n$ such that $t \in u \star v$. By the induction hypothesis $s \in \sim Q \subseteq \sim Q_n$. Hence $s \not\rightarrow u$ and $s \not\rightarrow v$, i.e. $s \in \sim\{u, v\}$. Since $t \in u \star v$, $s \not\rightarrow t$. For t is arbitrary, $s \in \sim Q_{n+1}$. Therefore, $\sim Q \subseteq \sim Q_{n+1}$. ⊣

The following lemma suggests a way to get a bigger bi-orthogonally closed set from a smaller one using linear closures.

2.3.22. LEMMA. *In a geometric frame $\mathfrak{F} = (\Sigma, \rightarrow)$, let $s \in \Sigma$ and $P \subseteq \Sigma$ be bi-orthogonally closed. Then $\mathcal{C}(\{s\} \cup P)$ is bi-orthogonally closed.*

Proof. Note that being bi-orthogonally closed in a geometric frame $\mathfrak{F}$ means the same as being a closed subspace in the pure orthogeometry $\mathbf{G}(\mathfrak{F})$ in the sense of Proposition 14.2.5 in [39]. Since P is bi-orthogonally closed, so is $\mathcal{C}(\{s\} \cup P)$ by this proposition. ⊣

In the following proposition I show that in a geometric frame for a finite set the linear closure coincides with the bi-orthogonal closure.

2.3.23. PROPOSITION. *In a geometric frame $\mathfrak{F} = (\Sigma, \rightarrow)$, for any $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$, $\mathcal{C}(\{s_1, ..., s_n\}) = \sim\sim\{s_1, ..., s_n\}$.*

Proof. I prove by induction that, for every $n \in \mathbb{N}$, $\mathcal{C}(\{s_1, ..., s_n\})$ is bi-orthogonally closed.

Base Step: $n = 0$. By convention $\{s_1, ..., s_n\}$ is the empty set, so $\mathcal{C}(\emptyset) = \emptyset$ is bi-orthogonally closed by Proposition 2.2.1.

Induction Step: $n = k + 1$. By the induction hypothesis $\mathcal{C}(\{s_1, ..., s_k\})$ is bi-orthogonally closed. Then by the above lemma $\mathcal{C}(\{s_{k+1}\} \cup \mathcal{C}(\{s_1, ..., s_k\}))$ is bi-orthogonally closed. By Corollary B.1.9 $\mathcal{C}(\{s_{k+1}\} \cup \mathcal{C}(\{s_1, ..., s_k\})) = \mathcal{C}(\{s_{k+1}\} \cup \{s_1, ..., s_k\}) = \mathcal{C}(\{s_1, ..., s_k, s_{k+1}\})$. Hence $\mathcal{C}(\{s_1, ..., s_k, s_{k+1}\})$ is bi-orthogonally closed. This finishes the proof by induction.

Now by Lemma 2.3.21 $\sim\sim\{s_1, ..., s_n\} = \sim\sim\mathcal{C}(\{s_1, ..., s_n\})$. As $\mathcal{C}(\{s_1, ..., s_n\})$ is bi-orthogonally closed, $\sim\sim\{s_1, ..., s_n\} = \mathcal{C}(\{s_1, ..., s_n\})$. ⊣

2.4 Complete Geometric Frames

In this section, I investigate the structure inside a complete geometric frame. I start with introducing saturated sets and studying their properties in geometric frames. Then I prove a correspondence between complete geometric frames and Hilbertian geometries. Finally I study the properties of the subsets which are the bi-orthogonal closures of finite sets and the consequences of finite-dimensionality.

2.4.1 Saturated Sets in Geometric Frames

In this subsection, I introduce saturated sets and study their properties in geometric frames. I start with the definition of saturated sets.

2.4.1. Definition. In a Kripke frame $\mathfrak{F} = (\Sigma, \to)$, $P \subseteq \Sigma$ is *saturated*, if every $s \in \Sigma \setminus \sim P$ has an approximation in P, i.e. an $s' \in P$ satisfying $s \approx_P s'$.

Note that in this terminology Property AL, Property AH and Property A say that every set of the form $\sim\sim\{s, t\}$, of the form $\sim\{s\}$ or being bi-orthogonally closed, respectively, is saturated.

For convenience, I fix a geometric frame $\mathfrak{F} = (\Sigma, \to)$ until the end of this subsection. The following proposition establishes that approximations in subspaces are unique, if they exist.

2.4.2. Proposition. *Let $s \in \Sigma$ and $P \subseteq \Sigma$ be a subspace. Then s has a unique approximation in P, i.e. if $t, t' \in P$ are such that $s \approx_P t$ and $s \approx_P t'$, $t = t'$.*

Proof. Suppose (towards a contradiction) that $t \neq t'$.

Observe that there is a $w \in t \star t'$ such that $w \to t'$ and $w \not\to t$. Since $t' \neq t$, by Lemma 2.3.3 there is a $w \in t \star t'$ such that $w \not\to t$. Since $w \in t \star t' = \sim\sim\{t, t'\}$ and $w \to w$ by Reflexivity, $w \notin \sim\{t, t'\}$. Hence $w \to t'$.

Now since $t, t' \in P$ and P is a subspace, $t \star t' \subseteq P$. It follows from $w \in t \star t'$ that $w \in P$. On the one hand, since $w \to t'$ and $s \approx_P t'$, $s \to w$. On the other hand, since $w \not\to t$ and $s \approx_P t$, $s \not\to w$. Hence a contradiction is derived. Therefore, $t = t'$. ⊣

Next I introduce the notion of orthogonal decompositions, which generalizes a notion in the theory of Hilbert spaces with the same name. This notion will be very useful in studying saturated sets.

2.4.3. Definition. An *orthogonal decomposition of* $s \in \Sigma$ *with respect to* $P \subseteq \Sigma$ is a pair $(s_\parallel, s_\perp) \in P \times {\sim}P$ such that $s \in {\sim}{\sim}\{s_\parallel, s_\perp\}$.

$P \subseteq \Sigma$ *admits orthogonal decomposition*, if every $s \in \Sigma \setminus (P \cup {\sim}P)$ has an orthogonal decomposition with respect to P.

The following proposition is a basic fact about the relation among saturated sets, bi-orthogonally closed sets and sets admitting orthogonal decomposition.

2.4.4. Proposition. *For every* $P \subseteq \Sigma$,

1. *if* P *admits orthogonal decomposition,* P *is bi-orthogonally closed;*
2. *if* P *is saturated,* P *admits orthogonal decomposition.*

Proof. **For 1:** Assume that P admits orthogonal decomposition. By Proposition 2.2.1 $P \subseteq {\sim}{\sim}P$, so it remains to show that ${\sim}{\sim}P \subseteq P$. I prove the contrapositive. Suppose that $s \notin P$. If $s \in {\sim}P$, $s \notin {\sim}{\sim}P$ since $s \to s$. If $s \notin {\sim}P$, by the assumption there is a pair $(s_\parallel, s_\perp) \in P \times {\sim}P$ such that $s \in {\sim}{\sim}\{s_\parallel, s_\perp\}$. Since $s \notin P$ and $s_\parallel \in P$, $s \neq s_\parallel$. By Corollary 2.3.5 it follows from $s \in {\sim}{\sim}\{s_\parallel, s_\perp\}$ that $s_\perp \in {\sim}{\sim}\{s, s_\parallel\}$. For $s_\perp \to s_\perp$, $s_\perp \to s$ or $s_\perp \to s_\parallel$. Since $s_\perp \not\to s_\parallel$ by definition, $s_\perp \to s$, and thus $s \to s_\perp$ by Symmetry. Since $s_\perp \in {\sim}P$, $s \notin {\sim}{\sim}P$. As a result, P is bi-orthogonally closed.

For 2: Assume that P is saturated and $s \in \Sigma \setminus (P \cup {\sim}P)$. Then s has an approximation s' in P, i.e. $s' \in P$ such that $s \approx_P s'$. Since $s \notin P$ and $s' \in P$, $s \neq s'$. Hence there is an $s_\perp \in {\sim}{\sim}\{s, s'\}$ such that $s_\perp \not\to s'$ by Lemma 2.3.3.

Observe that $s_\perp \in {\sim}P$. Suppose (towards a contradiction) that $s_\perp \notin {\sim}P$. By the assumption $s_\perp$ has an approximation in P, i.e. there is an $s'_\perp \in P$ such that $s_\perp \approx_P s'_\perp$. On the one hand, it follows from $s'_\perp \to s'_\perp$ that $s_\perp \to s'_\perp$. On the other hand, since $s_\perp \not\to s'$ and $s' \in P$, I derive that $s'_\perp \not\to s'$. Since $s \approx_P s'$ and $s'_\perp \in P$, $s'_\perp \not\to s$. Hence $s'_\perp \in {\sim}\{s, s'\}$. For $s_\perp \in {\sim}{\sim}\{s, s'\}$, $s_\perp \not\to s'_\perp$, contradicting that $s_\perp \to s'_\perp$. Therefore, $s_\perp \in {\sim}P$.

Moreover, since $s' \in P$ and $s_\perp \in {\sim}P$, s' and $s_\perp$ are orthogonal and thus distinct. Hence it follows from $s_\perp \in \{s, s'\}$ that $s \in \{s', s_\perp\}$.

I conclude that $(s', s_\perp) \in P \times {\sim}P$ satisfies $s \in {\sim}{\sim}\{s', s_\perp\}$, and thus is an orthogonal decomposition of s with respect to P. For s is arbitrary, P admits orthogonal decomposition. ⊣

It turns out that the converse of *2* in Proposition 2.4.4 also holds. Before proving this, a closer look at the notion of orthogonal decomposition is needed. The following is a useful technical lemma.

2.4.5. LEMMA. *$s \approx_Q s'$, if $Q \subseteq \Sigma$, $s \in \Sigma$, $t \in {\sim}Q$ and $s' \in {\sim}{\sim}\{s,t\} \cap Q$.*

Proof. Observe that $s \approx_{\sim\{t\}} s'$. Let $w \in {\sim}\{t\}$ be arbitrary. First assume that $w \not\to s$. Then $w \in {\sim}\{s,t\}$. Since $s' \in {\sim}{\sim}\{s,t\}$, $w \not\to s'$. Second assume that $w \not\to s'$. Since $s' \in Q$ and $t \in {\sim}Q$, $s' \neq t$. For $s' \in {\sim}{\sim}\{s,t\}$, by Corollary 2.3.5 $s \in {\sim}{\sim}\{s',t\}$. By the assumption $w \in {\sim}\{s',t\}$, so $w \not\to s$.

Since $t \in {\sim}Q$, $Q \subseteq {\sim}{\sim}Q \subseteq {\sim}\{t\}$. Then from $s \approx_{\sim\{t\}} s'$ it follows that $s \approx_Q s'$ by Remark 2.1.2. ⊣

The following proposition establishes some basic facts about orthogonal decompositions.

2.4.6. PROPOSITION. *Let $P \subseteq \Sigma$ admit orthogonal decomposition and $(s_\parallel, s_\perp) \in P \times {\sim}P$ be an orthogonal decomposition of $s \in \Sigma \setminus (P \cup {\sim}P)$ with respect to P.*

1. *s, $s_\parallel$ and $s_\perp$ are three distinct points.*
2. *$s \approx_P s_\parallel$.*
3. *$s \approx_{\sim P} s_\perp$.*
4. *$s \to s_\parallel$.*
5. *$s \to s_\perp$.*
6. *If $(t_\parallel, t_\perp) \in P \times {\sim}P$ is an orthogonal decomposition of s with respect to P, then $(s_\parallel, s_\perp) = (t_\parallel, t_\perp)$.*

Proof. For *1*, the three disjoint sets $\Sigma \setminus (P \cup {\sim}P)$, P and ${\sim}P$ separate s, $s_\parallel$ and $s_\perp$, so these three points are all different.

For *2*, since $s \in {\sim}{\sim}\{s_\parallel, s_\perp\}$, $s_\parallel \in {\sim}{\sim}\{s, s_\perp\}$ by *1* and Corollary 2.3.5. Then $s_\parallel \in {\sim}{\sim}\{s, s_\perp\} \cap P$. Since $s_\perp \in {\sim}P$, $s \approx_P s_\parallel$ by Lemma 2.4.5.

3 can be proved in a way similar to *2*.

4 follows from *2*, $s_\parallel \in P$ and $s_\parallel \to s_\parallel$; and *5* from *3*, $s_\perp \in {\sim}P$ and $s_\perp \to s_\perp$.

For *6*, assume that $(t_\parallel, t_\perp) \in P \times {\sim}P$ is an orthogonal decomposition of s with respect to P. By *2* and *3* $s \approx_P t_\parallel$ and $s \approx_{\sim P} t_\perp$. Note that, since P admits orthogonal decomposition, by Proposition 2.4.4 P is bi-orthogonally closed, and thus is a subspace by Lemma 2.3.20. Hence $s_\parallel = t_\parallel$ and $s_\perp = t_\perp$ follow from Proposition 2.4.2. Therefore, $(s_\parallel, s_\perp) = (t_\parallel, t_\perp)$. ⊣

Now I show that, in a geometric frame, the saturated sets coincide with the sets admitting orthogonal decomposition.

2.4.7. Theorem. *For every $P \subseteq \Sigma$, P is saturated, if and only if P admits orthogonal decomposition.*

Proof. **The 'Only If' Part:** This is *2* of Proposition 2.4.4.

The 'If' Part: Assume that P admits orthogonal decomposition and $s \notin {\sim}P$. If $s \in P$, it is easy to see by definition that s itself is an approximation of s in P. In the following, I focus on the case when $s \notin P$. Then by the assumption there is an orthogonal decomposition $(s_{\parallel}, s_{\perp}) \in P \times {\sim}P$ of s with respect to P. According to the above proposition, $s \approx_P s_{\parallel}$. Hence $s_{\parallel} \in P$ is an approximation of s in P. For s is arbitrary, P is saturated. $\dashv$

An important corollary about the structure formed by the saturated sets can be drawn from this theorem.

2.4.8. Corollary. *The set of all saturated sets in a geometric frame $\mathfrak{F} = (\Sigma, \to)$ forms an orthomodular poset*[8] *with $\subseteq$ as the partial order and ${\sim}(\cdot)$ the orthocomplementation. In particular, for any two saturated sets P and Q, all of the following hold:*

1. *if $P \subseteq {\sim}Q$, $P \sqcup Q \stackrel{\text{def}}{=} {\sim}{\sim}(P \cup Q)$ is saturated and is the least upper bound, or the join, of P and Q in the poset;*
2. *$P \subseteq Q$ implies that $P = Q \cap ({\sim}Q \sqcup P)$;*
3. *if $P \subseteq {\sim}Q$, $P \sqcup Q = \mathcal{C}(P \cup Q)$.*

Proof. In principle, this follows from Proposition 3.3 in [52], although there the irreflexive orthogonality relation is taken to be primitive.

First, I point out some correspondences between the terminologies in [52] and those in this thesis. Since $(\Sigma, \to)$ is a geometric frame, $(\Sigma, \not\to)$ is an orthogonality space satisfying (1°) and (3°) in [52]. For any $P, Q \subseteq \Sigma$, ${\sim}P$, ${\sim}{\sim}P$ and $\mathcal{C}(P \cup Q)$ in this thesis mean the same sets as P', P^- and $P + Q$ in [52], respectively.

Second, observe that, for every subspace P of $\mathfrak{F}$ with $P \neq \emptyset$ and ${\sim}P \neq \emptyset$, by Theorem B.1.8 and Lemma B.1.6

$$
\begin{aligned}
& P + P' = \Sigma \\
\Leftrightarrow\ & \mathcal{C}(P \cup {\sim}P) = \Sigma \\
\Leftrightarrow\ & \text{there are } s_{\parallel} \in \mathcal{C}(P) \text{ and } s_{\perp} \in \mathcal{C}({\sim}P) \text{ such that } s \in s_{\parallel} \star s_{\perp}, \text{ for all } s \in \Sigma \\
\Leftrightarrow\ & \text{there are } s_{\parallel} \in P \text{ and } s_{\perp} \in {\sim}P \text{ such that } s \in s_{\parallel} \star s_{\perp}, \text{ for every } s \in \Sigma \\
\Leftrightarrow\ & \text{there are } s_{\parallel} \in P \text{ and } s_{\perp} \in {\sim}P \text{ such that } s \in s_{\parallel} \star s_{\perp}, \\
& \qquad \text{for every } s \in \Sigma \setminus (P \cup {\sim}P) \\
\Leftrightarrow\ & P \text{ admits orthogonal decomposition}
\end{aligned}
$$

[8]Relevant notions about posets are reviewed in Appendix A.

Moreover, the equivalence between the first and the last statements also holds in the case when $P = \emptyset$ or $\sim P = \emptyset$. Therefore, $P + P' = \Sigma$ if and only if P admits orthogonal decomposition, for every subspace P of Σ.

Third, the linear subsets of $(\Sigma, \not\to)$ in the sense of [52], which coincide with the subspaces of $\mathfrak{F}$, form a modular lattice. The reason is that $(\Sigma, \star)$ is a projective geometry by Remark 2.3.8 and the subspaces of a projective geometry form a modular lattice by Proposition 2.4.6 in [39]. Proposition 3.3 in [52] says that under this condition the splitting sets of $(\Sigma, \not\to)$ form an orthomodular poset.

I claim that the splitting sets of $(\Sigma, \not\to)$ in the sense of [52] are exactly the sets admitting orthogonal decomposition in my terminology. Let $P \subseteq \Sigma$ be arbitrary. First assume that it is a splitting set, which is defined to be a linear subset P satisfying $P + P' = \Sigma$. Then P admits orthogonal decomposition by the above observation. Second assume that it admits orthogonal decomposition. Then it is a linear subset by Proposition 2.4.4 and Lemma 2.3.20. It follows from the above observation that $P + P' = \Sigma$, so P is a splitting set.

As a result, the saturated sets, which are the sets admitting orthogonal decomposition by the theorem and thus are the splitting sets, form an orthomodular poset. ⊣

2.4.2 Complete Geometric Frames and Hilbertian Geometries

In this subsection, I study complete geometric frames and show a correspondence between them and Hilbertian geometries.

Based on the results of the previous subsection, I prove a counterpart of the Orthogonal Decomposition Theorem in Hilbert space theory.

2.4.9. Theorem. *In a complete geometric frame $\mathfrak{F} = (\Sigma, \to)$, for every $P \subseteq \Sigma$, the following are equivalent:*

(i) P is bi-orthogonally closed;

(ii) P is saturated;

(iii) P admits orthogonal decomposition.

Proof. The equivalence of (i) and (ii) follows from Property A and Proposition 2.4.4, and that of (ii) and (iii) follows from Theorem 2.4.7. ⊣

Now I show a correspondence between complete geometric frames and Hilbertian geometries based on Corollary 2.3.18.

2.4.10. Theorem.

1. *For every geometric frame $\mathfrak{F}$, it is a complete geometric frame, if and only if $\mathbf{G}(\mathfrak{F})$ is a Hilbertian geometry.*

2. *For every pure orthogeometry $\mathcal{G}$, it is a Hilbertian geometry, if and only if $\mathbf{F}(\mathcal{G})$ is a complete geometric frame.*

Proof. For *1*, let $\mathfrak{F} = (\Sigma, \to)$ be an arbitrary geometric frame. Let $\mathbf{G}(\mathfrak{F}) = (\Sigma, \star, \not\to)$, as is defined in Corollary 2.3.18. According to Lemma 2.3.20, every bi-orthogonally closed subset of Σ is a subspace. Remember from the proof of Corollary 2.4.8 that, for every subspace $P \subseteq \Sigma$, $\mathcal{C}(P \cup {\sim}P) = \Sigma$, if and only if P admits orthogonal decomposition, and thus if and only if P is saturated by Theorem 2.4.7. Therefore, the following are equivalent:

(i) for every $P \subseteq \Sigma$ satisfying ${\sim\sim}P = P$, P is saturated;

(ii) for every $P \subseteq \Sigma$ satisfying ${\sim\sim}P = P$, $\mathcal{C}(P \cup {\sim}P) = \Sigma$.

According to the definitions, $\mathfrak{F}$ is a complete geometric frame if (i) holds, and $\mathbf{G}(\mathfrak{F})$ is a Hilbertian geometry if (ii) holds. As a result, $\mathfrak{F}$ is a complete geometric frame, if and only if $\mathbf{G}(\mathfrak{F})$ is a Hilbertian geometry.

2 follows from *1* and Corollary 2.3.18, which says that, for every pure orthogeometry $\mathcal{G}$, $\mathcal{G} = \mathbf{G} \circ \mathbf{F}(\mathcal{G})$. ⊣

2.4.3 Finite-Dimensionality

In this subsection, I study the properties of the subspaces generated by finite sets in geometric frames and the consequences of finite-dimensionality.

I start with a technical lemma, which gives a way to get a bigger saturated set from a smaller saturated set using linear closures.

2.4.11. Lemma. *In a geometric frame $\mathfrak{F} = (\Sigma, \to)$, let $P \subseteq \Sigma$ be saturated and $t \in {\sim}P$. Then $\mathcal{C}(\{t\} \cup P)$ is also saturated.*

Proof. On the one hand, $t \in {\sim}P$ implies that $\{t\} \subseteq {\sim}P$. Since P is saturated by the assumption and so is $\{t\}$ by definition, ${\sim\sim}(\{t\} \cup P)$ is saturated by Corollary 2.4.8. According to Lemma 2.3.21, ${\sim\sim}\mathcal{C}(\{t\} \cup P) = {\sim\sim}(\{t\} \cup P)$ is saturated. On the other hand, since P is saturated, it is bi-orthogonally closed by Theorem 2.4.7. Then, according to Lemma 2.3.22, $\mathcal{C}(\{t\} \cup P)$ is bi-orthogonally closed and thus $\mathcal{C}(\{t\} \cup P) = {\sim\sim}\mathcal{C}(\{t\} \cup P)$. Therefore, $\mathcal{C}(\{t\} \cup P)$ is saturated. ⊣

The following proposition says that the bi-orthogonal closure of a finite orthogonal set is saturated.

2.4.12. Proposition. *In a geometric frame $\mathfrak{F} = (\Sigma, \to)$, for every $n \in \mathbb{N}$ and every orthogonal set $\{s_1, ..., s_n\} \subseteq \Sigma$, ${\sim\sim}\{s_1, ..., s_n\}$ is saturated.*

Proof. Use induction on n.

Base Step: $n = 0$. In this case, the set is the empty set. Since, for every $s \in \Sigma$, $s \in \Sigma = {\sim}\emptyset$, vacuously the set is saturated.

Induction Step: $n = k + 1$. By the induction hypothesis ${\sim\sim}\{s_1, ..., s_k\}$ is saturated. Moreover, as $\{s_1, ..., s_k, s_{k+1}\}$ is orthogonal, $s_{k+1} \in {\sim}\{s_1, ..., s_k\} = {\sim\sim\sim}\{s_1, ..., s_k\}$. Then by Lemma 2.4.11 $\mathcal{C}(\{s_{k+1}\} \cup {\sim\sim}\{s_1, ..., s_k\})$ is saturated. According to Corollary B.1.9 and Proposition 2.3.23,

$$\begin{aligned}
\mathcal{C}(\{s_{k+1}\} \cup {\sim\sim}\{s_1, ..., s_{k+1}\}) &= \mathcal{C}(\{s_{k+1}\} \cup \mathcal{C}(\{s_1, ..., s_{k+1}\})) \\
&= \mathcal{C}(\{s_{k+1}\} \cup \{s_1, ..., s_{k+1}\}) \\
&= \mathcal{C}(\{s_1, ..., s_{k+1}\}) \\
&= {\sim\sim}\{s_1, ..., s_{k+1}\}
\end{aligned}$$

Therefore, ${\sim\sim}\{s_1, ..., s_k, s_{k+1}\}$ is saturated. ⊣

Next I prove the counterpart of the finite version of Gram-Schmidt Theorem for Hilbert spaces.

2.4.13. Theorem. *In a geometric frame $\mathfrak{F} = (\Sigma, \rightarrow)$, for any $n \in \mathbb{N}$ and any $s_1, ..., s_n \in \Sigma$, there are $m \leq n$ and $t_1, ..., t_m \in \Sigma$ such that $\{t_1, ..., t_m\}$ is an orthogonal set and ${\sim\sim}\{s_1, ..., s_n\} = {\sim\sim}\{t_1, ..., t_m\}$.*

Proof. Use induction on n.

Base Step: $n = 0$. In this case the set ${\sim\sim}\{s_1, ..., s_n\}$ is $\emptyset$. Since $\emptyset$ is orthogonal by definition, the result holds.

Induction Step: $n = k+1$. By the induction hypothesis there is an $l \leq k$ and $t_1, ..., t_l \in \Sigma$ such that $\{t_1, ..., t_l\}$ is orthogonal and ${\sim\sim}\{s_1, ..., s_k\} = {\sim\sim}\{t_1, ..., t_l\}$. It follows easily that ${\sim}\{s_1, ..., s_k\} = {\sim}\{t_1, ..., t_l\}$. Now consider three cases.

Case 1: $s_{k+1} \in {\sim\sim}\{s_1, ..., s_k\}$. Then it is easy to see that ${\sim\sim}\{s_1, ..., s_k\} = {\sim\sim}\{s_1, ..., s_k, s_{k+1}\}$, and hence the above l and $t_1, ..., t_l$ suffice.

Case 2: $s_{k+1} \in {\sim}\{s_1, ..., s_k\}$. Define t_{l+1} to be s_{k+1}. Then it is not hard to see that $\{t_1, ..., t_l, t_{l+1}\}$ is orthogonal, $l + 1 \leq k + 1 = n$ and

$$\begin{aligned}
{\sim\sim}\{s_1, ..., s_k, s_{k+1}\} &= \mathcal{C}(\{s_1, ..., s_k, s_{k+1}\}) \\
&= \mathcal{C}(\{s_{k+1}\} \cup \{s_1, ..., s_k\}) \\
&= \mathcal{C}(\{s_{k+1}\} \cup \mathcal{C}(\{s_1, ..., s_k\})) \\
&= \mathcal{C}(\{s_{k+1}\} \cup {\sim\sim}\{s_1, ..., s_k\}) \\
&= \mathcal{C}(\{t_{l+1}\} \cup {\sim\sim}\{t_1, ..., t_l\}) \\
&= \mathcal{C}(\{t_{l+1}\} \cup \mathcal{C}(\{t_1, ..., t_l\})) \\
&= \mathcal{C}(\{t_{l+1}\} \cup \{t_1, ..., t_l\}) \\
&= \mathcal{C}(\{t_1, ..., t_l, t_{l+1}\}) \\
&= {\sim\sim}\{t_1, ..., t_l, t_{l+1}\}
\end{aligned}$$

Case 3: $s_{k+1} \notin {\sim}\{s_1, ..., s_k\}$ and $s_{k+1} \notin {\sim\sim}\{s_1, ..., s_k\}$. It follows that $s_{k+1} \notin {\sim\sim\sim}\{t_1, ..., t_l\}$ and $s_{k+1} \notin {\sim\sim}\{t_1, ..., t_l\}$. Since ${\sim\sim}\{t_1, ..., t_l\}$ is saturated by the above proposition, it admits orthogonal decomposition by Theorem 2.4.7. Hence s_{k+1} has an orthogonal decomposition $(t_\|, t_{l+1}) \in {\sim\sim}\{t_1, ..., t_l\} \times {\sim\sim\sim}\{t_1, ..., t_l\}$ with respect to ${\sim\sim}\{t_1, ..., t_l\}$. By Proposition 2.4.6 $s_{k+1} \approx_{\sim\sim\sim\{t_1,...,t_l\}} t_{l+1}$, i.e. $s_{k+1} \approx_{\sim\{t_1,...,t_l\}} t_{l+1}$. Note that, since $t_{l+1} \in {\sim\sim\sim}\{t_1, ..., t_l\} = {\sim}\{t_1, ..., t_l\}$, $\{t_1, ..., t_l, t_{l+1}\}$ is orthogonal. Moreover, for every $x \in \Sigma$,

$$
\begin{aligned}
x \in {\sim}\{s_1, ..., s_k, s_{k+1}\} &\Leftrightarrow x \in {\sim}\{s_1, ..., s_k\} \text{ and } s_{k+1} \not\to x \\
&\Leftrightarrow x \in {\sim}\{t_1, ..., t_l\} \text{ and } t_{l+1} \not\to x \\
&\Leftrightarrow x \in {\sim}\{t_1, ..., t_{l+1}\}
\end{aligned}
$$

Therefore, ${\sim\sim}\{s_1, ..., s_k, s_{k+1}\} = {\sim\sim}\{t_1, ..., t_{l+1}\}$. Moreover, $l + 1 \leq k + 1 = n$ follows from $l \leq k$. ⊣

To draw an important corollary from the above results, I introduce the notion of finitely presentable sets and that of finite-dimensional Kripke frames.

2.4.14. DEFINITION. Let $\mathfrak{F} = (\Sigma, \to)$ be a Kripke frame.

$P \subseteq \Sigma$ is *finite-dimensional*, if there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = {\sim\sim}\{s_1, ..., s_n\}$; otherwise, it is *infinite-dimensional*.

$P \subseteq \Sigma$ is *finite-codimensional*, if there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = {\sim}\{s_1, ..., s_n\}$; otherwise, it is *infinite-codimensional*.

$P \subseteq \Sigma$ is *finitely presentable*, if it is finite-dimensional or finite-codimensional.

$\mathfrak{F}$ is *finite-dimensional*, if Σ is finite-dimensional; otherwise, it is *infinite-dimensional*.

2.4.15. COROLLARY. *In a geometric frame $\mathfrak{F} = (\Sigma, \to)$, every finitely presentable subset of Σ is saturated.*

Proof. Assume that P is a finitely presentable subset of Σ.

First consider the case when there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = {\sim\sim}\{s_1, ..., s_n\}$. By Theorem 2.4.13 $P = {\sim\sim}\{t_1, ..., t_m\}$ for some $m \leq n$ and some orthogonal set $\{t_1, ..., t_n\} \subseteq \Sigma$. Then by Proposition 2.4.12 P is saturated.

Second consider the case when there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = {\sim}\{s_1, ..., s_n\}$. According to the above case, ${\sim\sim}\{s_1, ..., s_n\}$ is saturated. By Corollary 2.4.8 ${\sim\sim\sim}\{s_1, ..., s_n\}$ is saturated. Hence $P = {\sim}\{s_1, ..., s_n\} = {\sim\sim\sim}\{s_1, ..., s_n\}$ is saturated. ⊣

Finally, I study the geometric frames which are finite-dimensional.

First, I prove that they correspond to finite-dimensional projective geometries with pure polarities, and to finite-dimensional pure orthogeometries.

2.4.16. Proposition. *A geometric frame $\mathfrak{F} = (\Sigma, \to)$ is finite-dimensional, if and only if the projective geometry with a pure polarity $\mathbf{G}'(\mathfrak{F})$ is finite-dimensional, if and only if the pure orthogeometry $\mathbf{G}(\mathfrak{F})$ is finite-dimensional.*

Proof. By definition a geometric frame $\mathfrak{F} = (\Sigma, \to)$ is finite-dimensional, if and only if there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $\Sigma = {\sim}{\sim}\{s_1, ..., s_n\}$. This is equivalent to that there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $\Sigma = \mathcal{C}(\{s_1, ..., s_n\})$, according to Proposition 2.3.23. Again this is equivalent to that $\mathbf{G}'(\mathfrak{F})$ has a finite generating set $\{s_1, ..., s_n\}$ and thus is finite-dimensional. Finally, this is the case, if and only if $\mathbf{G}(\mathfrak{F})$ is finite-dimensional, noticing that $\mathbf{G}(\mathfrak{F})$ and $\mathbf{G}'(\mathfrak{F})$ are related by the canonical bijection in Theorem B.1.19. ⊣

The following lemma is very useful.

2.4.17. Lemma. *In a finite-dimensional geometric frame $\mathfrak{F} = (\Sigma, \to)$, for every $P \subseteq \Sigma$, the following are equivalent:*

(i) P is bi-orthogonally closed;

(ii) P is a subspace of $\mathfrak{F}$;

(iii) there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = {\sim}{\sim}\{s_1, ..., s_n\}$.

Proof. **From (i) to (ii):** This is Lemma 2.3.20.

From (ii) to (iii): Since $\mathfrak{F}$ is finite-dimensional, by Proposition 2.4.16 the pure orthogeometry $\mathbf{G}(\mathfrak{F})$ is finite-dimensional. By Theorem B.1.14 P has a finite generating set. This means that there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = \mathcal{C}(\{s_1, ..., s_n\})$. Hence $P = {\sim}{\sim}\{s_1, ..., s_n\}$ by Proposition 2.3.23.

From (iii) to (i): This follows from Proposition 2.2.1. ⊣

Next I make a significant observation.

2.4.18. Proposition. *If a geometric frame $\mathfrak{F} = (\Sigma, \to)$ is finite-dimensional, Property A holds.*

Proof. Let $P \subseteq \Sigma$ be bi-orthogonally closed. By the previous lemma there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = {\sim}{\sim}\{s_1, ..., s_n\}$. Hence P is saturated by Corollary 2.4.15. Since P is arbitrary, Property A holds. ⊣

The significance of this proposition lies in the fact that, if finite-dimensionality (which is indeed second-order) is added, the second-order Property A in the definition of complete geometric frames and that of quantum Kripke frames can be replaced by the two first-order conditions Property AL and Property AH. I make this point clearer in the following theorem, which expresses the counterpart of the fact that every finite-dimensional pre-Hilbert space is a Hilbert space.

2.4.19. THEOREM. *Every finite-dimensional geometric frame is a complete geometric frame. Every finite-dimensional quasi-quantum Kripke frame is a quantum Kripke frame.*

Proof. This follows from Proposition 2.4.18 and the definitions. ⊣

I end this subsection with a discussion about the bi-orthogonally closed hyperplanes in geometric frames. Remember that by definition hyperplanes are maximal proper subspaces. I show that the bi-orthogonally closed hyperplanes in geometric frames all take a simple form.

2.4.20. PROPOSITION. *In a geometric frame $\mathfrak{F} = (\Sigma, \to)$, every bi-orthogonally closed hyperplane is of the form $\sim\{s\}$ for some $s \in \Sigma$.*

Proof.[9] Let P be a bi-orthogonally closed hyperplane of $\mathfrak{F}$. Since P is a hyperplane, $P \neq \Sigma$. By Proposition 2.2.1 $\sim P \neq \emptyset$.

Observe that $\sim P$ is a singleton. Suppose (towards a contradiction) that there are $u, v \in \sim P$ such that $u \neq v$. Since P is a hyperplane, by Theorem B.1.4 $(u \star v) \cap P$ is either $u \star v$ or a singleton. Since $u \in \sim P$, by Reflexivity $u \notin P$. Hence $(u \star v) \cap P$ can only be a singleton. Denote by w the unique element in this set. Since $\sim P$ is bi-orthogonally closed by Proposition 2.2.1, it is a subspace by Lemma 2.3.20. Since $u, v \in \sim P$, $w \in u \star v \subseteq \sim P$. Then $w \not\to w$ follows from $w \in P$, contradicting Reflexivity. Therefore, $\sim P$ must be a singleton. Denote by s the unique element in this singleton.

From $\sim P = \{s\}$ it follows that $P = \sim\sim P = \sim\{s\}$. ⊣

2.5 (Quasi-)Quantum Kripke Frames

In this section I investigate the relation between Superposition on geometric frames and the irreducibility of projective geometries, and then obtain correspondence results for quasi-quantum Kripke frames and quantum Kripke frames. Remember that a projective geometry is *irreducible*, if $a \star b$ contains at least three points for any distinct points a and b.

The following proposition relates Superposition with the number of elements in a set of the form $s \star t$ in a geometric frame.

2.5.1. PROPOSITION. *Let $\mathfrak{F} = (\Sigma, \to)$ be a geometric frame. Then the following are equivalent:*

(i) Superposition holds;

[9]In principle, this result follows from Theorem 2.3.17 in this thesis and Remark 11.3.6 in [39]. However, since it is not long, a direct proof may be more helpful in developing intuitions.

(ii) *for any* $s, t \in \Sigma$ *satisfying* $s \neq t$*, there are at least* 3 *elements in* $s \star t$*, i.e. there is a* $w \in s \star t$ *such that* $w \neq s$ *and* $w \neq t$*.*

Proof. **From (i) to (ii):** Let $s, t \in \Sigma$ satisfying $s \neq t$ be arbitrary. Two cases need to be considered.

Case 1: $s \to t$. Since $s \neq t$, by Lemma 2.3.3 there is a $w \in s \star t$ such that $s \not\to w$. Then $w \neq s$ by Reflexivity. Since $s \not\to w$ and $s \to t$, $w \neq t$.

Case 2: $s \not\to t$. According to (i), there is a $v \in \Sigma$ such that $v \to s$ and $v \to t$. It follows that $v \notin \sim\{s,t\}$. By Property AL there is a $w \in \sim\sim\{s,t\}$ such that $w \approx_{\sim\sim\{s,t\}} v$. This means that $w \in s \star t$ is such that $w \approx_{s\star t} v$. Note that $w \neq s$; otherwise, from $w \approx_{s\star t} v$, $t \in s \star t$ and $s \not\to t$ one can deduce that $v \not\to t$, contradicting that $v \to t$. Similarly, one can show that $w \neq t$.

From (ii) to (i): Let $s, t \in \Sigma$ be arbitrary. Two cases need to be considered.

Case 1: $s \to t$. Then take w to be s. Hence $w \to t$; and $w \to s$ by Reflexivity.

Case 2: $s \not\to t$. By Reflexivity $s \neq t$. By (ii) there is a $w \in s \star t$ such that $w \neq s$ and $w \neq t$. According to Lemma 2.3.4, $w \star t = s \star t$. Using Proposition 2.3.9 and Theorem B.1.4, one deduce that $(w \star t) \cap \sim\{s\}$ is either $w \star t$ or a singleton. Since $s \notin \sim\{s\}$ and $s \in s \star t = w \star t$, $(w \star t) \cap \sim\{s\}$ is a singleton. Since $t \in \sim\{s\}$, $(w \star t) \cap \sim\{s\} = \{t\}$. Therefore, $w \to s$ follows from $w \neq t$. Similarly one can deduce from $w \neq s$ that $w \to t$. ⊣

Using this proposition , I obtain the following correspondence results:

2.5.2. Theorem.

1. *For every geometric frame* $\mathfrak{F}$*, it is a quasi-quantum Kripke frame, if and only if* $\mathbf{G}'(\mathfrak{F})$ *is irreducible.*

   For every projective geometry with a pure polarity $\mathcal{G}$*, it is irreducible, if and only if* $\mathbf{F}'(\mathcal{G})$ *is a quasi-quantum Kripke frame.*

2. *For every geometric frame* $\mathfrak{F}$*, it is a quasi-quantum Kripke frame, if and only if* $\mathbf{G}(\mathfrak{F})$ *is irreducible.*

   For every pure orthogeometry $\mathcal{G}$*, it is irreducible, if and only if* $\mathbf{F}(\mathcal{G})$ *is a quasi-quantum Kripke frame.*

3. *For every geometric frame* $\mathfrak{F}$*, it is a quantum Kripke frame, if and only if* $\mathbf{G}(\mathfrak{F})$ *is an irreducible Hilbertian geometry.*

   For every pure orthogeometry $\mathcal{G}$*, it is an irreducible Hilbertian geometry, if and only if* $\mathbf{F}(\mathcal{G})$ *is a quantum Kripke frame.*

Proof. This follows from Theorem 2.3.17, Corollary 2.3.18, Theorem 2.4.10 and Proposition 2.5.1. ⊣

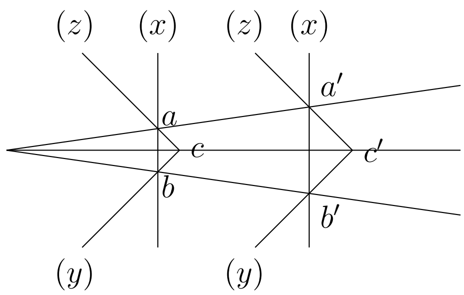
Since the analytic method plays an important role in this thesis, it is convenient to draw a corollary of this theorem, which is a representation theorem of quasi-quantum Kripke frames via some special vector spaces[10]. For this aim, I need to introduce the notion of arguesian geometric frames:

2.5.3. Definition. A quasi-quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$ is *arguesian*, if, for any $a, b, c, a', b', c' \in \Sigma$ such that 1 to 7 of the following hold:

1. a, b, c, a', b', c' are all distinct;
2. $c \notin {\sim}{\sim}\{a, b\}$;
3. $c' \notin {\sim}{\sim}\{a', b'\}$;
4. ${\sim}{\sim}\{a, a'\} \cap {\sim}{\sim}\{b, b'\} \cap {\sim}{\sim}\{c, c'\}$ is a singleton;
5. $x \in {\sim}{\sim}\{a, b\} \cap {\sim}{\sim}\{a', b'\}$;
6. $y \in {\sim}{\sim}\{b, c\} \cap {\sim}{\sim}\{b', c'\}$;
7. $z \in {\sim}{\sim}\{c, a\} \cap {\sim}{\sim}\{c', a'\}$;

then $s_1 \in {\sim}{\sim}\{s_2, s_3\}$, for some $s_1, s_2, s_3 \in \Sigma$ such that $\{s_1, s_2, s_3\} = \{x, y, z\}$.

This definition involves complicated configurations. The following picture of the analogue in an affine plane may help to make sense of it: (in the picture '(x)' means that x is not a point in the affine plane; instead, it is an imaginary point at infinity where parallel lines intersect.)



2.5.4. Remark. It is not hard to see that a quasi-quantum Kripke frame $\mathfrak{F}$ is arguesian, if and only if the irreducible pure orthogeometry $\mathbf{G}(\mathfrak{F})$ is arguesian in the sense of Definition B.3.3; and an irreducible pure orthogeometry $\mathcal{G}$ is arguesian, if and only if the quasi-quantum Kripke frame $\mathbf{F}(\mathcal{G})$ is arguesian.

2.5.5. Corollary.

[10]Relevant notions and results in linear algebra are reviewed in Appendix B.

1. *For every vector space V over a division ring $\mathcal{K}$ with an anisotropic Hermitian form Φ, define $\to_V \subseteq \Sigma(V) \times \Sigma(V)$ such that, for any $\mathbf{u}, \mathbf{v} \in V \setminus \{\mathbf{0}\}$, $\langle \mathbf{u} \rangle \to_V \langle \mathbf{v} \rangle$ if and only if $\Phi(\mathbf{u}, \mathbf{v}) \neq 0$, then $(\Sigma(V), \to_V)$ is a quasi-quantum Kripke frame, called* the quasi-quantum Kripke frame induced by V.

   Moreover, $\langle \mathbf{w} \rangle \in \sim\sim\{\langle \mathbf{u} \rangle, \langle \mathbf{v} \rangle\}$, if and only if $\mathbf{w} = x\mathbf{u} + y\mathbf{v}$ for some x, y in $\mathcal{K}$, for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V \setminus \{\mathbf{0}\}$.

2. *For every quasi-quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$, the following are equivalent:*

   (i) $\mathfrak{F}$ is arguesian;

   (ii) there is a vector space V of dimension at least 3 over a division ring $\mathcal{K}$ with an anisotropic Hermitian form Φ such that $(\Sigma(V), \to_V) \cong \mathfrak{F}$.

   Moreover, both V and $\mathcal{K}$ are unique up to isomorphism, if they exist.

3. *Every quasi-quantum Kripke frame having an orthogonal set of cardinality 4 is isomorphic to $(\Sigma(V), \to_V)$, for some vector space V over a division ring $\mathcal{K}$ equipped with an anisotropic Hermitian form Φ.*

Proof. **For *1*:** By Theorem B.3.8 $(\Sigma(V), *, \perp_V)$ is an irreducible pure orthogeometry. Then $\mathbf{F}((\Sigma(V), *, \perp_V)) = (\Sigma(V), \not\perp_V)$ is a quasi-quantum Kripke frame by Theorem 2.5.2. Note that, for any $\mathbf{u}, \mathbf{v} \in V \setminus \{\mathbf{0}\}$, $\langle \mathbf{u} \rangle \not\perp_V \langle \mathbf{v} \rangle \Leftrightarrow \Phi(\mathbf{u}, \mathbf{v}) \neq 0 \Leftrightarrow \langle \mathbf{u} \rangle \to_V \langle \mathbf{v} \rangle$; so $(\Sigma(V), \to_V) = (\Sigma(V), \not\perp_V)$ is a quasi-quantum Kripke frame.

Moreover, for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V \setminus \{\mathbf{0}\}$, by Lemma 2.3.13 and Theorem B.1.19

$$\langle \mathbf{w} \rangle \in \sim\sim\{\langle \mathbf{u} \rangle, \langle \mathbf{v} \rangle\} \quad \Leftrightarrow \quad \langle \mathbf{w} \rangle \in \langle \mathbf{u} \rangle * \langle \mathbf{v} \rangle \quad \Leftrightarrow \quad \mathbf{w} = x\mathbf{u} + y\mathbf{v} \text{ for some } x, y \text{ in } \mathcal{K}$$

For *2*: *From (i) to (ii):* According to Theorem 2.5.2 and the above remark, $\mathbf{G}(\mathfrak{F}) = (\Sigma, \star, \not\to)$ is an arguesian pure orthogeometry. By Theorem B.3.8 there is a vector space V over a division ring $\mathcal{K}$ equipped with an anisotropic Hermitian form Φ such that $i : (\Sigma(V), *, \perp_V) \cong (\Sigma, \star, \not\to)$. Then it follows easily that i is a bijection from $\Sigma(V)$ to Σ such that, for any $\mathbf{u}, \mathbf{v} \in V \setminus \{\mathbf{0}\}$, $\langle \mathbf{u} \rangle \to_V \langle \mathbf{v} \rangle \Leftrightarrow i(\langle \mathbf{u} \rangle) \to i(\langle \mathbf{v} \rangle)$. Hence i is an isomorphism from $(\Sigma(V), \to_V)$ to $\mathfrak{F} = (\Sigma, \to)$.

From (ii) to (i): Let $(\Sigma(V), *, \perp_V)$ be the arguesian pure orthogeometry of V given in Theorem B.3.8. By Theorem 2.5.2 and the above remark $\mathbf{F}((\Sigma(V), *, \perp_V)) = (\Sigma(V), \not\perp_V)$ is an arguesian quasi-quantum Kripke frame. Observe that, for any $\mathbf{u}, \mathbf{v} \in V \setminus \{\mathbf{0}\}$, $\langle \mathbf{u} \rangle \not\perp_V \langle \mathbf{v} \rangle \Leftrightarrow \langle \mathbf{u} \rangle \to_V \langle \mathbf{v} \rangle$. Therefore, $\mathfrak{F}$, being isomorphic to $(\Sigma(V), \to_V)$ and thus to $(\Sigma(V), \not\perp_V)$, is an arguesian quasi-quantum Kripke frame.

Finally, uniqueness follows from the uniqueness statement in Theorem B.3.8.

For *3*: By Theorem 2.5.2 $\mathbf{G}(\mathfrak{F}) = (\Sigma, \star, \not\to)$ is an irreducible pure orthogeometry. Since $\mathfrak{F}$ has an orthogonal set of cardinality 4, it is not hard to show that this set is an independent set of cardinality 4 in $(\Sigma, \star, \not\to)$, so the rank of this pure orthogeometry is at least 4. By Theorem B.3.5 $(\Sigma, \star, \not\to)$ is arguesian.

Hence $\mathfrak{F} = (\Sigma, \to)$ is also arguesian, according to the above remark. Then the conclusion follows from *2*. ⊣

Similarly, a representation theorem of arguesian quantum Kripke frames via generalized Hilbert spaces can be drawn as a corollary of Theorem 2.5.2.

2.5.6. Corollary.

1. *For every generalized Hilbert space V, $(\Sigma(V), \to_V)$ is a quantum Kripke frame, called* the quantum Kripke frame induced by V.
2. *For every quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$, the following are equivalent:*
   - *(i) $\mathfrak{F}$ is arguesian;*
   - *(ii) there is a generalized Hilbert space V of dimension at least 3 over a division ring $\mathcal{K}$ such that $(\Sigma(V), \to_V) \cong \mathfrak{F}$.*

   Moreover, both V and $\mathcal{K}$ are unique up to isomorphism, if they exist.
3. *Every quantum Kripke frame having an orthogonal set of cardinality 4 is isomorphic to $(\Sigma(V), \to_V)$, for some generalized Hilbert space V.*

Proof. The proof of this corollary is very similar to the previous one, and the only significant difference is that Theorem B.3.11 is used instead of Theorem B.3.8. ⊣

2.6 Subframes

In this section I study the special substructures of Kripke frames, which are called subframes. I show that every subframe of a geometric frame is again a geometric frame, and the same holds for a quasi-quantum Kripke frame and a quantum Kripke frame.

2.6.1. Definition. A *subframe* of a Kripke frame $\mathfrak{F} = (\Sigma, \to)$ is a substructure of $\mathfrak{F}$ (in the sense of universal algebra) whose underlying set is non-empty and saturated in $\mathfrak{F}$, i.e. a tuple $(P, \to_P)$ such that

1. $P \subseteq \Sigma$ is non-empty and saturated in $\mathfrak{F}$;
2. $\to_P$ is the restriction of $\to$ to P, i.e. $\to_P = \to \cap (P \times P)$.

2.6.2. Remark. Let $(P, \to_P)$ be a subframe of a Kripke frame $\mathfrak{F} = (\Sigma, \to)$. Then, for any $s, t \in P$, $s \to t$ if and only if $s \to_P t$. Moreover, for any $A \subseteq P$ and $s, t \in P$, s and t are indistinguishable with respect to A in $(P, \to_P)$ if and only if they are indistinguishable with respect to A in $\mathfrak{F}$.

Given this remark, in the following, I denote the restriction of $\to$ in $\mathfrak{F}$ to P by $\to$ instead of $\to_P$. I also use the same symbol $\approx_A$ indexed by $A \subseteq P$ to denote the indistinguishabililty relation with respect to A in $(P, \to_P)$ and that in $\mathfrak{F}$. On the contrary, I denote the orthocomplement of $A \subseteq P$ in $(P, \to)$ by $\sim_P A$, because the orthocomplements in the subframe is different from those in $\mathfrak{F}$.

2.6.1 Subframes of a Geometric Frame

In this subsection, I investigate the properties of the subframes of a geometric frame and arrive at the conclusion that every subframe of a geometric frame is a geometric frame.

I start from a simple characterization of the orthocomplements and the bi-orthogonal closures in a subframe of a geometric frame.

2.6.3. LEMMA. *Let $(P, \to)$ be a subframe of a geometric frame $\mathfrak{F} = (\Sigma, \to)$.*

1. *P is bi-orthogonally closed.*
2. *$\sim_P A = \sim A \cap P$, for every $A \subseteq P$.*
3. *$\sim_P \sim_P A = A$, for every $A \subseteq P$ which is saturated in $\mathfrak{F}$.*

Proof. **For *1*:** This follows from the definition and Proposition 2.4.4.

For *2*: Easy verification.

For *3*: It follows from *2* that

$$\begin{aligned}\sim_P \sim_P A &= \sim(\sim A \cap P) \cap P \\ &= \sim(\sim A \cap \sim\sim P) \cap P \\ &= \sim\sim(A \cup \sim P) \cap P \\ &= (A \sqcup \sim P) \cap P\end{aligned}$$

Since both A and P are saturated in $\mathfrak{F}$ and $A \subseteq P$, $(A \sqcup \sim P) \cap P = A$ by Corollary 2.4.8. Therefore, $\sim_P \sim_P A = A$. ⊣

Next, I study the conditions in the definition of geometric frames one by one.

2.6.4. LEMMA. *Let $(P, \to)$ be a subframe of a geometric frame $\mathfrak{F} = (\Sigma, \to)$. Reflexivity, Symmetry and Separation hold on $(P, \to)$, so $(P, \to)$ is a state space.*

Proof. It is obvious by the definition of subframes that Reflexivity and Symmetry hold on $(P, \to)$, since they hold on $\mathfrak{F}$.

For Separation, assume that $s, t \in P$ are such that $s \neq t$. Since $\mathfrak{F}$ satisfies Separation, there is a $w \in \Sigma$ such that $w \to s$ and $w \not\to t$. Since $w \to s$, $w \notin \sim P$. It follows from the saturation of P that there is a $w' \in P$ such that $w \approx_P w'$. Hence $w' \in P$ is such that $w' \to s$ and $w' \not\to t$. ⊣

2.6.5. LEMMA. *Let $(P, \to)$ be a subframe of a geometric frame $\mathfrak{F} = (\Sigma, \to)$. Then Property AL holds on $(P, \to)$.*

Proof. Assume that $s, t \in P$ are such that $s \neq t$ and $w \in P \setminus {\sim_P}\{s,t\}$. Then $w \notin {\sim}\{s,t\}$. By Property AL of $\mathfrak{F}$ there is a $w' \in {\sim\sim}\{s,t\}$ such that $w \approx_{\sim\sim\{s,t\}} w'$.

I claim that ${\sim_P}{\sim_P}\{s,t\} = {\sim\sim}\{s,t\}$. Since P is bi-orthogonally closed and $s, t \in P$, ${\sim\sim}\{s,t\} \subseteq P$. Now, on the one hand, by Lemma 2.6.3

$$\begin{aligned}
{\sim\sim}\{s,t\} &= {\sim\sim}\{s,t\} \cap P \\
&\subseteq {\sim\sim}(\{s,t\} \cup {\sim}P) \cap P \\
&= {\sim}({\sim}\{s,t\} \cap {\sim\sim}P) \cap P \\
&= {\sim}({\sim}\{s,t\} \cap P) \cap P \\
&= {\sim_P}{\sim_P}\{s,t\}
\end{aligned}$$

On the other hand, since $\{s,t\} \subseteq {\sim\sim}\{s,t\} \subseteq P$, by the above lemma and Proposition 2.2.1 ${\sim_P}{\sim_P}\{s,t\} \subseteq {\sim_P}{\sim_P}{\sim\sim}\{s,t\}$. By Corollary 2.4.15 ${\sim\sim}\{s,t\} \subseteq P$ is saturated in $\mathfrak{F}$. Hence ${\sim_P}{\sim_P}{\sim\sim}\{s,t\} = {\sim\sim}\{s,t\}$. It follows that ${\sim_P}{\sim_P}\{s,t\} \subseteq {\sim_P}{\sim_P}{\sim\sim}\{s,t\} = {\sim\sim}\{s,t\}$. Therefore, ${\sim_P}{\sim_P}\{s,t\} = {\sim\sim}\{s,t\}$.

As a result, $w' \in {\sim_P}{\sim_P}\{s,t\}$ is such that $w \approx_{\sim_P\sim_P\{s,t\}} w'$, so w' is an approximation of w in ${\sim_P}{\sim_P}\{s,t\}$. ⊣

2.6.6. LEMMA. *Let $(P, \to)$ be a subframe of a geometric frame $\mathfrak{F} = (\Sigma, \to)$. Then Property AH holds on $(P, \to)$.*

Proof. Assume that $s, w \in P$ are such that $w \notin {\sim_P}{\sim_P}\{s\}$. Using Lemma 2.6.4, it is not hard to show that $w \neq s$. Since $\mathfrak{F}$ is a geometric frame, by Remark 2.2.6 there is a $w'' \in {\sim}\{s\}$ such that $w \approx_{\sim\{s\}} w''$. Since $w'' \to w''$, $w'' \to w$, and thus $w'' \notin {\sim}P$. Since P is saturated in $\mathfrak{F}$, there is a $w' \in P$ such that $w'' \approx_P w'$.

I claim that w' has the required property, i.e. $w' \in {\sim_P}\{s\}$ and $w \approx_{\sim_P\{s\}} w'$. First, since $w'' \in {\sim}\{s\}$, it follows that $w' \in {\sim_P}\{s\}$. Second, for every $t \in {\sim_P}\{s\}$, $t \in P$ and $t \in {\sim}\{s\}$, so $t \to w \Leftrightarrow t \to w'' \Leftrightarrow t \to w'$. Therefore, $w \approx_{\sim_P\{s\}} w'$. ⊣

Now I can conclude the following:

2.6.7. PROPOSITION. *Every subframe of a geometric frame is a geometric frame.*

Proof. This follows from all lemmas proved in this subsection. ⊣

2.6.2 Subframes of a (Quasi-)Quantum Kripke Frame

In this subsection, I study the properties of the subframes of a (quasi-) quantum Kripke frame and conclude that every subframe of a (quasi-)quantum Kripke frame is a (quasi-)quantum Kripke frame.

Given the results in the previous subsection, for quasi-quantum Kripke frames, Superposition is the only concern.

2.6.8. PROPOSITION. *Let $(P, \to)$ be a subframe of a quasi-quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$. $(P, \to)$ satisfies Superposition and is a quasi-quantum Kripke frame.*

Proof. Let $s, t \in P$ be arbitrary. Since $\mathfrak{F}$ is a quasi-quantum Kripke frame, by Superposition there is a $w \in \Sigma$ such that $w \to s$ and $w \to t$. Since P is saturated, there is a $w' \in P$ such that $w \approx_P w'$. Therefore, $w' \in P$ is such that $w' \to s$ and $w' \to t$. As a result, $(P, \to)$ satisfies Superposition.

Since $\mathfrak{F}$ is a quasi-quantum Kripke frame, $(P, \to)$ is a geometric frame by Proposition 2.6.7, and thus is a quasi-quantum Kripke frame. ⊣

Now, for quantum Kripke frames, Property A is the remaining concern. I deal with this in the context of complete geometric frames first.

2.6.9. PROPOSITION. *Let $(P, \to)$ be a subframe of a complete geometric frame $\mathfrak{F} = (\Sigma, \to)$. $(P, \to)$ satisfies Property A, and thus is a complete geometric frame.*

Proof. Let $A \subseteq P$ and $s \in P$ be such that $A = {\sim_P}{\sim_P}A$ and $s \notin {\sim_P}A$. By Lemma 2.6.3 $A = {\sim_P}{\sim_P}A = {\sim}({\sim}A \cap P) \cap P$. Since both P and ${\sim}({\sim}A \cap P)$ are bi-orthogonally closed in $\mathfrak{F}$ by Proposition 2.2.1, $A = {\sim}({\sim}A \cap P) \cap P$ is also bi-orthogonally closed in $\mathfrak{F}$ by Proposition 2.2.3. Then it follows from Property A of $\mathfrak{F}$ that A is saturated. Moreover, it follows from $s \notin {\sim_P}A$ that $s \notin {\sim}A$. Hence there is an $s' \in A$ such that $s \approx_A s'$. As a result, $(P, \to)$ has Property A.

Since $\mathfrak{F}$ is a complete geometric frame, $(P, \to)$ is a geometric frame by Proposition 2.6.7. Since it has Property A, it is a complete geometric frame. ⊣

Finally, I study the subframes in a quantum Kripke frame.

2.6.10. PROPOSITION. *Let $\mathfrak{F} = (\Sigma, \to)$ be a quantum Kripke frame.*

1. *every subframe of $\mathfrak{F}$ is a quantum Kripke frame.*
2. *For every $P \subseteq \Sigma$, it is non-empty and bi-orthogonally closed, if and only if $(P, \to)$ is a subframe of $\mathfrak{F}$.*

Proof. **For *1*:** For every subframe of $\mathfrak{F}$, it is a state space by Proposition 2.6.7; it satisfies Property A by Proposition 2.6.9, and it satisfies Superposition by Proposition 2.6.8. Therefore, it is a quantum Kripke frame.

For *2*: For the 'If' part, since $(P, \to)$ is a subframe of $\mathfrak{F}$, P is non-empty and saturated, and thus it is bi-orthogonally closed by Theorem 2.4.9. The 'Only If' part follows from Property A and the definition of subframes. ⊣

2.7 Quantum Kripke Frames in Quantum Logic

In this section, I discuss the role played by quantum Kripke frames in quantum logic. I first investigate the relation between quantum Kripke frames and Hilbert spaces, which are exclusively used in the standard formalism of quantum mechanics. Then I discuss the relation between quantum Kripke frames and some quantum structures in the literature. Finally I discuss the difference between classical physics and quantum physics in the light of quantum Kripke frames.

2.7.1 Quantum Kripke Frames and Hilbert Spaces

In this subsection, I investigate the relation between quantum Kripke frames and Hilbert spaces.

I start from showing that every Hilbert space $\mathcal{H}$ over $\mathbb{C}$ gives rise to a quantum Kripke frame in a natural way.

2.7.1. Theorem. *For every Hilbert space $\mathcal{H}$ over $\mathbb{C}$, $(\Sigma(\mathcal{H}), \rightarrow_{\mathcal{H}})$ is a quantum Kripke frame.*

Proof. According to the theory of Hilbert spaces, $\mathcal{H}$ is a generalized Hilbert space with the inner product as the orthomodular Hermitian form. According to Theorem 2.5.6, $(\Sigma(\mathcal{H}), \rightarrow_{\mathcal{H}})$ is the quantum Kripke frame induced by $\mathcal{H}$. ⊣

2.7.2. Remark. By Proposition 2.2.4 Property A implies both Property AL and Property AH in a state space. Hence both Property AL and Property AH hold in quantum Kripke frames. It follows that $(\Sigma(\mathcal{H}), \rightarrow_{\mathcal{H}})$ satisfies both Property AL and Property AH. I conclude that every condition on Kripke frames introduced in Definition 2.1.3 holds in $(\Sigma(\mathcal{H}), \rightarrow_{\mathcal{H}})$ for every Hilbert space $\mathcal{H}$ over $\mathbb{C}$.

Now one may wonder whether the converse of this theorem holds, i.e. whether every quantum Kripke frame is of the form $(\Sigma(\mathcal{H}), \rightarrow_{\mathcal{H}})$ for some Hilbert space $\mathcal{H}$ over $\mathbb{C}$, or, more generally, some Hilbert space $\mathcal{H}$ over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$[11]. As it turns out, this is not true. The reason is as follows: Keller in [62] constructed an infinite-dimensional generalized Hilbert space which is not isomorphic to any Hilbert space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. It follows from Corollary 2.5.6 that the quantum Kripke frame induced by this generalized Hilbert space is not isomorphic to one of the form $(\Sigma(\mathcal{H}), \rightarrow_{\mathcal{H}})$ for any Hilbert space $\mathcal{H}$ over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

Then it makes sense and is natural to ask how to characterize the quantum Kripke frames induced by Hilbert spaces over $\mathbb{C}$. In the remaining part of this subsection, I present one solution to this.

My plan is as follows: Corollary 2.5.6 shows that all arguesian quantum Kripke frames are induced by generalized Hilbert spaces, which include Hilbert spaces.

[11] In this thesis, $\mathbb{H}$ denotes the division ring of quaternions.

Hence first I study the problem of characterizing Hilbert spaces in the class of generalized Hilbert spaces. Second I discuss how the conditions involved in this characterization in the language of vector spaces relate to the properties of the non-orthogonality relation in the language of Kripke frames. As a result of this, I eventually arrive at a characterization of the quantum Kripke frames induced by Hilbert spaces over $\mathbb{C}$.

In the literature there are two famous results on characterizing Hilbert spaces in the class of generalized Hilbert spaces. I would like to start with the Piron-Amemiya-Arakia Theorem, which goes as follows:

2.7.3. Theorem (Piron-Amemiya-Arakia Theorem). *Every Hilbert space over the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$ or the quaternions $\mathbb{H}$ is a generalized Hilbert space.*

For a generalized Hilbert space, it is a Hilbert space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, if the underlying division ring is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ and the accompanying involution of the Hermitian form is the identity in the case of $\mathbb{R}$, the complex conjugation in the case of $\mathbb{C}$ and the conjugation in the case of $\mathbb{H}$, respectively.

Proof. The first part of this theorem can be verified easily (Remark B.2.14). The second part, especially the completeness of the norm topology, is where the main difficulty lies, and it is first proved in [9]. The proof can also be found in [74] (Theorem 3.26) and [86] (Lemma 4.42). ⊣

This theorem means that the difference between generalized Hilbert spaces and Hilbert spaces completely lies in the underlying division rings. However, it does not give any hint at why these three division rings are so special. Hence it is not clear whether there is any characteristic property of the non-orthogonality relations in Hilbert spaces. In this respect, another theorem, Solèr's theorem, is an improvement. The theorem goes as follows:

2.7.4. Theorem (Solèr's Theorem). *Let V be a generalized Hilbert space with $\mathcal{K}$ as the underlying division ring and Φ as the orthomodular Hermitian form. V is a Hilbert space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, if there is a non-zero element k in $\mathcal{K}$ and a set $\{\mathbf{v}_i \mid i \in \mathbb{N}\} \subseteq V$ such that*

$$\Phi(\mathbf{v}_i, \mathbf{v}_j) = \begin{cases} k, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Proof. This theorem is first proved in [83]. A nice presentation of a proof can be found in [76]. ⊣

Note that the set $\{\mathbf{v}_i \mid i \in \mathbb{N}\}$ in this theorem is very similar to an infinite orthonormal set, and the theorem says that the existence of such a set is a sufficient condition for a generalized Hilbert space to be a Hilbert space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

An obvious nice feature of this theorem is that the additional assumption sounds very natural, at least mathematically, especially for those who are used to work with Hilbert spaces over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. A disadvantage, also obvious, is that it is a sufficient condition and only works for infinite-dimensional spaces. People who work on quantum computation and quantum information will turn their faces away due to this. Anyway, in this thesis, the foundations of quantum theory is a main concern, and for most of the time physicists work with infinite-dimensional Hilbert spaces. Hence Solèr's theorem is of great value and interest.

The condition in Solèr's theorem is in the language of vector spaces, hence there is the question whether it is equivalent to some property of the non-orthogonality relation in the language of Kripke frames. A positive answer to this question is hinted at in the literature. Holland in [55] showed that Solèr's condition is equivalent to a property of the orthogeometries induced by generalized Hilbert spaces. This implies that there is a condition of the non-orthogonality relation in the language of Kripke frames equivalent to Solèr's condition, given the correspondence between quantum Kripke frames and irreducible Hilbertian geometries. I first present Holland's result, and then present an equivalent version which is simpler and more useful.

2.7.5. Theorem. *In an irreducible Hilbertian geometry $\mathcal{G} = (G, \star, \perp)$, if there is an orthogonal set $\{a_i \mid i \in \mathbb{N}\}$ such that, for any $i, j \in \mathbb{N}$ with $i \neq j$, there is a $b_{ij} \in (a_i \star a_j) \setminus \{a_i, a_j\}$ such that the harmonic conjugate[12] of b_{ij} with respect to a_i and a_j is orthogonal to b_{ij}, then $\mathcal{G}$ is isomorphic to an irreducible Hilbertian geometry $(\Sigma(\mathcal{H}), *, \perp_{\mathcal{H}})$ of some Hilbert space $\mathcal{H}$ over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.*

Proof. This follows from Theorem 3.7 in [55]. ⊣

Given the correspondence between quasi-quantum Kripke frames and irreducible pure orthogeometries, the notion of harmonic conjugates can be introduced in quasi-quantum Kripke frames:

2.7.6. Definition. In a quasi-quantum Kripke frame $\mathfrak{F} = (\Sigma, \rightarrow)$, for any $s, t, u \in \Sigma$ such that $s \neq t$ and $u \in {\sim\sim}\{s, t\} \setminus \{s, t\}$, $v \in \Sigma$ is *a harmonic conjugate of u with respect to s and t*, if there are $a, b, c, d \in \Sigma$ satisfying all of the following:

1. a, b, c, d are distinct;
2. $w_1 \notin {\sim\sim}\{w_2, w_3\}$, for any $w_1, w_2, w_3 \in \{a, b, c, d\}$ such that $w_1 \neq w_2 \neq w_3 \neq w_1$;
3. $s \in {\sim\sim}\{a, b\} \cap {\sim\sim}\{c, d\}$;

[12]The notion of harmonic conjugates in projective geometry and its basic properties are reviewed in Appendix B.4.

4. $t \in \sim\sim\{a, c\} \cap \sim\sim\{b, d\}$;

5. $u \in \sim\sim\{a, d\} \cap \sim\sim\{s, t\}$;

6. $v \in \sim\sim\{b, c\} \cap \sim\sim\{s, t\}$.

This definition involves complicated configurations. The picture under Definition B.4.1 may help to make sense of it.

2.7.7. Remark. Given Theorem 2.5.2, it is easy to see that, in a quasi-quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$, for any $s, t, u, v \in \Sigma$, v is a harmonic conjugate of u with respect to s and t, if and only if v is a harmonic conjugate of u with respect to s and t in the irreducible pure orthogeometry $\mathbf{G}(\mathfrak{F})$.

Given Theorem 2.5.2 and Corollary 2.5.5, the results in Appendix B.4 can be applied to quasi-quantum Kripke frames as well.

Moreover, in a projective geometry or a quasi-quantum Kripke frame, for any three points u, s, t such that $u \in (s \star t) \setminus \{s, t\}$, u may not have a harmonic conjugate with respect to s and t, or it may have more than one. However, according to Lemma B.4.3, in an *arguesian* projective geometry or an *arguesian* quasi-quantum Kripke frame, for any three points u, s, t such that $u \in (s \star t) \setminus \{s, t\}$, u has a *unique* harmonic conjugate with respect to s and t. For this reason, it makes sense to talk about 'the harmonic conjugate' in Theorem 2.7.5.

With the notion of harmonic conjugates, I introduce a condition on a Kripke frame $\mathfrak{F} = (\Sigma, \to)$:

- **Existence of Orthogonal Harmonic Conjugate (OHC)**

  For any $s, t \in \Sigma$ with $s \not\to t$, there is a $w \in \sim\sim\{s, t\} \setminus \{s, t\}$ such that every harmonic conjugate of w with respect to s and t is orthogonal to w.

Next, I reveal the analytic content of this condition.

2.7.8. Lemma. *Let V be a vector space of dimension at least 3 over a division ring $\mathcal{K}$ with an anisotropic Hermitian form Φ. The following are equivalent:*

(i) $(\Sigma(V), \to_V)$ *is a quasi-quantum Kripke frame with Property OHC;*

(ii) for any $\mathbf{s}, \mathbf{t} \in V \setminus \{\mathbf{0}\}$, $\Phi(\mathbf{s}, \mathbf{t}) = 0$ *implies that there is an x in $\mathcal{K}$ such that* $\Phi(\mathbf{s}, \mathbf{s}) = \Phi(x\mathbf{t}, x\mathbf{t})$.

(iii) for any $\mathbf{s}, \mathbf{t} \in V \setminus \{\mathbf{0}\}$, *there is an x in $\mathcal{K}$ such that* $\Phi(\mathbf{s}, \mathbf{s}) = \Phi(x\mathbf{t}, x\mathbf{t})$.

Proof. I denote by $(\cdot)^*$ the accompanying involution on $\mathcal{K}$ of Φ.

From (i) to (ii): Let $\mathbf{s}, \mathbf{t} \in V \setminus \{\mathbf{0}\}$ be arbitrary such that $\Phi(\mathbf{s}, \mathbf{t}) = 0$. Then $\langle \mathbf{s} \rangle \not\to_V \langle \mathbf{t} \rangle$. By (i) there is a $\mathbf{w} \in V \setminus \{\mathbf{0}\}$ such that $\langle \mathbf{w} \rangle \in {\sim\sim}\{\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle\} \setminus \{\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle\}$ and every harmonic conjugate of $\langle \mathbf{w} \rangle$ with respect to $\langle \mathbf{s} \rangle$ and $\langle \mathbf{t} \rangle$ is orthogonal to $\langle \mathbf{w} \rangle$. Since $\langle \mathbf{w} \rangle \in {\sim\sim}\{\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle\} \setminus \{\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle\}$, by Corollary 2.5.5 there are non-zero y and z in $\mathcal{K}$ such that $\mathbf{w} = y\mathbf{s} + z\mathbf{t}$. Since V is of dimension at least 3, by Lemma B.4.3 a harmonic conjugate of $\langle \mathbf{w} \rangle$ with respect to $\langle \mathbf{s} \rangle$ and $\langle \mathbf{t} \rangle$ is $\langle y\mathbf{s} - z\mathbf{t} \rangle$. Since it is orthogonal to $\langle \mathbf{w} \rangle$, $\Phi(y\mathbf{s} + z\mathbf{t}, y\mathbf{s} - z\mathbf{t}) = 0$. It follows that $y \cdot \Phi(\mathbf{s}, \mathbf{s}) \cdot y^* - y \cdot \Phi(\mathbf{s}, \mathbf{t}) \cdot z^* + z \cdot \Phi(\mathbf{t}, \mathbf{s}) \cdot y^* - z \cdot \Phi(\mathbf{t}, \mathbf{t}) \cdot z^* = 0$. Since $\Phi(\mathbf{s}, \mathbf{t}) = 0$, $y \cdot \Phi(\mathbf{s}, \mathbf{s}) \cdot y^* - z \cdot \Phi(\mathbf{t}, \mathbf{t}) \cdot z^* = 0$. Take x to be $y^{-1} \cdot z$. Then it is easy to verify that $\Phi(x\mathbf{t}, x\mathbf{t}) = \Phi(\mathbf{s}, \mathbf{s})$.

From (ii) to (iii): Let $\mathbf{s}, \mathbf{t} \in V \setminus \{\mathbf{0}\}$ be arbitrary. Since V is of dimension at least 3, it is not hard to find $\mathbf{v} \in V \setminus \{\mathbf{0}\}$ such that $\Phi(\mathbf{s}, \mathbf{v}) = \Phi(\mathbf{v}, \mathbf{t}) = 0$. By (ii) there are y and z in $\mathcal{K}$ such that $\Phi(\mathbf{s}, \mathbf{s}) = \Phi(y\mathbf{v}, y\mathbf{v})$ and $\Phi(\mathbf{v}, \mathbf{v}) = \Phi(z\mathbf{t}, z\mathbf{t})$. Take $x = y \cdot z$. Then it is easy to verify that $\Phi(x\mathbf{t}, x\mathbf{t}) = \Phi(\mathbf{s}, \mathbf{s})$.

From (iii) to (i): By Corollary 2.5.5 $(\Sigma(V), \to_V)$ is a quasi-quantum Kripke frame. To prove that Property OHC holds, let $s, t \in \Sigma(V)$ be arbitrary such that $s \not\to_V t$. Let $\mathbf{s}, \mathbf{t} \in V \setminus \{\mathbf{0}\}$ be such that $s = \langle \mathbf{s} \rangle$ and $t = \langle \mathbf{t} \rangle$. Then $\Phi(\mathbf{s}, \mathbf{t}) = 0$. By (iii) there is an x in $\mathcal{K}$ such that $\Phi(\mathbf{s}, \mathbf{s}) = \Phi(x\mathbf{t}, x\mathbf{t})$. Since $\mathbf{s}, \mathbf{t} \in V \setminus \{\mathbf{0}\}$, $\Phi(\mathbf{s}, \mathbf{t}) = 0$ and Φ is anisotropic, $\mathbf{s}$ and $\mathbf{t}$ must be linearly independent, and thus neither $\mathbf{s} + x\mathbf{t}$ nor $\mathbf{s} - x\mathbf{t}$ is $\mathbf{0}$. Consider $\langle \mathbf{s} + x\mathbf{t} \rangle$. It is not hard to see that $\langle \mathbf{s} + x\mathbf{t} \rangle \notin \{\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle\}$. Let $\langle \mathbf{v} \rangle$ be an arbitrary harmonic conjugate of $\langle \mathbf{s} + x\mathbf{t} \rangle$ with respect to $\langle \mathbf{s} \rangle$ and $\langle \mathbf{t} \rangle$, where $\mathbf{v} \in V \setminus \{\mathbf{0}\}$. According to Lemma B.4.3, $\langle \mathbf{v} \rangle = \langle \mathbf{s} - x\mathbf{t} \rangle$. Moreover,

$$\Phi(\mathbf{s} + x\mathbf{t}, \mathbf{s} - x\mathbf{t}) = \Phi(\mathbf{s}, \mathbf{s}) - \Phi(\mathbf{s}, \mathbf{t}) \cdot x^* + x \cdot \Phi(\mathbf{t}, \mathbf{s}) - \Phi(x\mathbf{t}, x\mathbf{t}) = 0$$

Hence $\langle \mathbf{s} + x\mathbf{t} \rangle \not\to_V \langle \mathbf{s} - x\mathbf{t} \rangle$, i.e. $\langle \mathbf{s} + x\mathbf{t} \rangle \not\to_V \langle \mathbf{v} \rangle$. ⊣

2.7.9. Remark. Intuitively, by this lemma an arguesian quasi-quantum Kripke frame $(\Sigma(V), \to_V)$ has Property OHC, if and only if, for every non-zero vector $\mathbf{s}$, there is always a vector of the same 'length' of $\mathbf{s}$ in every one-dimensional subspace of V.

In particular, every quasi-quantum Kripke frame induced by a pre-Hilbert space $\mathcal{H}$ over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ has Property OHC. For any $\mathbf{s}, \mathbf{t} \in \mathcal{H} \setminus \{\mathbf{0}\}$, both $\Phi(\mathbf{s}, \mathbf{s})$ and $\Phi(\mathbf{t}, \mathbf{t})$ are positive real numbers, no matter what the underlying division ring of $\mathcal{H}$ is. Take $x = \sqrt{\Phi(\mathbf{s}, \mathbf{s}) \cdot \Phi(\mathbf{t}, \mathbf{t})^{-1}}$, which is also a positive real number. It is easy to verify that $\Phi(\mathbf{s}, \mathbf{s}) = \Phi(x\mathbf{t}, x\mathbf{t})$. Hence by the lemma $(\Sigma(\mathcal{H}), \to_{\mathcal{H}})$ satisfies Property OHC.

As a result, every quantum Kripke frame induced by a Hilbert space used in quantum theory has Property OHC.

Now I propose a condition which is equivalent to Holland's.

2.7.10. Theorem. *For a quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$, the following are equivalent:*

(i) *$\mathfrak{F}$ has an infinite orthogonal set and satisfies Property OHC;*

(ii) *$\mathfrak{F}$ satisfies Holland's condition, i.e. there is an orthogonal set $\{s_i \mid i \in \mathbb{N}\}$ such that, for any $i, j \in \mathbb{N}$ with $i \neq j$, there is a $t_{ij} \in (s_i \star s_j) \setminus \{s_i, s_j\}$ such that the harmonic conjugate of t_{ij} with respect to s_i and s_j is orthogonal to t_{ij};*

(iii) *$\mathfrak{F}$ is isomorphic to $(\Sigma(\mathcal{H}), \to_{\mathcal{H}})$, where $\mathcal{H}$ is an infinite-dimensional Hilbert space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.*

Proof. **From (i) to (ii):** Let $\{s_i \mid i \in \mathbb{N}\}$ be an infinite orthogonal set in $\mathfrak{F}$. Then, for any $i, j \in \mathbb{N}$ with $i \neq j$, $s_i \not\to s_j$, so by Property OHC there is a $t_{ij} \in {\sim}{\sim}\{s_i, s_j\} \setminus \{s_i, s_j\}$ such that every harmonic conjugate of t_{ij} with respect to s_i and s_j is orthogonal to t_{ij}. Moreover, since $\mathfrak{F}$ has an infinite orthogonal set, it follows from Theorem B.3.5 that $\mathfrak{F}$ is arguesian. It is not hard to derive from Remark B.4.4 that, for any $i, j \in \mathbb{N}$ with $i \neq j$, t_{ij} has exactly one harmonic conjugate with respect to s_i and s_j, which is orthogonal to t_{ij}.

From (ii) to (iii): This follows from the correspondence between quantum Kripke frames and irreducible Hilbertian geometries (Corollary 2.3.18), Corollary 2.5.6 and Theorem 2.7.5.

From (iii) to (i): Since $\mathcal{H}$ is an infinite-dimensional Hilbert space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, it has an infinite orthonormal basis $\{\mathbf{s}_i \mid i \in \mathbb{N}\}$ by the theory of Hilbert spaces. Then $\{\langle \mathbf{s}_i \rangle \mid i \in \mathbb{N}\}$ is an infinite orthogonal set in $(\Sigma(\mathcal{H}), \to_{\mathcal{H}})$. Moreover, $(\Sigma(\mathcal{H}), \to_{\mathcal{H}})$ has Property OHC, according to Remark 2.7.9. Since $\mathfrak{F}$ is isomorphic to $(\Sigma(\mathcal{H}), \to_{\mathcal{H}})$, it has an infinite orthogonal set and has Property OHC. ⊣

This proposition characterizes the quantum Kripke frames induced by infinite-dimensional Hilbert spaces over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ in the language of Kripke frames. Moreover, Property OHC involved in this proposition turns out to imply many nice properties of quantum Kripke frames. Proposition 4.1.6 and Remark 5.4.2 below are two examples.

In quantum physics, only Hilbert spaces over $\mathbb{C}$ are used. Hence one may wonder whether the above proposition can be improved to one characterizing quantum Kripke frames induced by infinite-dimensional Hilbert spaces over $\mathbb{C}$. The answer is yes. The key observation is that the three division rings can be distinguished by first-order properties of rings. Moreover, every line in an arguesian quantum Kripke frame, deleting one arbitrary point, has the structure of a division ring; and hence first-order properties of rings can be expressed in the language of Kripke frames.[13] This idea will be used in many places in this thesis, for example, in Chapter 4.

[13]This idea, as far as I know, dates back to the papers on projective geometry in 1856 and 1857 by K.G.C. von Staudt. For a modern and elegant exposition, please refer to [18].

Therefore, $\mathbb{H}$ can be excluded, if the multiplication is forced to be commutative. Geometrically, this can be done using Pappus's Hexagon Theorem. Further, $\mathbb{R}$ can be excluded, if it is forced that every element in the field has a square root. This idea was first proposed by Wilbur ([91]) in the setting of lattices, and it can be applied to quantum Kripke frames without much difficulty. In the following, I elaborate on these ideas.

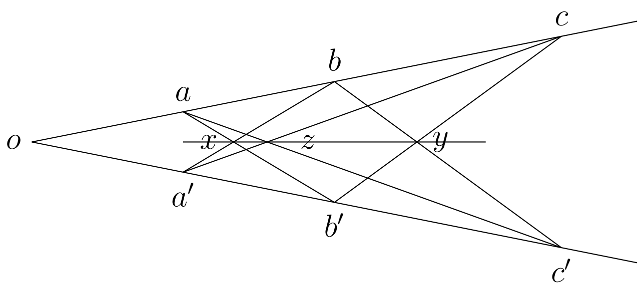
First, I consider a condition that helps to exclude $\mathbb{H}$ and to characterize the quantum Kripke frames induced by infinite-dimensional Hilbert spaces over $\mathbb{R}$ or $\mathbb{C}$ only. For this aim, I need to define the notion of Pappian quasi-quantum Kripke frames.

2.7.11. Definition. A quasi-quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$ is *Pappian*, if it has an orthogonal set of cardinality 3 and, for any $a, b, c, a', b', c' \in \Sigma$ satisfying all of the following:

1. a, b, c, a', b', c' are all distinct;
2. $c \in \sim\sim\{a, b\}$;
3. $c' \in \sim\sim\{a', b'\}$;
4. $\sim\sim\{a, b\} \cap \sim\sim\{a', b'\} = \{o\}$, for some $o \in \Sigma \setminus \{a, b, c, a', b', c'\}$;
5. $x \in \sim\sim\{a, b'\} \cap \sim\sim\{a', b\}$;
6. $y \in \sim\sim\{b, c'\} \cap \sim\sim\{b', c\}$;
7. $z \in \sim\sim\{c, a'\} \cap \sim\sim\{c', a\}$;

it holds that $s_1 \in \sim\sim\{s_2, s_3\}$, for some $s_1, s_2, s_3 \in \Sigma$ with $\{s_1, s_2, s_3\} = \{x, y, z\}$.

This definition involves complicated configurations. The following picture of the analogue in an affine plane may help to make sense of it:



2.7.12. Remark. It is not hard to verify that a quasi-quantum Kripke frame $\mathfrak{F}$ is Pappian, if and only if the irreducible pure orthogeometry $\mathbf{G}(\mathfrak{F})$ is Pappian in the sense of Definition B.3.6; and an irreducible pure orthogeometry $\mathcal{G}$ is Pappian, if and only if the quasi-quantum Kripke frame $\mathbf{F}(\mathcal{G})$ is Pappian.

The following proposition is a representation theorem of Pappian quasi-quantum Kripke frames via vector spaces over fields.

2.7.13. PROPOSITION. *For a quasi-quantum Kripke frame $\mathfrak{F} = (\Sigma, \rightarrow)$, the following are equivalent:*

(i) $\mathfrak{F}$ *is Pappian;*

(ii) there is a vector space V of dimension as least 3 over some field $\mathcal{K}$ *equipped with an anisotropic Hermitian form Φ such that $(\Sigma(V), \rightarrow_V) \cong \mathfrak{F}$.*

Moreover, both V and $\mathcal{K}$ are unique up to isomorphism, if they exist.

Proof. The proof is very similar to that of Corollary 2.5.5, and the only significant difference is that *3* in Theorem B.3.11 is used instead of Theorem B.3.8. ⊣

The following theorem characterizes the quantum Kripke frames which are induced by infinite-dimensional Hilbert spaces over $\mathbb{R}$ or $\mathbb{C}$.

2.7.14. THEOREM. *For a quantum Kripke frame $\mathfrak{F} = (\Sigma, \rightarrow)$, the following are equivalent:*

(i) $\mathfrak{F}$ *is Pappian, has an infinite orthogonal set and satisfies Property OHC;*

(ii) $\mathfrak{F}$ *is isomorphic to $(\Sigma(\mathcal{H}), \rightarrow_{\mathcal{H}})$, where $\mathcal{H}$ is an infinite-dimensional Hilbert space over $\mathbb{R}$ or $\mathbb{C}$.*

Proof. This theorem follows from Theorem 2.7.10 and Proposition 2.7.13. ⊣

Next, I further exclude $\mathbb{R}$ and characterize the quantum Kripke frames induced by infinite-dimensional Hilbert spaces over $\mathbb{C}$. For this aim, I introduce the following condition on a quasi-quantum Kripke frame $\mathfrak{F} = (\Sigma, \rightarrow)$:

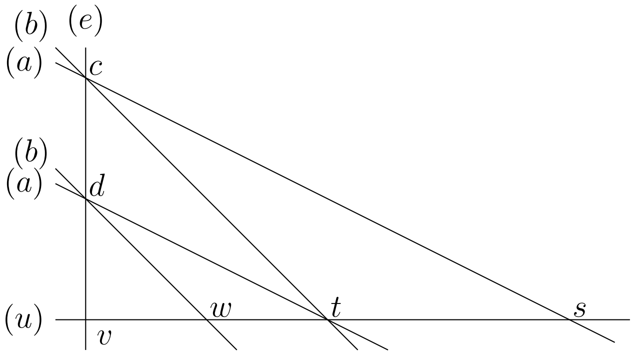
- **Existence of Square Roots (SR)**

  For any four distinct elements $u, v, w, s \in \Sigma$ such that $w, s \in {\sim\sim}\{u, v\}$, there are $t, a, b, c, d \in \Sigma$ such that all of the following hold:

  1. a, b, c, d are all distinct;
  2. $w_1 \notin {\sim\sim}\{w_2, w_3\}$, for any $w_1, w_2, w_3 \in \{a, b, c, d\}$ such that $w_1 \neq w_2 \neq w_3 \neq w_1$;
  3. $u \in {\sim\sim}\{a, b\}$;
  4. $v \in {\sim\sim}\{c, d\}$;
  5. $w \in {\sim\sim}\{b, d\}$;
  6. $s \in {\sim\sim}\{a, c\}$;

7. $t \in \sim\sim\{a, d\} \cap \sim\sim\{b, c\} \cap \sim\sim\{u, v\}$;

This definition involves complicated configurations. The following picture of the analogue in an affine plane may help to make sense of it: (in the picture '(u)' means that u is not a point in the affine plane; instead, it is an imaginary point at infinity where parallel lines intersect.)



