Cuts and Completions

Algebraic aspects of structural proof theory

Frederik Möllerström Lauridsen

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Til Jørgen, Margit, og Katrine

Contents

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Amsterdam
August, 2019.

Frederik Möllerström Lauridsen

Chapter 1

Introduction

This thesis examines different aspects of the interplay between proof theory and algebraic semantics for several non-classical propositional logics. We begin with a short and rather informal introduction providing some context and summarizing the contents of the thesis in some detail.

1.1 Background

Non-classical logic Much of what this thesis is about is inspired or motivated by the abstract study of logical systems which, in one way or another, deviate from so-called classical logic. Such systems are often referred to as non-classical and are all concerned with different ways in which classical logic may be seen as being deficient. For example, one can consider either what classical logic, arguably, gets "wrong" or what classical logic "leaves out". Examples of the first kind include intuitionistic logic and its extensions [51] as well as various types of substructural logics [211, 98, 199], including many-valued [144] and relevance logics [4, 5]. Examples of the second kind include various types of modal logics [51, 39], such as epistemic [155, 76], as well as temporal and computational logics [128]. Of course this distinction is by no means sharp, with some systems arguably fitting both, or neither, of these descriptions. For an overview and discussion of some of the philosophical motivations for a few of the most well-known non-classical logics, see, e.g., [143, 45].

Part of what makes the study of non-classical logic attractive from a mathematical point of view is that it allows us to see differences, similarities, and connections between concepts which are otherwise conflated or trivial in the classical setting.

Proof theory One approach for studying non-classical logics is via proof theory, see, e.g., [246, 204] for textbook accounts. Proof theory is primarily concerned with constructing and investigating formal calculi for deriving validities of a given logic. Firstly, we would like to have a calculus allowing us to derive all, and only those, formulas which belong to the logical system at hand. Secondly, we would also like to have access to a calculus that is flexible and transparent enough to allow for meaningful analysis of the structure of possible derivations. Such analysis may allow us to obtain additional knowledge of the logical system which we are, for whatever reason, interested in.

The proof formalisms that we shall consider in this thesis are all sequent-based. That is, we consider systems for manipulating expressions of the form $\Gamma \Rightarrow \Pi$ with Γ and Π finite sequences of formulas, or, alternatively, terms, possibly subject to some restrictions. Moreover, we will mainly consider so-called structural calculi, i.e., calculi extending some basic system with additional rules for directly manipulating the structure of the sequent-expressions as opposed to the logical connectives themselves.

In the sequent formalism the role of the rule of *modus ponens* is played by the so-called cut-rule

$$\frac{\Gamma_2 \Rightarrow \psi \quad \Gamma_1, \psi, \Gamma_3 \Rightarrow \Pi}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Pi} \text{ (CUT)}$$

expressing the "transitivity" of the sequent arrow " $\Rightarrow$ ". One of the most fundamental questions concerning any sequent-based calculus is whether the cut-rule can be dispensed with. If this rule is indeed redundant relative to the rest of the rules of the calculus this means that the question of whether or not a certain sequent $\Gamma \Rightarrow \Pi$ is derivable can, often but not always, be settled by a backwards proof-search which proceeds by analyzing the constituents of the sequent. Calculi with this property are often informally referred to as being *analytic* although in many contexts the term "analytic" may simply be taken to mean that the cut-rule is redundant.

Thus knowing that there is a cut-free calculus for a given logic can often lead to a (not necessarily optimal) decision procedure for the logic. Moreover, cut-free calculi can also often be used to deduce additional information about the logic such as disjunction, variable separation and (uniform) interpolation properties as well as conservativity results. We refer to [246, Chap. 4] and [98, Chap. 5] for text-book accounts and further references for some of these applications.[1] However, because cut-free derivations are, in the worst case, much "larger" than derivations which do make use of the cut-rule it may, as argued by Boolos and others [43, 1], still be desirable to have the cut-rule, or at least some restricted version of it, in the calculus.

Many proofs of analyticity for various formalisms are variants of Gentzen's original argument [116, 117]. To show that the cut-rule is redundant one proves that any application of the cut-rule can be replaced with one or more applications of the cut-rule which are however of lower "complexity" or which occur "earlier" in the derivation. Iterating this process of replacing cuts we eventually obtain a

[1] For uniform interpolation see also [213, 3, 163, 2].

derivation which does not make use of the cut-rule, see, e.g., [246, Chap. 4.1] for a detailed argument. These types of arguments are completely constructive in that they provide a concrete algorithm for transforming any derivation into a cut-free one. However, since such cut-elimination arguments proceed by considering all possible combinations of rules, they are generally not very modular in the sense that simply adding new rules leads to many new cases to be checked.

Unfortunately, the standard cut-elimination procedure sketched above is usually not very robust and the required symmetry of the rules is easily broken by adding additional rules to the system. Therefore, the sequent calculus formalism is not expressive enough to capture many, even relatively simple, logics in an analytic manner. As a consequence of this a plethora of alternative formalisms, all in some way building on Gentzen's sequent formalism, have been developed. As examples we mention hypersequent calculi [200, 215, 10, 12], nested sequent calculi [177, 90], labeled calculi [95, 248, 81] and display calculi [22, 249]. See also [63] for a survey of some of these systems, their relations, and history.