The following lemma about some basic geometric facts may also help with understanding the property.

2.7.15. LEMMA. *In a geometric frame $\mathfrak{F} = (\Sigma, \to)$, let $u, v, w, s \in \Sigma$ be distinct and such that $w, s \in \sim\sim\{u, v\}$. Also let $t, a, b, c, d \in \Sigma$ such that all of 1 to 7 in the above definition hold. Then*

(1) $\sim\sim\{a, b\} \cap \sim\sim\{c, d\} = \{e\}$, *for some* $e \in \Sigma$;

(2) $e \neq v$;

(3) $\sim\sim\{e, v\} = \sim\sim\{c, d\}$;

(4) $t \neq u$;

(5) $t \neq v$;

(6) $e \notin \sim\sim\{u, v\}$;

(7) $d \neq v$.

Proof. **For *(1)*:** First show that $\sim\sim\{a, b\} \cap \sim\sim\{c, d\} \neq \emptyset$. By 7 $t \in \sim\sim\{a, d\} \cap \sim\sim\{b, c\}$. Then $d \neq t$; otherwise, $d \in \sim\sim\{b, c\}$, contradicting 2. Hence $a \in \sim\sim\{d, t\}$ by Corollary 2.3.5. Since $a \in \sim\sim\{d, t\}$, $t \in \sim\sim\{b, c\}$ and $a \neq b$ by 1, $\sim\sim\{a, b\} \cap \sim\sim\{c, d\} \neq \emptyset$ by Proposition 2.3.7.

Second show that $\sim\sim\{a, b\} \cap \sim\sim\{c, d\}$ has at most one element. Suppose (towards a contradiction) that $x, y \in \sim\sim\{a, b\} \cap \sim\sim\{c, d\}$ and $x \neq y$. Then by Corollary 2.3.6 $\sim\sim\{a, b\} = \sim\sim\{x, y\} = \sim\sim\{c, d\}$. Hence $a \in \sim\sim\{a, b\} = \sim\sim\{c, d\}$, contradicting 2.

As a result, $\sim\sim\{a,b\} \cap \sim\sim\{c,d\} = \{e\}$, for some $e \in \Sigma$.

For *(2)*: Suppose (towards a contradiction) that $e = v$. Then $u, v \in \sim\sim\{a,b\}$. Since $u \neq v$, by Corollary 2.3.6 $\sim\sim\{u,v\} = \sim\sim\{a,b\}$. Since $a \neq b$, $t \neq a$ or $t \neq b$. Two cases need to be considered.

Case 1: $t \neq a$. Then $d \in \sim\sim\{a,t\}$ follows from $t \in \sim\sim\{a,d\}$ and Corollary 2.3.5. Since $a, t \in \sim\sim\{u,v\} = \sim\sim\{a,b\}$, $\sim\sim\{a,t\} = \sim\sim\{a,b\}$ by Corollary 2.3.6. It follows that $d \in \sim\sim\{a,t\} = \sim\sim\{a,b\}$, contradicting 2.

Case 2: $t \neq b$. Similar to Case 1, using $t \in \sim\sim\{b,c\}$ one can derive that $c \in \sim\sim\{a,b\}$, contradicting 2.

Therefore, $e \neq v$.

For *(3)*: It follows from *(2)*, $e, v \in \sim\sim\{c,d\}$ and Corollary 2.3.6.

For *(4)*: Suppose (towards a contradiction) that $t = u$. Since $a \neq b$, $t \neq a$ or $t \neq b$. Two cases need to be considered.

Case 1: $t \neq a$. Then it follows from $t \in \sim\sim\{a,d\}$ that $d \in \sim\sim\{t,a\}$. Since $t, a \in \sim\sim\{a,b\}$ by 3, $\sim\sim\{t,a\} = \sim\sim\{a,b\}$ by Corollary 2.3.6. Hence $d \in \sim\sim\{a,b\}$, contradicting 2.

Case 2: $t \neq b$. Similar to Case 1,using $t \in \sim\sim\{b,c\}$ one can derive that $c \in \sim\sim\{a,b\}$, contradicting 2.

Therefore, $t \neq u$.

For *(5)*: The proof is similar to that of *(4)*, the only significant difference is that $t \neq c$ or $t \neq d$ is used, instead of $t \neq a$ or $t \neq b$.

For *(6)*: Suppose (towards a contradiction) that $e \in \sim\sim\{u,v\}$. Since $e, v \in \sim\sim\{u,v\}$ and $e \neq v$ by *(2)*, $\sim\sim\{e,v\} = \sim\sim\{u,v\}$ by Corollary 2.3.6. Hence $\sim\sim\{u,v\} = \sim\sim\{e,v\} = \sim\sim\{c,d\}$. Since $c \neq d$, $t \neq c$ or $t \neq d$. Two cases need to be considered.

Case 1: $t \neq c$. Then $b \in \sim\sim\{c,t\}$ follows from $t \in \sim\sim\{b,c\}$ and Corollary 2.3.5. Since $c, t \in \sim\sim\{u,v\} = \sim\sim\{c,d\}$, $\sim\sim\{c,t\} = \sim\sim\{c,d\}$ by Corollary 2.3.6. It follows that $b \in \sim\sim\{c,t\} = \sim\sim\{c,d\}$, contradicting 2.

Case 2: $t \neq d$. Similar to Case 1, using $t \in \sim\sim\{a,d\}$ one can derive that $a \in \sim\sim\{c,d\}$, contradicting 2.

Therefore, $e \notin \sim\sim\{u,v\}$, so $e \notin \{u,v,w,s,t\}$.

For *(7)*: Suppose (towards a contradiction) that $d = v$. On the one hand, since $v \neq w$, $d \neq w$. Then $b \in \sim\sim\{d,w\}$ follows from $w \in \sim\sim\{b,d\}$. Since $d, w \in \sim\sim\{u,v\}$, $b \in \sim\sim\{d,w\} \subseteq \sim\sim\{u,v\}$. On the other hand, by *(5)* $t \neq v = d$. Then $a \in \sim\sim\{t,d\}$ follows from $t \in \sim\sim\{a,d\}$. Since $t, d \in \sim\sim\{u,v\}$, $a \in \sim\sim\{t,d\} \subseteq \sim\sim\{u,v\}$. Therefore, $a, b \in \sim\sim\{u,v\}$. Since $a \neq b$, $\sim\sim\{a,b\} = \sim\sim\{u,v\}$ by Corollary 2.3.6. Then $d = v \in \sim\sim\{u,v\} = \sim\sim\{a,b\}$, contradicting 2. As a result, $d \neq v$. ⊣

Next I reveal the analytic content of Property SR.

2.7.16. Lemma. *Let V be a vector space of dimension at least 3 over some division ring $\mathcal{K} = (K, +, \cdot, 0, 1)$ equipped with an anisotropic Hermitian form Φ. The* *following are equivalent:*

(i) $(\Sigma(V), \to_V)$ *is a quasi-quantum Kripke frame with Property SR;*

(ii) *For every* $x \in K$*, there is a* $y \in K$ *such that* $x = y \cdot y$.

Proof. **From (i) to (ii):** Let $x \in K$ be arbitrary. If $x \in \{0, 1\}$, then $x = x \cdot x$ and y can be defined to be x. In the following, I focus on the case when $x \notin \{0, 1\}$. Since V is of dimension at least 3, take two linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$. Then $u = \langle \mathbf{u} \rangle, v = \langle \mathbf{v} \rangle, w = \langle \mathbf{u} + \mathbf{v} \rangle, s = \langle x\mathbf{u} + \mathbf{v} \rangle$ satisfy that they are distinct and $s, w \in {\sim\sim}\{u, v\}$. Since $(\Sigma(V), \to_V)$ has Property SR, there are $t, a, b, c, d \in \Sigma(V)$ satisfying 1 to 7 in the definition of Property SR. By *(1)* in Lemma 2.7.15 let $e \in {\sim\sim}\{a, b\} \cap {\sim\sim}\{c, d\}$. By 2 in the definition of Property SR $e \notin \{a, b, c, d\}$.

Now I find the vectors that generate the involved one-dimensional subspaces.

For t, by *(4)* and *(5)* in the previous lemma $t \notin \{u, v\}$ and $t \in {\sim\sim}\{u, v\}$, so there is a $y \in K$ such that $t = \langle y\mathbf{u} + \mathbf{v} \rangle$.

For e, let $\mathbf{e} \in V \setminus \{\mathbf{0}\}$ be such that $\langle \mathbf{e} \rangle = e$. By *(6)* in the previous lemma $\mathbf{u}, \mathbf{v}, \mathbf{e}$ are linearly independent.

For d, $d \notin \{e, v\}$ by *(7)* in the previous lemma and the definition of e. Since $d \in {\sim\sim}\{c, d\} = {\sim\sim}\{e, v\}$ by *(3)*, without loss of generality I can let $d = \langle \mathbf{v} + \mathbf{e} \rangle$. It follows that $d \notin {\sim\sim}\{u, v\}$, and thus $d \notin \{u, v, w, s, t\}$.

For b, since $w \in {\sim\sim}\{b, d\}$ and $d \neq w$, $b \in {\sim\sim}\{w, d\}$. Since $e, u \in {\sim\sim}\{a, b\}$ and $e \neq u$, ${\sim\sim}\{e, u\} = {\sim\sim}\{a, b\}$. Hence $b \in {\sim\sim}\{e, u\} \cap {\sim\sim}\{w, d\}$. Therefore, it is not hard to see $b = \langle \mathbf{u} - \mathbf{e} \rangle$, so $b \notin \{u, v, w, s, t\}$.

For a, since $u \in {\sim\sim}\{a, b\}$ and $b \neq u$, $a \in {\sim\sim}\{b, u\}$. Since $t \in {\sim\sim}\{a, d\}$ and $t \neq d$, $a \in {\sim\sim}\{t, d\}$. Hence $a \in {\sim\sim}\{b, u\} \cap {\sim\sim}\{t, d\}$. Therefore, it is not hard to see $a = \langle y\mathbf{u} - \mathbf{e} \rangle$, so $a \notin \{u, v, w, s, t\}$.

For c, $c \in {\sim\sim}\{c, d\} = {\sim\sim}\{e, v\}$ by *(3)*. Since $s \in {\sim\sim}\{a, c\}$ and $s \neq a$, $c \in {\sim\sim}\{s, a\}$. Since $t \in {\sim\sim}\{b, c\}$ and $t \neq b$, $c \in {\sim\sim}\{b, t\}$. Hence I have two ways to find vectors that generate c. From $c \in {\sim\sim}\{e, v\} \cap {\sim\sim}\{s, a\}$, $c = \langle y\mathbf{v} + x\mathbf{e} \rangle$. From $c \in {\sim\sim}\{e, v\} \cap {\sim\sim}\{b, t\}$, $c = \langle \mathbf{v} + y\mathbf{e} \rangle$.

Hence $\langle y\mathbf{v} + x\mathbf{e} \rangle = c = \langle \mathbf{v} + y\mathbf{e} \rangle$. Then it is not hard to show that $x = y \cdot y$.

From (ii) to (i): By Corollary 2.5.5 it suffices to show that $(\Sigma(V), \to_V)$ has Property SR. Assume that $u, v, w, s \in \Sigma(V)$ are distinct such that $w, s \in {\sim\sim}\{u, v\}$. Let $\mathbf{u}, \mathbf{v} \in V \setminus \{\mathbf{0}\}$ be such that $u = \langle \mathbf{u} \rangle$ and $v = \langle \mathbf{v} \rangle$. Since $w, s \in {\sim\sim}\{u, v\}$ and $w, s \notin \{u, v\}$, without loss of generality suppose that $w = \langle \mathbf{u} + \mathbf{v} \rangle$ and $s = \langle x\mathbf{u} + \mathbf{v} \rangle$ with $x \notin \{0, 1\}$. By (ii) there is a $y \in K$ such that $x = y \cdot y$. Since V is of dimension at least 3, take $\mathbf{e} \in V \setminus \{\mathbf{0}\}$ such that $\mathbf{u}, \mathbf{v}, \mathbf{e}$ are linearly independent. Then let

1) $t = \langle y\mathbf{u} + \mathbf{v} \rangle$;

2) $a = \langle y\mathbf{u} - \mathbf{e} \rangle$;

3) $b = \langle \mathbf{u} - \mathbf{e} \rangle$;

4) $c = \langle \mathbf{v} + y\mathbf{e} \rangle$;

5) $d = \langle \mathbf{v} + \mathbf{e} \rangle$.

It is a straightforward verification that t, a, b, c, d satisfy 1 to 7 in the definition of Property SR. ⊣

2.7.17. Remark. Intuitively, according to this lemma, every arguesian quasi-quantum Kripke frame $(\Sigma(V), \to_V)$ has Property SR, if and only if every element of the underlying division ring of V has a square root.

The idea of Property SR comes from [91] (Lemma 6.2). I do not just take Axiom VI in the paper because it is too lattice-theoretic.

2.7.18. Theorem. *Let $\mathfrak{F} = (\Sigma, \to)$ be a quantum Kripke frame. The following are equivalent:*

(i) $\mathfrak{F}$ is Pappian, has an infinite orthogonal set and satisfies Property OHC and Property SR;

(ii) $\mathfrak{F} \cong (\Sigma(\mathcal{H}), \to_{\mathcal{H}})$, for some infinite-dimensional Hilbert space $\mathcal{H}$ over $\mathbb{C}$.

Proof. This theorem follows from Theorem 2.7.14 and Lemma 2.7.16. ⊣

This theorem characterizes the quantum Kripke frames induced by infinite-dimensional Hilbert spaces over $\mathbb{C}$. Moreover, this characterization is quite simple from a logical point of view. To be precise, note that it is natural to talk about Kripke frames using a predicate language with exactly one binary relation symbol. In this formal language, all of the conditions on Kripke frames involved in (i) of the theorem can be expressed in *first-order* formulas, except that Property A and the existence of an infinite orthogonal set are second-order. Existence of an infinite orthogonal set enforces infinite-dimensionality, so it is not a surprise that it is second-order. It would be nice to replace Property A by some first-order conditions, but this is impossible. I prove this in Subsection 4.1.4 (Theorem 4.1.15) where the setting is more formal.

However, the significance of Theorem 2.7.18 only lies in the possibility or existence of such a characterization. I have to confess that being Pappian, Property OHC and Property SR are proposed for purely mathematical considerations, and thus very complicated and ad hoc. It would be nice to find conditions which are equivalent but simpler and which make more sense from a physical point of view.

The above is just one way of characterizing the quantum Kripke frames induced by infinite-dimensional Hilbert spaces. Other ways are hinted at by results in the literature. For example, the results in [67] hint at the fact that the quantum Kripke frames induced by infinite-dimensional Hilbert spaces over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ can be characterized by the existence of an automorphism with certain properties, and the underlying division rings can be identified by the structure formed by the automorphisms on quantum Kripke frames. The results in [7] also hint at the fact that the structure formed by the automorphisms on quantum Kripke frames is useful in characterizing the underlying division rings. This way of proceeding makes physical sense, because by using Wigner's Theorem[14] one can show that the automorphisms on a quantum Kripke frame induced by a Hilbert space $\mathcal{H}$ over $\mathbb{C}$ are all induced by the unitary or anti-unitary operators on $\mathcal{H}$, which model the evolution of a quantum system, according to Subsection 1.2.3. However, since this idea uses automorphisms, it has the nature of a higher-order type when viewed from a logical perspective. Therefore, I do not present the details of this direction. In Proposition 4.1.6 I provide two conditions equivalent to Property OHC on quasi-quantum Kripke frames which involve automorphisms.

Using the notion of subframes, a corollary of Theorem 2.7.18 can be drawn, which characterizes the quantum Kripke frames induced by Hilbert spaces over $\mathbb{C}$ (in general, not just the infinite-dimensional ones).

2.7.19. Corollary. *Let $\mathfrak{F} = (\Sigma, \to)$ be a quantum Kripke frame. The following are equivalent:*

(i) $\mathfrak{F}$ is isomorphic to a subframe of a quantum Kripke frame which is Pappian, has an infinite orthogonal set and has Property OHC and Property SR;

(ii) $\mathfrak{F} \cong (\Sigma(\mathcal{H}), \to_{\mathcal{H}})$, for some Hilbert space $\mathcal{H}$ over $\mathbb{C}$.

Proof. **From (i) to (ii):** Assume that (i) holds. By Theorem 2.7.18, without loss of generality, assume that $\mathfrak{F} \cong (P, \to_{\mathcal{H}})$, where $(P, \to_{\mathcal{H}})$ is a subframe of the quantum Kripke frame $(\Sigma(\mathcal{H}), \to_{\mathcal{H}})$ induced by some infinite-dimensional Hilbert space $\mathcal{H}$ over $\mathbb{C}$. If $P = \Sigma(\mathcal{H})$, then $\mathfrak{F} \cong (\Sigma(\mathcal{H}), \to_{\mathcal{H}})$. In the following, I focus on the case when $P \neq \Sigma(\mathcal{H})$. Since $(P, \to_{\mathcal{H}})$ is a subframe of $(\Sigma(\mathcal{H}), \to_{\mathcal{H}})$, P is non-empty and bi-orthogonally closed by Proposition 2.6.10. Hence $\sim P \neq \emptyset$ follows from $P \neq \Sigma(\mathcal{H})$. By Lemma B.3.9 $\bigcup P = ((\bigcup P)^{\perp})^{\perp}$ in $\mathcal{H}$. According to the theory of Hilbert spaces, $\bigcup P$ is a closed linear subspace of $\mathcal{H}$, and it forms a Hilbert space equipped with the addition, the multiplications by scalars and the inner product inherited from $\mathcal{H}$. Note that by Lemma B.3.2 $P = \Sigma(\bigcup P)$ and by definition $\langle \mathbf{u} \rangle \to_{\mathcal{H}} \langle \mathbf{v} \rangle \Leftrightarrow \langle \mathbf{u}, \mathbf{v} \rangle \neq 0 \Leftrightarrow \langle \mathbf{u} \rangle \to_{\bigcup P} \langle \mathbf{v} \rangle$, for any $\mathbf{u}, \mathbf{v} \in (\bigcup P) \setminus \{\mathbf{0}\}$. Hence $(P, \to_{\mathcal{H}}) = (\Sigma(\bigcup P), \to_{\bigcup P})$. Since $\mathfrak{F}$ is isomorphic to $(P, \to_{\mathcal{H}})$, it is isomorphic to $(\Sigma(\bigcup P), \to_{\bigcup P})$, where $\bigcup P$ forms a Hilbert space over $\mathbb{C}$.

From (ii) to (i): Assume that (ii) holds. If $\mathcal{H}$ is infinite-dimensional, then by Theorem 2.7.18 it is easy to see that (i) holds. In the following, I focus on the case when $\mathcal{H}$ is finite-dimensional. Let l_2 be the separable Hilbert space of absolutely

[14] This theorem is proved in [89], of which [90] is the translation in English. In [38] a nice and elementary proof can be found.

squared summable sequences of complex numbers (Example 2.1.12 in [61]). By Theorem 2.7.18 $(\Sigma(l_2), \rightarrow_{l_2})$ is a quantum Kripke frame which is Pappian, has an infinite orthogonal set and satisfies Property OHC and Property SR. Now, on the one hand, since $\mathcal{H}$ is finite-dimensional and l_2 is countably infinite-dimensional, by the theory of Hilbert spaces $\mathcal{H}$ is isomorphic to a finite-dimensional closed linear subspace V of l_2, which form a Hilbert space equipped with the addition, the multiplications by scalars and the inner product inherited from l_2. Hence $(\Sigma(\mathcal{H}), \rightarrow_{\mathcal{H}}) \cong (\Sigma(V), \rightarrow_{l_2})$, and thus $\Sigma(V) \neq \emptyset$. On the other hand, since V is a closed linear subspace, by the theory of Hilbert spaces $V = (V^{\perp})^{\perp}$, and thus $\Sigma(V)$ is bi-orthogonally closed in $(\Sigma(l_2), \rightarrow_{l_2})$ by Lemma B.3.9. It follows from Proposition 2.6.10 that $(\Sigma(V), \rightarrow_{l_2})$ is a subframe of $(\Sigma(l_2), \rightarrow_{l_2})$. Therefore, $\mathfrak{F} \cong (\Sigma(V), \rightarrow_{l_2})$, which is a subframe of the quantum Kripke frame $(\Sigma(l_2), \rightarrow_{l_2})$ with the required properties. ⊣

According to this corollary, quantum Kripke frames are closely related to Hilbert spaces over $\mathbb{C}$, which are used in the standard formalism of quantum mechanics. Hence quantum Kripke frames are quantum structures and can be used to model quantum systems.

2.7.2 Piron Lattices and Other Quantum Structures

In the previous subsection, I have shown that quantum Kripke frames are quantum structures. In the literature on the foundations of quantum theory, there are many other quantum structures. In this subsection, I review two famous kinds of quantum structures, Piron lattices and quantum dynamic frames, and discuss the relation between them and quantum Kripke frames.

According to Subsection 1.1.1, quantum logic originates from the result of Birkhoff and von Neumann in [22] that the testable properties of a quantum system form an ortho-lattice which, unlike the case of classical systems, is not a Boolean algebra. Then it is natural to ask for a lattice-theoretic characterization of such lattices. In [74] Piron defined the special lattices called irreducible propositional systems and proved a representation theorem for them via generalized Hilbert spaces[15]. The definition of irreducible propositional systems is recalled in Definition A.0.7 in Appendix A. The representation theorem goes as follows:

2.7.20. THEOREM (PIRON'S THEOREM). *For a generalized Hilbert space V, the set of all (bi-orthogonally) closed subspaces of V, i.e. subspaces S satisfying $S = (S^{\perp})^{\perp}$, forms an irreducible propositional system with set-theoretic inclusion $\subseteq$ as the partial order and $(\cdot)^{\perp} : \wp(V) \rightarrow \wp(V)$ (restricted to this set) as the orthocomplementation.*

[15] Piron's result was first published in his PhD thesis [72] and in a journal [73], both in French. [74] is the first time when this theorem and its proof are available in English.

Every irreducible propositional system $\mathfrak{L} = (L, \leq, (\cdot)')$*, which is of rank at least 4, i.e. has four* distinct *elements* a_1, a_2, a_3, a_4 *such that* $O \neq a_1 \leq a_2 \leq a_3 \leq a_4$*, is isomorphic to an ortho-lattice of the above form.*

Proof. Proofs of this famous theorem can be found in [74] (Theorem 3.23), [86] (Theorem 4.40) and [84] (Theorem 81). ⊣

Considering the close relation between generalized Hilbert spaces and Hilbert spaces (Theorem 2.7.3), Piron's result is commonly regarded as a milestone in the foundations of quantum physics.[16] Hence irreducible propositional systems were later called *Piron lattices* (e.g. in [14] and [93]). I adopt this name in this thesis.

Further study on Piron lattices shows that they are in correspondence with irreducible Hilbertian geometries.

2.7.21. Theorem.

1. *For every irreducible Hilbertian geometry* $\mathcal{G} = (G, \star, \perp)$*, the set* $\mathcal{L}$ *of all closed subspaces of* $\mathcal{G}$*, i.e. subspaces* E *satisfying* $(E^{\perp})^{\perp} = E$*, forms a Piron lattice, denoted by* $\mathbf{P}(\mathcal{G})$*, with set-theoretic inclusion* $\subseteq$ *as the partial order and* $(\cdot)^{\perp} : \mathcal{L} \to \mathcal{L}$ *as the orthocomplementation.*

2. *For every Piron lattice* $\mathfrak{L} = (L, \leq, (\cdot)')$*, define* $\mathbf{Q}(\mathfrak{L}) = (At(\mathfrak{L}), \star, \perp)$*, where*

   - $At(\mathfrak{L})$ *is the set of all atoms of* $\mathfrak{L}$*;*
   - $\star : At(\mathfrak{L}) \times At(\mathfrak{L}) \to At(\mathfrak{L})$ *is a function such that, for any* $p, q \in At(\mathfrak{L})$*,* $p \star q \overset{\text{def}}{=} \{r \in At(\mathfrak{L}) \mid r \leq p \vee q\}$*;*
   - $\perp \subseteq At(\mathfrak{L}) \times At(\mathfrak{L})$ *is such that* $p \perp q \Leftrightarrow p \leq q'$*, for any* $p, q \in At(\mathfrak{L})$*.*

   Then $\mathbf{Q}(\mathfrak{L})$ *is an irreducible Hilbertian geometry.*

3. $\mathcal{G} \mapsto \mathbf{P}(\mathcal{G})$ *is a class function, denoted by* $\mathbf{P}$*, from the class of irreducible Hilbertian geometries to the class of Piron lattices.*

   $\mathfrak{L} \mapsto \mathbf{Q}(\mathfrak{L})$ *is a class function, denoted by* $\mathbf{Q}$*, from the class of Piron lattices to the class of irreducible Hilbertian geometries.*

   Moreover, for any irreducible Hilbertian geometry $\mathcal{G}$ *and Piron lattice* $\mathfrak{L}$*,*

$$\mathbf{Q} \circ \mathbf{P}(\mathcal{G}) \cong \mathcal{G} \qquad \mathbf{P} \circ \mathbf{Q}(\mathfrak{L}) \cong \mathfrak{L}$$

[16] For a more detailed discussion of Piron's contribution and its background, please refer to Subsection 1.1.1.

Proof. This follows from Theorem 66 and Proposition 79 in [84]. ⊣

Another important kind of quantum structure, called a quantum dynamic frame, was proposed relatively recently and was observed to be in correspondence with Piron lattices ([14]).[17] The definition of quantum dynamic frames is recalled in Definition A.0.8 in Appendix A. The correspondence is the following theorem:

2.7.22. Theorem.

1. *For every quantum dynamic frame* $\mathsf{F} = (\Sigma, \mathcal{L}, \{\stackrel{P?}{\to}\}_{P\in\mathcal{L}})$, $\mathcal{L}$ *forms a Piron lattice, denoted by* $\mathbf{U}(\mathsf{F})$, *with set-theoretic inclusion* $\subseteq$ *as the partial order and* $\sim(\cdot) : \mathcal{L} \to \mathcal{L}$ *as the orthocomplementation.*

2. *For every Piron lattice* $\mathfrak{L} = (L, \leq, (\cdot)')$, *define* $\mathbf{V}(\mathfrak{L}) = (At(\mathfrak{L}), \mathcal{L}, \{\stackrel{[\![a]\!]?}{\longrightarrow}\}_{a\in\mathcal{L}})$, *where*

   - $At(\mathfrak{L})$ *is the set of all atoms of* $\mathfrak{L}$;
   - $\mathcal{L} \stackrel{\text{def}}{=} \{[\![a]\!] \mid a \in L\}$, *in which* $[\![a]\!]$ *is the set of all atoms below* $a \in L$;
   - *for each* $[\![a]\!] \in \mathcal{L}$, $\stackrel{[\![a]\!]?}{\longrightarrow} \subseteq At(\mathfrak{L}) \times At(\mathfrak{L})$ *is such that for any* $p, q \in At(\mathfrak{L})$
   $$p \stackrel{[\![a]\!]?}{\longrightarrow} q \quad \Leftrightarrow \quad q = a \wedge (p \vee a')$$

   Then $\mathbf{V}(\mathfrak{L})$ *is a quantum dynamic frame.*

3. $\mathsf{F} \mapsto \mathbf{U}(\mathsf{F})$ *is a class function, denoted by* $\mathbf{U}$, *from the class of quantum dynamic frames to the class of Piron lattices.*

   $\mathfrak{L} \mapsto \mathbf{V}(\mathfrak{L})$ *is a class function, denoted by* $\mathbf{V}$, *from the class of Piron lattices to the class of quantum dynamic frames.*

   Moreover, for any quantum dynamic frame F *and Piron lattice* $\mathfrak{L}$,

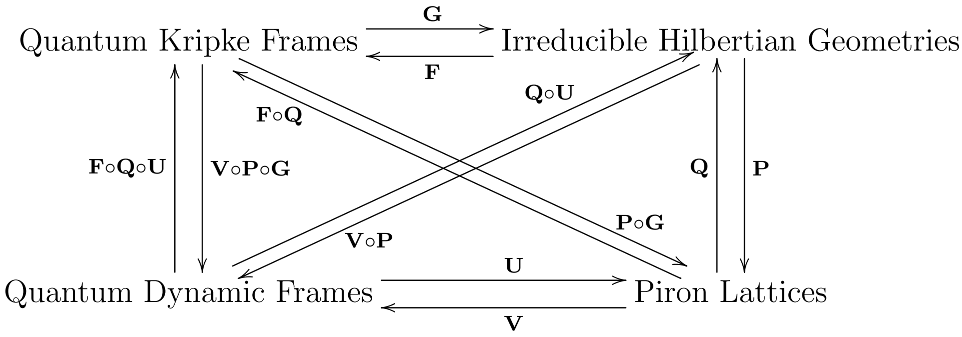
$$\mathbf{V} \circ \mathbf{U}(\mathsf{F}) \cong \mathsf{F} \qquad \mathbf{U} \circ \mathbf{V}(\mathfrak{L}) \cong \mathfrak{L}$$

Proof. Please refer to [20] (Theorem 3.24) for a detailed proof. ⊣

Combining the above two theorems with the correspondence between quantum Kripke frames and irreducible Hilbertian geometries (Theorem 2.5.2), a square of

[17] For more about the motivations and the background of quantum dynamic frames, please refer to Subsection 1.1.1.

mutually corresponding structures can be obtained:



I make the correspondences involving quantum Kripke frames, Piron lattices and quantum dynamic frames explicit by the following theorems.

2.7.23. THEOREM.

1. *For every quantum Kripke frame* $\mathfrak{F} = (\Sigma, \to)$, $\mathbf{P} \circ \mathbf{G}(\mathfrak{F}) = (\mathcal{L}_{\mathfrak{F}}, \subseteq, \sim(\cdot))$ *is a Piron lattice, where* $\mathcal{L}_{\mathfrak{F}}$ *is the set of all bi-orthogonally closed subsets,* $\subseteq$ *is set-theoretic inclusion, and* $\sim(\cdot)$ *is the orthocomplement with respect to* $\to$ *(restricted to* $\mathcal{L}_{\mathfrak{F}}$*).*

2. *For every Piron lattice* $\mathfrak{L} = (L, \leq, (\cdot)')$, $\mathbf{F} \circ \mathbf{Q}(\mathfrak{L}) = (At(\mathfrak{L}), \to)$ *is a quantum Kripke frame, where* $At(\mathfrak{L})$ *is the set of all atoms of* $\mathfrak{L}$ *and* $\to \subseteq At(\mathfrak{L}) \times At(\mathfrak{L})$ *is such that, for any* $p, q \in At(\mathfrak{L})$, $p \to q \Leftrightarrow p \not\leq q'$ *.*

3. $\mathbf{P} \circ \mathbf{G}$ *is a class function from the class of quantum Kripke frames to the class of Piron lattices.*

   $\mathbf{F} \circ \mathbf{Q}$ *is a class function from the class of Piron lattices to the class of quantum Kripke frames.*

   Moreover, for any quantum Kripke frame $\mathfrak{F}$ *and Piron lattice* $\mathfrak{L}$*,*

$$(\mathbf{F} \circ \mathbf{Q}) \circ (\mathbf{P} \circ \mathbf{G})(\mathfrak{F}) \cong \mathfrak{F} \qquad (\mathbf{P} \circ \mathbf{G}) \circ (\mathbf{F} \circ \mathbf{Q})(\mathfrak{L}) \cong \mathfrak{L}$$

Proof. This follows from Corollary 2.3.18, Theorem 2.5.2 and Theorem 2.7.21. ⊣

2.7.24. REMARK. There are some forerunners of *1* of this theorem in this thesis. For a quantum Kripke frame $\mathfrak{F}$, since it is a state space, by Proposition 2.2.3 $\mathcal{L}_{\mathfrak{F}}$ forms a complete atomistic ortho-lattice with $\subseteq$ as the partial order and $\sim(\cdot)$ the orthocomplementation. Moreover, since it is a geometric frame, by Corollary 2.4.8 the saturated subsets form an orthomodular poset under $\subseteq$ and $\sim(\cdot)$. By Theorem 2.4.9 $\mathcal{L}_{\mathfrak{F}}$ is the same as the set of saturated subsets, and thus it forms a complete atomistic ortho-lattice satisfying Weak Modularity. Moreover, for any $P, Q \in \mathcal{L}_{\mathfrak{F}}$, $P \vee Q = P \sqcup Q$. A closer examination yields that this lattice satisfies the Covering Law. Finally, irreducibility follows from Superposition.

2.7.25. Theorem.

1. *For every quantum Kripke frame* $\mathfrak{F} = (\Sigma, \to)$, $\mathbf{X}(\mathfrak{F}) = (\Sigma, \mathcal{L}_{\mathfrak{F}}, \{\stackrel{P?}{\to}\}_{P \in \mathcal{L}_{\mathfrak{F}}})$ *is a quantum dynamic frame, where* $\mathcal{L}_{\mathfrak{F}}$ *is the set of all bi-orthogonally closed subsets and, for every* $P \in \mathcal{L}_{\mathfrak{F}}$, $\stackrel{P?}{\to} \subseteq \Sigma \times \Sigma$ *is such that, for any* $s, t \in \Sigma$,
$$s \stackrel{P?}{\to} t \quad \Leftrightarrow \quad t \in P \text{ and } s \approx_P t.$$

2. *For a quantum dynamic frame* $\mathsf{F} = (\Sigma, \mathcal{L}, \{\stackrel{P?}{\to}\}_{P \in \mathcal{L}})$, $\mathbf{Y}(\mathsf{F}) = (\Sigma, \bigcup_{P \in \mathcal{L}} \stackrel{P?}{\to})$ *is a quantum Kripke frame.*

3. $\mathfrak{F} \mapsto \mathbf{X}(\mathfrak{F})$ *is a class function, denoted by* $\mathbf{X}$*, from the class of quantum Kripke frames to the class of quantum dynamic frames.*

   $\mathsf{F} \mapsto \mathbf{Y}(\mathsf{F})$ *is a class function from the class of quantum dynamic frames to the class of quantum Kripke frames.*

   Moreover, for any quantum Kripke frame $\mathfrak{F}$ *and quantum dynamic frame* F,
$$\mathbf{Y} \circ \mathbf{X}(\mathfrak{F}) = \mathfrak{F} \qquad\qquad \mathbf{X} \circ \mathbf{Y}(\mathsf{F}) = \mathsf{F}$$

Proof. For any quantum Kripke frame $\mathfrak{F}$ and quantum dynamic frame F, Theorem 2.7.22 and Theorem 2.7.23 imply that, $\mathbf{V} \circ \mathbf{P} \circ \mathbf{G}(\mathfrak{F})$ is a quantum dynamic frame and $\mathbf{F} \circ \mathbf{Q} \circ \mathbf{U}(\mathsf{F})$ is a quantum Kripke frame. Moreover, $(\mathbf{F} \circ \mathbf{Q} \circ \mathbf{U}) \circ (\mathbf{V} \circ \mathbf{P} \circ \mathbf{G})(\mathfrak{F}) \cong \mathfrak{F}$, and $(\mathbf{V} \circ \mathbf{P} \circ \mathbf{G}) \circ (\mathbf{F} \circ \mathbf{Q} \circ \mathbf{U})(\mathsf{F}) \cong \mathsf{F}$. Observe that $\mathbf{X}(\mathfrak{F}) \cong (\mathbf{V} \circ \mathbf{P} \circ \mathbf{G})(\mathfrak{F})$ and $\mathbf{Y}(\mathsf{F}) \cong (\mathbf{F} \circ \mathbf{Q} \circ \mathbf{U})(\mathsf{F})$. Hence $\mathbf{X}(\mathfrak{F})$ is a quantum dynamic frame and $\mathbf{Y}(\mathsf{F})$ is a quantum Kripke frame. Moreover, $\mathbf{Y} \circ \mathbf{X}(\mathfrak{F}) \cong \mathfrak{F}$ and $\mathbf{X} \circ \mathbf{Y}(\mathsf{F}) \cong \mathsf{F}$. A further examination of the involved Kripke frames shows that $\mathbf{Y} \circ \mathbf{X}(\mathfrak{F}) = \mathfrak{F}$ and $\mathbf{X} \circ \mathbf{Y}(\mathsf{F}) = \mathsf{F}$. $\dashv$

2.7.3 Classical Frames

In this subsection I make the only diversion in this thesis from quantum physics to classical physics. I propose special Kripke frames called classical frames, which are abstractions of the structure formed by the states of classical systems. I compare them with quantum Kripke frames and discuss the difference between quantum physics and classical physics.

According to Subsection 1.2.1, two states of a quantum system are orthogonal, if and only if no single execution of a measurement can trigger a transition from one of them to the other. In classical physics it is a sound physical idealization that measurements, if designed carefully enough, will not change the state of a system at all. Based on this intuition, I define classical frames as follows:

2.7.26. Definition. A *classical frame* is a Kripke frame $\mathfrak{F} = (\Sigma, \to)$ such that $\to$ is (the graph of) the identity map on Σ, i.e. $\to\, = Id_\Sigma = \{(s,t) \in \Sigma \times \Sigma \mid s = t\}$.

For any two elements in a classical frame, being orthogonal collapses to being distinct. I confess that classical frames are very abstract, but they are at the same level of abstraction as quantum Kripke frames.

The following are some basic properties of classical frames.

2.7.27. LEMMA. *In a classical frame* $\mathfrak{F} = (\Sigma, \to)$,

1. $\sim P = \Sigma \setminus P$, *for every* $P \subseteq \Sigma$;
2. $P = \sim\sim P$, *for every* $P \subseteq \Sigma$.
3. $\mathfrak{F}$ *is a complete geometric frame, i.e. all of Reflexivity, Symmetry, Separation and Property A hold.*

Proof. **For *1*:** $\sim P = \{s \in \Sigma \mid s \not\to t$, for every $t \in P\} = \{s \in \Sigma \mid s \neq t$, for every $t \in P\} = \{s \in \Sigma \mid s \notin P\} = \Sigma \setminus P$.

For *2*: By *1* $\sim\sim P = \Sigma \setminus (\sim P) = \Sigma \setminus (\Sigma \setminus P) = P$.

For *3*: It follows easily from *1*, *2* and the definition. ⊣

2.7.28. REMARK. For a quantum system modelled by a quantum dynamic frame, its testable properties are modelled by the bi-orthogonally closed subsets. This is the same in a classical frame. According to classical physics, for a classical system modelled by a classical frame, its testable properties are modelled by the subsets of the frame, which by *2* of this lemma are the bi-orthogonally closed subsets.

The following proposition shows the difference between quantum Kripke frames and classical frames.

2.7.29. PROPOSITION. *Let* $\mathfrak{F} = (\Sigma, \to)$ *be a complete geometric frame.*

1. $\mathfrak{F}$ *is a quantum Kripke frame, if and only if Superposition holds.*
2. $\mathfrak{F}$ *is a classical frame, if and only if Transitivity holds, i.e.* $s \to t$ *and* $t \to u$ *imply that* $s \to u$, *for any* $s, t, u \in \Sigma$.
3. *Both Superposition and Transitivity hold, if and only if* Σ *has exactly one element.*

Proof. **For *1*:** It follows directly from the definition of quantum Kripke frames.

For *2*: The 'Only If' part is obvious from the definition of classical frames. For the 'If' part, assume that Transitivity holds. Let $s, t \in \Sigma$ be arbitrary such that $s \to t$. Then, for each $w \in \Sigma$, $s \to w$ if and only if $t \to w$ by Symmetry and Transitivity. It follows from the contrapositive of Separation that $s = t$. Combining this with Reflexivity, I conclude that $\to = \{(s, t) \in \Sigma \times \Sigma \mid s = t\}$. Therefore, $\mathfrak{F}$ is a classical frame.

For *3*: The 'If' Part: Assume that Σ has exactly one element, namely s. Then it is not hard to see that $\mathfrak{F} = (\{s\}, \{(s,s)\})$, and thus both Superposition and Transitivity hold.

The 'Only If' Part: Assume that both Superposition and Transitivity hold. Suppose (towards a contradiction) that Σ does not have exactly one element. Since Σ cannot be empty by definition, it must have at least two elements. Let $s, t \in \Sigma$ be such that $s \neq t$. By Superposition there is a $w \in \Sigma$ such that $w \to s$ and $w \to t$. By Transitivity and *2* $\mathfrak{F}$ is a classical frame. It follows that $s = w$ and $t = w$, and thus $s = t$, contradicting that $s \neq t$. As a result, Σ has exactly one element. $\dashv$

This proposition suggests that the superposition of states is a characteristic feature of quantum mechanics.

2.8 Historical Notes

The general idea underlying the notion of quantum Kripke frames is to use Kripke frames, possibly the simplest mathematical structures, to model and study quantum systems. This idea is not new and has its own history, although lattices are originally and largely used in quantum logic. In this section, I give a survey of the work in quantum logic involving Kripke frames, which has inspired and is beneficial for the work in this thesis.

As far as I know, Kripke frames were first used in quantum logic in the mid-1960s by Foulis and his students, which is pointed out at the beginning of Section 1.4 in [92]. The results were first published in [29], which is now hard to assess.

In journal papers, the first time when Kripke frames are used in quantum logic traces back to [45] by Foulis and Randall. In this paper, an *orthogonality space* is defined to be a Kripke frame $(\Sigma, \perp)$ satisfying Symmetry and Irreflexivity, i.e. $s \perp s$ does not hold for every $s \in \Sigma$. $\perp$ is called *the orthogonality relation*.[18] The motivating example is that Σ is a join-dense set in a complete orthomodular lattice $\mathfrak{L}$, and $\perp$ is defined in such a way that, for any $s, t \in \Sigma$, $s \perp t$ if and only if $s \leq t'$, where $(\cdot)'$ is the orthocomplementation in $\mathfrak{L}$. Although the authors of [45] rarely mentioned quantum mechanics, from this motivating example it is obvious that Σ can be the set of all states of a quantum system and $\perp$ the orthogonality relation between states, as is defined in Subsection 1.2.1. In an orthogonality space $(\Sigma, \perp)$, the *orthocomplement of* $P \subseteq \Sigma$, denoted by $P^{\perp}$, is the set $\{s \in \Sigma \mid s \perp t$ for all $t \in P\}$; and $P \subseteq \Sigma$ is called *closed*, if $P = P^{\perp\perp}$. It is mentioned that in [21] Birkhoff proved that the set of all closed subsets of an orthogonality space

[18]Following the common practice in the literature, I use the symbol $\perp$ to denote the orthogonality relation in an orthogonality space in this section of the thesis. In other parts of this thesis, $\perp$ is reserved for the orthogonality relations in orthogeometries.

forms a complete ortho-lattice[19]; however, in [21] the significance of this result in quantum logic is not mentioned at all. An orthogonality space $(\Sigma, \perp)$ is a *complete orthomodular space*, if the set of all closed subsets forms a complete orthomodular lattice, i.e. a complete ortho-lattice satisfying Weak Modularity. In [45] complete orthomodular spaces are characterized in terms of the properties of the orthogonality relations (Theorem 1): an orthogonality space $(\Sigma, \perp)$ is a complete orthomodular space, if and only if $P = Q^{\perp\perp}$ for every closed subset P of Σ and every maximal subset Q of P with the property that any two distinct elements in Q are related by $\perp$.[20] The main result in [45] is that the free monoid of a complete orthomodular space can be organized into a complete orthomodular space in a natural way (Theorem 4).

Shortly after [45], two papers, [31] by Dishkant and [47] by Goldblatt, were published. Both authors proposed to analyse ortho-logic using Kripke frames, inspired by the increased attention for research on Kripke semantics for intensional logics around that period of time. Note that in ortho-lattices there is enough structure to interpret propositional logic: conjunctions are interpreted by meets, disjunctions by joins and negations by orthocomplements. As such, *ortho-logic* is the set of all propositional formulas valid in an algebraic semantics based on ortho-lattices. In [31] Dishkant used Kripke frames $(\Sigma, \to)$ satisfying the following:[21]

R0 $s \to s$, for every $s \in \Sigma$;

R1 for any $s, t \in \Sigma$, $s \to t$ implies that there is a $w \in \Sigma$ such that $t \to w$ and, for each $u \in \Sigma$, $w \to u$ implies that $s \to u$.

The notion of a propositional formula being semantically true can be defined on the basis of these Kripke frames. The main result is that a formula is semantically true if and only if it is in ortho-logic (the Corollary before Remark 2). Therefore, a relational semantics for ortho-logic is obtained. Finally, please note that the condition R1 follows from Symmetry, because one can just take w to be s.

For ortho-logic, Goldblatt devised in [47] a proof system and a relational semantics based on orthogonality spaces, which he called *orthoframes*. He proved that the proof system is sound and complete with respect to the relational semantics. Goldblatt's result has important consequences in the theory of ortho-lattices. In his words,

> *What appears to be novel is the idea of using such structures to provide models for a propositional language. Furthermore, an algebraic version of the completeness theorem set out below shows that every*

[19] Please refer to Appendix A for a review of the notions about lattices involved in this section.

[20] This result is ascribed to different people in different publications. In [45], it is stated and proved without referring to any source. In [52], it is ascribed to Finch ([42]), and [45] is also plainly referred to. In [92], it is ascribed to Dacey ([29]).

[21] The statements of these two conditions have been adapted to the terminology of this thesis, but the indices 'R0' and 'R1' are from the paper.

> *ortholattice is, within isomorphism, a subortholattice of the lattice of closed subsets of some orthogonality space. Previous results in this direction have either been confined to complete ortholattices (cf. [3]) or else have involved a somewhat different notion of orthogonality relation (MacLaren [4]).* (pp.24-25 in [47])

Moreover, Goldblatt proved the decidability of ortho-logic and found a translation from ortho-logic to the modal logic **B**, similar to the famous Gödel-Tarski translation from intuitionistic propositional logic to the modal logic **S4**. In [47] Goldblatt gave a similar analysis for *orthomodular logic*, the set of all propositional formulas valid in an algebraic semantics based on orthomodular lattices. However, the relational semantics for orthomodular logic involves some structures a bit more general than Kripke frames. Hence I do not go into the details.

For a long time after the publication of these three papers, no work on Kripke frames in quantum logic had been published, as far as I know. Moreover, all of these three papers deal with ortho-lattices and orthomodular lattices; and these lattices are too general for the foundations of quantum theory when viewed from the idea of the definition of Piron lattices and Piron's theorem. In my opinion, the first result on getting a Piron lattice from a Kripke frame with special properties appears essentially in [52] by Hedlíková and Pulmannová. In this paper, orthogonality spaces with various additional properties are investigated, and nice implications from the properties of orthogonality spaces to the properties of the lattices of closed subsets are proved, leading to a set of properties to get Piron lattices. I concentrate on these results from their paper. The definition of orthogonality spaces in [52] is slightly more general than that in [45], but in fact the focus goes to a special kind of orthogonality space $(\Sigma, \perp)$ (in the sense of [45]) satisfying the following:[22]

(I) $\{x\}^{\perp} \subseteq \{y\}^{\perp}$ implies that $x = y$, for any $x, y \in \Sigma$.

Note that this property is a separation property, because its contrapositive is the following:

(I)' $x \neq y$ implies that there is a $w \in \Sigma$ such that $w \perp s$ but *not* $w \perp t$, for any $x, y \in \Sigma$.

Remember that in [21] it is proved that the set of closed subsets of an orthogonality space forms a complete ortho-lattice. Proposition 1.4, together with the discussion in the last but one paragraph on p.11, in [52] implies that the set of closed subsets of an orthogonality space satisfying (I) forms a complete atomic ortho-lattice, and in fact it is a complete atomistic ortho-lattice (Lemma 5.5 in

[22]The index (I) of this condition and (II), (III), etc. below are indices used in this thesis, which are not used in [52].

[69])[23]. For the Covering Law to hold, ten sufficient and necessary conditions are listed in Theorem 2.2 in this paper. I present here (v), the condition of 3-minimal dependence, in the list which I find the simplest:

(II) for any $s, s_1, s_2, s_3 \in \Sigma$, $\{s, s_1\}^{\perp\perp} \cap \{s_2, s_3\}^{\perp\perp} \neq \emptyset$, if $s \in \{s_1, s_2, s_3\}^{\perp\perp}$ and $s \notin \{w_1, w_2\}^{\perp\perp}$ for any $w_1, w_2 \in \{s_1, s_2, s_3\}$.

For Weak Modularity, the result in [45] is mentioned. Besides, assuming that the Covering Law already holds, three sufficient and necessary conditions for Weak Modularity are listed in Theorem 3.5, which all involve the technical notion of splitting sets. A set $P \subseteq \Sigma$ is a *splitting set*, if both of the following hold:

1. P is *linear*, i.e. $\{s, t\}^{\perp\perp} \subseteq P$ for any $s, t \in \Sigma$;

2. Σ itself is the only linear subset of Σ containing both P and $P^{\perp}$.

Now I present (ii) in Theorem 3.5, which is the simplest according to my taste:

(III) every closed subset of Σ is a splitting set.

Finally, for irreducibility, Lemma 2.3 in [52] gives a sufficient and necessary condition:

(IV) for any two non-empty subsets P, Q of Σ, $P \cup Q = \Sigma$ implies that there are $s \in P$ and $t \in Q$ such that $s \perp t$ does not hold.

In the following picture I roughly summarize how the properties of an orthogonality space enforce the properties of the lattice of closed subsets of it, according to the results presented in [52]:

Condition (IV)	$\Rightarrow$	Irreducibility
$\uparrow$		$\uparrow$
Condition (III) or others	$\Rightarrow$	Weak Modularity
$\uparrow$		$\uparrow$
Condition (II) or others	$\Rightarrow$	Covering Law
$\uparrow$		$\uparrow$
Condition (I)	$\Rightarrow$	Atomisticity
$\uparrow$		$\uparrow$
Irreflexivity and Symmetry	$\Rightarrow$	Complete Ortho-Lattice
Kripke Frame $\mathfrak{F}$	|	Closed Subsets of $\mathfrak{F}$

In the picture, the arrows $\Rightarrow$ mean implication, and the arrows $\uparrow$ mean logical dependence. The idea is that, from a Kripke frame $\mathfrak{F}$, if the conditions on the

[23] In Proposition 1.4 in [52] the word 'atomistic' is used, but in this context it just means the same as 'atomic' in Definition A.0.5 in this thesis.

left are put one by one from the bottom to the top, i.e. from Irreflexivity and Symmetry to Condition (IV), then the lattice of closed subsets of $\mathfrak{F}$ will have the corresponding properties on the right one by one from the bottom to the top, and hence become a complete ortho-lattice until a Piron lattice. In a word, in [52] the conditions on the left are proposed and proved to be sufficient for the closed subsets of a Kripke frame to form a Piron lattice.

In [69] Moore investigated the orthogonality spaces satisfying the condition (I)' mentioned above. He called such structures *state spaces*, and this name is adopted in this thesis. Moore proved that state spaces and complete atomistic ortho-lattices, which he called *property lattices*, are in correspondence. Moreover, he defined morphisms between state spaces, and thus organized the class of state spaces into a category. He did the same for the class of property lattices, and proved that this category of property lattices and the category of state spaces form a duality under two functors which are defined in a natural way. Therefore, the physical state-property duality is captured in a duality in category theory. Note that the work of [69] also includes strong physical motivations for the definitions of state spaces and property lattices.

As far as I know, the above discussion lists all of the work in the area of quantum logic that makes use of Kripke frames. There is other work in quantum logic which uses mathematical structures slightly different from Kripke frames. Examples are orthomodular frames in [47], state property systems proposed in [4] and nicely surveyed in [5], and quantum dynamic frames in [14].

In the remaining part of this section, I discuss two points on how the work in this thesis relates to the approaches that are surveyed above.

The first point is about the difference in primitives. It is obvious that in most of the papers, e.g. [45], [47], [52] and [69], the irreflexive and symmetric orthogonality relation is taken to be primitive. However, in my work I take the reflexive and symmetric non-orthogonality relation to be primitive, and only Dishkant in [31] did the same. From the perspective of mathematics, on the same set Σ the orthogonality relation $\perp$ and the non-orthogonality relation $\to$ are set-theoretic complements of each other in $\Sigma \times \Sigma$, so it does not matter which should be taken as primitive. However, there is a small difference from the perspective of modal logic. In modal logic it is well known that from a binary relation on a set Σ a function, called *the universal Kripke modality of the relation*, on the power set $\wp(\Sigma)$ can be defined. For the non-orthogonality relation $\to$ on Σ, the universal Kripke modality of it, denoted by $\Box$, is defined as follows: for each $P \subseteq \Sigma$,

$$\Box P = \{s \in \Sigma \mid s \to t \text{ implies that } t \in P, \text{ for every } t \in \Sigma\}$$

Similarly, the universal Kripke modality of the orthogonality relation $\perp$, denoted by $\boxdot$, is defined as follows: for each $P \subseteq \Sigma$,

$$\boxdot P = \{s \in \Sigma \mid s \perp t \text{ implies that } t \in P, \text{ for every } t \in \Sigma\}$$

Then it is not hard to see that the orthocomplements can be defined using $\Box$:

$$\begin{aligned} P^{\perp} &= \{s \in \Sigma \mid s \perp t \text{ for every } t \in P\} \\ &= \{s \in \Sigma \mid s \to t \text{ does not hold for every } t \in P\} \ (= {\sim}P) \\ &= \Box(\Sigma \setminus P) \end{aligned}$$

There is not any nice result similar to this for $\boxdot$. Given the well-known fact that modal logic is very useful in describing relational structures ([23]), I prefer to take as primitive the non-orthogonality relation, whose universal Kripke modality is more useful in expressing the notions in quantum theory like orthocomplements. Moreover, it is also well known in modal logic that reflexive relations are easier to axiomatize than irreflexive relations. Another reason is that some maps between quantum Kripke frames have simpler and more natural definitions in terms of the non-orthogonality relation. This can be seen in the next chapter.

The second point I want to make is about the exact contribution of this thesis. Among the papers [45], [31], [47], [52] and [69], the results in [52] are the closest in spirit and strength to those in this thesis, in particular Theorem 2.7.23, in spite of the difference in primitives. While in [52] orthogonality spaces with different properties are studied in wider and deeper extent than in this thesis, Theorem 2.7.23 is still an improvement of the results in [52] in two aspects. First, in the light of Theorem 2.7.23, if the condition (III) holds in an orthogonality space satisfying the condition (I), condition (II) will also hold and the condition (IV) can be simplified a lot, perhaps with a modified definition of splitting sets. Second, Theorem 2.7.23 shows a correspondence between some special Kripke frames and Piron lattices, while the results in [52] are only about constructing Piron lattices from special Kripke frames.

Chapter 3

Maps between Quantum Kripke Frames

In the previous chapter, I investigated in detail the structure inside a single quantum Kripke frame. In this chapter, I introduce maps between quantum Kripke frames and study the structure formed by them.

The need of studying maps between quantum Kripke frames stems from the significance of the operators on Hilbert spaces. In quantum theory many important physical notions are modelled by the operators on Hilbert spaces, or, more precisely, the linear maps having adjoints. For example, the observables of a quantum system are modelled by the self-adjoint operators on the Hilbert space modelling the system; and the evolution is modelled by the unitary operators. For another example, a compound quantum system consisting of two subsystems is described by the tensor product of the Hilbert spaces modelling the two subsystems; and this tensor product can be constructed from the set of Hilbert-Schmidt operators from one of the Hilbert space to the other. Therefore, to reconstruct quantum theory in the framework of quantum Kripke frames, it is crucial to find the counterparts of these important kinds of operators on Hilbert spaces.

This chapter is organized as follows: In Section 3.1 I define continuous homomorphisms between quasi-quantum Kripke frames, and study their basic properties. Moreover, I study three special kinds of continuous homomorphisms defined parallel to the unitary operators, self-adjoint operators and projectors on Hilbert spaces, respectively. In Section 3.2 I define the notion of parties of continuous homomorphisms, and use it to solve a special case of an open problem in [39]. In Section 3.3 I define the non-orthogonality relation between the members of the same party of continuous homomorphisms, and show that under some assumptions a party of continuous homomorphisms equipped with this relation forms a quantum Kripke frame. I also argue that this construction is a counterpart of the tensor product construction on Hilbert spaces.

3.1 Continuous Homomorphisms

In this section, I introduce continuous homomorphisms between quasi-quantum Kripke frames. I show some useful properties of them. Moreover, I study some special continuous homomorphisms.

3.1.1 Definition and Basic Properties

In this subsection, I define continuous homomorphisms between quasi-quantum Kripke frames, and prove some useful results about them.

I start with the definition of continuous homomorphisms. The idea comes from the properties of the linear maps having adjoints on Hilbert spaces.

3.1.1. Definition. A *continuous homomorphism* from a quasi-quantum Kripke frame $\mathfrak{F}_1 = (\Sigma_1, \rightarrow_1)$ to a quasi-quantum Kripke frame $\mathfrak{F}_2 = (\Sigma_2, \rightarrow_2)$ is a partial function $F : \Sigma_1 \dashrightarrow \Sigma_2$ such that, for any $s_1 \in \Sigma_1$ and $s_2, t_2 \in \Sigma_2$, if $(s_1, s_2) \in F$ and $s_2 \rightarrow_2 t_2$, there is a $t_1 \in \Sigma_1$ satisfying the following:

(Ad)$_F$[1] for any $w_1 \in \Sigma_1$, $t_1 \rightarrow_1 w_1 \Leftrightarrow \big(F(w_1)$ is defined and $t_2 \rightarrow_2 F(w_1)\big)$

Moreover, $Ker(F) \stackrel{\text{def}}{=} \{s_1 \in \Sigma_1 \mid F(s_1) \text{ is undefined}\}$ is called the *kernel of* F.

3.1.2. Remark. For every $P_2 \subseteq \Sigma_2$, remember that *the inverse image of* P_2 *under* F, denoted by $F^{-1}[P_2]$, is $\{s_1 \in \Sigma_1 \mid (s_1, s_2) \in F, \text{ for some } s_2 \in P_2\}$. With this terminology, it is not hard to see that a partial function $F : \Sigma_1 \dashrightarrow \Sigma_2$ is a continuous homomorphism, if and only if, for every $t_2 \in \Sigma_2$, $Ker(F) \cup F^{-1}[\sim_2\{t_2\}] = \Sigma_1$ or $Ker(F) \cup F^{-1}[\sim_2\{t_2\}] = \sim_1\{t_1\}$ for some $t_1 \in \Sigma_1$.

Moreover, remember that, for every $t_1 \in \Sigma_1$, $\sim_1\{t_1\}$ is a hyperplane of $\mathfrak{F}_1$ (Proposition 2.3.9). Hence a partial function $F : \Sigma_1 \dashrightarrow \Sigma_2$ is a continuous homomorphism, if and only if $Ker(F) \cup F^{-1}[\sim_2\{t_2\}]$ is bi-orthogonally closed and includes $\sim_1\{t_1\}$ for some $t_1 \in \Sigma_1$.

Besides continuous homomorphisms, between two quantum Kripke frames other interesting (partial) functions with nice properties can be defined. Good examples are the weak maps and strong maps between quantum dynamic frames defined in Definitions 2.20 and 2.21 in [20]. Given the correspondence between quantum Kripke frames and quantum dynamic frames (Theorem 2.7.25), as well as the observation that both definitions essentially only involve the non-orthogonality relation, weak maps and strong maps can be defined in the setting of quantum Kripke frames and will have the same properties. I do not go into the details in this thesis.

[1] Please remember that (Ad)$_F$ is not a property of a quasi-quantum Kripke frame or a continuous homomorphism; instead it is a property between an (ordered) pair of two elements in two quasi-quantum Kripke frames, in this case, (t_1, t_2).

Now I prove some basic facts about continuous homomorphisms. First, I show that the t_1 in the above definition is unique, if it exists.

3.1.3. Lemma. *Let F be a continuous homomorphism from a quasi-quantum Kripke frame $\mathfrak{F}_1 = (\Sigma_1, \rightarrow_1)$ to a quasi-quantum Kripke frame $\mathfrak{F}_2 = (\Sigma_2, \rightarrow_2)$, and $t_2 \in \Sigma_2$. If it exists, $t_1 \in \Sigma_1$ with (t_1, t_2) satisfying $(Ad)_F$ is unique.*

Proof. Suppose (towards a contradiction) that $t'_1 \in \Sigma_1 \setminus \{t_1\}$ is also such that (t'_1, t_2) satisfies $(\mathrm{Ad})_F$. Since $t_1 \neq t'_1$, by Separation there is a $w_1 \in \Sigma_1$ such that $t_1 \not\rightarrow_1 w_1$ and $t'_1 \rightarrow_1 w_1$. Since (t'_1, t_2) satisfies $(\mathrm{Ad})_F$, $F(w_1)$ is defined and $t_2 \rightarrow_2 F(w_1)$. Since (t_1, t_2) satisfies $(\mathrm{Ad})_F$, it is not the case that $F(w_1)$ is defined and $t_2 \rightarrow_2 F(w_1)$. Hence a contradiction is derived. Therefore, if it exists, $t_1 \in \Sigma_1$ with (t_1, t_2) satisfying $(\mathrm{Ad})_F$ is unique. ⊣

Before investigating other properties of continuous homomorphisms, I introduce the useful notion of adjunctions between quasi-quantum Kripke frames.

3.1.4. Definition. An *adjunction* between two quasi-quantum Kripke frames $\mathfrak{F}_1 = (\Sigma_1, \rightarrow_1)$ and $\mathfrak{F}_2 = (\Sigma_2, \rightarrow_2)$ is a pair of partial functions (F, G) with $F : \Sigma_1 \dashrightarrow \Sigma_2$ and $G : \Sigma_2 \dashrightarrow \Sigma_1$ such that all of the following hold:

1. $Ker(F) = {\sim_1} G[\Sigma_2]$;
2. $Ker(G) = {\sim_2} F[\Sigma_1]$;
3. $s_1 \rightarrow_1 G(s_2) \Leftrightarrow F(s_1) \rightarrow_2 s_2$, if $s_1 \notin Ker(F)$ and $s_2 \notin Ker(G)$.

3.1.5. Remark. Note that, for two quasi-quantum Kripke frames $\mathfrak{F}_1$ and $\mathfrak{F}_2$, (F, G) is an adjunction between them, if and only if it is an adjunction between the irreducible pure orthogeometries $\mathbf{G}(\mathfrak{F}_1)$ and $\mathbf{G}(\mathfrak{F}_2)$ in the sense of Definition B.1.26. Moreover, for two irreducible pure orthogeometries $\mathcal{G}_1$ and $\mathcal{G}_2$, (F, G) is an adjunction between them, if and only if it is an adjunction between the quasi-quantum Kripke frames $\mathbf{F}(\mathcal{G}_1)$ and $\mathbf{F}(\mathcal{G}_2)$.

I show that, for two quasi-quantum Kripke frames $\mathfrak{F}_1$ and $\mathfrak{F}_2$, the continuous homomorphisms from $\mathfrak{F}_1$ to $\mathfrak{F}_2$ and the adjunctions between $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are in correspondence to each other.

First I show how to get an adjunction given a continuous homomorphism.

3.1.6. Proposition. *Let $\mathfrak{F}_1 = (\Sigma_1, \rightarrow_1)$ and $\mathfrak{F}_2 = (\Sigma_2, \rightarrow_2)$ be quasi-quantum Kripke frames and $F : \Sigma_1 \dashrightarrow \Sigma_2$ be a continuous homomorphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$. $F^\dagger \stackrel{\text{def}}{=} \{(s_2, s_1) \in \Sigma_2 \times \Sigma_1 \mid (s_1, s_2)$ satisfies $(Ad)_F \}$ is a partial function from Σ_2 to Σ_1, called* the adjoint of F, *and $(F, F^\dagger)$ is an adjunction between $\mathfrak{F}_1$ and $\mathfrak{F}_2$.*

Proof. By Lemma 3.1.3 $F^\dagger$ is a partial function from Σ_2 to Σ_1.

To show that $(F, F^\dagger)$ is an adjunction between $\mathfrak{F}_1$ and $\mathfrak{F}_2$, I prove three facts.

Fact 1: $Ker(F) = \sim_1 F^\dagger[\Sigma_2]$.

Let $w_1 \in \Sigma_1$ be arbitrary.

First assume that $w_1 \notin Ker(F)$. By Reflexivity $F(w_1) \to_2 F(w_1)$. Since F is a continuous homomorphism, there is a $w_1^+ \in \Sigma_1$ such that $(w_1^+, F(w_1))$ satisfies $(\mathrm{Ad})_F$. Since $F(w_1)$ is defined and $F(w_1) \to_2 F(w_1)$, $w_1^+ \to_1 w_1$. Hence $w_1 \to_1 w_1^+ = F^\dagger(F(w_1))$, and thus $w_1 \notin \sim_1 F^\dagger[\Sigma_2]$.

Second assume that $w_1 \notin \sim_1 F^\dagger[\Sigma_2]$. Then there are $s_1 \in \Sigma_1$ and $s_2 \in \Sigma_2$ such that $F^\dagger(s_2) = s_1$ and $s_1 \to_1 w_1$. By the definition of $F^\dagger$ $(\mathrm{Ad})_F$ holds for (s_1, s_2). Since $s_1 \to_1 w_1$, $F(w_1)$ is defined. Hence $w_1 \notin Ker(F)$.

Fact 2: $Ker(F^\dagger) = \sim_2 F[\Sigma_1]$.

Let $w_2 \in \Sigma_2$ be arbitrary.

First assume that $w_2 \notin Ker(F^\dagger)$. Then there is an $w_1 \in \Sigma_1$ such that $(w_2, w_1) \in F^\dagger$, i.e. $(\mathrm{Ad})_F$ holds for (w_1, w_2). By Reflexivity $w_1 \to_1 w_1$, so $F(w_1)$ is defined and $w_2 \to_2 F(w_1)$. Therefore, $w_2 \notin \sim_2 F[\Sigma_1]$.

Second assume that $w_2 \notin \sim_2 F[\Sigma_1]$. Then there are $s_1 \in \Sigma_1$ and $s_2 \in \Sigma_2$ such that $F(s_1) = s_2$ and $s_2 \to_2 w_2$. Since F is a continuous homomorphism, there is a $w_1 \in \Sigma_1$ such that $(\mathrm{Ad})_F$ holds for (w_1, w_2). By the definition of $F^\dagger$ $(w_2, w_1) \in F^\dagger$, so $w_2 \notin Ker(F^\dagger)$.

Fact 3: $s_1 \to_1 F^\dagger(s_2) \Leftrightarrow F(s_1) \to_2 s_2$, if $s_1 \notin Ker(F)$ and $s_2 \notin Ker(F^\dagger)$.

Let $s_1 \notin Ker(F)$ and $s_2 \notin Ker(F^\dagger)$ be arbitrary. By the definition of $F^\dagger$ $(\mathrm{Ad})_F$ holds for $(F^\dagger(s_2), s_2)$. Hence $F^\dagger(s_2) \to_1 s_1$, if and only if $F(s_1)$ is defined and $s_2 \to_2 F(s_1)$. Since $s_1 \notin Ker(F)$, the required equivalence follows easily.

As a result, $(F, F^\dagger)$ is an adjunction between $\mathfrak{F}_1$ and $\mathfrak{F}_2$. ⊣

3.1.7. Remark. Given two quasi-quantum Kripke frames $\mathfrak{F}_1$ and $\mathfrak{F}_2$, I denote by $\mathbf{A}_{\mathfrak{F}_2}^{\mathfrak{F}_1}$ the function from the set of continuous homomorphisms from $\mathfrak{F}_1$ to $\mathfrak{F}_2$ to the set of adjunctions between $\mathfrak{F}_1$ and $\mathfrak{F}_2$, which maps every continuous homomorphism F to the adjunction $(F, F^\dagger)$ defined as in this proposition. When the context is clear, I abbreviate $\mathbf{A}_{\mathfrak{F}_2}^{\mathfrak{F}_1}$ to $\mathbf{A}$.

Besides, note that, for any $s_1 \in \Sigma_1$ and $s_2 \in \Sigma_2$, $(\mathrm{Ad})_F$ holds for (s_1, s_2), if and only if $(s_2, s_1) \in F^\dagger$, i.e. $F^\dagger(s_2)$ is defined and equals to s_1.

I proceed to show how to get a continuous homomorphism given an adjunction.

3.1.8. Proposition. *Let $\mathfrak{F}_1 = (\Sigma_1, \to_1)$ and $\mathfrak{F}_2 = (\Sigma_2, \to_2)$ be two quasi-quantum Kripke frames and (F, G) an adjunction between $\mathfrak{F}_1$ and $\mathfrak{F}_2$ with $F : \Sigma_1 \dashrightarrow \Sigma_2$ and $G : \Sigma_2 \dashrightarrow \Sigma_1$. Then both F and G are continuous homomorphisms.*

Proof. I show that F is a continuous homomorphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$. Let $s_1 \in \Sigma_1$ and $s_2, t_2 \in \Sigma_2$ be arbitrary such that $(s_1, s_2) \in F$ and $s_2 \to t_2$. Then $t_2 \notin \sim_2 F[\Sigma_1] = Ker(G)$. I claim that $(G(t_2), t_2)$ satisfies $(\mathrm{Ad})_F$.

First assume that $G(t_2) \to_1 w_1$. Then $w_1 \notin {\sim_1} G[\Sigma_2] = Ker(F)$, so $F(w_1)$ is defined. Since $G(t_2) \to_1 w_1$, by the definition of adjunction and Symmetry $t_2 \to_2 F(w_1)$. Second assume that $F(w_1)$ is defined and $t_2 \to_2 F(w_1)$. It follows directly from the definition of adjunction and Symmetry that $G(t_2) \to_1 w_1$.

Since t_2 is arbitrary, F is a continuous homomorphism.

Finally, by definition (G, F) is an adjunction between $\mathfrak{F}_2$ and $\mathfrak{F}_1$. Hence by what was just proved G is a continuous homomorphism from $\mathfrak{F}_2$ to $\mathfrak{F}_1$. ⊣

3.1.9. Remark. Given two quasi-quantum Kripke frames $\mathfrak{F}_1$ and $\mathfrak{F}_2$, I denote by $\mathbf{C}^{\mathfrak{F}_1}_{\mathfrak{F}_2}$ the function from the set of adjunctions between $\mathfrak{F}_1$ and $\mathfrak{F}_2$ to the set of continuous homomorphisms from $\mathfrak{F}_1$ to $\mathfrak{F}_2$, which maps every adjunction (F, G) to the continuous homomorphism F from $\mathfrak{F}_1$ to $\mathfrak{F}_2$ defined as in this proposition. When the context is clear, I abbreviate $\mathbf{C}^{\mathfrak{F}_1}_{\mathfrak{F}_2}$ to $\mathbf{C}$.

Now I prove the correspondence between continuous homomorphisms and adjunctions.

3.1.10. Proposition. *Let $\mathfrak{F}_1 = (\Sigma_1, \to_1)$ and $\mathfrak{F}_2 = (\Sigma_2, \to_2)$ be two quasi-quantum Kripke frames.*

1. *For every continuous homomorphism F from $\mathfrak{F}_1$ to $\mathfrak{F}_2$, $\mathbf{C} \circ \mathbf{A}(F) = F$.*
2. *For every adjunction (F, G) between $\mathfrak{F}_1$ and $\mathfrak{F}_2$ with $F : \Sigma_1 \dashrightarrow \Sigma_2$ and $G : \Sigma_2 \dashrightarrow \Sigma_1$, $\mathbf{A} \circ \mathbf{C}((F, G)) = (F, G)$.*

Proof. **For *1*:** By the previous two propositions, $\mathbf{C} \circ \mathbf{A}(F) = \mathbf{C}((F, F^\dagger)) = F$.

For *2*: I start with proving that $G = F^\dagger$.

First note that $Ker(F^\dagger) = Ker(G)$. Since both (F, G) and $(F, F^\dagger)$ are adjunctions, by definition $Ker(F^\dagger) = {\sim_2} F[\Sigma_1] = Ker(G)$.

Second show that $G(s_2) = F^\dagger(s_2)$, for every $s_2 \notin Ker(F^\dagger) = Ker(G)$. Suppose (towards a contradiction) that this is not the case, as is witnessed by $s_2 \notin Ker(F^\dagger) = Ker(G)$. By Separation there is an $s_1 \in \Sigma_1$ such that $s_1 \to F^\dagger(s_2)$ and $s_1 \not\to G(s_2)$. On the one hand, $s_1 \to F^\dagger(s_2)$ implies that $s_1 \notin Ker(F)$ and $s_2 \to_2 F(s_1)$ by the definition of $F^\dagger$. On the other hand, since $s_1 \notin Ker(F)$ and $s_1 \not\to_1 G(s_2)$, by the definition of adjunctions $F(s_1) \not\to_2 s_2$. By Symmetry $s_2 \not\to_2 F(s_1)$. Therefore, a contradiction is derived.

As a result, $G = F^\dagger$, and thus $\mathbf{A} \circ \mathbf{C}((F, G)) = \mathbf{A}(F) = (F, F^\dagger) = (F, G)$. ⊣

This proposition means that the continuous homomorphisms and the adjunctions between two quasi-quantum Kripke frames are in correspondence witnessed by the functions $\mathbf{A}$ and $\mathbf{C}$. From this correspondence and the one between quasi-quantum Kripke frames and irreducible pure orthogeometries (Theorem 2.5.2), many results about the adjunctions between orthogeometries can be employed to show useful results about the continuous homomorphisms between quasi-quantum

Kripke frames[2]. I collect those highly related to this thesis in the following two corollaries. The first one is about some useful properties of continuous homomorphisms.

3.1.11. COROLLARY. *Let* $\mathfrak{F}_1 = (\Sigma_1, \rightarrow_1)$ *and* $\mathfrak{F}_2 = (\Sigma_2, \rightarrow_2)$ *be two quasi-quantum Kripke frames.*

1. *For every partial function* $F : \Sigma_1 \dashrightarrow \Sigma_2$*,* F *is a continuous homomorphism from* $\mathfrak{F}_1$ *to* $\mathfrak{F}_2$*, if and only if it is a continuous homomorphism from* $\mathbf{G}(\mathfrak{F}_1)$ *to* $\mathbf{G}(\mathfrak{F}_2)$ *(Definition B.1.25).*

2. *For every continuous homomorphism* F *from* $\mathfrak{F}_1$ *to* $\mathfrak{F}_2$*, there is a unique continuous homomorphism from* $\mathfrak{F}_2$ *to* $\mathfrak{F}_1$*, namely* $F^\dagger$*, the adjoint of* F*, such that* $(F, F^\dagger)$ *is an adjunction between* $\mathfrak{F}_1$ *and* $\mathfrak{F}_2$*.*

Proof. **For *1*:** First assume that F is a continuous homomorphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$, by Proposition 3.1.6 $(F, F^\dagger)$ is an adjunction between $\mathfrak{F}_1$ and $\mathfrak{F}_2$. According to Remark 3.1.5, $(F, F^\dagger)$ is an adjunction between $\mathbf{G}(\mathfrak{F}_1)$ and $\mathbf{G}(\mathfrak{F}_2)$. By Theorem B.1.27 F is a continuous homomorphism from $\mathbf{G}(\mathfrak{F}_1)$ to $\mathbf{G}(\mathfrak{F}_2)$.

Second assume that F is a continuous homomorphism from $\mathbf{G}(\mathfrak{F}_1)$ to $\mathbf{G}(\mathfrak{F}_2)$. According to Theorem B.1.27, there is a partial function G such that (F, G) is an adjunction between $\mathbf{G}(\mathfrak{F}_1)$ and $\mathbf{G}(\mathfrak{F}_2)$. By Remark 3.1.5 (F, G) is an adjunction between $\mathfrak{F}_1$ and $\mathfrak{F}_2$. It follows from Proposition 3.1.8 that F is a continuous homomorphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$.

For *2*: For existence, according to Proposition 3.1.6, $(F, F^\dagger)$ is an adjunction between $\mathfrak{F}_1$ and $\mathfrak{F}_2$; and $F^\dagger$ is a continuous homomorphism by Proposition 3.1.8.

For uniqueness, assume that (F, G) is an adjunction between $\mathfrak{F}_1$ and $\mathfrak{F}_2$. Then $(F, G) = \mathbf{A}(\mathbf{C}((F, G))) = \mathbf{A}(F) = \mathbf{A}(\mathbf{C}((F, F^\dagger))) = (F, F^\dagger)$ by the previous proposition. Therefore, $G = F^\dagger$. ⊣

The second corollary is a representation theorem for arguesian continuous homomorphisms via continuous quasi-linear maps. Before presenting it, I introduce the notion of arguesian continuous homomorphisms.

3.1.12. DEFINITION. A continuous homomorphism F from $\mathfrak{F}_1 = (\Sigma_1, \rightarrow_1)$ to $\mathfrak{F}_2 = (\Sigma_2, \rightarrow_2)$ is *non-degenerate*, if $F[\Sigma_1]$ has an orthogonal set of cardinality 3.

It is *arguesian*, if it is the composition of finitely many non-degenerate continuous homomorphisms. (These non-degenerate continuous homomorphisms may involve quasi-quantum Kripke frames other than $\mathfrak{F}_1$ and $\mathfrak{F}_2$.)

3.1.13. REMARK. Note that a partial function $F : \Sigma_1 \dashrightarrow \Sigma_2$ is an arguesian continuous homomorphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$, if and only if it is an arguesian continuous homomorphism from $\mathbf{G}(\mathfrak{F}_1)$ to $\mathbf{G}(\mathfrak{F}_2)$ in the sense of Definition B.3.12.

[2]For the involved definitions and results in projective geometry and linear algebra, please refer to Appendix B.

3.1.14. Corollary. *Let V_1 and V_2 be two vector spaces over two division rings $\mathcal{K}_1$ and $\mathcal{K}_2$ equipped with anisotropic Hermitian forms Φ_1 and Φ_2, respectively. For every partial function F from $\Sigma(V_1)$ to $\Sigma(V_2)$, the following are equivalent:*

(i) F is an arguesian continuous homomorphism from the quasi-quantum Kripke frame $(\Sigma(V_1), \to_{V_1})$ to $(\Sigma(V_2), \to_{V_2})$;

(ii) there is a continuous quasi-linear map $f : V_1 \to V_2$ such that $\mathcal{P}(f) = F$.

Proof. **From (i) to (ii):** Assume that F is an arguesian continuous homomorphism from $(\Sigma(V_1), \to_{V_1})$ to $(\Sigma(V_2), \to_{V_2})$. By Remark 3.1.13 F is an arguesian continuous homomorphism from $\mathbf{G}((\Sigma(V_1), \to_{V_1}))$ to $\mathbf{G}((\Sigma(V_2), \to_{V_2}))$. By Theorem B.3.14 there is a continuous quasi-linear map $f : V_1 \to V_2$ such that $\mathcal{P}(f) = F$.

From (ii) to (i): Assume that there is a continuous quasi-linear map $f : V_1 \to V_2$ such that $\mathcal{P}(f) = F$. By Theorem B.3.14 F is an arguesian continuous homomorphism from $(\Sigma(V_1), \star_{V_1}, \perp_{V_1})$ to $(\Sigma(V_2), \star_{V_2}, \perp_{V_2})$. Note that $(\Sigma(V_i), \star_{V_i}, \perp_{V_i}) = \mathbf{G}((\Sigma(V_i), \to_{V_i}))$ for $i = 1, 2$. Hence by Remark 3.1.13 F is an arguesian continuous homomorphism from $(\Sigma(V_1), \to_{V_1})$ to $(\Sigma(V_2), \to_{V_2})$. $\dashv$

3.1.15. Remark. Note that, on the one hand, the linear maps having adjoints on Hilbert spaces are continuous quasi-linear maps. On the other hand, for a continuous quasi-linear maps on Hilbert spaces, if it is a linear map, it will have an adjoint. Therefore, continuous homomorphisms are the counterparts of linear maps having adjoints on Hilbert spaces.

3.1.2 Special Continuous Homomorphisms

In the previous subsection, I show that continuous homomorphisms are the counterparts of linear maps having adjoints between Hilbert spaces. Many important operators on Hilbert spaces are special kinds of linear maps having adjoints. In this subsection I define three special kinds of continuous homomorphisms parallel to the unitary operators, self-adjoint operators and projectors on Hilbert spaces, and study their properties from the perspective of the non-orthogonality relation.

First, the unitary operators on Hilbert spaces model the evolution of quantum systems, and they are defined to be linear maps whose adjoints are their inverses. Hence it is interesting to study the continuous homomorphisms with a similar property. They turn out to closely relate to bisimulations in modal logic[3].

3.1.16. Proposition. *Let $\mathfrak{F}_1 = (\Sigma_1, \to_1)$ and $\mathfrak{F}_2 = (\Sigma_2, \to_2)$ be two quasi-quantum Kripke frames, and F a partial function from Σ_1 to Σ_2. The following are equivalent:*

[3] For a definition, please refer to Definition 2.16 in [23], ignoring the condition for unary predicates.

(i) F is a non-empty continuous homomorphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$ with $F^{-1} = F^\dagger$;

(ii) F is a one-to-one bisimulation[4] *from $\mathfrak{F}_1$ to $\mathfrak{F}_2$.*

Moreover, if either, and thus both, of (i) and (ii) holds, $F[\Sigma_1] = \Sigma_2$.[5]

Proof. **For (i) to (ii):** First I prove the forth condition. Assume that $(s_1, s_2) \in F$ and $s_1 \to_1 t_1$. Then $(s_2, s_1) \in F^{-1} = F^\dagger$. By Proposition 3.1.6 (s_1, s_2) satisfies $(\mathrm{AD})_F$. Since $s_1 \to_1 t_1$, there is a $t_2 \in \Sigma_2$ such that $(t_1, t_2) \in F$ and $s_2 \to_2 t_2$.

Second I prove the back condition. Assume that $(s_1, s_2) \in F$ and $s_2 \to_2 t_2$. Since F is a continuous homomorphism, there is a $t_1 \in \Sigma_1$ such that (t_1, t_2) satisfies $(\mathrm{Ad})_F$. By Proposition 3.1.6 $(t_2, t_1) \in F^\dagger$. Since $F^{-1} = F^\dagger$, $(t_2, t_1) \in F^{-1}$, and thus $(t_1, t_2) \in F$. Moreover, since $F(s_1)$ is defined and $t_2 \to_2 s_2 = F(s_1)$, by $(\mathrm{Ad})_F$ $s_1 \to_1 t_1$. Therefore, $t_1 \in \Sigma_1$ is such that $s_1 \to_1 t_1$ and $(t_1, t_2) \in F$.

Third, note that by Lemma 3.1.3 $F^{-1} = F^\dagger$ is a partial function, so the partial function F is a non-empty one-to-one relation.

As a result, F is a one-to-one bisimulation from $\mathfrak{F}_1$ to $\mathfrak{F}_2$.

From (ii) to (i): First, F is non-empty by the definition of bisimulations.

Second, F is a partial function, since F is a one-to-one relation.

Third, I show that F is a continuous homomorphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$. I start from proving the claim that every $(t_1, t_2) \in F$ satisfies $(\mathrm{Ad})_F$. Assume firstly that $t_1 \to_1 w_1$. By the forth condition $F(w_1)$ is defined and $t_2 = F(t_1) \to_2 F(w_1)$. Assume secondly that $F(w_1)$ is defined and $t_2 \to_2 F(w_1)$. By the back condition there is a $v_1 \in \Sigma_1$ such that $F(v_1) = F(w_1)$ and $t_1 \to_1 v_1$. Since F is one-to-one, $v_1 = w_1$. Hence $t_1 \to_1 w_1$. Therefore, $(t_1, t_2) \in F$ satisfies $(\mathrm{Ad})_F$. Now assume that $(s_1, s_2) \in F$ and $s_2 \to_2 t_2$. By the back condition there is a $t_1 \in \Sigma_1$ such that $(t_1, t_2) \in F$ and $s_1 \to_1 t_1$. By the claim (t_1, t_2) satisfies $(\mathrm{Ad})_F$.

Fourth, I show that $F^{-1} = F^\dagger$. It follows from the above claim and the definition of $F^\dagger$ that $F^{-1} \subseteq F^\dagger$. To show that $F^\dagger \subseteq F^{-1}$, assume that $(s_2, s_1) \in F^\dagger$. Then by definition (s_1, s_2) satisfies $(\mathrm{Ad})_F$. Since by Reflexivity $s_1 \to_1 s_1$, $F(s_1)$ is defined and $s_2 \to_2 F(s_1)$. Since $(s_1, F(s_1)) \in F$ and $F(s_1) \to_2 s_2$, by the back condition there is a $t_1 \in \Sigma_1$ such that $(t_1, s_2) \in F$ and $s_1 \to_1 t_1$. Then $(s_2, t_1) \in F^{-1} \subseteq F^\dagger$, according to what has been proved. Since $F^\dagger$ is a partial function by Proposition 3.1.6, $s_1 = t_1$, so $(s_2, s_1) \in F^{-1}$. Therefore, $F^{-1} = F^\dagger$.

Finally, I show that $F[\Sigma_1] = \Sigma_2$ using (ii). Since F is a partial function from Σ_1 to Σ_2, $F[\Sigma_1] \subseteq \Sigma_2$. To show that $\Sigma_2 \subseteq F[\Sigma_1]$, let $s_2 \in \Sigma_2$ be arbitrary. Since F is not empty, there are $u_1 \in \Sigma_1$ and $u_2 \in \Sigma_2$ such that $(u_1, u_2) \in F$. Since $\mathfrak{F}_2$ is a quasi-quantum Kripke frame, by Superposition there is a $v_2 \in \Sigma_2$ such that $v_2 \to_2 u_2$ and $v_2 \to_2 s_2$. Since $(u_1, u_2) \in F$ and $u_2 \to v_2$, by the back condition

[4] A relation $R \subseteq A \times B$ is *one-to-one*, if $a = c \Leftrightarrow b = d$, for any $(a, b), (c, d) \in R$.

[5] This surjectivity result and its proof are inspired by the similar case for quantum dynamic frames. The strong maps between quantum dynamic frames, for which the back condition holds, are surjective due to Proper Superposition (the sentence in brackets in the paragraph under Definition 2.21 in [20]). This was observed by one of my co-authors of [20].

there is a $v_1 \in \Sigma_1$ such that $(v_1, v_2) \in F$ and $u_1 \to_1 v_1$. Since $v_2 \to_2 s_2$, by the back condition again there is an $s_1 \in \Sigma_1$ such that $(s_1, s_2) \in F$ and $v_1 \to_1 s_1$. Therefore, $s_2 \in F[\Sigma_1]$. As a result, $F[\Sigma_1] = \Sigma_2$. ⊣

The unitary operators on Hilbert spaces are isomorphisms. Based on this proposition, I can similarly characterize the isomorphisms between quasi-quantum Kripke frames as continuous homomorphisms with special properties.

3.1.17. PROPOSITION. *Let $\mathfrak{F}_1 = (\Sigma_1, \to_1)$ and $\mathfrak{F}_2 = (\Sigma_2, \to_2)$ be two quasi-quantum Kripke frames, and F a partial function from Σ_1 to Σ_2. The following are equivalent:*

(i) F is a continuous homomorphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$ such that $F^{-1} = F^\dagger$ and $Ker(F) = \emptyset$;

(ii) F is an isomorphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$.

Proof. **From (i) to (ii):** By the previous proposition F is a one-to-one bisimulation from $\mathfrak{F}_1$ to $\mathfrak{F}_2$ satisfying $F[\Sigma_1] = \Sigma_2$.

First, I show that F is a bijection from Σ_1 to Σ_2. Since $Ker(F) = \emptyset$, F is a function from Σ_1 to Σ_2. Since $F[\Sigma_1] = \Sigma_2$, F is surjective. Since F is a one-to-one relation, F is injective. As a result, F is a bijection from Σ_1 to Σ_2.

Second, I show that $s_1 \to_1 t_1 \Leftrightarrow F(s_1) \to_2 F(t_1)$, for any $s_1, t_1 \in \Sigma_1$. Since $Ker(F) = \emptyset$, both $F(s_1)$ and $F(t_1)$ are defined, and thus $(s_1, F(s_1)) \in F$. Then $(F(s_1), s_1) \in F^{-1} = F^\dagger$. By Proposition 3.1.6 $(s_1, F(s_1))$ satisfies $(\mathrm{Ad})_F$. It follows that $s_1 \to_1 t_1 \Leftrightarrow F(s_1) \to_2 F(t_1)$.

As a result, F is an isomorphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$.

From (ii) to (i): Since F is an isomorphism, it is easy to see that it is a one-to-one bisimulation from $\mathfrak{F}_1$ to $\mathfrak{F}_2$ and $Ker(F) = \emptyset$. By Proposition 3.1.16 F is a non-empty continuous homomorphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$ such that $F^{-1} = F^\dagger$ and $Ker(F) = \emptyset$. ⊣

3.1.18. REMARK. According to this proposition, when $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are the same quantum Kripke frame $\mathfrak{F}$ induced by a Hilbert space $\mathcal{H}$ over $\mathbb{C}$, the continuous homomorphisms between them whose adjoints are their inverses and whose kernels are empty are the automorphisms on $\mathfrak{F}$. According to Wigner's theorem ([89]), these automorphisms are induced by the unitary or anti-unitary operators on $\mathcal{H}$. Therefore, such continuous homomorphisms are indeed the counterparts of unitary operators on Hilbert spaces.

Second, the self-adjoint operators on Hilbert spaces model the observables of quantum systems, and they are defined to be linear maps identical to their adjoints. Parallel to this, I define the notion of self-adjoint continuous homomorphisms as follows:

3.1.19. Definition. A continuous homomorphism F on a quasi-quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$ is *self-adjoint*, if $F = F^\dagger$.

The following proposition provides characterizations of self-adjoint continuous homomorphisms purely in terms of the non-orthogonality relation.

3.1.20. Proposition. *Let $\mathfrak{F} = (\Sigma, \to)$ be a quasi-quantum Kripke frame and $F \subseteq \Sigma \times \Sigma$. The following are equivalent:*

(i) F is a self-adjoint continuous homomorphism;

(ii) both of the following hold:[6]

1. *for any $s, t, u \in \Sigma$, if $(s, t) \in F$ and $t \to u$, there is a $v \in \Sigma$ such that $(u, v) \in F$ and $v \to s$;*
2. *for any $s, t, u, v \in \Sigma$, if $(s, t), (u, v) \in F$ and $t \to u$, then $v \to s$;*

(iii) for any $s, t \in \Sigma$, if $(s, t) \in F$, the following holds for every $u \in \Sigma$:

$$t \to u, \text{ if and only if there is a } v \in \Sigma \text{ satisfying } (u, v) \in F \text{ and } v \to s.$$

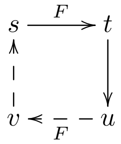
Proof. **From (i) to (ii):** For *1*, assume that $(s, t) \in F$ and $t \to u$. By (i) $(s, t) \in F = F^\dagger$, so (t, s) satisfies $(\mathrm{Ad})_F$. Since $t \to u$, $F(u)$ is defined and $F(u) \to s$. Taking v to be $F(u)$, $(u, v) \in F$ and $v \to s$.

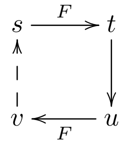
For *2*, assume that $(s, t), (u, v) \in F$ and $t \to u$. Similar to the above, I can derive that $F(u)$ is defined and $F(u) \to s$. Since $(u, v), (u, F(u)) \in F$ and F is a partial function, $F(u) = v$, so $v \to s$.

From (ii) to (iii): Let $s, t, u \in \Sigma$ be arbitrary such that $(s, t) \in F$. First assume that $t \to u$. By *1* there is a $v \in \Sigma$ such that $(u, v) \in F$ and $v \to s$. Second assume that there is a $v \in \Sigma$ satisfying $(u, v) \in F$ and $v \to s$. By *2* $t \to u$.

From (iii) to (i): First show that F is a partial function. Suppose (towards a contradiction) that $(s, t), (s, t') \in F$ and $t \neq t'$. By Separation there is a $u \in \Sigma$ such that $u \to t$ and $u \not\to t'$. Since $(s, t) \in F$ and $u \to t$, by (iii) there is a $v \in \Sigma$ such that $(u, v) \in F$ and $v \to s$. Since $(s, t') \in F$ and $u \not\to t'$, by (iii) there is no $v \in \Sigma$ such that $(u, v) \in F$ and $v \to s$. Hence a contradiction is derived. Therefore, F is a partial function.

[6] Both of *1* and *2* can be considered as intuitive square-completing conditions. These two diagrams may help to realize this. (The solid arrows are given in the antecedents and the dashed arrows are in the consequents.)





Second show that F is a continuous homomorphism. Let $u, v, s \in \Sigma$ be arbitrary such that $(u, v) \in F$ and $v \to s$. By (iii) $F(s)$ is defined. By (iii) again, for every $w \in \Sigma$, $F(s) \to w$, if and only if $F(w)$ is defined and $F(w) \to s$. Hence $(F(s), s)$ satisfies $(\mathrm{Ad})_F$. Therefore, F is a continuous homomorphism.

Third show that $F \subseteq F^\dagger$. Let $(s, t) \in F$ be arbitrary. By (iii), for every $w \in \Sigma$, $t \to w$, if and only if $F(w)$ is defined and $F(w) \to s$. Hence (t, s) satisfies $(\mathrm{Ad})_F$, and thus $(s, t) \in F^\dagger$. Therefore, $F \subseteq F^\dagger$.

Forth show that $F^\dagger \subseteq F$. Let $(s, t) \in F^\dagger$ be arbitrary. By definition (t, s) satisfies $(\mathrm{Ad})_F$. By Reflexivity $t \to t$, so by $(\mathrm{Ad})_F$ $F(t)$ is defined and $F(t) \to s$. By (iii) $F(s)$ is defined. By (iii) again, for every $w \in \Sigma$, $F(s) \to w$, if and only if $F(w)$ is defined and $F(w) \to s$. Hence $(F(s), s)$ satisfies $(\mathrm{Ad})_F$, and thus $(s, F(s)) \in F^\dagger$. Since $F^\dagger$ is a partial function, $F(s) = t$. Therefore, $(s, t) \in F$. ⊣

Third, the projectors on Hilbert spaces model the testable properties of quantum systems. They are defined to be idempotent (bounded) self-adjoint operators. Parallel to this, I define the notion of projections on quasi-quantum Kripke frames as follows:

3.1.21. Definition. A *projection* on a quasi-quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$ is a self-adjoint continuous homomorphism $F : \Sigma \dashrightarrow \Sigma$ such that $F \circ F \subseteq F$.

3.1.22. Remark. $F \circ F \subseteq F$ is equivalent to that $(u, v), (v, w) \in F \Rightarrow v = w$, for any $u, v, w \in \Sigma$.

Besides the similarity in the definitions, there is another reason for considering projections as the counterparts of projectors on Hilbert spaces. As it turns out, the projections are in correspondence with the bi-orthogonally closed subsets of a quantum Kripke frame. This is the counterpart of the correspondence between the projectors and the closed linear subspaces of a Hilbert space.

Before discussing this correspondence, I prove a lemma which shows that the image of an element under a projection F has a very special property.

3.1.23. Lemma. *Let $\mathfrak{F} = (\Sigma, \to)$ be a quasi-quantum Kripke frame and F a projection on $\mathfrak{F}$. For every $s \in \Sigma$, $F(s)$ is an approximation of s in $F[\Sigma]$, if it is defined.*

Proof. Assume that $F(s)$ is defined, by (iii) of Lemma 3.1.20, for every $t \in \Sigma$ such that $F(t)$ is defined, $F(s) \to F(t)$, if and only if $F(F(t))$ is defined and $F(F(t)) \to s$. Note that, for every $t \in \Sigma$ such that $F(t)$ is defined, by Reflexivity $F(t) \to F(t)$, so $F(F(t))$ is defined by (iii) in Lemma 3.1.20. Then it follows from $F \circ F \subseteq F$ that, for every $t \in \Sigma$ such that $F(t)$ is defined, $F(s) \to F(t)$ if and only if $F(t) \to s$. This means that $s \approx_{F[\Sigma]} F(s)$. Since $F(s) \in F[\Sigma]$, $F(s)$ is an approximation of s in $F[\Sigma]$. ⊣

Next I show a correspondence between the projections and the saturated subsets of a quasi-quantum Kripke frame, from which the desired one follows.

The following lemma shows how to get a saturated subset given a projection.

3.1.24. LEMMA. *Let $\mathfrak{F} = (\Sigma, \to)$ be a quasi-quantum Kripke frame and F a projection on $\mathfrak{F}$. Then $\mathbf{I}_{\mathfrak{F}}(F) = F[\Sigma]$ is a saturated subset of Σ.*

Moreover, $F \mapsto \mathbf{I}_{\mathfrak{F}}(F)$ is a function, denoted by $\mathbf{I}_{\mathfrak{F}}$, from the set of projections on $\mathfrak{F}$ to the set of saturated subsets of $\mathfrak{F}$.

Proof. Assume that $s \notin {\sim}F[\Sigma]$. Then there is a $t \in F[\Sigma]$ such that $t \to s$. Since $t \in F[\Sigma]$, by (iii) of Lemma 3.1.20 $F(s)$ is defined. By Lemma 3.1.23 $F(s)$ is an approximation of s in $F[\Sigma]$. As a result, $F[\Sigma]$ is saturated. ⊣

The following lemma shows how to get a projection given a saturated subset.

3.1.25. LEMMA. *Let $\mathfrak{F} = (\Sigma, \to)$ be a quasi-quantum Kripke frame and $P \subseteq \Sigma$ saturated. Define that*

$$\mathbf{J}_{\mathfrak{F}}(P) = \{(s, s') \in \Sigma \times \Sigma \mid s' \textit{ is an approximation of } s \textit{ in } P\}$$

Then $\mathbf{J}_{\mathfrak{F}}(P)$ is a projection.

Moreover, $P \mapsto \mathbf{J}_{\mathfrak{F}}(P)$ is a function, denoted by $\mathbf{J}_{\mathfrak{F}}$, from the set of saturated subsets of $\mathfrak{F}$ to the set of projections on $\mathfrak{F}$.

Proof. Note that $s \in {\sim}P$, if and only if there is no $s' \in \Sigma$ such that $(s, s') \in \mathbf{J}_{\mathfrak{F}}(P)$.

First show that $\mathbf{J}_{\mathfrak{F}}(P)$ is a self-adjoint continuous homomorphism by proving (iii) in Lemma 3.1.20. Let $s, s', t \in \Sigma$ be arbitrary such that $(s, s') \in \mathbf{J}_{\mathfrak{F}}(P)$. By definition $s' \in P$ and $s \approx_P s'$. Assume firstly that $s' \to t$. Then $t \notin {\sim}P$. Since P is saturated, there is a $t' \in \Sigma$ such that $t' \in P$ and $t \approx_P t'$. By definition $(t, t') \in \mathbf{J}_{\mathfrak{F}}(P)$. Since $s', t' \in P$, $t \approx_P t'$ and $s \approx_P s'$, I derive from $s' \to t$ that $t' \to s$. Assume secondly that there is a $t' \in \Sigma$ such that $(t, t') \in \mathbf{J}_{\mathfrak{F}}(P)$ and $t' \to s$. Since $(t, t') \in \mathbf{J}_{\mathfrak{F}}(P)$, $t' \in P$ and $t \approx_P t'$. Since $s', t' \in P$, $t \approx_P t'$ and $s \approx_P s'$, I derive from $t' \to s$ that $s' \to t$.

Second show that $\mathbf{J}_{\mathfrak{F}}(P) \circ \mathbf{J}_{\mathfrak{F}}(P) \subseteq \mathbf{J}_{\mathfrak{F}}(P)$. Assume that $(s, s'') \in \mathbf{J}_{\mathfrak{F}}(P) \circ \mathbf{J}_{\mathfrak{F}}(P)$. Then there is an $s' \in \Sigma$ such that $(s, s'), (s', s'') \in \mathbf{J}_{\mathfrak{F}}(P)$. By definition $s', s'' \in P$, $s \approx_P s'$ and $s' \approx_P s''$. Since $\approx_P$ is an equivalence relation by Remark 2.1.2, $s \approx_P s''$. Hence both s' and s'' are approximations of s in P. Since P is saturated, it is a subspace by Proposition 2.4.4 and Lemma 2.3.20. Hence by Proposition 2.4.2 $s' = s''$. It follows that $(s, s'') \in \mathbf{J}_{\mathfrak{F}}(P)$. ⊣

Now I prove the correspondence.

3.1.26. PROPOSITION. *Let $\mathfrak{F} = (\Sigma, \to)$ be a quasi-quantum Kripke frame.*

1. *For every projection F on $\mathfrak{F}$, $\mathbf{J}_{\mathfrak{F}} \circ \mathbf{I}_{\mathfrak{F}}(F) = F$.*

2. For every saturated subset P of $\mathfrak{F}$, $\mathbf{I}_{\mathfrak{F}} \circ \mathbf{J}_{\mathfrak{F}}(P) = P$.

Proof. **For *1*:** By definition $\mathbf{I}_{\mathfrak{F}}(F) = F[\Sigma]$.

First assume that $(s, s') \in \mathbf{J}_{\mathfrak{F}}(F[\Sigma])$. By definition s' is an approximation of s in $F[\Sigma]$. Since $s' \to s'$ by Reflexivity, $s' \to s$. Since $s' \in F[\Sigma]$, by (iii) of Lemma 3.1.20 $F(s)$ is defined. By Lemma 3.1.23 $F(s)$ is an approximation of s in $F[\Sigma]$. By Lemma 2.3.20 and Proposition 2.4.2 s has a unique approximation in $F[\Sigma]$. Hence $F(s) = s'$, and thus $(s, s') \in F$.

Second assume that $(s, s') \in F$. Then $F(s)$ is defined. By Lemma 3.1.23 s' is an approximation of s in $F[\Sigma]$. By the definition of $\mathbf{J}_{\mathfrak{F}}$ $(s, s') \in \mathbf{J}_{\mathfrak{F}}(F[\Sigma])$.

For *2*: By definition $\mathbf{I}_{\mathfrak{F}} \circ \mathbf{J}_{\mathfrak{F}}(P) = \mathbf{J}_{\mathfrak{F}}(P)[\Sigma]$. It follows easily from the definition of $\mathbf{J}_{\mathfrak{F}}(P)$ that $\mathbf{J}_{\mathfrak{F}}(P)[\Sigma] = P$. Therefore, $\mathbf{I}_{\mathfrak{F}} \circ \mathbf{J}_{\mathfrak{F}}(P) = P$. ⊣

Finally, the counterpart of the correspondence between the projectors and the closed linear subspaces of a Hilbert space follows as a corollary of this proposition.

3.1.27. Corollary. *Let $\mathfrak{F} = (\Sigma, \to)$ be a quantum Kripke frame. The set of projections on $\mathfrak{F}$ and the set of bi-orthogonally closed subsets of $\mathfrak{F}$ are in correspondence witnessed by $\mathbf{I}_{\mathfrak{F}}$ and $\mathbf{J}_{\mathfrak{F}}$.*

Proof. By Theorem 2.4.9 for $\mathfrak{F}$ the set of bi-orthogonally closed subsets coincides with the set of the saturated subsets. Hence the conclusion follows from the previous proposition. ⊣

In [14] Baltag and Smets also captured, in a relational setting, the essential properties of projectors on Hilbert spaces. It is interesting to compare their work with the analysis here.

3.2 Parties of Homomorphisms

The theme of this chapter is to study the structure formed by the continuous homomorphisms between two quasi-quantum Kripke frames. In this section, I move to the setting of projective geometries, which is more general than that of quasi-quantum Kripke frames, and study the structure formed by the arguesian homomorphisms between Pappian projective geometries. Certainly the results hold in the more specific setting of quasi-quantum Kripke frames.

In Subsection 3.2.1 I introduce the useful notion of rulers and some other related notions, and prove some basic facts about them. In Subsection 3.2.2 I define a binary relation between the arguesian homomorphisms from one Pappian projective geometry to another such that two arguesian homomorphisms are related if and only if they can be induced by quasi-linear maps with the same accompanying field isomorphisms. In Subsection 3.2.3 I study the structure of the set of arguesian homomorphisms all of which relate to a certain one by this relation, and show that there is a bijection from this set to the set of points in a projective geometry induced by a vector space of linear maps. This solves a special case of an open problem in [39].

For the involved definitions and results in projective geometry and linear algebra, please refer to Appendices B and C.

3.2.1 Rulers

In this subsection, I introduce the notion of rulers, as well as other related notions.

3.2.1. Definition. A *ruler* in a projective geometry $\mathcal{G} = (G, \star)$ is a tuple (r^o, r^∞, r^e) such that $r^o, r^\infty, r^e \in G$ are distinct and $r^e \in r^o \star r^\infty$.

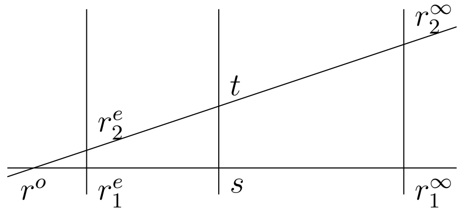
3.2.2. Remark. In a ruler (r^o, r^∞, r^e), r^o is called *the origin* and r^e *the unit*.

Now I use rulers to measure things.

3.2.3. Definition. Let $\mathcal{G} = (G, \star)$ be a projective geometry, and (r^o, r_1^∞, r_1^e) and (r^o, r_2^∞, r_2^e) be two rulers in $\mathcal{G}$.

- (r^o, r_1^∞, r_1^e) and (r^o, r_2^∞, r_2^e) are *in different directions*, if $r^o \star r_1^\infty \neq r^o \star r_2^\infty$.
- when (r^o, r_1^∞, r_1^e) and (r^o, r_2^∞, r_2^e) are in different directions, the two tuples $(r^o, r_1^\infty, r_1^e \mid s)$ and $(r^o, r_2^\infty, r_2^e \mid t)$ are *proportional*, denoted by $(r^o, r_1^\infty, r_1^e \mid s) \equiv (r^o, r_2^\infty, r_2^e \mid t)$, if one of the following holds:
  - $s = r^o = t$;
  - $s = r_1^\infty$ and $t = r_2^\infty$;
  - $s \in (r^o \star r_1^\infty) \setminus \{r^o, r_1^\infty\}$, $t \in (r^o \star r_2^\infty) \setminus \{r^o, r_2^\infty\}$ and $(s \star t) \cap (r_1^e \star r_2^e) \cap (r_1^\infty \star r_2^\infty)$ is a singleton.

    (The following picture of the analogue in an affine plane may help to understand this condition. Remember that parallel lines in an affine plane are considered to intersect at a point at infinity.)



The significance of this notion shall be made clear by the following proposition.

3.2.4. PROPOSITION. *Suppose that V is a vector space over a division ring $\mathcal{K} = (K, +, \cdot, 0, 1)$, and $(\langle \mathbf{w} \rangle, \langle \mathbf{u} \rangle, \langle \mathbf{w}+\mathbf{u} \rangle)$ and $(\langle \mathbf{w} \rangle, \langle \mathbf{v} \rangle, \langle \mathbf{w}+\mathbf{v} \rangle)$ two rulers in $\mathcal{P}(V)$ in different directions, $x, y, x', y' \in K$ are such that either x or y is not 0 and either x' or y' is not 0. The following are equivalent:*

(i) $(\langle \mathbf{w} \rangle, \langle \mathbf{u} \rangle, \langle \mathbf{w}+\mathbf{u} \rangle \mid \langle x\mathbf{w}+y\mathbf{u} \rangle) \equiv (\langle \mathbf{w} \rangle, \langle \mathbf{v} \rangle, \langle \mathbf{w}+\mathbf{v} \rangle \mid \langle x'\mathbf{w}+y'\mathbf{v} \rangle)$;

(ii) $y = y' = 0$, *or* $y^{-1} \cdot x = (y')^{-1} \cdot x'$.

Proof. **From (i) to (ii):** Since the two rulers are in different directions, $\mathbf{w}, \mathbf{u}, \mathbf{v}$ are linearly independent. By definition three cases need to be considered.

Case 1: $\langle x\mathbf{w}+y\mathbf{u} \rangle = \langle \mathbf{w} \rangle = \langle x'\mathbf{w}+y'\mathbf{v} \rangle$. Since $\mathbf{w}, \mathbf{u}, \mathbf{v}$ are linearly independent, $y = y' = 0$.

Case 2: $\langle x\mathbf{w}+y\mathbf{u} \rangle = \langle \mathbf{u} \rangle$ and $\langle x'\mathbf{w}+y'\mathbf{v} \rangle = \langle \mathbf{v} \rangle$. Then $x = x' = 0$. Since either x or y is not 0 and either x' or y' is not 0, $y, y' \notin \{0\}$. Hence $y^{-1} \cdot x = 0 = (y')^{-1} \cdot x'$.

Case 3: $\langle x\mathbf{w}+y\mathbf{u} \rangle \notin \{\langle \mathbf{w} \rangle, \langle \mathbf{u} \rangle\}$, $\langle x'\mathbf{w}+y'\mathbf{v} \rangle \notin \{\langle \mathbf{w} \rangle, \langle \mathbf{v} \rangle\}$ and $(\langle \mathbf{u} \rangle \star \langle \mathbf{v} \rangle) \cap (\langle \mathbf{w}+\mathbf{u} \rangle \star \langle \mathbf{w}+\mathbf{v} \rangle) \cap (\langle x\mathbf{w}+y\mathbf{u} \rangle \star \langle x'\mathbf{w}+y'\mathbf{v} \rangle)$ is a singleton. Since the point in the singleton is in $\langle \mathbf{u} \rangle \star \langle \mathbf{v} \rangle$ and $\langle \mathbf{w}+\mathbf{u} \rangle \star \langle \mathbf{w}+\mathbf{v} \rangle$, it is not hard to see that it is $\langle \mathbf{u}-\mathbf{v} \rangle$. Since $\langle \mathbf{u}-\mathbf{v} \rangle$ is in $\langle x\mathbf{w}+y\mathbf{u} \rangle \star \langle x'\mathbf{w}+y'\mathbf{v} \rangle$, it follows that there are $a, b \in K$ such that

$$\mathbf{u}-\mathbf{v} = a(x\mathbf{w}+y\mathbf{u}) + b(x'\mathbf{w}+y'\mathbf{v}) = (a \cdot x + b \cdot x')\mathbf{w} + (a \cdot y)\mathbf{u} + (b \cdot y')\mathbf{v}$$

Since $\mathbf{w}, \mathbf{u}, \mathbf{v}$ are linearly independent, it follows that

$$\begin{aligned} a \cdot x + b \cdot x' &= 0 \\ a \cdot y - 1 &= 0 \\ b \cdot y' + 1 &= 0 \end{aligned}$$

Then it is not hard to deduce that $y \neq 0$, $y' \neq 0$ and $y^{-1} \cdot x = (y')^{-1} \cdot x'$.

From (ii) to (i): It is a straightforward verification. ⊣

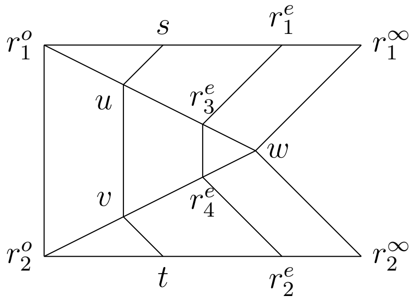
I continue to introduce the notion of harmonicity, which generalizes that of proportionality.

3.2.5. DEFINITION. Let $\mathcal{G} = (G, \star)$ be a projective geometry, and $(r_1^o, r_1^\infty, r_1^e)$ and $(r_2^o, r_2^\infty, r_2^e)$ be two rulers in $\mathcal{G}$. The tuples $(r_1^o, r_1^\infty, r_1^e \mid s)$ and $(r_2^o, r_2^\infty, r_2^e \mid t)$ are *harmonic*, denoted by $(r_1^o, r_1^\infty, r_1^e \mid s) - (r_2^o, r_2^\infty, r_2^e \mid t)$, if $s \in r_1^o \star r_1^\infty$, $t \in r_2^o \star r_2^\infty$, and one of the following holds:

- $r_1^o \neq r_2^o$, and there are points $w \notin (r_1^o \star r_2^o) \cup (r_1^o \star r_1^\infty) \cup (r_2^o \star r_2^\infty)$, $u \in r_1^o \star w$ and $v \in r_2^o \star w$ and four rulers (r_1^o, w, r_3^e), (w, r_1^o, r_3^e), (w, r_2^o, r_4^e), (r_2^o, w, r_4^e) such that all of the following hold:

1. $(r_1^o, r_1^\infty, r_1^e \mid s) \equiv (r_1^o, w, r_3^e \mid u)$;
2. $(w, r_1^o, r_3^e \mid u) \equiv (w, r_2^o, r_4^e \mid v)$;
3. $(r_2^o, w, r_4^e \mid v) \equiv (r_2^o, r_2^\infty, r_2^e \mid t)$;

(The following picture of the analogue in an affine plane may help to understand these conditions:)



- $r_1^o = r_2^o$, and there is a ruler $(r_3^o, r_3^\infty, r_3^e)$ and $p \in r_3^o \star r_3^\infty$ such that $r_3^o \neq r_1^o$ and
$$(r_1^o, r_1^\infty, r_1^e \mid s) - (r_3^o, r_3^\infty, r_3^e \mid p) - (r_2^o, r_2^\infty, r_2^e \mid t)$$
in the sense of the above case.
- $s = r_1^o$ and $t = r_2^o$;
- $s = r_1^\infty$ and $t = r_2^\infty$.

The following proposition explains the meaning of the word 'harmonic':

3.2.6. PROPOSITION. *Let V be a vector space of dimension at least 3 over a division ring $\mathcal{K} = (K, +, \cdot, 0, 1)$, $(\langle \mathbf{u} \rangle, \langle \mathbf{v} \rangle, \langle \mathbf{u} + \mathbf{v} \rangle)$ and $(\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle, \langle \mathbf{s} + \mathbf{t} \rangle)$ two rulers in $\mathcal{P}(V)$, and $x, y, x', y' \in K$ be such that either x or y is not 0 and either x' or y' is not 0. The following are equivalent:*

(i) $(\langle \mathbf{u} \rangle, \langle \mathbf{v} \rangle, \langle \mathbf{u} + \mathbf{v} \rangle \mid \langle x\mathbf{u} + y\mathbf{v} \rangle) - (\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle, \langle \mathbf{s} + \mathbf{t} \rangle \mid \langle x'\mathbf{s} + y'\mathbf{t} \rangle)$;

(ii) $y = y' = 0$, *or* $y^{-1} \cdot x = z^{-1} \cdot ((y')^{-1} \cdot x') \cdot z$ *for some* $z \in K \setminus \{0\}$.

Moreover, when $\mathcal{K}$ is a field, the above are also equivalent to the following:

(iii) $y = y' = 0$, *or* $y^{-1} \cdot x = (y')^{-1} \cdot x'$.

Proof. **From (i) to (ii):** According to the definition, consider four cases.

Case 1: $\langle \mathbf{u} \rangle \neq \langle \mathbf{s} \rangle$, and there is a $\langle \mathbf{w} \rangle \in \Sigma(V)$ such that $\langle \mathbf{w} \rangle \notin (\langle \mathbf{u} \rangle \star \langle \mathbf{s} \rangle) \cup (\langle \mathbf{u} \rangle \star \langle \mathbf{v} \rangle) \cup (\langle \mathbf{s} \rangle \star \langle \mathbf{t} \rangle)$, and

1. $(\langle \mathbf{u} \rangle, \langle \mathbf{v} \rangle, \langle \mathbf{u} + \mathbf{v} \rangle \mid \langle x\mathbf{u} + y\mathbf{v} \rangle) \equiv (\langle \mathbf{u} \rangle, \langle \mathbf{w} \rangle, \langle \mathbf{u} + e\mathbf{w} \rangle \mid \langle a\mathbf{u} + b\mathbf{w} \rangle)$;

2. $(\langle \mathbf{w} \rangle, \langle \mathbf{u} \rangle, \langle \mathbf{u} + e\mathbf{w} \rangle \mid \langle a\mathbf{u} + b\mathbf{w} \rangle) \equiv (\langle \mathbf{w} \rangle, \langle \mathbf{s} \rangle, \langle \mathbf{s} + e'\mathbf{w} \rangle \mid \langle c\mathbf{s} + d\mathbf{w} \rangle)$;

3. $(\langle \mathbf{s} \rangle, \langle \mathbf{w} \rangle, \langle \mathbf{s} + e'\mathbf{w} \rangle \mid \langle c\mathbf{s} + d\mathbf{w} \rangle) \equiv (\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle, \langle \mathbf{s} + \mathbf{t} \rangle \mid \langle x'\mathbf{s} + y'\mathbf{t} \rangle)$;

for some $a, b, c, d \in K$ and $e, e' \in K \setminus \{0\}$ such that either a or b is not 0 and either c or d is not 0. By 1 and Proposition 3.2.4 $(\langle \mathbf{u} \rangle, \langle \mathbf{v} \rangle, \langle \mathbf{u} + \mathbf{v} \rangle \mid \langle x\mathbf{u} + y\mathbf{v} \rangle) \equiv (\langle \mathbf{u} \rangle, \langle e\mathbf{w} \rangle, \langle \mathbf{u} + e\mathbf{w} \rangle \mid \langle a\mathbf{u} + (b \cdot e^{-1})e\mathbf{w} \rangle)$, so $y = b = 0$, or $y^{-1} \cdot x = e \cdot b^{-1} \cdot a$. By 2 and Proposition 3.2.4

$$\begin{aligned}&(\langle \mathbf{w} \rangle, \langle \mathbf{u} \rangle, \langle \mathbf{w} + e^{-1}\mathbf{u} \rangle \mid \langle b\mathbf{w} + (a \cdot e)e^{-1}\mathbf{u} \rangle)\\ \equiv\ &(\langle \mathbf{w} \rangle, \langle \mathbf{s} \rangle, \langle \mathbf{w} + e'^{-1}\mathbf{s} \rangle \mid \langle d\mathbf{w} + (c \cdot e')e'^{-1}\mathbf{s} \rangle)\end{aligned}$$

so $a = c = 0$, or $e^{-1} \cdot a^{-1} \cdot b = e'^{-1} \cdot c^{-1} \cdot d$. By 3 and Proposition 3.2.4 $(\langle \mathbf{s} \rangle, \langle e'\mathbf{w} \rangle, \langle \mathbf{s} + e'\mathbf{w} \rangle \mid \langle c\mathbf{s} + (d \cdot e'^{-1})e'\mathbf{w} \rangle) \equiv (\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle, \langle \mathbf{s} + \mathbf{t} \rangle \mid \langle x'\mathbf{s} + y'\mathbf{t} \rangle)$, so $d = y' = 0$, or $e' \cdot d^{-1} \cdot c = y'^{-1} \cdot x'$. It follows that $y = y' = 0$, or $y^{-1} \cdot x = z^{-1} \cdot (y'^{-1} \cdot x') \cdot z$, where $z = e' \cdot e^{-1} \neq 0$.

Case 2: $\langle \mathbf{u} \rangle = \langle \mathbf{s} \rangle$ and there is a ruler $(\langle \mathbf{p} \rangle, \langle \mathbf{q} \rangle, \langle \mathbf{p} + \mathbf{q} \rangle)$ such that $\langle \mathbf{p} \rangle \neq \langle \mathbf{u} \rangle = \langle \mathbf{s} \rangle$ and

$$\begin{aligned}&(\langle \mathbf{u} \rangle, \langle \mathbf{v} \rangle, \langle \mathbf{u} + \mathbf{v} \rangle \mid \langle x\mathbf{u} + y\mathbf{v} \rangle)\\ -\ &(\langle \mathbf{p} \rangle, \langle \mathbf{q} \rangle, \langle \mathbf{p} + \mathbf{q} \rangle \mid \langle a\mathbf{p} + b\mathbf{q} \rangle)\\ -\ &(\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle, \langle \mathbf{s} + \mathbf{t} \rangle \mid \langle x'\mathbf{s} + y'\mathbf{t} \rangle)\end{aligned}$$

in the sense of the above case, for $a, b \in K$ of which at least one is not 0. By what has been proved in Case 1, both of the following hold:

1. $y = b = 0$, or $y^{-1} \cdot x = c^{-1} \cdot (b^{-1} \cdot a) \cdot c$ for some $c \in K \setminus \{0\}$;

2. $y' = b = 0$, or $y'^{-1} \cdot x' = d^{-1} \cdot (b^{-1} \cdot a) \cdot d$ for some $d \in K \setminus \{0\}$.

Therefore, $y = y' = 0$, or $y^{-1} \cdot x = z^{-1} \cdot (y'^{-1} \cdot x') \cdot z$ for $z = d^{-1} \cdot c \neq 0$.

Case 3: $\langle \mathbf{u} \rangle = \langle x\mathbf{u} + y\mathbf{v} \rangle$ and $\langle \mathbf{s} \rangle = \langle x'\mathbf{s} + y'\mathbf{t} \rangle$. It follows that $y = y' = 0$.

Case 4: $\langle \mathbf{v} \rangle = \langle x\mathbf{u} + y\mathbf{v} \rangle$ and $\langle \mathbf{t} \rangle = \langle x'\mathbf{s} + y'\mathbf{t} \rangle$. It follows that $x = x' = 0$ and $y, y' \notin \{0\}$. Take $z = 1 \neq 0$, then $y^{-1} \cdot x = 0 = z^{-1} \cdot (y'^{-1} \cdot x') \cdot z$.

From (ii) to (i): Again four cases need to be considered.

Case 1: $y = 0$ or $y' = 0$. Then by (ii) $y = y' = 0$. Hence $\langle x\mathbf{u} + y\mathbf{v} \rangle = \langle \mathbf{u} \rangle$ and $\langle x'\mathbf{s} + y'\mathbf{t} \rangle = \langle \mathbf{s} \rangle$. By definition $(\langle \mathbf{u} \rangle, \langle \mathbf{v} \rangle, \langle \mathbf{u} + \mathbf{v} \rangle \mid \langle x\mathbf{u} + y\mathbf{v} \rangle) - (\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle, \langle \mathbf{s} + \mathbf{t} \rangle \mid \langle x'\mathbf{s} + y'\mathbf{t} \rangle)$.

Case 2: $y, y' \notin \{0\}$ and $x = 0$. Since in this case $y^{-1} \cdot x = z^{-1} \cdot (y'^{-1} \cdot x') \cdot z$ for some $z \in K \setminus \{0\}$, $x' = 0$. Hence $\langle x\mathbf{u} + y\mathbf{v} \rangle = \langle \mathbf{v} \rangle$ and $\langle x'\mathbf{s} + y'\mathbf{t} \rangle = \langle \mathbf{t} \rangle$. By definition $(\langle \mathbf{u} \rangle, \langle \mathbf{v} \rangle, \langle \mathbf{u} + \mathbf{v} \rangle \mid \langle x\mathbf{u} + y\mathbf{v} \rangle) - (\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle, \langle \mathbf{s} + \mathbf{t} \rangle \mid \langle x'\mathbf{s} + y'\mathbf{t} \rangle)$.

Case 3: $y, y' \notin \{0\}$, $x \neq 0$ and $\langle \mathbf{u} \rangle \neq \langle \mathbf{s} \rangle$. Since in this case $y^{-1} \cdot x = z^{-1} \cdot (y'^{-1} \cdot x') \cdot z$ for some $z \in K \setminus \{0\}$, $x' \neq 0$. Since $(\langle \mathbf{u} \rangle, \langle \mathbf{v} \rangle, \langle \mathbf{u} + \mathbf{v} \rangle)$ is a ruler, $\langle \mathbf{u} \rangle \neq \langle \mathbf{v} \rangle$. Similarly, $\langle \mathbf{s} \rangle \neq \langle \mathbf{t} \rangle$. By Lemma B.2.5 there is a $\mathbf{w} \in V \setminus \{\mathbf{0}\}$ such that $\langle \mathbf{w} \rangle \notin (\langle \mathbf{u} \rangle \star \langle \mathbf{s} \rangle) \cup (\langle \mathbf{u} \rangle \star \langle \mathbf{v} \rangle) \cup (\langle \mathbf{s} \rangle \star \langle \mathbf{t} \rangle)$. By Proposition 3.2.4 it is easy to see that:

1. $(\langle \mathbf{u} \rangle, \langle \mathbf{v} \rangle, \langle \mathbf{u}+\mathbf{v} \rangle \mid \langle x\mathbf{u}+y\mathbf{v} \rangle) \equiv (\langle \mathbf{u} \rangle, \langle \mathbf{w} \rangle, \langle \mathbf{u}+\mathbf{w} \rangle \mid \langle x\mathbf{u}+y\mathbf{w} \rangle)$;
2. $(\langle \mathbf{w} \rangle, \langle \mathbf{u} \rangle, \langle \mathbf{u}+\mathbf{w} \rangle \mid \langle x\mathbf{u}+y\mathbf{w} \rangle) \equiv (\langle \mathbf{w} \rangle, \langle \mathbf{s} \rangle, \langle \mathbf{s}+z\mathbf{w} \rangle \mid \langle x'\mathbf{s}+(y'\cdot z)\mathbf{w} \rangle)$;
3. $(\langle \mathbf{s} \rangle, \langle \mathbf{w} \rangle, \langle \mathbf{s}+z\mathbf{w} \rangle \mid \langle x'\mathbf{s}+(y'\cdot z)\mathbf{w} \rangle) \equiv (\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle, \langle \mathbf{s}+\mathbf{t} \rangle \mid \langle x'\mathbf{s}+y'\mathbf{t} \rangle)$;

By definition $(\langle \mathbf{u} \rangle, \langle \mathbf{v} \rangle, \langle \mathbf{u}+\mathbf{v} \rangle \mid \langle x\mathbf{u}+y\mathbf{v} \rangle) - (\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle, \langle \mathbf{s}+\mathbf{t} \rangle \mid \langle x'\mathbf{s}+y'\mathbf{t} \rangle)$.

Case 4: $y, y' \notin \{0\}$, $x \neq 0$ and $\langle \mathbf{u} \rangle = \langle \mathbf{s} \rangle$. Since in this case $y^{-1} \cdot x = z^{-1} \cdot (y'^{-1} \cdot x') \cdot z$ for some $z \in K \setminus \{0\}$, $x' \neq 0$. Since V is of dimension at least 3, take a $\mathbf{w} \in V \setminus \{\mathbf{0}\}$ such that $\langle \mathbf{w} \rangle \neq \langle \mathbf{u} \rangle = \langle \mathbf{s} \rangle$. By the previous case

$$\begin{aligned} &(\langle \mathbf{u} \rangle, \langle \mathbf{v} \rangle, \langle \mathbf{u}+\mathbf{v} \rangle \mid \langle x\mathbf{u}+y\mathbf{v} \rangle) \\ -\ &(\langle \mathbf{w} \rangle, \langle \mathbf{u} \rangle, \langle \mathbf{w}+\mathbf{u} \rangle \mid \langle x\mathbf{w}+y\mathbf{u} \rangle) \\ -\ &(\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle, \langle \mathbf{s}+\mathbf{t} \rangle \mid \langle x'\mathbf{s}+y'\mathbf{t} \rangle) \end{aligned}$$

By definition $(\langle \mathbf{u} \rangle, \langle \mathbf{v} \rangle, \langle \mathbf{u}+\mathbf{v} \rangle \mid \langle x\mathbf{u}+y\mathbf{v} \rangle) - (\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle, \langle \mathbf{s}+\mathbf{t} \rangle \mid \langle x'\mathbf{s}+y'\mathbf{t} \rangle)$.

Finally, when $\mathcal{K}$ is a field, it is easy to see that (iii) is equivalent to (ii), and thus to (i). ⊣

3.2.7. Remark. This proposition implies that, for any two rulers $(r_1^o, r_1^\infty, r_1^e)$ and $(r_2^o, r_2^\infty, r_2^e)$ in a projective geometry $\mathcal{P}(V)$ and $s \in r_1^o \star r_1^\infty$, a $t \in r_2^o \star r_2^\infty$ can always be found such that $(r_1^o, r_1^\infty, r_1^e \mid s) - (r_2^o, r_2^\infty, r_2^e \mid t)$.

However, this t may not be unique.

When V is a vector space over a division ring $\mathcal{K}$ which is not a field, even with a ruler $(\langle \mathbf{v}^o \rangle, \langle \mathbf{v}^\infty \rangle, \langle \mathbf{v}^e \rangle)$ in $\mathcal{P}(V)$, where $\mathbf{v}^e = \mathbf{v}^o + \mathbf{v}^\infty$, and a non-zero x in $\mathcal{K}$ fixed, there are many points $t \in \langle \mathbf{v}^o \rangle \star \langle \mathbf{v}^\infty \rangle$ such that $(\langle \mathbf{v}^o \rangle, \langle \mathbf{v}^\infty \rangle, \langle \mathbf{v}^e \rangle \mid t)$ is harmonic with $(\langle \mathbf{v}^o \rangle, \langle \mathbf{v}^\infty \rangle, \langle \mathbf{v}^e \rangle \mid \langle \mathbf{v}^o + x\mathbf{v}^\infty \rangle)$. To be precise, for every non-zero z in $\mathcal{K}$, by Proposition 3.2.6

$$(\langle \mathbf{v}^o \rangle, \langle \mathbf{v}^\infty \rangle, \langle \mathbf{v}^e \rangle \mid \langle \mathbf{v}^o + (z^{-1}\cdot x\cdot z)\mathbf{v}^\infty \rangle) - (\langle \mathbf{v}^o \rangle, \langle \mathbf{v}^\infty \rangle, \langle \mathbf{v}^e \rangle \mid \langle \mathbf{v}^o + x\mathbf{v}^\infty \rangle)$$

Since $\mathcal{K}$ is not a field, for a different z, $\langle \mathbf{v}^o + (z^{-1}\cdot x\cdot z)\mathbf{v}^\infty \rangle$ is different. On the one hand, this could be understood, for $(\langle \mathbf{v}^o \rangle, \langle \mathbf{v}^\infty \rangle, \langle \mathbf{v}^e \rangle \mid \langle \mathbf{v}^o + (z^{-1}\cdot x\cdot z)\mathbf{v}^\infty \rangle)$ is the same as $(\langle z\mathbf{v}^o \rangle, \langle z\mathbf{v}^\infty \rangle, \langle z\mathbf{v}^e \rangle \mid \langle z\mathbf{v}^o + x(z\mathbf{v}^\infty) \rangle)$. On the other hand, this is unfavourable, because this means that, even after fixing the origin, the direction and the unit, the point on the ruler with 'coordinate' x is still undetermined; and it is determined only after the representative vectors generating the points in $\mathcal{P}(V)$ are fixed. This is not good from the perspective of projective geometry.

When $\mathcal{K}$ is a field, this unfavourable fact diminishes. In this case, for a different z, $\langle \mathbf{v}^o + (z^{-1}\cdot x\cdot z)\mathbf{v}^\infty \rangle$ is the same point $\langle \mathbf{v}^o + x\mathbf{v}^\infty \rangle$ in $\mathcal{P}(V)$, which intuitively is *the* point on the line $\langle \mathbf{v}^o \rangle \star \langle \mathbf{v}^\infty \rangle$ with 'coordinate' x with respect to the unit $\langle \mathbf{v}^e \rangle$. Please be reminded by this of the idea that every line in an arguesian projective geometry, deleting one arbitrary point, has the structure of a division ring, which was used already in Subsection 2.7.1.

Because of the reasons mentioned above, in the remaining part of this chapter, I mostly focus on vector spaces (of dimension at least 3) over *fields*. Remember that the projective geometries induced by them are Pappian (Theorem B.3.7), and the quasi-quantum Kripke frames induced by them are also Pappian (Proposition 2.7.13).

3.2.2 The Comrades of a Homomorphism

Every arguesian homomorphism from one Pappian projective geometry to another is induced by a quasi-linear map between the vector spaces inducing the two projective geometries (Theorem B.3.13). In this subsection, I aim at a geometric way to judge, for any two arguesian homomorphisms between two Pappian projective geometries, whether there are quasi-linear maps that induce them and have the same accompanying field isomorphism. I start from the context of linear algebra, and arrive at a criterion in the context of projective geometries in the form of a binary relation at the end.

I start from a quite general situation.

3.2.8. LEMMA. *Suppose that V_1 and V_2 are two vector spaces over two* fields $\mathcal{K}_1 = (K_1, +, \cdot, 0, 1)$[7] *and $\mathcal{K}_2 = (K_2, +, \cdot, 0, 1)$ such that both are of dimension at least 3, $f, g : V_1 \to V_2$ are two quasi-linear maps such that both of their images are at least two-dimensional, and σ and τ are accompanying field isomorphisms of f and g, respectively. Then the following are equivalent:*

(i) $\sigma = \tau$;

(ii) for any rulers $(\langle \mathbf{v}_1^o \rangle, \langle \mathbf{v}_1^\infty \rangle, \langle \mathbf{v}_1^e \rangle)$ and $(\langle \mathbf{w}_1^o \rangle, \langle \mathbf{w}_1^\infty \rangle, \langle \mathbf{w}_1^e \rangle)$ in $\mathcal{P}(V_1)$ such that both $(\langle f(\mathbf{v}_1^o) \rangle, \langle f(\mathbf{v}_1^\infty) \rangle, \langle f(\mathbf{v}_1^e) \rangle)$ and $(\langle g(\mathbf{w}_1^o) \rangle, \langle g(\mathbf{w}_1^\infty) \rangle, \langle g(\mathbf{w}_1^e) \rangle)$ are rulers in $\mathcal{P}(V_2)$, $\langle \mathbf{v}_1 \rangle \in \langle \mathbf{v}_1^o \rangle \star \langle \mathbf{v}_1^\infty \rangle$ and $\langle \mathbf{w}_1 \rangle \in \langle \mathbf{w}_1^o \rangle \star \langle \mathbf{w}_1^\infty \rangle$,

$$(\langle f(\mathbf{v}_1^o) \rangle, \langle f(\mathbf{v}_1^\infty) \rangle, \langle f(\mathbf{v}_1^e) \rangle \mid \langle f(\mathbf{v}_1) \rangle) - (\langle g(\mathbf{w}_1^o) \rangle, \langle g(\mathbf{w}_1^\infty) \rangle, \langle g(\mathbf{w}_1^e) \rangle \mid \langle g(\mathbf{w}_1) \rangle),$$

if $(\langle \mathbf{v}_1^o \rangle, \langle \mathbf{v}_1^\infty \rangle, \langle \mathbf{v}_1^e \rangle \mid \langle \mathbf{v}_1 \rangle) - (\langle \mathbf{w}_1^o \rangle, \langle \mathbf{w}_1^\infty \rangle, \langle \mathbf{w}_1^e \rangle \mid \langle \mathbf{w}_1 \rangle)$.

Proof. **From (i) to (ii):** Let $(\langle \mathbf{v}_1^o \rangle, \langle \mathbf{v}_1^\infty \rangle, \langle \mathbf{v}_1^e \rangle)$, $(\langle \mathbf{w}_1^o \rangle, \langle \mathbf{w}_1^\infty \rangle, \langle \mathbf{w}_1^e \rangle)$ be two rulers in $\mathcal{P}(V_1)$ such that $(\langle f(\mathbf{v}_1^o) \rangle, \langle f(\mathbf{v}_1^\infty) \rangle, \langle f(\mathbf{v}_1^e) \rangle)$ and $(\langle g(\mathbf{w}_1^o) \rangle, \langle g(\mathbf{w}_1^\infty) \rangle, \langle g(\mathbf{w}_1^e) \rangle)$ are rulers in $\mathcal{P}(V_2)$. Also let $\langle \mathbf{v}_1 \rangle \in \langle \mathbf{v}_1^o \rangle \star \langle \mathbf{v}_1^\infty \rangle$ and $\langle \mathbf{w}_1 \rangle \in \langle \mathbf{w}_1^o \rangle \star \langle \mathbf{w}_1^\infty \rangle$ be arbitrary such that

$$(\langle \mathbf{v}_1^o \rangle, \langle \mathbf{v}_1^\infty \rangle, \langle \mathbf{v}_1^e \rangle \mid \langle \mathbf{v}_1 \rangle) - (\langle \mathbf{w}_1^o \rangle, \langle \mathbf{w}_1^\infty \rangle, \langle \mathbf{w}_1^e \rangle \mid \langle \mathbf{w}_1 \rangle).$$

[7] In the last subsection, only one projective geometry or vector space is involved, so subscripts are used to index objects in the same structure. In this subsection, typically two projective geometries or vector spaces are involved. I use subscripts $_1$ and $_2$ to distinguish between objects from the two structures, and superscripts will be used to index objects in the same structure. Moreover, I do not use subscripts $_1$ and $_2$ to distinguish between the operations in the two division rings involved. This is for simplicity of the notations, and no ambiguity should arise given the context. This usage of superscripts and subscripts applies to the whole thesis.

Since $\langle \mathbf{v}_1^e \rangle \in \langle \mathbf{v}_1^o \rangle \star \langle \mathbf{v}_1^\infty \rangle$, without loss of generality, assume that $\langle \mathbf{v}_1^e \rangle = \langle \mathbf{v}_1^o + \mathbf{v}_1^\infty \rangle$. Since $\langle \mathbf{v}_1 \rangle \in \langle \mathbf{v}_1^o \rangle \star \langle \mathbf{v}_1^\infty \rangle$, there are $x, y \in K_1$ such that at least one of them is not 0 and $\langle \mathbf{v}_1 \rangle = \langle x\mathbf{v}_1^o + y\mathbf{v}_1^\infty \rangle$. Similarly, assume that $\langle \mathbf{w}_1^e \rangle = \langle \mathbf{w}_1^o + \mathbf{w}_1^\infty \rangle$, and there are $x', y' \in K_1$ such that at least one of them is not 0 and $\langle \mathbf{w}_1 \rangle = \langle x'\mathbf{w}_1^o + y'\mathbf{w}_1^\infty \rangle$. Since $(\langle \mathbf{v}_1^o \rangle, \langle \mathbf{v}_1^\infty \rangle, \langle \mathbf{v}_1^e \rangle \mid \langle \mathbf{v}_1 \rangle) - (\langle \mathbf{w}_1^o \rangle, \langle \mathbf{w}_1^\infty \rangle, \langle \mathbf{w}_1^e \rangle \mid \langle \mathbf{w}_1 \rangle)$, by Proposition 3.2.6 $y = y' = 0$ or $y^{-1} \cdot x = (y')^{-1} \cdot x'$. Hence by (i) either $\sigma(y) = \tau(y') = 0$, or

$$\begin{aligned}(\sigma(y))^{-1} \cdot \sigma(x) &= \sigma(y^{-1} \cdot x) \\ &= \tau(y^{-1} \cdot x) \\ &= \tau((y')^{-1} \cdot x') \\ &= (\tau(y'))^{-1} \cdot \tau(x')\end{aligned}$$

Then calculate

$$\begin{aligned}&(\langle f(\mathbf{v}_1^o) \rangle, \langle f(\mathbf{v}_1^\infty) \rangle, \langle f(\mathbf{v}_1^e) \rangle \mid \langle f(\mathbf{v}_1) \rangle) \\ =&(\langle f(\mathbf{v}_1^o) \rangle, \langle f(\mathbf{v}_1^\infty) \rangle, \langle f(\mathbf{v}_1^o + \mathbf{v}_1^\infty) \rangle \mid \langle f(x\mathbf{v}_1^o + y\mathbf{v}_1^\infty) \rangle) \\ =&(\langle f(\mathbf{v}_1^o) \rangle, \langle f(\mathbf{v}_1^\infty) \rangle, \langle f(\mathbf{v}_1^o) + f(\mathbf{v}_1^\infty) \rangle \mid \langle \sigma(x) f(\mathbf{v}_1^o) + \sigma(y) f(\mathbf{v}_1^\infty) \rangle)\end{aligned}$$

$$\begin{aligned}&(\langle g(\mathbf{w}_1^o) \rangle, \langle g(\mathbf{w}_1^\infty) \rangle, \langle g(\mathbf{w}_1^e) \rangle \mid \langle g(\mathbf{w}_1) \rangle) \\ =&(\langle g(\mathbf{w}_1^o) \rangle, \langle g(\mathbf{w}_1^\infty) \rangle, \langle g(\mathbf{w}_1^o + \mathbf{w}_1^\infty) \rangle \mid \langle g(x'\mathbf{w}_1^o + y'\mathbf{w}_1^\infty) \rangle) \\ =&(\langle g(\mathbf{w}_1^o) \rangle, \langle g(\mathbf{w}_1^\infty) \rangle, \langle g(\mathbf{w}_1^o) + g(\mathbf{w}_1^\infty) \rangle \mid \langle \tau(x') g(\mathbf{w}_1^o) + \tau(y') g(\mathbf{w}_1^\infty) \rangle)\end{aligned}$$

Therefore, by Proposition 3.2.6

$$(\langle f(\mathbf{v}_1^o) \rangle, \langle f(\mathbf{v}_1^\infty) \rangle, \langle f(\mathbf{v}_1^e) \rangle \mid \langle f(\mathbf{v}_1) \rangle) - (\langle g(\mathbf{w}_1^o) \rangle, \langle g(\mathbf{w}_1^\infty) \rangle, \langle g(\mathbf{w}_1^e) \rangle \mid \langle g(\mathbf{w}_1) \rangle).$$

From (ii) to (i): Since both images of f and g are at least two-dimensional, there are $\mathbf{v}_1^o, \mathbf{v}_1^\infty, \mathbf{w}_1^o, \mathbf{w}_1^\infty \in V_1 \setminus \{\mathbf{0}_1\}$ satisfying that both $(\langle \mathbf{v}_1^o \rangle, \langle \mathbf{v}_1^\infty \rangle, \langle \mathbf{v}_1^e \rangle)$ and $(\langle \mathbf{w}_1^o \rangle, \langle \mathbf{w}_1^\infty \rangle, \langle \mathbf{w}_1^e \rangle)$ are rulers in $\mathcal{P}(V_1)$, where $\mathbf{v}_1^e = \mathbf{v}_1^o + \mathbf{v}_1^\infty$ and $\mathbf{w}_1^e = \mathbf{w}_1^o + \mathbf{w}_1^\infty$, and both $(\langle f(\mathbf{v}_1^o) \rangle, \langle f(\mathbf{v}_1^\infty) \rangle, \langle f(\mathbf{v}_1^e) \rangle)$ and $(\langle g(\mathbf{w}_1^o) \rangle, \langle g(\mathbf{w}_1^\infty) \rangle, \langle g(\mathbf{w}_1^e) \rangle)$ are rulers in $\mathcal{P}(V_2)$.

Now let $x \in K_1$ be arbitrary. According to Proposition 3.2.6,

$$(\langle \mathbf{v}_1^o \rangle, \langle \mathbf{v}_1^\infty \rangle, \langle \mathbf{v}_1^e \rangle \mid \langle x\mathbf{v}_1^o + \mathbf{v}_1^\infty \rangle) - (\langle \mathbf{w}_1^o \rangle, \langle \mathbf{w}_1^\infty \rangle, \langle \mathbf{w}_1^e \rangle \mid \langle x\mathbf{w}_1^o + \mathbf{w}_1^\infty \rangle)$$

By (ii)

$$\begin{aligned}&(\langle f(\mathbf{v}_1^o) \rangle, \langle f(\mathbf{v}_1^\infty) \rangle, \langle f(\mathbf{v}_1^e) \rangle \mid \langle f(x\mathbf{v}_1^o + \mathbf{v}_1^\infty) \rangle) \\ -\ &(\langle g(\mathbf{w}_1^o) \rangle, \langle g(\mathbf{w}_1^\infty) \rangle, \langle g(\mathbf{w}_1^e) \rangle \mid \langle g(x\mathbf{w}_1^o + \mathbf{w}_1^\infty) \rangle)\end{aligned}$$

Since both f and g are quasi-linear,

$$(\langle f(\mathbf{v}_1^o) \rangle, \langle f(\mathbf{v}_1^\infty) \rangle, \langle f(\mathbf{v}_1^e) \rangle \mid \langle \sigma(x) f(\mathbf{v}_1^o) + f(\mathbf{v}_1^\infty) \rangle)$$

$$-\ (\langle g(\mathbf{w}_1^o)\rangle\,, \langle g(\mathbf{w}_1^\infty)\rangle\,, \langle g(\mathbf{w}_1^e)\rangle \mid \langle \tau(x)g(\mathbf{w}_1^o) + g(\mathbf{w}_1^\infty)\rangle)$$

According to Proposition 3.2.6, $\sigma(x) = \tau(x)$. Since x is arbitrary, $\sigma = \tau$. ⊣

Next I deal with the degenerate case.

3.2.9. Lemma. *Let V_1 and V_2 be two vector spaces over two division rings $\mathcal{K}_1 = (K_1, +, \cdot, 0, 1)$ and $\mathcal{K}_2 = (K_2, +, \cdot, 0, 1)$, $g : V_1 \to V_2$ a quasi-linear map whose image is at most one-dimensional, and $\sigma : \mathcal{K}_1 \to \mathcal{K}_2$ a division ring isomorphism. Then there is a quasi-linear map $f : V_1 \to V_2$ such that σ is an accompanying isomorphism of f and $\mathcal{P}(f) = \mathcal{P}(g)$.*

Proof. Let τ be an accompanying division ring isomorphism of g. Two cases need to be considered.

Case 1: $g[V_1] = \{\mathbf{0}_2\}$. In this case g is the zero map, and it is not hard to verify that g is a quasi-linear map with σ as an accompanying division ring isomorphism, and trivially $\mathcal{P}(g) = \mathcal{P}(g)$.

Case 2: $g[V_1]$ is one-dimensional. By Lemma B.2.19 there is a Hamel basis $\{\mathbf{v}_1^i \mid I \cup \{e\}\}$ of V_1 such that $e \notin I$ and $g(\mathbf{v}_1^i) \neq \mathbf{0}_2$ if and only if $i = e$, for every $i \in I \cup \{e\}$. For every vector $\mathbf{v}_1 \in V_1$, if $\mathbf{v}_1 = \sum_{i\in J} x_i \mathbf{v}_1^i$ under the basis $\{\mathbf{v}_1^i \mid i \in I \cup \{e\}\}$, define $f(\mathbf{v}_1)$ to be $\sum_{i\in J} \sigma(x_i)g(\mathbf{v}_1^i)$. Then f is a function from V_1 to V_2. I claim that f has the required properties.

First show that f is additive. For any two vectors $\mathbf{s}_1, \mathbf{t}_1 \in V$, suppose that under the basis $\mathbf{s}_1 = \sum_{i\in J_1} x_i \mathbf{v}_1^i$ and $\mathbf{t}_1 = \sum_{i\in J_2} y_i \mathbf{v}_1^i$. Then

$$\begin{aligned}
f(\mathbf{s}_1 + \mathbf{t}_1) &= f(\sum_{i\in J_1} x_i\mathbf{v}_1^i + \sum_{i\in J_2} y_i\mathbf{v}_1^i) \\
&= f(\sum_{i\in J_1\cap J_2} (x_i + y_i)\mathbf{v}_1^i + \sum_{i\in J_1\setminus(J_1\cap J_2)} x_i\mathbf{v}_1^i + \sum_{i\in J_2\setminus(J_1\cap J_2)} y_i\mathbf{v}_1^i) \\
&= \sum_{i\in J_1\cap J_2} \sigma(x_i + y_i)g(\mathbf{v}_1^i) + \sum_{i\in J_1\setminus(J_1\cap J_2)} \sigma(x_i)g(\mathbf{v}_1^i) + \sum_{i\in J_2\setminus(J_1\cap J_2)} \sigma(y_i)g(\mathbf{v}_1^i) \\
&= \sum_{i\in J_1} \sigma(x_i)g(\mathbf{v}_1^i) + \sum_{i\in J_2} \sigma(y_i)g(\mathbf{v}_1^i) \\
&= f(\mathbf{s}_1) + f(\mathbf{t}_1)
\end{aligned}$$

Second show that $f(x\mathbf{u}_1) = \sigma(x)f(\mathbf{u}_1)$, for any $x \in K_1$ and $\mathbf{u}_1 \in V_1$. Suppose that under the basis $\mathbf{u}_1 = \sum_{i\in J} x_i\mathbf{v}_1^i$. Then

$$\begin{aligned}
f(x\mathbf{u}_1) &= f(x\sum_{i\in J} x_i\mathbf{v}_1^i) \\
&= f(\sum_{i\in J} (x \cdot x_i)\mathbf{v}_1^i)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i\in J} \sigma(x \cdot x_i) g(\mathbf{v}_1^i) \\
&= \sigma(x) \sum_{i\in J} \sigma(x_i) g(\mathbf{v}_1^i) \\
&= \sigma(x) f(\sum_{i\in J} x_i \mathbf{v}_1^i) \\
&= \sigma(x) f(\mathbf{u}_1)
\end{aligned}
$$

It can now be concluded that f is a quasi-linear map from V_1 to V_2 with σ as an accompanying division ring isomorphism. It remains to show that $\mathcal{P}(f) = \mathcal{P}(g)$. Let $\mathbf{u}_1 \in V_1 \setminus \{\mathbf{0}_1\}$ be arbitrary. Suppose that $\mathbf{u}_1 = \sum_{i\in J} x_i \mathbf{v}_1^i$ under the basis. When $e \notin J$,

$$
f(\mathbf{u}_1) = \sum_{i\in J} \sigma(x_i) g(\mathbf{v}_1^i) = \mathbf{0}_2, \qquad g(\mathbf{u}_1) = \sum_{i\in J} \tau(x_i) g(\mathbf{v}_1^i) = \mathbf{0}_2
$$

When $e \in J$,

$$
\begin{aligned}
f(\mathbf{u}_1) &= \sum_{i\in J} \sigma(x_i) g(\mathbf{v}_1^i) = \sigma(x_e) g(\mathbf{v}_1^e) \neq \mathbf{0}_2 \\
g(\mathbf{u}_1) &= \sum_{i\in J} \tau(x_i) g(\mathbf{v}_1^i) = \tau(x_e) g(\mathbf{v}_1^e) \neq \mathbf{0}_2,
\end{aligned}
$$

It follows that $\mathcal{P}(f) = \mathcal{P}(g)$. ⊣

Now I return to the context of projective geometry. I introduce a binary relation on the set of homomorphisms from a projective geometry $\mathcal{G}_1$ to a projective geometry $\mathcal{G}_2$.

3.2.10. Definition. Two homomorphisms F and G from a projective geometry $\mathcal{G}_1 = (G_1, \star_1)$ to a projective geometry $\mathcal{G}_2 = (G_2, \star_2)$ are *comrades*, denoted as $F \frown G$, if one of the following holds:

- the image of either F or G is empty or a singleton;
- for any two rulers $(s_1^o, s_1^\infty, s_1^e)$ and $(t_1^o, t_1^\infty, t_1^e)$ in $\mathcal{G}_1$ satisfying that both $(F(s_1^o), F(s_1^\infty), F(s_1^e))$ and $(G(t_1^o), G(t_1^\infty), G(t_1^e))$ are rulers in $\mathcal{G}_2$, and any $s_1 \in s_1^o \star_1 s_1^\infty$ and $t_1 \in t_1^o \star_1 t_1^\infty$, $(s_1^o, s_1^\infty, s_1^e \mid s_1) - (t_1^o, t_1^\infty, t_1^e \mid t_1)$ implies that $(F(s_1^o), F(s_1^\infty), F(s_1^e) \mid F(s_1)) - (G(t_1^o), G(t_1^\infty), G(t_1^e) \mid G(t_1))$.

The following proposition is a characterization of this relation in the language of linear algebra.

3.2.11. Proposition. *Let V_1 and V_2 be two vector spaces over two* fields $\mathcal{K}_1 = (K_1, +, \cdot, 0, 1)$ *and* $\mathcal{K}_2 = (K_2, +, \cdot, 0, 1)$ *such that both are of dimension at least* 3, *F and G two arguesian homomorphisms from $\mathcal{P}(V_1)$ to $\mathcal{P}(V_2)$. The following are equivalent:*

(i) $F \Bumpeq G$*;*

(ii) there are a field isomorphism $\theta : \mathcal{K}_1 \to \mathcal{K}_2$ *and two quasi-linear maps* $f, g : V_1 \to V_2$ *such that* $F = \mathcal{P}(f)$*,* $G = \mathcal{P}(g)$ *and* θ *is an accompanying isomorphism of both* f *and* g*.*

Proof. Since both F and G are arguesian homomorphisms, by Theorem B.3.13 there are two field isomorphisms $\sigma, \tau : \mathcal{K}_1 \to \mathcal{K}_2$ and two quasi-linear maps $f, g : V_1 \to V_2$ such that $F = \mathcal{P}(f)$, $G = \mathcal{P}(g)$ and σ and τ are accompanying isomorphisms of f and g, respectively.

From (i) to (ii): Consider two cases.

Case 1: The image of one of F and G is empty or a singleton. Without loss of generality, assume that the image of G is empty or a singleton. Then the image of g is at most one-dimensional. By Lemma 3.2.9 there is a quasi-linear map $g' : V_1 \to V_2$ such that σ is an accompanying isomorphism of g' and $\mathcal{P}(g') = \mathcal{P}(g) = G$. Therefore, let θ be σ.

Case 2: None of the images of F and G is empty or a singleton. Then both of the images of f and g are at least 2-dimensional. By assumption, for any two rulers $(\langle \mathbf{s}_1^o \rangle, \langle \mathbf{s}_1^\infty \rangle, \langle \mathbf{s}_1^e \rangle)$ and $(\langle \mathbf{t}_1^o \rangle, \langle \mathbf{t}_1^\infty \rangle, \langle \mathbf{t}_1^e \rangle)$ in $\mathcal{P}(V_1)$ such that both $(F(\langle \mathbf{s}_1^o \rangle), F(\langle \mathbf{s}_1^\infty \rangle), F(\langle \mathbf{s}_1^e \rangle))$ and $(G(\langle \mathbf{t}_1^o \rangle), G(\langle \mathbf{t}_1^\infty \rangle), G(\langle \mathbf{t}_1^e \rangle))$ are rulers in $\mathcal{P}(V_2)$, $\langle \mathbf{s}_1 \rangle \in \langle \mathbf{s}_1^o \rangle \star_1 \langle \mathbf{s}_1^\infty \rangle$ and $\langle \mathbf{t}_1 \rangle \in \langle \mathbf{t}_1^o \rangle \star_1 \langle \mathbf{t}_1^\infty \rangle$,

$$(\langle \mathbf{s}_1^o \rangle, \langle \mathbf{s}_1^\infty \rangle, \langle \mathbf{s}_1^e \rangle \mid \langle \mathbf{s}_1 \rangle) - (\langle \mathbf{t}_1^o \rangle, \langle \mathbf{t}_1^\infty \rangle, \langle \mathbf{t}_1^e \rangle \mid \langle \mathbf{t}_1 \rangle)$$

implies that

$$(F(\langle \mathbf{s}_1^o \rangle), F(\langle \mathbf{s}_1^\infty \rangle), F(\langle \mathbf{s}_1^e \rangle) \mid F(\langle \mathbf{s}_1 \rangle)) - (G(\langle \mathbf{t}_1^o \rangle), G(\langle \mathbf{t}_1^\infty \rangle), G(\langle \mathbf{t}_1^e \rangle) \mid G(\langle \mathbf{t}_1 \rangle)),$$

and thus

$$(\langle f(\mathbf{s}_1^o) \rangle, \langle f(\mathbf{s}_1^\infty) \rangle, \langle f(\mathbf{s}_1^e) \rangle \mid \langle f(\mathbf{s}_1) \rangle) - (\langle g(\mathbf{t}_1^o) \rangle, \langle g(\mathbf{t}_1^\infty) \rangle, \langle g(\mathbf{t}_1^e) \rangle \mid \langle g(\mathbf{t}_1) \rangle).$$

It is not hard to deduce from Lemma 3.2.8 that $\sigma = \tau$. Therefore, let θ be σ.

From (ii) to (i): By (ii) I can assume that $\theta = \sigma = \tau$. Consider two cases.

Case 1: The image of one of F and G is empty or a singleton. By definition $F \Bumpeq G$.

Case 2: Neither of the images of F and G is empty or a singleton. Then both of the images of f and g are at least 2-dimensional. By (ii) and Lemma 3.2.8, for any two rulers $(\langle \mathbf{s}_1^o \rangle, \langle \mathbf{s}_1^\infty \rangle, \langle \mathbf{s}_1^e \rangle)$ and $(\langle \mathbf{t}_1^o \rangle, \langle \mathbf{t}_1^\infty \rangle, \langle \mathbf{t}_1^e \rangle)$ in $\mathcal{P}(V_1)$ such that both $(F(\langle \mathbf{s}_1^o \rangle), F(\langle \mathbf{s}_1^\infty \rangle), F(\langle \mathbf{s}_1^e \rangle))$ and $(G(\langle \mathbf{t}_1^o \rangle), G(\langle \mathbf{t}_1^\infty \rangle), G(\langle \mathbf{t}_1^e \rangle))$ are rulers in $\mathcal{P}(V_2)$, $\langle \mathbf{s}_1 \rangle \in \langle \mathbf{s}_1^o \rangle \star_1 \langle \mathbf{s}_1^\infty \rangle$ and $\langle \mathbf{t}_1 \rangle \in \langle \mathbf{t}_1^o \rangle \star_1 \langle \mathbf{t}_1^\infty \rangle$,

$$(\langle \mathbf{s}_1^o \rangle, \langle \mathbf{s}_1^\infty \rangle, \langle \mathbf{s}_1^e \rangle \mid \langle \mathbf{s}_1 \rangle) - (\langle \mathbf{t}_1^o \rangle, \langle \mathbf{t}_1^\infty \rangle, \langle \mathbf{t}_1^e \rangle \mid \langle \mathbf{t}_1 \rangle)$$

implies that

$$(\langle f(\mathbf{s}_1^o) \rangle, \langle f(\mathbf{s}_1^\infty) \rangle, \langle f(\mathbf{s}_1^e) \rangle \mid \langle f(\mathbf{s}_1) \rangle) - (\langle g(\mathbf{t}_1^o) \rangle, \langle g(\mathbf{t}_1^\infty) \rangle, \langle g(\mathbf{t}_1^e) \rangle \mid \langle g(\mathbf{t}_1) \rangle),$$

and thus

$$(F(\langle \mathbf{s}_1^o \rangle), F(\langle \mathbf{s}_1^\infty \rangle), F(\langle \mathbf{s}_1^e \rangle) \mid F(\langle \mathbf{s}_1 \rangle)) - (G(\langle \mathbf{t}_1^o \rangle), G(\langle \mathbf{t}_1^\infty \rangle), G(\langle \mathbf{t}_1^e \rangle) \mid G(\langle \mathbf{t}_1 \rangle)).$$

Hence by the definition of comrades $F \Bumpeq G$. ⊣

3.2.3 Parties of Homomorphisms

In this subsection, I study the set $\Sigma(V_1) \otimes_G \Sigma(V_2)$ defined as follows:

$$\{F \mid F \text{ is a non-empty arguesian homomorphism from } \mathcal{P}(V_1) \text{ to } \mathcal{P}(V_2), F \Bumpeq G\},$$

where V_1 and V_2 are two vector spaces over two fields $\mathcal{K}_1$ and $\mathcal{K}_2$ such that both are of dimension at least 3, G is a non-degenerate homomorphism from $\mathcal{P}(V_1)$ to $\mathcal{P}(V_2)$. Since $\Sigma(V_1) \otimes_G \Sigma(V_2)$ contains the comrades of G except the empty function, it can be called a *party*.

I start with a useful technical lemma.

3.2.12. Lemma. *Let V_1 and V_2 be two vector spaces over two division rings $\mathcal{K}_1 = (K_1, +, \cdot, 0, 1)$ and $\mathcal{K}_2 = (K_2, +, \cdot, 0, 1)$, $f, g : V_1 \to V_2$ two quasi-linear maps neither of which is the zero map, and σ and τ be accompanying division ring isomorphisms of f and g, respectively. If there is an $a \in K_2 \setminus \{0\}$ such that $g = af$, then $\tau(x) = a \cdot \sigma(x) \cdot a^{-1}$ for every $x \in K_1$; in particular, when $\mathcal{K}_2$ is a field, $\tau = \sigma$.*

Proof. Let $x \in K_1$ be arbitrary. If $x = 0$ or $x = 1$, $\tau(x) = a \cdot \sigma(x) \cdot a^{-1}$ by definition. In the following, I focus on the case when $x \notin \{0, 1\}$. Since f is not the zero map, there is a $\mathbf{v}_1 \in V_1$ such that $f(\mathbf{v}_1) \neq \mathbf{0}_2$. Since $g = af$

$$g(x\mathbf{v}_1) = af(x\mathbf{v}_1) = (a \cdot \sigma(x))f(\mathbf{v}_1)$$
$$g(x\mathbf{v}_1) = \tau(x)g(\mathbf{v}_1) = (\tau(x) \cdot a)f(\mathbf{v}_1)$$

Hence $(\tau(x) \cdot a)f(\mathbf{v}_1) = (a \cdot \sigma(x))f(\mathbf{v}_1)$. Since $f(\mathbf{v}_1) \neq \mathbf{0}_2$, it follows that $\tau(x) \cdot a = a \cdot \sigma(x)$, and thus $\tau(x) = a \cdot \sigma(x) \cdot a^{-1}$. ⊣

Next, I prove an important result.

3.2.13. Theorem. *Let V_1 and V_2 be two vector spaces over two* fields *$\mathcal{K}_1 = (K_1, +, \cdot, 0, 1)$ and $\mathcal{K}_2 = (K_2, +, \cdot, 0, 1)$ such that both are of dimension at least 3, $g : V_1 \to V_2$ a quasi-linear map with σ as an accompanying field isomorphism such that $g[V_1]$ is at least two-dimensional. For every arguesian homomorphism F from $\mathcal{P}(V_1)$ to $\mathcal{P}(V_2)$, the following are equivalent:*

(i) $F \Bumpeq \mathcal{P}(g)$;

(ii) there is a quasi-linear map $f : V_1 \to V_2$ such that $F = \mathcal{P}(f)$ and σ is an accompanying isomorphism of f.

Moreover, such a quasi-linear map f is unique up to a constant multiple, if it exists.

Proof. **From (i) to (ii):** By Proposition 3.2.11 there are a field isomorphism τ and two quasi-linear maps $f, g' : V_1 \to V_2$ such that $\mathcal{P}(f) = F$, $\mathcal{P}(g') = \mathcal{P}(g)$ and τ is an accompanying isomorphism of both f and g'. Since $\mathcal{P}(g) = \mathcal{P}(g')$ and $g[V_1]$ is at least two-dimensional, by Proposition B.2.18 there is a unique $a \in K_2$ such that $g' = ag$. Obviously g' is not the zero map, so $a \neq 0$. By the previous lemma $\tau = \sigma$. Therefore, f is a quasi-linear map such that σ is an accompanying isomorphism and $\mathcal{P}(f) = F$.

From (ii) to (i): By Proposition 3.2.11 $F = \mathcal{P}(f) \Bumpeq \mathcal{P}(g)$.

Uniqueness: Assume that $f, f' : V_1 \to V_2$ are two quasi-linear maps such that $\mathcal{P}(f) = \mathcal{P}(f')$ and σ is an accompanying isomorphism of both f and f'. Consider three cases.

Case 1: Both f and f' are the zero map. Then trivially $f = f'$.

Case 2: $f[V_1]$ is one-dimensional. Then there is a $\mathbf{v}_2 \in V_2 \setminus \{\mathbf{0}_2\}$ such that $f[V_1] = \langle \mathbf{v}_2 \rangle$. Moreover, it follows easily from $\mathcal{P}(f) = \mathcal{P}(f')$ that $f'[V_1] = \langle \mathbf{v}_2 \rangle$ and $f(\mathbf{v}_1) = \mathbf{0}_2 \Leftrightarrow f'(\mathbf{v}_1) = \mathbf{0}_2$ for every $\mathbf{v}_1 \in V_1$, i.e. the null spaces of f and f' are the same. By Lemma B.2.19 there is a Hamel basis $\{\mathbf{u}_1^i \in V_1 \mid i \in I \cup \{e\}\}$ of V_1 such that $e \notin I$ and $\{\mathbf{u}_1^i \in V_1 \mid i \in I\}$ is a Hamel basis of the null space of f and f'. Then there are $a, a' \in K_2 \setminus \{0\}$ such that $f(\mathbf{u}_1^e) = a\mathbf{v}_2$ and $f'(\mathbf{u}_1^e) = a'\mathbf{v}_2$. Now let $\mathbf{v}_1 \in V_1$ be arbitrary. Suppose that $\mathbf{v}_1 = \sum_{i \in J} x_i \mathbf{u}_1^i$ under this basis. If $e \notin J$, $f(\mathbf{v}_1) = \mathbf{0}_2 = (a \cdot (a')^{-1})\mathbf{0}_2 = (a \cdot (a')^{-1}) f'(\mathbf{v}_1)$. If $e \in J$,

$$f(\mathbf{v}_1) = f(\sum_{i \in J} x_i \mathbf{u}_1^i) = \sum_{i \in J} \sigma(x_i) f(\mathbf{u}_1^i) = (\sigma(x_e) \cdot a)\mathbf{v}_2$$
$$f'(\mathbf{v}_1) = f'(\sum_{i \in J} x_i \mathbf{u}_1^i) = \sum_{i \in J} \sigma(x_i) f'(\mathbf{u}_1^i) = (\sigma(x_e) \cdot a')\mathbf{v}_2$$

Since $\mathcal{K}_2$ is a field, $f(\mathbf{v}_1) = (a \cdot (a')^{-1}) f'(\mathbf{v}_1)$. Therefore, $f = (a \cdot (a')^{-1}) f'$.

Case 3: $f[V_1]$ is at least two-dimensional. Then it follows easily from $\mathcal{P}(f) = \mathcal{P}(f')$ that $f'[V_1]$ is also at least two-dimensional. Since $\mathcal{P}(f) = \mathcal{P}(f')$, by Proposition B.2.18 there is a unique $a \in K_2$ such that $f = af'$. ⊣

This theorem characterizes the arguesian homomorphisms induced by the quasi-linear maps with the same accompanying field isomorphism, assuming that one of these arguesian homomorphisms with an image of rank at least 2 is given. I show in the following that the corollaries of this theorem are *partial* solutions to Problem 4 in the List of Problems in [39] (p.345), i.e. to 'characterize geometrically arguesian (endo)morphisms that are induced by a linear map'. These solutions are partial because they are only about the special case when the arguesian (endo)morphisms are from one Pappian projective geometry to another. In the general case, the arguesian (endo)morphisms should be from one *arguesian* projective geometry to another. This is much harder, if there can be a solution (please refer to Remark 3.2.7). Moreover, as far as I know, there is no solution even for the special case in the literature. In [39] the authors pointed to 5° of Exercise 9.7.6 as a complication of this problem, which involves the Pappian projective geometry induced by $\mathbb{R}^{(\mathbb{N})}$.

I start with the case when only one vector space or Pappian projective geometry is involved.

3.2.14. Corollary. *Let V be a vector space over a* field $\mathcal{K} = (K, +, \cdot, 0, 1)$ *of dimension at least* 3, *and F an arguesian homomorphism from $\mathcal{P}(V)$ to itself. The following are equivalent:*

(i) $F \doteqdot Id$, *i.e.* $F[\Sigma(V_1)]$ *is empty or a singleton, or, for any ruler* $(r_1^o, r_1^\infty, r_1^e)$ *in* $\Sigma(V_1)$ *such that* $(F(r_1^o), F(r_1^\infty), F(r_1^e))$ *is also a ruler, and any* $s_1 \in r_1^o \star r_1^\infty$,

$$(F(r_1^o), F(r_1^\infty), F(r_1^e) \mid F(s_1)) - (r_1^o, r_1^\infty, r_1^e \mid s_1);$$

(ii) there is a linear map $f : V \to V$ *such that* $F = \mathcal{P}(f)$.

Proof. Consider the identity map id_V on V, which is a linear map satisfying $\mathcal{P}(id_V) = Id$. Then the result follows directly from Theorem 3.2.13. ⊣

I proceed to the case when two (possibly different) vector spaces or Pappian projective geometries are involved. Unlike the previous case, there is no natural or canonical linear map in this case. In the good cases when a linear map is given, I can conclude the following:

3.2.15. Corollary. *Let V_1 and V_2 be two vector spaces over two* fields $\mathcal{K}_1 = (K_1, +, \cdot, 0, 1)$ *and* $\mathcal{K}_2 = (K_2, +, \cdot, 0, 1)$ *such that both are of dimension at least* 3, $g : V_1 \to V_2$ *a linear map such that* $g[V_1]$ *is at least* 2*-dimensional. For every arguesian homomorphism F from $\mathcal{P}(V_1)$ to $\mathcal{P}(V_2)$, the following are equivalent:*

(i) $F \doteqdot \mathcal{P}(g)$;

(ii) there is a linear map $f : V_1 \to V_2$ *such that* $F = \mathcal{P}(f)$.

Proof. This is a special case of Theorem 3.2.13. ⊣

In the bad cases when no linear map is given, I argue that Theorem 3.2.13 is already a solution to this special case of Problem 4 in [39]. In these cases, the subtlety is the following: When it is known only that there is a quasi-linear map between two vector spaces or a non-degenerate homomorphism between two

Pappian projective geometries, the only conclusion one can draw is that the fields underlying these two vector spaces are isomorphic; and the identity of the fields cannot be concluded from any geometric or algebraic assumptions, unless it is directly given. In other words, whether a quasi-linear map is a linear map does not depend on the properties of the map and the vector spaces; instead, it depends on how the fields are represented or interpreted.

Let me illustrate this with a concrete example. Assume that it is given that there is a quasi-linear map g, other than the zero map, from the vector space $\mathbb{C}^4$ to a 5-dimensional vector space V over a field $\mathcal{K} = (K, +_K, \cdot_K, 0_K, 1_K)$, where $K = \mathbb{R}^2$, and, for any $(a, b), (c, d) \in \mathbb{R}^2$,

$$\begin{aligned} (a, b) +_K (c, d) &\stackrel{\text{def}}{=} (a + c, b + d) \\ (a, b) \cdot_K (c, d) &\stackrel{\text{def}}{=} (ac - bd, ad + bc) \\ 0_K &\stackrel{\text{def}}{=} (0, 0) \\ 1_K &\stackrel{\text{def}}{=} (1, 0) \end{aligned}$$

Then by the definition of quasi-linear maps there is a field isomorphism $\sigma : \mathbb{C} \cong \mathcal{K}$ being an accompanying isomorphism of g. For convenience, I further assume that $\sigma(a + bi) = (a, b)$, for every $a + bi \in \mathbb{C}$. It is easy to see that such a σ is indeed a field isomorphism, and thus this assumption is acceptable. However, σ cannot be the identity map because $\mathbb{R}^2$ and $\mathbb{C}$ are different. Now I make some stipulations about the field $\mathcal{K}$. First, I stipulate that $K = \mathbb{R}^2$ just means a way of abbreviating the notations of complex numbers, and $(a, b) \in \mathbb{R}^2$ abbreviates $a + bi$. In this case, σ is the identity map, and thus g is a linear map. Second, instead, I stipulate that $K = \mathbb{R}^2$ means another way of abbreviating the notations of complex numbers, and $(a, b) \in \mathbb{R}^2$ abbreviates $a + b(-i)$. In this case, σ is an isomorphism different from the identity map. By Remark B.2.17 the identity map cannot be an accompanying isomorphism of g, and thus g is not a linear map.

In this discussion, g can be stipulated to be a linear map and also not to be a linear map. These stipulations have nothing to do with g and the two vector spaces, but are only about the set-theoretic nature of the fields. Therefore, when two (possibly different) vector spaces or Pappian projective geometries are involved, the best one can do is to characterize geometrically the arguesian homomorphisms that are induced by, instead of the linear maps, the *quasi-linear* maps which can have the same accompanying field isomorphism. Then one can assume without loss of generality that this field isomorphism is the identity. As a result, in my opinion, Theorem 3.2.13 is already a solution to this special case of Problem 4 in [39].