Because of their relative simplicity and well-understood connection with algebra, we shall in this thesis primarily be concerned with sequent and hypersequent calculi in the context of intermediate logics, viz., (consistent) extensions of intuitionistic propositional logic, and substructural logics. Other formalisms and logics will only be mentioned in passing. Of course, the general approach taken in this thesis applies equally well, in one form or another, to the study of other formalisms or other types of non-classical logics. For examples see [182, 203, 81, 188, 192, 62, 141, 243, 140].

Algebraic semantics Instead of working directly with the syntax, another approach for studying various types of non-classical logics is to consider their models. Of course different systems may have very different kinds of "intended" or "standard" models. However, one uniform approach to provide models for many different non-classical logics is via algebra [40, 91]. In such cases we may identify logics with the equational theory[2] of their algebraic models. In fact, through the lens of duality theory the algebraic models can be seen as providing the point of contact between a logic and its spatio-relational models. For just some examples of this connection see [127, 80, 108, 109, 70].

In most cases where a non-classical logic includes some form of conjunction and disjunction the corresponding algebras will all be lattice-based. That is, they will be lattices possibly equipped with additional operations for interpreting any additional logical connectives. Concretely, intuitionistic propositional logic, as well as all of its extensions, has a sound and complete algebraic semantics in the form of Heyting algebras, see, e.g., [51, Chap. 7]. Similarly, the basic substructural logic, called the full Lambek calculus, as well as its extensions, has a sound and

[2]And often also the quasi-equational theory depending on whether a logic is considered to be a collection of formulas or a consequence relation between formulas.

complete algebraic semantics in the form of pointed residuated lattices also known as FL-algebras, see, e.g., [98, Chap. 2.6.1].

Knowing that a given logic is sound and complete with respect to a given class of algebras can be helpful in answering specific questions about it. Especially if the algebras involved have already been studied in some other context. Furthermore, because of the uniformity of the algebraic approach we may also establish general theorems about the relationships between the syntax and the semantics. For example in the form of so-called bridge theorems showing that a logic enjoys a certain property such as interpolation, Beth definability, or the disjunction property if, and only if, the corresponding class of algebras enjoys some associated property, see, e.g., [96] and [98, Chap. 5] for textbook treatments of this topic[3].

Thus from one perspective we can establish syntactic properties of logics by reasoning about their algebraic models with all the freedom and flexibility this entails. However, this perspective may also be turned around. That is, we may also see the close connections between logic and algebra as allowing us to apply syntactic analysis to obtain insights about classes of algebras, many of which are of independent interest outside of logic.

Algebraic proof theory The extension of the relationship between logic and algebra sketched above also covering different types of proof calculi was named algebraic proof theory by Ciabattoni, Galatos, and Terui [59]. One example of this is the approach for establishing not the *eliminability* of the cut-rule from a sequent-based calculus but rather the *admissibility* of the cut-rule in the calculus without the cut-rule. That is, one shows that exactly the same sequents are derivable in the two systems. This is something which can be done by appealing to the semantics. For example by showing that the calculus with the cut-rule and the calculus without the cut-rule are both sound and complete with respect to the same class of structures.

Concretely, in the case of the sequent calculus FL for basic substructural logic this can be done by introducing some "non-transitive" structures, called Gentzen frames, for which the calculus without the cut-rule is complete. Then one shows that any Gentzen frame $\mathbf{W}$ can be "embedded" into a corresponding FL-algebra $\mathbf{W}^+$. As the calculus FL may be soundly interpreted in the class of FL-algebras and as the "embedding" of a Gentzen frame $\mathbf{W}$ into $\mathbf{W}^+$ preserves validity of sequents it follows that a sequent is derivable using the cut-rule if, and only if, it is also derivable without using the cut-rule. Then in order to check that adding a new rule to the basic system preserves the redundancy of the cut-rule one simply has to check if the validity of this new rule is preserved when passing from any Gentzen frame $\mathbf{W}$ to its corresponding algebra $\mathbf{W}^+$. Moreover, since each algebra $\mathbf{A}$ determines a Gentzen frame $\mathbf{W_A}$ the passage from $\mathbf{W_A}$ to $\mathbf{W_A^+}$ can be viewed as

[3]Other examples include: admissibility [227, 162], unification [118] and uniform interpolation [131, 121].

an operation on algebras. In fact, in this particular case this operation turns out to be the well-known MacNeille completion [21, 59, 97]. Similarly, in the context of hypersequent calculi an analogous approach can be followed [57, 60]. Again this leads to a completion of algebras which is appropriately referred to as the hyper-MacNeille completion.

The strategy of making use of the algebraic semantics to establish analyticity of proof calculi has a long history going back, in one form or another, to at least the early 1990s [60, 97, 59, 101, 57, 64, 251, 21, 209, 207, 206, 208, 197], see also [98, Chap. 7] for a textbook treatment of these techniques and for further context and references.[4] In fact, this method may even be seen as a variant of the "completeness-via-canonicity" method known from modal logic, e.g., [39, Chap. 4].

To systematize the approach sketched above, Ciabattoni, Galatos, and Terui [57] defined a collection $\langle \mathcal{P}_n, \mathcal{N}_n \rangle_{n \in \omega}$ of sets of pointed residuated lattice terms (or equivalently formulas of the full Lambek calculus), and hence by extension also equations, called the *substructural hierarchy*, see Figure 1.1. The different levels measure the nesting of terms having different polarities, with $\mathcal{P}_n$ containing the terms with leading operation having positive polarity, and $\mathcal{N}_n$ containing the terms with leading operation having negative polarity.[5]

$$\begin{tikzcd}
\mathcal{P}_0 \arrow[r] \arrow[d, leftrightarrow] \arrow[dr] & \mathcal{P}_1 \arrow[r] \arrow[dr] & \mathcal{P}_2 \arrow[r] \arrow[dr] & \mathcal{P}_3 \arrow[r] \arrow[dr] & \dots \\
\mathcal{N}_0 \arrow[r] \arrow[ur] & \mathcal{N}_1 \arrow[r] \arrow[ur] & \mathcal{N}_2 \arrow[r] \arrow[ur] & \mathcal{N}_3 \arrow[r] \arrow[ur] & \dots
\end{tikzcd}$$

Figure 1.1: The substructural hierarchy. Arrows denote set-theoretic inclusion.

Each of the equations associated with the level $\mathcal{N}_2$ can effectively be transformed into a set of equivalent structural sequent rules. Similarly each set of equations associated with the level $\mathcal{P}_3$ can effectively be transformed into a set of equivalent structural hypersequent rules.[6] Conversely, every structural rule is equivalent to some equations associated with the level $\mathcal{N}_2$. Moreover, at least in the presence of exchange, contraction and weakening, every structural hypersequent rule is equivalent to some equation associated with the level $\mathcal{P}_3$. Thus the levels $\mathcal{N}_2$ and $\mathcal{P}_3$ can be said to capture syntactically the expressive power of structural sequent and hypersequent rules. However, there is also an interesting algebraic perspective. Ciabattoni, Galatos, and Terui [59, Thm. 6.3] have

[4] We also mention [243, 140] for recent applications of this method in the context of higher-order intuitionistic logic and display calculi for lattice-based logics respectively.

[5] An operation has positive (negative) polarity if its corresponding left (right) logical rule is invertible, see, e.g., [59, Sec. 2.2].

[6] This is only true if we consider the derived level $\mathcal{P}_3^\flat$ or assume that the equations imply commutativity. In particular, it will be true in the context of intermediate logics.

shown that if E is a set of equations associated with the level $\mathcal{N}_2$, then E can effectively be transformed into an equivalent set of analytic structural sequent rules if, and only if, the variety of FL-algebras axiomatized by E is closed under MacNeille completions. Moreover, in case E implies integrality[7] both parts of this equivalence must be true.

An analogous result holds for the level $\mathcal{P}_3$ in the presence of commutativity or, equivalently, the exchange rule [60, Thm. 7.3]. Namely, if E is a set of equations associated with the level $\mathcal{P}_3$ which implies commutativity, then E can effectively be transformed into an equivalent set of analytic structural hypersequent rules if, and only if, variety of FL-algebras axiomatized by E is closed under hyper-MacNeille completions. Again, in case E implies integrality both parts of this equivalence must be true

Thus the substructural hierarchy provides both axiomatic characterizations regarding the expressibility of certain formalisms as well as bridge-like theorems linking existence of proof calculi for some logic to properties of the corresponding algebraic semantics. However, as noted by Jeřábek [167], in the presence of commutativity the "hierarchy" collapses at the level $\mathcal{N}_3$ in the sense that all varieties of commutative pointed residuated lattices can be axiomatized by equations associated with terms belonging to this level. This suggests that in order to extend the approach of [57, 59, 60] further a more fine-grained stratification will be necessary.

MacNeille completions As the results of [57, 58, 59, 60, 21] make clear, the existence of a structural sequent and hypersequent calculus for a logic is closely related to the corresponding class of algebras being closed under some type of completion. Interestingly, the completions involved are, variants of, the well-known MacNeille completion.[8] This type of completion was introduced by MacNeille [196] as a generalization, to arbitrary partially ordered sets, of Dedekind's "completion by cuts" method [75] for obtaining the real line[9] $\mathbb{R}$ from the rational line $\mathbb{Q}$. Moreover, just as the operations of addition, subtraction and multiplication may be extended from $\mathbb{Q}$ to $\mathbb{R}$, the operations on any lattice-based algebra $\mathbf{A}$ may be extended to the MacNeille completion of the underlying lattice of $\mathbf{A}$ to give an algebra of the same type. The MacNeille completion $\overline{\mathbf{L}}$ of any lattice $\mathbf{L}$ is uniquely determined, up to isomorphism, by the property that each element of $\overline{\mathbf{L}}$ is both the infimum of the element in $\mathbf{L}$ above it and the supremum of the element in $\mathbf{L}$ below it. Thus the MacNeille completion can be viewed as the "smallest" completion.[10] Together these facts make the MacNeille completion a

[7] That is, the equation $x \leq \mathrm{e}$, ensuring that the monoidal unit is the greatest element.