Finally, the uniqueness part of Theorem 3.2.13 actually yields a stronger result. Remember that, for two vector spaces V_1 and V_2 over a field $\mathcal{K}$, by Theorem C.3.1 $Hom(V_1, V_2)$ is a vector space over $\mathcal{K}$, and thus $\Sigma(Hom(V_1, V_2))$ forms a projective geometry by Theorem B.3.1. Based on Theorem 3.2.13, a geometric characterization of the points in this projective geometry can be obtained.

3.2.16. Corollary. *Let V_1 and V_2 be vector spaces over a* field $\mathcal{K} = (K, +, \cdot, 0, 1)$ *such that both are of dimension at least* 3*. For every homomorphism G from $\mathcal{P}(V_1)$ to $\mathcal{P}(V_2)$ induced by a linear map and with a range of rank at least* 2*, there is a bijection $\mathbf{I}_G$ from $\Sigma(V_1) \otimes_G \Sigma(V_2)$ to $\Sigma(Hom(V_1, V_2))$, where $\Sigma(V_1) \otimes_G \Sigma(V_2)$ is*

$$\{F \mid F \text{ is a non-empty arguesian homomorphism from } \mathcal{P}(V_1) \text{ to } \mathcal{P}(V_2), F \not\perp G\}$$

Proof. Define that

$$\mathbf{I}_G = \{(F, \langle f \rangle) \mid F \in \Sigma(V_1) \otimes_G \Sigma(V_2), f \in Hom(V_1, V_2), F = \mathcal{P}(f)\}$$

$\mathbf{I}_G$ is a partial function by the uniqueness part of Theorem 3.2.13. It is defined on every element in $\Sigma(V_1) \otimes_G \Sigma(V_2)$ by the direction from (i) to (ii) in Theorem 3.2.13. Moreover, it is surjective by the direction from (ii) to (i) in Theorem 3.2.13. Finally, it is injective: for any $(F, \langle f \rangle), (F', \langle f \rangle) \in \mathbf{H}_G$, $F = \mathcal{P}(f) = F'$. As a result, $\mathbf{I}_G$ is a bijection from $\Sigma(V_1) \otimes_G \Sigma(V_2)$ to $\Sigma(Hom(V_1, V_2))$. ⊣

3.2.17. Remark. Note that, when a quasi-linear map g from a vector space V_1 over a field to another vector space is given such that $g[V_1]$ is at least two-dimensional, one can just stipulate that the accompanying field isomorphism of g is the identity map, then this corollary can be applied.

3.3 Orthogonal Continuous Homomorphisms

In this section, I return to the setting of quasi-quantum Kripke frames. I define a non-orthogonality relation on a party of continuous homomorphisms between two quasi-quantum Kripke frames under three assumptions. I also show that, equipped with this relation, the party forms a quasi-quantum Kripke frame. This construction turns out to be the geometric counterpart of the tensor product construction on vector spaces.

3.3.1 Uniformly Scaled Rulers

In this subsection, I introduce the notion of uniformly scaled rulers and study its properties, which will be of great use later.

First of all, please note that the notion of rulers and its properties discussed in Subsection 3.2.1 in the more general setting of projective geometries apply to the setting of quasi-quantum Kripke frames.

Now I define uniformly scaled rulers.

3.3.1. DEFINITION. Let $\mathfrak{F} = (\Sigma, \to)$ be a quasi-quantum Kripke frame, F a continuous endo-homomorphism on $\mathfrak{F}$, $w \in \Sigma$ and $n \in \mathbb{N}^+$. *A set of rulers uniformly scaled by F and w* is a set of tuple $\{(F(s_i), s_i, r_i^e) \mid i = 1, ..., n\}$ satisfying all of the following:

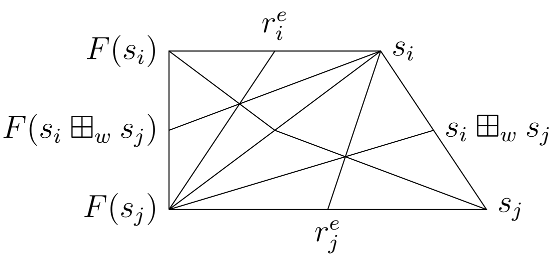
1. $\{s_1, ..., s_n\} \subseteq \Sigma$ is an orthogonal set, $w \in {\sim}{\sim}\{s_1, ..., s_n\}$, and $w \to s_i$ for $i = 1, ..., n$;

2. for $i = 1, ..., n$, $(F(s_i), s_i, r_i^e)$ is a ruler, i.e. $F(s_i), s_i, r_i^e$ are distinct and $r_i^e \in F(s_i) \star s_i$;

3. for any $i, j \in \{1, ..., n\}$ such that $i \neq j$, $s_i \notin F(s_j) \star s_j$ and $F(s_j) \notin s_i \star F(s_i)$, the intersection of the following three sets is a singleton:

   - $s_i \star F(s_j)$,
   - $F(s_i) \star \big((F(s_i \boxplus_w s_j) \star s_i) \cap (r_i^e \star F(s_j))\big)$,
   - $s_j \star \big((F(s_j) \star (s_i \boxplus_w s_j)) \cap (r_j^e \star s_i)\big)$;

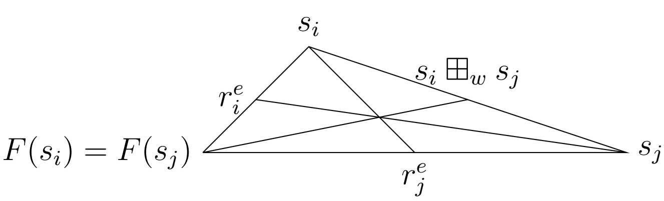
   where $s_i \boxplus_w s_j$ is the approximation of w in $s_i \star s_j$. (The following picture of the analogue in an affine plane may help to understand this condition:)



4. for any $i, j \in \{1, ..., n\}$ such that $i \neq j$, $s_i \notin F(s_j) \star s_j$ and $F(s_i) = F(s_j)$, the intersection of the following three sets is a singleton:

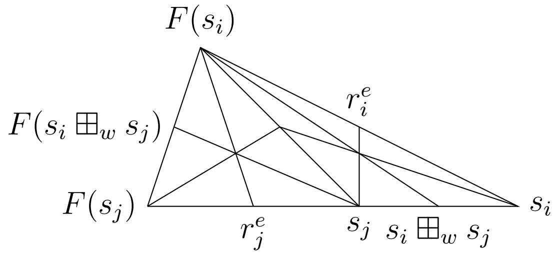
   - $F(s_i) \star (s_i \boxplus_w s_j)$,
   - $r_i^e \star s_j$,
   - $r_j^e \star s_i$;

   (The following picture of the analogue in an affine plane may help to understand this condition:)



5. for any $i, j \in \{1, ..., n\}$ such that $i \neq j$, $s_i \in F(s_j) \star s_j$ and $F(s_i) \notin F(s_j) \star s_j$, the intersection of the following three sets is a singleton:
   - $s_j \star F(s_i)$,
   - $F(s_j) \star \big((F(s_j \boxplus_w s_i) \star s_j) \cap (r_j^e \star F(s_i))\big)$,
   - $s_i \star \big((F(s_i) \star (s_j \boxplus_w s_i)) \cap (r_i^e \star s_j)\big)$.

   (The following picture of the analogue in an affine plane may help to understand this condition:)

This definition involves complicated configurations. In the remaining part of this subsection, I study this definition in the context of quasi-quantum Kripke frames of the form $(\Sigma(V), \to_V)$ for a vector space V over a field equipped with an anisotropic Hermitian form. The goal is two-fold: One is to facilitate the use of the analytic method. The other is to prove the existence of uniformly scaled rulers in such quasi-quantum Kripke frames.

I discuss the involved configurations one by one. I start from the configuration prescribed by 1 in the above definition.

3.3.2. LEMMA. *Let V be a vector space over a division ring $\mathcal{K} = (K, +, \cdot, 0, 1)$ equipped with an anisotropic Hermitian form Φ. If $\{s_1, ..., s_n\}$ for some $n \in \mathbb{N}^+$ is an orthogonal set in $(\Sigma(V), \to_V)$, there is a $w \in \Sigma(V)$ such that $w \in {\sim\sim}\{s_1, ..., s_n\}$, and $w \to s_i$ for $i = 1, ..., n$.*

Moreover, for such orthogonal set $\{s_1, ..., s_n\}$ and $w \in \Sigma(V)$, there are pairwise orthogonal vectors $\mathbf{s}_1, ..., \mathbf{s}_n$, unique up to a common constant multiple, such that $s_i = \langle \mathbf{s}_i \rangle$ for $i = 1, ..., m$ and $w = \langle \sum_{i=1}^n \mathbf{s}_i \rangle$. In addition, $s_i \boxplus_w s_j = \langle \mathbf{s}_i + \mathbf{s}_j \rangle$, for any $1 \leq i < j \leq n$.

Proof. For the first part, assume that $\{s_1, ..., s_n\}$ for some $n \in \mathbb{N}^+$ is an orthogonal set in $(\Sigma(V), \to_V)$. For $i = 1, ..., n$, let $\mathbf{s}_i \in V \setminus \{\mathbf{0}\}$ be such that $s_i = \langle \mathbf{s}_i \rangle$. Since $\{s_1, ..., s_n\}$ is orthogonal, $\Phi(\mathbf{s}_i, \mathbf{s}_j) = 0$ when $1 \leq i < j \leq n$. Consider $w = \langle \sum_{i=1}^n \mathbf{s}_i \rangle$. It is easy to verify that $\sum_{i=1}^n \mathbf{s}_i \neq \mathbf{0}$, $w \in {\sim\sim}\{s_1, ..., s_n\}$ and $w \to_V s_i$ for $i = 1, ..., n$.

For the second part, assume that $\{s_1, ..., s_n\}$ for some $n \in \mathbb{N}^+$ is an orthogonal set in $(\Sigma(V), \to_V)$ and $w \in \Sigma(V)$ is such that $w \in {\sim\sim}\{s_1, ..., s_n\}$ and $w \to_V s_i$ for $i = 1, ..., n$. For $i = 1, ..., n$, let $\mathbf{v}_i \in V \setminus \{\mathbf{0}\}$ be such that $s_i = \langle \mathbf{v}_i \rangle$. Since

$\{s_1, ..., s_n\}$ is orthogonal, $\Phi(\mathbf{v}_i, \mathbf{v}_j) = 0$ when $1 \leq i < j \leq n$. By Lemma 2.3.23 $w \in {\sim\sim}\{s_1, ..., s_n\} = \mathcal{C}(\{s_1, ..., s_n\})$. Hence there are $x_1, ..., x_n \in K$ such that $w = \langle \sum_{i=1}^n x_i \mathbf{v}_i \rangle$. For each $i = 1, ..., n$, since $w \rightarrow_V s_i$, $0 \neq \Phi(\sum_{j=1}^n x_j \mathbf{v}_j, \mathbf{v}_i) = \sum_{j=1}^n x_j \cdot \Phi(\mathbf{v}_j, \mathbf{v}_i) = x_i \Phi(\mathbf{v}_i, \mathbf{v}_i)$, and thus $x_i \neq 0$. Let $\mathbf{s}_i = x_i \mathbf{v}_i$ for $i = 1, ..., n$. Then it follows easily that $\mathbf{s}_1, ..., \mathbf{s}_n$ are pairwise orthogonal, $s_i = \langle \mathbf{s}_i \rangle$ for $i = 1, ..., n$ and $w = \langle \sum_{i=1}^n \mathbf{s}_i \rangle$.

For uniqueness, assume that $\mathbf{u}_1, ..., \mathbf{u}_n$ are pairwise orthogonal vectors such that $s_i = \langle \mathbf{u}_i \rangle$ for $i = 1, ..., n$ and $w = \langle \sum_{i=1}^n \mathbf{u}_i \rangle$. For each i, since $\langle \mathbf{s}_i \rangle = s_i = \langle \mathbf{u}_i \rangle$, $\mathbf{u}_i = y_i \mathbf{s}_i$ for some $y_i \in K$. Since $\langle \sum_{i=1}^n \mathbf{s}_i \rangle = w = \langle \sum_{i=1}^n \mathbf{u}_i \rangle$, $\sum_{i=1}^n \mathbf{u}_i = y \sum_{i=1}^n \mathbf{s}_i$ for some $y \in K$. It follows that $\sum_{i=1}^n y_i \mathbf{s}_i = \sum_{i=1}^n y \mathbf{s}_i$, and hence $\sum_{i=1}^n (y_i - y)\mathbf{s}_i = \mathbf{0}$. Since $\mathbf{s}_1, ..., \mathbf{s}_n$ are pairwise orthogonal, $y = y_1 = ... = y_n$.

Finally, to show that $s_i \boxplus_w s_j = \langle \mathbf{s}_i + \mathbf{s}_j \rangle$ for $1 \leq i < j \leq n$, let $u \in s_i \star s_j$ be arbitrary. Then there are $x, y \in K$ such that $\langle x\mathbf{s}_i + y\mathbf{s}_j \rangle = u$. Moreover,

$$
\begin{aligned}
u \rightarrow_V w &\Leftrightarrow \langle x\mathbf{s}_i + y\mathbf{s}_j \rangle \rightarrow_V \left\langle \sum_{k=1}^n \mathbf{s}_k \right\rangle \\
&\Leftrightarrow \Phi\left(x\mathbf{s}_i + y\mathbf{s}_j, \sum_{k=1}^n \mathbf{s}_k \right) \neq 0 \\
&\Leftrightarrow x \cdot \sum_{k=1}^n \Phi(\mathbf{s}_i, \mathbf{s}_k) + y \cdot \sum_{k=1}^n \Phi(\mathbf{s}_j, \mathbf{s}_k) \neq 0 \\
&\Leftrightarrow x \cdot \Phi(\mathbf{s}_i, \mathbf{s}_i) + y \cdot \Phi(\mathbf{s}_j, \mathbf{s}_j) \neq 0 \\
&\Leftrightarrow x \cdot \Phi(\mathbf{s}_i, \mathbf{s}_i) + x \cdot \Phi(\mathbf{s}_i, \mathbf{s}_j) + y \cdot \Phi(\mathbf{s}_j, \mathbf{s}_i) + y \cdot \Phi(\mathbf{s}_j, \mathbf{s}_j) \neq 0 \\
&\Leftrightarrow \Phi(x\mathbf{s}_i + y\mathbf{s}_j, \mathbf{s}_i + \mathbf{s}_j) \neq 0 \\
&\Leftrightarrow \langle x\mathbf{s}_i + y\mathbf{s}_j \rangle \rightarrow_V \langle \mathbf{s}_i + \mathbf{s}_j \rangle \\
&\Leftrightarrow u \rightarrow_V \langle \mathbf{s}_i + \mathbf{s}_j \rangle
\end{aligned}
$$

Since $\langle \mathbf{s}_i + \mathbf{s}_j \rangle \in s_i \star s_j$, it is the approximation of w in $s_i \star s_j$, i.e. $s_i \boxplus_w s_j$. ⊣

Next, I consider the configuration prescribed by 2 in the definition.

3.3.3. Lemma. *Let V be a vector space, of finite dimension $n \geq 3$, over a* field *$\mathcal{K} = (K, +, \cdot, 0, 1)$ equipped with an anisotropic Hermitian form Φ, F an arguesian, continuous endo-homomorphism on $(\Sigma(V), \rightarrow_V)$ such that $F \not\approx Id$. There is an orthogonal set $\{s_1, ..., s_n\} \subseteq \Sigma(V)$ such that $\Sigma(V) = {\sim\sim}\{s_1, ..., s_n\}$ and $F(s_i) \neq s_i$ for $i = 1, ..., n$.*

Proof. Since F is a continuous endo-homomorphism on $(\Sigma(V), \rightarrow_V)$, by Corollary 3.1.11 F is a homomorphism on the pure orthogeometry $\mathbf{G}((\Sigma(V), \rightarrow_V))$. Since F is arguesian and $F \not\approx Id$, by Corollary 3.2.14 there is a linear map $f : V \rightarrow V$ such that $F = \mathcal{P}(f)$. Since V is of finite dimension $n \geq 3$ and equipped with an anisotropic Hermitian form, K is infinite by Proposition B.2.11. According to

Proposition C.4.2, there is an orthogonal basis $\{\mathbf{s}_1, ..., \mathbf{s}_n\}$ of V which does not include any eigenvectors of f. Let $s_i = \langle \mathbf{s}_i \rangle$ for $i = 1, ..., n$. Since $\{\mathbf{s}_1, ..., \mathbf{s}_n\}$ is an orthogonal basis, $\{s_1, ..., s_n\}$ is an orthogonal set and $\Sigma(V) = \mathcal{C}(\{s_1, ..., s_n\}) = {\sim}{\sim}\{s_1, ..., s_n\}$. Moreover, for $i = 1, ..., n$, since $\mathbf{s}_i$ is not an eigenvector of f, $F(s_i) = \mathcal{P}(f)(\langle \mathbf{s}_i \rangle) = \langle f(\mathbf{s}_i) \rangle \neq \langle \mathbf{s}_i \rangle = s_i$. ⊣

I proceed to show the analytic significance of uniformly scaled rulers.

3.3.4. Lemma. *Let V be a vector space of dimension at least 3 over a field $\mathcal{K} = (K, +, \cdot, 0, 1)$ equipped with an anisotropic Hermitian form Φ, F an continuous endo-homomorphism on $(\Sigma(V), \rightarrow_V)$, $\{s_1, ..., s_n\} \subseteq \Sigma(V)$ an orthogonal set and $w \in \Sigma(V)$. Suppose that 1 and 2 in Definition 3.3.1 are satisfied, $F = \mathcal{P}(f)$ for some linear map $f : V \to V$ and $\mathbf{s}_1, ..., \mathbf{s}_n$ are non-zero vectors such that $s_i = \langle \mathbf{s}_i \rangle$ for $i = 1, ..., n$ and $w = \langle \sum_{i=1}^n \mathbf{s}_i \rangle$. Then the following are equivalent:*

(i) $\{(F(s_i), s_i, r_i^e) \mid i = 1, ..., n\}$ is a set of rulers uniformly scaled by F and w;

(ii) there is an $x \in K \setminus \{0\}$ such that $r_i^e = \langle f(\mathbf{s}_i) + x\mathbf{s}_i \rangle$ for $i = 1, ..., n$.

Proof. For $i = 1, ..., n$, it follows from $F = \mathcal{P}(f)$ and $s_i = \langle \mathbf{s}_i \rangle$ that $F(s_i) = \langle f(\mathbf{s}_i) \rangle$. Moreover, for any distinct $i, j \in \{1, ..., n\}$, since $w = \langle \sum_{i=1}^n \mathbf{s}_i \rangle$, by Lemma 3.3.2 $s_i \boxplus_w s_j = \langle \mathbf{s}_i + \mathbf{s}_j \rangle$, and thus $F(s_i \boxplus_w s_j) = \langle f(\mathbf{s}_i) + f(\mathbf{s}_j) \rangle$.

From (i) to (ii): For $i = 1, ..., n$, since $(F(s_i), s_i, r_i^e)$ is a ruler, there is an $x_i \in K \setminus \{0\}$ such that $r_i^e = \langle f(\mathbf{s}_i) + x_i \mathbf{s}_i \rangle$. I claim the following:

(♣) for every $i, j \in \{1, ..., n\}$, $x_i = x_j$, if $i \neq j$ and $\{F(s_i), s_i\} \nsubseteq F(s_j) \star s_j$.

To prove this claim, four cases need to be considered.

Case 1: $s_i \notin F(s_j) \star s_j$ and $F(s_j) \notin F(s_i) \star s_i$. Then both $\{f(\mathbf{s}_j), f(\mathbf{s}_i), \mathbf{s}_i\}$ and $\{f(\mathbf{s}_j), \mathbf{s}_i, \mathbf{s}_j\}$ are linearly independent. By 3 in Definition 3.3.1 the intersection of the following three sets is a singleton $\{t\}$:

- $s_i \star F(s_j)$,
- $F(s_i) \star \big((F(s_i \boxplus_w s_j) \star s_i) \cap (r_i^e \star F(s_j))\big)$,
- $s_j \star \big((F(s_j) \star (s_i \boxplus_w s_j)) \cap (r_j^e \star s_i)\big)$.

Find vectors generating t in two ways:

$$\begin{aligned} a =& (F(s_i \boxplus_w s_j) \star s_i) \cap (r_i^e \star F(s_j)) \\ =& \big(\, \langle f(\mathbf{s}_i) + f(\mathbf{s}_j) \rangle \star \langle \mathbf{s}_i \rangle \,\big) \cap \big(\, \langle f(\mathbf{s}_i) + x_i \mathbf{s}_i \rangle \star \langle f(\mathbf{s}_j) \rangle \,\big) \\ =& \langle f(\mathbf{s}_i) + x_i \mathbf{s}_i + f(\mathbf{s}_j) \rangle \qquad (\{f(\mathbf{s}_j), f(\mathbf{s}_i), \mathbf{s}_i\} \text{ is linearly independent}) \end{aligned}$$

$$t = (s_i \star F(s_j)) \cap (F(s_i) \star a)$$

$$\begin{aligned}
&= \big(\,\langle \mathbf{s}_i\rangle \star \langle f(\mathbf{s}_j)\rangle\,\big) \cap \big(\,\langle f(\mathbf{s}_i)\rangle \star \langle f(\mathbf{s}_i) + x_i\mathbf{s}_i + f(\mathbf{s}_j)\rangle\,\big)\\
&= \langle f(\mathbf{s}_j) + x_i\mathbf{s}_i\rangle \qquad (\{f(\mathbf{s}_j), f(\mathbf{s}_i), \mathbf{s}_i\} \text{ is linearly independent})
\end{aligned}$$

$$\begin{aligned}
b &= (F(s_j) \star (s_i \boxplus_w s_j)) \cap \big(r_j^e \star s_i\big)\\
&= \big(\,\langle f(\mathbf{s}_j)\rangle \star \langle \mathbf{s}_i + \mathbf{s}_j\rangle\,\big) \cap \big(\,\langle f(\mathbf{s}_j) + x_j\mathbf{s}_j\rangle \star \langle \mathbf{s}_i\rangle\,\big)\\
&= \langle f(\mathbf{s}_j) + x_j\mathbf{s}_j + x_j\mathbf{s}_i\rangle \qquad (\{f(\mathbf{s}_j), \mathbf{s}_i, \mathbf{s}_j\} \text{ is linearly independent})
\end{aligned}$$

$$\begin{aligned}
t &= (s_i \star F(s_j)) \cap (s_j \star b)\\
&= \big(\,\langle \mathbf{s}_i\rangle \star \langle f(\mathbf{s}_j)\rangle\,\big) \cap \big(\,\langle \mathbf{s}_j\rangle \star \langle f(\mathbf{s}_j) + x_j\mathbf{s}_j + x_j\mathbf{s}_i\rangle\,\big)\\
&= \langle f(\mathbf{s}_j) + x_j\mathbf{s}_i\rangle \qquad (\{f(\mathbf{s}_j), \mathbf{s}_i, \mathbf{s}_j\} \text{ is linearly independent})
\end{aligned}$$

Then $\langle f(\mathbf{s}_j) + x_i\mathbf{s}_i\rangle = t = \langle f(\mathbf{s}_j) + x_j\mathbf{s}_i\rangle$. It follows that $x_i = x_j$.

Case 2: $s_i \notin F(s_j) \star s_j$, $F(s_j) \in F(s_i) \star s_i$ and $F(s_i) \neq F(s_j)$. Note that $s_j \notin F(s_i) \star s_i$; otherwise, by (P8) of Lemma B.1.2 $\{s_j, F(s_j)\} \subseteq F(s_i) \star s_i$ implies that $F(s_j) \star s_j = F(s_i) \star s_i$, and thus $\{s_i, F(s_i)\} \subseteq F(s_j) \star s_j$, contradicting $\{s_i, F(s_i)\} \nsubseteq F(s_j) \star s_j$. Also note that $F(s_i) \notin F(s_j) \star s_j$; otherwise, $F(s_i), F(s_j) \in F(s_j) \star s_j$ and $s_i \in F(s_j) \star F(s_i)$ imply that $s_i \in F(s_j) \star s_j$, contradicting the assumption of this case. Then both $\{\mathbf{s}_j, f(\mathbf{s}_i), \mathbf{s}_i\}$ and $\{f(\mathbf{s}_i), f(\mathbf{s}_j), \mathbf{s}_j\}$ are linearly independent. By 3 in Definition 3.3.1 the intersection of the following three sets is a singleton $\{t\}$:

- $s_j \star F(s_i)$,
- $F(s_j) \star \big((F(s_j \boxplus_w s_i) \star s_j) \cap (r_j^e \star F(s_i))\big)$,
- $s_i \star \big((F(s_i) \star (s_j \boxplus_w s_i)) \cap (r_i^e \star s_j)\big)$.

Find vectors generating t in two ways:

$$\begin{aligned}
a &= (F(s_j \boxplus_w s_i) \star s_j) \cap (r_j^e \star F(s_i))\\
&= \big(\,\langle f(\mathbf{s}_j) + f(\mathbf{s}_i)\rangle \star \langle \mathbf{s}_j\rangle\,\big) \cap \big(\,\langle f(\mathbf{s}_j) + x_j\mathbf{s}_j\rangle \star \langle f(\mathbf{s}_i)\rangle\,\big)\\
&= \langle f(\mathbf{s}_j) + x_j\mathbf{s}_j + f(\mathbf{s}_i)\rangle \qquad (\{f(\mathbf{s}_i), f(\mathbf{s}_j), \mathbf{s}_j\} \text{ is linearly independent})
\end{aligned}$$

$$\begin{aligned}
t &= (s_j \star F(s_i)) \cap (F(s_j) \star a)\\
&= \big(\,\langle \mathbf{s}_j\rangle \star \langle f(\mathbf{s}_i)\rangle\,\big) \cap \big(\,\langle f(\mathbf{s}_j)\rangle \star \langle f(\mathbf{s}_j) + x_j\mathbf{s}_j + f(\mathbf{s}_i)\rangle\,\big)\\
&= \langle f(\mathbf{s}_i) + x_j\mathbf{s}_j\rangle \qquad (\{f(\mathbf{s}_i), f(\mathbf{s}_j), \mathbf{s}_j\} \text{ is linearly independent})
\end{aligned}$$

$$\begin{aligned}
b &= (F(s_i) \star (s_j \boxplus_w s_i)) \cap (r_i^e \star s_j)\\
&= \big(\,\langle f(\mathbf{s}_i)\rangle \star \langle \mathbf{s}_j + \mathbf{s}_i\rangle\,\big) \cap \big(\,\langle f(\mathbf{s}_i) + x_i\mathbf{s}_i\rangle \star \langle \mathbf{s}_j\rangle\,\big)\\
&= \langle f(\mathbf{s}_i) + x_i\mathbf{s}_i + x_i\mathbf{s}_j\rangle \qquad (\{f(\mathbf{s}_i), \mathbf{s}_j, \mathbf{s}_i\} \text{ is linearly independent})
\end{aligned}$$

$$\begin{aligned} t &= (s_j \star F(s_i)) \cap (s_i \star b) \\ &= \big(\langle \mathbf{s}_j \rangle \star \langle f(\mathbf{s}_i) \rangle \big) \cap \big(\langle \mathbf{s}_i \rangle \star \langle f(\mathbf{s}_i) + x_i \mathbf{s}_i + x_i \mathbf{s}_j \rangle \big) \\ &= \langle f(\mathbf{s}_i) + x_i \mathbf{s}_j \rangle \qquad (\{f(\mathbf{s}_i), \mathbf{s}_j, \mathbf{s}_i\} \text{ is linearly independent}) \end{aligned}$$

Then $\langle f(\mathbf{s}_i) + x_i \mathbf{s}_j \rangle = t = \langle f(\mathbf{s}_i) + x_j \mathbf{s}_j \rangle$. It follows that $x_i = x_j$.

Case 3: $s_i \notin F(s_j) \star s_j$ and $F(s_i) = F(s_j)$. Then $\{\mathbf{s}_i, f(\mathbf{s}_j), \mathbf{s}_j\}$ is linearly independent. By 4 in Definition 3.3.1 the intersection of the following three sets is a singleton $\{t\}$:

- $F(s_j) \star (s_j \boxplus_w s_i)$;
- $r_j^e \star s_i$;
- $r_i^e \star s_j$.

Find vectors generating t in two ways:

$$\begin{aligned} t &= (F(s_j) \star (s_j \boxplus_w s_i)) \cap \big(r_j^e \star s_i\big) \\ &= \big(\langle f(\mathbf{s}_j) \rangle \star \langle \mathbf{s}_j + \mathbf{s}_i \rangle \big) \cap \big(\langle f(\mathbf{s}_j) + x_j \mathbf{s}_j \rangle \star \langle \mathbf{s}_i \rangle \big) \\ &= \langle f(\mathbf{s}_j) + x_j \mathbf{s}_j + x_j \mathbf{s}_i \rangle \qquad (\{f(\mathbf{s}_j), \mathbf{s}_j, \mathbf{s}_i\} \text{ is linearly independent}) \end{aligned}$$

$$\begin{aligned} t &= (F(s_j) \star (s_j \boxplus_w s_i)) \cap (r_i^e \star s_j) \\ &= \big(\langle f(\mathbf{s}_j) \rangle \star \langle \mathbf{s}_j + \mathbf{s}_i \rangle \big) \cap \big(\langle f(\mathbf{s}_i) + x_i \mathbf{s}_i \rangle \star \langle \mathbf{s}_j \rangle \big) \\ &= \langle f(\mathbf{s}_j) + x_i \mathbf{s}_j + x_i \mathbf{s}_i \rangle \qquad (\{f(\mathbf{s}_j), \mathbf{s}_j, \mathbf{s}_i\} \text{ is linearly independent}) \end{aligned}$$

Then $\langle f(\mathbf{s}_j) + x_i \mathbf{s}_j + x_i \mathbf{s}_i \rangle = t = \langle f(\mathbf{s}_j) + x_j \mathbf{s}_j + x_j \mathbf{s}_i \rangle$. It follows that $x_i = x_j$.

Case 4: $s_i \in F(s_j) \star s_j$. Since $\{F(s_i), s_i\} \nsubseteq F(s_j) \star s_j$, $F(s_i) \notin F(s_j) \star s_j$. Then $\{f(\mathbf{s}_i), f(\mathbf{s}_j), \mathbf{s}_j\}$ is linearly independent. By 5 in Definition 3.3.1 the intersection of the following three sets is a singleton $\{t\}$:

- $s_j \star F(s_i)$,
- $F(s_j) \star \big((F(s_j \boxplus_w s_i) \star s_j) \cap (r_j^e \star F(s_i))\big)$,
- $s_i \star \big((F(s_i) \star (s_j \boxplus_w s_i)) \cap (r_i^e \star s_j)\big)$.

Find vectors generating t in two ways:

$$\begin{aligned} a &= (F(s_j \boxplus_w s_i) \star s_j) \cap \big(r_j^e \star F(s_i)\big) \\ &= \big(\langle f(\mathbf{s}_j) + f(\mathbf{s}_i) \rangle \star \langle \mathbf{s}_j \rangle \big) \cap \big(\langle f(\mathbf{s}_j) + x_j \mathbf{s}_j \rangle \star \langle f(\mathbf{s}_i) \rangle \big) \\ &= \langle f(\mathbf{s}_j) + x_j \mathbf{s}_j + f(\mathbf{s}_i) \rangle \end{aligned}$$

$$
\begin{aligned}
t &= (s_j \star F(s_i)) \cap (F(s_j) \star a) \\
&= \big(\langle \mathbf{s}_j \rangle \star \langle f(\mathbf{s}_i) \rangle\big) \cap \big(\langle f(\mathbf{s}_j) \rangle \star \langle f(\mathbf{s}_j) + x_j \mathbf{s}_j + f(\mathbf{s}_i) \rangle\big) \\
&= \langle f(\mathbf{s}_i) + x_j \mathbf{s}_j \rangle
\end{aligned}
$$

$$
\begin{aligned}
b &= (F(s_i) \star (s_j \boxplus_w s_i)) \cap (r_i^e \star s_j) \\
&= \big(\langle f(\mathbf{s}_i) \rangle \star \langle \mathbf{s}_i + \mathbf{s}_j \rangle\big) \cap \big(\langle f(\mathbf{s}_i) + x_i \mathbf{s}_i \rangle \star \langle \mathbf{s}_j \rangle\big) \\
&= \langle f(\mathbf{s}_i) + x_i \mathbf{s}_i + x_i \mathbf{s}_j \rangle
\end{aligned}
$$

$$
\begin{aligned}
t &= (s_j \star F(s_i)) \cap (s_i \star b) \\
&= \big(\langle \mathbf{s}_j \rangle \star \langle f(\mathbf{s}_i) \rangle\big) \cap \big(\langle \mathbf{s}_i \rangle \cap \langle f(\mathbf{s}_i) + x_i \mathbf{s}_i + x_i \mathbf{s}_j \rangle\big) \\
&= \langle f(\mathbf{s}_i) + x_i \mathbf{s}_j \rangle
\end{aligned}
$$

Then $\langle f(\mathbf{s}_i) + x_i \mathbf{s}_j \rangle = t = \langle f(\mathbf{s}_i) + x_j \mathbf{s}_j \rangle$. It follows that $x_i = x_j$.

This finishes the proof of (♣).

Now let $i \in \{2, ..., n\}$ be arbitrary. If $\{F(s_i), s_i\} \not\subseteq F(s_1) \star s_1$, $x_i = x_1$ by (♣). If $\{F(s_i), s_i\} \subseteq F(s_1) \star s_1$, since V is of dimension at least 3, there is a $j \notin \{1, i\}$ such that $s_j \notin s_1 \star s_i$. Since $\{F(s_i), s_i\} \subseteq F(s_1) \star s_1$, $F(s_1) \star s_1 = F(s_i) \star s_i = s_1 \star s_i$ by (P8) in Lemma B.1.2. Hence it follows from $s_j \notin s_1 \star s_i$ that $\{F(s_j), s_j\} \not\subseteq F(s_1) \star s_1$ and $\{F(s_j), s_j\} \not\subseteq F(s_i) \star s_i$. By (♣) $x_i = x_j = x_1$. Therefore, let x be x_1, and then it holds that $r_i^e = \langle f(\mathbf{s}_i) + x \mathbf{s}_i \rangle$ for $i = 1, ..., n$.

From (ii) to (i): Tedious but straightforward verification. ⊣

The significance of this lemma is that, in a set of uniformly scaled rulers, the units of the rulers are uniquely determined by any one of them according to the geometric constraints indicated Definition 3.3.1. This is the reason why these rulers are called uniformly scaled.

3.3.2 Trace of a Continuous Homomorphism

In this subsection, I define the notion of traces of continuous endo-homomorphisms and study its basic properties.

I start with the definition of traces.

3.3.5. Definition. Let $\mathfrak{F} = (\Sigma, \to)$ be a quasi-quantum Kripke frame, $F : \Sigma \dashrightarrow \Sigma$ a continuous endo-homomorphism on $\mathfrak{F}$, $\{(F(s_i), s_i, r_i^e) \mid i = 1, ..., n\}$ a set of rulers uniformly scaled by F and $w \in \Sigma$ such that $\Sigma = \sim\sim\{s_1, ..., s_n\}$. *The trace of F with respect to the set of uniformly scaled rulers* $\{(F(s_i), s_i, r_i^e) \mid i = 1, ..., n\}$ is a sequence $(t_1, ..., t_n)$ in Σ defined recursively as follows:

- t_1 is the element s_1^* in the singleton $\sim\{s_1\} \cap (F(s_1) \star s_1)$;
- to define t_{k+1}, a series of auxiliary points is needed:

– t^-_{k+1}: the point on $F(s_{k+1}) \star s_{k+1}$ such that

$$(F(s_k), s_k, r^e_k \mid t_k) - (F(s_{k+1}), s_{k+1}, r^e_{k+1} \mid t^-_{k+1}),$$

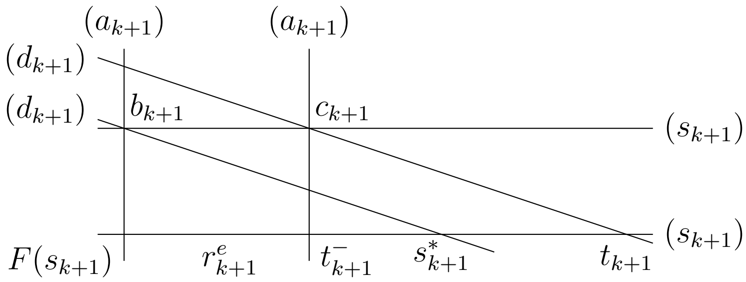
– a_{k+1}: an arbitrary point not on $F(s_{k+1}) \star s_{k+1}$,

– b_{k+1}: an arbitrary third point on $a_{k+1} \star F(s_{k+1})$,

– c_{k+1}: the intersecting point of the lines $a_{k+1} \star t^-_{k+1}$ and $b_{k+1} \star s_{k+1}$,

– d_{k+1}: the intersecting point of the lines $b_{k+1} \star s^*_{k+1}$ and $a_{k+1} \star s_{k+1}$, where s^*_{k+1} is the element in the singleton $\sim\{s_{k+1}\} \cap (F(s_{k+1}) \star s_{k+1})$.

Finally t_{k+1} is defined to be the intersecting point of the lines $c_{k+1} \star d_{k+1}$ and $F(s_{k+1}) \star s_{k+1}$. (The following picture of the analogue in an affine plane may help to understand this construction. Remember that, in the picture, for example, '(a_{k+1})' means that a_{k+1} is not a point in the affine plane; instead, it is an imaginary point at infinity where parallel lines intersect.)



Note that the trace of a continuous homomorphism depends on the order of the rulers in the set.

I show that the notion of traces defined above is closely related to the traces of linear maps. Before showing this, I prove a technical lemma.

3.3.6. Lemma. *Let V be a vector space of finite dimension $n \geq 3$ over a field $\mathcal{K} = (K, +, \cdot, 0, 1)$ equipped with an anisotropic Hermitian form Φ, $F : \Sigma \dashrightarrow \Sigma$ a continuous endo-homomorphism on $(\Sigma(V), \rightarrow_V)$, $s \in \Sigma$ satisfying $F(s) \neq s$.*

1. *$\sim\{s\} \cap (F(s) \star s)$ is a singleton.*

2. *If $f : V \to V$ is a linear map satisfying $F = \mathcal{P}(f)$, $\mathbf{s} \in V \setminus \{\mathbf{0}\}$ satisfies $s = \langle \mathbf{s} \rangle$ and $\{s^*\} = \sim\{s\} \cap (F(s) \star s)$, then*

$$s^* = \big\langle f(\mathbf{s}) - (\Phi(f(\mathbf{s}), \mathbf{s}) \cdot \Phi(\mathbf{s}, \mathbf{s})^{-1})\mathbf{s} \big\rangle$$

Proof. **For *1*:** It follows from Proposition 2.3.9, Theorem B.1.4 and Reflexivity.

For *2*: It follows from $F = \mathcal{P}(f)$ and $s = \langle \mathbf{s} \rangle$ that $F(s) = \langle f(\mathbf{s}) \rangle$. Because $s^* \in F(s) \star s$, $s^* = \langle x\mathbf{s} + yf(\mathbf{s}) \rangle$ for some $x, y \in K$ which cannot be both 0. Since $s^* \not\rightarrow_V s$, $0 = \Phi(x\mathbf{s} + yf(\mathbf{s}), \mathbf{s}) = x \cdot \Phi(\mathbf{s}, \mathbf{s}) + y \cdot \Phi(f(\mathbf{s}), \mathbf{s})$. Since Φ is anisotropic, $\Phi(\mathbf{s}, \mathbf{s}) \neq 0$. Hence $y \neq 0$ and $x = -y \cdot \Phi(f(\mathbf{s}), \mathbf{s}) \cdot \Phi(\mathbf{s}, \mathbf{s})^{-1}$. Then

$$s^* = \langle y\,(f(\mathbf{s}) - (\Phi(f(\mathbf{s}), \mathbf{s}) \cdot \Phi(\mathbf{s}, \mathbf{s})^{-1})\mathbf{s})\rangle = \langle f(\mathbf{s}) - (\Phi(f(\mathbf{s}), \mathbf{s}) \cdot \Phi(\mathbf{s}, \mathbf{s})^{-1})\mathbf{s}\rangle \quad \dashv$$

Now I can show the relation between the traces of continuous homomorphisms and those of linear maps. The idea is again the one of K.G.C. von Staudt: every line in an arguesian quasi-quantum Kripke frame, deleting one arbitrary point, has the structure of a division ring. The definition of the traces of continuous homomorphisms in fact mimics in geometric means the calculation of the traces of linear maps.

3.3.7. Proposition. *Let V be a vector space of finite dimension $n \geq 3$ over a field $\mathcal{K} = (K, +, \cdot, 0, 1)$ equipped with an anisotropic Hermitian form Φ, $F : \Sigma \dashrightarrow \Sigma$ a continuous endo-homomorphism on $(\Sigma(V), \rightarrow_V)$, $\{(F(s_i), s_i, r_i^e) \mid i = 1, ..., n\}$ a set of rulers uniformly scaled by F and $w \in \Sigma(V)$ such that $\Sigma(V) = {\sim}{\sim}\{s_1, ..., s_n\}$, $(t_1, ..., t_n)$ be the trace of F with respect to $\{(F(s_i), s_i, r_i^e) \mid i = 1, ..., n\}$.*

If $f : V \to V$ is a linear map satisfying $F = \mathcal{P}(f)$, $\mathbf{s}_i \in V$ for $i = 1, ..., n$ are such that $\Phi(\mathbf{s}_i, \mathbf{s}_i) = 1$, $s_i = \langle \mathbf{s}_i\rangle$, $r_i^e = \langle f(\mathbf{s}_i) + x\mathbf{s}_i\rangle$ for some $x \in K \setminus \{0\}$ and $w = \langle \sum_{i=1}^n \mathbf{s}_i\rangle$, then, for each $i = 1, ..., n$,

$$t_i = \langle f(\mathbf{s}_i) - A_i\mathbf{s}_i\rangle\,,\ \textit{where}\ A_i = \sum_{j=1}^{i} \Phi(f(\mathbf{s}_j), \mathbf{s}_j)$$

In particular, $t_n = \langle f(\mathbf{s}_n) - tr(f)\mathbf{s}_n\rangle$.

Proof. For t_1, according to the definition, the previous lemma and $\Phi(\mathbf{s}_1, \mathbf{s}_1) = 1$,

$$t_1 = s_1^* = \langle f(\mathbf{s}_1) - \Phi(f(\mathbf{s}_1), \mathbf{s}_1)\mathbf{s}_1\rangle = \langle f(\mathbf{s}_1) - A_1\mathbf{s}_1\rangle$$

For t_{k+1}, find non-zero vectors generating the involved points one by one. Since $t_{k+1}^- \in F(s_{k+1}) \star s_{k+1}$, $t_{k+1}^- = \langle yf(\mathbf{s}_{k+1}) + z\mathbf{s}_{k+1}\rangle$ for some $y, z \in K$ which cannot be both 0. Since $(F(s_k), s_k, r_k^e \mid t_k) - (F(s_{k+1}), s_{k+1}, r_{k+1}^e \mid t_{k+1}^-)$,

$$\begin{aligned}&(\langle f(\mathbf{s}_k)\rangle\,, \langle x\mathbf{s}_k\rangle\,, \langle f(\mathbf{s}_k) + x\mathbf{s}_k\rangle \mid \langle f(\mathbf{s}_i) - (A_k \cdot x^{-1})(x\mathbf{s}_i)\rangle)\\ &- (\langle f(\mathbf{s}_{k+1})\rangle\,, \langle x\mathbf{s}_{k+1}\rangle\,, \langle f(\mathbf{s}_{k+1}) + x\mathbf{s}_{k+1}\rangle \mid \langle yf(\mathbf{s}_{k+1}) + (z \cdot x^{-1})(x\mathbf{s}_{k+1})\rangle)\end{aligned}$$

By Proposition 3.2.6 $A_k = z = 0$ or $-A_k^{-1} = z^{-1} \cdot y$. Hence $y \neq 0$ and $-A_k = y^{-1} \cdot z$. Hence $t_{k+1}^- = \langle f(\mathbf{s}_{k+1}) - A_k\mathbf{s}_{k+1}\rangle$. Since V is at least three-dimensional, I can take an $a_{k+1} = \langle \mathbf{u}_{k+1}\rangle$ such that $\mathbf{u}_{k+1}, f(\mathbf{s}_{k+1}), \mathbf{s}_{k+1}$ are linearly independent. Then a_{k+1} is not on the line $F(s_{k+1}) \star s_{k+1}$. Also take $b_{k+1} = \langle \mathbf{u}_{k+1} + f(\mathbf{s}_{k+1})\rangle$, then b_{k+1} is a third point on $F(s_{k+1}) \star a_{k+1}$. Continue to find non-zero vectors generating the points c_{k+1}, d_{k+1} and t_{k+1}:

$$\begin{aligned}c_{k+1} &= (a_{k+1} \star t_{k+1}^-) \cap (b_{k+1} \star s_{k+1})\\ &= \big(\,\langle \mathbf{u}_{k+1}\rangle \star \langle f(\mathbf{s}_{k+1}) - A_k\mathbf{s}_{k+1}\rangle\,\big) \cap \big(\,\langle \mathbf{u}_{k+1} + f(\mathbf{s}_{k+1})\rangle \star \langle \mathbf{s}_{k+1}\rangle\,\big)\end{aligned}$$

$$= \langle \mathbf{u}_{k+1} + f(\mathbf{s}_{k+1}) - A_k \mathbf{s}_{k+1} \rangle$$

$$\begin{aligned} d_{k+1} &= (b_{k+1} \star s^*_{k+1}) \cap (a_{k+1} \star s_{k+1}) \\ &= \big(\langle \mathbf{u}_{k+1} + f(\mathbf{s}_{k+1}) \rangle \star \langle f(\mathbf{s}_{k+1}) - \Phi(f(\mathbf{s}_{k+1}), \mathbf{s}_{k+1}) \mathbf{s}_{k+1} \rangle \big) \cap \big(\langle \mathbf{u}_{k+1} \rangle \star \langle \mathbf{s}_{k+1} \rangle \big) \\ &= \langle \mathbf{u}_{k+1} + \Phi(f(\mathbf{s}_{k+1}), \mathbf{s}_{k+1}) \mathbf{s}_{k+1} \rangle \end{aligned}$$

$$\begin{aligned} t_{k+1} &= (c_{k+1} \star d_{k+1}) \cap (F(s_{k+1}) \star s_{k+1}) \\ &= \big(\langle \mathbf{u}_{k+1} + f(\mathbf{s}_{k+1}) - A_k \mathbf{s}_{k+1} \rangle \star \langle \mathbf{u}_{k+1} + \Phi(f(\mathbf{s}_{k+1}), \mathbf{s}_{k+1}) \mathbf{s}_{k+1} \rangle \big) \\ &\quad \cap \big(\langle f(\mathbf{s}_{k+1}) \rangle \star \langle \mathbf{s}_{k+1} \rangle \big) \\ &= \langle f(\mathbf{s}_{k+1}) - A_k \mathbf{s}_{k+1} - \Phi(f(\mathbf{s}_{k+1}), \mathbf{s}_{k+1}) \mathbf{s}_{k+1} \rangle \\ &= \langle f(\mathbf{s}_{k+1}) - A_{k+1} \mathbf{s}_{k+1} \rangle \end{aligned}$$

By definition $tr(f) = A_n$, so $t_n = \langle f(\mathbf{s}_n) - tr(f)\mathbf{s}_n \rangle$. ⊣

3.3.8. Remark. This proposition implies that, in a Pappian quasi-quantum Kripke frame, the choice of a_i and b_i for $i = 1, ..., n$ has no effect on the trace of a continuous endo-homomorphism.

3.3.3 Three Assumptions

In this subsection, I introduce three assumptions on two quasi-quantum Kripke frames, and investigate their consequences. These three assumptions will be assumed until the end of this chapter.

For convenience, throughout this subsection I fix two quasi-quantum Kripke frames $\mathfrak{F}_1 = (\Sigma_1, \to_1)$ and $\mathfrak{F}_2 = (\Sigma_2, \to_2)$.

Assumption 1: Both $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are Pappian and finite-dimensional.

First, remember that Pappian quasi-quantum Kripke frames are defined in Definition 2.7.11, and finite-dimensional geometric frames in Definition 2.4.14.

Second, since both $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are finite-dimensional, by Theorem 2.4.19 they are quantum Kripke frames.

Third, since $\mathfrak{F}_1$ is a Pappian quantum Kripke frame, by Proposition 2.7.13 there is a vector space V_1 of dimension as least 3 over some *field* $\mathcal{K}_1$ equipped with an anisotropic Hermitian form Φ_1 such that $(\Sigma(V_1), \to_{V_1}) \cong \mathfrak{F}_1$. Moreover, since $\mathfrak{F}_1$ is finite-dimensional, V_1 is finite-dimensional and Φ_1 is orthomodular by Lemma B.2.15. Finally, the same holds for $\mathfrak{F}_2$, i.e. there is a vector space V_2 of finite-dimension as least 3 over some *field* $\mathcal{K}_2$ equipped with an orthomodular Hermitian form Φ_2 such that $(\Sigma(V_2), \to_{V_2}) \cong \mathfrak{F}_2$.

Assumption 2: Both $\mathfrak{F}_1$ and $\mathfrak{F}_2$ have Property OHC.

Remember that Property OHC is defined immediately after Definition 2.7.6. Since $\mathfrak{F}_1$ has Property OHC, by Lemma 2.7.8 for any $\mathbf{s}_1, \mathbf{t}_1 \in V_1 \setminus \{\mathbf{0}_1\}$ there is an x in $\mathcal{K}_1$ such that $\Phi(\mathbf{s}_1, \mathbf{s}_1) = \Phi(x\mathbf{t}_1, x\mathbf{t}_1)$. I suppose that $\Phi_1(\mathbf{v}_1, \mathbf{v}_1) = 1$ for some $\mathbf{v}_1 \in V_1$. Then, for every $\mathbf{s}_1 \in V_1 \setminus \{\mathbf{0}_1\}$, there is an x in $\mathcal{K}_1$ such that $\Phi_1(x\mathbf{s}_1, x\mathbf{s}_1) = 1$, i.e. V_1 admits normalization (Definition C.2.4). This supposition is reasonable for the following reason[8]: Given Φ_1 take a $\mathbf{v}_1 \in V_1 \setminus \{\mathbf{0}_1\}$ and define $\Psi_1 = \Phi_1(\mathbf{v}_1, \mathbf{v}_1)^{-1}\Phi_1$. Then it is easy to verify that:

1. $\Psi_1(\mathbf{v}_1, \mathbf{v}_1) = \Phi_1(\mathbf{v}_1, \mathbf{v}_1)^{-1} \cdot \Phi_1(\mathbf{v}_1, \mathbf{v}_1) = 1$;

2. Ψ_1 is also an orthomodular Hermitian form on V_1, because Φ_1 is an orthomodular Hermitian form and $\mathcal{K}_1$ is a field;

3. equipped with Ψ_1 instead of Φ_1, V_1 induces the same quasi-quantum Kripke frame, because $\Psi_1(\mathbf{s}_1, \mathbf{t}_1) = 0 \Leftrightarrow \Phi_1(\mathbf{v}_1, \mathbf{v}_1)^{-1} \cdot \Phi_1(\mathbf{s}_1, \mathbf{t}_1) = 0 \Leftrightarrow \Phi_1(\mathbf{s}_1, \mathbf{t}_1) = 0$, for any $\mathbf{s}_1, \mathbf{t}_1 \in V_1$.

Therefore, I can always replace Φ_1 by such a Ψ_1, so the supposition is reasonable. The same comments apply equally well to $\mathfrak{F}_2$, and thus V_2 also admits normalization.

> **Assumption 3:** There is a continuous homomorphism G from $\mathfrak{F}_1$ to $\mathfrak{F}_2$ whose image contains an orthogonal set of cardinality 3.

This means that G is non-degenerate, and thus arguesian. By Corollary 3.1.14 there is a continuous quasi-linear map $g : V_1 \to V_2$ such that $\mathcal{P}(g) = G$. Since g is a quasi-linear map, by definition it has an accompanying field isomorphism $\sigma : \mathcal{K}_1 \to \mathcal{K}_2$. In the following, usually I assume without loss of generality and for simplicity that $\mathcal{K}_1 = \mathcal{K}_2$ and σ is the identity map.

These three assumptions are important for the technical results that I am going to prove in the following subsection.

Before ending this subsection, I show a close connection between the continuous homomorphisms between finite-dimensional quasi-quantum Kripke frames and the homomorphisms between projective geometries so that the results in the previous section about the latter can be used in the study of the former.

3.3.9. Proposition. *Let $\mathfrak{F}_1 = (\Sigma_1, \to_1)$ and $\mathfrak{F}_2 = (\Sigma_2, \to_2)$ be quasi-quantum Kripke frames such that $\mathfrak{F}_1$ is* finite-dimensional. *For every partial map $F : \Sigma_1 \dashrightarrow \Sigma_2$, the following are equivalent:*

(i) F is a continuous homomorphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$;

(ii) F is a homomorphism from $\mathbf{G}(\mathfrak{F}_1)$ to $\mathbf{G}(\mathfrak{F}_2)$;

[8]This idea is from the second paragraph below the theorem on p 207 of [55].

(iii) F is a continuous homomorphism from $\mathbf{G}(\mathfrak{F}_1)$ to $\mathbf{G}(\mathfrak{F}_2)$.

Proof. **From (i) to (ii):** This follows from Corollary 3.1.11 and Definition B.1.25.

From (ii) to (iii): It suffices to show that F satisfies (CON). Assume that $A_2 \subseteq \Sigma_2$ satisfies $A_2 = A_2^{\perp\perp}$. Using Corollary 2.3.18 and Lemma 2.3.20 it is easy to show that A_2 is a subspace of $\mathbf{G}(\mathfrak{F}_2)$. Since F is a homomorphism, by the definition and Lemma B.1.23 $Ker(F) \cup F^{-1}[A_2]$ is a subspace of $\mathbf{G}(\mathfrak{F}_1)$. Since $\mathfrak{F}_1$ is finite-dimensional, using Corollary 2.3.18 and Lemma 2.4.17 it is not hard to show that $Ker(F) \cup F^{-1}[A_2] = (Ker(F) \cup F^{-1}[A_2])^{\perp\perp}$. Therefore, F satisfies (CON). By definition F is a continuous homomorphism from $\mathbf{G}(\mathfrak{F}_1)$ to $\mathbf{G}(\mathfrak{F}_2)$.

From (iii) to (i): This follows from Corollary 3.1.11. ⊣

3.3.4 (Non-)Orthogonality between Continuous Homomorphisms

In this subsection, I define a non-orthogonality relation between continuous homomorphisms. As a small remark about the notations, since two quasi-quantum Kripke frames will be involved, I use the subscripts $_1$ and $_2$ to distinguish between them, and use superscripts to index the objects in the same Kripke frame.

3.3.10. Definition. Let $\mathfrak{F}_1 = (\Sigma_1, \to_1)$ and $\mathfrak{F}_2 = (\Sigma_2, \to_2)$ be two quasi-quantum Kripke frames. Two continuous homomorphisms F and G from $\mathfrak{F}_1$ to $\mathfrak{F}_2$ are *non-orthogonal*, denoted by $F \to G$, if both of the following hold:

1. $F \approx G$;
2. $F^\dagger \circ G = Id$, or there is an orthogonal basis $\{s_1^1, ..., s_1^n\}$ of Σ_1 and a set of rulers $\{(F^\dagger \circ G(s_1^i), s_1^i, r_1^i) \mid i = 1, .., n\}$ such that the rulers are uniformly scaled by $F^\dagger \circ G$ and some $w \in \Sigma_1$ and $t_1^n \neq F^\dagger \circ G(s_1^n)$, where t_1^n is the last element in the trace $(t_1^1, ..., t_1^n)$ of $F^\dagger \circ G$ with respect to this set of rulers.

The following lemma shows the close relation between the non-orthogonality of two continuous homomorphisms and the traces of linear maps.

3.3.11. Lemma. *Suppose that V_1 and V_2 are two vector spaces over two* fields *$\mathcal{K}_1 = (K_1, +, \cdot, 0, 1)$ and $\mathcal{K}_2 = (K_2, +, \cdot, 0, 1)$ equipped with anisotropic Hermitian forms Φ_1 and Φ_2, respectively. Moreover, both are of finite dimension at least 3 and admit normalization. Let $h : V_1 \to V_2$ be a quasi-linear map such that $h[V_1]$ is at least 2-dimensional, and σ an accompanying field isomorphism of h. For any two arguesian continuous homomorphisms F and G from $(\Sigma(V_1), \to_{V_1})$ to $(\Sigma(V_2), \to_{V_2})$ such that $F \approx \mathcal{P}(h)$ and $G \approx \mathcal{P}(h)$, the following are equivalent:*

(i) $F \to G$;

(ii) $tr(f^\dagger \circ g) \neq 0$*, for any two quasi-linear maps* $f, g : V_1 \to V_2$ *such that* σ *is an accompanying field isomorphism of both of them,* $F = \mathcal{P}(f)$ *and* $G = \mathcal{P}(g)$*.*

Proof. I start with some useful observations.

Denote the dimension of V_1 by n. According to Proposition B.2.10, V_1 has an orthogonal basis. Since V_1 admits normalization, V_1 has an orthonormal basis. Therefore, the notion of traces is well-defined for linear maps on V_1.

Since both V_1 and V_2 are finite-dimensional and over fields, by Proposition C.1.1 every linear map from V_1 to V_2 is continuous, and thus has an adjoint.

Moreover, Since V_1 is a vector space of dimension at least 3 over a field $\mathcal{K}_1$ equipped with an anisotropic Hermitian form, by Proposition B.2.11 $\mathcal{K}_1$ is an infinite field. By Proposition C.4.2, for every linear map on V_1 which is not a constant multiple of id_{V_1}, there is an orthogonal basis of V_1 which doe not involve any eigenvectors of this linear map.

From (i) to (ii): Let $f, g : V_1 \to V_2$ be two quasi-linear maps such that σ is an accompanying field isomorphism of both of them, $F = \mathcal{P}(f)$ and $G = \mathcal{P}(g)$. By Lemma B.2.20 σ^{-1} is an accompanying field isomorphism of $f^\dagger$. It follows that $f^\dagger \circ g$ is a linear map on V_1. Consider two cases.

Case 1: $f^\dagger \circ g = x\ id_{V_1}$ for some $x \in K_1$. Then $x \neq 0$; otherwise, $F^\dagger \circ G = (\mathcal{P}(f))^\dagger \circ \mathcal{P}(g) = \mathcal{P}(f^\dagger \circ g)$ is the empty function, which is not Id but could not have a set of rulers uniformly scaled by it, contradicting $F \to G$. Hence $tr(f^\dagger \circ g) = tr(x\ id_{V_1}) = n \cdot x \neq 0$.

Case 2: $f^\dagger \circ g$ is not a constant multiple of id_{V_1}. Then there is an orthogonal basis $\{\mathbf{v}_1^1, ..., \mathbf{v}_1^n\}$ of V_1 which does not involve any eigenvector of $f^\dagger \circ g$. Since V_1 admits normalization, I assume that $\{\mathbf{v}_1^1, ..., \mathbf{v}_1^n\}$ is an orthonormal basis. By Lemma 3.3.4 $\{(F^\dagger \circ G(\langle \mathbf{v}_1^i \rangle), \langle \mathbf{v}_1^i \rangle, \langle f^\dagger \circ g(\mathbf{v}_1^i) + \mathbf{v}_1^i \rangle) \mid i = 1, ..., n\}$, which is the same as $\{(\langle f^\dagger \circ g(\mathbf{v}_1^i) \rangle, \langle \mathbf{v}_1^i \rangle, \langle f^\dagger \circ g(\mathbf{v}_1^i) + \mathbf{v}_1^i \rangle) \mid i = 1, ..., n\}$, is a set of rulers uniformly scaled by $F^\dagger \circ G$ and $\langle \sum_{i=1}^n \mathbf{v}_1^i \rangle$. Let $(t_1^1, ..., t_1^n)$ be the trace of $F^\dagger \circ G$ with respect to this set of rulers. By Proposition 3.3.7 $t_1^n = \langle f^\dagger \circ g(\mathbf{v}_1^n) + tr(f^\dagger \circ g)\mathbf{v}_1^n \rangle$. Since $F \to G$, $t_1^n \neq \langle f^\dagger \circ g(\mathbf{v}_1^n) \rangle$. Therefore, $tr(f^\dagger \circ g) \neq 0$.

From (ii) to (i): Since $F \approx \mathcal{P}(h)$, by Theorem 3.2.13 there is a quasi-linear map $f : V_1 \to V_2$ such that $F = \mathcal{P}(f)$ and σ is an accompanying field isomorphism. Similarly, there is a quasi-linear map $g : V_1 \to V_2$ such that $G = \mathcal{P}(g)$ and σ is an accompanying field isomorphism. By (ii) $tr(f^\dagger \circ g) \neq 0$. By Proposition 3.2.11 $F \approx G$. Consider two cases.

Case 1: $f^\dagger \circ g = x\ id_{V_1}$ for some $x \in K_1$. Since $tr(f^\dagger \circ g) \neq 0$, $x \neq 0$. Then $F^\dagger \circ G = (\mathcal{P}(f))^\dagger \circ \mathcal{P}(g) = \mathcal{P}(f^\dagger \circ g) = \mathcal{P}(x\ id_{V_1}) = Id$. Hence $F \to G$.

Case 2: $f^\dagger \circ g$ is not a constant multiple of id_{V_1}. Then there is an orthogonal basis $\{\mathbf{v}_1^1, ..., \mathbf{v}_1^n\}$ of V_1 which does not involve any eigenvector of $f^\dagger \circ g$. Since V_1 admits normalization, I assume that $\{\mathbf{v}_1^1, ..., \mathbf{v}_1^n\}$ is an orthonormal basis. By Lemma 3.3.4 $\{(F^\dagger \circ G(\langle \mathbf{v}_1^i \rangle), \langle \mathbf{v}_1^i \rangle, \langle f^\dagger \circ g(\mathbf{v}_1^i) + \mathbf{v}_1^i \rangle) \mid i = 1, ..., n\}$, which is the same as $\{(\langle f^\dagger \circ g(\mathbf{v}_1^i) \rangle, \langle \mathbf{v}_1^i \rangle, \langle f^\dagger \circ g(\mathbf{v}_1^i) + \mathbf{v}_1^i \rangle) \mid i = 1, ..., n\}$, is a set of rulers uniformly scaled by $F^\dagger \circ G$ and $\langle \sum_{i=1}^n \mathbf{v}_1^i \rangle$. Let $(t_1^1, ..., t_1^n)$ be the trace of $F^\dagger \circ G$ with respect to this set of rulers. By Proposition 3.3.7 $t_1^n = \langle f^\dagger \circ g(\mathbf{v}_1^n) + tr(f^\dagger \circ g)\mathbf{v}_1^n \rangle$. Since $tr(f^\dagger \circ g) \neq 0$, $t_1^n \neq \langle f^\dagger \circ g(\mathbf{v}_1^n) \rangle$, so $F \rightarrow G$. ⊣

Finally I show that two quasi-quantum Kripke frames satisfying Assumptions 1 to 3 in the previous subsection can be amalgamated to one.

3.3.12. THEOREM. *Let $\mathfrak{F}_1 = (\Sigma_1, \rightarrow_1)$ and $\mathfrak{F}_2 = (\Sigma_2, \rightarrow_2)$ be two quasi-quantum Kripke frames satisfying Assumptions 1, 2 and 3, and H is the non-degenerate continuous homomorphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$ granted by Assumption 3. $(\Sigma_1 \otimes_H \Sigma_2, \rightarrow)$ is a quasi-quantum Kripke frame, where $\Sigma_1 \otimes_H \Sigma_2$ is defined to be*

$$\{F : \Sigma_1 \dashrightarrow \Sigma_2 \mid F \text{ is a non-empty arguesian continuous homomorphism, } F \leftrightharpoons H\},$$

and $\rightarrow$ is defined as in Definition 3.3.10.

Proof. Since $\mathfrak{F}_1$ and $\mathfrak{F}_2$ satisfy Assumptions 1 and 2 in the previous subsection, by the discussion there there are two vector spaces V_1 and V_2 both of finite dimension at least 3 over two fields $\mathcal{K}_1$ and $\mathcal{K}_2$ equipped with two anisotropic Hermitian forms Φ_1 and Φ_2, respectively, such that both Φ_1 and Φ_2 are chosen in such a way that both V_1 and V_2 admits normalization, $\mathfrak{F}_1 \cong (\Sigma(V_1), \rightarrow_1)$ and $\mathfrak{F}_2 \cong (\Sigma(V_2), \rightarrow_2)$. Without loss of generality, I identify $\mathfrak{F}_1$ and $\mathfrak{F}_2$ with $(\Sigma(V_1), \rightarrow_1)$ and $(\Sigma(V_2), \rightarrow_2)$, respectively.

Moreover, by Assumption 3 H is a non-degenerate continuous homomorphism from $(\Sigma(V_1), \rightarrow_1)$ to $(\Sigma(V_2), \rightarrow_2)$, by Corollary 3.1.14 there is a continuous quasi-linear map $h : V_1 \rightarrow V_2$ such that $H = \mathcal{P}(h)$. According to the discussion in Subsection 3.2.3, without loss of generality, I identify $\mathcal{K}_1$ with $\mathcal{K}_2$ (and thus drop the subscripts below), and assume that h is a linear map. Now, on the one hand, by Proposition 3.3.9 continuous homomorphisms between finite-dimensional quasi-quantum Kripke frames are the same as homomorphisms between the corresponding pure orthogeometries; on the other hand, by Proposition C.1.1 linear maps between two finite-dimensional vector spaces over fields are all continuous. Hence Corollary 3.2.16 gives a bijection $\mathbf{I}_H : \Sigma(V_1) \otimes_H \Sigma(V_2) \rightarrow \Sigma(Hom(V_1, V_2))$.

Define $\twoheadrightarrow \subseteq \Sigma(Hom(V_1, V_2)) \times \Sigma(Hom(V_1, V_2))$ such that, for any $f, g \in Hom(V_1, V_2)$ neither of which is the zero map, $\langle f \rangle \twoheadrightarrow \langle g \rangle$, if and only if $tr(f^\dagger \circ g) \neq 0$. According to Theorem C.3.1 and Theorem C.3.3, $Hom(V_1, V_2)$ is a vector space over $\mathcal{K}$ and $tr((\cdot)^\dagger \circ \cdot)$ is an anisotropic Hermitian form over this vector space. Therefore, $(\Sigma(Hom(V_1, V_2)), \twoheadrightarrow)$ is a quasi-quantum Kripke frame.

Finally, to show that $(\Sigma(V_1) \otimes_H \Sigma(V_2), \rightarrow)$ is a quasi-quantum Kripke frame, I prove that it is isomorphic to $(\Sigma(Hom(V_1, V_2)), \twoheadrightarrow)$ via $\mathbf{I}_H$. Let $F, G \in \Sigma(V_1) \otimes_H \Sigma(V_2)$ be arbitrary. Since both F and G are comrades of $H = \mathcal{P}(h)$, by Lemma 3.3.11 the following are equivalent:

(i) $F \rightarrow G$.

(ii) $tr(f^\dagger \circ g) \neq 0$, for any $f, g \in Hom(V_1, V_2)$ with $F = \mathcal{P}(f)$ and $G = \mathcal{P}(g)$.

Since neither of F and G is the empty function, by the definition of $\twoheadrightarrow$ (ii) is equivalent to the following:

(iii) $\langle f \rangle \twoheadrightarrow \langle g \rangle$, for any $f, g \in Hom(V_1, V_2)$ with $F = \mathcal{P}(f)$ and $G = \mathcal{P}(g)$.

By Corollary 3.2.16 $\mathbf{I}_H(F) = \langle f' \rangle$, for every $f' \in Hom(V_1, V_2)$ with $F = \mathcal{P}(f')$. The same holds for G. Hence (iii) is equivalent to the following:

(iv) $\mathbf{I}_H(F) \twoheadrightarrow \mathbf{I}_H(G)$.

Therefore, $F \to G$, if and only if $\mathbf{I}_H(F) \twoheadrightarrow \mathbf{I}_H(G)$.

As a result, $(\Sigma(V_1) \otimes_H \Sigma(V_2), \to)$ is isomorphic to the quasi-quantum Kripke frame $(\Sigma(Hom(V_1, V_2)), \twoheadrightarrow)$, and thus is a quasi-quantum Kripke frame. $\dashv$

Remember that the tensor product of two vector spaces V_1 and V_2 can be constructed from $Hom(V_1, V_2)$. Therefore, this theorem shows that under some assumptions two quasi-quantum Kripke frames can be amalgamated to one in a way closely related to the tensor product construction of vector spaces. However, it is not completely satisfactory. First, it is only about finite-dimensional quasi-quantum Kripke frames, and it is not clear how to generalize it to the infinite-dimensional case. Second, it is not intuitively clear why some of the assumptions are needed, for example, the property of being Pappian and Assumption 2. Third, even one buys all three assumptions, the way how the amalgamation is defined is too complicated, and thus not satisfactory. The definition of being comrades and that of traces involve very complicated geometric configurations which do not have any operational or physical meaning. A way of amalgamating quantum Kripke frames, which leads to a good understanding of quantum entanglement, is still looked forward to.

Chapter 4

Logics of Quantum Kripke Frames

In the previous two chapters I have shown that quantum Kripke frames are useful in modelling quantum systems and their behaviour. In this chapter, I study the logical aspects of quantum Kripke frames. To be precise, I study the descriptions of quantum Kripke frames in formal languages. Although they are less expressive than natural languages, formal languages are useful because they may highlight some hidden structural features and also facilitate automated reasoning. Therefore, the study in this chapter aims to lay the foundation for automated reasoning about quantum Kripke frames and thus about the quantum phenomena that they model.

Every quantum Kripke frame consists of a set equipped with a binary relation, so there are at least two natural candidates of formal languages to describe them: the first-order language and the modal language.

The first-order language used in this chapter is a predicate language with a binary relation symbol R in its signature to describe the non-orthogonality relations in quantum Kripke frames. The first-order language has quantifications over states but not quantifications over sets of states, sets of sets of states, etc. For this reason, it is called first-order. First-order languages are important because they are expressive enough to formalize the axioms of many important mathematical theories, including set theory, group theory, etc. First-order languages also have many nice properties, like compactness and the Löwenheim-Skolem property. However, the great expressive power and nice properties of first-order languages come at a price: logics in these languages may have high computational complexity or even be undecidable.

To find languages of low computational complexity, people have studied many fragments of first-order languages and languages with the same expressive power. An important kind of them is (propositional) modal languages. In some technical sense they are equivalent in expressive power to a useful fragment of first-order languages. The modal language used in this thesis is a propositional language equipped with, besides conjunction, disjunction and negation, a unary modal operator $\Box$. $\Box$ denotes the universal Kripke modality with respect to the non-orthogonality relations in quantum Kripke frames.[1] Intuitively, if p is a property of a quantum system, $\Box p$ is the property processed by the states which are only non-orthogonal to the states satisfying p. The important feature is that the Kripke modality only involves a restricted quantification, i.e. it is restricted to the states that are non-orthogonal to the current state. The compensation of this loss in expressive power is a considerable decrease in complexity: in this modal language many classes of Kripke frames with important properties can be axiomatized in logics which are decidable and even of low complexity. Therefore, modal logic is widely used in formal philosophy and computer science. For more details on modal logic, please refer to [23].

In this chapter, on the one hand, I show that the first-order theory of quantum Kripke frames is undecidable. Hence, for automated reasoning, it seems infertile to consider formal languages that are more expressive than first-order languages, although the definition of quantum Kripke frames is second-order. On the other hand, I only manage to axiomatize in the modal language state spaces, which are simpler than quantum Kripke frames. This is partly due to the deficiency in expressive power of the language. Hence it does not seem like a good idea to consider formal languages that are even less expressive than modal languages.

This chapter is organized as follows: In the first two sections I study the first-order logic of quantum Kripke frames. In Section 4.1 I give a characterization of the first-order definable, bi-orthogonally closed subsets in quasi-quantum Kripke frames. The techniques are also used to show that quantum Kripke frames are not first-order definable. In Section 4.2 I shows that the first-order theory of quantum Kripke frames is undecidable. In Section 4.3, I consider the modal language and provide sound, strongly complete and decidable axiomatizations of state spaces and of state spaces satisfying Superposition, respectively.

4.1 First-Order Definability in Quantum Kripke Frames

In this section, I tackle the problem of characterizing first-order definable (with parameters), bi-orthogonally closed subsets in quasi-quantum Kripke frames. The techniques developed will also be used to show that quantum Kripke frames are not first-order definable in the class of quasi-quantum Kripke frames.

Observe that in the special case of finite-dimensional quasi-quantum Kripke frames (Definition 2.4.14) first-order definable, bi-orthogonally closed subsets have a simple characterization.

4.1.1. PROPOSITION. *In a finite-dimensional quasi-quantum Kripke frame* $\mathfrak{F} = (\Sigma, \to)$, *for every* $P \subseteq \Sigma$, *the following are equivalent:*

[1] This has been mentioned at the end of Section 2.8.

(i) P is first-order definable and bi-orthogonally closed;

(ii) there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = {\sim}{\sim}\{s_1, ..., s_n\}$.

Proof. **From (i) to (ii):** Assume that P is bi-orthogonally closed. Since $\mathfrak{F}$ is a quasi-quantum Kripke frame, it is a geometric frame. Then, according to Lemma 2.4.17, there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = {\sim}{\sim}\{s_1, ..., s_n\}$.

From (ii) to (i): Assume that there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = {\sim}{\sim}\{s_1, ..., s_n\}$. By Proposition 2.2.1 $P = {\sim}{\sim}\{s_1, ..., s_n\}$ is bi-orthogonally closed. Moreover, P is definable by the following first-order formula, where R is interpreted by $\to$ and $x_1, ..., x_n$ by $s_1, ..., s_n$, respectively:

$$\forall y(xRy \to yRx_1 \vee ... \vee yRx_n)$$

Therefore, P is first-order definable and bi-orthogonally closed. ⊣

One may wonder whether this also holds in the infinite-dimensional case. The bad news is that the answer is no, but the good news is that there is something similar. As it turns out, for the infinite-dimensional case, two things are needed. First, (ii) in the above proposition need to be generalized to finitely presentable subsets (Definition 2.4.14). Second, Property OHC, introduced below Remark 2.7.7 in Subsection 2.7.1, is needed as an additional assumption.

This section is devoted to the details of generalizing the above proposition to the infinite-dimensional case and the applications of this generalization. The organization is as follows: Subsection 4.1.1 is devoted to a study of infinite-dimensional or infinite-codimensional subsets. Automorphisms, which play an important role in determining first-order definable sets, are studied in Subsection 4.1.2. The main theorem is stated and proved in Subsection 4.1.3. In Subsection 4.1.4, as an application of the techniques developed in the previous subsections, quantum Kripke frames are shown not to be first-order definable in the class of quasi-quantum Kripke frames following the idea of Goldblatt's argument in [48].

4.1.1 Infinite-Dimensional or -Codimensional Sets

In this subsection, I study the infinite-dimensional or infinite-codimensional subsets in quasi-quantum Kripke frames and prove three technical lemmas.

The first one shows the desirable result that every infinite-dimensional subspace has an infinite orthogonal subset.

4.1.2. Lemma. *Let $\mathfrak{F} = (\Sigma, \to)$ be a quasi-quantum Kripke frame and $P \subseteq \Sigma$ be an infinite-dimensional subspace.*

1. *If Q is a finite orthogonal subset of P, then there is an $s \in P$ such that $Q \cup \{s\}$ is also an orthogonal subset of P.*

2. *For every $n \in \mathbb{N}$, there is an orthogonal subset of P of cardinality n.*

3. P has an infinite orthogonal subset.

Proof. **For *1*:** Let Q be an arbitrary finite orthogonal subset of P. Then $\sim\sim Q = \mathcal{C}(Q) \subseteq P$ by Proposition 2.3.23. Since P is infinite-dimensional, this inclusion is proper, so there is a $w \in P$ such that $w \notin \sim\sim Q$. If $w \in \sim Q$, then define s to be w, and thus it is obvious that $Q \cup \{s\}$ is an orthogonal subset of P. If $w \notin \sim Q$, since $\mathfrak{F}$ is a geometric frame, by Corollary 2.4.15 and Proposition 2.4.4 there are $w_\| \in \sim\sim Q$ and $w_\perp \in \sim\sim\sim Q = \sim Q$ such that $w \in \sim\sim\{w_\|, w_\perp\}$. Since by Proposition 2.4.6 $w \neq w_\|$, and thus $w_\perp \in \sim\sim\{w, w_\|\}$ by Corollary 2.3.5. Since $w, w_\| \in P$ and P is a subspace, $\sim\sim\{w, w_\|\} \subseteq P$, so $w_\perp \in P$. Define s to be $w_\perp$. From $w_\perp \in P$ and $w_\perp \in \sim Q$, it is easy to conclude that $Q \cup \{s\}$ is an orthogonal subset of P.

For *2*: Observe that $\emptyset$ is an orthogonal subset of P by definition. Using *1* an easy induction shows that one can build a sequence $\{Q_n\}_{n\in\mathbb{N}}$ of orthogonal subsets of P such that Q_n is of cardinality n for each $n \in \mathbb{N}$ and $Q_i \subseteq Q_j$ for any $i, j \in \mathbb{N}$ with $i \leq j$.

For *3*: Take $\bigcup_{n\in\mathbb{N}} Q_n$ of the above chain $\{Q_n\}_{n\in\mathbb{N}}$ of orthogonal subsets of P. It is easy to verify that $\bigcup_{n\in\mathbb{N}} Q_n$ is an infinite orthogonal subset of P. ⊣

The second lemma shows that, for any infinite-codimensional, bi-orthogonally closed subset P and finite-codimensional subset $\sim\{s_1, ..., s_n\}$, there is an element which is in $\sim\{s_1, ..., s_n\}$ but not in P.

4.1.3. Lemma. *Let $\mathfrak{F} = (\Sigma, \to)$ be a quasi-quantum Kripke frame and $P \subseteq \Sigma$ be bi-orthogonally closed and infinite-codimensional. For any $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$, there is a $v \in \sim\{s_1, ..., s_n\}$ such that $v \notin P$.*

Proof. Suppose (towards a contradiction) that $\sim\{s_1, ..., s_n\} \subseteq P$. It follows that $\sim P \subseteq \sim\sim\{s_1, ..., s_n\}$. According to Proposition 2.3.23, $\sim P \subseteq \mathcal{C}(\{s_1, ..., s_n\})$. It follows from Theorem B.1.14 that there are $m \leq n$ and $w_1, ..., w_m \in \Sigma$ such that $\sim P = \mathcal{C}(\{w_1, ..., w_m\})$. By Proposition 2.3.23 again $\sim P = \sim\sim\{w_1, ..., w_m\}$. Then $P = \sim\sim P = \sim\sim\sim\{w_1, ..., w_m\} = \sim\{w_1, ..., w_m\}$, contradicting that P is infinite-codimensional. Therefore, $\sim\{s_1, ..., s_n\} \nsubseteq P$.

It follows that there is a $v \in \sim\{s_1, ..., s_n\}$ such that $v \notin P$. ⊣

The third lemma shows that, roughly speaking, for any infinite-dimensional subspace P and finite-codimensional subset $\sim\{s_1, ..., s_n\}$, there are two elements in $\sim\{s_1, ..., s_n\}$ such that they are orthogonal, one is in P but the other is not.

4.1.4. Lemma. *Let $\mathfrak{F} = (\Sigma, \to)$ be a quasi-quantum Kripke frame, $P \subseteq \Sigma$ be an infinite-dimensional subspace, $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$. If $v \in \sim\{s_1, ..., s_n\}$ and $v \notin P$, then there is a $u \in \sim\{s_1, ..., s_n\}$ such that $u \in P$ and $u \not\to v$.*

Proof. Take an orthogonal subset $\{t_1, ..., t_{n+2}\}$ of P, whose existence is granted by Lemma 4.1.2. Without loss of generality, assume that there is a $k \in \{0, 1, ..., n\}$ such that $s_i \in \sim\{t_1, ..., t_{n+2}\}$ if and only if $i > k$. Then write $s'_1, ..., s'_k$ for the approximations of $s_1, ..., s_k$ in $\sim\sim\{t_1, ..., t_{n+2}\}$, respectively, given by Corollary 2.4.15. Now two cases need to be considered.

- **Case 1:** $v \notin \sim\{t_1, ..., t_{n+2}\}$.

  v has an approximation v' in $\sim\sim\{t_1, ..., t_{n+2}\}$, according to Corollary 2.4.15. By definition $\{s'_1, ..., s'_k, v'\} \subseteq \sim\sim\{t_1, ..., t_{n+2}\}$, and thus $\sim\sim\{s'_1, ..., s'_k, v'\} \subseteq \sim\sim\{t_1, ..., t_{n+2}\}$. By Proposition 2.3.23 $\mathcal{C}(\{s'_1, ..., s'_k, v'\}) \subseteq \mathcal{C}(\{t_1, ..., t_{n+2}\})$. Since $\{t_1, ..., t_{n+2}\}$ is an orthogonal subset larger than $\{s'_1, ..., s'_k, v'\}$, this inclusion must be proper, according to Theorem B.1.14. Hence there is an $x \in \mathcal{C}(\{t_1, ..., t_{n+2}\})$ such that $x \notin \mathcal{C}(\{s'_1, ..., s'_k, v'\})$. By Proposition 2.3.23 $x \in \sim\sim\{t_1, ..., t_{n+2}\}$ and $x \notin \sim\sim\{s'_1, ..., s'_k, v'\}$.

  If $x \in \sim\{s'_1, ..., s'_k, v'\}$, define u to be x.

  If $x \notin \sim\{s'_1, ..., s'_n, v'\}$, by Corollary 2.4.15 and Proposition 2.4.4 there are $x' \in \sim\sim\{s'_1, ..., s'_k, v'\}$ and $x_\perp \in \sim\sim\sim\{s'_1, ..., s'_k, v'\} = \sim\{s'_1, ..., s'_k, v'\}$ such that $x \in \sim\sim\{x', x_\perp\}$. Since $x \neq x'$ by Proposition 2.4.6, $x_\perp \in \sim\sim\{x, x'\}$ follows from $x \in \sim\sim\{x', x_\perp\}$ and Corollary 2.3.5. Note that $x \in \sim\sim\{t_1, ..., t_{n+2}\}$ and $x' \in \sim\sim\{s'_1, ..., s'_k, v'\} \subseteq \sim\sim\{t_1, ..., t_{n+2}\}$. Hence $x, x' \in \sim\sim\{t_1, ..., t_{n+2}\}$, and thus $\sim\sim\{x, x'\} \subseteq \sim\sim\{t_1, ..., t_{n+2}\}$. Hence $x_\perp \in \sim\sim\{t_1, ..., t_{n+2}\}$. Define u to be $x_\perp$.

  In both cases, the chosen u is such that $u \in \sim\sim\{t_1, ..., t_{n+2}\}$ and $u \in \sim\{s'_1, ..., s'_k, v'\}$. Then $u \in \sim\sim\{t_1, ..., t_{n+2}\} = \mathcal{C}(\{t_1, ..., t_{n+2}\}) \subseteq P$, since $\{t_1, ..., t_{n+2}\} \subseteq P$. Moreover, by definition of v' and s'_i for $i \in \{1, ..., k\}$, $u \not\to v$ and $u \not\to s_i$ for $i \in \{1, ..., k\}$. For every $k < i \leq n$, since $s_i \in \sim\{t_1, ..., t_{n+2}\}$ and $u \in \sim\sim\{t_1, ..., t_{n+2}\}$, $u \not\to s_i$. As a result, $u \in P$, $u \in \sim\{s_1, ..., s_n\}$ and $u \not\to v$.

- **Case 2:** $v \in \sim\{t_1, ..., t_{n+2}\}$.

  In this case, the argument is simpler, for I do not need to care about v. Apply an argument similar to the above to the set $\{s'_1, ..., s'_k\}$ instead of $\{s'_1, ..., s'_k, v'\}$. Then one gets a u such that $u \in \sim\sim\{t_1, ..., t_{n+2}\} \subseteq P$ and $u \in \sim\{s'_1, ..., s'_k\}$. Since $v \in \sim\{t_1, ..., t_{n+2}\}$ and $u \in \sim\sim\{t_1, ..., t_{n+2}\}$, $u \not\to v$. Moreover, one can prove that $u \in \sim\{s_1, ..., s_n\}$ by the same argument as the above.

In both cases, I find a $u \in \sim\{s_1, ..., s_n\}$ such that $u \in P$ and $u \not\to v$. ⊣

4.1.2 Automorphisms of a Quasi-Quantum Kripke Frame

In Subsection 2.7.1 (below Remark 2.7.7) I introduce Property OHC on Kripke frames. In this subsection I have a close look at this condition because it has important implications on the existence of automorphisms on a quasi-quantum Kripke frame.

First I show that Property OHC is consistent with and independent of the conditions in the definition of quantum Kripke frames.

4.1.5. Proposition. *There is a quantum Kripke frame satisfying Property OHC, and there is a quantum Kripke frame which does not satisfy Property OHC.*

Proof. For the first part, it is not hard to see that the vector space $\mathbb{R}^3$ equipped with the inner product defined as usual induces a quantum Kripke frame satisfying Property OHC by Corollary 2.5.6 and Remark 2.7.9.

For the second part, I consider the infinite-dimensional vector space V over a division ring $\mathcal{K}$, different from $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, equipped with an orthomodular Hermitian form Φ defined by Keller in [62]. By Corollary 2.5.6 $(\Sigma(V), \to_V)$ is a quantum Kripke frame. Suppose (towards a contradiction) that the quantum Kripke frame $(\Sigma(V), \to)$ satisfies Property OHC. Since V is of dimension at least 3, by Lemma 2.7.8 (iii) in the lemma holds in V. Since V is infinite-dimensional, it is not hard to see that it has an infinite orthogonal sequence. (For example, one may derive that $(\Sigma(V), \to_V)$ is infinite-dimensional and take a sequence of vectors, each of which generates an element in the infinite orthogonal subset given in *3* of Lemma 4.1.2.) Without loss of generality, assume that one vector $\mathbf{v} \in V$ satisfies $\Phi(\mathbf{v}, \mathbf{v}) = 1$. Then an infinite orthonormal sequence can be obtained by normalization, which is made possible by (iii). By Solèr's Theorem $\mathcal{K}$ is one of $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, contradicting that $\mathcal{K}$ is different from $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. As a result, the quantum Kripke frame $(\Sigma(V), \to)$ does not satisfy Property OHC. ⊣

Second, a useful result about the existence of automorphisms can be obtained.