[8] See, however, also [141] which links the existence of structural display calculi with closure under canonical completions of the corresponding algebraic semantics.

[9] Technically we should say the extended real line, i.e., the real line together with a least and greatest element.

[10] See, e.g., [16, Thm. XII.2.8] for one way to make this statement precise.

rather natural type of completion to consider. Moreover, viewed as a completion of partially ordered sets the MacNeille completion enjoys a number of good categorical properties attesting to its naturality [19].

Nevertheless, in many cases the MacNeille completion does not interact well with even very simple properties of lattices and lattice-based algebras, see, e.g., [94, 71, 145, 151, 152, 154] for examples of this fact.[11] This should be contrasted with the fact that many lattice-based algebras are closed under canonical completions, see, e.g., [228, 173, 111, 132, 68] as well as [51, Chap. 10] and [39, Chap. 4–5] for just some examples of this phenomenon.[12] In fact many of these approaches for establishing canonicity are syntactic in nature and in some ways not unlike the syntactic methods explored in the context of algebraic proof theory.

Thus the approach of algebraic proof theory establishes a surprising and interesting link between proof theory and completions of lattice-based algebras. Namely, that in some cases, necessary, and possibly even sufficient, conditions for a given logic to admit an analytic sequent or hyper-sequent calculus is that the corresponding class of algebras is closed under MacNeille or hyper-MacNeille completions. Conversely, turning this perspective on its head, we see that knowing that a given class of algebras is such that its corresponding logic admits an analytic sequent or hyper-sequent calculus can enable us to establish that this class of algebras is closed under completions.

1.2 What is this thesis about?

This thesis further explores some of the connections between structural proof theory and algebra. As outlined above, such connections are particularly strong when it comes to logics associated with the levels $\mathcal{P}_3$ and $\mathcal{N}_2$ of the substructural hierarchy. However, as of yet, such logics, and their corresponding classes of algebras, cannot be said to be fully understood. In fact, many interesting algebraic properties related to the levels $\mathcal{N}_2$ and $\mathcal{P}_3$ remain largely unexplored. The aim of this thesis is to gain a better understanding of these two levels. Chapters 2, 3, and 4 are concerned with various aspects of the level $\mathcal{P}_3$ in the context of intermediate logics and Heyting algebras, while Chapter 5 looks at a specific equation associated with the level $\mathcal{N}_2$ in the context of residuated lattices.

Structural hypersequent calculi for intermediate logics

One difficulty related to the substructural hierarchy is that because the levels are defined completely syntactically they are not closed under logical equivalence. Thus it is possible that a logic may be axiomatized both by formulas belonging

[11] Of course, as shown by Givant and Venema [124] and Theunissen and Venema [245] there are also exceptions to this phenomenon.

[12] But see also, e.g., [51, Chap. 6] and [150] for some exceptions.

to a given level but also by formulas not belonging to that same level. This means that in general it is not straightforward to show that a logic can, or cannot, be axiomatized by formulas belonging to a given level.[13] Consequently, in order to understand better the expressive power of each of the levels and hence to be able to distinguish them, semantic characterizations will be helpful.

In Chapter 2 we look closer at this issue as it applies to the level $\mathcal{P}_3$ in the context of intermediate logics, viz., consistent extensions of intuitionistic propositional logic. We note that since no proper intermediate logic can be axiomatized by $\mathcal{N}_2$-formulas [59, Prop. 7.3] and since every intermediate logic can be axiomatized by $\mathcal{N}_3$-formulas [167], in the context of algebraic proof theory for intermediate logics the level $\mathcal{P}_3$ is the most relevant level to consider. Furthermore, because no non-trivial proper intermediate logic admits a structural sequent calculus, the structural hypersequent calculus formalism is arguably one of the simplest frameworks in which large classes of intermediate logics can be given a minimally satisfactory proof theoretic treatment.

As Ciabattoni, Galatos and Terui [57, 60] have shown any intermediate logic axiomatizable by $\mathcal{P}_3$-formulas admits an analytic structural hypersequent calculus. The primary aim of this chapter is to classify the intermediate logics which admit such hypersequent calculi in terms of properties of their corresponding classes of Heyting algebras. This is done by considering the notion of $(\wedge, 0, 1)$-stability which is a stronger version of the notion of $(\wedge, \vee, 0, 1)$-stability first introduced by Guram and Nick Bezhanishvili [25] and further studied in [29, 34, 164]. We show that for an intermediate logic L the following are equivalent.

1. The logic L admits an analytic structural hypersequent calculus.
2. The logic L can be axiomatized by $\mathcal{P}_3$-formulas.
3. The logic L is $(\wedge, 0, 1)$-stable.

This characterization allows us to prove that certain well-known intermediate logics, such as the logics of bounded depth BD_n, for $n \geq 2$, do not admit a structural hypersequent calculus.

We also draw connections to the work of Lahav [188] and Lellmann [192, Sec. 6.1] who used the relational semantics to construct cut-free hypersequent calculi for modal logics characterized by Kripke frames defined by so-called simple sentences. Concretely, we show that all $(\wedge, 0, 1)$-stable intermediate logics are elementary and indeed determined by posets definable by so-called simple geometric implications which are a certain type of Π_2-sentences closely related to the simple sentences considered by Lahav and Lellmann.

Finally, we compare the $(\wedge, 0, 1)$-stable intermediate logics with the $(\wedge, \vee, 0, 1)$-stable intermediate logics. This is done by showing that a $(\wedge, \vee, 0, 1)$-stable intermediate logic L is a $(\wedge, 0, 1)$-stable intermediate logic if, and only if, L can be

[13] See, however, [59, Sec. 7], [167], and [60, Sec. 8] for a discussion of some results related to this.

axiomatized by $\vee$-free formulas. Thus the $(\wedge, 0, 1)$-stable intermediate logics may alternatively be described as the $(\wedge, \vee, 0, 1)$-stable intermediate logics which are also cofinal subframe logics.

MacNeille transferability

As reported in the previous section there is an interesting link between completions of lattice-based algebras and proof theory. For example, analytic structural hypersequent calculi give rise to universal classes of lattice-based algebras closed under MacNeille completions. In particular, if $\mathcal{V}$ is any variety of Heyting algebras axiomatized by $\mathcal{P}_3$-equations then the universal class $\mathcal{V}_{fsi}$ of its finitely subdirectly irreducible members is closed under MacNeille completions [58, Thm. 4.3]. This is particularly interesting since the only non-trivial varieties of Heyting algebras closed under MacNeille completions are the variety of all Heyting algebras and the variety of all Boolean algebras [151]. Drawing on the connection between the level $\mathcal{P}_3$ and the notion of stability established in Chapter 2, in Chapter 3 we look, from a purely algebraic point of view, at the phenomenon of universal classes of lattices and lattice-based algebras being closed under MacNeille completions.

Our starting point is the notion of (ideal) transferability originally introduced by Grätzer [133, Sec. 10(ii)]. A finite lattice $\mathbf{L}$ is (ideal) transferable if for any lattice $\mathbf{K}$, the lattice $\mathbf{L}$ is a sublattice of the lattice of ideals of $\mathbf{K}$ only if $\mathbf{L}$ is a sublattice of $\mathbf{K}$. We introduce analogous notions of MacNeille and canonical transferability and show how finite transferable lattices give rise to universal classes of lattices which are closed under completions. Thus the problem of finding universal classes of lattice-based algebras closed under MacNeille completions can in some cases be reduced to the problem of finding finite MacNeille transferable lattices.

While we are mainly interested in MacNeille transferability we also explore the relationships between ideal, MacNeille, and canonical transferability. Concretely, we show that under mild assumptions MacNeille transferability entails canonical transferability which in turn entails ideal transferability.

We provide necessary conditions for a finite lattice to be MacNeille transferable for the class of all lattices. In particular any such lattice must be distributive. This highlights some of the crucial differences between ideal and MacNeille transferability. Nevertheless, we show that, just as in the case of ideal transferability, the concept of (weak) projectivity plays an important role in understanding the concept of MacNeille transferability.

Using the connection between MacNeille transferability and projectivity, we obtain an alternative proof of the fact, first established by purely syntactic methods, that if $\mathcal{V}$ is any variety of Heyting algebras axiomatized by $\mathcal{P}_3$-equations then the universal class $\mathcal{V}_{fsi}$ is closed under MacNeille completions.

We then focus on MacNeille transferability with respect to the class of Heyting algebras and the class of bi-Heyting algebras. In this setting we are able to say much more about necessary and sufficient conditions for different types of MacNeille transferability. In particular, we show that all finite distributive lattices are MacNeille transferable with respect to the class of bi-Heyting algebras.

Finally, we discuss how canonical and MacNeille transferability of finite distributive lattices relates to intermediate logics. In particular we consider the problem of whether all $(\wedge, \vee, 0, 1)$-stable logics are (i) canonical, and (ii) elementary [164, Chap. 3]. In this respect a number of partial results of a positive nature are obtained.

Hyper-MacNeille completions

As mentioned above in connection with their proof of the admissibility of the cut-rule in certain types of structural hypersequent calculi, Ciabattoni, Galatos, and Terui [60] introduced a new type of completion of (pointed) residuated lattices which they called the hyper-MacNeille completion. Among other things they established that any variety of Heyting algebras axiomatized by $\mathcal{P}_3$-equations must be closed under this type of completion. Thus closure under hyper-MacNeille completions is a necessary condition for a variety of Heyting algebras to be determined by $\mathcal{P}_3$-equations and consequently for the corresponding intermediate logic to admit an analytic structural hypersequent calculus.

In Chapter 4 we consider more closely this type of completion in the context of Heyting algebras. We first identify the concept of a De Morgan supplemented Heyting algebra as being helpful for understanding the hyper-MacNeille completions of Heyting algebras. These algebras may be viewed as Heyting algebras equipped with a "co-negation" satisfying both of the De Morgan laws. We prove that for De Morgan supplemented Heyting algebras the MacNeille and hyper-MacNeille completions coincide. This generalizes the fact that the MacNeille and hyper-MacNeille completions coincide for subdirectly irreducible algebras [60, Prop. 6.6], at least in the context of Heyting algebras.