4.1.6. Proposition. *Let $\mathfrak{F} = (\Sigma, \to)$ be a quasi-quantum Kripke frame having an orthogonal set of cardinality* 4. *Then the following are equivalent:*

(i) $\mathfrak{F}$ has Property OHC;

(ii) for any $s, t \in \Sigma$ with $s \not\to t$ there is an automorphism F of $\mathfrak{F}$ such that $F(s) = t$, $F(t) = s$ and F restricted to $\sim\{s, t\}$ is the identity;

(iii) for any $s, t \in \Sigma$ with $s \not\to t$, there is an automorphism F of $\mathfrak{F}$ such that $F(s) = t$ and F restricted to $\sim\sim\{u, v\}$ is the identity for some $u, v \in \Sigma$ with $u \neq v$.[2]

Proof. I use the analytic method. Since $\mathfrak{F}$ has an orthogonal set of cardinality 4, by Corollary 2.5.5 there is a vector space V over some division ring

[2]Note that this is the version in the framework of quantum Kripke frames of the Axiom of Plane Transitivity in [7].

$\mathcal{K} = (K, +, \cdot, 0, 1)$ equipped with an anisotropic Hermitian form Φ such that $(\Sigma(V), \rightarrow_V)$ is isomorphic to $\mathfrak{F}$. Since $\mathfrak{F}$ has an orthogonal set of cardinality 4, it is easy to see that V is of dimension at least 4. For simplicity, I identify $(\Sigma(V), \rightarrow_V)$ with $\mathfrak{F}$.

From (i) to (ii): Let $\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle \in \Sigma(V)$ be such that $\langle \mathbf{s} \rangle \not\rightarrow_V \langle \mathbf{t} \rangle$, where $\mathbf{s}, \mathbf{t} \in V \setminus \{\mathbf{0}\}$. Then $\Phi(\mathbf{s}, \mathbf{t}) = 0$. Since $\mathfrak{F}$ satisfies Property OHC, by Lemma 2.7.8 I assume that $\Phi(\mathbf{s}, \mathbf{s}) = \Phi(\mathbf{t}, \mathbf{t})$ without loss of generality. Using the idea from Gram-Schmidt Theorem, for every $\mathbf{u} \in V$, it is not hard to see that $\mathbf{u}$ can be uniquely decomposed as follows:

$$\begin{aligned}\mathbf{u} = \; & \left(\Phi(\mathbf{u}, \mathbf{s}) \cdot \Phi(\mathbf{s}, \mathbf{s})^{-1}\right) \mathbf{s} \\ & + \left(\Phi(\mathbf{u}, \mathbf{t}) \cdot \Phi(\mathbf{t}, \mathbf{t})^{-1}\right) \mathbf{t} \\ & + (\mathbf{u} - \left(\Phi(\mathbf{u}, \mathbf{s}) \cdot \Phi(\mathbf{s}, \mathbf{s})^{-1}\right) \mathbf{s} - \left(\Phi(\mathbf{u}, \mathbf{t}) \cdot \Phi(\mathbf{t}, \mathbf{t})^{-1}\right) \mathbf{t})\end{aligned}$$

with the last component $\mathbf{u} - (\Phi(\mathbf{u}, \mathbf{s}) \cdot \Phi(\mathbf{s}, \mathbf{s})^{-1})\, \mathbf{s} - (\Phi(\mathbf{u}, \mathbf{t}) \cdot \Phi(\mathbf{t}, \mathbf{t})^{-1})\, \mathbf{t}$ in the orthocomplement of $\{\mathbf{s}, \mathbf{t}\}$. In the following, for simplicity, I denote $\Phi(\mathbf{u}, \mathbf{s}) \cdot \Phi(\mathbf{s}, \mathbf{s})^{-1}$ by u_s, $\Phi(\mathbf{u}, \mathbf{t}) \cdot \Phi(\mathbf{t}, \mathbf{t})^{-1}$ by u_t and $\mathbf{u} - (\Phi(\mathbf{u}, \mathbf{s}) \cdot \Phi(\mathbf{s}, \mathbf{s})^{-1})\, \mathbf{s} - (\Phi(\mathbf{u}, \mathbf{t}) \cdot \Phi(\mathbf{t}, \mathbf{t})^{-1})\, \mathbf{t}$ by $\mathbf{u}_\perp$. Hence under this notation $\mathbf{u} = u_s \mathbf{s} + u_t \mathbf{t} + \mathbf{u}_\perp$.

Define a map $f : V \to V$ such that every $\mathbf{u} = u_s \mathbf{s} + u_t \mathbf{t} + \mathbf{u}_\perp$ is mapped to $f(\mathbf{u}) = u_s \mathbf{t} + u_t \mathbf{s} + \mathbf{u}_\perp$. It follows from existence and uniqueness of the decomposition that f is a bijection, and linearity of f can be verified easily as follows: let $\mathbf{u} = u_s \mathbf{s} + u_t \mathbf{t} + \mathbf{u}_\perp$ and $\mathbf{v} = v_s \mathbf{s} + v_t \mathbf{t} + \mathbf{v}_\perp$ be arbitrary, then

$$\begin{aligned} f(a\mathbf{u} + b\mathbf{v}) &= f\Big(a(u_s \mathbf{s} + u_t \mathbf{t} + \mathbf{u}_\perp) + b(v_s \mathbf{s} + v_t \mathbf{t} + \mathbf{v}_\perp)\Big) \\ &= f\Big((au_s + bv_s)\mathbf{s} + (au_t + bv_t)\mathbf{t} + (a\mathbf{u}_\perp + b\mathbf{v}_\perp)\Big) \\ &= (au_s + bv_s)\mathbf{t} + (au_t + bv_t)\mathbf{s} + (a\mathbf{u}_\perp + b\mathbf{v}_\perp) \\ &= a(u_s \mathbf{t} + u_t \mathbf{s} + \mathbf{u}_\perp) + b(v_s \mathbf{t} + v_t \mathbf{s} + \mathbf{v}_\perp) \\ &= af(\mathbf{u}) + bf(\mathbf{v}) \end{aligned}$$

Moreover, f preserves Φ, for

$$\begin{aligned} \Phi(f(\mathbf{u}), f(\mathbf{v})) &= \Phi(u_s \mathbf{t} + u_t \mathbf{s} + \mathbf{u}_\perp, v_s \mathbf{t} + v_t \mathbf{s} + \mathbf{v}_\perp) \\ &= \quad u_s \Phi(\mathbf{t}, \mathbf{t}) v_s^* + u_s \Phi(\mathbf{t}, \mathbf{s}) v_t^* + u_s \Phi(\mathbf{t}, \mathbf{v}_\perp) \\ &\quad + u_t \Phi(\mathbf{s}, \mathbf{t}) v_s^* + u_t \Phi(\mathbf{s}, \mathbf{s}) v_t^* + u_t \Phi(\mathbf{s}, \mathbf{v}_\perp) \\ &\quad + \Phi(\mathbf{u}_\perp, \mathbf{t}) v_s^* + \Phi(\mathbf{u}_\perp, \mathbf{s}) v_t^* + \Phi(\mathbf{u}_\perp, \mathbf{v}_\perp) \\ &= \quad u_s \Phi(\mathbf{s}, \mathbf{s}) v_s^* + u_s \Phi(\mathbf{s}, \mathbf{t}) v_t^* + u_s \Phi(\mathbf{s}, \mathbf{v}_\perp) \\ &\quad + u_t \Phi(\mathbf{t}, \mathbf{s}) v_s^* + u_t \Phi(\mathbf{t}, \mathbf{t}) v_t^* + u_t \Phi(\mathbf{t}, \mathbf{v}_\perp) \\ &\quad + \Phi(\mathbf{u}_\perp, \mathbf{s}) v_s^* + \Phi(\mathbf{u}_\perp, \mathbf{t}) v_t^* + \Phi(\mathbf{u}_\perp, \mathbf{v}_\perp) \\ &= \Phi(u_s \mathbf{s} + u_t \mathbf{t} + \mathbf{u}_\perp, v_s \mathbf{s} + v_t \mathbf{t} + \mathbf{v}_\perp) \\ &= \Phi(\mathbf{u}, \mathbf{v}) \end{aligned}$$

Therefore, it is not hard to verify that the map $F : \Sigma(V) \to \Sigma(V)$ defined by $F(\langle \mathbf{u} \rangle) = \langle f(\mathbf{u}) \rangle$ for every $\mathbf{u} \in V \setminus \{\mathbf{0}\}$ is an automorphism of $\mathfrak{F}$. By definition

$$F(\langle \mathbf{s} \rangle) = \langle f(\mathbf{s}) \rangle = \langle \mathbf{t} \rangle\,,$$
$$F(\langle \mathbf{t} \rangle) = \langle f(\mathbf{t}) \rangle = \langle \mathbf{s} \rangle\,,$$
$$F(\langle \mathbf{v} \rangle) = \langle f(\mathbf{v}) \rangle = \langle \mathbf{v} \rangle\,, \text{ for every } \mathbf{v} \neq \mathbf{0} \text{ in the orthocomplement of } \{\mathbf{s}, \mathbf{t}\}.$$

Therefore, F is an automorphism of $\mathfrak{F}$ with the required property.

From (ii) to (iii): It suffices to show that there are $u, v \in {\sim}\{s, t\}$ such that $u \neq v$. Since $\mathfrak{F}$ has an orthogonal set of cardinality 4, ${\sim}{\sim}\{s, t\} \neq \Sigma$. Hence there is a $u' \in \Sigma$ such that $u' \notin {\sim}{\sim}\{s, t\}$. By the proof of Theorem 2.4.13 there is a $u \in {\sim}{\sim}\{s, t, u'\}$ such that $\{s, t, u\}$ is orthogonal. Again since $\mathfrak{F}$ has an orthogonal set of cardinality 4, ${\sim}{\sim}\{s, t, u\} \neq \Sigma$. Hence there is a $v' \in \Sigma$ such that $v' \notin {\sim}{\sim}\{s, t, u\}$. By the proof of Theorem 2.4.13 there is a $v \in {\sim}{\sim}\{s, t, u, v'\}$ such that $\{s, t, u, v\}$ is orthogonal. Therefore, $u, v \in {\sim}\{s, t\}$ and $u \not\to v$, so $u \neq v$.

From (iii) to (i):[3] By Lemma 2.7.8 it suffice to show that, for any $\mathbf{s}, \mathbf{t} \in V \setminus \{\mathbf{0}\}$, $\Phi(\mathbf{s}, \mathbf{t}) = 0$ implies that there is an x in $\mathcal{K}$ such that $\Phi(\mathbf{s}, \mathbf{s}) = \Phi(x\mathbf{t}, x\mathbf{t})$.

Let $\mathbf{s}, \mathbf{t} \in V \setminus \{\mathbf{0}\}$ be arbitrary such that $\Phi(\mathbf{s}, \mathbf{t}) = 0$. Then $\langle \mathbf{s} \rangle \to_V \langle \mathbf{t} \rangle$. By (iii) there is an automorphism F of $\mathfrak{F}$ such that $F(\langle \mathbf{s} \rangle) = \langle \mathbf{t} \rangle$ and F restricted to ${\sim}{\sim}\{\langle \mathbf{u} \rangle, \langle \mathbf{v} \rangle\}$ is the identity for some $\mathbf{u}, \mathbf{v} \in V \setminus \{\mathbf{0}\}$ with $\langle \mathbf{u} \rangle \neq \langle \mathbf{v} \rangle$. Then $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Since F is an automorphism, it is a continuous homomorphism by Proposition 3.1.17 and an orthogonal morphism on $\mathbf{G}(\mathfrak{F})$ (Definition 14.3.1 in [39]). By Theorem 14.3.4 in [39] there is a quasi-linear map $f : V \to V$ such that $F = \mathcal{P}(f)$ and, for some k in $\mathcal{K}$, $\Phi(f(\mathbf{p}), f(\mathbf{q})) = \sigma(\Phi(\mathbf{p}, \mathbf{q}) \cdot k)$, where σ is the accompanying field isomorphism of f.

I claim that there is a $z \in K \setminus \{0\}$ such that $\sigma(y) = z \cdot y \cdot z^{-1}$ for every y in $\mathcal{K}$. Since F restricted to ${\sim}{\sim}\{\langle \mathbf{u} \rangle, \langle \mathbf{v} \rangle\}$ is the identity, $f(\mathbf{p}) \in \langle id(\mathbf{p}) \rangle$ for every $\mathbf{p} \in L(\{\mathbf{u}, \mathbf{v}\})$, where id is the identity map on $L(\{\mathbf{u}, \mathbf{v}\})$. Since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, $L(\{\mathbf{u}, \mathbf{v}\})$, as the image of id, is two-dimensional. By Proposition B.2.18 there is a unique $z \in K \setminus \{0\}$ such that $f(\mathbf{p}) = z\mathbf{p}$ for every $\mathbf{p} \in L(\{\mathbf{u}, \mathbf{v}\})$. Now let y in $\mathcal{K}$ be arbitrary. Consider $\mathbf{u} + y\mathbf{v} \in L(\{\mathbf{u}, \mathbf{v}\})$. On the one hand, $f(\mathbf{u} + y\mathbf{v}) = f(\mathbf{u}) + \sigma(y) f(\mathbf{v}) = z\mathbf{u} + (\sigma(y) \cdot z)\mathbf{v}$. On the other hand, $f(\mathbf{u} + y\mathbf{v}) = z\mathbf{u} + (z \cdot y)\mathbf{v}$. Therefore, $z\mathbf{u} + (\sigma(y) \cdot z)\mathbf{v} = z\mathbf{u} + (z \cdot y)\mathbf{v}$, and thus $\sigma(y) = z \cdot y \cdot z^{-1}$. Since y is arbitrary, $\sigma(y) = z \cdot y \cdot z^{-1}$ for every y in $\mathcal{K}$.

I claim that $k = z^* \cdot z$. On the one hand, $\Phi(f(\mathbf{u}), f(\mathbf{u})) = \sigma(\Phi(\mathbf{u}, \mathbf{u}) \cdot k) = z \cdot \Phi(\mathbf{u}, \mathbf{u}) \cdot k \cdot z^{-1}$. On the other hand, $\Phi(f(\mathbf{u}), f(\mathbf{u})) = \Phi(z\mathbf{u}, z\mathbf{u}) = z \cdot \Phi(\mathbf{u}, \mathbf{u}) \cdot z^*$, where $(\cdot)^*$ is the accompanying involution of Φ. Therefore, $z \cdot \Phi(\mathbf{u}, \mathbf{u}) \cdot z^* = z \cdot \Phi(\mathbf{u}, \mathbf{u}) \cdot k \cdot z^{-1}$, and so $k = z^* \cdot z$.

Since $\langle f(\mathbf{s}) \rangle = F(\langle \mathbf{s} \rangle) = \langle \mathbf{t} \rangle$, there is an $l \in K \setminus \{0\}$ satisfying $f(\mathbf{s}) = l\mathbf{t}$, so $\Phi(l\mathbf{t}, l\mathbf{t}) = \Phi(f(\mathbf{s}), f(\mathbf{s})) = \sigma(\Phi(\mathbf{s}, \mathbf{s}) \cdot k) = z \cdot \Phi(\mathbf{s}, \mathbf{s}) \cdot z^* \cdot z \cdot z^{-1} = z \cdot \Phi(\mathbf{s}, \mathbf{s}) \cdot z^*$

Therefore, taking x to be $z^{-1} \cdot l$, $\Phi(\mathbf{s}, \mathbf{s}) = \Phi(x\mathbf{t}, x\mathbf{t})$. ⊣

[3] This result and its proof are inspired by Proposition 1 in [7].

4.1.7. REMARK. This proposition gives two conditions equivalent to Property OHC on quasi-quantum Kripke frames. Although they involve automorphisms, they seem simpler than Property OHC.

4.1.3 The Main Theorem

In this subsection I state and prove the theorem characterizing the first-order definable, bi-orthogonally closed subsets in quasi-quantum Kripke frames.

4.1.8. THEOREM. *In a quasi-quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$ satisfying Property OHC, for every $P \subseteq \Sigma$, the following are equivalent:*

(i) P is first-order definable and bi-orthogonally closed;

(ii) P is finitely presentable.

Proof. If $\mathfrak{F} = (\Sigma, \to)$ is finite-dimensional, then the conclusion follows from Proposition 4.1.1. In the following, I focus on the case when $\mathfrak{F}$ is infinite-dimensional. Let $P \subseteq \Sigma$ be arbitrary.

From (i) to (ii): I prove the contrapositive. Assume that P is bi-orthogonally closed but not finitely presentable. I need to show that P is not first-order definable. Let $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ be arbitrary. Since P is bi-orthogonally closed, it is a subspace by Lemma 2.3.20. Since P is not finitely presentable, by Lemma 4.1.3 there is a $v \in {\sim}\{s_1, ..., s_n\}$ such that $v \notin P$. By Lemma 4.1.4 there is a $u \in {\sim}\{s_1, ..., s_n\}$ such that $u \in P$ and $u \not\to v$. Since $\mathfrak{F}$ is infinite-dimensional, it has an orthogonal set of cardinality 4 by Lemma 4.1.2. Then by Proposition 4.1.6 there is an automorphism F of $\mathfrak{F}$ such that $F(u) = v$, $F(v) = u$ and F restricted to ${\sim}\{u, v\}$ is the identity. It follows that, for every $i \in \{1, ..., n\}$, $F(s_i) = s_i$, because $s_i \in {\sim}\{u, v\}$. This means that F fixes the set $\{s_1, ..., s_n\}$ pointwise. However, F does not fix P setwise, for $u \in P$ and $F(u) = v \notin P$. Hence P is not definable by any first-order formula with parameters from $\{s_1, ..., s_n\}$, according to a well-known result in model theory (Lemma 2.1.1 in [54], Proposition 1.3.5 in [66]). Since $s_1, ..., s_n$ are arbitrary, P is not first-order definable.

From (ii) to (i): Assume that P is finitely presentable. By definition there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = {\sim}\{s_1, ..., s_n\}$ or $P = {\sim}{\sim}\{s_1, ..., s_n\}$. According to Proposition 2.2.1, P is bi-orthogonally closed. To see that P is first-order definable, two cases need to be considered. If $P = {\sim}\{s_1, ..., s_n\}$, then P can be defined by the following first-order formula with R interpreted by $\to$ and $x_1, ..., x_n$ by $s_1, ..., s_n$, respectively:

$$\neg\,(xRx_1 \vee ... \vee xRx_n)$$

If $P = {\sim}{\sim}\{s_1, ..., s_n\}$, then P can be defined by the following first-order formula with R interpreted by $\to$ and $x_1, ..., x_n$ by $s_1, ..., s_n$, respectively:

$$\forall y(xRy \to yRx_1 \vee ... \vee yRx_n)$$

Therefore, P is first-order definable. $\dashv$

4.1.9. Remark. In a pre-Hilbert space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, two vectors $\mathbf{u}, \mathbf{v}$ are orthogonal, if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. The idea of proving this theorem can be used to show that a set of vectors is a first-order definable (with parameters and in terms of orthogonality only), closed linear subspaces, if and only if it is the orthocomplement of a finite set of vectors or the orthocomplement of the orthocomplement of a finite set of vectors. This is a characterization of the closed linear subspaces which are first-order definable in terms of orthogonality.

To show an interesting consequence of this theorem, I introduce the notion of first-order quantum Kripke frames.

4.1.10. Definition. A *first-order quantum Kripke frame* is a Kripke frame $\mathfrak{F} = (\Sigma, \rightarrow)$ such that $\mathfrak{F} \models T$, where T is a first-order theory in a formal language with exactly one binary relation symbol R consisting of the following formulas:

- **Reflexivity:** $\forall x(xRx)$
- **Symmetry:** $\forall x \forall y(xRy \rightarrow yRx)$
- **Separation:** $\forall x \forall y\big(x \neq y \rightarrow \exists z(zRx \wedge \neg zRy)\big)$
- **Superposition:** $\forall x \forall y \exists z(zRx \wedge zRy)$
- **Schema for Existence of Approximation:**

  For every first-order formula $\varphi(y, \overline{w})$,

$$\forall \overline{w}\Big[BOC(\varphi(y,\overline{w})) \rightarrow \forall x\Big(\exists y(xRy \wedge \varphi(y,\overline{w})) \rightarrow \exists z\big(\varphi(z,\overline{w}) \wedge \forall u(\varphi(u,\overline{w}) \rightarrow (uRz \leftrightarrow uRx))\big)\Big)\Big],$$

  where $\overline{w}$ is a finite tuple of variables and $BOC(\varphi(y, \overline{w}))$ is a formula defined as follows, saying that $\varphi(y, \overline{w})$ defines a bi-orthogonally closed set:

$$\forall x\Big[\forall z\big(xRz \rightarrow \exists u(zRu \wedge \varphi(u,\overline{w}))\big) \rightarrow \varphi(x,\overline{w})\Big]$$

Intuitively, Schema for Existence of Approximation says that every first-order definable, bi-orthogonally closed subset is saturated, while Property A says that every bi-orthogonally closed subset is saturated. Therefore, a quantum Kripke frame is a first-order quantum Kripke frame. In fact, the definition of first-order quantum Kripke frames is obtained by just replacing Property A, which is second-order, in the definition of quantum Kripke frames by a first-order schema. The relation between first-order quantum Kripke frames and quantum Kripke frames is similar to that between first-order arithmetic and Peano arithmetic.

The following corollary can be drawn from the theorem:

4.1.11. Corollary. *For a Kripke frame $\mathfrak{F} = (\Sigma, \to)$ satisfying Property OHC, the following are equivalent:*

(i) $\mathfrak{F}$ is a quasi-quantum Kripke frame;

(ii) $\mathfrak{F}$ is a first-order quantum Kripke frame.

Proof. **From (i) to (ii):** Assume that $\mathfrak{F}$ is a quasi-quantum Kripke frame. By Corollary 2.4.15 every finitely presentable subsets of Σ is saturated. For $\mathfrak{F}$ satisfies Property OHC, by the theorem every first-order definable, bi-orthogonally closed subset is finitely presentable. Hence every first-order definable, bi-orthogonally closed subset is saturated. Therefore, $\mathfrak{F}$ is a first-order quantum Kripke frame.

From (ii) to (i): Assume that $\mathfrak{F}$ is a first-order quantum Kripke frame. It suffices to derive Property AL and Property AH from the schema. Note that the set $\sim\sim\{s,t\}$ for $s, t \in \Sigma$ can be defined by the following formula with R interpreted by $\to$ and x_1, x_2 by s, t, respectively:

$$\forall y\,(xRy \to (yRx_1 \vee yRx_2))$$

and the set $\sim\{s\}$ for $s \in \Sigma$ can be defined by the following formula with R interpreted by $\to$ and y by s, respectively:

$$\neg xRy$$

Then it is not hard to see that Property AL and Property AH follow from the schema. Therefore, $\mathfrak{F}$ is a quasi-quantum Kripke frame. $\dashv$

4.1.12. Remark. Because by definition quasi-quantum Kripke frames are finitely first-order axiomatizable and Property OHC is first-order, according to this corollary, first-order quantum Kripke frames satisfying Property OHC are also finitely first-order axiomatizable.

4.1.4 Application

In this subsection, as an application of the techniques developed above, I show that quantum Kripke frames are not first-order definable in the class of quasi-quantum Kripke frames, employing Goldblatt's idea in [48].

The following proposition generalizes Theorem 1 in [48] in two aspects. First, this proposition is about the quantum Kripke frames satisfying Property OHC, while Theorem 1 in [48] is only about Hilbert spaces, which only induce a specific kind of quantum Kripke frame. Second, this proposition is about the quantum Kripke frames of arbitrary infinite dimension, while Theorem 1 in [48] is only about the Hilbert spaces being separable, i.e. countably infinite-dimensional.

4.1.13. Proposition. *Let $\mathfrak{F} = (\Sigma, \to)$ be a quantum Kripke frame satisfying Property OHC and $P \subseteq \Sigma$ be an infinte-dimensional subspace. If $s_1, ..., s_n \in P$ and $t \in \Sigma$, there exists an automorphism F of $\mathfrak{F}$ such that $F(s_i) = s_i$ for $i = 1, ..., n$ and $F(t) \in P$.*

Proof. By Proposition 2.3.23 $\sim\sim\{s_1, ..., s_n\} = \mathcal{C}(\{s_1, ..., s_n\}) \subseteq P$. Moreover, since $\mathfrak{F}$ has an infinite-dimensional subspace, it follows from Lemma 4.1.2 that it has an orthogonal subset of cardinality 4. Now three cases need to be considered.

Case 1: $t \in P$. Then the identity map on Σ is an automorphism of $\mathfrak{F}$ with the required property.

Case 2: $t \notin P$ and $t \in \sim\{s_1, ..., s_n\}$. By Lemma 4.1.4 there is an $s \in \sim\{s_1, ..., s_n\}$ such that $s \in P$ and $t \not\to s$. Hence $s, t \in \sim\{s_1, ..., s_n\}$ and $t \not\to s$. By Proposition 4.1.6 there is an automorphism F of $\mathfrak{F}$ such that $F(t) = s$, $F(s) = t$ and F restricted to $\sim\{s, t\}$ is the identity. Since $\{s_1, ..., s_n\} \subseteq \sim\sim\{s_1, ..., s_n\} \subseteq \sim\{s, t\}$, $F(s_i) = s_i$ for $i = 1, ..., n$. Moreover, $F(t) = s \in P$. Therefore, F is an automorphism of $\mathfrak{F}$ with the required property.

Case 3: $t \notin P$ and $t \notin \sim\{s_1, ..., s_n\}$. Then $t \notin \sim\sim\sim\{s_1, ..., s_n\}$. Moreover, since $\sim\sim\{s_1, ..., s_n\} \subseteq P$, $t \notin \sim\sim\{s_1, ..., s_n\}$. Hence by Proposition 2.4.4 and Corollary 2.4.15 there are $t_{\|} \in \sim\sim\{s_1, ..., s_n\}$ and $t_{\perp} \in \sim\sim\sim\{s_1, ..., s_n\} = \sim\{s_1, ..., s_n\}$ such that $t \in \sim\sim\{t_{\|}, t_{\perp}\}$. Note that $t_{\perp} \notin P$; otherwise, since P is a subspace, $t_{\|}, t_{\perp} \in P$ implies that $t \in \sim\sim\{t_{\|}, t_{\perp}\} \subseteq P$, contradicting that $t \notin P$. Since $t_{\perp} \notin P$ and $t_{\perp} \in \sim\{s_1, ..., s_n, t_{\|}\}$, by Lemma 4.1.4 there is an $s \in \sim\{s_1, ..., s_n, t_{\|}\}$ such that $s \in P$ and $s \not\to t_{\perp}$. It follows that $s, t_{\perp} \in \sim\{s_1, ..., s_n, t_{\|}\}$ and $t_{\perp} \not\to s$. By Proposition 4.1.6 there is an automorphism F of $\mathfrak{F}$ such that $F(t_{\perp}) = s$, $F(s) = t_{\perp}$ and F restricted to $\sim\{s, t_{\perp}\}$ is the identity. Since $\{s_1, ..., s_n, t_{\|}\} \subseteq \sim\sim\{s_1, ..., s_n, t_{\|}\} \subseteq \sim\{s, t_{\perp}\}$, $F(s_i) = s_i$ for $i = 1, ..., n$ and $F(t_{\|}) = t_{\|}$. Moreover, from $t \in \sim\sim\{t_{\|}, t_{\perp}\}$ it is not hard to verify that $F(t) \in \sim\sim\{F(t_{\|}), F(t_{\perp})\}$. Since $F(t_{\|}) = t_{\|} \in P$, $F(t_{\perp}) = s \in P$ and P is a subspace, $F(t) \in \sim\sim\{F(t_{\|}), F(t_{\perp})\} \subseteq P$. Therefore, F is an automorphism of $\mathfrak{F}$ with the required property. ⊣

The following corollary gives a lot of elementary substructures in an infinite-dimensional quantum Kripke frame satisfying Property OHC.

4.1.14. Corollary. *Let $\mathfrak{F} = (\Sigma, \to)$ be a quantum Kripke frame satisfying Property OHC and $P \subseteq \Sigma$ be an infinite-dimensional subspace. Then $(P, \to)$ is an elementary substructure of $\mathfrak{F}$, where I use the same symbol for the relation $\to$ on Σ and its restriction to P.*

Proof. I use the Tarski-Vaught Test. Let $n \in \mathbb{N}$ be arbitrary and φ an arbitrary first-order formula with exactly one binary relation symbol and at most $n+1$ free variables. Assume that $\mathfrak{F} \models \varphi[s_1, ..., s_n, t]$ for some $s_1, ..., s_n \in P$ and $t \in \Sigma$. By the previous proposition there exists an automorphism F of $\mathfrak{F}$ such that $F(s_i) = s_i$ for $i = 1, ..., n$ and $F(t) \in P$. Hence $\mathfrak{F} \models \varphi[F(s_1), ..., F(s_n), F(t)]$ (Theorem 2.4.3 (c) in [54], the proof of Theorem 1.1.10 in [66]), and thus $\mathfrak{F} \models \varphi[s_1, ..., s_n, F(t)]$. It follows from the Tarski-Vaught Theorem (Theorem 2.5.1 in [54], Proposition 2.3.5 in [66]) that $(P, \to)$ is an elementary substructure of $(\Sigma, \to)$. ⊣

Now I am ready to prove the undefinability result.

4.1.15. THEOREM. *Quantum Kripke frames are not first-order definable in the class of quasi-quantum Kripke frames.*

Proof. I use the same counterexample as in [48]. Let l_2 be the separable Hilbert space of absolutely squared summable sequences of complex numbers, and $l_2^{(0)}$ the pre-Hilbert space of finitely non-zero sequences of complex numbers. It can be proved that $l_2^{(0)}$ is a subspace of l_2, which is infinite-dimensional but not closed. (Please refer to, for example, Example 2.1.13 in [61].) Based on the above, the following can be derived:

1. $(\Sigma(l_2), \rightarrow_{l_2})$ is a quantum Kripke frame by Corollary 2.5.6.
2. $\Sigma(l_2^{(0)})$ is an infinite-dimensional subspace of $(\Sigma(l_2), \rightarrow_{l_2})$ by Lemma B.3.2.
3. $(\Sigma(l_2^{(0)}), \rightarrow_{l_2})$ is a quasi-quantum Kripke frame, according to Corollary 2.5.5; but it is not a quantum Kripke frame, according to the Piron-Amemiya-Araki Theorem in [9].
4. $(\Sigma(l_2), \rightarrow_{l_2})$ satisfies Property OHC, according to Remark 2.7.9.

Now, on the one hand, by 1 and 3 $(\Sigma(l_2), \rightarrow_{l_2})$ is a quantum Kripke frame, and $(\Sigma(l_2^{(0)}), \rightarrow_{l_2})$ is a quasi-quantum Kripke frame but not a quantum Kripke frame. On the other hand, by 2, 4 and the previous corollary $(\Sigma(l_2^{(0)}), \rightarrow_{l_2})$ is an elementary substructure of $(\Sigma(l_2), \rightarrow_{l_2})$. Hence no first-order formula can distinguish between them. $\dashv$

4.2 Undecidability in Quantum Kripke Frames

In this section, I show that the first-order theories of quasi-quantum Kripke frames, quantum Kripke frames and Piron lattices, respectively, are all undecidable.[4] The general strategy is to show that some undecidable first-order theories of fields can be interpreted in the first-order theories of these three kinds of quantum structures.

4.2.1 The Decision Problem for Fields

The undecidablity results will be based on the following theorem of J. Robinson:

4.2.1. THEOREM. *The first-order theory of a class of fields is undecidable, if the field of rational numbers $\mathbb{Q}$ is in this class. ([79])*

[4]The topics investigated in this section are suggested to me by Prof. Johan van Benthem.

By Proposition 2.7.13 every Pappian quasi-quantum Kripke frame $\mathfrak{F}$ is isomorphic to one induced by a vector space over a field equipped with an anisotropic Hermitian form. Moreover, the field is uniquely determined up to isomorphism, which I call *the field associated to* $\mathfrak{F}$ in the following. Moreover, I denote by $\mathcal{F}$ the class of fields each of which is associated to some Pappian quasi-quantum Kripke frame.

I observe the following:

4.2.2. LEMMA. *The first-order theory of $\mathcal{F}$ is undecidable.*

Proof. Given the previous theorem, it suffices to prove that $\mathbb{Q} \in \mathcal{F}$. I equip the vector space $\mathbb{Q}^4$ with an inner product Φ defined in a similar way to that in $\mathbb{R}^4$ as follows: for any $(q_1, q_2, q_3, q_4), (q'_1, q'_2, q'_3, q'_4) \in \mathbb{Q}^4$,

$$\Phi((q_1, q_2, q_3, q_4), (q'_1, q'_2, q'_3, q'_4)) \stackrel{\text{def}}{=} q_1 \cdot q'_1 + q_2 \cdot q'_2 + q_3 \cdot q'_3 + q_4 \cdot q'_4$$

Then it is not hard to show that Φ is an anisotropic Hermitian form on $\mathbb{Q}^4$. By Corollary 2.5.5 $(\mathcal{P}(\mathbb{Q}^4), \to_{\mathbb{Q}^4})$ is a quasi-quantum Kripke frame. Since $\mathbb{Q}$ is a field, by Proposition 2.7.13 $(\mathcal{P}(\mathbb{Q}^4), \to_{\mathbb{Q}^4})$ is Pappian. Hence $\mathbb{Q}$ is the field associated to this Pappian quasi-quantum Kripke frame, and thus $\mathbb{Q} \in \mathcal{F}$. ⊣

4.2.2 The Decision Problem for Quantum Kripke Frames

In this subsection, I show that the first-order theories of quasi-quantum Kripke frames and quantum Kripke frames, respectively, are undecidable. I first show the undecidability of the first-order theory of quasi-quantum Kripke frames, from which the similar result for quantum Kripke frames follows as a corollary. The main task is to interpret the first-order theory of $\mathcal{F}$ to that of quasi-quantum Kripke frames.

To describe quasi-quantum Kripke frames, I use a formal predicate language with exactly one binary relation symbol R. The basic idea for interpreting the first-order theory of $\mathcal{F}^-$ to that of quasi-quantum Kripke frames is again the one of K.G.C. von Staudt: every line in a quasi-quantum Kripke frame, deleting one arbitrary point, has the structure of a division ring. The sums and products are defined in geometric means similar to concatenating line segments and drawing proportional segments in Euclidean geometry. For convenience, I define several abbreviations of first-order formulas in the language with R. The notations and intuitive meanings are as follows:

- $l(x, y, z)$ is a formula with three free variables x, y, z saying that (the denotation of) z is on the line determined by (the denotations of) x and y;
- $XY(o, e, e', x, y)$ is a formula with five free variables o, e, e', x, y saying that the denotations of these variables set an xy-coordinate system with o as the origin, the line ox as the x-axis, the line oy as the y-axis, e as the unit on the x-axis and e' as the unit on the y-axis;

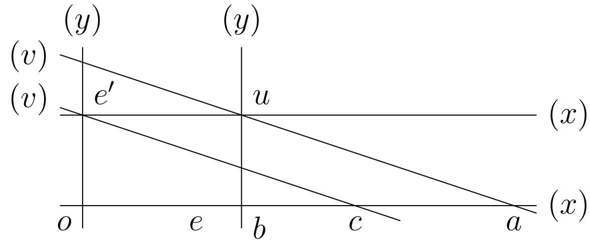
- $X(a)$ is a formula with one free variable a saying that a is on the x-axis of a fixed xy-coordinate system;
- $SUM(a; b, c)$ is a formula with three free variables a, b, c saying that, under the above xy-coordinate, the 'sum' of b and c is a;
- $PROD(a; b, c)$ is a formula with three free variables a, b, c saying that, under the above xy-coordinate system, the 'product' of b and c is a;
- PP is a sentence saying that a quasi-quantum Kripke frame is Pappian.

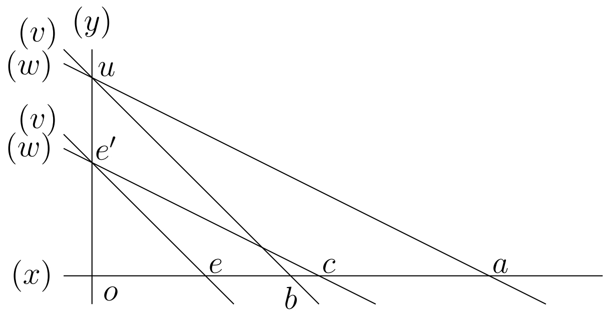
Here comes the formal definition. (To make the notations succinct, I always abbreviate $\neg(x = y)$ to $x \neq y$.)

4.2.3. Definition.

- $l(x, y, z) \stackrel{\text{def}}{=} \forall w(zRw \to wRx \vee wRy)$
- $$\begin{aligned} XY(o, e, e', x, y) \stackrel{\text{def}}{=} \quad & o \neq e \wedge o \neq e' \wedge o \neq x \wedge o \neq y \\ & \wedge e \neq e' \wedge e \neq x \wedge e \neq y \\ & \wedge e' \neq x \wedge e' \neq y \wedge x \neq y \\ & \wedge l(o, e, x) \wedge l(o, e', y) \\ & \wedge \neg l(o, x, y) \end{aligned}$$
- $X(a) \stackrel{\text{def}}{=} a \neq x \wedge l(a, o, x)$
- $SUM(a; b, c) \stackrel{\text{def}}{=} \forall u \forall v \Big(l(u, e', x) \wedge l(u, b, y) \wedge l(v, c, e') \wedge l(x, y, v) \to l(u, v, a) \Big)$
- $PROD(a; b, c)$ is defined as follows:
  $$\begin{aligned} \forall u \forall v \forall w \Big(\; & l(v, e, e') \wedge l(x, y, v) \wedge l(v, u, b) \wedge l(u, o, y) \wedge l(e', c, w) \wedge l(x, y, w) \\ & \to l(u, w, a) \Big) \end{aligned}$$
- PP is defined as follows:
  $$\begin{aligned} & \exists x \exists y \exists z (\neg(xRy) \wedge \neg(yRz) \wedge \neg(zRx)) \\ & \wedge \forall o \forall a \forall b \forall c \forall a' \forall b' \forall c' \forall x \forall y \forall z \\ & \quad \Big(\quad o \neq a \wedge o \neq a' \wedge a \neq a' \wedge \neg l(o, a, a') \\ & \qquad \wedge a \neq b \wedge b \neq c \wedge c \neq a \wedge o \neq b \wedge o \neq c \wedge l(b, o, a) \wedge l(c, o, a) \\ & \qquad \wedge a' \neq b' \wedge b' \neq c' \wedge c' \neq a' \wedge o \neq b' \wedge o \neq c' \wedge l(b', o, a') \wedge l(c', o, a') \\ & \qquad \wedge l(x, a, b') \wedge l(x, b, a') \\ & \qquad \wedge l(y, a, c') \wedge l(y, c, a') \\ & \qquad \wedge l(z, c, b') \wedge l(z, b, c') \\ & \quad \to l(x, y, z) \Big) \end{aligned}$$

The formulas $SUM(a; b, c)$ and $PROD(a; b, c)$ look complicated. The following pictures of the analogues of these constructions in an affine plane may help to make sense of the formulas: (in the picture '(x)' means that x is not a point in the affine plane; instead, it is an imaginary point at infinity where parallel lines intersect.)





The sentence PP also seems complicated, but it is just the definition expressed in a formal language.

The following lemma explains the meaning of these formulas in analytic terms.

4.2.4. Lemma. *Let $\mathbb{F} \in \mathcal{F}$, V a vector space over $\mathbb{F}$ equipped with an anisotropic Hermitian form Φ, $\mathbf{s}, \mathbf{t}, \mathbf{u}, \mathbf{v}, \mathbf{w} \in V \setminus \{\mathbf{0}\}$.*

1. $(\Sigma(V), \rightarrow_V) \models l(x, y, z)[x: \langle \mathbf{u} \rangle, y: \langle \mathbf{v} \rangle, z: \langle \mathbf{w} \rangle]$, *if and only if there are k_1 and k_2 in $\mathbb{F}$ such that $\mathbf{w} = k_1\mathbf{u} + k_2\mathbf{v}$.*

2. $(\Sigma(V), \rightarrow_V) \models XY(o, e, e', x, y)[o: \langle \mathbf{w} \rangle, e: \langle \mathbf{s} \rangle, e': \langle \mathbf{t} \rangle, x: \langle \mathbf{u} \rangle, y: \langle \mathbf{v} \rangle]$, *if and only if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ span a three-dimensional subspace and there are non-zero k_1, k_2, l_1, l_2 in $\mathbb{F}$ such that $\mathbf{s} = k_1\mathbf{w} + k_2\mathbf{u}$ and $\mathbf{t} = l_1\mathbf{w} + l_2\mathbf{v}$.*

Moreover, assuming that

$$(\Sigma(V), \rightarrow_V) \models XY(o, e, e', x, y)[o: \langle \mathbf{w} \rangle, e: \langle \mathbf{w} + \mathbf{u} \rangle, e': \langle \mathbf{w} + \mathbf{v} \rangle, x: \langle \mathbf{u} \rangle, y: \langle \mathbf{v} \rangle]$$

and the assignments of o, e, e', x, y are fixed, the following holds:

1. $(\Sigma(V), \rightarrow_V) \models X(a)[a: \langle \mathbf{a} \rangle]$, *if and only if there is a k in $\mathbb{F}$ such that $\langle \mathbf{a} \rangle = \langle \mathbf{w} + k\mathbf{u} \rangle$;*

2. $(\Sigma(V), \rightarrow_V) \models SUM(a; b, c)[a: \langle \mathbf{w} + \mathrm{a}\mathbf{u} \rangle, b: \langle \mathbf{w} + \mathrm{b}\mathbf{u} \rangle, c: \langle \mathbf{w} + \mathrm{c}\mathbf{u} \rangle]$ *if and only if $\mathrm{a} = \mathrm{b} + \mathrm{c}$, for any $\mathrm{a}, \mathrm{b}, \mathrm{c}$ in $\mathbb{F}$;*

3. $(\Sigma(V), \to_V) \models PROD(a; b, c)[a \colon \langle \mathbf{w} + \mathrm{a}\mathbf{u} \rangle, b \colon \langle \mathbf{w} + \mathrm{b}\mathbf{u} \rangle, c \colon \langle \mathbf{w} + \mathrm{c}\mathbf{u} \rangle]$ *if and only if* $\mathrm{a} = \mathrm{b} \cdot \mathrm{c}$*, for any* $\mathrm{a}, \mathrm{b}, \mathrm{c}$ *in* $\mathbb{F}$.

Proof. Easy Verifications. ⊣

Now I am ready to define the interpretation. I denote the set of all first-order formulas in the language of fields by $\mathcal{L}_f$ and the similar set in the language with a binary relation symbol R by $\mathcal{L}$. The interpretation consists of several steps. First, according to Corollary 2.6.2 in [54], every formula φ in $\mathcal{L}_f$ is logically equivalent to an unnested one, i.e. one without iterations of function symbols. This means that unnested formulas only contain as subformulas the atomic formulas of the form $u = v$, $u = 0$, $v = 1$, $u + v = w$ or $u \cdot v = w$. I denote the set of unnested formulas in the language of fields by $\mathcal{L}_f^-$. Formally, I express this point as follows:

4.2.5. Lemma. *There is a function* $PRE : \mathcal{L}_f \to \mathcal{L}_f^-$ *such that, for every* $\varphi \in \mathcal{L}_f$*, for every structure* $\mathcal{M}$ *in the appropriate signature and every assignment* I *on* $\mathcal{M}$*, the following are equivalent:*

(i) $\mathcal{M}, I \models \varphi$;

(ii) $\mathcal{M}, I \models PRE(\varphi)$,

In particular, when φ *is a sentence,* $\mathcal{M} \models \varphi \Leftrightarrow \mathcal{M} \models PRE(\varphi)$*, for every structure* $\mathcal{M}$ *in the appropriate signature.*

Proof. This follows from Corollary 2.6.2 in [54]. ⊣

Now I only need to focus on how to interpret formulas in $\mathcal{L}_f^-$ into formulas in $\mathcal{L}$. Without loss of generality, I assume that the symbols o, e, e', x, y do not occur in any formula in $\mathcal{L}_f$, and thus also do not occur in $\mathcal{L}_f^-$.

4.2.6. Definition. The translation $T : \mathcal{L}_f^- \to \mathcal{L}$ is defined recursively as follows:

1. $T(x_1 = x_2)$ is $x_1 = x_2$;
2. $T(x_1 = 0)$ is $x_1 = o$;
3. $T(x_1 = 1)$ is $x_1 = e$;
4. $T(x_1 + x_2 = x_3)$ is $SUM(x_3; x_1, x_2)$;
5. $T(x_1 \cdot x_2 = x_3)$ is $PROD(x_3; x_1, x_2)$;
6. $T(\neg\varphi)$ is $\neg T(\varphi)$;
7. $T(\varphi \wedge \psi)$ is $T(\varphi) \wedge T(\psi)$;

8. $T(\forall x_1 \varphi(x_1))$ is $\forall x_1 \Big(X(x_1) \to T\big(\varphi(x_1)\big)\Big)$.

The following lemma shows that T is faithful:

4.2.7. Lemma. *Let $\mathbb{F} \in \mathcal{F}$, V a vector space over $\mathbb{F}$ equipped with an anisotropic Hermitian form Φ, $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ span a 3-dimensional subspace. For every formula $\varphi(x_1, ..., x_n)$ in $\mathcal{L}_f^-$ and every assignment I on $\mathbb{F}$, the following are equivalent:*

(i) $\mathbb{F} \models \varphi[x_1{:}I(x_1), ..., x_n{:}I(x_n)]$;

(ii) $(\Sigma(V), \to_V) \models T(\varphi)[o{:}\, \langle \mathbf{w} \rangle, x{:}\, \langle \mathbf{u} \rangle, e{:}\, \langle \mathbf{w}+\mathbf{u} \rangle, y{:}\, \langle \mathbf{v} \rangle, e'{:}\, \langle \mathbf{w}+\mathbf{v} \rangle,$
$x_1{:}\, \langle \mathbf{w}+I(x_1)\mathbf{u} \rangle, ..., x_n{:}\, \langle \mathbf{w}+I(x_n)\mathbf{u} \rangle]$

Proof. Note that, since $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ span a 3-dimensional subspace and $\mathbb{F}$ is a field, by Proposition 2.7.13 $(\Sigma(V), \to_V)$ is a Pappian quasi-quantum Kripke frame. I use induction on the structure of formula.

For the Base Step, I need to consider 5 cases. The three cases for atomic formulas of the form $x_1 = x_2$, $x_1 = 0$ and $x_1 = 1$ are easy verifications, and the remaining two cases follow from Lemma 4.2.4.

For the Induction Step, I need to consider 3 cases. The two cases for negation and conjunction are routine. I only elaborate the case for universal quantifiers in detail. Consider the formula $\forall y_1 \varphi(y_1, x_1, ..., x_n)$. If y_1 does not occur free in $\varphi(y_1, x_1, ..., x_n)$, then by first-order logic $\forall y_1 \varphi(y_1, x_1, ..., x_n)$ is logically equivalent to $\varphi(y_1, x_1, ..., x_n)$, and thus the required conclusion follows easily from the Induction Hypothesis. In the following, I focus on the case when y_1 occurs free in $\varphi(y_1, x_1, ..., x_n)$

First assume that $\mathbb{F} \models \forall y_1 \varphi[y_1{:}I(y_1), x_1{:}I(x_1), ..., x_n{:}I(x_n)]$. Let $\mathbf{s} \in V \setminus \{\mathbf{0}\}$ be arbitrary such that $(\Sigma(V), \to_V) \models X(y_1)[y_1{:}\, \langle \mathbf{s} \rangle]$. By the definition of the formula X, there is a k in $\mathbb{F}$ such that $\mathbf{s} = \mathbf{w} + k\mathbf{u}$. Let J be an assignment on $\mathbb{F}$ obtained from I by just changing the denotation of y_1 from $I(y_1)$ to k. Then

$$\mathbb{F} \models \varphi\big[y_1{:}k, x_1{:}I(x_1), ..., x_n{:}I(x_n)\big], \tag{4.1}$$

if and only if

$$\mathbb{F} \models \varphi\big[y_1{:}J(y_1), x_1{:}J(x_1), ..., x_n{:}J(x_n)\big]. \tag{4.2}$$

Moreover,

$$\begin{aligned}(\Sigma(V), \to_V) \models T(\varphi)\big[&o{:}\, \langle \mathbf{w} \rangle, x{:}\, \langle \mathbf{u} \rangle, e{:}\, \langle \mathbf{w}+\mathbf{u} \rangle, y{:}\, \langle \mathbf{v} \rangle, e'{:}\, \langle \mathbf{w}+\mathbf{v} \rangle, \\ &y_1{:}\, \langle \mathbf{w}+J(y_1)\mathbf{u} \rangle, x_1{:}\, \langle \mathbf{w}+J(x_1)\mathbf{u} \rangle, ..., x_n{:}\, \langle \mathbf{w}+J(x_n)\mathbf{u} \rangle\big]\end{aligned} \tag{4.3}$$

if and only if

$$\begin{aligned}(\Sigma(V), \to_V) \models T(\varphi)\big[&o{:}\, \langle \mathbf{w} \rangle, x{:}\, \langle \mathbf{u} \rangle, e{:}\, \langle \mathbf{w}+\mathbf{u} \rangle, y{:}\, \langle \mathbf{v} \rangle, e'{:}\, \langle \mathbf{w}+\mathbf{v} \rangle, \\ &y_1{:}\, \langle \mathbf{w}+k\mathbf{u} \rangle, x_1{:}\, \langle \mathbf{w}+I(x_1)\mathbf{u} \rangle, ..., x_n{:}\, \langle \mathbf{w}+I(x_n)\mathbf{u} \rangle\big].\end{aligned} \tag{4.4}$$

By the Induction Hypothesis (4.2) holds if and only if (4.3) holds. By the assumption (4.1) holds, so (4.4) holds by all equivalences mentioned above, i.e.

$$(\Sigma(V), \to_V) \models T(\varphi)\big[o\colon \langle \mathbf{w}\rangle, x\colon \langle \mathbf{u}\rangle, e\colon \langle \mathbf{w}+\mathbf{u}\rangle, y\colon \langle \mathbf{v}\rangle, e'\colon \langle \mathbf{w}+\mathbf{v}\rangle, \\ y_1\colon \langle \mathbf{s}\rangle, x_1\colon \langle \mathbf{w}+I(x_1)\mathbf{u}\rangle, ..., x_n\colon \langle \mathbf{w}+I(x_n)\mathbf{u}\rangle\big]$$

Since $\mathbf{s}$ is arbitrary,

$$(\Sigma(V), \to_V) \models \forall y_1\big(X(y_1) \to T(\varphi)\big)\big[o\colon \langle \mathbf{w}\rangle, x\colon \langle \mathbf{u}\rangle, e\colon \langle \mathbf{w}+\mathbf{u}\rangle, y\colon \langle \mathbf{v}\rangle, e'\colon \langle \mathbf{w}+\mathbf{v}\rangle, \\ x_1\colon \langle \mathbf{w}+I(x_1)\mathbf{u}\rangle, ..., x_n\colon \langle \mathbf{w}+I(x_n)\mathbf{u}\rangle\big],$$

i.e.

$$(\Sigma(V), \to_V) \models T(\forall y_1\varphi)\big[o\colon \langle \mathbf{w}\rangle, x\colon \langle \mathbf{u}\rangle, e\colon \langle \mathbf{w}+\mathbf{u}\rangle, y\colon \langle \mathbf{v}\rangle, e'\colon \langle \mathbf{w}+\mathbf{v}\rangle, \\ x_1\colon \langle \mathbf{w}+I(x_1)\mathbf{u}\rangle, ..., x_n\colon \langle \mathbf{w}+I(x_n)\mathbf{u}\rangle\big].$$

Second assume that

$$(\Sigma(V), \to_V) \models \forall y_1\big(X(y_1) \to T(\varphi)\big)\big[o\colon \langle \mathbf{w}\rangle, x\colon \langle \mathbf{u}\rangle, e\colon \langle \mathbf{w}+\mathbf{u}\rangle, y\colon \langle \mathbf{v}\rangle, e'\colon \langle \mathbf{w}+\mathbf{v}\rangle, \\ x_1\colon \langle \mathbf{w}+I(x_1)\mathbf{u}\rangle, ..., x_n\colon \langle \mathbf{w}+I(x_n)\mathbf{u}\rangle\big],$$

J be an arbitrary assignment on $\mathbb{F}$ which agrees with I on $x_1, ..., x_n$. Then it is not hard to check that $(\Sigma(V), \to_V) \models X(y_1)\big[y_1\colon \langle \mathbf{w}+J(y_1)\mathbf{u}\rangle\big]$. Hence by the assumption,

$$(\Sigma(V), \to_V) \models T(\varphi)\big[o\colon \langle \mathbf{w}\rangle, x\colon \langle \mathbf{u}\rangle, e\colon \langle \mathbf{w}+\mathbf{u}\rangle, y\colon \langle \mathbf{v}\rangle, e'\colon \langle \mathbf{w}+\mathbf{v}\rangle, \\ y_1\colon \langle \mathbf{w}+J(y_1)\mathbf{u}\rangle, x_1\colon \langle \mathbf{w}+I(x_1)\mathbf{u}\rangle, ..., x_n\colon \langle \mathbf{w}+I(x_n)\mathbf{u}\rangle\big]$$

By the definition of J it follows that

$$(\Sigma(V), \to_V) \models T(\varphi)\big[o\colon \langle \mathbf{w}\rangle, x\colon \langle \mathbf{u}\rangle, e\colon \langle \mathbf{w}+\mathbf{u}\rangle, y\colon \langle \mathbf{v}\rangle, e'\colon \langle \mathbf{w}+\mathbf{v}\rangle, \\ y_1\colon \langle \mathbf{w}+J(y_1)\mathbf{u}\rangle, x_1\colon \langle \mathbf{w}+J(x_1)\mathbf{u}\rangle, ..., x_n\colon \langle \mathbf{w}+J(x_n)\mathbf{u}\rangle\big]$$

Applying the Induction Hypothesis, $\mathbb{F} \models \varphi\big[y_1{:}J(y_1), x_1{:}J(x_1), ..., x_n{:}J(x_n)\big]$. Since J is arbitrary, $\mathbb{F} \models \forall y_1\varphi\big[x_1{:}I(x_1), ..., x_n{:}I(x_n)\big]$. ⊣

This lemma helps to prove the following theorem, which gives a faithful translation from sentences in $\mathcal{L}_f$ to those in $\mathcal{L}$.

4.2.8. Theorem. *For every* sentence φ *in* $\mathcal{L}_f$*, the following are equivalent:*

(i) φ *is valid in the class* $\mathcal{F}$*;*

(ii) $PP \to \forall o\forall x\forall y\forall e\forall e'\Big(XY(o,x,y,e,e') \to T \circ PRE(\varphi)\Big)$ *is valid in the class of quasi-quantum Kripke frames;*

(iii) $PP \to \forall o \forall x \forall y \forall e \forall e' \Big(XY(o,x,y,e,e') \to T \circ PRE(\varphi) \Big)$ *is valid in the class of quantum Kripke frames.*

Proof. **From (i) to (ii):** I prove the contrapositive. Assume that the formula $PP \to \forall o \forall x \forall y \forall e \forall e' \Big(XY(o,x,y,e,e') \to T \circ PRE(\varphi) \Big)$ is not valid in the class of all quasi-quantum Kripke frames. Then there is a quasi-quantum Kripke frame $\mathfrak{F}$ such that $\mathfrak{F} \models PP$ but $\mathfrak{F} \not\models \forall o \forall x \forall y \forall e \forall e' \Big(XY(o,x,y,e,e') \to T \circ PRE(\varphi) \Big)$. Since $\mathfrak{F} \models PP$, $\mathfrak{F}$ is Pappian, and thus by Proposition 2.7.13 $\mathfrak{F} \cong (\Sigma(V), \to_V)$ for some vector space V over some field $\mathbb{F}$ equipped with an anisotropic Hermitian form. Then $\mathbb{F} \in \mathcal{F}$. Without loss of generality, I identify $\mathfrak{F}$ with $(\Sigma(V), \to_V)$. Since $\mathfrak{F} \not\models \forall o \forall x \forall y \forall e \forall e' \Big(XY(o,x,y,e,e') \to T \circ PRE(\varphi) \Big)$, it is not hard to show that there are $\mathbf{w}, \mathbf{u}, \mathbf{v} \in V$ such that they span a 3-dimensional subspace and

$$(\Sigma(V), \to_V) \not\models T \circ PRE(\varphi)[o\colon \langle \mathbf{w} \rangle, x\colon \langle \mathbf{u} \rangle, e\colon \langle \mathbf{w}+\mathbf{u} \rangle, y\colon \langle \mathbf{v} \rangle, e'\colon \langle \mathbf{w}+\mathbf{v} \rangle]$$

Then by Lemma 4.2.7 $\mathbb{F} \not\models PRE(\varphi)$, and thus $\mathbb{F} \not\models \varphi$ by Lemma 4.2.5. Since $\mathbb{F} \in \mathcal{F}$, φ is not valid in the class $\mathcal{F}$.

From (ii) to (iii): Trivial.

From (iii) to (i): Again I prove the contrapositive. Assume that φ is not valid in the class $\mathcal{F}$. Then there is an $\mathbb{F} \in \mathcal{F}$ such that $\mathbb{F} \not\models \varphi$. Since $\mathbb{F} \in \mathcal{F}$, it is associated to some Pappian quasi-quantum Kripke frame $\mathfrak{F}$. By Proposition 2.7.13 there is a vector space V, of dimension at least 3, over $\mathbb{F}$ equipped with an anisotropic Hermitian form Φ such that $\mathfrak{F} \cong (\Sigma(V), \to_V)$. Without loss of generality, I identify $\mathfrak{F}$ with $(\Sigma(V), \to_V)$. Hence $(\Sigma(V), \to_V) \models PP$. Since V is at least 3-dimensional, it is not hard to see that there are $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ such that

$$(\Sigma(V), \to_V) \models XY(o,x,y,e,e')[o\colon \langle \mathbf{w} \rangle, x\colon \langle \mathbf{u} \rangle, e\colon \langle \mathbf{w}+\mathbf{u} \rangle, y\colon \langle \mathbf{v} \rangle, e'\colon \langle \mathbf{w}+\mathbf{v} \rangle]$$

However, since $\mathbb{F} \not\models \varphi$, by Lemma 4.2.5 $\mathbb{F} \not\models PRE(\varphi)$. Then by Lemma 4.2.7

$$(\Sigma(V), \to_V) \not\models T \circ PRE(\varphi)[o\colon \langle \mathbf{w} \rangle, x\colon \langle \mathbf{u} \rangle, e\colon \langle \mathbf{w}+\mathbf{u} \rangle, y\colon \langle \mathbf{v} \rangle, e'\colon \langle \mathbf{w}+\mathbf{v} \rangle]$$

Now consider the 3-dimensional vector space $V' = L(\{\mathbf{u}, \mathbf{v}, \mathbf{w}\})$ equipped with the anisotropic Hermitian form Φ' which is the restriction of Φ to $V' \times V'$. By Lemma B.2.15 Φ' is orthomodular. Then $(\Sigma(V'), \to_{V'})$ is a Pappian quantum Kripke frame by Corollary 2.5.6 and Proposition 2.7.13. Hence $(\Sigma(V'), \to_{V'}) \models PP$. Moreover, it is not hard to verify that $(\Sigma(V'), \to_{V'})$ is a substructure of $(\Sigma(V), \to_V)$. Since $XY(o,x,y,e,e')$ is a Π^0_1-formula,

$$(\Sigma(V'), \to'_V) \models XY(o,x,y,e,e')[o\colon \langle \mathbf{w} \rangle, x\colon \langle \mathbf{u} \rangle, e\colon \langle \mathbf{w}+\mathbf{u} \rangle, y\colon \langle \mathbf{v} \rangle, e'\colon \langle \mathbf{w}+\mathbf{v} \rangle]$$

Moreover, by the definition of T every quantifier in $T \circ PRE(\varphi)$ only quantifies over $\Sigma(L(\{\mathbf{u}, \mathbf{w}\})) \subseteq \Sigma(V')$. Therefore,

$$(\Sigma(V'), \to_{V'}) \not\models T \circ PRE(\varphi)[o\colon \langle \mathbf{w} \rangle, x\colon \langle \mathbf{u} \rangle, e\colon \langle \mathbf{w}+\mathbf{u} \rangle, y\colon \langle \mathbf{v} \rangle, e'\colon \langle \mathbf{w}+\mathbf{v} \rangle]$$

As a result, $(\Sigma(V'), \rightarrow_{V'}) \not\models \forall o \forall x \forall y \forall e \forall e' \Big(XY(o, x, y, e, e') \rightarrow T \circ PRE(\varphi) \Big)$. Therefore, the formula $PP \rightarrow \forall o \forall x \forall y \forall e \forall e' \Big(XY(o, x, y, e, e') \rightarrow T \circ PRE(\varphi) \Big)$ is not valid in the class of quantum Kripke frames. ⊣

4.2.9. Corollary. *The first-order theory of quasi-quantum Kripke frames and that of quantum Kripke frames are both undecidable.*

Proof. For the first-order theory of quasi-quantum Kripke frames, suppose (towards a contradiction) that it is decidable. Then there is a mechanism to decide, for every sentence φ in $\mathcal{L}_f$, whether $PP \rightarrow \forall o \forall x \forall y \forall e \forall e' \Big(XY(o, x, y, e, e') \rightarrow T \circ PRE(\varphi) \Big)$ is valid in the class of all quasi-quantum Kripke frames or not. By the equivalence of (i) and (ii) in the previous theorem this means that I can decide whether φ is valid in $\mathcal{F}$ or not, contradicting the undecidability of the first-order theory of $\mathcal{F}$. As a result, the first-order theory of quasi-quantum Kripke frames is undecidable.

Using the equivalence of (i) and (iii) in the previous theorem, undecidability of the first-order theory of quantum Kripke frames can be proved similarly. ⊣

Finally, I show that the first-order theories of two simpler kinds of Kripke frames are also undecidable.

4.2.10. Corollary. *The first-order theory of state spaces and that of state spaces satisfying Superposition are both decidable.*

Proof. For the first-order theory of state spaces satisfying Superposition, suppose (towards a contradiction) that it is decidable. Note that Property AL and Property AH can be expressed as first-order sentences in $\mathcal{L}$. Denote these two sentences by AL and AH, respectively. Also note that by the Deduction Theorem, for every first-order sentence φ in $\mathcal{L}$, it is valid in the class of quasi-quantum Kripke frames, if and only if $AL \wedge AH \rightarrow \varphi$ is valid in the class of state spaces satisfying Superposition. Hence by the supposition the first-order theory of quasi-quantum Kripke frames is decidable, contradicting the previous corollary. Therefore, the first-order theory of state spaces satisfying Superposition is undecidable.

For the first-order theory of state spaces, note that Superposition can be expressed as a first-order sentence SP in $\mathcal{L}$. Also note that by the Deduction Theorem, for every first-order sentence φ in $\mathcal{L}$, it is valid in the class of quasi-quantum Kripke frames, if and only if $AL \wedge AH \wedge SP \rightarrow \varphi$ is valid in the class of state spaces. By an argument similar to the above the undecidability of the first-order theory of state spaces can be proved. ⊣

4.2.3 The Decision Problem for Piron Lattices

In this subsection, I consider Piron lattices[5] and prove that the first-order theory of them is undecidable. I describe Piron lattices in a formal language with one binary relation symbol $\leq$ for the partial order and one unary function symbol $(\cdot)'$ for the orthocomplementation. Moreover, I denote the set of formulas in this language by $\mathcal{L}_{PL}$.

The strategy of proving the undecidability result is the same as that in the previous subsection. I will show that a translation from $\mathcal{L}$ to $\mathcal{L}_{PL}$ can be defined such that the first-order theory of quantum Kripke frames is interpreted in that of Piron lattices. This implies that, if the first-order theory of Piron lattices is decidable, that of quantum Kripke frames will be decidable. Hence the undecidability of the first-order theory of Piron lattices follows from Corollary 4.2.9. Note that the key step is to define the translation from $\mathcal{L}$ to $\mathcal{L}_{PL}$.

4.2.11. THEOREM. *The first-order theory of Piron lattices is undecidable.*

Proof. I start from defining the translation from $\mathcal{L}$ to $\mathcal{L}_{PL}$. Note that there is a first-order formula $atom(a)$ with one free variable a in $\mathcal{L}_{PL}$ defining the atoms of a lattice:

$$atom(a) \stackrel{\text{def}}{=} \forall x\Big(x \neq a \wedge x \leq a \to \forall y(x \leq y)\Big)$$

There is also a first-order formula $NO(x, y)$ with two free variables in $\mathcal{L}_{PL}$ saying that (the denotations of) x and y are non-orthogonal:

$$NO(x, y) \stackrel{\text{def}}{=} \neg(x \leq y')$$

Hence I can define a translation $I : \mathcal{L} \to \mathcal{L}_{PL}$ recursively as follows:

1. $I(x = y)$ is $x = y$;
2. $I(xRy)$ is $NO(x, y)$;
3. $I(\neg\varphi)$ is $\neg I(\varphi)$;
4. $I(\varphi \wedge \psi)$ is $I(\varphi) \wedge I(\psi)$;
5. $I(\forall x\varphi(x))$ is $\forall x\Big(atom(x) \to I(\varphi(x))\Big)$.

According to Theorem 2.7.23, the points in a quantum Kripke frame $\mathfrak{F}$ correspond to the atoms in the corresponding Piron lattice $\mathbf{P} \circ \mathbf{Q}(\mathfrak{F})$. Hence it is not hard to prove that, for every sentence $\varphi \in \mathcal{L}$, the following are equivalent:

(i) φ is valid in the class of quantum Kripke frames;

[5] Please refer to Appendix A for the definition and to Subsection 1.1.1 for their significance in quantum logic.

(ii) $I(\varphi)$ is valid in the class of Piron lattices.

Suppose (towards a contradiction) that the first-order theory of Piron lattices is decidable. Then, for every sentence φ in $\mathcal{L}$, I can decide whether it is valid in the class of quantum Kripke frames by using the hypothetical decision procedure to decide whether $I(\varphi)$ is valid in the class of Piron lattices. This means that the first-order theory of quantum Kripke frames is decidable, contradicting Corollary 4.2.9. As a result, the first-order theory of Piron lattices is undecidable. $\dashv$

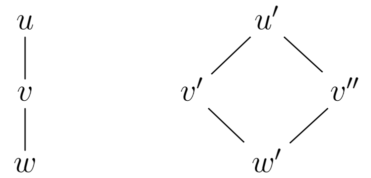
4.3 Modal Logics of State Spaces

In this section I axiomatize in the modal language state spaces and state spaces satisfying Superposition (Subsection 4.3.1 and Subsection 4.3.2). Moreover, I prove that these two axiomatizations are decidable (Subsection 4.3.3).[6]

4.3.1 Axiomatization of State Spaces

In this subsection, I axiomatize state spaces in modal logic.

There are three conditions in the definition of state spaces. For Reflexivity and Symmetry, it is well known that there are modal formulas that are canonical with respect to them. On the contrary, I observe that Separation is not definable by any modal formulas, even in Kripke frames satisfying Reflexivity and Symmetry. The argument is a routine one via bisimulation. Consider the following two Kripke frames satisfying Reflexivity and Symmetry:



In these two graphs, I omit the reflexive arrows, and the edges are not directed because they are symmetric. On the one hand, it is easy to see that these two Kripke frames are bisimular, where the bisimulation is $\{(u, u'), (v, v'), (v, v''), (w, w')\}$. Note that every valuation on the Kripke frame on the left determines a valuation on the Kripke frame on the right such that the two models are bisimular. Then it follows from Theorem 2.20 in [23] that a modal formula is valid in the Kripke frame on the left, if it is valid in the Kripke frame on the right. On the other hand, in the Kripke frame on the left, $u \neq v$, but there is no point s in the frame such that $s \to u$ and $s \not\to v$. Hence the Kripke frame on the left does not satisfy Separation, but it is not hard to verify that the one on the right does. As a result, Separation is not definable by any modal formulas, even in Kripke frames satisfying Reflexivity and Symmetry.

[6]The topics investigated in this section are suggested to me by Prof. Johan van Benthem.

Next, I introduce a construction called mirror union on Kripke frames. It is a slight variant of the disjoint union of a Kripke frame with an isomorphic copy of itself.

4.3.1. DEFINITION. The *mirror union* of a Kripke frame $\mathfrak{F} = (\Sigma, \to)$ is a Kripke frame $M(\mathfrak{F}) = (M(\Sigma), \rightarrowtail)$ such that

1. $M(\Sigma) = \Sigma \times \{0, 1\}$;
2. $\rightarrowtail \subseteq M(\Sigma) \times M(\Sigma)$ consists of four disjoint parts:

$$\begin{aligned} \rightarrowtail \;=\; & \{((s,0),(t,0)) \mid s \to t\} \\ \cup\; & \{((s,1),(t,1)) \mid s \to t\} \\ \cup\; & \{((s,0),(t,1)) \mid s \to t \text{ and } s \neq t\} \\ \cup\; & \{((s,1),(t,0)) \mid s \to t \text{ and } s \neq t\} \end{aligned}$$

The significance of mirror union is shown by the following two propositions. The first one says that this construction turns Kripke frames satisfying Reflexivity and Symmetry into state spaces.

4.3.2. PROPOSITION. *For every Kripke frame $\mathfrak{F} = (\Sigma, \to)$ satisfying Reflexivity and Symmetry, its mirror union $M(\mathfrak{F})$ is a state space.*

Proof. **Reflexivity:** Note that, for each $s \in \Sigma$, by Reflexivity of $\mathfrak{F}$ $s \to s$; and thus by the definition of $M(\mathfrak{F})$ $(s,0) \rightarrowtail (s,0)$ and $(s,1) \rightarrowtail (s,1)$.

Symmetry: Take two arbitrary elements in $M(\Sigma)$ that are related by $\rightarrowtail$. Four cases need to be considered.

Case 1: $(s,0) \rightarrowtail (t,0)$. By definition $s \to t$. By Symmetry of $\mathfrak{F}$ $t \to s$. Hence $(t,0) \rightarrowtail (s,0)$.

Case 2: $(s,1) \rightarrowtail (t,1)$. Symmetric to Case 1.

Case 3: $(s,0) \rightarrowtail (t,1)$. By definition $s \to t$ and $t \neq s$. By Symmetry of $\mathfrak{F}$ $t \to s$. Hence $(t,1) \rightarrowtail (s,0)$ by definition.

Case 4: $(s,1) \rightarrowtail (t,0)$. Symmetric to Case 3.

Separation: Take two arbitrary distinct elements in $M(\Sigma)$. Four cases need to be considered.

Case 1: $(s,0) \neq (t,0)$. It follows that $s \neq t$. Consider two subcases.

(A) $s \not\to t$. By definition $(s,0) \rightarrowtail (s,0)$ and $(s,0) \not\rightarrowtail (t,0)$. Moreover, by Symmetry of $\mathfrak{F}$ $t \not\to s$. Hence $(t,0) \not\rightarrowtail (s,0)$ and $(t,0) \rightarrowtail (t,0)$.

(B) $s \to t$. By definition $(s,1) \not\rightarrowtail (s,0)$ and $(s,1) \rightarrowtail (t,0)$. Moreover, by Symmetry of $\mathfrak{F}$ $t \to s$. Hence $(t,1) \rightarrowtail (s,0)$ and $(t,1) \not\rightarrowtail (t,0)$.

Case 2: $(s,1) \neq (t,1)$. Symmetric to Case 1.

Case 3: $(s,0) \neq (t,1)$. Consider two subcases.

(A) $s = t$. By definition $(s,0) \rightarrowtail (s,0)$ and $(s,0) \not\rightarrowtail (s,1)$. Moreover, $(s,1) \not\rightarrowtail (s,0)$ and $(s,1) \rightarrowtail (s,1)$.

(B) $s \neq t$. Consider two subsubcases.

(a) $s \not\to t$. By definition $(s,0) \rightarrowtail (s,0)$ and $(s,0) \not\rightarrowtail (t,1)$. Moreover, by Symmetry of $\mathfrak{F}$ $t \not\to s$. Hence $(t,1) \not\rightarrowtail (s,0)$ and $(t,1) \rightarrowtail (t,1)$.

(b) $s \to t$. By definition $(s,1) \not\rightarrowtail (s,0)$ and $(s,1) \rightarrowtail (t,1)$. Moreover, by Symmetry of $\mathfrak{F}$ $t \to s$. Hence $(t,0) \rightarrowtail (s,0)$ and $(t,0) \not\rightarrowtail (t,1)$.

Case 4: $(s,1) \neq (t,0)$. Symmetric to Case 3. $\dashv$

The second one shows that $\mathfrak{F}$ and $M(\mathfrak{F})$ are bisimular.

4.3.3. Proposition. *Let* $\mathfrak{F} = (\Sigma, \to)$ *be a Kripke frame. For every* $s \in \Sigma$, $\mathfrak{F}, s \mathrel{\underline{\leftrightarrow}} M(\mathfrak{F}), (s,0)$, *i.e.* s *and* $(s,0)$ *are bisimilar.*

Proof. Define a relation $Z \subseteq \Sigma \times M(\Sigma)$ as follows:

$$Z = \{(s,(s,0)) \mid s \in \Sigma\} \cup \{(s,(s,1)) \mid s \in \Sigma\}$$

I show that Z is a bisimulation. Observe that Z is non-empty.

Forth: Assume that $s \to t$ and sZs'. Consider two cases according to the definition of Z.

(a) s' is $(s,0)$. It is easy to see that $(t,0)$ is such that $tZ(t,0)$ and $(s,0) \rightarrowtail (t,0)$.

(b) s' is $(s,1)$. It is easy to see that $(t,1)$ is such that $tZ(t,1)$ and $(s,1) \rightarrowtail (t,1)$.

Back: Assume that $s' \to t'$ in $M(\mathfrak{F})$ and sZs'. I need to consider four cases.

(a) $s' = (s,0)$ and $t' = (t,0)$.

It is easy to see that $tZ(t,0)$ and $s \to t$ by the definitions.

(b) $s' = (s,0)$ and $t' = (t,1)$.

It is easy to see that $tZ(t,1)$ and $s \to t$ by the definitions.

(c) $s' = (s,1)$ and $t' = (t,1)$. Symmetric to (a).

(d) $s' = (s,1)$ and $t' = (t,0)$. Symmetric to (b).

Therefore, Z is a bisimulation between $\mathfrak{F}$ and $M(\mathfrak{F})$. Since it is obvious that $sZ(s,0)$, $\mathfrak{F}, s \underline{\leftrightarrow} M(\mathfrak{F}), (s,0)$. ⊣

Now I consider the modal logic **B**, i.e. the normal modal logic with axiom T, $\Box p \to p$, and axiom B, $p \to \Box\Diamond p$, as characteristic axioms. It is well known in modal logic that axiom T is canonical with respect to Reflexivity and axiom B to Symmetry, and thus **B** is sound and strongly complete with respect to the class of Kripke frames satisfying Reflexivity and Symmetry[7]. I show that the same modal logic also axiomatizes state spaces.

4.3.4. THEOREM. **B** *is sound and strongly complete with respect to the class of all state spaces.*

Proof. **Soundness:** Since axiom T is canonical with respect to Reflexivity and axiom B to Symmetry, **B** is sound with respect to the class of all Kripke frames satisfying Reflexivity and Symmetry. By definition the class of all state spaces is a subclass of that of all Kripke frames satisfying Reflexivity and Symmetry, so **B** is sound with respect to the class of all state spaces.

Strong Completeness: Let S be an arbitrary **B**-consistent set. By Lindenbaum's Lemma (Lemma 4.17 in [23]) S can be extended to a maximal **B**-consistent set S^+. By the Truth Lemma (Lemma 4.21 in [23]) in the canonical model $\mathfrak{M}^{\mathbf{B}} = (\mathfrak{F}^{\mathbf{B}}, V^{\mathbf{B}}) = (W^{\mathbf{B}}, R^{\mathbf{B}}, V^{\mathbf{B}})$, it holds that $\mathfrak{M}^{\mathbf{B}}, S^+ \Vdash S$. Since axiom T is canonical with respect to Reflexivity and axiom B to Symmetry, $\mathfrak{F}^{\mathbf{B}}$ is a Kripke frame satisfying Reflexivity and Symmetry. Now I consider $M(\mathfrak{F}^{\mathbf{B}})$. By Proposition 4.3.3 $\mathfrak{F}^{\mathbf{B}}, S^+ \underline{\leftrightarrow} M(\mathfrak{F}^{\mathbf{B}}), (S^+, 0)$. I define a valuation V on $M(\mathfrak{F}^{\mathbf{B}})$ such that, for each propositional variable p,

$$V(p) = \{(s,0) \in M(W^{\mathbf{B}}) \mid s \in V^{\mathbf{B}}(p)\} \cup \{(s,1) \in M(W^{\mathbf{B}}) \mid s \in V^{\mathbf{B}}(p)\}$$

It follows easily that $(\mathfrak{F}^{\mathbf{B}}, V^{\mathbf{B}}), S^+ \underline{\leftrightarrow} (M(\mathfrak{F}^{\mathbf{B}}), V), (S^+, 0)$. Since $(\mathfrak{F}^{\mathbf{B}}, V^{\mathbf{B}}), S^+ \Vdash S$, by Theorem 2.20 in [23] $(M(\mathfrak{F}^{\mathbf{B}}), V), (S^+, 0) \Vdash S$. Moreover, by Proposition 4.3.2 $M(\mathfrak{F}^{\mathbf{B}})$ is a state space. Hence S is satisfiable in a state space. As a result, **B** is strongly complete with respect to the class of all state spaces. ⊣

4.3.2 Axiomatization of State Spaces with Superposition

In this subsection, I axiomatize the state spaces satisfying Superposition in modal logic.

I start from studying the modal formula $\Box\Box p \to \Box\Box\Box p$. It is not hard to see that this is a very simple Sahlqvist formula in the sense of Definition 3.41 in [23], and hence, according to Theorem 4.42 in the same book, it is canonical with respect to the following first-order frame condition:

[7]For a proof of this result, please refer to Section 4.3 in [23]. Note that in this book the modal logic **B** is called **KB**.

$$(\clubsuit) \qquad \forall x \forall y \forall u \forall v \exists w. xRu \wedge uRv \wedge vRy \to xRw \wedge wRy$$

This means that x can reach y by the relation R in two steps, whenever x can reach y in three steps. Together with Reflexivity, this frame condition implies that x can reach y by the relation R in two steps, whenever x can reach y in finitely many steps. I consider the modal logic Λ obtained by adding to **B** the formula $\Box\Box p \to \Box\Box\Box p$ as another characteristic axiom. I show that Λ is sound and strongly complete with respect to the class of all state spaces satisfying Superposition.

4.3.5. Proposition. Λ *is sound with respect to the class of all state spaces satisfying Superposition.*

Proof. Given Theorem 4.3.4, it suffices to prove that the formula $\Box\Box p \to \Box\Box\Box p$ is valid in every state space satisfying Superposition.

Let a state space satisfying Superposition $\mathfrak{F} = (\Sigma, \to)$, $s \in \Sigma$ and a valuation V on $\mathfrak{F}$ be all arbitrary such that $(\mathfrak{F}, V), s \Vdash \Box\Box p$. Let $u, v, t \in \Sigma$ also be arbitrary such that $s \to u$, $u \to v$ and $v \to t$. For s and t, by Superposition there is a $w \in \Sigma$ such that $w \to s$ and $w \to t$. Since $\mathfrak{F}$ is a state space, by Symmetry $s \to w$. Since $s \to w$, $w \to t$ and $(\mathfrak{F}, V), s \Vdash \Box\Box p$, $(\mathfrak{F}, V), t \Vdash p$. By arbitrariness of t, $(\mathfrak{F}, V), v \Vdash \Box p$. By arbitrariness of v, $(\mathfrak{F}, V), u \Vdash \Box\Box p$. By arbitrariness of u, $(\mathfrak{F}, V), s \Vdash \Box\Box\Box p$. Therefore, $(\mathfrak{F}, V), s \Vdash \Box\Box p \to \Box\Box\Box p$. As a result, the formula $\Box\Box p \to \Box\Box\Box p$ is valid in every state space satisfying Superposition. $\dashv$

Before proving completeness, I prove a technical lemma. It says that, for a Kripke frame $\mathfrak{F}$ satisfying Reflexivity and Symmetry, if it satisfies Superposition and some special properties, $M(\mathfrak{F})$ will not only be a state space but also satisfy Superposition.

4.3.6. Lemma. *Let $\mathfrak{F} = (\Sigma, \to)$ be a Kripke frame satisfying Reflexivity, Symmetry and Superposition. If Σ is not a singleton and $\mathfrak{F}$ is point-generated, i.e. there is an $r \in \Sigma$ such that, for each $s \in \Sigma$, there are $n \in \mathbb{N}^+$ and $t_1, ..., t_n \in \Sigma$ satisfying $r = t_1 \to t_2 \to ... \to t_n = s$, then $M(\mathfrak{F})$ is a state space satisfying Superposition.*

Proof. Assume that Σ is not a singleton and $\mathfrak{F}$ is point-generated. Since $\mathfrak{F}$ is a Kripke frame satisfying Reflexivity and Symmetry, by Proposition 4.3.2 $M(\mathfrak{F})$ is a state space. It remains to show that it satisfies Superposition. Let s' and t' be two arbitrary elements in $M(\Sigma)$. I need to consider four cases.

Case 1: $s' = (s, 0)$ and $t' = (t, 0)$. Since $\mathfrak{F}$ satisfies Superposition, there is a $w \in \Sigma$ such that $w \to s$ and $w \to t$. It follows that $(w, 0) \rightarrowtail (s, 0)$ and $(w, 0) \rightarrowtail (t, 0)$.

Case 2: $s' = (s, 1)$ and $t' = (t, 1)$. Symmetric to Case 1.

Case 3: $s' = (s, 0)$ and $t' = (t, 1)$. Consider two subcases.

(a) $s \neq t$.

Since $\mathfrak{F}$ satisfies Superposition, there is a $w \in \Sigma$ such that $w \to s$ and $w \to t$. Moreover, $w \neq s$ or $w \neq t$. If $w \neq s$, then $(w, 1) \rightarrowtail (s, 0)$ and $(w, 1) \rightarrowtail (t, 1)$. If $w \neq t$, then $(w, 0) \rightarrowtail (s, 0)$ and $(w, 0) \rightarrowtail (t, 1)$.

(b) $s = t$.

Since Σ is not a singleton, I claim that there is a $w \in \Sigma$ such that $w \neq s$ and $w \to s$. Since $\mathfrak{F}$ is point-generated, I denote this point by r. Then there are $n \in \mathbb{N}^+$ and $u_1, .., u_n \in \Sigma$ such that $r = u_1 \to ... \to u_n = s$. If $r \neq s$, it is not hard to see that $n \geq 2$ and I can always make it so that $u_{n-1} \neq s$. Then $w = u_{n-1}$ has the required property. If $r = s$, since Σ is not a singleton, there are $u \in \Sigma \setminus \{r\}$, $m \in \mathbb{N}^+$ and $u_1, .., u_m \in \Sigma$ such that $r = u_1 \to ... \to u_m = u$. Since $u \neq r$, it is not hard to see that $m \geq 2$ and I can always make it so that $u_2 \neq r$. Then $w = u_2$ has the required property.

Having a w satisfying $w \neq s$ and $w \to s$, it is easy to see that $(w, 0) \rightarrowtail (s, 0)$ and $(w, 0) \rightarrowtail (s, 1) = (t, 1)$.

Case 4: $s' = (s, 1)$ and $t' = (t, 0)$. Symmetric to Case 3. $\dashv$

Now I am ready to prove the completeness of Λ.

4.3.7. Proposition. *Λ is strongly complete with respect to the class of all state spaces satisfying Superposition.*

Proof. Let S be an arbitrary Λ-consistent set of formulas. By Lindenbaum's Lemma (Lemma 4.17 in [23]) S can be extended to a maximal Λ-consistent set S^+. By the Truth Lemma (Lemma 4.21 in [23]) in the canonical model $\mathfrak{M}^\Lambda = (W^\Lambda, R^\Lambda, V^\Lambda)$ it holds that $\mathfrak{M}^\Lambda, S^+ \Vdash S$. Since Λ contains axiom T, axiom B and the formula $\Box\Box\Box p \to \Box\Box p$, which are canonical with respect to Reflexivity, Symmetry and the frame condition (♣), respectively, (W^Λ, R^Λ) is a Kripke frame satisfying Reflexivity, Symmetry and the frame condition (♣). I find a model for S based on a state space satisfying Superposition by transforming the canonical model in several steps.

In the first step, I take the submodel of $\mathfrak{M}^\Lambda$ generated by S^+, denoted by $\mathfrak{M}^\Lambda_r = (W^\Lambda_r, R^\Lambda_r, V^\Lambda_r)$. It satisfies all of the following:

1. $\mathfrak{M}^\Lambda_r, S^+ \Vdash S$;
2. $\mathfrak{F}^\Lambda_r = (W^\Lambda_r, R^\Lambda_r)$ is a Kripke frame satisfying Reflexivity and Symmetry;
3. $\mathfrak{F}^\Lambda_r$ is point-generated with S^+ as a generating point, i.e. for each $T \in W^\Lambda_r$, there are $n \in \mathbb{N}^+$ and $U_1, .., U_n \in \Sigma$ such that $S^+ = U_1 R^\Lambda_r ... R^\Lambda_r U_n = T$;
4. $\mathfrak{F}^\Lambda_r$ satisfies Superposition.

1 follows from $\mathfrak{M}^\Lambda, S^+ \Vdash S$ and Proposition 2.6 in [23]. 2 is easy to verify. 3 follows from the definition of generated submodels.

The proof of 4 is a bit involved. I first claim that $\mathfrak{F}_r^\Lambda$ satisfies the frame condition (♣). Let $X, Y \in W_r^\Lambda$ be arbitrary such that there are $U, V \in W_r^\Lambda$ satisfying $XR_r^\Lambda UR_r^\Lambda VR_r^\Lambda Y$. Then $XR^\Lambda UR^\Lambda VR^\Lambda Y$ holds as well. Since (W^Λ, R^Λ) satisfies the frame condition (♣), there is a $Z \in W^\Lambda$ such that $XR^\Lambda ZR^\Lambda Y$. By the definition of generated submodels $Z \in W_r^\Lambda$ and $XR_r^\Lambda ZR_r^\Lambda Y$. Therefore, $\mathfrak{F}_r^\Lambda$ satisfies the frame condition (♣). Then, given Reflexivity of $\mathfrak{F}_r^\Lambda$, it is not hard to prove by induction that, for any $X, Y \in W_r^\Lambda$, X can reach Y by the relation R_r^Λ in two steps, whenever X can reach Y in finitely many steps.

Now I am ready to show that $\mathfrak{F}_r^\Lambda$ satisfies Superposition. Let $X, Y \in W_r^\Lambda$ be arbitrary. By the above discussion and the definition of generated submodels $(S^+, X), (S^+, Y) \in R_r^\Lambda \circ R_r^\Lambda$. By Symmetry of $\mathfrak{F}_r^\Lambda$ $(X, Y) \in R_r^\Lambda \circ R_r^\Lambda \circ R_r^\Lambda \circ R_r^\Lambda$. By the above discussion $(X, Y) \in R_r^\Lambda \circ R_r^\Lambda$, i.e. there is a $U \in W_r^\Lambda$ such that $XR_r^\Lambda UR_r^\Lambda Y$. By Symmetry of $\mathfrak{F}_r^\Lambda$ $UR_r^\Lambda X$ and $UR_r^\Lambda Y$. As a result, $\mathfrak{F}_r^\Lambda$ satisfies Superposition.

The proof is nearly finished. In the case when $W_r^\Lambda = \{S^+\}$, it is easy to see that $\mathfrak{F}_r^\Lambda$ is a state space satisfying Superposition. Since $\mathfrak{M}_r^\Lambda, S^+ \Vdash S$, S is satisfiable in this state space. In the case when W_r^Λ is not a singleton, one further transformation is needed. By Lemma 4.3.6 $M(\mathfrak{F}_r^\Lambda)$ is a state space satisfying Superposition. Moreover, by Proposition 4.3.3 $\mathfrak{F}_r^\Lambda, S^+ \underline{\leftrightarrow} M(\mathfrak{F}_r^\Lambda), (S^+, 0)$. I define a valuation V on $M(\mathfrak{F}_r^\Lambda)$ such that, for each propositional variable p,

$$V(p) = \{(T, 0) \in M(W_r^\Lambda) \mid T \in V_r^\Lambda(p)\} \cup \{(T, 1) \in M(W_r^\Lambda) \mid T \in V_r^\Lambda(p)\}$$

It follows easily that $(\mathfrak{F}_r^\Lambda, V_r^\Lambda), S^+ \underline{\leftrightarrow} (M(\mathfrak{F}_r^\Lambda), V), (S^+, 0)$. Since $(\mathfrak{F}_r^\Lambda, V_r^\Lambda), S^+ \Vdash S$, by Theorem 2.20 in [23] $(M(\mathfrak{F}_r^\Lambda), V), (S^+, 0) \Vdash S$. Hence S is satisfiable in the state space $M(\mathfrak{F}_r^\Lambda)$ satisfying Superposition.

As a result, Λ is strongly complete with respect to the class of all state spaces satisfying Superposition. ⊣

4.3.3 Decidability

In this subsection, I discuss the decision problem for the modal logics **B** and Λ introduced above. For it is well known that **B** is decidable (Corollary 6.8 in [23]), I concentrate on the modal logic Λ and show that it is decidable using finite models via filtrations.

The following lemma shows that the filtration construction can result in Kripke frames with nice properties.

4.3.8. Lemma. *Let $\mathfrak{M} = (W, R, V)$ be a model such that (W, R) is a point-generated Kripke frame satisfying Reflexivity, Symmetry and Superposition, and Φ be a finite, subformula closed set of formulas. As in Definition 2.36 of [23], denote by W^f the set of equivalence classes induced on $\mathfrak{M}$ by $\leftrightsquigarrow_\Phi$, R^s the smallest filtration relation, i.e.*

$R^s|u||v|$, if and only if $Ru'v'$ for some $u' \in |u|$ and $v' \in |v|$.

Then (W^f, R^s) is a point-generated Kripke frame satisfying Reflexivity, Symmetry and Superposition with $card(W^f) \leq 2^{card(\Phi)}$.[8]

Proof. Since $card(\Phi)$ is finite, by definition $card(W^f) \leq 2^{card(\Phi)}$.

For Reflexivity, let $|w| \in W^f$ be arbitrary. By Reflexivity of (W, R) Rww. Since $w \in |w|$, $R^s|w||w|$.

For Symmetry, let $|u|, |v| \in W^f$ be arbitrary such that $R^s|u||v|$. Then $Ru'v'$ for some $u' \in |u|$ and $v' \in |v|$ by definition. By Symmetry of (W, R) $Rv'u'$. Hence by definition $R^s|v||u|$.

For Superposition, let $|u|, |v| \in W^f$ be arbitrary. Since (W, R) satisfies Superposition, there is a $w \in W$ such that Rwu and Rwv. Hence by definition $R^s|w||u|$ and $R^s|w||v|$.

For point-generatedness, since (W, R) is point-generated, it has a root, which I denote by r. Consider $|r|$. Let $|w| \in W^f$ be arbitrary. Since $w \in W$ and r is a root, there are $n \in \mathbb{N}^+$ and $u_1, ..., u_n \in W$ such that $r = u_1 R ... R u_n = w$. Then $|r| = |u_1| R^s ... R^s |u_n| = |w|$. For $|w|$ is arbitrary, $|r|$ is a root in (W^f, R^s). ⊣

I denote by $\mathcal{M}$ the class of models whose underlying Kripke frames are state spaces satisfying Superposition. The following proposition shows that the satisfiability in $\mathcal{M}$ is equivalent to the satisfiability in the models in $\mathcal{M}$ of some finite bounded size.

4.3.9. Proposition. *For every formula φ, φ is satisfiable in $\mathcal{M}$, if and only if there is a model (W', R', V') in $\mathcal{M}$ such that φ is satisfiable and $card(W') \leq 2^{card(\Phi)+1}$, where Φ is the set of all subformulas of φ.*

Proof. The 'if' direction is trivial, so I focus on the 'only if' direction.

Assume that φ is satisfiable in $\mathcal{M}$. Then there is a model $(W, R, V) \in \mathcal{M}$ and $w \in W$ such that $(W, R, V), w \Vdash \varphi$. By the definition of $\mathcal{M}$ (W, R) is a state space satisfying Superposition. I will transform the model (W, R, V) in several steps so that I end up with a model with the required properties. In each of these steps, I take care of different properties, but please keep in mind that Reflexivity, Symmetry and Superposition will always be preserved, and each property, once established, will be preserved in the following transformations.

Step 1: I take care of point-generatedness.

I take the submodel of (W, R, V) generated by w, and denote this model by (W^-, R^-, V^-). By Proposition 2.6 in [23] $(W^-, R^-, V^-), w \Vdash \varphi$, and it is not hard to see that (W^-, R^-) is a point-generated Kripke frame satisfying Reflexivity, Symmetry and Superposition.[9]

[8]For a set A, $card(A)$ denotes the cardinality of A.

[9]To see that Superposition is preserved under generated submodel, note that if x is the point such that Rxu and Rxv for u and v in the generated submodel, then x will be in the generated submodel as well and thus witnesses Superposition.

Step 2: I take care of finiteness.

Since Φ is subformula closed, I take the filtration of (W^-, R^-, V^-) through Φ. I denote by W^f the set of equivalence classes induced on (W^-, R^-, V^-) by $\leftrightsquigarrow_\Phi$, R^s the smallest filtration relation, and V^f the filtration valuation defined as usual (Definition 2.36 in [23]). Then $(W^f, R^s, V^f), |w| \Vdash \varphi$ by Filtration Theorem (Theorem 2.39 in [23]). Moreover, according to the previous lemma, (W^f, R^s) is a point-generated Kripke frame satisfying Reflexivity, Symmetry and Superposition with $card(W^f) \leq 2^{card(\Phi)}$.

Step 3: I take care of Separation.

Two cases need to be considered.

Case 1: $W^f = \{|w|\}$. It is easy to see that (W^f, R^s) is a state space satisfying Superposition. Since $1 = card(W^f) \leq 2^{card(\Phi)} < 2^{card(\Phi)+1}$, (W^f, R^s, V^f) has the required properties.

Case 2: $W^f \neq \{|w|\}$. By Lemma 4.3.6 $(W', R') = M(W^f, R^s)$ is a state space satisfying Superposition and $card(W') = 2 \cdot card(W^f) \leq 2^{card(\Phi)+1}$. Moreover, by Proposition 4.3.3 $(W', R'), (|w|, 0) \underline{\leftrightarrow} (W^f, R^s), |w|$. I define a valuation V' on (W', R') such that, for each propositional variable p,

$$V'(p) = \{(|u|, 0) \mid |u| \in V^f(p)\} \cup \{(|u|, 1) \mid |u| \in V^f(p)\}$$

Then it is easy to see that $(W', R', V'), (|w|, 0) \underline{\leftrightarrow} (W^f, R^s, V^f), |w|$. By Theorem 2.20 in [23] $(W^f, R^s, V^f), |w| \Vdash \varphi$ implies that $(W', R', V'), (|w|, 0) \Vdash \varphi$. Therefore, (W', R', V') has the required properties. ⊣

4.3.10. Theorem. *$\{\varphi \mid \varphi$ is satisfiable in $\mathcal{M}\}$ is a recursive set.*

Proof. Given a formula φ, I take the set Φ of all subformulas of φ. Then it follows that Φ is finite and subformula closed.

I find out whether φ is in the set by the following process: First, construct all models whose underlying frames are state spaces satisfying Superposition and have at most $2^{card(\Phi)+1}$ elements. Second, on each of these models, test whether φ is true at some point in the model. If a model among the above and a point in the model are found such that φ is true, then φ is satisfiable in $\mathcal{M}$, because this models is in $\mathcal{M}$. If nothing is found, then conclude that φ is not satisfiable in $\mathcal{M}$, according to the previous proposition.

It is not hard to see that the above process is computable, because only finitely many finite models are involved and checking the conditions of being a state space satisfying Separation is computable in finite Kripke frames. As a result, $\{\varphi \mid \varphi$ is satisfiable in $\mathcal{M}\}$ is a recursive set. ⊣

4.3.11. Corollary. *Λ is decidable.*

Proof. Given a formula φ, by the previous theorem there is an algorithm to decide whether $\neg\varphi$ is satisfiable in $\mathcal{M}$. If the answer is yes, then it follows that φ is not valid in the class of all state spaces satisfying Superposition. Hence $\nvdash_\Lambda \varphi$ by the soundness of Λ. If the answer is no, then it follows that φ is valid in the class of all state spaces satisfying Superposition. Hence $\vdash_\Lambda \varphi$ by the completeness of Λ. Therefore, Λ is decidable. ⊣

In the following table I summarize the complexity of the modal logics and the first-order theories of the various classes of Kripke frames known so far:

	Modal Language	First-Order Language
State Space	decidable	recursively enumerable
State Space Satisfying Superposition	decidable	recursively enumerable
Quasi-Quantum Kripke Frame	unknown	recursively enumerable
Quantum Kripke Frame	unknown	undecidable

The first-order theories of the first three kinds of Kripke frames are recursively enumerable, because they are finitely first-order axiomatized by definition and undecidable by Corollary 4.2.9 and Corollary 4.2.10. The first-order theory of quantum Kripke frames is undecidable by Corollary 4.2.9.

To complete this table, a solution to the interesting problem of axiomatizing quasi-quantum Kripke frames and quantum Kripke frames in the modal language will be helpful. Moreover, it's interesting to see whether the first-order theory of quantum Kripke frames satisfying Property OHC is the same as that of first-order quantum Kripke frames satisfying Property OHC, which is finitely axiomatizable by Remark 4.1.12. If the answer turns out to be yes, the first-order theory of these special quantum Kripke frames will be recursively enumerable.

This table shows that the proper formal language for the automated reasoning of quantum Kripke frames should be some fragment of the first-order language. It also suggests that the modal language is a promising option. Another option would be hybrid languages. Hybrid languages are the extensions of modal languages which have nominals to refer to the states, as well as some related modal operators. These slight extensions turn out to have a considerable increase in expressive power while maintaining a reasonable level in computational complexity. For more details about hybrid logic, please refer to Section 7.3 of [23]. Drs. Jort Bergfeld has devised a hybrid logic which is sound and complete with respect to the class of quantum Kripke frames having orthogonal sets of cardinalities at most n. For the details, please pay attention to his future publications.

Chapter 5

Probabilistic Quantum Kripke Frames

The previous chapters center on quantum Kripke frames, which are qualitative structures. It is desirable and natural to extend this framework to a quantitative one. On the one hand, according to Subsection 1.2.2, the measurements in quantum physics are intrinsically indeterminate. Probability is a powerful tool for describing indeterminate phenomena, and it is specially useful and important in quantum theory. Moreover, probabilities in quantum theory are considered to satisfy axioms that are different from the classical Kolmogorov axioms. Therefore, the extension to a quantitative framework not only increases our modelling power but also may shed light on the nature of quantum probability. On the other hand, adding probabilities to quantum Kripke frames is in fact not as hard as it appears. The reason is that the primitives in quantum Kripke frames, the non-orthogonality relations, have natural counterparts in quantum theory, that is the transition probabilities between the states. According to quantum theory, two states are non-orthogonal, if and only if the transition probabilities between them is not 0. Similar to the non-orthogonality relations, the transition probabilities are both physically intuitive and mathematically simple. According to quantum physics, the probability of getting a certain result in a measurement is equal to the transition probability between the initial state and the final state after the measurement yielding the result. From a mathematical point of view, the transition probabilities can be modelled by a set and a binary function from the set to the interval $[0, 1]$. Such structures are usually called probabilistic transition systems, and are extensively used and studied in computer science. As a result, this chapter will be devoted to extending the framework of quantum Kripke frames by adding the transition probabilities.

This chapter is organized as follows: In Section 5.1 I introduce the notion of probabilistic quantum Kripke frames and prove some elementary results. In Section 5.2 and Section 5.3, to show that probabilistic quantum Kripke frames are useful, I investigate how they relate to Hilbert spaces and to quantum probability measures, respectively. Section 5.4 is devoted to the special properties of those quantum Kripke frames which can be extended to a probabilistic one. Finally, in Section 5.5 I present a kind of quantum structure with only the transition probabilities between states as primitive.

5.1 Definition and Basic Properties

I start with the definition of probabilistic quantum Kripke frames.

5.1.1. Definition. A *probabilistic quantum Kripke frame* is a tuple $(\mathfrak{F}, \rho)$ such that $\mathfrak{F} = (\Sigma, \to)$ is a quantum Kripke frame and ρ, called a *transition probability function*, is a function from $\Sigma \times \Sigma$ to $[0, +\infty)$ satisfying the following:

(ρ1) $\rho(s,t) = \rho(t,s)$;

(ρ2) $\rho(s,t) = 0$, if and only if $s \not\to t$;

(ρ3) if $\{t_i \mid i \in I\}$ is an orthogonal subset of Σ and $s \in {\sim\sim}\{t_i \mid i \in I\}$, then $\sum_{i\in I} \rho(s, t_i) = 1$;

(ρ4) for any $s, u, v \in \Sigma$, if $u \neq v$ and s' is an approximation of s in ${\sim\sim}\{u, v\}$, then $\rho(s, w) = \rho(s, s') \cdot \rho(s', w)$ for every $w \in {\sim\sim}\{u, v\}$.

This definition involves the infinite sums of non-negative real numbers. Remember that, for a set $\{a_i \mid i \in I\}$ of non-negative real numbers, the sum of them is defined as follows:[1]

$$\sum_{i\in I} a_i \stackrel{\text{def}}{=} sup\{\sum_{i\in J} a_i \mid J \text{ is a finite subset of } I\}$$

Next, I present two lemmas. The first one is about the range of ρ in a probabilistic quantum Kripke frame.

5.1.2. Lemma. *Let $\mathfrak{F} = (\Sigma, \to)$ be a quantum Kripke frame and $\rho : \Sigma \times \Sigma \to [0, +\infty)$ is a function satisfying (ρ2) and (ρ3). For any $s, t \in \Sigma$, $\rho(s,t) \leq 1$, and $\rho(s,t) = 1$ if and only if $s = t$.*

Hence in a probabilistic quantum Kripke frame $(\mathfrak{F}, \rho)$ the range of ρ is $[0, 1]$.

Proof. Let $s, t \in \Sigma$ be arbitrary. Consider two cases.

Case 1: $s \neq t$. By Lemma 2.3.3 there is a $w \in {\sim\sim}\{s, t\}$ such that $w \not\to t$. It follows easily that $s \to w$. Since $w \not\to t$, by Reflexivity $w \neq t$. Since $w \in {\sim\sim}\{s, t\}$, by Corollary 2.3.5 $s \in {\sim\sim}\{w, t\}$. By (ρ3) $\rho(s, w) + \rho(s, t) = 1$. By definition $\rho(s, w) \geq 0$. Since $s \to w$, by (ρ2) $\rho(s, w) > 0$. Hence $\rho(s,t) = 1 - \rho(s, w) < 1$.

Case 2: $s = t$. Since $\{s\}$ is an orthogonal subset of Σ and $s \in {\sim\sim}\{s\}$, $\rho(s, s) = 1$ by (ρ3). Hence $\rho(s,t) = \rho(s, s) = 1$. $\dashv$

[1] Please refer to, for example, Page 28 in [61].

The second one is about a useful generalization of (ρ4) in probabilistic quantum Kripke frames.

5.1.3. Lemma. *Let $\mathfrak{F} = (\Sigma, \to)$ be a quantum Kripke frame and $\rho : \Sigma \times \Sigma \to [0, +\infty)$ is a function satisfying (ρ1) to (ρ3). The following are equivalent:*[2]

(ρ4) for any $s, u, v \in \Sigma$, if $u \neq v$ and s' is an approximation of s in $\sim\sim\{u, v\}$, then $\rho(s, w) = \rho(s, s') \cdot \rho(s', w)$ for every $w \in \sim\sim\{u, v\}$;

(ρ4') for any $s \in \Sigma$ and $P \subseteq \Sigma$, if $P = \sim\sim P$ and $P?(s)$ is an approximation of s in P, then $\rho(s, w) = \rho(s, P?(s)) \cdot \rho(P?(s), w)$ for every $w \in P$.

In particular, (ρ4') holds in every probabilistic quantum Kripke frame.

Proof. **From (ρ4) to (ρ4'):** Let $w \in P$ be arbitrary. Consider two cases.

- *Case 1:* $w = P?(s)$. Since $\{w\}$ is an orthogonal subset of Σ and $w \in \sim\sim\{w\}$, $\rho(w, w) = 1$ by (ρ3). Hence $\rho(P?(s), w) = 1$. Therefore, $\rho(s, w) = \rho(s, P?(s)) \cdot 1 = \rho(s, P?(s)) \cdot \rho(P?(s), w)$.
- *Case 2:* $w \neq P?(s)$. By Proposition 2.2.1 $\sim\sim\{w, P?(s)\}$ is bi-orthogonally closed. By the definition of $P?(s)$ and Reflexivity $s \to P?(s)$, and thus $s \notin \sim\{w, P?(s)\} = \sim\sim\sim\{w, P?(s)\}$. By Property A there is an $s' \in \sim\sim\{w, P?(s)\}$ which is an approximation of s in $\sim\sim\{w, P?(s)\}$. By (ρ4) $\rho(s, w) = \rho(s, s') \cdot \rho(s', w)$ for this $w \in \sim\sim\{w, P?(s)\}$.

  I claim that $P?(s) = s'$. By definition $s \approx_P P?(s)$ and $s \approx_{\sim\sim\{w,P?(s)\}} s'$. Since $w, P?(s) \in P$ by definition and $P = \sim\sim P$, $\sim\sim\{w, P?(s)\} \subseteq P$. By Remark 2.1.2 $\approx_P \subseteq \approx_{\sim\sim\{w,P?(s)\}}$, so $s \approx_{\sim\sim\{w,P?(s)\}} P?(s)$. By Lemma 2.3.20 and Proposition 2.4.2 the approximation of s in $\sim\sim\{w, P?(s)\}$ is unique. Hence $P?(s) = s'$.

  Therefore, $\rho(s, w) = \rho(s, P?(s)) \cdot \rho(P?(s), w)$.

Since $w \in P$ is arbitrary, $\rho(s, w) = \rho(s, P?(s)) \cdot \rho(P?(s), w)$ for every $w \in P$.

From (ρ4') to (ρ4): By Proposition 2.2.1 $\sim\sim\{u, v\}$ is bi-orthogonally closed. Hence (ρ4) is a special case of (ρ4'). ⊣

5.2 Probabilistic Quantum Kripke Frames and Hilbert Spaces

In this section I discuss the relation between probabilistic quantum Kripke frames and Hilbert spaces. First, I prove that every Hilbert space over $\mathbb{C}$ induces a probabilistic quantum Kripke frame.

[2]Originally I proposed (ρ4') in the definition of probabilistic quantum Kripke frames. Dr. Alexandru Baltag observed that it can be replaced by (ρ4) which is simpler.

For the discussion, I fix an arbitrary Hilbert space $\mathcal{H}$ over $\mathbb{C}$. I define a function $F : \mathcal{H} \times \mathcal{H} \to [0, +\infty)$ as follows: for any two vectors $\mathbf{v}, \mathbf{w} \in \mathcal{H}$,

$$F(\mathbf{v}, \mathbf{w}) = \frac{\langle \mathbf{v}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle \langle \mathbf{w}, \mathbf{w} \rangle}$$

Define a function $\rho_{\mathcal{H}} : \Sigma(\mathcal{H}) \times \Sigma(\mathcal{H}) \to [0, +\infty)$ such that, for any $s, t \in \Sigma(\mathcal{H})$,

$$\rho_{\mathcal{H}}(s, t) = F(\mathbf{s}, \mathbf{t}), \text{ for some } \mathbf{s} \in s \setminus \{\mathbf{0}\} \text{ and } \mathbf{t} \in t \setminus \{\mathbf{0}\}$$

I call $\rho_{\mathcal{H}}$ *the transition probability function of* $\mathcal{H}$. It is easy to see that it is well defined. Remember that by Corollary 2.5.6 $\mathfrak{F}_{\mathcal{H}} = (\Sigma(\mathcal{H}), \to_{\mathcal{H}})$ is a quantum Kripke frame.

5.2.1. Proposition. *In the quantum Kripke frame $\mathfrak{F}_{\mathcal{H}}$ and with $\rho_{\mathcal{H}} : \Sigma(\mathcal{H}) \times \Sigma(\mathcal{H}) \to [0, +\infty)$ defined as above, conditions (ρ1) to (ρ4) hold, and thus the tuple $(\mathfrak{F}_{\mathcal{H}}, \rho_{\mathcal{H}})$ is a probabilistic quantum Kripke frame.*

Proof. **For (ρ1):** Whenever $\mathbf{s} \in s \setminus \{\mathbf{0}\}$ and $\mathbf{t} \in t \setminus \{\mathbf{0}\}$,

$$\rho_{\mathcal{H}}(s, t) = F(\mathbf{s}, \mathbf{t}) = \frac{\langle \mathbf{s}, \mathbf{t} \rangle \langle \mathbf{t}, \mathbf{s} \rangle}{\langle \mathbf{s}, \mathbf{s} \rangle \langle \mathbf{t}, \mathbf{t} \rangle} = \frac{\langle \mathbf{t}, \mathbf{s} \rangle \langle \mathbf{s}, \mathbf{t} \rangle}{\langle \mathbf{t}, \mathbf{t} \rangle \langle \mathbf{s}, \mathbf{s} \rangle} = F(\mathbf{t}, \mathbf{s}) = \rho_{\mathcal{H}}(t, s)$$

For (ρ2): This follows easily from the definition of $\to_{\mathcal{H}}$.

For (ρ3): Assume that $s \in \Sigma(\mathcal{H})$ and $\{t_i \mid i \in I\}$ is an orthogonal subset of $\Sigma(\mathcal{H})$. Without loss of generality, assume that I is a cardinal. Let $\mathbf{s}$ be a unit vector such that $s = \langle \mathbf{s} \rangle$, and $\mathbf{t}_i$ a unit vector such that $t_i = \langle \mathbf{t}_i \rangle$ for each $i \in I$. By Theorem 2.2.10 in [61] there is a cardinal $\beta \geq I$ and an orthonormal basis of $\mathcal{H}$, $\{\mathbf{a}_i \mid i \in \beta\}$, such that $\mathbf{a}_i = \mathbf{t}_i$, for each $i \in I$. Hence by Parseval's identity (Theorem 2.2.9 in [61]) $\langle \mathbf{s}, \mathbf{s} \rangle = \sum_{i \in \beta} \langle \mathbf{s}, \mathbf{a}_i \rangle \langle \mathbf{a}_i, \mathbf{s} \rangle$. It follows that

$$\begin{aligned} 1 &= \frac{\langle \mathbf{s}, \mathbf{s} \rangle}{\langle \mathbf{s}, \mathbf{s} \rangle} \\ &= \frac{1}{\langle \mathbf{s}, \mathbf{s} \rangle} \cdot \sum_{i \in \beta} \langle \mathbf{s}, \mathbf{a}_i \rangle \langle \mathbf{a}_i, \mathbf{s} \rangle \\ &= \sum_{i \in \beta} \frac{\langle \mathbf{s}, \mathbf{a}_i \rangle \langle \mathbf{a}_i, \mathbf{s} \rangle}{\langle \mathbf{s}, \mathbf{s} \rangle \langle \mathbf{a}_i, \mathbf{a}_i \rangle} \\ &= \sum_{i \in \beta} F(\mathbf{s}, \mathbf{a}_i) \end{aligned}$$

Since $s \in {\sim}{\sim}\{t_i \mid i \in I\}$, $s \not\to_{\mathcal{H}} t_i$ for each $i \in \beta \setminus I$. Hence $\rho_{\mathcal{H}}(s, t_i) = 0$ for each $i \in \beta \setminus I$. Therefore, $\sum_{i \in I} \rho(s, t_i) = \sum_{i \in I} F(\mathbf{s}, \mathbf{a}_i) = \sum_{i \in \beta} F(\mathbf{s}, \mathbf{a}_i) = 1$.

For (ρ4): By Lemma 5.1.3 it suffices to prove (ρ4'). Let $s \in \Sigma$ and $P \subseteq \Sigma(\mathcal{H})$ be arbitrary such that $P = {\sim}{\sim}P$ and $P?(s)$ is an approximation of s in P.

It follows from the definition of approximations and Reflexivity that $s \notin {\sim}P$. Moreover, let $\mathbf{s}$ be a unit vector such that $s = \langle \mathbf{s} \rangle$.

If $P = \Sigma$, s is an approximation of s in P. By Proposition 2.4.2 $P?(s) = s$. It is easy to verify from the definition that $\rho_{\mathcal{H}}(s, P?(s)) = \rho_{\mathcal{H}}(s, s) = F(\mathbf{s}, \mathbf{s}) = 1$. Hence $\rho_{\mathcal{H}}(s, w) = 1 \cdot \rho_{\mathcal{H}}(s, w) = \rho_{\mathcal{H}}(s, P?(s)) \cdot \rho_{\mathcal{H}}(P?(s), w)$ for every $w \in P$. In the following, I focus on the case when $P \neq \Sigma$.

Since $P \neq \Sigma$ and $P = {\sim}{\sim}P$, ${\sim}P \neq \emptyset$. Moreover, by definition $P?(s) \in P$, so $P \neq \emptyset$. Hence by *4* in Lemma B.3.9 $((\bigcup P)^{\perp})^{\perp} = \bigcup P$. It follows from Corollary 2.2.4 in [61] that $\bigcup P$ is a closed linear subspace of $\mathcal{H}$. By Proposition 2.5.1 in [61] there is a projector P from $\mathcal{H}$ onto $\bigcup P$. Since $s \notin {\sim}P$, it is not hard to show that $\mathsf{P}(\mathbf{s}) \neq \mathbf{0}$. Now let $w \in P$ be arbitrary. Suppose that $\mathbf{w}$ is a unit vector in $\mathcal{H}$ such that $w = \langle \mathbf{w} \rangle$. Since $w \in P$, $\mathsf{P}(\mathbf{w}) = \mathbf{w}$. Then

$$
\begin{aligned}
& F(\mathbf{s}, \mathsf{P}(\mathbf{s})) \cdot F(\mathsf{P}(\mathbf{s}), \mathbf{w}) \\
= & \frac{\langle \mathbf{s}, \mathsf{P}(\mathbf{s}) \rangle \langle \mathsf{P}(\mathbf{s}), \mathbf{s} \rangle}{\langle \mathbf{s}, \mathbf{s} \rangle \langle \mathsf{P}(\mathbf{s}), \mathsf{P}(\mathbf{s}) \rangle} \cdot \frac{\langle \mathsf{P}(\mathbf{s}), \mathbf{w} \rangle \langle \mathbf{w}, \mathsf{P}(\mathbf{s}) \rangle}{\langle \mathsf{P}(\mathbf{s}), \mathsf{P}(\mathbf{s}) \rangle \langle \mathbf{w}, \mathbf{w} \rangle} \\
= & \frac{\langle \mathbf{s}, \mathsf{P} \circ \mathsf{P}(\mathbf{s}) \rangle \langle \mathsf{P} \circ \mathsf{P}(\mathbf{s}), \mathbf{s} \rangle}{\langle \mathbf{s}, \mathbf{s} \rangle \langle \mathsf{P}(\mathbf{s}), \mathsf{P}(\mathbf{s}) \rangle} \cdot \frac{\langle \mathsf{P}(\mathbf{s}), \mathbf{w} \rangle \langle \mathbf{w}, \mathsf{P}(\mathbf{s}) \rangle}{\langle \mathsf{P}(\mathbf{s}), \mathsf{P}(\mathbf{s}) \rangle \langle \mathbf{w}, \mathbf{w} \rangle} && (\mathsf{P} \text{ is idempotent}) \\
= & \frac{\langle \mathsf{P}(\mathbf{s}), \mathsf{P}(\mathbf{s}) \rangle \langle \mathsf{P}(\mathbf{s}), \mathsf{P}(\mathbf{s}) \rangle}{\langle \mathbf{s}, \mathbf{s} \rangle \langle \mathsf{P}(\mathbf{s}), \mathsf{P}(\mathbf{s}) \rangle} \cdot \frac{\langle \mathbf{s}, \mathsf{P}(\mathbf{w}) \rangle \langle \mathsf{P}(\mathbf{w}), \mathbf{s} \rangle}{\langle \mathsf{P}(\mathbf{s}), \mathsf{P}(\mathbf{s}) \rangle \langle \mathbf{w}, \mathbf{w} \rangle} && (\mathsf{P} \text{ is self-adjoint}) \\
= & \frac{\langle \mathbf{s}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{s} \rangle}{\langle \mathbf{s}, \mathbf{s} \rangle \langle \mathbf{w}, \mathbf{w} \rangle} && (\mathsf{P}(\mathbf{w}) = \mathbf{w}) \\
= & F(\mathbf{s}, \mathbf{w})
\end{aligned}
$$

Since w is arbitrary, $\rho_{\mathcal{H}}(s, w) = \rho_{\mathcal{H}}(s, \langle \mathsf{P}(\mathbf{s}) \rangle) \cdot \rho_{\mathcal{H}}(\langle \mathsf{P}(\mathbf{s}) \rangle, w)$ for every $w \in P$.

It follows that $\langle \mathsf{P}(\mathbf{s}) \rangle$ is an approximation of s in P. Since $P = {\sim}{\sim}P$, by Lemma 2.3.20 and Proposition 2.4.2 $P?(s) = \langle \mathsf{P}(\mathbf{s}) \rangle$. As a result, for every $w \in P$, $\rho_{\mathcal{H}}(s, w) = \rho_{\mathcal{H}}(s, P?(s)) \cdot \rho_{\mathcal{H}}(P?(s), w)$. ⊣

Now one may wonder whether the converse of this theorem holds, i.e. whether every probabilistic quantum Kripke frame is induced by a Hilbert space over $\mathbb{C}$, or more generally, one over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. This is not obvious. Anyway, every probabilistic quantum Kripke frame is based on a quantum Kripke frame. Hence Corollary 2.5.6 and the results in Section 2.7.1 can be used to characterize those probabilistic quantum Kripke frames whose underlying quantum Kripke frames are induced by Hilbert spaces over $\mathbb{C}$. One may note an important question here: In a probabilistic quantum Kripke frame $(\mathfrak{F}, \rho)$, if $i : \mathfrak{F} \cong (\Sigma(\mathcal{H}), \to_{\mathcal{H}})$ for some Hilbert space $\mathcal{H}$ over $\mathbb{C}$, does it hold that $\rho(s, t) = \rho_{\mathcal{H}}(i(s), i(t))$ for any s, t in $\mathfrak{F}$? I analyse this question in detail at the end of this chapter.

5.3 Connection with Quantum Probability Measures

In this section I present a natural way to get quantum probability measures from probabilistic quantum Kripke frames. First I recall the definition of quantum probability measures in [74].

5.3.1. Definition. A *quantum probability measure* on a Piron lattice $(L, \leq, (\cdot)')$ is a function $p : L \to [0, 1]$ satisfying the following:

1. (Normality) $p(I) = 1$;
2. (Coherence) $p(b) = p(c) = 0$ implies that $p(b \vee c) = 0$;
3. (σ-Additivity) if I is a set at most countable and $\{b_i\}_{i \in I} \subseteq L$ satisfies $b_i \leq b'_j$ whenever $i \neq j$, then $\sum_{i \in I} p(b_i) = p(\bigvee_{i \in I} b_i)$.

In a quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$, every $s \in \Sigma$ is interpreted as a state of the quantum system and every bi-orthogonally closed subset of Σ as a testable property. In quantum mechanics, every state should induce a probability measure on the set of testable properties. I show that this is the case in the setting of probabilistic quantum Kripke frames. This fact supports that the notion of probabilistic quantum Kripke frames makes sense.

For the discussion, I fix a probabilistic quantum Kripke frame $(\mathfrak{F}, \rho)$, where $\mathfrak{F} = (\Sigma, \to)$. By Theorem 2.7.23 $\mathcal{L}_{\mathfrak{F}}$, the set of bi-orthogonally closed subsets of Σ, forms a Piron lattice under $\subseteq$ and $\sim(\cdot)$. For any $s \in \Sigma$ and $P \in \mathcal{L}_{\mathfrak{F}}$, by Lemma 2.3.20 and Proposition 2.4.2 the approximation of s in P is unique, and I denote it by $P?(s)$, if it exists. For every $s \in \Sigma$, define a map $\mu_s : \mathcal{L}_{\mathfrak{F}} \to [0, 1]$ as follows:

$$\mu_s(P) = \begin{cases} 0 & \text{if } s \in {\sim}P, \\ \rho(s, P?(s)) & \text{otherwise.} \end{cases}$$

I show that, for every $s \in \Sigma$, μ_s is a quantum probability measure.

5.3.2. Proposition. *For every $s \in \Sigma$, μ_s satisfies Normality, i.e. $\mu_s(\Sigma) = 1$.*

Proof. Obviously $s \notin {\sim}\Sigma$, by Property A and the definition of $\Sigma?(s)$ $s = \Sigma?(s)$. Therefore, $\mu_s(\Sigma) = \rho(s, \Sigma?(s)) = \rho(s, s) = 1$, according to Lemma 5.1.2. ⊣

For simplicity, I write ${\sim}{\sim}\bigcup_{i \in I} P_i$, i.e. ${\sim}\bigcap_{i \in I} {\sim}P_i$, as $\bigsqcup_{i \in I} P_i$, for any $\{P_i \mid i \in I\} \subseteq \wp(\Sigma)$. Note that $\bigsqcup_{i \in I} P_i$ is always bi-orthogonally closed. Moreover, when $P_i \in \mathcal{L}_{\mathfrak{F}}$ for every $i \in I$, by Proposition 2.2.3 $\bigsqcup_{i \in I} P_i = \bigvee_{i \in I} P_i$, where the notation $\bigvee$ is defined in the same proposition. To simplify the notation, when $P_i = \{s_i\}$ for every $i \in I$, I omit the brackets and write $\bigsqcup_{i \in I} s_i$ for $\bigsqcup_{i \in I} \{s_i\}$.

5.3.3. Proposition. *For every $s \in \Sigma$, μ_s satisfies Coherence. To be precise, if $P, Q \in \mathcal{L}_{\mathfrak{F}}$ are such that $\mu_s(P) = 0$ and $\mu_s(Q) = 0$, then $\mu_s(P \sqcup Q) = 0$.*

Proof. First note that $s \in {\sim}P$. Suppose (towards a contradiction) that $s \notin {\sim}P$. Then by Property A $P?(s)$ is defined and by Reflexivity $s \to P?(s)$. By (ρ2) $\mu_s(P) = \rho(s, P?(s)) \neq 0$, contradicting that $\mu_s(P) = 0$. Therefore, $s \in {\sim}P$. Similarly, I can show that $s \in {\sim}Q$. It follows that $s \in {\sim}P \cap {\sim}Q = {\sim}(P \sqcup Q)$. By definition $\mu_s(P \sqcup Q) = 0$. ⊣

For σ-Additivity, first I prove the following property:

(Additivity) If $\{P_i\}_0^n \subseteq \mathcal{L}_{\mathfrak{F}}$ satisfies $P_i \subseteq {\sim}P_j$ when $i \neq j$, $\sum_0^n \mu_s(P_i) = \mu_s(\bigsqcup_0^n P_i)$

This property is weaker than σ-Additivity, but the reason for proving this is two-fold. First, the proof involves some nice results and useful lemmas. Second, the proof is constructive, i.e. it does not need the Axiom of Choice or its equivalences in ZF. On the contrary, I can only find a proof of σ-Additivity (the proof of Corollary 5.3.11) which involves the Axiom of Choice.

The following lemma shows that $(P \sqcup Q)?(s)$, $P?(s)$ and $Q?(s)$ are collinear when $P \subseteq {\sim}Q$ and all three of them exist.

5.3.4. Lemma. *Let $s \in \Sigma$ and $P, Q \in \mathcal{L}_{\mathfrak{F}}$ be such that $P \subseteq {\sim}Q$, $s \notin {\sim}P$ and $s \notin {\sim}Q$. Then $(P \sqcup Q)?(s)$ exists and $(P \sqcup Q)?(s) \in P?(s) \sqcup Q?(s)$.*

Proof. First I show that $(P \sqcup Q)?(s)$ exists. Since $s \notin {\sim}P$, there is an $x \in P$ such that $s \to x$. Then $s \to x$ for this $x \in P \subseteq P \sqcup Q$. Hence $s \notin {\sim}(P \sqcup Q)$, and thus $(P \sqcup Q)?(s)$ exists by Property A. To simplify the notation, I use t to denote $(P \sqcup Q)?(s)$ in the following.

Second I show that $P?(t)$ and $({\sim}P)?(t)$ are defined and $t \in P?(t) \sqcup ({\sim}P)?(t)$. Since $s \notin {\sim}P$, $P?(s)$ is defined. Since $s \to P?(s)$ and $P?(s) \in P \subseteq P \sqcup Q$, $t \to P?(s)$ by definition of $(P \sqcup Q)?(s)$, and thus $t \notin {\sim}P$. Similarly, I can show that $t \notin {\sim}Q$. Note that $t \notin P$; otherwise $t \in {\sim}Q$, since $P \subseteq {\sim}Q$, which contradicts $t \notin {\sim}Q$. Since $t \notin P \cup {\sim}P$, by Property A, Proposition 2.4.4 and Proposition 2.4.6 $P?(t)$ and $({\sim}P)?(t)$ are defined and $t \in P?(t) \sqcup ({\sim}P)?(t)$. Moreover, by Proposition 2.4.6 t, $P?(t)$ and $({\sim}P)?(t)$ are distinct. Hence by Corollary 2.3.5 $({\sim}P)?(t) \in t \sqcup P?(t)$.

Third I show that $P?(t) = P?(s)$. Since $P \subseteq P \sqcup Q$ and $s \approx_{P \sqcup Q} t$, by Remark 2.1.2 $s \approx_P t$. Since $t \approx_P P?(t)$, by Remark 2.1.2 $s \approx_P P?(t)$. Since $P?(t) \in P$, $P?(t)$ is an approximation of s in P, so by Lemma 2.3.20 and Proposition 2.4.2 $P?(s) = P?(t)$.

Fourth I show that $({\sim}P)?(t) = Q?(s)$. I claim that $({\sim}P)?(t) \in Q$. Since $t \in P \sqcup Q$ and $P?(t) \in P \subseteq P \sqcup Q$, it is not hard to see that $t \sqcup P?(t) \subseteq P \sqcup Q$. Hence $({\sim}P)?(t) \in P \sqcup Q$. Since $P \subseteq {\sim}Q$, it is easy to prove that $Q = {\sim}{\sim}Q \subseteq {\sim}P$. For $P, Q \in \mathcal{L}_{\mathfrak{F}}$, $Q = {\sim}P \cap ({\sim}{\sim}P \sqcup Q) = {\sim}P \cap (P \sqcup Q)$ by Corollary 2.4.8. Since $({\sim}P)?(t) \in {\sim}P$ and $({\sim}P)?(t) \in P \sqcup Q$, $({\sim}P)?(t) \in Q$.

Now, since $s \approx_{P \sqcup Q} t$ and $Q \subseteq P \sqcup Q$, by Remark 2.1.2 $s \approx_Q t$. Similarly, since $t \approx_{\sim P} (\sim P)?(t)$ and $Q \subseteq \sim P$, by Remark 2.1.2 $t \approx_Q (\sim P)?(t)$. Hence by Remark 2.1.2 $s \approx_Q (\sim P)?(t)$. Since $(\sim P)?(t) \in Q$, $(\sim P)?(t)$ is an approximation of s in Q, so $Q?(s) = (\sim P)?(t)$ by Lemma 2.3.20 and Proposition 2.4.2.

It follows from $t \in P?(t) \sqcup (\sim P)?(t)$, $P?(t) = P?(s)$ and $(\sim P)?(t) = Q?(s)$ that $(P \sqcup Q)?(s) = t \in P?(s) \sqcup Q?(s)$. ⊣

The following lemma deals with an important special case of Additivity when only two bi-orthogonally closed subsets are involved.

5.3.5. Lemma. *Let $s \in \Sigma$ and $P, Q \in \mathcal{L}_{\mathfrak{F}}$ satisfy $P \subseteq \sim Q$. Then $\mu_s(P \sqcup Q) = \mu_s(P) + \mu_s(Q)$.*

Proof. Consider four cases.

Case 1: $s \in \sim P$ and $s \in \sim Q$. By definition $\mu_s(P) = \mu_s(Q) = 0$. Moreover, $s \in \sim P \cap \sim Q = \sim(P \sqcup Q)$, so $\mu_s(P \sqcup Q) = 0$. Therefore, $\mu_s(P \sqcup Q) = 0 = \mu_s(P) + \mu_s(Q)$.

Case 2: $s \notin \sim P$ and $s \in \sim Q$. By definition $\mu_s(Q) = 0$. Since $s \notin \sim P$, there is a $t \in P$ such that $s \to t$. Then $s \notin \sim(P \sqcup Q)$, for this t satisfies $t \in P \subseteq P \sqcup Q$. Then by Property A $P?(s)$ and $(P \sqcup Q)?(s)$ are both defined.

I claim that $(P \sqcup Q)?(s) \in P$. It follows from $s \approx_{P \sqcup Q} (P \sqcup Q)?(s)$ and $s \in \sim Q$ that $(P \sqcup Q)?(s) \in \sim Q$. Since $P \subseteq \sim Q$, $P = \sim Q \cap (P \sqcup \sim\sim Q) = \sim Q \cap (P \sqcup Q)$ by Corollary 2.4.8. Then $(P \sqcup Q)?(s) \in P$ by the fact that $(P \sqcup Q)?(s) \in \sim Q$ and $(P \sqcup Q)?(s) \in P \sqcup Q$.

Now, since $s \approx_{P \sqcup Q} (P \sqcup Q)?(s)$ and $P \subseteq P \sqcup Q$, by Remark 2.1.2 $s \approx_P (P \sqcup Q)?(s)$. Since $(P \sqcup Q)?(s) \in P$, $(P \sqcup Q)?(s)$ is an approximation of s in P, so $P?(s) = (P \sqcup Q)?(s)$ by Lemma 2.3.20 and Proposition 2.4.2. It follows that $\mu_s(P \sqcup Q) = \rho(s, (P \sqcup Q)?(s)) = \rho(s, P?(s)) = \mu_s(P) = \mu_s(P) + \mu_s(Q)$.

Case 3: $s \in \sim P$ and $s \notin \sim Q$. From $P \subseteq \sim Q$, it is easy to deduce that $\sim Q \subseteq P$, so this case can be dealt symmetrically to Case 2.

Case 4: $s \notin \sim P$ and $s \notin \sim Q$. By Property A both $P?(s)$ and $Q?(s)$ are defined. By Lemma 5.3.4 $(P \sqcup Q)?(s)$ exists and $(P \sqcup Q)?(s) \in P?(s) \sqcup Q?(s)$. Since $P?(s) \in P \subseteq P \sqcup Q$, by ($\rho$4')

$$\begin{aligned} \mu_s(P) &= \rho(s, P?(s)) \\ &= \rho(s, (P \sqcup Q)?(s))\rho((P \sqcup Q)?(s), P?(s)) \\ &= \mu_s(P \sqcup Q)\rho((P \sqcup Q)?(s), P?(s)). \end{aligned}$$

Similarly, $\mu_s(Q) = \mu_s(P \sqcup Q)\rho((P \sqcup Q)?(s), Q?(s))$. Adding these two, I get

$$\mu_s(P) + \mu_s(Q) = \mu_s(P \sqcup Q)\Big(\rho((P \sqcup Q)?(s), P?(s)) + \rho((P \sqcup Q)?(s), Q?(s))\Big)$$

Since $P \subseteq \sim Q$, $P?(s) \not\to Q?(s)$. Together with $(P \sqcup Q)?(s) \in P?(s) \sqcup Q?(s)$, I have $\rho((P \sqcup Q)?(s), P?(s)) + \rho((P \sqcup Q)?(s), Q?(s)) = 1$, according to ($\rho$3). Therefore, $\mu_s(P) + \mu_s(Q) = \mu_s(P \sqcup Q)$. ⊣

Now I am ready to prove Additivity.

5.3.6. Proposition. *For every $s \in \Sigma$, μ_s satisfies Additivity, i.e. if $\{P_i\}_0^n \subseteq \mathcal{L}_{\mathfrak{F}}$ are such that $P_i \subseteq {\sim}P_j$ whenever $i \neq j$, then $\mu_s(\bigsqcup_0^n P_i) = \sum_0^n \mu_s(P_i)$.*

Proof. Use induction on n.

Base Step: $n = 0$. In this case, trivially $\mu_s(\bigsqcup_0^0 P_i) = \mu_s(P_0) = \sum_0^0 \mu_s(P_i)$.

Induction Step: $n = k + 1$. Let $\{P_i\}_0^{k+1} \subseteq \mathcal{L}_{\mathfrak{F}}$ be such that $P_i \subseteq {\sim}P_j$ whenever $i \neq j$. By the induction hypothesis $\mu_s(\bigsqcup_0^k P_i) = \sum_0^k \mu_s(P_i)$. I consider the relation between $\bigsqcup_0^k P_i$ and P_{k+1}. Since $P_{k+1} \subseteq {\sim}P_i$ for each $i \leq k$, $P_{k+1} \subseteq \bigcap_0^k {\sim}P_i = {\sim}\bigsqcup_0^k P_i$. By Lemma 5.3.5 $\mu_s(\bigsqcup_0^{k+1} P_i) = \mu_s(P_{k+1} \sqcup \bigsqcup_0^k P_i) = \mu_s(P_{k+1}) + \mu_s(\bigsqcup_0^k P_i)$. It follows that $\mu_s(\bigsqcup_0^{k+1} P_i) = \sum_0^{k+1} \mu_s(P_i)$. ⊣

Now I prove σ-Additivity using a non-constructive method. In fact, I can prove the following, which is stronger than σ-Additivity:

(Arbitrary Additivity) if $\{P_i \mid i \in I\} \subseteq \mathcal{L}_{\mathfrak{F}}$ satisfies $P_i \subseteq {\sim}P_j$ when $i \neq j$, $\sum_{i \in I} \mu_s(P_i) = \mu_s(\bigsqcup_{i \in I} P_i)$.

I start from the notion of maximal orthogonal subset: Q is a *maximal orthogonal subset of* $P \subseteq \Sigma$, if Q is an orthogonal subset of P and $s \notin P$ for every $s \in {\sim}Q$. The following two lemmas exhibit some nice properties of this kind of subset. The first one is about the existence of maximal orthogonal subsets.

5.3.7. Lemma. *For every subset P of Σ and every orthogonal subset Q of P, Q can be extended to a maximal orthogonal subset of P. In particular, every subset P of Σ has a maximal orthogonal subset.*

Proof. Let $\mathfrak{P}$ be the set of all orthogonal subsets of P which includes Q. Since Q is an orthogonal subset of P, $Q \in \mathfrak{P}$, so $\mathfrak{P}$ is non-empty. It is easy to see that $\mathfrak{P}$ is partially ordered by inclusion $\subseteq$.

I show that every chain in $\mathfrak{P}$ has an upper bound. Let $\{Q_j \mid j \in \beta\}$ be a chain in $\mathfrak{P}$ such that $Q_j \subseteq Q_{j'}$ whenever $j \leq j'$. Consider $\bigcup_{j \in \beta} Q_j$. Since $\{Q_j \mid j \in \beta\}$ is a chain in $\mathfrak{P}$, $Q_j \subseteq P$ for each $j \in \beta$, so $\bigcup_{j \leq \beta} Q_j \subseteq P$. And $Q \subseteq \bigcup_{j \leq \beta} Q_j$, because $Q \subseteq Q_j$ for each $j \in \beta$. Moreover, for any distinct $u, v \in \bigcup_{j \in \beta} Q_j$, there must be $j_u, j_v \in \beta$ such that $u \in Q_{j_u}$ and $v \in Q_{j_v}$, and thus $u, v \in Q_{j^*}$ where $j^* = max\{j_u, j_v\}$. Since $Q_{j^*} \in \mathfrak{P}$, $u \not\to v$, so $\bigcup_{j \in \beta} Q_j$ is an orthogonal set. Therefore, $\bigcup_{j \in \beta} Q_j \in \mathfrak{P}$ and it is an upper bound of the chain $\{Q_j \mid j \in \beta\}$.

Using Zorn's Lemma (Theorem 5.4 in [60]), I conclude that the set $\mathfrak{P}$ contains a maximal element Q^*. Hence, for every $s \in {\sim}Q^*$, $Q^* \cup \{s\} \notin \mathfrak{P}$ by maximality. Since $Q^* \cup \{s\}$ is an orthogonal set, it cannot be a subset of P and thus $s \notin P$. Therefore, Q^* is a maximal orthogonal subset of P.

By definition $\emptyset$ is always an orthogonal subset, so I conclude that every subset P of Σ has a maximal orthogonal subset. ⊣

The second lemma about maximal orthogonal subsets shows the close relation between a bi-orthogonally closed set and a maximal orthogonal subset of it.

5.3.8. LEMMA. *For every $P \in \mathcal{L}_{\mathfrak{F}}$ and every maximal orthogonal subset Q of P, $P = {\sim}{\sim}Q$.*

Proof. Since $Q \subseteq P$, ${\sim}{\sim}Q \subseteq {\sim}{\sim}P = P$.

To show that $P \subseteq {\sim}{\sim}Q$, I suppose (towards a contradiction) that there is an $s \in \Sigma$ such that $s \in P$ and $s \notin {\sim}{\sim}Q$. Since ${\sim}Q$ is bi-orthogonally closed, by Property A there is an $s' \in {\sim}Q$ such that $s \approx_{\sim Q} s'$. I observe that $s' \in {\sim}{\sim}P$. Let $t \in {\sim}P$ be arbitrary. Since $Q \subseteq P$, ${\sim}P \subseteq {\sim}Q$ and thus $t \in {\sim}Q$. For $s \in P$ and $t \in {\sim}P$, $s \not\to t$. Then it follows from $s \approx_{\sim Q} s'$ that $s' \not\to t$. Therefore, $s' \in {\sim}{\sim}P = P$. Since $s' \in P$ and $s' \in {\sim}Q$, $Q \cup \{s'\}$ is an orthogonal subset of P. This contradicts the maximality of Q. As a result, $P \subseteq {\sim}{\sim}Q$. ⊣

Now I am ready to prove the transfinite version of Lemma 5.3.4.

5.3.9. LEMMA. *Let $s \in \Sigma$ and $\{P_i \mid i \in I\} \subseteq \mathcal{L}_{\mathfrak{F}}$ satisfy that $I \neq \emptyset$, $P_i \subseteq {\sim}P_j$ whenever $i \neq j$, and $s \notin {\sim}P_i$ for each $i \in I$. Then $P_i?(s)$ is defined for each $i \in I$ and $(\bigsqcup_{i\in I} P_i)?(s) \in \bigsqcup_{i\in I} P_i?(s)$.*

Proof. For each $i \in I$, $P_i?(s)$ is defined, since $s \notin {\sim}P_i$. By the previous lemmas I can find a cardinal α_i and a maximal orthogonal subset $\{t^i_j \mid j \in \alpha_i\}$ of P_i such that $\bigsqcup_{j\in\alpha_i} t^i_j = P_i$ and $t^i_0 = P_i?(s)$. Using the Axiom of Choice, I do this for each $i \in I$. To simplify the notation, I denote $(\bigsqcup_{i\in I} P_i)?(s)$ by w.

I claim that $w \not\to t^i_j$, for each $j \in \alpha_i$ with $j \neq 0$ and $i \in I$. Since $\{t^i_j \mid j \in \alpha_j\}$ is an orthogonal set, $t^i_0 \not\to t^i_j$ with $j \neq 0$. Since t^i_0 is $P_i?(s)$ and $t^i_j \in P_i$ for all $j \in \alpha_i$, $s \not\to t^i_j$ with $j \neq 0$. Since w is $(\bigsqcup_{i\in I} P_i)?(s)$ and $t^i_j \in P_i \subseteq \bigsqcup_{i\in I} P_i$ for all $j \in \alpha_i$, $w \not\to t^i_j$ with $j \neq 0$.

To further simplify the notation, I write Q' for ${\sim}{\sim}\{t^i_j \mid i \in I,\ j \in \alpha_i,\ j \neq 0\}$ and Q for $\bigsqcup_{i\in I} t^i_0$. It follows from the above claim that $w \in {\sim}Q'$. Moreover, note that by Proposition 2.2.3,

$$
\begin{aligned}
\bigsqcup_{i\in I} P_i &= \bigsqcup_{i\in I} {\sim}{\sim}\{t^i_j \mid j \in \alpha_i\} \\
&= {\sim}\bigcap_{i\in I} {\sim}\{t^i_j \mid j \in \alpha_i\} \\
&= {\sim}{\sim}\bigcup_{i\in I} \{t^i_j \mid j \in \alpha_i\} \\
&= {\sim}{\sim}\left(\{t^i_j \mid i \in I,\ j \in \alpha_i,\ j \neq 0\} \cup \{t^i_0 \mid i \in I\}\right) \\
&= {\sim}\left({\sim}\{t^i_j \mid i \in I,\ j \in \alpha_i,\ j \neq 0\} \cap {\sim}\{t^i_0 \mid i \in I\}\right) \\
&= {\sim}{\sim}\{t^i_j \mid i \in I,\ j \in \alpha_i,\ j \neq 0\} \sqcup {\sim}{\sim}\{t^i_0 \mid i \in I\}
\end{aligned}
$$

$$= Q' \sqcup Q$$

By the definition of the sets $\{t^i_j \mid j \in \alpha_j\}$ $Q \subseteq {\sim}Q'$. Since $Q, Q' \in \mathcal{L}_{\mathfrak{F}}$, by Corollary 2.4.8 $Q = {\sim}Q' \cap (Q \sqcup {\sim}{\sim}Q') = {\sim}Q' \cap (Q \sqcup Q')$. Since $w \in {\sim}Q'$ and $w \in \bigsqcup_{i\in I} P_i = Q \sqcup Q'$, $w \in Q$. This means that $(\bigsqcup_{i\in I} P_i)?(s) \in \bigsqcup_{i\in I} P_i?(s)$. ⊣

The following lemma deals with a special but substantial case of Arbitrary Additivity.

5.3.10. LEMMA. *Let $s \in \Sigma$ and $\{P_i\}_{i\in I} \subseteq \mathcal{L}_{\mathfrak{F}}$ satisfy that $I \neq \emptyset$, $P_i \subseteq {\sim}P_j$ whenever $i \neq j$ and $s \notin {\sim}P_i$ for each $i \in I$. $\sum_{i\in I} \mu_s(P_i) = \mu_s(\bigsqcup_{i\in I} P_i)$.*

Proof. Since $I \neq \emptyset$, there is an $i^* \in I$. It follows from $P_{i^*} \subseteq \bigsqcup_{i\in I} P_i$ and $s \notin {\sim}P_{i^*}$ that $s \notin {\sim}\bigsqcup_{i\in I} P_i$. Hence by Property A $(\bigsqcup_{i\in I} P_i)?(s)$ is defined. For $i \in I$, since $s \notin {\sim}P_i$, $P_i?(s)$ is defined and in $\bigsqcup_{i\in I} P_i$. Hence by (ρ4') $\rho(s, P_i?(s)) = \rho(s, (\bigsqcup_{i\in I} P_i)?(s))\rho((\bigsqcup_{i\in I} P_i)?(s), P_i?(s))$. Since $P_i \subseteq {\sim}P_j$ whenever $i \neq j$, $\{P_i?(s) \mid i \in I\}$ is an orthogonal set. Then it follows from Lemma 5.3.7, Lemma 5.3.8 and (ρ3) that both $\sum_{i\in I} \rho(s, P_i?(s))$ and $\sum_{i\in I} \rho((\bigsqcup_{i\in I} P_i)?(s), P_i?(s))$ exist. Hence

$$\begin{aligned}\sum_{i\in I} \rho(s, P_i?(s)) &= \sum_{i\in I} \rho(s, (\bigsqcup_{i\in I} P_i)?(s))\rho((\bigsqcup_{i\in I} P_i)?(s), P_i?(s)) \\ &= \rho(s, (\bigsqcup_{i\in I} P_i)?(s)) \sum_{i\in I} \rho((\bigsqcup_{i\in I} P_i)?(s), P_i?(s))\end{aligned}$$

By definition $\sum_{i\in I} \mu_s(P_i) = \mu_s(\bigsqcup_{i\in I} P_i) \sum_{i\in I} \rho((\bigsqcup_{i\in I} P_i)?(s), P_i?(s))$. By the previous lemma $(\bigsqcup_{i\in I} P_i)?(s) \in \bigsqcup_{i\in I} P_i?(s)$, so $\sum_{i\in I} \rho((\bigsqcup_{i\in I} P_i)?(s), P_i?(s)) = 1$ by (ρ3). Therefore, $\sum_{i\in I} \mu_s(P_i) = \mu_s(\bigsqcup_{i\in I} P_i)$. ⊣

Now Arbitrary Additivity follows from this lemma as a corollary.

5.3.11. COROLLARY. *For every $s \in \Sigma$, μ_s satisfies Arbitrary Additivity, i.e if $s \in \Sigma$ and $\{P_i\}_{i\in I} \subseteq \mathcal{L}_{\mathfrak{F}}$ satisfy $P_i \subseteq {\sim}P_j$ whenever $i \neq j$, then $\sum_{i\in I} \mu_s(P_i) = \mu_s(\bigsqcup_{i\in I} P_i)$.*

Proof. If $I = \emptyset$, then $\bigsqcup_{i\in I} P_i = \emptyset$ and $\sum_{i\in I} \mu_s(P_i) = 0 = \mu_s(\emptyset) = \mu_s(\bigsqcup_{i\in I} P_i)$ by definition. In the following, I focus on the case when $I \neq \emptyset$.

I denote by I_0 the set $\{i \in I \mid s \in {\sim}P_i\}$ and by I_1 the set $I \backslash I_0$. By the previous lemma $\sum_{i\in I_1} \mu_s(P_i) = \mu_s(\bigsqcup_{i\in I_1} P_i)$. For each $i \in I_0$, since $s \in {\sim}P_i$, $\mu_s(P_i) = 0$ by definition. Moreover, since $P_i \subseteq {\sim}P_j$ whenever $i \neq j$, $\bigsqcup_{i\in I_0} P_i \subseteq {\sim}\bigsqcup_{i\in I_1} P_i$. Thus $\mu_s(\bigsqcup_{i\in I} P_i) = \mu_s(\bigsqcup_{i\in I_0} P_i \sqcup \bigsqcup_{i\in I_1} P_i) = \mu_s(\bigsqcup_{i\in I_0} P_i) + \mu_s(\bigsqcup_{i\in I_1} P_i)$ by Lemma 5.3.5. Also note that by the definition of I_0 $s \in {\sim}\bigsqcup_{i\in I_0} P_i$, so $\mu_s(\bigsqcup_{i\in I_0} P_i) = 0$ by definition. Therefore,

$$\sum_{i\in I}\mu_s(P_i)$$
$$= sup\{\sum_{i\in J}\mu_s(P_i) \mid J \text{ is a finite subset of } I\}$$
$$= sup\{\sum_{i\in J\cap I_0}\mu_s(P_i) + \sum_{i\in J\cap I_1}\mu_s(P_i) \mid J \text{ is a finite subset of } I\}$$
$$= sup\{\sum_{i\in J_0}\mu_s(P_i)+\sum_{i\in J_1}\mu_s(P_i) | J_0 \text{ is a finite subset of } I_0,\ J_1 \text{ is a finite subset of } I_1\}$$
$$= sup\{\sum_{i\in J_0}\mu_s(P_i) \mid J_0 \text{ is a finite subset of } I_0\}$$
$$+sup\{\sum_{i\in J_1}\mu_s(P_i) \mid J_1 \text{ is a finite subset of } I_1\}$$
$$= 0 + \sum_{i\in I_1}\mu_s(P_i)$$
$$= \mu_s(\bigsqcup_{i\in I_0} P_i) + \mu_s(\bigsqcup_{i\in I_1} P_i)$$
$$= \mu_s(\bigsqcup_{i\in I} P_i)$$

⊣

5.3.12. Theorem. *For every $s \in \Sigma$, μ_s is a quantum probability measure on the Piron lattice $(\mathcal{L}_{\mathfrak{F}}, \subseteq, \sim(\cdot))$.*

Proof. Proposition 5.3.2 and Proposition 5.3.3 show Normality and Coherence. σ-Additivity is implied by the previous corollary. ⊣

According to Gleason's theorem ([46]), in a Hilbert space not every quantum probability measure is induced in this way from a pure state. Therefore, it remains an interesting question to find a counterpart of Gleason's theorem in the abstract setting of probabilistic quantum Kripke frames and characterize the quantum probability measures on the Piron lattices of bi-orthogonally closed subsets of quantum Kripke frames.

5.4 Quantum Kripke Frames That Can Be Probabilistic

In this section, I observe that the quantum Kripke frames on which probabilistic ones can be built have a special property. This implies that not every quantum Kripke frame can be equipped with a transition probability function and extended to a probabilistic quantum Kripke frame. Moreover, this property has important consequences.

5.4.1. PROPOSITION. *If $(\mathfrak{F}, \rho)$ is a probabilistic quantum Kripke frame where $\mathfrak{F} = (\Sigma, \to)$, then $\mathfrak{F}$ has the following property:*

- ***Countable Non-Orthogonality (CNO)***

  If $s \in \Sigma$ and $\{t_i \mid i \in I\}$ is an orthogonal subset of Σ such that $s \in {\sim\sim}\{t_i \mid i \in I\}$, then the set $\{i \in I \mid s \to t_i\}$ is at most countable.[3]

Proof. By (ρ3) $1 = \sum_{i \in I} \rho(s, t_i) = sup\{\sum_{i \in J} \rho(s, t_i) \mid J \text{ is a finite subset of } I\}$. By the definition of supremum, for each $n \in \mathbb{N}^+$, there is a finite subset J_n of I such that $\sum_{i \in J_n} \rho(s, t_i) > 1 - \frac{1}{2n}$. Then, for each finite subset K of I, $K \cap J_n = \emptyset$ implies that

$$\begin{aligned}
|\sum_{i \in K} \rho(s, t_i)| &= |\sum_{i \in K \cup J_n} \rho(s, t_i) - \sum_{i \in J_n} \rho(s, t_i)| \\
&= |\big(\sum_{i \in I} \rho(s, t_i) - \sum_{i \in J_n} \rho(s, t_i)\big) - \big(\sum_{i \in I} \rho(s, t_i) - \sum_{i \in K \cup J_n} \rho(s, t_i)\big)| \\
&\leq |\sum_{i \in I} \rho(s, t_i) - \sum_{i \in J_n} \rho(s, t_i)| + |\sum_{i \in I} \rho(s, t_i) - \sum_{i \in K \cup J_n} \rho(s, t_i)| \\
&= |1 - \sum_{i \in J_n} \rho(s, t_i)| + |1 - \sum_{i \in K \cup J_n} \rho(s, t_i)| \\
&= \big(1 - \sum_{i \in J_n} \rho(s, t_i)\big) + \big(1 - \sum_{i \in K \cup J_n} \rho(s, t_i)\big) \\
&< 2 \cdot \big(1 - \sum_{i \in J_n} \rho(s, t_i)\big) \\
&< 2 \cdot \frac{1}{2n} = \frac{1}{n}
\end{aligned}$$

Since J_n is finite for each $n \in \mathbb{N}^+$, $\bigcup_{n \in \mathbb{N}^+} J_n$ is at most countable. Moreover, for $i \in I \setminus \bigcup_{n \in \mathbb{N}^+} J_n$, $\rho(s, t_i) < \frac{1}{n}$ for each $n \in \mathbb{N}^+$, so $\rho(s, t_i) = 0$ and thus $s \not\to t_i$ by (ρ2). Hence $\{i \in I \mid s \to t_i\}$ is a subset of $\bigcup_{n \in \mathbb{N}^+} J_n$, and thus is at most countable. ⊣

5.4.2. REMARK. In a quantum Kripke frame $\mathfrak{F}$ Property OHC implies Property CNO. The reason is as follows: In the case when $\mathfrak{F}$ is finite-dimensional, Property CNO holds trivially. In the case when it is infinite-dimensional, by Theorem 2.7.10 $\mathfrak{F} \cong (\Sigma(\mathcal{H}), \to_{\mathcal{H}})$, where $\mathcal{H}$ is an infinite-dimensional Hilbert space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and then Property CNO can be derived in a proof similar to the above.

Using this proposition I can prove the following important proposition. I explain its significance after the proof.

[3] In principle, this proposition and the following one can be derived from some results in Section 5.5 and in [68]. However, since it is not very long, a direct proof may be more helpful in developing intuitions than introducing Mielnik's terminologies and applying his results.

5.4.3. PROPOSITION. *Let $(\mathfrak{F}, \rho)$ be a probabilistic quantum Kripke frame, where $\mathfrak{F} = (\Sigma, \to)$. For every $P \in \mathcal{L}_{\mathfrak{F}}$ and two maximal orthogonal subset Q and Q' of P, Q and Q' have the same cardinality.*

Proof. Consider two cases.

Case 1: There is an $n \in \mathbb{N}$ and an independent set $\{s_1, ..., s_n\} \subseteq P$ such that $P = \mathcal{C}(\{s_1, ..., s_n\})$. Since Q is an orthogonal set, it is easy to see that it is independent. By Theorem B.1.11 it can be extended to a basis $\overline{Q}$ of P. Then $\mathcal{C}(\overline{Q}) = P = \mathcal{C}(\{s_1, ..., s_n\})$. By Theorem B.1.12 $\overline{Q}$ and $\{s_1, ..., s_n\}$ must have the same cardinality, so the cardinality of Q is at most n. Then by Proposition 2.3.23 $\mathcal{C}(Q) = {\sim\sim}Q$. Hence by Lemma 5.3.8 $\mathcal{C}(Q) = P$. Theorem B.1.12 implies that Q and $\{s_1, ..., s_n\}$ must have the same cardinality, so the cardinality of Q is n. A similar argument applies to Q', so the cardinalities of Q and Q' are the same.

Case 2: There is no $n \in \mathbb{N}$ and independent set $\{s_1, ..., s_n\} \subseteq P$ such that $P = \mathcal{C}(\{s_1, ..., s_n\})$. In this case, Q must be infinite; otherwise, $P = {\sim\sim}Q = \mathcal{C}(Q)$, contradicting the assumption of this case. Similarly I conclude that Q' is infinite. I denote the cardinalities of Q and Q' by κ and λ, respectively. Since $Q \subseteq P = {\sim\sim}Q'$ with Q' being orthogonal, I have $\bigcup_{u \in Q}\{v \in Q' \mid u \to v\} \subseteq Q'$. According to Proposition 5.4.1, $\{v \in Q' \mid u \to v\}$ is at most countable for each $u \in Q$. Hence the above inclusion gives rise to the cardinality inequality $\kappa \cdot \aleph_0 \leq \lambda$. Since Q is infinite, $\aleph_0 \leq \kappa$, so $\kappa \leq \lambda$. Symmetrically, I can deduce $\lambda \leq \kappa$. It follows from Cantor-Bernstein Theorem (Theorem 3.2 in [60]) that $\kappa = \lambda$. ⊣

To understand the significance of this result, I introduce some terminology about quantum Kripke frames. Given a subset P of Σ, paralleling to Hilbert space theory, I call by *Hamel basis* an independent set whose linear closure is P, and by *Schauder basis* an orthogonal set whose bi-orthogonal closure is P. In Subsection 2.3.3 by importing the dimension theory of projective geometries I know that Hamel bases exist for every subspaces, and the Hamel bases of the same subspace have the same cardinality. Hence a dimension theory of quantum Kripke frames is built. Now, according to Lemma 5.3.7, Schauder bases exist for every bi-orthogonally closed subsets, and the Schauder bases of the same bi-orthogonally closed subset have the same cardinality. Based on this, another dimension theory of quantum Kripke frames can be built. Remember that a bi-orthogonally closed subset is always a subspace (Lemma 2.3.20), so it can have bases in both of the above senses. Moreover, it is not hard to see from the proof of the proposition that a bi-orthogonally closed subset P has a finite Hamel basis if and only if it has a finite Schauder basis, and in this case all bases, Hamel and Schauder, have the same finite cardinality. However, if a bi-orthogonally closed subset has an infinite basis, Hamel or Schauder, then the cardinality of a Hamel basis will not be the same as that of a Schauder basis in general.[4]

[4]For example, it is well known that, for the Hilbert space $\mathcal{H}$ of square-integrable real-valued functions on $\mathbb{R}$, every Schauder basis is countable, but every Hamel basis is uncountable. This

5.5 Quantum Transition Probability Spaces

In some sense probabilistic quantum Kripke frames are 'quasi-quantitative', because each of them has two primitive structures, the non-orthogonality relation and the transition probability function, although they are closely related. It is somehow plausible and desirable that these two should be combined into one.[5] In this section, I define and study quantum transition probability spaces, which only have the transition probability functions as primitive.

Taking the transition probabilities as primitive is not at all a new idea in the foundations of quantum theory. Mielnik in [68] argued for the significance of the geometric structure imposed on the pure states of a quantum system by the transition probabilities, and he thought that the failure to capture this geometric structure is a main disadvantage of traditional quantum logic proposed by Birkhoff, von Neumann, Piron, etc. To highlight transition probabilities, he defined *probability spaces*, later commonly called *transition probability spaces*, each of which consists of a set and a binary real-valued function on it. He also proved many elementary and important results about them. In particular, he characterized the two-dimensional transition probability spaces which can be embedded in Hilbert spaces. In [94] and [77] efforts are made to connect transition probability spaces with Jauch and Piron's work in the reconstruction of quantum mechanics. Finally, for a nice survey on transition probability spaces in foundations of quantum theory, please refer to Chapter 18 in [17].

I start with the definition of quantum transition probability spaces.

5.5.1. Definition. A *quantum transition probability space* $\mathcal{S}$ is a tuple (Σ, p) such that Σ is a non-empty set and p, called a *transition probability function*, is a function from $\Sigma \times \Sigma$ to $[0, +\infty)$ satisfying the following, where, for every $P \subseteq \Sigma$,

$$\backsim P = \{s \in \Sigma \mid p(s,t) = 0, \text{ for every } t \in P\}$$

(TP1) $p(s,t) = p(t,s)$;

(TP2) if $s \neq t$, then there is a $w \in \Sigma$ such that $p(s,w) = 0$ and $p(t,w) \neq 0$;

(TP3) $\sum_{i \in I} p(s,t_i) = 1$, if $\{t_i \mid i \in I\} \subseteq \Sigma$ satisfies $p(t_i,t_j) = 0$ when $i \neq j$ and $s \in \backsim\backsim\{t_i \mid i \in I\}$;

(TP4) if $s \in \Sigma \setminus \backsim P$ and $P \subseteq \Sigma$ satisfies $P = \backsim\backsim P$, there is a $t \in P$ such that $p(s,w) = p(s,t) \cdot p(t,w)$ for every $w \in P$;

(TP5) for any $s,t \in \Sigma$, there is a $w \in \Sigma$ satisfying $p(s,w) \neq 0$ and $p(t,w) \neq 0$.

implies that, for the bi-orthogonally closed set $\Sigma(\mathcal{H})$ in the quantum Kripke frame $(\Sigma(\mathcal{H}), \rightarrow_{\mathcal{H}})$, every Schauder basis is countable, but every Hamel basis is uncountable.

[5]This question was suggested to me by Dr. Alessandra Palmigiano.

I present a correspondence between probabilistic quantum Kripke frames and quantum transition probability spaces.

First I show how to get a probabilistic quantum Kripke frame from a quantum transition probability space.

5.5.2. Lemma. *Let $\mathcal{S} = (\Sigma, p)$ be a quantum transition probability space. Define a tuple $\mathbf{T}(\mathcal{S}) = ((\Sigma, \to), p)$, where $\to \subseteq \Sigma \times \Sigma$ is such that, for any $s, t \in \Sigma$, $s \to t \Leftrightarrow p(s,t) \neq 0$. Then $\mathbf{T}(\mathcal{S})$ is a probabilistic quantum Kripke frame.*

Moreover, $\mathcal{S} \mapsto \mathbf{T}(\mathcal{S})$ is a class function, denoted by $\mathbf{T}$, from the class of quantum transition probability spaces to the class of probabilistic quantum Kripke frames.

Proof. First I show that $(\Sigma, \to)$ is a quantum Kripke frame.

Note that with the definition of $\to$, Symmetry follows from (TP1), Separation follows from (TP2), and Superposition follows from (TP5).

For Reflexivity, since $s \in \backsim\backsim\{s\}$, $p(s,s) = 1$ by (TP3). Hence $s \to s$.

For Property A, assume that $s \notin \sim P$ and $P \subseteq \Sigma$ satisfies $\sim\sim P = P$. By the definition of $\to$ and $\backsim$ it is easy to see that $\backsim\backsim P = P$ and $s \notin \backsim P$. By (TP4) there is a $t \in P$ such that $p(s,w) = p(s,t) \cdot p(t,w)$ for every $w \in P$. Note that $p(s,t) \neq 0$; otherwise, $p(s,w) = 0$ for every $w \in P$, contradicting that $s \notin \backsim P$. It follows that $p(s,w) = 0 \Leftrightarrow p(t,w) = 0$ for every $w \in P$. According to the definition of $\to$, $s \to w \Leftrightarrow t \to w$ for every $w \in P$.

Second I show that p satisfies (ρ1) to (ρ4). (ρ1) follows from (TP1). (ρ2) follows from the definition of $\to$. (ρ3) follows from (TP3). (ρ4') follows from (TP4). Hence by Lemma 5.1.3 (ρ4) holds.

As a result, $\mathbf{T}(\mathcal{S})$ is a probabilistic quantum Kripke frame. ⊣

Next, I show how to get a quantum transition probability space from a probabilistic quantum Kripke frame.

5.5.3. Lemma. *Let $(\mathfrak{F}, \rho)$ be a probabilistic quantum Kripke frame, where $\mathfrak{F} = (\Sigma, \to)$. Then the tuple $\mathbf{S}(\mathfrak{F}) = (\Sigma, \rho)$ is a quantum transition probability space.*

Moreover, $\mathfrak{F} \mapsto \mathbf{S}(\mathfrak{F})$ is a class function, denoted by $\mathbf{S}$, from the class of probabilistic quantum Kripke frames to the class of quantum transition probability spaces.

Proof. (TP1) follows from (ρ1). (TP2) follows from Separation and (ρ2). (TP3) follows from (ρ3). (TP4) follows from (ρ4'), which can be derived from (ρ4) and Lemma 5.1.3. (TP5) follows from Superposition and (ρ2). ⊣

Now I am ready to prove the correspondence.

5.5.4. Proposition.

1. *For every quantum transition probability space $\mathcal{S}$, $\mathbf{S} \circ \mathbf{T}(\mathcal{S}) = \mathcal{S}$.*

2. *For every probabilistic quantum Kripke frame $\mathfrak{F}$, $\mathbf{T} \circ \mathbf{S}(\mathfrak{F}) = \mathfrak{F}$.*

Proof. **For *1*:** Let $\mathcal{S} = (\Sigma, p)$ be a quantum transition probability space. Then $\mathbf{S} \circ \mathbf{T}(\mathcal{S}) = \mathbf{S}\big(((\Sigma, \to), p)\big) = (\Sigma, p) = \mathcal{S}$.

For *2*: Let $(\mathfrak{F}, \rho)$ be a probabilistic quantum Kripke frame, where $\mathfrak{F} = (\Sigma, \to)$. Then $\mathbf{T} \circ \mathbf{S}(\mathfrak{F}) = \mathbf{T}\big((\Sigma, \rho)\big) = ((\Sigma, \rightsquigarrow), \rho)$. Note that by the definition of $\rightsquigarrow$ and (ρ2), for any $s, t \in \Sigma$, $s \rightsquigarrow t \;\Leftrightarrow\; \rho(s,t) \neq 0 \;\Leftrightarrow\; s \to t$. Therefore, $\mathbf{T} \circ \mathbf{S}(\mathfrak{F}) = ((\Sigma, \to), \rho) = \mathfrak{F}$. ⊣

Given this correspondence between quantum transition probability spaces and probabilistic quantum Kripke frames, the results in Subsections 2.7.1 and 2.7.2 can be employed to show close relations which quantum transition probability spaces have with Hilbert spaces and other quantum structures. Therefore, quantum transition probability spaces can be useful in modelling quantum systems and their behaviour. In particular, the following proposition can be proved:

5.5.5. PROPOSITION. *Let $\mathcal{S} = (\Sigma, p)$ be a quantum transition probability space.*

1. *Define $l \subseteq \Sigma \times \Sigma \times \Sigma$ and $\perp \subseteq \Sigma \times \Sigma$ such that*

   $l(u, v, w) \Leftrightarrow p(s, u) = p(s, v) = 0$ *implies* $p(s, w) = 0$*, for every* $s \in \Sigma$

   $s \perp t \Leftrightarrow p(s,t) = 0$

   Then $(\Sigma, l, \perp)$ is an irreducible Hilbertian geometry.

2. *If there are $s_i \in \Sigma$, $i = 1, 2, 3, 4$, satisfying $p(s_i, s_j) = 0$ whenever $i \neq j$, then there is a generalized Hilbert space V and a bijection $F : \Sigma \to \Sigma(V)$ such that, for any $s, t \in \Sigma$,*

$$p(s,t) \neq 0 \;\Leftrightarrow\; F(s) \to_V F(t)$$

3. *Define $\to \subseteq \Sigma \times \Sigma$ such that $s \to t \Leftrightarrow p(s,t) \neq 0$. If $(\Sigma, \to)$ satisfies the conditions in (i) of Theorem 2.7.18, then there is an infinite-dimensional Hilbert space $\mathcal{H}$ over $\mathbb{C}$ and a bijection $F : \Sigma \to \Sigma(\mathcal{H})$ such that, for any $s, t \in \Sigma$,*

$$p(s,t) \neq 0 \;\Leftrightarrow\; s \to t \;\Leftrightarrow\; F(s) \to_{\mathcal{H}} F(t)$$

Proof. By Lemma 5.5.2 $(\Sigma, \to)$ is a quantum Kripke frame, where $\to$ is defined in the same way as in *3*. For *1*, it is not hard to see that $(\Sigma, l, \perp)$ is identical to $\mathbf{G}((\Sigma, \to))$, and thus is an irreducible Hilbertian geometry by Theorem 2.5.2. *2* follows from Corollary 2.5.6, and *3* follows from Theorem 2.7.18. ⊣

This proposition is helpful, but it also hints at important open questions. For *2*, note that two division rings are involved here. One is the field $\mathbb{R}$ of the real numbers, and the other is the division ring over which the generalized Hilbert space V is. It is not clear what the relation is between them. It is an interesting problem to characterize the cases when these two division rings are homomorphic. For *3*, given that $\mathbb{R}$ is a sub-field of $\mathbb{C}$, a more specific question can be raised, that is whether $p = \rho_{\mathcal{H}}$ and thus $\mathcal{S} \cong (\Sigma(\mathcal{H}), \rho_{\mathcal{H}})$ where $\rho_{\mathcal{H}}$ is defined in the same way as that at the beginning of Section 5.2. Given the correspondence between probabilistic quantum Kripke fames and quantum transition probability spaces, this question is essentially the same as the one raised at the end of Section 5.2. These are all important questions about how probabilistic quantum Kripke fames and quantum transition probability spaces relate to Hilbert spaces. The answers to them will be the cornerstones of reconstructing quantum theory from transition probabilities.

Chapter 6

Conclusion

6.1 Summary

In this thesis I study in depth the non-orthogonality relation in quantum theory. I show that the Kripke frames whose binary relations possess some basic properties of this relation are useful in modelling quantum systems and promising for a relational reconstruction of quantum theory. In the following I summarize the main results in each technical chapter of this thesis and discuss their significance.

Chapter 2 Quantum Kripke Frames. In Chapter 2 I introduce quantum Kripke frames, the protagonists of this thesis. The main results are listed below. In this list, Item 1 is the starting point of this chapter as well as of the whole thesis. Items 2 and 4 provide a sophisticated understanding of the structure of quantum Kripke frames and other related kinds of Kripke frames. Powerful tools from geometry are also introduced into the study of these Kripke frames. As a result, Item 3 shows the close relation between quantum Kripke frames and some important quantum structures, and is also an improvement of some results in the literature. Moreover, Item 5 strongly hints that quantum Kripke frames are useful in modelling quantum systems and their behaviour.

1. Inspired by the properties of the non-orthogonality relation in quantum theory, I define five special kinds of Kripke frames, from general ones to more specific ones: state spaces, geometric frames, complete geometric frames, quasi-quantum Kripke frames and quantum Kripke frames (Definition 2.1.4).

2. I study the structure of each of these kinds of Kripke frames from a geometric perspective. In particular, I prove four correspondences between Kripke frames and projective geometries (Theorem 2.3.17, Corollary 2.3.18, Theorem 2.4.10 and Theorem 2.5.2). As far as I know, this is the first time

when some kinds of projective geometries are proved to be Kripke frames in disguise. They are the following:

geometric frame – projective geometry with a pure polarity
geometric frame – pure orthogeometry
complete geometric frame – Hilbertian geometry
quasi-quantum Kripke frame – irreducible pure orthogeometry
quantum Kripke frame – irreducible Hilbertian geometry

3. Based on Item 2, I prove a correspondence between quantum Kripke frames and Piron lattices (Theorem 2.7.23). This result gives a sufficient and necessary set of conditions on a Kripke frame for the bi-orthogonally closed subsets to form a Piron lattice. In the literature, for example [52], there are only sufficient conditions, and they are more complicated than those in the definition of quantum Kripke frames.

4. I prove a representation theorem of quasi-quantum Kripke frames via vector spaces over a division ring equipped with an anisotropic Hermitian form (Corollary 2.5.5), and one of quantum Kripke frames via generalized Hilbert spaces (Corollary 2.5.6). They make it possible to study these two important kinds of Kripke frames using the analytic method, which is very useful.

5. I prove that the quantum Kripke frames induced by the infinite-dimensional Hilbert spaces over $\mathbb{C}$ can be characterized as those being Pappian and having an infinite orthogonal set, Property OHC and Property SR (Theorem 2.7.18). This characterization is simple from a logical point of view. Except that Property A and the existence of an infinite orthogonal set are second-order, the other conditions on Kripke frames are all first-order. This result shows that quantum Kripke frames are closely related to Hilbert spaces.

Chapter 2 is about Research Question 1 in Subsection 1.1.3. According to Item 3, the answer to this question is yes.

Chapter 3 Maps between Quantum Kripke Frames. The linear maps between Hilbert spaces play an important role in the formalism of quantum theory, so the maps between quantum Kripke frames are worth studying. This is undertaken Chapter 3. The main results are listed below. In this list, Item 1 is the starting point of the chapter. Continuous homomorphisms, the protagonists, are defined and shown to be closely related to the linear maps having adjoints on Hilbert spaces. Item 2 shows that the important kinds of linear maps in the Hilbert space formalism of quantum theory have counterparts in the framework of quantum Kripke frames. Item 4 shows that the tensor product construction of Hilbert spaces based on linear maps also has a counterpart in a special case.

Items 2 and 4 together show that quantum Kripke frames and the continuous homomorphisms between them are the abstractions from Hilbert spaces which are significant from a physical perspective. Item 3 is not only an interesting result in itself but also an important preliminary to Item 4.

1. I define continuous homomorphisms between quasi-quantum Kripke frames (Definition 3.1.1), and prove a representation theorem of them via continuous quasi-linear maps (Corollary 3.1.14). Note that continuous quasi-linear maps are mild generalizations of linear maps having adjoints between Hilbert spaces, which are used to model quantum entanglement. Hence this result shows that continuous homomorphisms can be useful in modelling quantum entanglement. Moreover, it paves the way for introducing the useful analytic method to study this significant kind of map between quasi-quantum Kripke frames.

2. I study three special kinds of continuous homomorphisms between quantum Kripke frames, which are defined parallel to the unitary operators, self-adjoint operators and projectors, respectively, on Hilbert spaces. I prove simple characterizations of them in terms of the non-orthogonality relation (Proposition 3.1.17, Proposition 3.1.20, Remark 3.1.22).