We also show that the De Morgan supplemented Heyting algebras are precisely the Heyting algebras which are isomorphic to Boolean products of finitely subdirectly irreducible Heyting algebras. This connection allows us to draw inspiration from previous work on MacNeille completions of Boolean products of lattices [146, 73]. Concretely, we establish the following close connection between MacNeille and hyper-MacNeille completions of Heyting algebras.

1. The hyper-MacNeille completion of a Heyting algebra $\mathbf{A}$ is the MacNeille completion of some De Morgan supplemented Heyting algebra $Q(\mathbf{A})$.

2. A variety $\mathcal{V}$ of Heyting algebras is closed under hyper-MacNeille completions if, and only if, the class of De Morgan supplemented members of $\mathcal{V}$ is closed under MacNeille completions.

Specifically, the last item allows us to turn the question of which varieties of Heyting algebras are closed under hyper-MacNeille completions into the question of which varieties of De Morgan supplemented Heyting algebras are closed under MacNeille completions.

Our analysis also allows us to show that any finitely generated variety of Heyting algebras is closed under hyper-MacNeille completions. From this and the results of Chapter 2 it follows that being axiomatizable by $\mathcal{P}_3$-equations is not a necessary condition for being closed under hyper-MacNeille completions. In fact, we obtain that there are varieties of Heyting algebras, such as the variety $\mathcal{BD}_2$ corresponding to the logic of posets having depth at most 2, which are closed under hyper-MacNeille completions but which are neither axiomatizable by $\mathcal{P}_3$-equations nor finitely generated.

Finally, we show that the sufficient conditions for the hyper-MacNeille completion to be regular, identified by Ciabattoni, Galatos, and Terui [60, Thm. 6.11], are in fact also necessary, at least in the context of Heyting algebras.

Integrally closed residuated lattices

In Chapter 5 we change perspective in two respects. First, we switch from considering Heyting algebras to considering residuated lattices and various closely related types of algebras. Second, instead of being concerned with properties related to hypersequent calculi and $\mathcal{P}_3$-equations we look at a specific $\mathcal{N}_2$-equation and an equivalent non-standard sequent calculus for the equational theory of the residuated lattices determined by this equation. As we will see, even though this equation belongs to the level $\mathcal{N}_2$, the approach of Ciabattoni, Galatos, and Terui [59] cannot be applied to obtain an equivalent cut-free structural sequent calculus. Nevertheless, we show that algebraic methods can still yield some proof-theoretical insights, although of a different type than those found in [59].

Concretely, we look at residuated lattices satisfying the equation $x\backslash x \approx \mathrm{e}$, or equivalently the equation $x/x \approx \mathrm{e}$, viz., the so-called integrally closed residuated lattices [92, Chap. XII.3]. These structures encompass a large number of well-known residuated lattices, such as integral residuated lattices, ℓ-groups [6], cancellative residuated lattices [13], and GBL-algebras [102]. Moreover, as we will show, integrally closed residuated lattices are also connected to Dubreil-Jacotin semi-groups [78, 93, 218, 41], pseudo BCI-algebras [165, 178, 181, 79], sirmonoids [219, 83] and algebras for Casari's comparative logic [48, 49, 50, 210, 198].

By considering a variant of the well-known double negation nucleus on Heyting algebras, we show that the variety of integrally closed residuated lattices admits a Glivenko theorem [126, 100] with respect to the variety of ℓ-groups. This allows us to establish the soundness of a non-standard version of the weakening rule. Variants of this rule have already been considered before in the context of BCI-algebras [178] and Casari's comparative logic [198]. Adding this rule to the ordinary sequent calculus for the equational theory of residuated lattices we obtain a sound and complete calculus for the equational theory of integrally closed residuated lattices. Furthermore, using a standard argument we show that the cut-rule is eliminable in this calculus. From this the decidability of the equational theory of integrally closed residuated lattices follows.

Finally, we use the cut-free calculus for the equational theory of integrally closed residuated lattices to obtain conservativity results concerning the equational theories of pseudo BCI-algebras, sirmonoids, and the algebras for Casari's comparative logic.

What is this thesis not about?

In an effort to make it absolutely clear what the scope of this thesis is we will also say a few words about what is not included in the thesis.

1. The term "proof theory" is used with a wide range of different meanings. We are in this thesis only concerned with proof theory as it applies to the study of propositional non-classical logics. In particular, there will be no mention of type theory, arithmetic, consistency strength, ordinal analysis, etc.

2. As we are only concerned with propositional logics there will be no discussion of first- or higher-order versions of non-classical logics.

3. Although the general approach followed here can also be pursued in the context of other proof formalisms the only formalism that we consider will be the (structural) sequent and hypersequent calculus formalism.