3. I characterize geometrically the arguesian homomorphisms between two Pappian projective geometries that are induced by linear maps (Theorem 3.2.13 and Corollary 3.2.14). This solves a special case of Problem 4 in the List of Problems in [39] (p 345).

4. Using the analytic method, I prove that, if there is a non-degenerate continuous homomorphism between them, two finite-dimensional Pappian quantum Kripke frames satisfying Property OHC can be amalgamated into one quantum Kripke frame (Theorem 3.3.12). Moreover, if the two quantum Kripke frames are represented by two generalized Hilbert spaces, their amalgamation will be represented by the tensor product of the two spaces. This is a solution to a special case of the tensor product problem in the framework of quantum Kripke frames.

Chapter 3 is about Research Question 2 in Subsection 1.1.3. According to Items 2 and 4, the objects and structure of Hilbert spaces used in modelling evolution, observables, testable properties and compound systems have counterparts in quantum Kripke frames.

Chapter 4 Logics of Quantum Kripke Frames. Since quantum Kripke frames are shown to be useful in the previous chapters, it is natural to consider the automated reasoning about them. To pave the way for this, Chapter 4 studies the logics of quantum Kripke frames in some formal languages. The main results are listed below. In this list, Item 1 and Item 2 provide decidable, sound and strongly complete axiomatizations in modal logic of state spaces and of state spaces satisfying Superposition, respectively. The proofs of these results hint that the modal language is not expressive enough for useful descriptions of quantum Kripke frames. Item 3 shows that the first-order theory of quantum Kripke frames is undecidable. Hence, as far as automated reasoning is concerned, it is not wise to consider formal languages that are more expressive than the first-order language. Therefore, Items 1, 2 and 3 suggest that the right language for automated reasoning about quantum Kripke frames should be some fragment of the first-order language. Item 4 not only is an interesting result in itself, but also gives a negative answer to the question whether Property A, the second-order condition in the definition of quantum Kripke frames, can be replaced by a first-order one. Moreover, it raises an interesting problem about axiomatization, which I will discuss later.

1. I prove that the decidable modal logic **B** is sound and strongly complete with respect to the class of state spaces (Theorem 4.3.4).

2. I prove that the modal logic Λ obtained by adding the axiom $\Box\Box p \to \Box\Box\Box p$ to **B** is sound and strongly complete with respect to the class of state spaces satisfying Superposition (Proposition 4.3.5 and Proposition 4.3.7). Moreover, this logic is decidable (Corollary 4.3.11).

3. I prove that the first-order theories of quasi-quantum Kripke frames, quantum Kripke frames and Piron lattices, respectively, are undecidable (Corollary 4.2.9 and Theorem 4.2.11).

4. I characterize the first-order definable, bi-orthogonally closed subsets of the quasi-quantum Kripke frames satisfying Property OHC (Theorem 4.1.8). Moreover, two important corollaries follow. One is that first-order quantum Kripke frames satisfying Property OHC are finitely first-order axiomatizable (Corollary 4.1.11). The other is that quantum Kripke frames are not first-order definable in the class of quasi-quantum Kripke frames (Theorem 4.1.15).

Chapter 4 is about Research Question 3 in Subsection 1.1.3. Item 3 shows that the first-order theory of quantum Kripke frames is undecidable. Items 1 and 2 are helpful for the axiomatization of quantum Kripke frames in modal logic, while Item 4 gives some hints at the axiomatization of quantum Kripke frames in first-order logic.

Chapter 5 Probabilistic Quantum Kripke Frames. Chapter 5 is a pilot study of the transition probabilities between pure states, which are the more fine-grained version of the non-orthogonality relation. The main results are listed below. In this list, Item 1 is the starting point of the chapter where probabilistic quantum Kripke frames are defined. Items 2 and 3 show the significance of probabilistic quantum Kripke frames. Item 2 means that the conditions in the definition are sound with respect to the transition probabilities between the pure states of quantum systems. Item 3 shows that probabilistic quantum Kripke frames give rise to quantum probability measures in an expected way, and thus can be helpful in studying quantum probability theory. Item 4 resolves the dualism of the primitives in probabilistic quantum Kripke frames, and leads to the nice notion of quantum transition probability spaces, in which only the transition probabilities are primitive.

1. Inspired by the properties of the transition probabilities in quantum theory, I define the notion of probabilistic quantum Kripke frames (Definition 5.1.1).

2. I show that a probabilistic quantum Kripke frame can be abstracted from every Hilbert space over $\mathbb{C}$ (Proposition 5.2.1).

3. I show that in a probabilistic quantum Kripke frame every element gives rise in a natural way to a quantum probability measure on the Piron lattice of bi-orthogonally closed subsets (Theorem 5.3.12).

4. I discover a special property of the quantum Kripke frames underlying probabilistic quantum Kripke frames: if an element is in the bi-orthogonal closure of an infinite orthogonal set, it is non-orthogonal to at most countably many of the elements in the set (Proposition 5.4.1). I explain that this property is crucial to build a dimension theory of quantum Kripke frames based on orthogonal sets and bi-orthogonal closures.

5. I define quantum transition probability spaces (Definition 5.5.1), and prove a correspondence between them and probabilistic quantum Kripke frames (Proposition 5.5.4).

Chapter 5 is about Research Question 4 in Subsection 1.1.3. Quantum transition probability spaces are shown to have not only the same nice structure as quantum Kripke frames but also quantitative modelling power. Therefore, their further study will be promising.

6.2 Future Work

The results in this thesis answer many questions, but they also raise a lot of questions for future work. I discuss them in detail.

Questions for Foundations of Quantum Theory

Quantum Kripke Frames Induced by Hilbert Spaces over $\mathbb{C}$. This is about improving Item 5 in the list of results of Chapter 1. Progress in this line will explain in terms of the properties of the non-orthogonality relation why Hilbert spaces over $\mathbb{C}$ are exclusively used in the formalism of quantum theory. This would also resolve the surprise that the complex numbers are used in quantum physics. Therefore, it will be a cornerstone of the relational reconstruction of quantum theory based on the non-orthogonality relation. To be precise, the problem is as follows:

> **Problem 1:** Find a set of simple, intuitive conditions to characterize the quantum Kripke frames induced by Hilbert spaces over $\mathbb{C}$.

Theorem 2.7.18 and Corollary 2.7.19 show the possibility to characterize the quantum Kripke frames induced by Hilbert spaces over $\mathbb{C}$. However, the conditions involved are too technical and are a bit remote from direct physical intuition. Therefore, it is interesting to find simpler and more intuitive conditions.

This characterization may not be achieved in such a way that simpler but equivalent conditions are found for each of the conditions in Theorem 2.7.18. However, some conditions in Theorem 2.7.18 are interesting from other perspectives, and finding an equivalent condition of each of them separately would be an interesting problem. Hence I raise the following two sub-problems related to Problem 1.

> **Sub-Problem 1.1:** Find a simple and intuitive condition on quasi-quantum Kripke frames which is equivalent to being Pappian.

According to Theorem 3.3.12, the tensor-product-like amalgamation of quantum Kripke frames can be applied to the quantum Kripke frames which are more general than those induced by Hilbert spaces. However, being Pappian is one of the restrictions which cannot be dropped. In mathematics being Pappian of a quantum Kripke frame is equivalent to being able to be represented by a vector space over a field. From Appendix C one can see that the commutativity of fields is crucial to the tensor product construction from linear maps. In particular, according to the proof of Theorem C.3.1, the linear maps between two vector spaces may not form a vector space if the vector spaces are not over fields. Given this importance, it is interesting to find a condition simpler but equivalent to being Pappian. This would help to understand the physics behind the tensor product construction.

> **Sub-Problem 1.2:** Find a simple and intuitive condition on quasi-quantum Kripke frames which is equivalent to Property OHC.

According to the results in Chapter 4 and in Section 5.4 Property OHC implies many nice properties of quantum Kripke frames. Therefore, it is interesting to find a condition that is simpler but equivalent to Property OHC. Proposition 4.1.6 gives a promising start, but it will be good to find a first-order condition which does not involve automorphisms. Such a condition may be found by trying to prove this proposition using the geometric method instead of the analytic one.

Tensor-Product-Like Amalgamation of Quantum Kripke Frames. This is about improving Item 4 in the list of results of Chapter 3. Progress in this line leads to an understanding of quantum entanglement from the perspective of the non-orthogonality relation, which is indispensable for a relational reconstruction of quantum theory. To be precise, there are two problems.

> **Problem 2:** Define the non-orthogonality relation between the continuous homomorphisms from one quantum Kripke frame to another in a way which is simpler but still equivalent to Definition 3.3.10.

Defining the non-orthogonality relation between the continuous homomorphisms is a crucial step for Theorem 3.3.12. However, Definition 3.3.10 involves many complicated geometric constructions and thus is highly unsatisfactory. Therefore, it is desirable to replace Definition 3.3.10 with a simpler and more intuitive one. One useful observation is that to define this relation, instead of the full strength of traces, what is needed is only a characterization of the linear maps with trace zero. Therefore, it may be possible to find a simpler definition by using some deep results in linear algebra. An example could be the following, which is first proved by K. Shoda [81] for the fields of characteristic 0 and then generalized to arbitrary fields by A. Albert and B. Muckenhoupt [8]:

6.2.1. THEOREM (SHODA-ALBERT-MUCKENHOUPT). *For each n-rowed square matrix M over some* field $\mathcal{K}$, *the following are equivalent:*

- *(i) the trace of M is* 0*;*
- *(ii) there are two n-rowed square matrices A and B such that $M = AB - BA$.*

Although it may only work for finite-dimensional quantum Kripke frames, a solution to this problem is still valuable for understanding the structure of quantum entanglement. In quantum computation and quantum information only the quantum systems described by finite-dimensional Hilbert spaces are involved, but the entanglement between them is important and useful.

> **Problem 3:** Generalize and extend Theorem 3.3.12 to the infinite-dimensional case.

A limitation of Theorem 3.3.12, due to Assumption 1, is that the construction only works for finite-dimensional quantum Kripke frames. It is desirable to resolve this limitation and extend the theorem, but trying to do this leads to many difficulties. One of the difficulties is that continuous homomorphisms may no longer work in the general case. In Hilbert space theory, the tensor product of two infinite-dimensional Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is built from the set of Hilbert-Schmidt operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ (Section 2.6 in [61]). A *Hilbert-Schmidt operator from* $\mathcal{H}_1$ *to* $\mathcal{H}_2$ is a bounded linear map $f : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\sum_{\mathbf{x}\in X} \sum_{\mathbf{y}\in Y} |\langle f(\mathbf{x}), \mathbf{y}\rangle|^2$ is finite, for any orthonormal bases X and Y of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. According to Corollary 3.1.14, the arguesian continuous homomorphisms between two quantum Kripke frames induced by Hilbert spaces over $\mathbb{C}$ correspond to the linear maps between the two Hilbert spaces which have adjoints. Therefore, it is not clear whether continuous homomorphisms are the geometric counterparts of Hilbert-Schmidt operators. It is an interesting and challenging problem in mathematics to characterize the Hilbert-Schmidt operators in terms of the non-orthogonality relation. Another difficulty is to define the non-orthogonality relation. Finite-dimensionality is crucial in Definition 3.3.10 and the Shoda-Albert-Muckenhoupt Theorem mentioned above. It is not clear how to characterize the linear maps between infinite-dimensional Hilbert spaces with trace zero and to define the non-orthogonality relation.

Quantum Transition Probability Spaces Induced by Hilbert Spaces. Corollary 2.7.19 characterizes the quantum Kripke frames induced by Hilbert spaces over $\mathbb{C}$, and it is desirable to find a similar result for quantum transition probability spaces. A characterization of the quantum transition probability spaces induced by Hilbert spaces over $\mathbb{C}$ will be not only useful in modelling quantum systems but also a cornerstone of reconstructing quantum theory from the transition probabilities. To be precise, the problem is the following:

> **Problem 4:** Find some conditions such that, if a quantum transition probability space $\mathcal{S} = (\Sigma, p)$ satisfies all of them, there will be a Hilbert space $\mathcal{H}$ over $\mathbb{C}$ and a bijection $i : \Sigma \to \Sigma(\mathcal{H})$ such that $p(s,t) = \rho_{\mathcal{H}}(i(s), i(t))$ holds for any $s, t \in \Sigma$.

Proposition 5.5.5 only gives a set of conditions which guarantees the existence of a Hilbert space $\mathcal{H}$ over $\mathbb{C}$ and a bijection $i : \Sigma \to \Sigma(\mathcal{H})$ such that $p(s,t) = 0 \Leftrightarrow \rho_{\mathcal{H}}(i(s), i(t)) = 0$ holds for any $s, t \in \Sigma$. It is an interesting direction to strengthen the consequent of this proposition and also find a simpler and more intuitive set of conditions for the antecedent.

This problem can be considered from a more general perspective. Proposition 5.5.5 gives a set of conditions which guarantees the existence of a *generalized Hilbert space* V *over a division ring* $\mathcal{K}$ and a bijection $i : \Sigma \to \Sigma(V)$ such that $p(s,t) = 0 \Leftrightarrow i(s) \to_V i(t)$ holds for any $s, t \in \Sigma$. Here two division rings are involved: the field $\mathbb{R}$ of the real numbers and the division ring $\mathcal{K}$. It is interesting to investigate the relation between these two division rings and find conditions to make them relate in nice ways like being homomorphic. Progress on this problem may help us solve Problem 4.

Questions for Logics of Kripke Frames

Axiomatization of Quantum Kripke Frames in Modal Logic. This is about improving Item 2 in the list of results of Chapter 4. To be precise, the problem is the following:

> **Problem 5:** Axiomatize quantum Kripke frames in some propositional modal language under some modal semantics.

As is shown in Section 4.3, there are modal formulas which are canonical with respect to Reflexivity, Symmetry and Superposition, respectively. Separation is not modal definable, but it can be imposed by some model transformations like mirror union. In fact, Property A is the hardest to capture in the modal language. The reason is that it concerns a special relation between two points which is defined by how they connect to other points. On the contrary, intuitively speaking, a Kripke modality always 'centers' on one point at a time. This makes Property A hard to capture.

I can conceive of two ways to overcome this. One way is to stick to the Kripke modality of the non-orthogonality relation and look for some modal logic, maybe Λ in Subsection 4.3.2, which axiomatizes quantum Kripke frames due to the lack of expressive power of the modal language. Efforts in this line will lead to a deeper understanding of quantum Kripke frames and of bisimulations, perhaps also to new constructions on Kripke frames. The other way is to define a kind of modality which is different from the Kripke modality and tailored to express the conditions defining quantum Kripke frames. This new modal semantics will become a new branch and foster the development of modal logic.

Axiomatization of the First-Order Theory of Quantum Kripke Frames. Although quantum Kripke frames are defined using a second-order condition and the first-order theory of them is proved to be undecidable, it is still interesting to see whether this theory can be finitely axiomatized and thus be recursively enumerable. The hope comes from Corollary 4.1.11, i.e. Item 4 in the list of results of Chapter 4. It implies that the first-order quantum Kripke frames satisfying Property OHC are finitely first-order axiomatizable. Considering that the first-order quantum Kripke frames are the first-order 'approximations' of quantum Kripke frames, I raise the following problem:

> **Problem 6:** Prove that the first-order theory of quasi-quantum Kripke frames satisfying Property OHC is complete with respect to the class of quantum Kripke frames, or prove that this is not the case.

I will elaborate on the strategy of proving completeness. Following the usual idea, the main task is the following: given a set of first-order formulas true in a first-order quantum Kripke frame satisfying Property OHC, which is just a quasi-quantum Kripke frame by Corollary 4.1.11, construct a quantum Kripke frame in which the formulas are also true. Note that, if this quasi-quantum Kripke frame is finite-dimensional, it is done according to Theorem 2.4.19. Hence the only interesting case is when this quasi-quantum Kripke frame is infinite-dimensional. Then by Corollary 2.5.5 it is isomorphic to $(\Sigma(V), \to_V)$ for some vector space V over a division ring equipped with an anisotropic Hermitian form. Moreover, note that the quantum Kripke frame being constructed must have Property OHC and can be infinite-dimensional without loss of generality, so it will be isomorphic to $(\Sigma(\mathcal{H}), \to_{\mathcal{H}})$ for some Hilbert space $\mathcal{H}$ over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ by Theorem 2.7.10. To make the formulas true in $(\Sigma(\mathcal{H}), \to_{\mathcal{H}})$, a coarse strategy is to show that $(\Sigma(V), \to_V)$ and $(\Sigma(\mathcal{H}), \to_{\mathcal{H}})$ are elementary equivalent. Then the crux becomes finding the right Hilbert space $\mathcal{H}$ and the right division ring among $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ such that the Verifier has a winning strategy in the EhrenfeuchtFraïssé game between $(\Sigma(V), \to_V)$ and $(\Sigma(\mathcal{H}), \to_{\mathcal{H}})$. This is a very interesting problem.

Questions about Quantum Kripke Frames

Quantum Probability Measures in Quantum Kripke Frames. This is an extension of Item 3 in the list of results of Chapter 5. According to this result, in a probabilistic quantum Kripke frame every element, interpreted as a pure state, gives rise to a quantum probability measure on the Piron lattice of bi-orthogonally closed subsets. According to quantum theory, quantum probability measures are induced by not only the pure states but also the mixed states. Therefore, it is interesting to consider the following problem:

> **Problem 7:** Define in a probabilistic quantum Kripke frame the notion of mixed states in terms of the transition probabilities, and thus characterize the quantum probability measures on the Piron lattice of bi-orthogonally closed subsets.

Research in this line will be helpful both in modelling the mixed states using probabilistic quantum Kripke frames and in the study of quantum probability theory.

Topology in Quantum Kripke Frames. The norm topology and its properties are very crucial to the structure of Hilbert spaces. However, in the thesis the topological perspective is completely absent. In future work it is interesting to find some topological structures inside quantum Kripke frames and study their properties. Here I raise two specific questions. For the first one, remember that, according to the Piron-Amemiya-Arakia Theorem, in a pre-Hilbert space Weak Modularity of the lattice of closed linear subspaces is equivalent to the completeness of the norm topology. From Section 2.4.1 it is clear that in a quasi-quantum Kripke frame Property A is crucial to Weak Modularity of the lattice of bi-orthogonally closed subsets. Hence it is natural to raise the following problem:

> **Problem 8:** Define a topology on a quasi-quantum Kripke frame in such a way that an analogue of the Piron-Amemiya-Arakia Theorem can be proved, i.e. the completeness of this topology is equivalent to Property A.

For the second question, note that the real numbers are used in the definition of probabilistic quantum Kripke frames and thus can help to define a topology. In a probabilistic quantum Kripke frame $(\mathfrak{F}, \rho)$ where $\mathfrak{F} = (\Sigma, \to)$, for every $s \in \Sigma$ and $\epsilon \in [0, 1]$ define $B_\epsilon(s)$ to be $\{t \in \Sigma \mid \rho(s, t) > \epsilon\}$. Then the problem is the following:

> **Problem 9:** Find out whether $\{B_\epsilon(s) \subseteq \Sigma \mid s \in \Sigma \text{ and } \epsilon \in [0, 1]\}$ generates the trivial topology. If not, reveal the properties and significance of this topology.

As a concluding remark, the work in this thesis shows that the study of quantum Kripke frames is promising towards a relational reconstruction of quantum theory. It provides a simple and intuitive perspective on quantum theory. It also raises many interesting questions, and a lot of development is on the way. Moreover, it inspires new directions for research in logic and mathematics.

Appendix A

Quantum Structures

The main goal of this appendix is to review the definitions of two kinds of quantum structures in the literature - Piron lattices and quantum dynamic frames.

First, I recall the notion of Piron lattices and some related notions. This is just for the convenience of the readers of this thesis. For a more detailed discussion about these notions and their properties, please refer to Section 1 in [92] and [84]. For the significance of these notions in physics, I recommend [59] and [74].

I start from the notion of posets.

A.0.2. Definition. A *partially ordered set*, or for short *poset*, is a tuple $\mathfrak{P} = (P, \leq)$ such that P is a non-empty set and $\leq\ \subseteq P \times P$ satisfies all of the following:

1. (Reflexivity) $a \leq a$, for every $a \in P$;
2. (Anti-Symmetry) $a \leq b$ and $b \leq a$ imply that $a = b$, for any $a, b \in P$;
3. (Transitivity) $a \leq b$ and $b \leq c$ imply that $a \leq c$, for any $a, b, c \in P$.

The following definition includes many useful notions on a poset.

A.0.3. Definition. Let $\mathfrak{P} = (P, \leq)$ be a poset.

- A *top in* $\mathfrak{P}$ is an $I \in P$ such that $a \leq I$ for every $a \in P$.
- A *bottom in* $\mathfrak{P}$ is an $O \in P$ such that $O \leq a$ for every $a \in P$.
- The *(arbitrary) meet*, or *greatest lower bound, of* $A \subseteq P$ is an element $\bigwedge A$ in P such that both of the following hold:
  1. $\bigwedge A \leq a$, for every $a \in A$;
  2. For every $b \in P$, $b \leq a$ for all $a \in A$ implies that $b \leq \bigwedge A$.

  The meet of $a, b \in P$ is usually written as $a \wedge b$.

- The *(arbitrary) join*, or *least upper bound, of* $A \subseteq P$ is an element $\bigvee A$ in P such that both of the following hold:

  1. $a \leq \bigvee A$, for every $a \in A$;
  2. For every $b \in P$, $a \leq b$ for all $a \in A$ implies that $\bigvee A \leq b$.

  The join of $a, b \in P$ is usually written as $a \vee b$.

- If $\mathfrak{P}$ has a bottom O, an *atom of* $\mathfrak{P}$ is a $p \in P$ such that

  1. $O \neq p$; and
  2. $O \leq a \leq p$ implies that $a = O$ or $a = p$, for every $a \in P$.

- If $\mathfrak{P}$ has a top I and a bottom O, an *orthocomplementation on* $\mathfrak{P}$ is a function $(\cdot)' : P \to P$ such that all of the following hold:

  1. (Complement Law) $a \vee a' = I$ and $a \wedge a' = O$, for every $a \in P$;
  2. (Order-Reversing) $a \leq b$ implies that $b' \leq a'$, for any $a, b \in P$;
  3. (Involution Law) $a'' = a$, for every $a \in P$.

  For $a \in P$, a' is called *the orthocomplement of* a.

Posets may have special names when objects for these notions exist.

A.0.4. Definition.

- A poset is *bounded*, if it has a top and a bottom.
- A bounded poset equipped with an orthocomplementation is called an *ortho-poset*.
- An ortho-poset $\mathfrak{P} = (P, \leq, (\cdot)')$ is an *orthomodular poset*, if

  1. $a \vee b$ exists, for any $a, b \in P$ with $a \leq b'$; and
  2. $a \leq b$ implies that $a = b \wedge (a \vee b')$, for any $a, b \in P$.

- A poset $\mathfrak{P} = (P, \leq)$ is a *lattice*, if both $a \wedge b$ and $a \vee b$ exist for any $a, b \in P$.
- A lattice is *bounded*, if it is a bounded poset.
- An *orthocomplemented lattice*, or for short *ortho-lattice*, is an ortho-poset forming a lattice.

I list some properties of lattices that are often used in quantum logic.

A.0.5. Definition. Let $\mathfrak{L} = (L, \leq)$ be a lattice.

- Assuming that $\mathfrak{L}$ has a bottom O, it is *atomic*, if there is an atom $p \leq a$ for every $a \in L \setminus \{O\}$.
- Assuming that $\mathfrak{L}$ has a bottom O, it is *atomistic*, if, for every $a \in L$, $a = \bigvee\{p \in L \mid p \text{ is an atom and } p \leq a\}$.
- $\mathfrak{L}$ is *complete*, if both $\bigwedge A$ and $\bigvee A$ exist for every $A \subseteq L$.
- Assuming that $\mathfrak{L}$ has a bottom O, it satisfies *the Covering Law*, if, for every atom p of $\mathfrak{L}$ and $b \in L$ satisfying $p \wedge b = O$, $p \vee b$ covers b, i.e. $c = b$ or $c = p \vee b$ for every $c \in L$ with $b \leq c \leq p \vee b$.
- Assuming that $\mathfrak{L}$ is an ortho-lattice with $(\cdot)' : L \to L$ as the orthocomplementation, it satisfies *Weak Modularity*, if $a \leq b$ implies that $a = b \wedge (a \vee b')$ for any $a, b \in L$.

Next I introduce a useful construction on ortho-lattices.

A.0.6. Definition. Given two ortho-lattices $\mathfrak{L}_1 = (L_1, \leq_1, (\cdot)^{\perp_1})$ and $\mathfrak{L}_2 = (L_2, \leq_2, (\cdot)^{\perp_2})$, the *direct product of* $\mathfrak{L}_1$ *and* $\mathfrak{L}_2$ is a tuple $(L, \leq, (\cdot)')$ such that:

1. $L = L_1 \times L_2$;
2. for any $(a_1, a_2), (b_1, b_2) \in L$, $(a_1, a_2) \leq (b_1, b_2)$, if $a_1 \leq_1 b_1$ and $a_2 \leq_2 b_2$;
3. for any $(a_1, a_2) \in L$, $(a_1, a_2)' = (a_1^{\perp_1}, a_2^{\perp_2})$.

Now I am ready to define Piron lattices.

A.0.7. Definition. A *propositional system* is a complete, atomistic ortho-lattice satisfying the Covering Law and Weak Modularity.

A propositional system is *irreducible*, if it is not isomorphic to the direct product of two propositional systems, both of which have at least two elements.

A *Piron lattice* is an irreducible propositional system.

Second, I recall the notion of quantum dynamic frames. They are first proposed in [14], and their mathematical properties and relation with Piron lattices are studied in detail in [20].[1]

A.0.8. Definition. A *quantum dynamic frame* is a tuple $\mathsf{F} = (\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P \in \mathcal{L}})$, where

1. Σ is a non-empty set; and

[1]The definition of quantum dynamic frames in this thesis is from [20], which is different from the one in [14]. For the reasons, please refer to the explanation after Definition 2.7 in [20].

2. $\mathcal{L} \subseteq \wp(\Sigma)$; and

3. $\stackrel{P?}{\rightarrow} \subseteq \Sigma \times \Sigma$, for every $P \in \mathcal{L}$;

such that all of the following are satisfied, where $\rightarrow \stackrel{\text{def}}{=} \bigcup_{P \in \mathcal{L}} \stackrel{P?}{\rightarrow}$:

1. $\mathcal{L}$ is closed under arbitrary intersection, and, for every $P \in \mathcal{L}$, $\sim P \stackrel{\text{def}}{=} \{s \in \Sigma \mid s \rightarrow t \text{ does not hold, for every } t \in P\} \in \mathcal{L}$;

2. (Atomicity) $\{s\} \in \mathcal{L}$, for every $s \in \Sigma$;

3. (Adequacy) $s \in P$ implies that $s \stackrel{P?}{\rightarrow} s$, for any $s \in \Sigma$ and $P \in \mathcal{L}$;

4. (Repeatability) $s \stackrel{P?}{\rightarrow} t$ implies that $t \in P$, for any $s, t \in \Sigma$ and $P \in \mathcal{L}$;

5. (Self-Adjointness) if $s \stackrel{P?}{\rightarrow} t$ and $t \rightarrow u$, then there is a $v \in \Sigma$ such that $u \stackrel{P?}{\rightarrow} v$ and $v \rightarrow s$, for any $s, t, u \in \Sigma$ and $P \in \mathcal{L}$;

6. (Covering Property) if $s \stackrel{P?}{\rightarrow} t$, $t \neq u$ and $u \in P$, then there is a $v \in P$ such that $u \rightarrow v$ and $v \not\rightarrow s$, for any $s, t, u \in \Sigma$ and $P \in \mathcal{L}$;

7. (Proper Superposition) for any $s, t \in \Sigma$, there is a $w \in \Sigma$ such that $w \rightarrow s$ and $w \rightarrow t$.

Appendix B

Geometry and Algebra

In this appendix, I review the close relation between projective geometry and linear algebra. In [11] this is presented for the first time in a systematic way, and [39] is an up-to-date monograph.

B.1 Projective Geometry

In this section I review some elements of projective geometry. If without explanation, the definitions are from [39].

B.1.1 Basic Notions in Projective Geometry

In this subsection I review the definition of projective geometries and some relevant definitions and results.

B.1.1. Definition. A *projective geometry* is a tuple $\mathcal{G} = (G, \star)$, where G is a non-empty set, whose elements are called *points*, and $\star : G \times G \to \wp(G)$ is a function such that all of the following hold:

(P1) $a \star a = \{a\}$, for every $a \in G$;

(P2) $a \in b \star a$, for any $a, b \in G$;

(P3) $a \in b \star r$, $r \in c \star d$ and $a \neq c$ imply that $(a \star c) \cap (b \star d) \neq \emptyset$, for all $a, b, c, d, r \in G$.

Please refer to Proposition 2.2.3 and Exercise 2.8.1 in [39] for the equivalence of this definition to the classical ones, which are in terms of a ternary collinearity relation and lines, respectively. The following lemma collects some useful properties following from this definition:

B.1.2. Lemma. *In a projective geometry $\mathcal{G} = (G, \star)$, the following hold, for any $a, b, c \in G$:*

(P4) if $a \in b \star c$ and $a \neq b$, then $c \in a \star b$;

(P5) if $a \in b \star c$, then $a \star b \subseteq b \star c$;

(P6) $a \star b = b \star a$;

(P7) if $a \in b \star c$ and $a \neq b$, then $a \star b = b \star c$.

(P8) if $c, d \in a \star b$ and $c \neq d$, then $c \star d = a \star b$.

Proof. For (P4) to (P7), please refer to the proof of Proposition 2.2.2 in [39]. For (P8), please refer to the proof of Proposition 2.2.5 in the same book. ⊣

The following are some important notions in projective geometries.

B.1.3. Definition. Let $\mathcal{G} = (G, \star)$ be a projective geometry.

- $a, b, c \in G$ are *collinear*, if $s_1 \in s_2 \star s_3$ for some s_1, s_2, s_3 such that $\{s_1, s_2, s_3\} = \{a, b, c\}$.[1]
- A *subspace* F of $\mathcal{G}$ is a subset of G satisfying that $a, b \in F$ implies that $a \star b \subseteq F$, for any $a, b \in G$.
- A *hyperplane* H of $\mathcal{G}$ is a subspace of $\mathcal{G}$ satisfying both of the following:
    1. H is proper, i.e. $H \neq G$;
    2. H is maximal, i.e. $H \subseteq F$ implies that $F = H$ or $F = G$ for every subspace F of $\mathcal{G}$.

The following theorem is about an important property of hyperplanes.

B.1.4. Theorem. *In a projective geometry $\mathcal{G} = (G, \star)$, for any hyperplane H and $a, b \in G$ satisfying $a \neq b$, $(a \star b) \cap H$ is $a \star b$ or a singleton.*

Proof. See Proposition 2.4.12 and Remark 2.4.13 (1°) of [39]. ⊣

[1] It is not hard to show that, in a projective geometry, three points a, b, c are collinear, if and only if $a \neq b$ implies that $c \in a \star b$. Although the statement involved is simpler, it hides the fact that the collinearity relation is cyclic. This is why I do not use this as a definition.

B.1.2 Dimensions in Projective Geometry

In this subsection, I recall the notion of dimensions in projective geometries. For convenience, I fix a projective geometry $\mathcal{G} = (G, \star)$ throughout this subsection.

The notion of dimensions in projective geometries is built on the notion of linear closures.

B.1.5. Definition. Given $A \subseteq G$, the *linear closure*[2] *of* A, denoted by $\mathcal{C}(A)$, is defined to be the set $\bigcap\{E \in \wp(G) \mid A \subseteq E$ and E is a subspace of $\mathcal{G}\}$.

The following lemma collects some useful properties of linear closures.

B.1.6. Lemma.

1. *For every subspace* E *of* $\mathcal{G}$, $\mathcal{C}(E) = E$.
2. $\mathcal{C}(A)$ *is a subspace, for each* $A \subseteq G$.
3. $\mathcal{C}(\cdot)$ *is a closure operator on* G.

Proof. *1* is obvious from the definition. *2* follows from the definition together with the fact that any arbitrary intersection of subspaces of $\mathcal{G}$ is still a subspace (Proposition 2.3.3 in [39]). *3* is implied by Corollary 3.3.8 in [39]. ⊣

The following proposition gives a recursive characterization of linear closures.

B.1.7. Proposition. *For* $A \subseteq G$, *define a sequence* $\{A_i\}_{i\in\mathbb{N}}$ *of subsets of* G *as follows:*

- $A_0 = A$;
- $A_{n+1} = \bigcup\{a \star b \mid a, b \in A_n\}$.

Then $\mathcal{C}(A) = \bigcup_{i\in\mathbb{N}} A_i$.

Proof. First I prove by induction that $A_i \subseteq \mathcal{C}(A)$, for every $i \in \mathbb{N}$.

Base Step: $i = 0$. By the definition of linear closures $A_0 = A \subseteq \mathcal{C}(A)$.

Induction Step: $i = n + 1$. Let $c \in A_{n+1}$ be arbitrary. By the definition of A_{n+1} there are $a, b \in A_n$ such that $c \in a \star b$. By the induction hypothesis $a, b \in A_n \subseteq \mathcal{C}(A)$, so $c \in a \star b \subseteq \mathcal{C}(A)$ since $\mathcal{C}(A)$ is a subspace.

This finishes the proof by induction. Therefore, $\bigcup_{i\in\mathbb{N}} A_i \subseteq \mathcal{C}(A)$.

Second I prove that $\bigcup_{i\in\mathbb{N}} A_i$ is a subspace including A, and thus $\mathcal{C}(A) \subseteq \bigcup_{i\in\mathbb{N}} A_i$. By definition $A = A_0 \subseteq \bigcup_{i\in\mathbb{N}} A_i$. Now let $a, b \in \bigcup_{i\in\mathbb{N}} A_i$ be arbitrary. Then there are $n, n' \in \mathbb{N}$ such that $a \in A_n$ and $b \in A_{n'}$. Note that by definition

[2] In [39] this notion is just called a closure. In this thesis, to distinguish from the notion of bi-orthogonal closures, it is called a linear closure.

$A_i \subseteq A_{i+1}$, for every $i \in \mathbb{N}$. Hence $a, b \in A_m$, where $m = max\{n, n'\}$. Therefore, $a \star b \subseteq A_{m+1} \subseteq \bigcup_{i\in\mathbb{N}} A_i$. As a result, $\bigcup_{i\in\mathbb{N}} A_i$ is a subspace. ⊣

The following is a very important and useful result in projective geometry called *the projective law*.

B.1.8. Theorem. *For any non-empty sets $A, B \subseteq G$,*

$$\mathcal{C}(A \cup B) = \bigcup\{a \star b \mid a \in \mathcal{C}(A),\ b \in \mathcal{C}(B)\}.$$

In particular, if E is a non-empty subspace of $\mathcal{G}$ and $a \in G$, then

$$\mathcal{C}(\{a\} \cup E) = \bigcup\{a \star b \mid b \in E\}.$$

Proof. Please refer to the proof of Corollary 2.4.5 in [39]. ⊣

B.1.9. Corollary. *For any $a \in G$ and $A \subseteq G$, $\mathcal{C}(\{a\} \cup \mathcal{C}(A)) = \mathcal{C}(\{a\} \cup A)$.*

Proof. It is easy to see from the definition that $\{a\}$ is a subspace, and thus $\mathcal{C}(\{a\}) = \{a\}$. Using the projective law,

$$\begin{aligned}
\mathcal{C}(\{a\} \cup A) &= \bigcup\{c \star d \mid c \in \mathcal{C}(\{a\}),\ d \in \mathcal{C}(A)\} \\
&= \bigcup\{c \star d \mid c \in \{a\},\ d \in \mathcal{C}(A)\} \\
&= \bigcup\{a \star d \mid d \in \mathcal{C}(A)\} \\
&= \mathcal{C}(\{a\} \cup \mathcal{C}(A))
\end{aligned}$$

⊣

Based on the notion of linear closures, the notions of independent sets, generating sets, bases and finite dimensionality can be defined.

B.1.10. Definition.

- $A \subseteq G$ is *independent*, if $a \notin \mathcal{C}(A \setminus \{a\})$ for every $a \in A$.
- $A \subseteq G$ *generates a subspace*, or *is a generating set of a subspace*, E of $\mathcal{G}$, if $E = \mathcal{C}(A)$.
- A *basis* of a subspace E of $\mathcal{G}$ is a set $A \subseteq G$ which is independent and generates E.
- $\mathcal{G} = (G, \star)$ is called *finite-dimensional*, if G, as a subspace, has a finite generating set.

Next I cite some important theorems for defining the notion of dimensions.

B.1.11. THEOREM. *Let E be a subspace of $\mathcal{G}$ and $A \subseteq D \subseteq E$ be such that A is independent and D generates E. Then there exists a basis B of E with $A \subseteq B \subseteq D$. In particular, every subspace of $\mathcal{G}$ has a basis.*

Proof. By Proposition 3.1.13 in [39] $\mathcal{G}$ is a geometry and thus a matroid. Then the conclusion is implied by Theorem 4.1.9 in [39]. ⊣

B.1.12. THEOREM. *Let E be a subspace of $\mathcal{G}$, $A \uplus B_1$ and $A \uplus B_2$ be two bases of E. Then B_1 and B_2 are of the same cardinality. In particular, any two bases of a subspace are of the same cardinality.*

Proof. This is similar to the proof of the above theorem, but Theorem 4.2.2 in [39] is applied instead of Theorem 4.1.9. ⊣

Now I am ready to define the notion of ranks.

B.1.13. DEFINITION. The *rank* of a subspace E of $\mathcal{G}$, denoted by $r(E)$, is the cardinality of one (and thus any) basis of E.

Intuitively, in projective geometries, the rank of a subspace is the smallest number of independent points needed to generate the subspace. Unfortunately, this natural notion does not match the ordinary conception of dimensions, which intuitively is the (cardinal) number of degrees of freedom. This mismatch is the reason why the new term 'rank' is needed. For example, a line in a projective geometry is generated by two distinct points, and thus is of rank 2; but in a line there is only one degree of freedom, and thus normally a line is said to be of dimension 1. In general, if a subspace is of finite rank $n \geq 1$, it is said to be of dimension $n - 1$.

I cite an important property of ranks.

B.1.14. THEOREM. *For two subspaces E and F of $\mathcal{G}$, if $E \subseteq F$, then $r(E) \leq r(F)$.*

Proof. Please refer to the proof of Proposition 4.3.1 in [39]. ⊣

B.1.3 Projective Geometries with Additional Structures

In this subsection, I discuss three kinds of projective geometries with additional structures: projective geometries with polarities, orthogeomtries and Hibertian geometries.

I start from projective geometries with polarities.

B.1.15. Definition. A *projective geometry with a polarity* is a tuple $\mathcal{G} = (G, \star, p)$, where $(G, \star)$ is a projective geometry and p is a function from G to the set of all hyperplanes of $(G, \star)$, called a *polarity* on $(G, \star)$, such that $a \in p(b) \Leftrightarrow b \in p(a)$, for any $a, b \in G$.

Moreover, a polarity p is *pure*, if $a \notin p(a)$, for every $a \in G$.

Sometimes a polarity is required to be surjective (e.g. Definition on p 154 in [85]), but I do not make this requirement in this thesis.

The following proposition collects some useful results about projective geometries with polarities.

B.1.16. Proposition. *Let $\mathcal{G} = (G, \star, p)$ be a projective geometry with a polarity.*

1. *For any $a, b, c \in G$, $c \in a \star b \Rightarrow p(a) \cap p(b) \subseteq p(c)$.*
2. *p is injective.*
3. *For any $a, b, c \in G$, $c \in a \star b \Leftarrow p(a) \cap p(b) \subseteq p(c)$.*

Proof.[3] For *1*, assume that $c \in a \star b$. Let $x \in p(a) \cap p(b)$ be arbitrary. For $x \in p(a)$, $a \in p(x)$ by the definition of a polarity. Similarly one can deduce $b \in p(x)$ from $x \in p(b)$. Since $p(x)$ is a hyperplane, $a \star b \subseteq p(x)$. Hence $c \in a \star b \subseteq p(x)$. By the definition of a polarity $x \in p(c)$. Therefore, $p(a) \cap p(b) \subseteq p(c)$.

For *2*, suppose (towards a contradiction) that there are $a, a' \in G$ such that $a \neq a'$ but $p(a) = p(a')$. Since $p(a)$ is a hyperplane, there is a $b \in G$ such that $b \notin p(a)$. Since $a \neq a'$, by Theorem B.1.4 $(a \star a') \cap p(b)$ is non-empty. Take $x \in (a \star a') \cap p(b)$. Since $x \in a \star a'$, by *1* and supposition $p(a) = p(a) \cap p(a') \subseteq p(x)$. Since both $p(a)$ and $p(x)$ are hyperplanes, by definition $p(a) = p(x)$. Now it follows from $x \in p(b)$ that $b \in p(x) = p(a)$, contradicting that $b \notin p(a)$. Therefore, p is injective.

For *3*, assume that $p(a) \cap p(b) \subseteq p(c)$. As a start, note that there are two easy cases: when $c \in \{a, b\}$ the conclusion follows easily from the definition; and when $a = b$ the conclusion follows easily with the help of *2*. In the following, I focus on the case where a, b, c are distinct. Then by *2* $p(a)$ and $p(b)$ are different hyperplanes of $(G, \star)$. Hence there is an $x \in G$ such that $x \in p(b)$ and $x \notin p(a)$.

Observe that $(a \star c) \cap p(x)$ is a singleton. Since $p(x)$ is a hyperplane and $a \neq c$, by Theorem B.1.4 $(a \star c) \cap p(x)$ is either $a \star c$ or a singleton. Since $x \notin p(a)$, $a \notin p(x)$ and thus $(a \star c) \cap p(x) \neq a \star c$. Therefore, $(a \star c) \cap p(x)$ is a singleton. Denote by y the unique element in it.

For this y, observe that $p(b) \subseteq p(y)$. Let $r \in p(b)$ be arbitrary. If $r = x$, then $r = x \in p(y)$ since $y \in p(x)$. If $r \neq x$, one can show from $x \notin p(a)$ that

[3] The proof of *2* is inspired by that of Proposition 11.3.3 in [39], and the proof of *3* by that of Proposition 14.2.5 in the same book. One could arrive at the same conclusions by introducing some terminologies from this book and applying these propositions. However, since they are not very long, direct proofs may be more helpful in developing intuitions.

$(x \star r) \cap p(a)$ is a singleton. Denote by u the unique element in it. Since $x, r \in p(b)$, $u \in x \star r \subseteq p(b)$. Hence $u \in p(a) \cap p(b) \subseteq p(c)$ by the assumption. It follows that $u \in p(a) \cap p(c)$. Since $y \in a \star c$, $u \in p(a) \cap p(c) \subseteq p(y)$ by *1.* Since $x \notin p(a)$ and $u \in p(a)$, $u \neq x$. Then $r \in x \star u$ follows from $u \in x \star r$ and (P4). Since $x, u \in p(y)$, $r \in x \star u \subseteq p(y)$. For r is arbitrary, $p(b) \subseteq p(y)$.

Now I am ready to show that $c \in a \star b$. Since both $p(b)$ and $p(y)$ are hyperplanes, $p(b) = p(y)$. By *2* $b = y$. Then $b \in a \star c$ follows from $y \in a \star c$. Since $a \neq b$, $c \in a \star b$ by (P4). $\dashv$

B.1.17. Remark. The above proposition shows that a polarity p on a projective geometry $\mathcal{G} = (G, \star)$ is an injection with the property that $c \in a \star b \Leftrightarrow p(a) \cap p(b) \subseteq p(c)$, for any $a, b, c \in G$. One can define the *dual geometry* $\mathcal{G}^*$ *of* $\mathcal{G}$ to be the tuple $(G^*, \circledast)$, where G^* is the set of all hyperplanes of $\mathcal{G}$ and $H \in E \circledast F \Leftrightarrow E \cap F \subseteq H$ for any $E, F, H \in G^*$. It can be proved that $\mathcal{G}^*$ is a projective geometry (Proposition 11.2.3 of [39]). Then the above proposition means that p is an embedding of $\mathcal{G}$ into its dual $\mathcal{G}^*$.

Another special kind of projective geometry, called an orthogeometry, is closely related to projective geometries with polarities.

B.1.18. Definition. An *orthogeometry* is a tuple $\mathcal{G} = (G, \star, \perp)$, where $(G, \star)$ is a projective geometry and $\perp \subseteq G \times G$, called *the orthogonality relation*, satisfies all of the following properties:

(O1) $a \perp b$ implies that $b \perp a$, for any $a, b \in G$;

(O2) if $a \perp q$, $b \perp q$ and $c \in a \star b$, then $c \perp q$, for any $a, b, c, q \in G$;

(O3) if $a, b, c \in G$ and $b \neq c$, then there is a $q \in b \star c$ such that $q \perp a$;

(O4) for every $a \in G$, there is a $b \in G$ such that $a \not\perp b$.

The orthogonality relation is *pure*, if $a \not\perp a$ for every $a \in G$. An orthogeometry is *pure*, if the orthogonality relation in it is pure.

For $E \subseteq G$, $E^{\perp} \overset{\text{def}}{=} \{a \in G \mid a \perp b, \text{ for every } b \in E\}$ is called *the orthocomplement of* E.

The close relation between projective geometries with polarities and orthogeometries is revealed by the following theorem:

B.1.19. Theorem. *For every projective geometry* $\mathcal{G} = (G, \star)$*, there is a canonical bijection from the set of all polarities on* $\mathcal{G}$ *to the set of all orthogonality relations on* $\mathcal{G}$*. To be precise, a polarity* $p : G \to \wp(G)$ *on* $\mathcal{G}$ *is mapped by this bijection to the orthogonality relation* $\perp \subseteq G \times G$ *satisfying that* $a \perp b \Leftrightarrow a \in p(b)$*, for any* $a, b \in G$*.*

Moreover, a polarity p *on* $\mathcal{G}$ *is pure, if and only if its image under this bijection is a pure orthogonality relation.*

Proof. This is implied by Proposition 14.1.3 in [39]. ⊣

Finally I discuss Hilbertian geometries.

B.1.20. Definition. A *Hilbertian geometry* is an orthogeometry $\mathcal{G} = (G, \star, \perp)$ satisfying the following condition:

$$\text{for every } E \subseteq G \text{ satisfying } (E^{\perp})^{\perp} = E,\ \mathcal{C}(E \cup E^{\perp}) = G.$$

I take a close look at the orthogonality relations in Hilbertian geometries.

B.1.21. Lemma. *Every Hilbertian geometry is a pure orthogeometry.*

Proof. Let $\mathcal{G} = (G, \star, \perp)$ be a Hilbertian geometry. By definition it suffices to show that $\perp$ is pure. Note that by Theorem B.1.19 $\{\cdot\}^{\perp} : G \to \wp(G)$ is a polarity on $(G, \star)$.

As a preparation, I show that $(\{a\}^{\perp})^{\perp} = \{a\}$, for every $a \in G$. First I show that $\{a\} \subseteq (\{a\}^{\perp})^{\perp}$. For every $b \in \{a\}^{\perp}$, $b \perp a$ by definition, so $a \perp b$ by (O1). Hence $\{a\} \subseteq (\{a\}^{\perp})^{\perp}$. Second I show that $(\{a\}^{\perp})^{\perp} \subseteq \{a\}$. Suppose (towards a contradiction) that $b \in (\{a\}^{\perp})^{\perp}$ and $a \neq b$. For $\{\cdot\}^{\perp}$ is a polarity on $(G, \star)$, by *2* of Proposition B.1.16 $\{a\}^{\perp}$ and $\{b\}^{\perp}$ are different hyperplanes of $(G, \star)$. Then there is a $c \in \{a\}^{\perp}$ such that $c \notin \{b\}^{\perp}$, i.e. $c \in \{a\}^{\perp}$ but $b \not\perp c$, contradicting that $b \in (\{a\}^{\perp})^{\perp}$. Therefore, $(\{a\}^{\perp})^{\perp} \subseteq \{a\}$, and thus $(\{a\}^{\perp})^{\perp} = \{a\}$.

Now suppose (towards a contradiction) that $\perp$ is not pure, i.e. $a \perp a$ for some $a \in G$. Since $(\{a\}^{\perp})^{\perp} = \{a\}$, by the definition of Hilbertian geometries $G = \mathcal{C}(\{a\} \cup \{a\}^{\perp})$. Since $a \in \{a\}^{\perp}$, $\{a\} \cup \{a\}^{\perp} = \{a\}^{\perp}$. Hence $G = \mathcal{C}(\{a\}^{\perp}) = \{a\}^{\perp}$, contradicting that $\{a\}^{\perp}$ is a hyperplane. Therefore, $\perp$ is pure. ⊣

B.1.4 Maps between Projective Geometries

In this subsection, I review some maps between projective geometries.

B.1.22. Definition. A *homomorphism* from a projective geometry $\mathcal{G}_1 = (G_1, \star_1)$ to a projective geometry $\mathcal{G}_2 = (G_2, \star_2)$ is a partial function[4] $F : G_1 \dashrightarrow G_2$ satisfying all of the following:

(M1) $Ker(F) \stackrel{\text{def}}{=} \{a_1 \in G_1 \mid F(a_1) \text{ is undefined}\}$ is a subspace of $\mathcal{G}_1$;

(M2) if $a_1, b_1 \notin Ker(F)$, $c_1 \in Ker(F)$ and $a_1 \in b_1 \star_1 c_1$, then $F(a_1) = F(b_1)$;

(M3) if $a_1, b_1, c_1 \notin Ker(F)$ and $a_1 \in b_1 \star_1 c_1$, then $F(a_1) \in F(b_1) \star_2 F(c_1)$;

(M4) if $a_1, b_1 \notin Ker(F)$ and $F(a_1) \neq F(b_1)$, then $F(a_1) \star_2 F(b_1) \subseteq F[a_1 \star_1 b_1]$;

[4] A *partial function* f from a set A to a set B is a subset of $A \times B$ such that $(x, y), (x, y') \in f$ implies that $y = y'$.

(M5) if $a_1 \neq b_1$, $a_1, b_1 \notin Ker(F)$ and $F(a_1) = F(b_1)$, then $(a_1 \star_1 b_1) \cap Ker(F) \neq \emptyset$.

It turns out that (M1), (M2) and (M3) together can be characterized in terms of subspaces.

B.1.23. Lemma. *Let $\mathcal{G}_1 = (G_1, \star_1)$ and $\mathcal{G}_2 = (G_2, \star_2)$ be two projective geometries. For every partial function $F : G_1 \dashrightarrow G_2$, the following are equivalent:*

(i) F is a morphism from $\mathcal{G}_1$ to $\mathcal{G}_2$, *i.e. it satisfies (M1), (M2) and (M3);*

(ii) for every subspace E_2 of $\mathcal{G}_2$, $Ker(F) \cup F^{-1}[E_2]$ is a subspace of $\mathcal{G}_1$.

Proof. This follows from Proposition 6.2.3 in [39]. ⊣

In fact, according to Proposition 6.2.3 in [39], (ii) is equivalent to its restriction to the subspaces of rank at most 2, so it is not essentially second-order. Similarly, homomorphisms can be characterized in terms of hyperplanes.

B.1.24. Lemma. *Let $\mathcal{G}_1 = (G_1, \star_1)$ and $\mathcal{G}_2 = (G_2, \star_2)$ be two projective geometries. For every partial function $F : G_1 \dashrightarrow G_2$, the following are equivalent:*

(i) F is a homomorphism from $\mathcal{G}_1$ to $\mathcal{G}_2$;

(ii) for every hyperplane H_2 of $\mathcal{G}_2$, $Ker(F) \cup F^{-1}[H_2]$ is either G_1 or a hyperplane of $\mathcal{G}_1$.

Proof. This is Proposition 6.5.10 in [39]. ⊣

Next I proceed to maps between orthogeometries.

B.1.25. Definition. A *continuous homomorphism from an orthogeometry $\mathcal{G}_1 = (G_1, \star_1, \perp_1)$ to an orthogeometry $\mathcal{G}_2 = (G_2, \star_2, \perp_2)$* is a homomorphism F from $(G_1, \star_1)$ to $(G_2, \star_2)$ that satisfies the following:

(CON) $A_2^{\perp\perp} = A_2$ implies that $(Ker(F) \cup F^{-1}[A_2])^{\perp\perp} = Ker(F) \cup F^{-1}[A_2]$, for every $A_2 \subseteq G_2$.

I also introduce the related notion of adjunctions.

B.1.26. Definition. An *adjunction between orthogeometries $\mathcal{G}_1 = (G_1, \star_1, \perp_1)$ and $\mathcal{G}_2 = (G_2, \star_2, \perp_2)$* is a pair of partial functions $F : G_1 \dashrightarrow G_2$ and $G : G_2 \dashrightarrow G_1$ satisfying both of the following:

(A1) $Ker(F) = G[G_2]^{\perp}$ and $Ker(G) = F[G_1]^{\perp}$;

(A2) for any $a_1 \notin Ker(F)$ and $a_2 \notin Ker(G)$, $F(a_1) \perp_2 a_2 \Leftrightarrow a_1 \perp_1 G(a_2)$.

It turns out that there is a correspondence between continuous homomorphisms and adjunctions.

B.1.27. THEOREM. *Let $\mathcal{G}_1 = (G_1, \star_1, \perp_1)$ and $\mathcal{G}_2 = (G_2, \star_2, \perp_2)$ be two orthogeometries. For every partial function $F : G_1 \dashrightarrow G_2$, the following are equivalent:*

(i) F is a continuous homomorphism from $\mathcal{G}_1$ to $\mathcal{G}_2$;

(ii) there is a partial function $F^\dagger : G_2 \dashrightarrow G_1$ such that $(F, F^\dagger)$ forms an adjunction between $\mathcal{G}_1$ and $\mathcal{G}_2$.

Moreover, if it exists, the partial function $F^\dagger$ in (ii) is uniquely defined by $Ker(F^\dagger) = F[G_1]^\perp$ and $F^\dagger(a_2) = (Ker(F) \cup F^{-1}[\{a_2\}^\perp])^\perp$ for every $a_2 \notin Ker(F^\dagger)$. $F^\dagger$ is called the adjoint *of F.*

Proof. This follows from Proposition 14.4.4 in [39]. ⊣

B.2 Linear Algebra

In this section, I would like to review some relevant elements of linear algebra. The point of this review lies in its generality: vector spaces over division rings are investigated, instead of just those over $\mathbb{R}$ or $\mathbb{C}$. If without explanation, the definitions are from [39].

B.2.1 Division Rings and Vector Spaces

In this subsection, I review the notion of vector spaces and some relevant notions.

First I recall the notion of division rings.

B.2.1. DEFINITION. A *division ring* $\mathcal{K}$ is a tuple $(K, +, \cdot, 0, 1)$, where K is a non-empty set, $+, \cdot : K \times K \to K$ are two functions and $0, 1 \in K$, such that:

1. $0 \neq 1$; and
2. $(K, +, 0)$ is an Abelian group; and
3. $(K \setminus \{0\}, \cdot, 1)$ is a group; and
4. $\cdot$ distributes over $+$, i.e. for any $x, y, z \in K$,

$$x \cdot (y + z) = x \cdot y + x \cdot z, \qquad (x + y) \cdot z = x \cdot z + y \cdot z.$$

If $(K \setminus \{0\}, \cdot, 1)$ forms an Abelian group, then $\mathcal{K}$ is called a *field*.

Please note that in [39] the word 'field' actually means a division ring. However, I stick to the common usage of this word in this thesis.

Second, I introduce the notion of vector spaces over division rings.

B.2.2. Definition. A *(left) vector space over a division ring* $\mathcal{K} = (K, +, \cdot, 0, 1)$ is a set V equipped with functions $+ : V \times V \to V$, called *addition*, and $x(\cdot) : V \to V$, called *multiplication by the scalar* x, for every $x \in K$, as well as $\mathbf{0} \in V$[5] such that all of the following hold:

1. $(V, +, \mathbf{0})$ is an Abelian group;
2. $x(\mathbf{v} + \mathbf{w}) = x\mathbf{v} + x\mathbf{w}$, for any $x \in K$ and $\mathbf{v}, \mathbf{w} \in V$;
3. $(x + y)\mathbf{v} = x\mathbf{v} + y\mathbf{v}$, for any $x, y \in K$ and $\mathbf{v} \in V$;
4. $(x \cdot y)\mathbf{v} = x(y\mathbf{v})$, for any $x, y \in K$ and $\mathbf{v} \in V$;
5. $1\mathbf{v} = \mathbf{v}$, for every $\mathbf{v} \in V$.

Next, I review some important notions about vector spaces.

B.2.3. Definition. Let V be a vector space over some division ring $\mathcal{K}$.

- $E \subseteq V$ is a *subspace of* V, if E is non-empty and closed under addition and scalar multiplication, where being closed under scalar multiplication means that $x\mathbf{v} \in E$ whenever $\mathbf{v} \in E$ and x is an element in $\mathcal{K}$.
- For every $A \subseteq V$, *the linear span of* A, denoted by $L(A)$, is the set $\{\sum_{i=1}^{n} x_i \mathbf{v}_i \in V \mid n \in \mathbb{N}^+$ and, for $i = 1, ..., n,\ x_i$ is in $\mathcal{K}$ and $\mathbf{v}_i \in A \cup \{\mathbf{0}\}\}$.
- *The sum of two subspaces* E *and* F *of* V, denoted by $E + F$, is the set $L(E \cup F)$.

B.2.4. Remark. The following results are well known and easy to verify:

- for every $A \subseteq V$, $L(A)$ is the smallest subspace of V which includes A.
- for any two subspaces E and F of V, $E + F = \{\mathbf{u} + \mathbf{v} \mid \mathbf{u} \in E \text{ and } \mathbf{v} \in F\}$.

Finally, I prove a technical result.

B.2.5. Lemma. *Let* V *be a vector space of dimension at least* 3 *over a division ring* $\mathcal{K} = (K, +, \cdot, 0, 1)$, $\mathbf{u}, \mathbf{v}, \mathbf{s}, \mathbf{t} \in V \setminus \{\mathbf{0}\}$ *such that* $\langle \mathbf{u} \rangle \neq \langle \mathbf{v} \rangle$, $\langle \mathbf{s} \rangle \neq \langle \mathbf{t} \rangle$ *and* $\langle \mathbf{u} \rangle \neq \langle \mathbf{s} \rangle$. *There is a* $\mathbf{w} \in V \setminus \{\mathbf{0}\}$ *such that* $\langle \mathbf{w} \rangle \notin \big(\langle \mathbf{u} \rangle \star \langle \mathbf{s} \rangle\big) \cup \big(\langle \mathbf{u} \rangle \star \langle \mathbf{v} \rangle\big) \cup \big(\langle \mathbf{s} \rangle \star \langle \mathbf{t} \rangle\big)$.

[5] In this thesis, when discussing about vector spaces, I am not going to present them in a rigorous set-theoretic way, which is tedious and unusual. Instead, I just follow the usual way how mathematicians talk about them.

Proof. Since $\langle \mathbf{u} \rangle \neq \langle \mathbf{s} \rangle$, $L(\{\mathbf{u}, \mathbf{v}, \mathbf{s}, \mathbf{t}\})$ is at least two-dimensional. Consider three cases, depending on the dimension of this linear span.

Case 1: $L(\{\mathbf{u}, \mathbf{v}, \mathbf{s}, \mathbf{t}\})$ is two-dimensional. Since V is of dimension at least 3, there is a $\mathbf{w} \in V \setminus \{\mathbf{0}\}$ such that $\mathbf{w} \notin L(\{\mathbf{u}, \mathbf{v}, \mathbf{s}, \mathbf{t}\})$. Hence $\langle \mathbf{w} \rangle \notin \big(\langle \mathbf{u} \rangle \star \langle \mathbf{s} \rangle\big) \cup \big(\langle \mathbf{u} \rangle \star \langle \mathbf{v} \rangle\big) \cup \big(\langle \mathbf{s} \rangle \star \langle \mathbf{t} \rangle\big)$.

Case 2: $L(\{\mathbf{u}, \mathbf{v}, \mathbf{s}, \mathbf{t}\})$ is three-dimensional. Then there are $a, b, c, d \in K$ such that $\{a, b, c, d\} \nsubseteq \{0\}$ and $a\mathbf{u} + b\mathbf{v} + c\mathbf{s} + d\mathbf{t} = \mathbf{0}$. Consider four subcases.

- *Subcase 1:* Exactly three of a, b, c, d is 0.

  This is impossible. For instance, $a \neq 0$ and $b = c = d = 0$. Then $a\mathbf{u} = \mathbf{0}$, and thus $\mathbf{u} = \mathbf{0}$, contradicting that $\mathbf{u} \neq \mathbf{0}$. The other cases are similar.

- *Subcase 2:* Exactly two of a, b, c, d is 0.

  Since $\langle \mathbf{u} \rangle \neq \langle \mathbf{v} \rangle$, $\langle \mathbf{s} \rangle \neq \langle \mathbf{t} \rangle$ and $\langle \mathbf{u} \rangle \neq \langle \mathbf{s} \rangle$, the only possibilities are $b, d \notin \{0\}$, $a, d \notin \{0\}$ and $b, c \notin \{0\}$. The proofs for these three possibilities are similar, and for an example I show the possibility when $b, d \notin \{0\}$. Then $\langle \mathbf{v} \rangle = \langle \mathbf{t} \rangle$. Since $L(\{\mathbf{u}, \mathbf{v}, \mathbf{s}, \mathbf{t}\})$ is three-dimensional, $\{\mathbf{u}, \mathbf{v}, \mathbf{s}\}$ is linearly independent. Let $\mathbf{w} = \mathbf{u} + \mathbf{v} + \mathbf{s}$. It is easy to see that $\mathbf{w} \neq \mathbf{0}$ and $\langle \mathbf{w} \rangle \notin \big(\langle \mathbf{u} \rangle \star \langle \mathbf{s} \rangle\big) \cup \big(\langle \mathbf{u} \rangle \star \langle \mathbf{v} \rangle\big) \cup \big(\langle \mathbf{s} \rangle \star \langle \mathbf{t} \rangle\big)$.

- *Subcase 3:* Exactly one of a, b, c, d is 0.

  Then there are four possibilities. The proofs for these four possibilities are similar, and for an example I show the possibility when $a = 0$. Then $\mathbf{t} = -(d^{-1} \cdot b)\mathbf{v} - (d^{-1} \cdot c)\mathbf{s}$, and thus $L(\{\mathbf{u}, \mathbf{v}, \mathbf{s}, \mathbf{t}\}) = L(\{\mathbf{u}, \mathbf{v}, \mathbf{s}\})$. Since $L(\{\mathbf{u}, \mathbf{v}, \mathbf{s}, \mathbf{t}\})$ is three-dimensional, $\{\mathbf{u}, \mathbf{v}, \mathbf{s}\}$ is linearly independent. Let $\mathbf{w} = \mathbf{u} + \mathbf{v} + \mathbf{s}$. It is easy to see that $\mathbf{w} \neq \mathbf{0}$ and $\langle \mathbf{w} \rangle \notin \big(\langle \mathbf{u} \rangle \star \langle \mathbf{s} \rangle\big) \cup \big(\langle \mathbf{u} \rangle \star \langle \mathbf{v} \rangle\big) \cup \big(\langle \mathbf{s} \rangle \star \langle \mathbf{t} \rangle\big)$.

- *Subcase 4:* None of a, b, c, d is 0.

  I claim that $\{\mathbf{u}, \mathbf{v}, \mathbf{s}\}$ is linearly independent. Suppose (towards a contradiction) that it is linearly dependent. Then there are $k, l, m \in K$ such that $\{k, l, m\} \nsubseteq \{0\}$ and $k\mathbf{u} + l\mathbf{v} + m\mathbf{s} = \mathbf{0}$. Since $\langle \mathbf{u} \rangle \neq \langle \mathbf{v} \rangle$, $m \neq 0$, and thus $\mathbf{s} \in L(\{\mathbf{u}, \mathbf{v}\})$. Since $a\mathbf{u} + b\mathbf{v} + c\mathbf{s} + d\mathbf{t} = \mathbf{0}$, $\mathbf{t} = -d^{-1}(a\mathbf{u} + b\mathbf{v} + c\mathbf{s}) \in L(\{\mathbf{u}, \mathbf{v}\})$. Hence $L(\{\mathbf{u}, \mathbf{v}, \mathbf{s}, \mathbf{t}\}) \subseteq L(\{\mathbf{u}, \mathbf{v}\})$, contradicting that $L(\{\mathbf{u}, \mathbf{v}, \mathbf{s}, \mathbf{t}\})$ is three-dimensional. Therefore, $\{\mathbf{u}, \mathbf{v}, \mathbf{s}\}$ is linearly independent.

  Let $\mathbf{w} = b\mathbf{v} + c\mathbf{s}$. Since $\{\mathbf{u}, \mathbf{v}, \mathbf{s}\}$ is linearly independent and $b, c \notin \{0\}$, $\mathbf{w} \neq \mathbf{0}$. Moreover, $\langle \mathbf{w} \rangle \notin \big(\langle \mathbf{u} \rangle \star \langle \mathbf{s} \rangle\big) \cup \big(\langle \mathbf{u} \rangle \star \langle \mathbf{v} \rangle\big)$.

  Finally, I claim that $\langle \mathbf{w} \rangle \notin \langle \mathbf{s} \rangle \star \langle \mathbf{t} \rangle$. Suppose (towards a contradiction) that $\langle \mathbf{w} \rangle \in \langle \mathbf{s} \rangle \star \langle \mathbf{t} \rangle$. Then there are $m, n \in K$ such that $\mathbf{w} = m\mathbf{s} + n\mathbf{t}$. It follows that $(n \cdot d^{-1} \cdot a)\mathbf{u} + (b + n \cdot d^{-1} \cdot b)\mathbf{v} + (c - m + n \cdot d^{-1} \cdot c)\mathbf{s} = \mathbf{0}$. Since $\{\mathbf{u}, \mathbf{v}, \mathbf{s}\}$ is linearly independent, $n \cdot d^{-1} \cdot a = 0$ and $b + n \cdot d^{-1} \cdot b = 0$. Since $a \neq 0$, $n \cdot d^{-1} = 0$. Since $b \neq 0$, $1 + n \cdot d^{-1} = 0$. It follows that $0 = 1$, contradicting the definition of division rings. Therefore, $\langle \mathbf{w} \rangle \notin \langle \mathbf{s} \rangle \star \langle \mathbf{t} \rangle$.

Case 3: $L(\{\mathbf{u}, \mathbf{v}, \mathbf{s}, \mathbf{t}\})$ is four-dimensional. Then $\{\mathbf{u}, \mathbf{v}, \mathbf{s}\}$ is linearly independent. Let $\mathbf{w} = \mathbf{u} + \mathbf{v} + \mathbf{s}$. It is easy to see that $\mathbf{w} \neq \mathbf{0}$ and $\langle \mathbf{w} \rangle \notin (\langle \mathbf{u} \rangle \star \langle \mathbf{s} \rangle) \cup (\langle \mathbf{u} \rangle \star \langle \mathbf{v} \rangle) \cup (\langle \mathbf{s} \rangle \star \langle \mathbf{t} \rangle)$. ⊣

B.2.2 Hermitian Forms on Vector Spaces

In this subsection, I review the notion of Hermitian forms on vector spaces, some relevant notions and their basic properties.

I start from Hermitian forms on vector spaces over division rings.

B.2.6. DEFINITION. An *Hermitian form* on a vector space V over a division ring $\mathcal{K} = (K, +, \cdot, 0, 1)$ is a function $\Phi : V \times V \to K$ such that there is a function $\mu : K \to K$ satisfying all of the following:

1. μ is an involution on $\mathcal{K}$, i.e. all of the following hold:
   (a) μ is bijective;
   (b) $\mu(x + y) = \mu(x) + \mu(y)$ and $\mu(x \cdot y) = \mu(y) \cdot \mu(x)$, for any $x, y \in K$;
   (c) $\mu \circ \mu(x) = x$, for every $x \in K$;
2. $\Phi(\mathbf{u} + \mathbf{v}, \mathbf{w}) = \Phi(\mathbf{u}, \mathbf{w}) + \Phi(\mathbf{v}, \mathbf{w})$, for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$;
3. $\Phi(x\mathbf{v}, \mathbf{w}) = x \cdot \Phi(\mathbf{v}, \mathbf{w})$, for any $\mathbf{v}, \mathbf{w} \in V$;
4. $\Phi(\mathbf{v}, \mathbf{w}) = \mu(\Phi(\mathbf{w}, \mathbf{v}))$, for any $\mathbf{v}, \mathbf{w} \in V$.

μ is called *the accompanying involution* of Φ.

An Hermitian form is *anisotropic*, if, for every $\mathbf{v} \in V$, $\Phi(\mathbf{v}, \mathbf{v}) = 0$ implies $\mathbf{v} = \mathbf{0}$.

B.2.7. REMARK. First, note that Hermitian forms and inner products on Hilbert spaces are very much alike. For example, they are both additive in each argument while the other is fixed. However, a notational difference is that an Hermitian form is linear in the first argument, while an inner product is linear in the second argument.

Second, from the above definition, for any $x, y \in K$ and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$,

$$\begin{aligned} \Phi(x\mathbf{u} + y\mathbf{v}, \mathbf{w}) &= x \cdot \Phi(\mathbf{u}, \mathbf{w}) + y \cdot \Phi(\mathbf{v}, \mathbf{w}) \\ \Phi(\mathbf{w}, x\mathbf{u} + y\mathbf{v}) &= \Phi(\mathbf{w}, \mathbf{u}) \cdot \mu(x) + \Phi(\mathbf{w}, \mathbf{v}) \cdot \mu(y) \end{aligned}$$

Hermitian forms make it possible to introduce orthocomplements.

B.2.8. DEFINITION. Let V be a vector space over a division ring $\mathcal{K}$ equipped with an Hermitian form Φ. *The orthocomplement of* $A \subseteq V$, denoted by $A^{\perp}$, is the set $\{\mathbf{v} \in V \mid \Phi(\mathbf{v}, \mathbf{u}) = 0, \text{ for every } \mathbf{u} \in A\}$.

It turns out that the orthocomplement of any set is a subspace.

B.2.9. LEMMA. *Let V be a vector space over a division ring $\mathcal{K}$ equipped with an Hermitian form Φ. $A^\perp$ is a subspace, for every $A \subseteq V$.*

Proof. Easy verification. ⊣

Next, I cite two properties of vector spaces with anisotropic Hermitian forms.

B.2.10. PROPOSITION. *Every finite-dimensional vector space over a division ring equipped with an anisotropic Hermitian form has an orthogonal basis.*

Proof. Since the Hermitian form is anisotropic, it is not alternate in the sense of Subsection 1.1.4 (p 10) in [49]. By Corollary 1 in Section 2.2 (p 65) in the same book, the vector space has an orthogonal basis. ⊣

B.2.11. PROPOSITION. *If a vector space of dimension at least 3 on some division ring $\mathcal{K}$ can be equipped with an anisotropic Hermitian form, $\mathcal{K}$ must be an infinite division ring.*

Proof. Please refer to Proposition 14.1.12 in [39]. ⊣

Moreover, observe that, in a vector space equipped with an anisotropic Hermitian form, linear dependence of vectors can be characterized in terms of the Hermitian form. To be precise, I prove the following proposition:

B.2.12. PROPOSITION. *Let V be a vector space over a division ring $\mathcal{K}$ equipped with an anisotropic Hermitian form Φ. Then, for any $\mathbf{u} \in V \setminus \{\mathbf{0}\}$ and $\mathbf{v}, \mathbf{w} \in V$, the following are equivalent:*

(i) there are x and y in $\mathcal{K}$ such that $\mathbf{w} = x\mathbf{u} + y\mathbf{v}$;

(ii) For every $\mathbf{w}' \in V$, $\Phi(\mathbf{u}, \mathbf{w}') = 0$ and $\Phi(\mathbf{v}, \mathbf{w}') = 0$ imply $\Phi(\mathbf{w}, \mathbf{w}') = 0$.

Proof. **From (i) to (ii):** Assume that $\mathbf{w}' \in V$ is such that $\Phi(\mathbf{u}, \mathbf{w}') = 0$ and $\Phi(\mathbf{v}, \mathbf{w}') = 0$. Then

$$\Phi(\mathbf{w}, \mathbf{w}') = \Phi(x\mathbf{u} + y\mathbf{v}, \mathbf{w}') = x \cdot \Phi(\mathbf{u}, \mathbf{w}') + y \cdot \Phi(\mathbf{v}, \mathbf{w}') = x \cdot 0 + y \cdot 0 = 0$$

From (ii) to (i): If $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent, then it is easy to see that the conclusion holds. In the following, I focus on the case when $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

Note that $\Phi(\mathbf{u}, \mathbf{u}) \neq 0$, since Φ is anisotropic and $\mathbf{u} \neq \mathbf{0}$. Define two vectors

$$\begin{aligned} \mathbf{v}' &= \mathbf{v} - \left(\Phi(\mathbf{v}, \mathbf{u}) \cdot \Phi(\mathbf{u}, \mathbf{u})^{-1}\right) \mathbf{u} \\ \mathbf{x} &= \mathbf{w} - \left(\Phi(\mathbf{w}, \mathbf{u}) \cdot \Phi(\mathbf{u}, \mathbf{u})^{-1}\right) \mathbf{u} - \left(\Phi(\mathbf{w}, \mathbf{v}') \cdot \Phi(\mathbf{v}', \mathbf{v}')^{-1}\right) \mathbf{v}' \end{aligned}$$

Note that, since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, $\mathbf{v}' \neq \mathbf{0}$, and thus $\Phi(\mathbf{v}', \mathbf{v}')^{-1}$ is well-defined. A tedious calculation yields that $\Phi(\mathbf{u}, \mathbf{x}) = \Phi(\mathbf{v}, \mathbf{x}) = 0$. By (ii) $\Phi(\mathbf{w}, \mathbf{x}) = 0$. A further calculation shows that $\Phi(\mathbf{x}, \mathbf{x}) = 0$. Since Φ is anisotropic, $\mathbf{x} = \mathbf{0}$. Therefore, it's easy to see that $\mathbf{w} = x\mathbf{u} + y\mathbf{v}$, where

$$x = \Phi(\mathbf{w}, \mathbf{u}) \cdot \Phi(\mathbf{u}, \mathbf{u})^{-1} - \Phi(\mathbf{w}, \mathbf{v}') \cdot \Phi(\mathbf{v}', \mathbf{v}')^{-1} \cdot \Phi(\mathbf{v}, \mathbf{u}) \cdot \Phi(\mathbf{u}, \mathbf{u})^{-1}$$
$$y = \Phi(\mathbf{w}, \mathbf{v}') \cdot \Phi(\mathbf{v}', \mathbf{v}')^{-1}$$ ⊣

Finally, I review orthomodular Hermitian forms and generalized Hilbert spaces.

B.2.13. Definition. An Hermitian form Φ on a vector space V over some division ring $\mathcal{K}$ is *orthomodular*, if, for every subspace E of V, $E = (E^{\perp})^{\perp}$ implies that $E + E^{\perp} = V$.

A vector space V over some division ring $\mathcal{K}$ equipped with an orthomodular Hermitian form Φ is called a *generalized Hilbert space*. $\mathcal{K}$ is called *the underlying division ring of* V and Φ is called *the underlying Hermitian form of* V.

B.2.14. Remark. Generalized Hilbert spaces indeed generalize Hilbert spaces over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, because every Hilbert space is a generalized Hilbert space. The inner product serves as an Hermitian form, and its being orthomodular can be easily proved using the Orthogonal Decomposition Theorem in the theory of Hilbert spaces.

For Hermitian forms, being orthomodular is closely related to being anisotropic.

B.2.15. Lemma. *Every orthomodular Hermitian form is anisotropic. On finite-dimensional vector spaces, every anisotropic Hermitian form is orthomodular.*

Proof. Please refer to the last but one paragraph on p 206 of [55]. ⊣

B.2.3 Maps between Vector Spaces

In this subsection, I consider maps between vector spaces.

B.2.16. Definition. A *semi-linear map* from a vector space V_1 over a division ring $\mathcal{K}_1 = (K_1, +, \cdot, 0, 1)$ to a vector space V_2 over a division ring $\mathcal{K}_2$ is a function $f : V_1 \to V_2$ such that both of the following conditions are satisfied:

1. (Additivity) $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$, for any $\mathbf{u}, \mathbf{v} \in V_1$;
2. (Homogeneity) f has an *accompanying homomorphism*, i.e. there is a division ring homomorphism σ from $\mathcal{K}_1$ to $\mathcal{K}_2$ such that $f(x\mathbf{v}) = \sigma(x)f(\mathbf{v})$, for any $x \in K_1$ and $\mathbf{v} \in V_1$.

f is *quasi-linear*, if f has an accompanying homomorphism which is a division ring isomorphism; it is *linear*, if $\mathcal{K}_1 = \mathcal{K}_2$ and the identity map on this division ring is an accompanying homomorphism of f.

The *null space of a semi-linear map* $f : V_1 \to V_2$ is $\{\mathbf{v}_1 \in V_1 \mid f(\mathbf{v}_1) = \mathbf{0}_2\}$.

The *zero map* is a function $f : V_1 \to V_2$ such that $f[V_1] = \{\mathbf{0}_2\}$.

For a detailed study of semi-linear maps, please refer to Section 6.3 in [39]. Here I just mention some results used in this thesis. I start with some small points in the form of a remark:

B.2.17. Remark.

- A semi-linear map f has a unique accompanying homomorphism, if it is not the zero map;
- The zero map is semi-linear, if and only if there exists a homomorphism between the two division rings involved.

Next, I present two useful propositions.

B.2.18. Proposition. *Let f and g be two additive maps from a vector space V_1 over a division ring $\mathcal{K}_1$ to a vector space V_2 over a division ring $\mathcal{K}_2$. If $g(\mathbf{v}_1) \in \langle f(\mathbf{v}_1) \rangle$ for all $\mathbf{v}_1 \in V_1$ and the image of f is at least two-dimensional, there exists a unique element x in $\mathcal{K}_2$ such that $g = xf$, i.e. $g(\mathbf{v}_1) = xf(\mathbf{v}_1)$ for every $\mathbf{v}_1 \in V_1$.*

Proof. Please refer to the proof of Lemma 6.3.4 in [39]. $\dashv$

B.2.19. Proposition. *Let f be a quasi-linear map from a vector space V_1 over a division ring $\mathcal{K}_1 = (K_1, +, \cdot, 0, 1)$ to a vector space V_2 over a division ring $\mathcal{K}_2 = (K_2, +, \cdot, 0, 1)$, whose image is one-dimensional. Then there is a Hamel basis $\{\mathbf{v}_1^i \mid i \in I \cup \{e\}\}$ of V_1 with $e \notin I$ such that $f(\mathbf{v}_1^i) \neq \mathbf{0}_2$ if and only if $i = e$, for every $i \in I \cup \{e\}$.*

Proof. It is not hard to note that the null space of f, denoted by $\mathfrak{N}$, is a subspace of V_1. By linear algebra there is a Hamel basis $\{\mathbf{v}_1^i \mid i \in I\}$ of this subspace, i.e., for every $\mathbf{v}_1 \in \mathfrak{N}$, there is a unique finite set $J \subseteq I$ and a unique set $\{x_i \in K_1 \setminus \{0\} \mid i \in J\}$ such that $\mathbf{v}_1 = \sum_{i \in J} x_i \mathbf{v}_1^i$.[6] Using Zorn's lemma it is not hard to show that there is a set $A \subseteq V \setminus \mathfrak{N}$ such that $A \cup \mathfrak{N}$ is a Hamel basis of V_1. Since $f[V_1]$ is one-dimensional, there is a $\mathbf{v}_2 \in V_2 \setminus \{\mathbf{0}_2\}$ such that $f[V_1] = \langle \mathbf{v}_2 \rangle$.

I claim that A is a singleton. Since $f[V_1]$ is one-dimensional, $\mathfrak{N} \neq V_1$, so A is non-empty. Now suppose (towards a contradiction) that $\mathbf{v}_1$ and $\mathbf{v}_1'$ are two distinct elements in A. By definition there are $x, x' \in K_2 \setminus \{0\}$ such that $f(\mathbf{v}_1) = x\mathbf{v}_2$ and $f(\mathbf{v}_1') = x'\mathbf{v}_2$. It follows that $x^{-1}f(\mathbf{v}_1) - (x')^{-1}f(\mathbf{v}_1) = \mathbf{v}_2 - \mathbf{v}_2 = \mathbf{0}_2$. Since f is a quasi-linear map, let σ be an accompanying division ring isomorphism. Then $\sigma^{-1}(x^{-1}), \sigma^{-1}((x')^{-1}) \notin \{0\}$ and $f\left(\sigma^{-1}(x^{-1})\mathbf{v}_1 - \sigma^{-1}(x'^{-1})\mathbf{v}_1'\right) = \mathbf{0}_2$. Hence $\sigma^{-1}(x^{-1})\mathbf{v}_1 - \sigma^{-1}(x'^{-1})\mathbf{v}_1' \in \mathfrak{N}$. Then it is routine to derive that $\{\mathbf{v}_1, \mathbf{v}_1'\} \cup \mathfrak{N}$ is not independent, contradicting that $A \cup \mathfrak{N}$ is a Hamel basis of V_1. Therefore, A is a singleton. I denote by $\mathbf{v}_1^e$ the unique element in this set, assuming that $e \notin I$.

[6] The existence of Hamel bases can also be derived from Theorem B.1.11 and the correspondence between vector spaces and arguesian projective geometries (Theorem B.3.1 and Theorem B.3.4).

Hence $\{\mathbf{v}_1^i \mid i \in I \cup \{e\}\}$ is a Hamel basis of V_1 with the required property. ⊣

Finally I introduce the notion of continuous quasi-linear maps between vector spaces equipped with anisotropic Hermitian forms.

B.2.20. Lemma. *Let V_1 and V_2 be two vector spaces over division rings $\mathcal{K}_1$ and $\mathcal{K}_2$ equipped with two anisotropic Hermitian forms Φ_1 and Φ_2, respectively, and $f : V_1 \to V_2$ be a quasi-linear map with $\sigma : \mathcal{K}_1 \to \mathcal{K}_2$ as an accompanying division ring isomorphism. Then the following are equivalent:*

(i) for every $\mathbf{v}_2 \in V_2$ there exists a vector $\mathbf{v}_1 \in V_1$ such that $\Phi_2(f(\mathbf{w}_1), \mathbf{v}_2) = \sigma(\Phi_1(\mathbf{w}_1, \mathbf{v}_1))$ for every $\mathbf{w}_1 \in V_1$;

(ii) there exists a quasi-linear map $f^\dagger : V_2 \to V_1$ such that, for any $\mathbf{w}_1 \in V_1$ and $\mathbf{v}_2 \in V_2$, $\Phi_2(f(\mathbf{w}_1), \mathbf{v}_2) = \sigma(\Phi_1(\mathbf{w}_1, f^\dagger(\mathbf{v}_2)))$.

f is called continuous*, if it satisfies any one, and thus both, of the above conditions. Moreover, $f^\dagger$, called* the adjoint of f*, is unique and with σ^{-1} as an accompanying division ring isomorphism, if it exists.*

Proof. Note that anisotropic Hermitian forms are non-singular in the sense of Definition 14.1.5 in [39]. The conclusion follows from Lemma 14.4.9 in [39]. ⊣

For examples of continuous linear maps, please refer to Proposition C.1.1, which shows that every linear map between two finite-dimensional vector spaces over fields equipped with anisotropic Hermitian forms are continuous.

B.3 Correspondences

In this section, I review the intimate relation between projective geometry and linear algebra. If without explanation, the definitions are from [39].

B.3.1 Projective Geometries and Vector Spaces

In this subsection, I discuss the correspondence between arguesian projective geometries and vector spaces over division rings.

I start from the important observation that every vector space over a division ring gives rise to a projective geometry in a canonical way.

B.3.1. Theorem. *Let V be a vector space over a division ring $\mathcal{K} = (K, +, \cdot, 0, 1)$. For every $E \subseteq V$, denote by $\Sigma(E)$ the set $\{\langle \mathbf{v} \rangle \mid \mathbf{v} \in E \setminus \{\mathbf{0}\}\}$, where $\langle \mathbf{v} \rangle = \{x\mathbf{v} \mid x \in K\}$ with $\mathbf{v} \neq \mathbf{0}$ is called an* one-dimensional subspace of V*. Define a function $* : \Sigma(V) \times \Sigma(V) \to \wp(\Sigma(V))$ such that, for any $\mathbf{u}, \mathbf{v} \in V \setminus \{\mathbf{0}\}$,*

$$\langle\mathbf{u}\rangle * \langle\mathbf{v}\rangle = \{\langle\mathbf{w}\rangle \mid \mathbf{w} \in V \setminus \{\mathbf{0}\} \text{ and } \mathbf{w} = x\mathbf{u} + y\mathbf{v} \text{ for some } x, y \in K\}.$$

Then $(\Sigma(V), *)$ *is an irreducible projective geometry, called* the projective geometry of V *and denoted by* $\mathcal{P}(V)$*.*

Proof. This follows from Proposition 2.1.6 and Proposition 2.2.3 in [39]. ⊣

Moreover, the subspaces of a vector space V and the subspaces of the projective geometry of V are closely related.

B.3.2. Lemma. *Let* V *be a vector space over some division ring* $\mathcal{K}$*.*

1. *For every* $\mathbf{v} \in V \setminus \{\mathbf{0}\}$ *and subspace* E *of* V, $\mathbf{v} \in E$ *if and only if* $\langle\mathbf{v}\rangle \in \Sigma(E)$*.*
2. *For every* $\mathbf{v} \in V \setminus \{\mathbf{0}\}$ *and* $P \subseteq \Sigma(V)$, $\mathbf{v} \in \bigcup P$ *if and only if* $\langle\mathbf{v}\rangle \in P$*.*
3. *For every subspace* E *of* V, $\Sigma(E)$ *is a subspace of* $\mathcal{P}(V)$*.*
4. *For every* $P \subseteq \Sigma(V)$, $\Sigma(\bigcup P) = P$*.*
5. P *is a non-empty subspace of* $\mathcal{P}(V)$*, if and only if* $\bigcup P$ *is a subspace of* V *different from* $\{\mathbf{0}\}$*.*

Proof. **For *1*:** First assume that $\mathbf{v} \in E$. Then by definition $\langle\mathbf{v}\rangle \in \Sigma(E)$.

Second assume that $\langle\mathbf{v}\rangle \in \Sigma(E)$. Then there is a $\mathbf{v}' \in E \setminus \{\mathbf{0}\}$ such that $\langle\mathbf{v}\rangle = \langle\mathbf{v}'\rangle$. It follows that $\mathbf{v} \in \langle\mathbf{v}\rangle = \langle\mathbf{v}'\rangle$. Hence there is an x in $\mathcal{K}$ such that $\mathbf{v} = x\mathbf{v}'$. Since $\mathbf{v}' \in E$ and E is a subspace, $\mathbf{v} \in E$.

For *2*: First assume that $\mathbf{v} \in \bigcup P$. Then there is a $\mathbf{v}' \in V \setminus \{\mathbf{0}\}$ such that $\mathbf{v} \in \langle\mathbf{v}'\rangle \in P$. Since $\mathbf{v} \in \langle\mathbf{v}'\rangle$, $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{v}' \neq \mathbf{0}$, it is not hard to deduce from the definition that $\langle\mathbf{v}\rangle = \langle\mathbf{v}'\rangle$. Hence $\langle\mathbf{v}\rangle = \langle\mathbf{v}'\rangle \in P$.

Second assume that $\langle\mathbf{v}\rangle \in P$. Then $\mathbf{v} \in \langle\mathbf{v}\rangle \subseteq \bigcup P$.

For *3*: Let $\langle\mathbf{u}\rangle, \langle\mathbf{v}\rangle \in \Sigma(E)$ be arbitrary, where $\mathbf{u}, \mathbf{v} \in V \setminus \{\mathbf{0}\}$. By *1* $\mathbf{u}, \mathbf{v} \in E$. Then, whenever $\mathbf{w} \in V \setminus \{\mathbf{0}\}$ is such that $\mathbf{w} = x\mathbf{u} + y\mathbf{v}$ for some $x, y \in K$, $\mathbf{w} \in E$, and thus $\langle\mathbf{w}\rangle \in \Sigma(E)$. It follows that $\langle\mathbf{u}\rangle * \langle\mathbf{v}\rangle \subseteq \Sigma(E)$. Therefore, $\Sigma(E)$ is a subspace of $\mathcal{P}(V)$.

For *4*: To show $\Sigma(\bigcup P) \subseteq P$, assume that $\langle\mathbf{v}\rangle \in \Sigma(\bigcup P)$, where $\mathbf{v} \in V \setminus \{\mathbf{0}\}$. By definition there is a $\mathbf{v}' \in \bigcup P \setminus \{\mathbf{0}\}$ such that $\langle\mathbf{v}\rangle = \langle\mathbf{v}'\rangle$. Since $\mathbf{v}' \in \bigcup P$, by *2* $\langle\mathbf{v}'\rangle \in P$. Hence $\langle\mathbf{v}\rangle = \langle\mathbf{v}'\rangle \in P$.

To show $P \subseteq \Sigma(\bigcup P)$, assume that $\langle\mathbf{v}\rangle \in P$, where $\mathbf{v} \in V \setminus \{\mathbf{0}\}$. By definition $\mathbf{v} \in \langle\mathbf{v}\rangle \subseteq \bigcup P$, and thus $\langle\mathbf{v}\rangle \in \Sigma(\bigcup P)$.

For *5*: First assume that $P \subseteq \Sigma(V)$ is a non-empty subspace of $\mathcal{P}(V)$. Then there is a $\mathbf{w} \in V \setminus \{\mathbf{0}\}$ such that $\langle\mathbf{w}\rangle \in P$. It follows that $\mathbf{0} = 0\mathbf{w} \in \langle\mathbf{w}\rangle \subseteq \bigcup P$, so $\bigcup P$ is non-empty. Let $\mathbf{u}, \mathbf{v} \in \bigcup P$ be arbitrary and x, y two arbitrary elements in $\mathcal{K}$. If $x\mathbf{u} + y\mathbf{v} = \mathbf{0}$, by the above $x\mathbf{u} + y\mathbf{v} = \mathbf{0} \in \bigcup P$. If $x\mathbf{u} + y\mathbf{v} \neq \mathbf{0}$, then three cases need to be considered.

Case 1: $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} = \mathbf{0}$. Since $\mathbf{u} \in \bigcup P$, by *2* $\langle \mathbf{u} \rangle \in P$. Hence $x\mathbf{u} + y\mathbf{v} = x\mathbf{u} \in \langle \mathbf{u} \rangle \subseteq \bigcup P$.

Case 2: $\mathbf{u} = \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$. Symmetric to Case 1.

Case 3: $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$. Since $\mathbf{u}, \mathbf{v} \in \bigcup P$, by *2* $\langle \mathbf{u} \rangle, \langle \mathbf{v} \rangle \in P$. Since P is a subspace, $\langle x\mathbf{u} + y\mathbf{v} \rangle \in \langle \mathbf{u} \rangle * \langle \mathbf{v} \rangle \subseteq P$. Hence $x\mathbf{u} + y\mathbf{v} \in \langle x\mathbf{u} + y\mathbf{v} \rangle \subseteq \bigcup P$.

Therefore, $\bigcup P$ is closed under addition and scalar multiplication, and thus is a subspace of V. Since $\mathbf{w} \neq \mathbf{0}$ and $\mathbf{w} \in \langle \mathbf{w} \rangle \subseteq \bigcup P$, $\bigcup P \neq \{\mathbf{0}\}$.

Second assume that $\bigcup P$ is a subspace of V different from $\{\mathbf{0}\}$. By *3* and *4* $P = \Sigma(\bigcup P)$ is a subspace of $\mathcal{P}(V)$. Moreover, since $\bigcup P \neq \{\mathbf{0}\}$, there is a non-zero vector $\mathbf{w}$ such that $\mathbf{w} \in \bigcup P$. By *2* $\langle \mathbf{w} \rangle \in P$, so P is a non-empty subspace of $\mathcal{P}(V)$. ⊣

Now I consider the converse of Theorem B.3.1. As it turns out, it is not always possible to get a vector space over a division ring from a projective geometry. To be able to do so, the geometry needs to have a special property, defined as follows:

B.3.3. Definition. $\mathcal{G} = (G, \star)$ has *Desargues' property*, if, for any six distinct points $a, b, c, a', b', c' \in G$ such that $c \notin a \star b$, $c' \notin a' \star b'$, if the lines $a \star a'$, $b \star b'$ and $c \star c'$ intersect at one point, then the three points $(a \star b) \cap (a' \star b')$, $(b \star c) \cap (b' \star c')$ and $(c \star a) \cap (c' \star a')$ are collinear.

$\mathcal{G} = (G, \star)$ is *irreducible*, if $a \star b$ contains at least three points for any $a, b \in G$ with $a \neq b$.

$\mathcal{G} = (G, \star)$ is *arguesian*, if $\mathcal{G}$ is irreducible and has Desargues' property.

The definition of Desargues' property involves complicated configurations. The picture under Definition 2.5.3 may help to make sense of it.

The importance of being arguesian is manifested in the following theorem:

B.3.4. Theorem. *For a projective geometry $\mathcal{G} = (G, \star)$, the following are equivalent:*

(i) $\mathcal{G}$ is arguesian;

(ii) there is a vector space V of dimension at least 3 over a division ring $\mathcal{K}$ such that $\mathcal{P}(V) \cong \mathcal{G}$.

Both V and $\mathcal{K}$ are unique up to isomorphism, when they exist.

Proof. This follows from Proposition 9.2.6 and Theorem 9.4.4 in [39]. ⊣

Desargues' property has nice and important consequences, but its statement is complicated. However, there is a simple sufficient condition for it to hold.

B.3.5. Theorem. *An irreducible projective geometry has Desargues' property, and thus is arguesian, if it is of rank at least 4.*

Proof. Please refer to the proof of Theorem 8.4.6 in [39]. ⊣

Finally I discuss a property of projective geometries, called Pappus' property, which is stronger than Desargues' property.

B.3.6. Definition. A projective geometry $\mathcal{G} = (G, \star)$ satisfies *Pappus' property*, if, for any six distinct points $a, b, c, a', b', c' \in G$ such that $c \in a \star b$, $c' \in a' \star b'$ and $(a \star b) \cap (a' \star b') = \{o\}$, for some $o \in \Sigma \setminus \{a, b, c, a', b', c'\}$, then the three points $(a \star b') \cap (a' \star b)$, $(a \star c') \cap (a' \star c)$ and $(b \star c') \cap (b' \star c)$ are collinear.

A projective geometry $\mathcal{G} = (G, \star)$ is *Pappian*, if it is irreducible, has Pappus' property and three non-collinear points.

The definition of Pappus' property involves complicated configurations. The picture under Definition 2.7.11 may help to make sense of it.

The importance of being Pappian is manifested by the following theorem:

B.3.7. Theorem. *For a projective geometry $\mathcal{G} = (G, \star)$, the following are equivalent:*

(i) $\mathcal{G}$ is Pappian;

(ii) there is a vector space V of dimension at least 3 over a field *$\mathcal{K}$ such that $\mathcal{P}(V) \cong \mathcal{G}$.*

Both V and $\mathcal{K}$ are unique up to isomorphism, when they exist.

Proof. By Hessenberg's theorem[7] every Pappian projective geometry is arguesian, so this theorem follows from Theorem B.3.4 above and Theorem 9.6.4 in [39]. ⊣

B.3.2 Orthogonality and Hermitian Forms

In this subsection I discuss the correspondence between arguesian pure orthogeometries and vector spaces equipped with anisotropic Hermitian forms, and that between arguesian Hilbertian geometries and vector spaces equipped with orthomodular Hermitian forms (generalized Hilbert spaces).

B.3.8. Theorem.

1. *For every vector space V over some division ring $\mathcal{K}$ with an anisotropic Hermitian form Φ, $(\Sigma(V), *, \perp_V)$ is an irreducible pure orthogeometry, where $\perp_V \subseteq \Sigma(V) \times \Sigma(V)$ is defined so that $\langle \mathbf{u} \rangle \perp_V \langle \mathbf{v} \rangle \Leftrightarrow \Phi(\mathbf{u}, \mathbf{v}) = 0$, for any $\mathbf{u}, \mathbf{v} \in V \setminus \{\mathbf{0}\}$.*

2. *For every pure orthogeometry $\mathcal{G} = (G, \star, \perp)$, the following are equivalent:*

[7] This result appears originally in [53]. Please refer to [28] for a complete proof.

(i) $\mathcal{G}$ is arguesian;

(ii) there is a vector space V of dimension at least 3 over a division ring $\mathcal{K}$ with an anisotropic Hermitian form Φ such that $(\Sigma(V), *, \perp_V) \cong \mathcal{G}$.

Moreover, both V and $\mathcal{K}$ are unique up to isomorphism, if they exist.

3. *If V is a vector space, of dimension at least 3, over a* field *with an anisotropic Hermitian form, $(\Sigma(V), *, \perp_V)$ is a Pappian pure orthogeometry; and every Pappian pure orthogeometry is isomorphic to one of this form.*

Proof. **For *1*:** By Proposition 14.1.6 in [39] $(\Sigma(V), *, \perp_V)$ is an orthogeometry. Irreducibility is obvious, and being pure follows easily from Φ being anisotropic.

For *2*: *From (i) to (ii):* Since $(G, \star)$ is arguesian, by Theorem B.3.4 there is a vector space V over a division ring $\mathcal{K}$ such that $i : \mathcal{P}(V) \cong (G, \star)$. Note that a pure orthogeometry is non-null in the sense of Definition 14.1.7 in [39]. Hence Theorem 14.1.8 in [39] implies that there is a non-singular Hermitian form Φ on V such that $i(\langle \mathbf{u} \rangle) \perp i(\langle \mathbf{v} \rangle) \Leftrightarrow \Phi(\mathbf{u}, \mathbf{v}) = 0$, for any $\mathbf{u}, \mathbf{v} \in V \setminus \{\mathbf{0}\}$. Finally, Φ being anisotropic follows easily from $\perp$ being pure.

From (ii) to (i): By *1* $(\Sigma(V), *, \perp_V)$ is a pure orthogeometry. By Theorem B.3.4 $(\Sigma(V), *, \perp_V)$ is arguesian. Hence $\mathcal{G}$, being isomorphic to $(\Sigma(V), *, \perp_V)$, is an arguesian pure orthogeometry.

Uniqueness of $\mathcal{K}$ and V follows from the uniqueness in Theorem B.3.4.

3 follows from *1*, *2* and Theorem B.3.7. ⊣

A lemma similar to Lemma B.3.2 is in order.

B.3.9. Lemma. *Let V be a vector space over some division ring $\mathcal{K}$ equipped with an anisotropic Hermitian form Φ.*

1. *For every $E \subseteq V$, $\Sigma(E^\perp) = (\Sigma(E))^\perp$*[8].

2. *For every $E \subseteq V$, if $(E^\perp)^\perp = E$, then $((\Sigma(E))^\perp)^\perp = \Sigma(E)$.*

3. *For every subset P of $\Sigma(V)$ with $P^\perp \neq \emptyset$, $\bigcup(P^\perp) = (\bigcup P)^\perp$.*

4. *For every subset P of $\Sigma(V)$ with $P \neq \emptyset$ and $P^\perp \neq \emptyset$, $(P^\perp)^\perp = P$, if and only if $((\bigcup P)^\perp)^\perp = \bigcup P$.*

Proof. *1* is not hard to verify, and *2* follows easily from *1*.

For *3*: To show $\bigcup(P^\perp) \subseteq (\bigcup P)^\perp$, assume that $\mathbf{v} \in \bigcup(P^\perp)$. Then there is a $\mathbf{u} \in V \setminus \{\mathbf{0}\}$ such that $\mathbf{v} \in \langle \mathbf{u} \rangle \in P^\perp$. Hence there is an x in $\mathcal{K}$ such that $\mathbf{v} = x\mathbf{u}$. Let $\mathbf{w} \in \bigcup P$ be arbitrary. If $\mathbf{w} = \mathbf{0}$, then by definition $\Phi(\mathbf{v}, \mathbf{w}) = 0$. If $\mathbf{w} \neq \mathbf{0}$,

[8] I use the same symbol $(\cdot)^\perp$ for orthocomplements in orthogeometries and those in vector spaces. Due to the context, no ambiguity should arise.

then it follows from $\mathbf{w} \in \bigcup P$ and Lemma B.3.2 that $\langle \mathbf{w} \rangle \in P$. As $\langle \mathbf{u} \rangle \in P^{\perp}$, $\langle \mathbf{u} \rangle \perp \langle \mathbf{w} \rangle$. By definition $\Phi(\mathbf{u}, \mathbf{w}) = 0$. Therefore, $\Phi(\mathbf{v}, \mathbf{w}) = \Phi(x\mathbf{u}, \mathbf{w}) = x\Phi(\mathbf{u}, \mathbf{w}) = 0$. Since $\mathbf{w}$ is arbitrary, $\mathbf{v} \in (\bigcup P)^{\perp}$.

To show $(\bigcup P)^{\perp} \subseteq \bigcup(P^{\perp})$, assume that $\mathbf{v} \in (\bigcup P)^{\perp}$. If $\mathbf{v} = \mathbf{0}$, then $\mathbf{0} \in \langle \mathbf{u} \rangle \subseteq \bigcup(P^{\perp})$, where $\mathbf{u} \neq \mathbf{0}$ and $\langle \mathbf{u} \rangle \in P^{\perp}$ witnesses that $P^{\perp} \neq \emptyset$. In the following, I focus on the case when $\mathbf{v} \neq \mathbf{0}$. Let $\langle \mathbf{w} \rangle \in P$ be arbitrary, where $\mathbf{w} \in V \setminus \{\mathbf{0}\}$. By Lemma B.3.2 $\mathbf{w} \in \bigcup P$. Since $\mathbf{v} \in (\bigcup P)^{\perp}$, $\Phi(\mathbf{v}, \mathbf{w}) = 0$. Hence by definition $\langle \mathbf{v} \rangle \perp \langle \mathbf{w} \rangle$. Since $\langle \mathbf{w} \rangle$ is arbitrary, $\langle \mathbf{v} \rangle \in P^{\perp}$, so $\mathbf{v} \in \langle \mathbf{v} \rangle \subseteq \bigcup(P^{\perp})$.

For *4*: First assume that $P = (P^{\perp})^{\perp}$. Since $(P^{\perp})^{\perp} = P \neq \emptyset$, by *3* $\bigcup((P^{\perp})^{\perp}) = (\bigcup(P^{\perp}))^{\perp}$. For $P^{\perp} \neq \emptyset$, by *3* $\bigcup(P^{\perp}) = (\bigcup P)^{\perp}$. Hence $\bigcup((P^{\perp})^{\perp}) = (\bigcup(P^{\perp}))^{\perp} = ((\bigcup P)^{\perp})^{\perp}$. Therefore, $\bigcup P = \bigcup((P^{\perp})^{\perp}) = ((\bigcup P)^{\perp})^{\perp}$.

Second assume that $\bigcup P = ((\bigcup P)^{\perp})^{\perp}$. Then $\Sigma(\bigcup P) = \Sigma(((\bigcup P)^{\perp})^{\perp})$. On the one hand, by Lemma B.3.2 $\Sigma(\bigcup P) = P$. On the other hand, by *1* and Lemma B.3.2 $\Sigma(((\bigcup P)^{\perp})^{\perp}) = (\Sigma((\bigcup P)^{\perp}))^{\perp} = ((\Sigma(\bigcup P))^{\perp})^{\perp} = (P^{\perp})^{\perp}$. Therefore, $P = \Sigma(\bigcup P) = \Sigma(((\bigcup P)^{\perp})^{\perp}) = (P^{\perp})^{\perp}$. ⊣

Next, I show a correspondence between arguesian Hilbertian geometries and generalized Hilbert spaces using Theorem B.3.8.

B.3.10. PROPOSITION. *Let V be a vector space over a division ring $\mathcal{K}$ equipped with an anisotropic Hermitian form Φ. Then the following are equivalent:*

(i) *Φ is orthomodular, and thus V is a generalized Hilbert space;*

(ii) *$(\Sigma(V), *, \perp_V)$ is an arguesian Hilbertian geometry.*

Proof. **From (i) to (ii):** This is well known in the literature, e.g. Example 53 in [84], but no detailed proof is available. Here I give a proof using the techniques developed above.

According to Theorem B.3.8, $(\Sigma(V), *, \perp_V)$ is an arguesian pure orthogeometry. By definition it suffices to show that, for every $P \subseteq \Sigma(V)$, $P = (P^{\perp})^{\perp}$ implies that $\Sigma(V) = \mathcal{C}(P \cup P^{\perp})$.

Let $P \subseteq \Sigma(V)$ satisfying $P = (P^{\perp})^{\perp}$ be arbitrary. If $P = \emptyset$ or $P^{\perp} = \emptyset$, $P^{\perp} = \Sigma(V)$ or $P = \Sigma(V)$, respectively. In both cases, $\Sigma(V) = \mathcal{C}(P \cup P^{\perp})$. In the following, I focus on the case when $P \neq \emptyset$ and $P^{\perp} \neq \emptyset$.

Then by the previous lemma $\bigcup P = ((\bigcup P)^{\perp})^{\perp}$, and $\bigcup P$ is a subspace by Lemma B.2.9. Since Φ is orthomodular, $V = (\bigcup P) + (\bigcup P)^{\perp}$. Let $\mathbf{v} \in V \setminus \{\mathbf{0}\}$ be arbitrary. Then there are $\mathbf{v}_{\parallel} \in \bigcup P$ and $\mathbf{v}_{\perp} \in (\bigcup P)^{\perp}$ such that $\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$. Note that, whenever $\mathbf{v}_{\parallel} = \mathbf{0}$ or $\mathbf{v}_{\perp} = \mathbf{0}$, $\langle \mathbf{v} \rangle \in P^{\perp}$ or $\langle \mathbf{v} \rangle \in P$, respectively, so $\langle \mathbf{v} \rangle \in \mathcal{C}(P \cup P^{\perp})$ for obvious reasons. When $\mathbf{v}_{\parallel} \neq \mathbf{0}$ and $\mathbf{v}_{\perp} \neq \mathbf{0}$, $\langle \mathbf{v} \rangle \in \langle \mathbf{v}_{\parallel} \rangle * \langle \mathbf{v}_{\perp} \rangle$ with $\langle \mathbf{v}_{\parallel} \rangle \in P$ and $\langle \mathbf{v}_{\perp} \rangle \in P^{\perp}$. It follows that $\langle \mathbf{v} \rangle \in \mathcal{C}(P \cup P^{\perp})$. Since $\mathbf{v}$ is arbitrary, $\Sigma(V) \subseteq \mathcal{C}(P \cup P^{\perp})$, so $\Sigma(V) = \mathcal{C}(P \cup P^{\perp})$.

From (ii) to (i): Let a subspace E of V with $E = (E^{\perp})^{\perp}$ and $\mathbf{w} \in V$ be both arbitrary. To prove that Φ is orthomodular, it suffices to show that $\mathbf{w} \in E + E^{\perp}$.

If $\mathbf{w} \in E$ or $\mathbf{w} \in E^\perp$, it is obvious that $\mathbf{w} \in E + E^\perp$. In the following, I focus on the case when $\mathbf{w} \notin E \cup E^\perp$.

It follows that $\mathbf{w} \neq \mathbf{0}$. Since $E = (E^\perp)^\perp$, both E and $E^\perp$ are subspaces of V by Lemma B.2.9, so both $\Sigma(E)$ and $(\Sigma(E))^\perp$ are subspaces of $\mathcal{P}(V)$ and $\Sigma(E) = ((\Sigma(E))^\perp)^\perp$ by Lemma B.3.2 and Lemma B.3.9. Then it follows from (ii) that $\Sigma(V) = \mathcal{C}\left(\Sigma(E) \cup (\Sigma(E))^\perp\right)$. By the projective law there are $w_\parallel \in \Sigma(E)$ and $w_\perp \in (\Sigma(E))^\perp$ such that $\langle \mathbf{w} \rangle \in w_\parallel * w_\perp$. Let $\mathbf{w}_\parallel, \mathbf{w}_\perp \in V \setminus \{\mathbf{0}\}$ be such that $w_\parallel = \langle \mathbf{w}_\parallel \rangle$ and $w_\perp = \langle \mathbf{w}_\perp \rangle$. By Lemma B.3.2 and Lemma B.3.9 $\mathbf{w}_\parallel \in E$ and $\mathbf{w}_\perp \in E^\perp$. Since $\mathbf{w} \in \langle \mathbf{w} \rangle \in w_\parallel * w_\perp = \langle \mathbf{w}_\parallel \rangle * \langle \mathbf{w}_\perp \rangle$, there are x, y in $\mathcal{K}$ such that $\mathbf{w} = x\mathbf{w}_\parallel + y\mathbf{w}_\perp$. Since both E and $E^\perp$ are subspaces of V, $x\mathbf{w}_\parallel \in E$ and $y\mathbf{w}_\perp \in E^\perp$, I conclude that $\mathbf{w} \in E + E^\perp$. ⊣

B.3.11. THEOREM.

1. *For every generalized Hilbert space V, $(\Sigma(V), *, \perp_V)$ is an irreducible Hilbertian geometry.*

2. *For every Hilbertian geometry $\mathcal{G} = (G, \star, \perp)$, the following are equivalent:*

   (i) $\mathcal{G}$ is arguesian;

   (ii) there is a generalized Hilbert space V of dimension at least 3 over a division ring $\mathcal{K}$ such that $(\Sigma(V), *, \perp_V) \cong \mathcal{G}$.

   Moreover, both V and $\mathcal{K}$ are unique up to isomorphism, if they exist.

3. *If V is a generalized Hilbert space of dimension at least 3 over a field, $(\Sigma(V), *, \perp_V)$ is a Pappian Hilbertian geometry; and every Pappian Hilbertian geometry is isomorphic to one of this form.*

Proof. Combine Theorem B.3.8, Proposition B.3.10 and Theorem B.3.7. ⊣

B.3.3 Homomorphisms and Linear Maps

In this subsection I discuss some correspondences involving maps.

I consider a special kind of homomorphism which is called an arguesian homomorphism.

B.3.12. DEFINITION. A homomorphism F from $\mathcal{G}_1 = (G_1, \star_1)$ to $\mathcal{G}_2 = (G_2, \star_2)$ is *non-degenerate*, if $F[G_1]$ contains three non-collinear points.

It is *arguesian*, if it is the composition of finitely many non-degenerate homomorphisms. (These non-degenerate homomorphisms may involve projective geometries other than $\mathcal{G}_1$ and $\mathcal{G}_2$.)

Arguesian homomorphisms are important because of the following theorem:

B.3.13. Theorem. *Let V_1 and V_2 be two vector spaces over two division rings $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. For every partial function $F : \Sigma(V_1) \dashrightarrow \Sigma(V_2)$, the following are equivalent:*

(i) F is an arguesian homomorphism from $\mathcal{P}(V_1)$ to $\mathcal{P}(V_2)$;

(ii) there is a quasi-linear map $f : V_1 \to V_2$ such that $F = \mathcal{P}(f)$, where $\mathcal{P}(f) : \Sigma(V_1) \dashrightarrow \Sigma(V_2)$ is such that $Ker(\mathcal{P}(f)) = \{\langle \mathbf{v}_1 \rangle \mid \mathbf{v}_1 \in V_1 \setminus \{\mathbf{0}_1\},\ f(\mathbf{v}_1) = \mathbf{0}_2\}$ and $\mathcal{P}(f)(\langle \mathbf{v}_1 \rangle) = \langle f(\mathbf{v}_1) \rangle$, for every $\mathbf{v}_1 \in V_1 \setminus \{\mathbf{0}_1\}$ with $\langle \mathbf{v}_1 \rangle \notin Ker(\mathcal{P}(f))$.

Proof. For the direction from *1* to *2*, combine Theorem 10.1.4 and Theorem 10.3.1 in [39]. For the other direction, please refer to Proposition 6.6.12 in [39]. ⊣

Next I consider a similar result for continuous homomorphisms between pure orthogeometries and continuous quasi-linear maps between vector spaces equipped with an anisotropic Hermitian form.

B.3.14. Theorem. *Let V_1 and V_2 be two vector spaces over two division rings $\mathcal{K}_1$ and $\mathcal{K}_2$ equipped with two anisotropic Hermitian forms Φ_1 and Φ_2, respectively. For every partial function $F : \Sigma(V_1) \dashrightarrow \Sigma(V_2)$, the following are equivalent:*

(i) F is an arguesian continuous homomorphism from $\mathcal{P}(V_1)$ to $\mathcal{P}(V_2)$;

(ii) there is a continuous quasi-linear map $f : V_1 \to V_2$ such that $F = \mathcal{P}(f)$.

Moreover, $F^\dagger = (\mathcal{P}(f))^\dagger = \mathcal{P}(f^\dagger)$, when f exists.

Proof. Note that every anisotropic Hermitian form is a non-singular reflexive sesquilinear form in the sense of Definition 14.1.5 in [39]. Then the conclusion follows from Theorem B.3.13 and Proposition 14.4.10 in [39]. ⊣

I summarize all mentioned correspondences between projective geometry and linear algebra in the following table:

Projective Geometry	Linear Algebra
arguesian projective geometry	vector space over a division ring
arguesian pure orthogeometry	vector space over a division ring + an anisotropic Hermitian form
arguesian Hilbertian geometry	vector space over a division ring + an orthomodular Hermitian form
Pappian projective geometry	vector space over a field
Pappian pure orthogeometry	vector space over a field + an anisotropic Hermitian form
Pappian Hilbertian geometry	vector space over a field + an orthomodular Hermitian form
arguesian homomorphism	quasi-linear map
arguesian continuous homomorphism	continuous quasi-linear map

B.4 Harmonic Conjugate

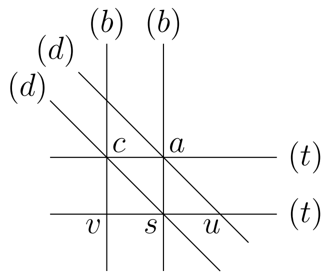
In this section, I would like to review the notion of harmonic conjugates, which is important in projective geometry.

I adopt the definition of harmonic conjugates from Section 2.5 in [27] instead of Exercise 9.7.3 in [39].

B.4.1. Definition. In a projective geometry $\mathcal{G} = (G, \star)$, for any $s, t, u \in G$ such that $s \neq t$ and $u \in (s \star t) \setminus \{s, t\}$, $v \in G$ is *a harmonic conjugate of u with respect to s and t*, if there are $a, b, c, d \in G$ satisfying all of the following:

1. no three of a, b, c, d are collinear;
2. $s \in a \star b \cap c \star d$;
3. $t \in a \star c \cap b \star d$;
4. $u \in a \star d \cap s \star t$;
5. $v \in b \star c \cap s \star t$.

The definition of harmonic conjugates seems complicated. The following picture of the analogue in an affine plane may help to make sense of it: (in the picture '(t)' means that t is not a point in the affine plane; instead, it is an imaginary point at infinity where parallel lines intersect.)



The following lemma reveals a geometric fact about harmonic conjugates.

B.4.2. Lemma. *Let $\mathcal{G} = (G, \star)$ be a projective geometry and $s, t, u, v \in G$ be such that $s \neq t$, $u \in s \star t$ and v is a harmonic conjugate of u with respect to s and t. Moreover, let $a, b, c, d \in G$ witness this, i.e. 1 to 5 in the above definition are satisfied. Then $b \notin s \star t$.*

Proof. Observe that $b \neq s$; otherwise, $b = s \in c \star d$, contradicting that b, c, d are not collinear. Also observe that $b \neq t$; otherwise, $b = t \in a \star c$, contradicting that a, b, c are not collinear.

Now suppose (towards a contradiction) that $b \in s \star t$. Then $b, s \in (a \star b) \cap (s \star t)$. Since $s \neq b$, by (P8) in Lemma B.1.2 $s \star t = s \star b = a \star b$. Moreover, by the supposition $b, t \in (b \star d) \cap (s \star t)$. Since $t \neq b$, by (P8) $s \star t = t \star b = b \star d$. Therefore, $a \in a \star b = s \star t = b \star d$, contradicting that a, b, d are not collinear. As a result, $b \notin s \star t$. ⊣

The following is a characterization of the harmonic conjugates in projective geometries of the form $\mathcal{P}(V)$ for some vector space V over some division ring. In the literature, its proof is considered as a direct computation (e.g. [55]). However, due to the existential definition that I adopt, it may not be obvious how to compute. Hence I provide a detailed proof below.

B.4.3. LEMMA. *Let V be a vector space, of dimension at least 3, over some division ring $\mathcal{K}$, $\mathbf{s}, \mathbf{t} \in V \setminus \{\mathbf{0}\}$ be linearly independent and $\mathbf{v} \in V \setminus \{\mathbf{0}\}$. In the projective geometry $\mathcal{P}(V)$, $\langle \mathbf{v} \rangle$ is a harmonic conjugate of $\langle \mathbf{s} + \mathbf{t} \rangle$ with respect to $\langle \mathbf{s} \rangle$ and $\langle \mathbf{t} \rangle$, if and only if $\langle \mathbf{v} \rangle = \langle \mathbf{s} - \mathbf{t} \rangle$.*

Proof. Since $\mathbf{s}$ and $\mathbf{t}$ are linearly independent, $\langle \mathbf{s} \rangle$, $\langle \mathbf{t} \rangle$ and $\langle \mathbf{s} + \mathbf{t} \rangle$ are distinct.

The 'Only If' part: Assume that $\langle \mathbf{v} \rangle$ is a harmonic conjugate of $\langle \mathbf{s} + \mathbf{t} \rangle$ with respect to $\langle \mathbf{s} \rangle$ and $\langle \mathbf{t} \rangle$. By definition there are $a, b, c, d \in \Sigma(V)$ such that

1. no three of a, b, c, d are collinear;
2. $\langle \mathbf{s} \rangle \in (a * b) \cap (c * d)$;
3. $\langle \mathbf{t} \rangle \in (a * c) \cap (b * d)$;
4. $\langle \mathbf{s} + \mathbf{t} \rangle \in (a * d) \cap (\langle \mathbf{s} \rangle * \langle \mathbf{t} \rangle)$;
5. $\langle \mathbf{v} \rangle \in (b * c) \cap (\langle \mathbf{s} \rangle * \langle \mathbf{t} \rangle)$.

Let $\mathbf{b} \in V \setminus \{\mathbf{0}\}$ be such that $\langle \mathbf{b} \rangle = b$. By the above lemma $b \notin \langle \mathbf{s} \rangle * \langle \mathbf{t} \rangle$, so $b \neq \langle \mathbf{s} \rangle$, $b \neq \langle \mathbf{t} \rangle$ and $\mathbf{b}, \mathbf{s}, \mathbf{t}$ are linearly independent.

Now I find vectors which generate a, c and d, respectively.

For a, since $\langle \mathbf{s} \rangle \in a * b$ and $\langle \mathbf{s} \rangle \neq b$, by (P4) in Lemma B.1.2 $a \in \langle \mathbf{s} \rangle * b$. Then there are x, y in $\mathcal{K}$, which can not be both 0, such that $a = \langle x\mathbf{s} + y\mathbf{b} \rangle$. Note that $x \neq 0$; otherwise, $a = b$, and thus a, b, c would be collinear. Also note that $y \neq 0$; otherwise, $a = s$, and thus $a \in c \star d$, contradicting that a, c, d are non-collinear. Hence, without loss of generality, I assume that $x = y = 1$, so $a = \langle \mathbf{s} + \mathbf{b} \rangle$.

For d, since $\langle \mathbf{t} \rangle \in b * d$ and $\langle \mathbf{t} \rangle \neq b$, by (P4) $d \in \langle \mathbf{t} \rangle * b$. Since $a = \langle \mathbf{s} + \mathbf{b} \rangle \neq \langle \mathbf{s} + \mathbf{t} \rangle$ and $\langle \mathbf{s} + \mathbf{t} \rangle \in a * d$, by (P4) $d \in a * \langle \mathbf{s} + \mathbf{t} \rangle$. Hence $d \in (\langle \mathbf{t} \rangle * b) \cap (a * \langle \mathbf{s} + \mathbf{t} \rangle)$. It follows that $d = \langle \mathbf{b} - \mathbf{t} \rangle$.

For c, since $a = \langle \mathbf{s} + \mathbf{b} \rangle \neq \langle \mathbf{t} \rangle$ and $\langle \mathbf{t} \rangle \in a * c$, by (P4) $c \in a * \langle \mathbf{t} \rangle$. Since $d = \langle \mathbf{b} - \mathbf{t} \rangle \neq \langle \mathbf{s} \rangle$ and $\langle \mathbf{s} \rangle \in c * d$, by (P4) $c \in d * \langle \mathbf{s} \rangle$. Hence $c \in (a * \langle \mathbf{t} \rangle) \cap (d * \langle \mathbf{s} \rangle)$. It follows that $c = \langle \mathbf{b} + \mathbf{s} - \mathbf{t} \rangle$.

Finally, since $\langle \mathbf{v} \rangle \in (b * c) \cap (\langle \mathbf{s} \rangle * \langle \mathbf{t} \rangle) = (\langle \mathbf{b} \rangle * \langle \mathbf{b} + \mathbf{s} - \mathbf{t} \rangle) \cap (\langle \mathbf{s} \rangle * \langle \mathbf{t} \rangle)$, $\langle \mathbf{v} \rangle = \langle \mathbf{s} - \mathbf{t} \rangle$.

The 'If' Part: Assume that $\langle \mathbf{v} \rangle = \langle \mathbf{s} - \mathbf{t} \rangle$. Since V is of dimension at least 3, there is a $\mathbf{b} \in V$ such that $\mathbf{b}, \mathbf{s}, \mathbf{t}$ are linearly independent. Let $a = \langle \mathbf{s} + \mathbf{b} \rangle$, $b = \langle \mathbf{b} \rangle$, $c = \langle \mathbf{b} + \mathbf{s} - \mathbf{t} \rangle$ and $d = \langle \mathbf{b} - \mathbf{t} \rangle$. Then it is easy to check that 1 to 5 in the definition of harmonic conjugates hold. Therefore, $\langle \mathbf{v} \rangle = \langle \mathbf{s} - \mathbf{t} \rangle$ is a harmonic conjugate of $\langle \mathbf{s} + \mathbf{t} \rangle$ with respect to $\langle \mathbf{s} \rangle$ and $\langle \mathbf{t} \rangle$. $\dashv$

B.4.4. REMARK. From this lemma and Theorem B.3.4 it is clear that, in an arguesian projective geometry, for every three points s, t and w such that $w \in (s \star t) \setminus \{s, t\}$, w has *exactly one* harmonic conjugate with respect to s and t.

Appendix C

Linear Maps

In this appendix, I investigate some notions and properties of linear maps, which are used in this thesis. The point of this appendix lies in its generality. These linear maps are not between vector spaces over $\mathbb{R}$ or $\mathbb{C}$ as in many textbooks of linear algebra. Instead, they are between two finite-dimensional vector spaces over the same field equipped with anisotropic Hermitian forms.

C.1 Adjoint of a Linear Map

In this section, I prove that every linear map between finite-dimensional vector spaces over the same field equipped with anisotropic Hermitian forms is continuous, and thus has an adjoint.

C.1.1. Proposition. *Let V_1 and V_2 be two vector spaces over a field $\mathcal{K} = (K, +, \cdot, 0, 1)$ equipped with anisotropic Hermitian forms Φ_1 and Φ_2, respectively. Moreover, V_1 and V_2 are of finite dimensions n and m, respectively. Then every linear map $f : V_1 \to V_2$ is continuous, and thus has an adjoint $f^\dagger$.*

Proof. Since V_1 is of finite dimension n and equipped with an anisotropic Hermitian form Φ_1, by Proposition B.2.10 V_1 has an orthogonal (not necessarily orthonormal) basis $\{\mathbf{v}_1^1, ..., \mathbf{v}_1^n\}$. Similarly, V_2 has an orthogonal basis $\{\mathbf{w}_2^1, ..., \mathbf{w}_2^m\}$. Moreover, let a_{ij} for $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$ be such that $f(\mathbf{v}_1^i) = \sum_{j=1}^m a_{ij}\mathbf{w}_2^j$ for $i = 1, ..., n$. Consider the function $f^\dagger : V_1 \to V_2$ such that, for every $\mathbf{u}_2 \in V_2$, if $\mathbf{u}_2 = \sum_{j=1}^m y_j \mathbf{w}_2^j$ under the basis, then $f^\dagger(\mathbf{u}_2) = \sum_{j=1}^m \sum_{i=1}^n (y_j \cdot \Phi_2(\mathbf{w}_2^j, \mathbf{w}_2^j) \cdot a_{ij}^* \cdot \Phi_1(\mathbf{v}_1^i, \mathbf{v}_1^i)^{-1})\mathbf{v}_1^i$, where $(\cdot)^*$ is the accompanying involution on $\mathcal{K}$ of Φ_2. Since $\mathcal{K}$ is a field, it is not hard to verify that $f^\dagger$ is a linear map.

I claim that, for any $\mathbf{s}_1 \in V_1$ and $\mathbf{t}_2 \in V_2$, $\Phi_2(f(\mathbf{s}_1), \mathbf{t}_2) = \Phi_1(\mathbf{s}_1, f^\dagger(\mathbf{t}_2))$. Assume that under the bases $\mathbf{s}_1 = \sum_{i=1}^n x_i \mathbf{v}_1^i$ and $\mathbf{t}_2 = \sum_{j=1}^m y_j \mathbf{w}_2^j$. Calculate

$$\Phi_2(f(\mathbf{s}_1), \mathbf{t}_2) = \Phi_2\Big(f(\sum_{i=1}^n x_i \mathbf{v}_1^i), \sum_{j=1}^m y_j \mathbf{w}_2^j\Big)$$

$$
\begin{aligned}
&= \Phi_2\Big(\sum_{i=1}^{n} x_i \sum_{k=1}^{m} a_{ik}\mathbf{w}_2^k, \sum_{j=1}^{m} y_j \mathbf{w}_2^j\Big) \\
&= \sum_{i=1}^{n}\sum_{j,k=1}^{m} x_i \cdot a_{ik} \cdot \Phi_2(\mathbf{w}_2^k, \mathbf{w}_2^j) \cdot y_j^* \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} x_i \cdot a_{ij} \cdot \Phi_2(\mathbf{w}_2^j, \mathbf{w}_2^j) \cdot y_j^*
\end{aligned}
$$

$$
\begin{aligned}
\Phi_1(\mathbf{s}_1, f^\dagger(\mathbf{t}_2)) &= \Phi_1\Big(\sum_{i=1}^{n} x_i \mathbf{v}_1^i, f^\dagger(\sum_{j=1}^{m} y_j \mathbf{w}_2^j)\Big) \\
&= \Phi_1\Big(\sum_{i=1}^{n} x_i \mathbf{v}_1^i, \sum_{j=1}^{m}\sum_{l=1}^{n}(y_j \cdot \Phi_2(\mathbf{w}_2^j, \mathbf{w}_2^j) \cdot a_{lj}^* \cdot \Phi_1(\mathbf{v}_1^l, \mathbf{v}_1^l)^{-1})\mathbf{v}_1^l\Big) \\
&= \sum_{i,l=1}^{n}\sum_{j=1}^{m} x_i \cdot \Phi_1(\mathbf{v}_1^i, \mathbf{v}_1^l) \cdot \Phi_1(\mathbf{v}_1^l, \mathbf{v}_1^l)^{-1} \cdot a_{lj} \cdot \Phi_2(\mathbf{w}_2^j, \mathbf{w}_2^j) \cdot y_j^* \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} x_i \cdot \Phi_1(\mathbf{v}_1^i, \mathbf{v}_1^i) \cdot \Phi_1(\mathbf{v}_1^i, \mathbf{v}_1^i)^{-1} \cdot a_{ij} \cdot \Phi_2(\mathbf{w}_2^j, \mathbf{w}_2^j) \cdot y_j^* \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} x_i \cdot a_{ij} \cdot \Phi_2(\mathbf{w}_2^j, \mathbf{w}_2^j) \cdot y_j^*
\end{aligned}
$$

Therefore, $\Phi_2(f(\mathbf{s}_1), \mathbf{t}_2) = \Phi_1(\mathbf{s}_1, f^\dagger(\mathbf{t}_2))$.

By Lemma B.2.20 f is a continuous linear map, and $f^\dagger$ is its adjoint. ⊣

C.2 Trace of a Linear Map

In this section, I discuss the notion of traces of linear maps, and show that it makes sense and has nice properties in settings more general than $\mathbb{R}^n$ and $\mathbb{C}^n$. I start with a technical lemma, which guarantees that the definition of traces below is legitimate.

C.2.1. LEMMA. *Let V be a vector space of finite dimension n over a* field $\mathcal{K} = (K, +, \cdot, 0, 1)$ *equipped with an anisotropic Hermitian form Φ, $f : V \to V$ a linear map, $\{\mathbf{u}_1, ..., \mathbf{u}_n\} \subseteq V$ an orthonormal basis of V, i.e. a basis of V satisfying*

$$\Phi(\mathbf{u}_i, \mathbf{u}_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

and $\{\mathbf{v}_1, ..., \mathbf{v}_n\} \subseteq V$ also an orthonormal basis of V. Then

$$\sum_{i=1}^{n} \Phi(f(\mathbf{u}_i), \mathbf{u}_i) = \sum_{i=1}^{n} \Phi(f(\mathbf{v}_i), \mathbf{v}_i)$$

Proof. Since $\{\mathbf{u}_1, ..., \mathbf{u}_n\}$ is a basis of V, for each $i = 1, ..., n$, there are $a_{i1}, ..., a_{in} \in K$ such that $\mathbf{v}_i = \sum_{k=1}^{n} a_{ik}\mathbf{u}_k$. Then, for any $i, j \in \{1, ..., n\}$,

$$\Phi(\mathbf{v}_i, \mathbf{v}_j) = \Phi(\sum_{k=1}^{n} a_{ik}\mathbf{u}_k, \sum_{l=1}^{n} a_{jl}\mathbf{u}_l) = \sum_{k,l=1}^{n} a_{ik} \cdot \Phi(\mathbf{u}_k, \mathbf{u}_l) \cdot a_{jl}^* = \sum_{k=1}^{n} a_{ik} \cdot a_{jk}^*$$

where $(\cdot)^*$ is the accompanying involution of Φ, and thus

$$\sum_{k=1}^{n} a_{ik} \cdot a_{jk}^* = \Phi(\mathbf{v}_i, \mathbf{v}_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Similarly, for each $i = 1, ..., n$, there are $b_{i1}, ..., b_{in} \in K$ such that $\mathbf{u}_i = \sum_{k=1}^{n} b_{ik}\mathbf{v}_k$. Then, for any $i, j \in \{1, ..., n\}$,

$$\Phi(\mathbf{u}_i, \mathbf{u}_j) = \Phi(\sum_{k=1}^{n} b_{ik}\mathbf{v}_k, \sum_{l=1}^{n} b_{jl}\mathbf{v}_l) = \sum_{k,l=1}^{n} b_{ik} \cdot \Phi(\mathbf{v}_k, \mathbf{v}_l) \cdot b_{jl}^* = \sum_{k=1}^{n} b_{ik} \cdot b_{jk}^*$$

and thus

$$\sum_{k=1}^{n} b_{ik} \cdot b_{jk}^* = \Phi(\mathbf{u}_i, \mathbf{u}_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Moreover, for any $i = \{1, ..., n\}$,

$$\mathbf{u}_i = \sum_{k=1}^{n} b_{ik}\mathbf{v}_k = \sum_{k=1}^{n} b_{ik} \left(\sum_{l=1}^{n} a_{kl}\mathbf{u}_l \right) = \sum_{l=1}^{n} \left(\sum_{k=1}^{n} b_{ik} \cdot a_{kl} \right) \mathbf{u}_l$$

Hence

$$\sum_{k=1}^{n} b_{ik} \cdot a_{kl} = \begin{cases} 1, & \text{if } i = l \\ 0, & \text{if } i \neq l \end{cases}$$

Then, for any $i, j \in \{1, ..., n\}$,

$$b_{ij} = \sum_{k=1}^{n} b_{ik} \cdot \left(\sum_{l=1}^{n} a_{kl} \cdot a_{jl}^* \right) = \sum_{l=1}^{n} \left(\sum_{k=1}^{n} b_{ik} \cdot a_{kl} \right) \cdot a_{jl}^* = a_{ji}^*$$

Therefore,

$$\begin{aligned} \sum_{i=1}^{n} \Phi(f(\mathbf{v}_i), \mathbf{v}_i) &= \sum_{i=1}^{n} \Phi(f(\sum_{k=1}^{n} a_{ik}\mathbf{u}_k), \sum_{l=1}^{n} a_{il}\mathbf{u}_l) \\ &= \sum_{i=1}^{n} \Phi(\sum_{k=1}^{n} a_{ik} f(\mathbf{u}_k), \sum_{l=1}^{n} a_{il}\mathbf{u}_l) \\ &= \sum_{i,k,l=1}^{n} a_{ik} \cdot \Phi(f(\mathbf{u}_k), \mathbf{u}_l) \cdot a_{il}^* \end{aligned}$$

$$\begin{aligned}
&= \sum_{k,l=1}^{n} \left(\sum_{i=1}^{n} a_{il}^* \cdot a_{ik} \right) \cdot \Phi(f(\mathbf{u}_k), \mathbf{u}_l) && (\mathcal{K} \text{ is a field}) \\
&= \sum_{k,l=1}^{n} \left(\sum_{i=1}^{n} b_{li} \cdot a_{ik} \right) \cdot \Phi(f(\mathbf{u}_k), \mathbf{u}_l) \\
&= \sum_{i=1}^{n} \Phi(f(\mathbf{u}_i), \mathbf{u}_i)
\end{aligned}$$

⊣

Based on this lemma, the notion of traces can be defined.

C.2.2. Definition. Let V be an n-dimensional vector space over a *field* $\mathcal{K}$ equipped with an anisotropic Hermitian form Φ, which has at least one orthonormal basis $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$. Then *the trace of a linear map f on V*, denoted by $tr(f)$, is $\sum_{i=1}^{n} \Phi(f(\mathbf{v}_i), \mathbf{v}_i)$.

In the remaining part of this section, I show that, under some natural constraints, the trace of any linear map of the form $f^\dagger \circ f$ is never 0. Before proving this, I need a technical lemma and a definition.

C.2.3. Lemma. *Let V be a vector space of dimension at least 2 over a division ring $\mathcal{K} = (K, +, \cdot, 0, 1)$ equipped with an anisotropic Hermitian form Φ, which satisfies the following:*

($\bigstar$) *for any $\mathbf{s}, \mathbf{t} \in V \setminus \{\mathbf{0}\}$, there is an x in $\mathcal{K} \setminus \{0\}$ such that $\Phi(\mathbf{s}, \mathbf{s}) = \Phi(x\mathbf{t}, x\mathbf{t})$.*

Then, for any $n \in \mathbb{N}^+$ and $\{\mathbf{w}_1, ..., \mathbf{w}_n\} \subseteq V \setminus \{\mathbf{0}\}$, there is a $\mathbf{w} \in V \setminus \{\mathbf{0}\}$ such that $\Phi(\mathbf{w}, \mathbf{w}) = \sum_{i=1}^{n} \Phi(\mathbf{w}_i, \mathbf{w}_i)$. In particular, $\sum_{i=1}^{n} \Phi(\mathbf{w}_i, \mathbf{w}_i) \neq 0$.

Proof.[1] Use induction on n.

Base Step: $n = 1$. The conclusion follows from Φ being anisotropic.

Induction Step: $n = k > 1$. By the induction hypothesis there is a $\mathbf{w}' \in V \setminus \{\mathbf{0}\}$ such that $\Phi(\mathbf{w}', \mathbf{w}') = \sum_{i=1}^{k-1} \Phi(\mathbf{w}_i, \mathbf{w}_i)$. Since V is of dimension at least 2, it is not hard to find $\mathbf{u}, \mathbf{v} \in V \setminus \{\mathbf{0}\}$ such that $\Phi(\mathbf{u}, \mathbf{v}) = 0$. By ($\bigstar$) there are x, y in $K \setminus \{0\}$ such that $\Phi(\mathbf{w}', \mathbf{w}') = \Phi(x\mathbf{u}, x\mathbf{u})$ and $\Phi(\mathbf{w}_k, \mathbf{w}_k) = \Phi(y\mathbf{v}, y\mathbf{v})$. Let $\mathbf{w} = x\mathbf{u} + y\mathbf{v}$. Since x, y are in $K \setminus \{0\}$ and $\mathbf{u}, \mathbf{v} \in V \setminus \{\mathbf{0}\}$ are such that $\Phi(\mathbf{u}, \mathbf{v}) = 0$, $\mathbf{w} \neq \mathbf{0}$. Moreover,

$$\begin{aligned}
\Phi(\mathbf{w}, \mathbf{w}) &= \Phi(x\mathbf{u} + y\mathbf{v}, x\mathbf{u} + y\mathbf{v}) \\
&= \Phi(x\mathbf{u}, x\mathbf{u}) + x \cdot \Phi(\mathbf{u}, \mathbf{v}) \cdot y^* + y \cdot \Phi(\mathbf{v}, \mathbf{u}) \cdot x^* + \Phi(y\mathbf{v}, y\mathbf{v}) \\
&= \Phi(x\mathbf{u}, x\mathbf{u}) + \Phi(y\mathbf{v}, y\mathbf{v})
\end{aligned}$$

[1] The idea of this proof is from that of Corollary 4 on p 379 in [76].

$$= \Phi(\mathbf{w}', \mathbf{w}') + \Phi(\mathbf{w}_k, \mathbf{w}_k)$$
$$= \sum_{i=1}^{k} \Phi(\mathbf{w}_i, \mathbf{w}_i),$$

where $(\cdot)^*$ is the accompanying involution of Φ on $\mathcal{K}$. Since Φ is anisotropic, $\sum_{i=1}^{k} \Phi(\mathbf{w}_i, \mathbf{w}_i) = \Phi(\mathbf{w}, \mathbf{w}) \neq \mathbf{0}$. ⊣

C.2.4. Definition. Let V be a vector space over a division ring $\mathcal{K}$ equipped with an anisotropic Hermitian form Φ. V *admits normalization*, if for every $\mathbf{v} \in V \setminus \{\mathbf{0}\}$ there is an x in $\mathcal{K}$ such that $\Phi(x\mathbf{v}, x\mathbf{v}) = 1$.

C.2.5. Remark. Note that (★) is the same as (iii) in Lemma 2.7.8, and admitting normalization is equivalent to (★) together with the fact that $\Phi(\mathbf{v}, \mathbf{v}) = 1$ for some $\mathbf{v} \in V$.

C.2.6. Proposition. *Let V and W be two vector spaces over the same* field $\mathcal{K} = (K, +, \cdot, 0, 1)$ *equipped with anisotropic Hermitian forms Φ and Ψ, respectively, such that V is of finite dimension $n \geq 2$ and admits normalization. Then, for every linear map $f : V \to W$ other than the zero map, $tr(f^\dagger \circ f) \neq 0$.*

Proof. By Proposition B.2.10 there is an orthogonal basis $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ of V. Since V admits normalization, there are $x_1, ..., x_n \in K$ such that $\Phi(x_i\mathbf{v}_i, x_i\mathbf{v}_i) = 1$ for $i = 1, ..., n$. Then $\{x_1\mathbf{v}_1, ..., x_n\mathbf{v}_n\}$ is an orthonormal basis of V. Hence the notion of traces is well-defined. Without loss of generality and for simplicity of the notation, I assume that $x_i = 1$ for $i = 1, ..., n$, i.e. $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is an *orthonormal basis*. Moreover, by Proposition C.1.1 every linear map $f : V \to W$ has an adjoint.

Now let $f : V \to W$ be an arbitrary linear map which is not the zero map. Then, for at least one $i \in \{1, ..., n\}$, $f(\mathbf{v}_i) \neq \mathbf{0}$. Without loss of generality, I assume that there is an $m \in \{1, ..., n\}$ such that $f(\mathbf{v}_i) = \mathbf{0}$ if and only if $i > m$. By Lemma C.2.3 $\sum_{i=1}^{m} \Phi(f(\mathbf{v}_i), f(\mathbf{v}_i)) \neq 0$. Therefore, using Lemma B.2.20,

$$tr(f^\dagger \circ f) = \sum_{i=1}^{n} \Phi(f^\dagger \circ f(\mathbf{v}_i), \mathbf{v}_i) = \sum_{i=1}^{n} \Phi(f(\mathbf{v}_i), f(\mathbf{v}_i)) = \sum_{i=1}^{m} \Phi(f(\mathbf{v}_i), f(\mathbf{v}_i)) \neq 0$$

⊣

C.3 The Structure of $Hom(V_1, V_2)$

Let V_1 and V_2 be two finite-dimensional vector spaces over the same *field* $\mathcal{K}$ equipped with anisotropic Hermitian forms Φ_1 and Φ_2, respectively. In this section I investigate the structure of $Hom(V_1, V_2)$, the set of all linear maps from V_1 to V_2, and show that it forms in a natural way a vector space over $\mathcal{K}$ equipped with an anisotropic Hermitian form.

C.3.1. THEOREM. *Let V_1 and V_2 be two finite-dimensional vector spaces over the same* field $\mathcal{K} = (K, +, \cdot, 0, 1)$ *equipped with anisotropic Hermitian forms Φ_1 and Φ_2, respectively. For any $f, g \in Hom(V_1, V_2)$ and $x \in K$,*

- *define $f + g$ to be the function from V_1 to V_2 such that $(f + g)(\mathbf{v}_1) = f(\mathbf{v}_1) + g(\mathbf{v}_1)$ for every $\mathbf{v}_1 \in V_1$;*
- *define xf to be the function from V_1 to V_2 such that $(xf)(\mathbf{v}_1) = xf(\mathbf{v}_1)$ for every $\mathbf{v}_1 \in V_1$.*

Then $Hom(V_1, V_2)$ forms a vector space over $\mathcal{K}$ with the above functions as addition and multiplications by scalars.

Proof. Since both V_1 and V_2 are vector spaces over the field $\mathcal{K}$, they are both modules over the commutative ring $\mathcal{K}$. By Theorem 3 in Section V.2 in [63] $Hom(V_1, V_2)$ forms a module over the commutative ring $\mathcal{K}$ with the functions defined above as addition and multiplications by scalars. Since $\mathcal{K}$ is a field, $Hom(V_1, V_2)$ is a vector space. ⊣

C.3.2. REMARK. Commutativity of $\cdot$ is needed to prove that xf is a linear map for every $x \in K$ and linear map $f : V_1 \to V_2$.

C.3.3. THEOREM. *Let V_1 and V_2 be two finite-dimensional vector spaces over the same* field $\mathcal{K} = (K, +, \cdot, 0, 1)$ *equipped with anisotropic Hermitian forms Φ_1 and Φ_2, respectively. If V_1 admits normalization, the map*

$$\Omega(\cdot, \cdot) : Hom(V_1, V_2) \times Hom(V_1, V_2) \to K :: (f, g) \mapsto tr(g^\dagger \circ f)$$

is an anisotropic Hermitian form on $Hom(V_1, V_2)$, whose accompanying involution on $\mathcal{K}$ is the same as that of Φ_1.

Proof. By Proposition C.1.1 every $f \in Hom(V_1, V_2)$ has an adjoint. By Proposition B.2.10 V_1 has an orthogonal basis. Since V_1 admits normalization, it is easy to see that V_1 has an orthonormal basis. Hence the notion of traces is well-defined for linear maps on V_1. I denote by $(\cdot)^*$ the accompanying involution of Φ_1 on $\mathcal{K}$, and let $\{\mathbf{v}_1^1, ..., \mathbf{v}_1^n\}$ be an orthonormal basis of V_1.

It is not hard to verify that Ω is an Hermitian form on $Hom(V_1, V_2)$, whose accompanying involution on $\mathcal{K}$ is the same as that of Φ_1. As an example, I show that $\Omega(f, g)^* = \Omega(g, f)$, for any linear maps $f, g : V_1 \to V_2$.

$$\begin{aligned}\Omega(f, g)^* &= tr(g^\dagger \circ f)^* \\ &= \left(\sum_{i=1}^{n} \Phi_1 \left(g^\dagger \circ f(\mathbf{v}_1^i), \mathbf{v}_1^i \right) \right)^* \\ &= \sum_{i=1}^{n} \Phi_1 \left(g^\dagger \circ f(\mathbf{v}_1^i), \mathbf{v}_1^i \right)^*\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \Phi_1 \left(f(\mathbf{v}_1^i), g(\mathbf{v}_1^i) \right)^* \\
&= \sum_{i=1}^{n} \Phi_1 \left(g(\mathbf{v}_1^i), f(\mathbf{v}_1^i) \right) \\
&= \sum_{i=1}^{n} \Phi_1 \left(f^\dagger \circ g(\mathbf{v}_1^i), \mathbf{v}_1^i \right) \\
&= tr(f^\dagger \circ g) \\
&= \Omega(g, f)
\end{aligned}
$$

Finally, by Proposition C.2.6 the Hermitian form $\Omega(\cdot, \cdot)$ is anisotropic. $\dashv$

C.4 Orthogonal Basis without Eigenvectors

Remember that in linear algebra the eigenvectors of a linear map $f : V \to V$ are very important in studying the properties of f, because they contain a lot of information in a very concise way. It is also warmly welcome if the eigenvectors of f form a basis of V. Then f can be represented by a diagonalized matrix under this basis, which is very elegant.

However, looking from the perspective of projective geometry, i.e. $\mathcal{P}(f)$ acting on $\mathcal{P}(V)$, the eigenvectors cause disasters. This is because in general, for an eigenvector $\mathbf{v}$ of f, $\langle \mathbf{v} \rangle$ and $\langle f(\mathbf{v}) \rangle$ is the same point and the information about the eigenvalue is lost! Therefore, in this section, I tackle the opposite of the diagonalization problem in linear algebra, and I show that in general orthogonal bases, which do not include any eigenvectors of f, always exist.

I start from the following lemma, which is about the two-dimensional case.

C.4.1. Lemma. *Let $\mathcal{K} = (K, +, \cdot, 0, 1)$ be a* field *with at least* 4 *elements, V a two-dimensional vector space over $\mathcal{K}$ equipped with an anisotropic Hermitian form Φ, and $f : V \to V$ a linear map. If f is not a constant multiple of the identity map, then there is an orthogonal basis of V which does not include any eigenvectors of f. (Note that the zero map is the identity map multiplied by* 0*, and the vectors in the null space are the eigenvectors with eigenvalue* 0*.)*

Proof. According to Proposition B.2.10, there is an orthogonal basis $\{\mathbf{u}, \mathbf{v}\}$ of V. If neither of them is an eigenvector of f, then it is done. Hence I only need to consider two cases.

Case 1: Both vectors are eigenvectors of f. Without loss of generality, assume that $f(\mathbf{u}) = x\mathbf{u}$ and $f(\mathbf{v}) = y\mathbf{v}$. Consider the vectors

$$\mathbf{w}_1 = \mathbf{u} + \mathbf{v} \text{ and } \mathbf{w}_2 = \mathbf{u} - (\Phi(\mathbf{u}, \mathbf{u}) \cdot \Phi(\mathbf{v}, \mathbf{v})^{-1})\mathbf{v},$$

Calculate that

$$\begin{aligned}
&\Phi(\mathbf{w}_1, \mathbf{w}_2)\\
&= \Phi(\mathbf{u}+\mathbf{v}, \mathbf{u} - (\Phi(\mathbf{u},\mathbf{u})\cdot\Phi(\mathbf{v},\mathbf{v})^{-1})\mathbf{v})\\
&= \Phi(\mathbf{u},\mathbf{u}) - \Phi(\mathbf{u},\mathbf{v})\cdot\Phi(\mathbf{v},\mathbf{v})^{-1}\cdot\Phi(\mathbf{u},\mathbf{u}) + \Phi(\mathbf{v},\mathbf{u}) - \Phi(\mathbf{v},\mathbf{v})\cdot\Phi(\mathbf{v},\mathbf{v})^{-1}\cdot\Phi(\mathbf{u},\mathbf{u})\\
&= 0
\end{aligned}$$

$$\begin{aligned}
f(\mathbf{w}_1) &= f(\mathbf{u}+\mathbf{v}) = x\mathbf{u} + y\mathbf{v}\\
f(\mathbf{w}_2) &= f(\mathbf{u} - (\Phi(\mathbf{u},\mathbf{u})\cdot\Phi(\mathbf{v},\mathbf{v})^{-1})\mathbf{v}) = \ x\mathbf{u} - \left(\Phi(\mathbf{u},\mathbf{u})\cdot\Phi(\mathbf{v},\mathbf{v})^{-1}\cdot y\right)\mathbf{v}
\end{aligned}$$

Hence $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal; and, since $\mathcal{K}$ is a field,

$$f(\mathbf{w}_1) \in \langle \mathbf{w}_1 \rangle \ \Leftrightarrow\ x = y \ \Leftrightarrow\ f(\mathbf{w}_2) \in \langle \mathbf{w}_2 \rangle$$

Note that $x \neq y$; otherwise, since $\mathcal{K}$ is a field, $f : z_1\mathbf{u} + z_2\mathbf{v} \mapsto z_1(x\mathbf{u}) + z_2(x\mathbf{v}) = x(z_1\mathbf{u} + z_2\mathbf{v})$ would be a constant multiple of the identity map. Therefore, $\{\mathbf{w}_1, \mathbf{w}_2\}$ is an orthogonal basis of V which does not include any eigenvectors of f.

Case 2: Exactly one of the two vectors is an eigenvector of f. Without loss of generality, assume that $f(\mathbf{u}) = x\mathbf{u}$ and $f(\mathbf{v}) = \mathbf{u} + y\mathbf{v}$. Consider the vectors

$$\mathbf{w}_1 = k\mathbf{u} + \mathbf{v} \text{ and } \mathbf{w}_2 = \mathbf{u} - (k^* \cdot \Phi(\mathbf{u},\mathbf{u})\cdot\Phi(\mathbf{v},\mathbf{v})^{-1})\mathbf{v},$$

where $k \in K$ is to be determined and $(\cdot)^*$ is the accompanying involution of Φ on $\mathcal{K}$. Calculate that

$$\begin{aligned}
\Phi(\mathbf{w}_1, \mathbf{w}_2) &= \Phi\left(k\mathbf{u}+\mathbf{v}, \mathbf{u} - (k^*\cdot\Phi(\mathbf{u},\mathbf{u})\cdot\Phi(\mathbf{v},\mathbf{v})^{-1})\mathbf{v}\right)\\
&= k\cdot\Phi(\mathbf{u},\mathbf{u}) - k\cdot\Phi(\mathbf{u},\mathbf{v})\cdot\Phi(\mathbf{v},\mathbf{v})^{-1}\cdot\Phi(\mathbf{u},\mathbf{u})\cdot k\\
&\quad +\Phi(\mathbf{v},\mathbf{u}) - \Phi(\mathbf{v},\mathbf{v})\cdot\Phi(\mathbf{v},\mathbf{v})^{-1}\cdot\Phi(\mathbf{u},\mathbf{u})\cdot k\\
&= 0 \qquad\qquad (\mathcal{K}\text{ is a field})
\end{aligned}$$

$$\begin{aligned}
f(\mathbf{w}_1) &= f(k\mathbf{u}+\mathbf{v}) = (k\cdot x)\mathbf{u} + (\mathbf{u}+y\mathbf{v}) = (k\cdot x+1)\mathbf{u} + y\mathbf{v}\\
f(\mathbf{w}_2) &= f(\mathbf{u} - (k^*\cdot\Phi(\mathbf{u},\mathbf{u})\cdot\Phi(\mathbf{v},\mathbf{v})^{-1})\mathbf{v})\\
&= x\mathbf{u} - (k^*\cdot\Phi(\mathbf{u},\mathbf{u})\cdot\Phi(\mathbf{v},\mathbf{v})^{-1})(\mathbf{u}+y\mathbf{v})\\
&= \left(x - k^*\cdot\Phi(\mathbf{u},\mathbf{u})\cdot\Phi(\mathbf{v},\mathbf{v})^{-1}\right)\mathbf{u} - \left(k^*\cdot\Phi(\mathbf{u},\mathbf{u})\cdot\Phi(\mathbf{v},\mathbf{v})^{-1}\cdot y\right)\mathbf{v}
\end{aligned}$$

Hence $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal no matter what k is. Moreover, since $\mathcal{K}$ is a field,

$$f(\mathbf{w}_1) \in \langle \mathbf{w}_1 \rangle\text{, if and only if } k\cdot(y-x) = 1$$

and

$$f(\mathbf{w}_2) \in \langle \mathbf{w}_2 \rangle\text{, if and only if } k = 0 \text{ or } k^*\cdot\Phi(\mathbf{u},\mathbf{u})\cdot\Phi(\mathbf{v},\mathbf{v})^{-1} = x - y$$

Hence, to avoid $\mathbf{w}_1$ and $\mathbf{w}_2$ being eigenvectors of f, a sufficient condition is:

1. when $x = y$, $k \notin \{0\}$; and

2. when $x \neq y$, $k \notin \{(y-x)^{-1}, 0, \Phi(\mathbf{u},\mathbf{u})^{-1} \cdot \Phi(\mathbf{v},\mathbf{v}) \cdot (x^* - y^*)\}$.

Since K has at least 4 elements, it is easy to find such a $k \in K$. Then $\{\mathbf{w}_1, \mathbf{w}_2\}$ is an orthogonal basis of V which does not include any eigenvectors of f. ⊣

Now I prove the main result.

C.4.2. PROPOSITION. *Let $\mathcal{K} = (K, +, \cdot, 0, 1)$ be a* field *with at least 4 elements, V an n-dimensional vector space over $\mathcal{K}$ equipped with an anisotropic Hermitian form Φ, and $f : V \to V$ a linear map. If f is not a constant multiple of the identity map, then there is an orthogonal basis of V which does not include any eigenvectors of f.*

Proof. According to Proposition B.2.10, there is an orthogonal basis $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ of V. Without loss of generality, assume that there is an $m \in \{0, ..., n\}$ such that $\mathbf{v}_i$ is an eigenvector of f, if and only if $i \leq m$. Use induction on m. In the Base Step $m = 0$, and thus it is done. In the Induction Step when $m = p+1$, consider two cases.

Case 1: Among $\{\mathbf{v}_1, ..., \mathbf{v}_{p+1}\}$, there are two of them, say $\mathbf{v}_p$ and $\mathbf{v}_{p+1}$, such that they are eigenvectors of f with different eigenvalues. Then f restricted to the linear span $L(\{\mathbf{v}_p, \mathbf{v}_{p+1}\})$ is a linear map from $L(\{\mathbf{v}_p, \mathbf{v}_{p+1}\})$ to itself which is not a constant multiple of the identity map. By the previous lemma there is an orthogonal basis $\{\mathbf{w}_p, \mathbf{w}_{p+1}\}$ of $L(\{\mathbf{v}_p, \mathbf{v}_{p+1}\})$ which does not include any eigenvectors of f (restricted to this subspace). Then it is not hard to see that $\{\mathbf{v}_1, ..., \mathbf{v}_{p-1}, \mathbf{w}_p, \mathbf{w}_{p+1}, \mathbf{v}_{p+2}, ..., \mathbf{v}_n\}$ is an orthogonal basis of V such that only the first $p-1$ vectors are eigenvectors of f. Then the conclusion follows from the induction hypothesis.

Case 2: All of $\{\mathbf{v}_1, ..., \mathbf{v}_{p+1}\}$ are eigenvectors of f with eigenvalue x. In this case, $p+1 < n$, since f is not a constant multiple of the identity map. Then $\mathbf{v}_{p+2}$ is not an eigenvector of f. Hence $f(\mathbf{v}_{p+2}) = a\mathbf{v}_{p+1} + b\mathbf{v}_{p+2} + c\mathbf{u}$, where $\mathbf{u}$ is a linear combination of vectors in $\{\mathbf{v}_1, ..., \mathbf{v}_p, \mathbf{v}_{p+3}, ..., \mathbf{v}_n\}$ such that $c\mathbf{u} = \mathbf{0}$ and $a = 0$ can *not* hold simultaneously. Consider the vectors

$$\mathbf{w}_{p+1} = k\mathbf{v}_{p+1} + \mathbf{v}_{p+2} \text{ and } \mathbf{w}_{p+2} = \mathbf{v}_{p+1} - (k^* \cdot \Phi(\mathbf{v}_{p+1}, \mathbf{v}_{p+1}) \cdot \Phi(\mathbf{v}_{p+2}, \mathbf{v}_{p+2})^{-1})\mathbf{v}_{p+2},$$

where $k \in K$ is to be determined and $(\cdot)^*$ is the accompanying involution of Φ on $\mathcal{K}$. Calculate that

$$\begin{aligned}
&\Phi(\mathbf{w}_{p+1}, \mathbf{w}_{p+2}) \\
&= \Phi(k\mathbf{v}_{p+1} + \mathbf{v}_{p+2}, \mathbf{v}_{p+1} - (k^* \cdot \Phi(\mathbf{v}_{p+1}, \mathbf{v}_{p+1}) \cdot \Phi(\mathbf{v}_{p+2}, \mathbf{v}_{p+2})^{-1})\mathbf{v}_{p+2}) \\
&= k \cdot \Phi(\mathbf{v}_{p+1}, \mathbf{v}_{p+1}) - k \cdot \Phi(\mathbf{v}_{p+1}, \mathbf{v}_{p+2}) \cdot \Phi(\mathbf{v}_{p+2}, \mathbf{v}_{p+2})^{-1} \cdot \Phi(\mathbf{v}_{p+1}, \mathbf{v}_{p+1}) \cdot k \\
&\quad + \Phi(\mathbf{v}_{p+2}, \mathbf{v}_{p+1}) - \Phi(\mathbf{v}_{p+2}, \mathbf{v}_{p+2}) \cdot \Phi(\mathbf{v}_{p+2}, \mathbf{v}_{p+2})^{-1} \cdot \Phi(\mathbf{v}_{p+1}, \mathbf{v}_{p+1}) \cdot k \\
&= 0 \qquad (\mathcal{K} \text{ is a field})
\end{aligned}$$

$$\begin{aligned}
f(\mathbf{w}_{p+1}) &= f(k\mathbf{v}_{p+1} + \mathbf{v}_{p+2}) \\
&= kf(\mathbf{v}_{p+1}) + f(\mathbf{v}_{p+2}) \\
&= (k \cdot x)\mathbf{v}_{p+1} + a\mathbf{v}_{p+1} + b\mathbf{v}_{p+2} + c\mathbf{u} \\
&= (k \cdot x + a)\mathbf{v}_{p+1} + b\mathbf{v}_{p+2} + c\mathbf{u} \\
f(\mathbf{w}_{p+2}) &= f\left(\mathbf{v}_{p+1} - (k^* \cdot \Phi(\mathbf{v}_{p+1}, \mathbf{v}_{p+1}) \cdot \Phi(\mathbf{v}_{p+2}, \mathbf{v}_{p+2})^{-1})\mathbf{v}_{p+2}\right) \\
&= f(\mathbf{v}_{p+1}) - (k^* \cdot \Phi(\mathbf{v}_{p+1}, \mathbf{v}_{p+1}) \cdot \Phi(\mathbf{v}_{p+2}, \mathbf{v}_{p+2})^{-1})f(\mathbf{v}_{p+2}) \\
&= x\mathbf{v}_{p+1} - (k^* \cdot \Phi(\mathbf{v}_{p+1}, \mathbf{v}_{p+1}) \cdot \Phi(\mathbf{v}_{p+2}, \mathbf{v}_{p+2})^{-1})(a\mathbf{v}_{p+1} + b\mathbf{v}_{p+2} + c\mathbf{u}) \\
&= (x - k^* \cdot \Phi(\mathbf{v}_{p+1}, \mathbf{v}_{p+1}) \cdot \Phi(\mathbf{v}_{p+2}, \mathbf{v}_{p+2})^{-1} \cdot a)\mathbf{v}_{p+1} \\
&\quad -(k^* \cdot \Phi(\mathbf{v}_{p+1}, \mathbf{v}_{p+1}) \cdot \Phi(\mathbf{v}_{p+2}, \mathbf{v}_{p+2})^{-1} \cdot b)\mathbf{v}_{p+2} \\
&\quad -(k^* \cdot \Phi(\mathbf{v}_{p+1}, \mathbf{v}_{p+1}) \cdot \Phi(\mathbf{v}_{p+2}, \mathbf{v}_{p+2})^{-1} \cdot c)\mathbf{u}
\end{aligned}$$

Hence $\mathbf{w}_{p+1}$ and $\mathbf{w}_{p+2}$ are orthogonal no matter what k is. Moreover,

$$f(\mathbf{w}_{p+1}) \in \langle \mathbf{w}_{p+1} \rangle \text{ , if and only if } c\mathbf{u} = \mathbf{0} \text{ and } k \cdot (b - x) = a$$

and $f(\mathbf{w}_{p+2}) \in \langle \mathbf{w}_{p+2} \rangle$, if and only if

$$k = 0 \text{ or } (c\mathbf{u} = \mathbf{0} \text{ and } k^* \cdot \Phi(\mathbf{v}_{p+1}, \mathbf{v}_{p+1}) \cdot \Phi(\mathbf{v}_{p+2}, \mathbf{v}_{p+2})^{-1} \cdot a = x - b)$$

Hence, to avoid $\mathbf{w}_{p+1}$ and $\mathbf{w}_{p+2}$ being eigenvectors of f, a sufficient condition is:

1. when $a = 0$ or $x = b$, $k \notin \{0\}$;
2. when $a \neq 0$ and $x \neq b$,

$$k \notin \{a \cdot (b - x)^{-1}, 0, \Phi(\mathbf{v}_{p+1}, \mathbf{v}_{p+1})^{-1} \cdot \Phi(\mathbf{v}_{p+2}, \mathbf{v}_{p+2}) \cdot (a^*)^{-1} \cdot (x^* - b^*)\};$$

Since K has at least 4 elements, in both cases it is not hard to find such a $k \in K$. Then both $\mathbf{w}_{p+1}$ and $\mathbf{w}_{p+2}$ are not eigenvectors of f. Moreover, since both $\mathbf{v}_{p+1}$ and $\mathbf{v}_{p+2}$ are orthogonal to every vector in the set $\{\mathbf{v}_1, ..., \mathbf{v}_p, \mathbf{v}_{p+3}, ..., \mathbf{v}_n\}$, so are $\mathbf{w}_{p+1}$ and $\mathbf{w}_{p+2}$. Combining with the fact that $\mathbf{w}_{p+1}$ and $\mathbf{w}_{p+2}$ are orthogonal, the set $\{\mathbf{v}_1, ..., \mathbf{v}_p, \mathbf{w}_{p+1}, \mathbf{w}_{p+2}, \mathbf{v}_{p+3}, ..., \mathbf{v}_n\}$ is an orthogonal basis of V in which only the first p vectors are eigenvectors of f. Then the conclusion follows from the induction hypothesis.

This finishes the proof by induction. $\dashv$

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Index

Samenvatting

Dit proefschrift is een uitvoerige wiskundige studie van de non-orthogonaliteitsrelatie in de kwantumtheorie.

De kwantummechanica is een cruciaal onderdeel van de moderne natuurkunde. Hoewel deze theorie zeer succesvol microscopische verschijnselen kan beschrijven, is de onderliggende conceptuele essentie nog steeds niet goed begrepen. Het standaard formalisme, dat gebruik maakt van Hilbert ruimtes, is daarvoor een van de redenen. Dit formalisme is goed voor berekenen en voorspellen, maar het verduistert in zekere zin ook het conceptuele beeld achter de kwantumtheorie. In de kwantumlogica nemen onderzoekers daarom de natuurkundige intuïties als uitgangspunt en modelleren deze in simpele wiskundige structuren die kwantumeigenschappen kunnen verhelderen. Vervolgens kan men de standaard kwantumtheorie weer reconstrueren door aan te tonen dat deze structuren worden gerepresenteerd door Hilbert ruimtes. Op deze manier wordt de kwantumtheorie in duidelijke structuren vervat.

In dit promotieonderzoek neem ik de relatie van niet-orthogonaliteit (nietloodrechtheid) tussen (zuivere) kwantum toestanden als uitgangspunt. Hiervoor zijn meerdere redenen. Ten eerste, niet-orthogonaliteit is een binaire relatie tussen toestanden en dus wiskundig zeer eenvoudig. Ten tweede, deze relatie beschrijft de toestandsveranderingen die worden veroorzaakt wordt door metingen aan een fysisch systeem. En ten derde, bestaand technisch werk heeft het belang van deze relatie in de kwantumtheorie al bewezen. Daarom kan een reconstructie in termen van deze relatie licht werpen op de grondslagen van de kwantumtheorie. De resultaten in dit proefschrift tonen de mogelijkheid aan van zo'n reconstructie, en vormen daarmee een veelbelovend begin.

In hoofdstuk 2 definieer ik kwantum Kripke frames, de hoofdrolspeler van mijn proefschrift. Een kwantum Kripke frame is een Kripke frame waarbij de binaire relatie enkele simpele eigenschappen bezit van de non-orthogonaliteitsrelatie in de kwantumtheorie. De structuur van kwantum Kripke frames wordt uitgebreid bestudeerd vanuit een meetkundig standpunt. Van daaruit bewijs ik een stellling die kwantum Kripke frames representeert door algemene Hilbert ruimtes. Bovendien bepaal ik de essentile kenmerken van kwantum Kripke frames die worden gerepresenteerd door Hilbert ruimtes over de complexe getallen. Deze stelling suggereert dat kwantum Kripke frames gebruikt kunnen worden om kwantumsystemen te modelleren. Tegelijkertijd wordt in mijn analyse aangetoond dat verschillende projectieve meetkundes eigenlijk vermomde Kripke frames zijn.

Verschillende operatoren op Hilbert ruimtes zijn essentieel voor het formaliseren van de kwantumtheorie, dus functies tussen kwantum Kripke frames zijn het bestuderen waard. Dit wordt gedaan in hoofdstuk 3. Ik definieer continue homomorfismen tussen kwantum Kripke frames en bewijs dat deze worden gerepresenteerd door continue quasi-lineaire functies. Ik definieer drie speciale continue homomorfismen die overeenkomen met projecties, unitaire, en Hermitische operatoren op Hilbert ruimtes. Ik bestudeer hun eigenschappen vanuit het perspectief van de non-orthogonaliteitsrelatie. Verder geef ik condities waaronder twee kwantum Kripke frames kunnen worden samengevoegd tot één, wat de tegenhanger is van het tensor product van Hilbert ruimtes. Een vereiste voor dit resultaat is een karakterisering van endomorfismen op een Pappiaanse projectieve meetkunde voortgebracht door lineaire functies. Deze karakterisering geeft een oplossing voor een speciaal geval van een open probleem in de projectieve meetkunde.

Hoofdstuk 4 gaat over geautomatiseerd logisch redeneren over kwantum Kripke frames. Ik geef een beslisbare, correcte en sterk volledige axiomatisering in de modale logica voor toestandsruimtes en toestandsruimtes met superposities. Deze structuren zijn algemener dan kwantum Kripke frames. Ik bewijs dat de eerste-orde theorie van kwantum Kripke frames onbeslisbaar is. Daarnaast geef ik een karakterisering van eerste-orde definieerbare, dubbel-orthogonaal gesloten deelverzamelingen van een speciaal soort kwantum Kripke frames, waaronder de frames die worden voortgebracht door Hilbert ruimtes. Deze karakterisering suggereert dat deze frames eindig axiomatiseerbaar zijn. De resultaten in dit hoofdstuk laten zien dat een adequate formele taal voor geautomatiseerd redeneren over kwantum Kripke frames een deeltaal is van de eerste-orde logica.

Hoofdstuk 5 is een voorstudie naar de kansen van toestandsveranderingen in kwantum systemen. Dit leidt tot een kwantitatieve, meer precieze versie van de non-orthogonaliteitsrelatie. Een probabilistisch kwantum Kripke frame is een kwantum Kripke frame waarin ieder paar van toestanden een kans tussen de 0 en 1 krijgt toegeschreven. De verdeling van deze kansen weerspiegelt eigenschappen van daadwerkelijke kansverdelingen in de kwantumtheorie. Ik laat zien dat in een probabilistisch kwantum Kripke frame ieder element wordt voortgebracht door een kwantum kansverdeling op de meest voor de hand liggende manier. Ik identificeer ook een kenmerkende eigenschap van de non-orthogonaliteitsrelatie in probabilistische kwantum Kripke frames. Daarnaast definieer ik probabilistische kwantum transitieruimtes met de veranderingskansen als basis. Ik laat zien dat deze ruimtes equivalent zijn met probabilistische kwantum Kripke frames. Aldus laat dit hoofdstuk zien dat probabilistische kwantum Kripke frames of probabilistische kwantum transitieruimtes wellicht nuttig gebruikt kunnen worden om kwantum systemen kwantitatief te modelleren.

Abstract

This thesis is an in-depth mathematical study of the non-orthogonality relation in quantum theory.

Quantum mechanics is a crucial part of modern physics. Although it is very successful in describing microscopic phenomena, its conceptual essence is still not well understood. Its standard formalism in Hilbert spaces is one of the reasons. This formalism is good for doing calculation and making precise prediction, but to some extent it obscures the conceptual picture of quantum theory. In quantum logic researchers take physically intuitive notions as the primitives and model them in simple mathematical structures which make their quantum features transparent. Then quantum theory can be reconstructed by proving representation theorems of these simple mathematical structures via Hilbert spaces. In this way the conceptual essence of quantum theory is captured and made clear.

In this thesis I mainly take as primitive the non-orthogonality relation between the (pure) states of quantum systems. There are several reasons. First, the non-orthogonality relation is a binary relation on the states, and thus is simple from a mathematical perspective. Second, this relation describes the transitions between states triggered by measurements, and thus is intuitive from a physical perspective. Third, it proves important in quantum theory according to existing work in the literature. Therefore, a relational reconstruction of quantum theory based on this relation will be able to reveal the conceptual essence of quantum theory in a clear form. The results in this thesis show the possibility of such a reconstruction, and constitute a promising start.

In Chapter 2, I define quantum Kripke frames, the protagonists of this thesis. A quantum Kripke frame is a Kripke frame in which the binary relation possesses some simple properties of the non-orthogonality relation in quantum theory. The structure of quantum Kripke frames is studied extensively from a geometric perspective. Based on this, I prove a representation theorem of quantum Kripke frames via generalized Hilbert spaces. Moreover, the quantum Kripke frames induced by Hilbert spaces over the complex numbers are characterized.

This suggests that quantum Kripke frames can be useful in modelling quantum systems. In the meantime, several kinds of projective geometries are discovered to be Kripke frames in disguise.

Many operators on Hilbert spaces are crucial in formalizing quantum theory, so maps between quantum Kripke frames are worth studying. This is undertaken in Chapter 3. I define continuous homomorphisms between quantum Kripke frames and prove a representation theorem of them via continuous quasi-linear maps. Parallel to the unitary operators, self-adjoint operators and projectors on Hilbert spaces, I define three special kinds of continuous homomorphisms and study their properties from the perspective of the non-orthogonality relation. Moreover, I prove that under some conditions two quantum Kripke frames can be amalgamated into one, and this is the counterpart of the tensor product construction on Hilbert spaces. A preliminary to this result is a characterization of the endo-homomorphisms on a Pappian projective geometry induced by the linear maps. This solves a special case of an open problem in projective geometry.

Chapter 4 concerns the automated reasoning of quantum Kripke frames. I provide decidable, sound and strongly complete axiomatizations in modal logic for state spaces and for state spaces satisfying Superposition. These are Kripke frames more general than quantum Kripke frames. I prove that the first-order theory of quantum Kripke frames is undecidable. Moreover, I characterize the first-order definable, bi-orthogonally closed subsets in a special kind of quantum Kripke frame. This kind of quantum Kripke frame includes those induced by Hilbert spaces. This characterization hints that the first-order theory of such quantum Kripke frames may be finitely axiomatizable. The results in this chapter show that the appropriate formal language for the automated reasoning of quantum Kripke frames should be some fragment of the first-order language.

Chapter 5 is a pilot study of the transition probabilities between the states of quantum systems. They are the quantitative, more fine-grained version of the non-orthogonality relation. A probabilistic quantum Kripke frame is defined to be a quantum Kripke frame in which every pair of elements is assigned a real number between 0 and 1. The assignment of these numbers captures some properties of the transition probabilities in quantum theory. I show that in a probabilistic quantum Kripke frame every element induces a quantum probability measure in an expected way. I also discover an important property of the non-orthogonality relations in probabilistic quantum Kripke frames. Moreover, I define quantum transition probability spaces in which only the transition probabilities are primitive. I show that they correspond to probabilistic quantum Kripke frames. The results in this chapter suggest that probabilistic quantum Kripke frames, or quantum transition probability spaces, can be useful in the quantitative modelling of quantum systems and thus are worth further study.

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