Furthermore, we wish to stress that despite the fact that the work of Ciabattoni, Galatos, and Terui [57, 58, 59, 60], from which much of the work in this thesis draws inspiration, applies in the setting of substructural logic we are mainly concerned with this work as it applies to the special case of intermediate logics and hence also Heyting algebras. Substructural logics are only considered in Chapter 5. This is not to say that considering the more general setting of substructural logics and pointed residuated lattices is not worthwhile, but simply that it has proven fruitful to first try to understand what can be said in the somewhat simple setting of intermediate logics and their corresponding classes of Heyting algebras.

Prerequisites This thesis is by no means self-contained. Throughout it is assumed that the reader already knows about, or at least is familiar with, various concepts and techniques. Therefore, for easy reference, we have included an appendix covering some of the most central technical preliminaries and providing references for more details and context.

1.3 Sources of the material

Much of the material which is found in this thesis has been obtained in close collaboration with others and parts of it has already appeared elsewhere.

1. Chapter 2 is a reworked version of the paper [190].
2. Chapter 3 is an expanded and slightly reworked version of joint work with Guram Bezhansihvili, John Harding, and Julia Ilin published as [31].
3. Chapter 4 is based on joint work with John Harding [153].
4. Chapter 5 is a slightly expanded and reworked version of joint work with José Gil-Férez and George Metcalfe [122] under submission.

Chapter 2

Structural hypersequent calculi for intermediate logics

In this chapter, based on [190], we look closer at the level $\mathcal{P}_3$ of the substructural hierarchy in the context of intermediate logics, viz., consistent extensions of intuitionistic propositional logic. As Ciabattoni, Galatos and Terui [57, 60] have shown any intermediate logic axiomatizable by $\mathcal{P}_3$-formulas admits an analytic structural hypersequent calculus. We note that no proper intermediate logic can be axiomatized by $\mathcal{N}_2$-formulas [59, Prop. 7.3] and moreover that every intermediate logic can be axiomatized by $\mathcal{N}_3$-formulas [167]. Thus for these reasons, in the context of algebraic proof theory for intermediate logics, the level $\mathcal{P}_3$ is the most relevant level to consider. Furthermore, because no non-trivial proper intermediate logic admits a structural sequent calculus, the structural hypersequent calculus formalism is arguably one of the simplest frameworks in which large classes of intermediate logics can be given a minimally satisfactory proof theoretic treatment.[1]

The primary aim of this chapter is to classify the intermediate logics which admit such hypersequent calculi in terms of properties of their corresponding classes of Heyting algebras. This is done by considering the notion of $(\wedge, 0, 1)$-stability which is a stronger version of the notion of $(\wedge, \vee, 0, 1)$-stability first introduced by Guram and Nick Bezhanishvili [25] and further studied in [29, 34, 164]. We show that for an intermediate logic L the following are equivalent.

1. The logic L admits an analytic structural hypersequent calculus.
2. The logic L can be axiomatized by $\mathcal{P}_3$-formulas.
3. The logic L is $(\wedge, 0, 1)$-stable.

[1] However, in [69, 235] a sequent calculus for the logic LC is obtained by adding infinitely many rules to a multi-succedent sequent calculus for IPC, and [161] gives a Gentzen-like calculus for KC in terms of finitely many rules which are, however, non-local. Finally, [9] gives tableau calculi for the seven intermediate logics with interpolation.

This characterization allows us to prove that certain well-known intermediate logics, such as the logics of bounded depth BD_n, for $n \geq 2$, do not admit a structural hypersequent calculus.

We also draw connections to the work of Lahav [188] and Lellmann [192, Sec. 6.1] who used the relational semantics to construct cut-free hypersequent calculi for modal logics characterized by Kripke frames defined by so-called simple sentences. Concretely, we show that all $(\wedge, 0, 1)$-stable intermediate logics are elementary and indeed determined by posets definable by so-called simple geometric implications which are a certain type of Π_2-sentences closely related to the simple sentences considered by Lahav and Lellmann.

Finally, we compare the $(\wedge, 0, 1)$-stable intermediate logics with the $(\wedge, \vee, 0, 1)$-stable intermediate logics. This is done by showing that a $(\wedge, \vee, 0, 1)$-stable intermediate logic L is a $(\wedge, 0, 1)$-stable intermediate logic if, and only if, L can be axiomatized by $\vee$-free formulas. Thus the $(\wedge, 0, 1)$-stable intermediate logics may alternatively be described as the $(\wedge, \vee, 0, 1)$-stable intermediate logics which are also cofinal subframe logics.

Outline The chapter is structured as follows: Section 2.1 contains an introduction to hypersequent calculi and their algebraic interpretation. Section 2.2 contains the algebraic characterization of intermediate logics with a (cut-free) structural hypersequent calculus. A number of applications of this characterization is provided in Section 2.3. In Section 2.4 the first-order conditions defining posets associated with the intermediate logics admitting a cut-free structural hypersequent calculi are determined and in Section 2.5 this class of intermediate logics is compared with the class of $(\wedge, \vee, 0, 1)$-stable logics. Finally, Section 2.6 contains a brief discussion of some possible directions for further work.

2.1 Hypersequent calculi

In this section we will briefly review the necessary background on structural hypersequent calculi. This section is primarily based on [57, 59, 60] but in part also on [101, 163] and [246, Chap. 3].

By a *single-succedent sequent* we shall in this chapter understand an expression of the form $s_1, \ldots, s_n \Rightarrow t$, or $s_1, \ldots, s_n \Rightarrow 0$, where $s_1, \ldots, s_n, t$ are terms in the language of Heyting algebras. Thus we may think of the right-hand side as a sequence of terms of length at most one. We will refer to such sequences as *stoups*. A *hypersequent* is simply a finite multiset of sequents written as

$$\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n,$$

where the sequents $\Gamma_i \Rightarrow \Pi_i$ are called the *components* of the hypersequent. One may think of a hypersequent as a "meta-disjunction" of sequents. The hypersequent formalism can therefore be thought of as a proof-theoretic framework that allows for the manipulation of sequents in parallel, see [11] and in particular [7, 8] for an interpretation of hypersequent rules as protocols for communication during concurrent computation.

In order to introduce sequent and hypersequent rules in a systematic manner it is helpful to work with *meta-variables*, see, e.g., [101, Sec. 2.1.3] and [163, Sec. 2.2–2.3] for a good exposition. More precisely we will use

s, t, u, v and their indexed versions	as variables for terms,
Γ, Δ and their indexed versions	as variables for sequences,
$\Pi, \Pi_0, \Pi_1, \ldots$	as variables for stoups,
$S, S_0, S_1, \ldots$	as variables for sequents,
$H, H_0, H_1, \ldots$	as variables for hypersequents.

A *meta-term* is a member of the least set $\underline{\mathrm{Tm}}$ which is closed under the rules

$$\text{if } \zeta_1, \zeta_2 \in \underline{\mathrm{Tm}} \quad \text{then} \quad \zeta_1 * \zeta_2 \in \underline{\mathrm{Tm}}, \qquad \text{for each } * \in \{\wedge, \vee, \to\},$$

and which contains all term variables as well as the symbols $\underline{0}$ and $\underline{1}$. A *meta-sequent* is an expression of the form $\Upsilon \Rightarrow \Psi$ where Υ is a finite sequence consisting of sequence-variables and meta-terms and Ψ is either empty, a meta-term or a stoup variable. A *meta-hypersequent* is a finite multiset of meta-sequents and hypersequent variables. A *sequent rule* is an expression

$$\frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \ldots \quad \Upsilon_m \Rightarrow \Psi_m}{\Upsilon_{m+1} \Rightarrow \Psi_{m+1}}\ (r)$$

where $\Upsilon_{m+1} \Rightarrow \Psi_{m+1}$ is a meta-sequent, which is called the *conclusion*, and $\Upsilon_1 \Rightarrow \Psi_1, \ldots, \Upsilon_m \Rightarrow \Psi_m$ are meta-sequents, which are called the *premises*. A *hypersequent rule* is an expression of the form

$$\frac{H \mid \Xi_1 \quad \ldots \quad H \mid \Xi_m}{H \mid \Xi_{m+1}}\ (r)$$

where H is a hypersequent variable and $\Xi_1, \ldots, \Xi_m, \Xi_{m+1}$ are meta-hypersequents. We call $H \mid \Xi_1, \ldots, H \mid \Xi_m$ the *premises* of (r), and $H \mid \Xi_{m+1}$ the *conclusion* or (r). A *reduced hypersequent rule* is a hypersequent rule of the form,

$$\frac{H \mid \Upsilon_1 \Rightarrow \Psi_1 \quad \ldots \quad H \mid \Upsilon_m \Rightarrow \Psi_m}{H \mid \Upsilon_{m+1} \Rightarrow \Psi_{m+1} \mid \ldots \mid \Upsilon_n \Rightarrow \Psi_n}\ (r)$$

where H is a hypersequent variable and $\Upsilon_1 \Rightarrow \Psi_1, \ldots, \Upsilon_n \Rightarrow \Psi_n$ are meta-sequents. Thus in a reduced hypersequent rule the type of meta-hypersequents allowed as premises is restricted.

By an *application* or *instance* of a (hyper)sequent rule (r) we understand a ordered pair consisting of a finite set of (hyper)sequents, which we call premises, and a single (hyper)sequent, which we call the conclusion, obtained by uniformly instantiating meta-objects occurring in (r) with object of the appropriate type.

For what follows it will be convenient to have fixed a notion of hypersequent calculus. Therefore, we call a *hypersequent calculus* any set of rules of the form $\mathsf{HLJ} + \mathscr{R}$, with $\mathscr{R}$ a set of reduced hypersequent rules and HLJ the set of rules displayed in Figure 2.1.

Identity Axioms

$$\frac{}{H \mid s \Rightarrow s}\ (\text{ID}) \qquad \frac{H \mid \Gamma_2 \Rightarrow s \quad H \mid \Gamma_1, s, \Gamma_3 \Rightarrow \Pi}{H \mid \Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Pi}\ (\text{CUT})$$

External Structural rules

$$\frac{H \mid \Gamma \Rightarrow \Pi \mid \Gamma \Rightarrow \Pi}{H \mid \Gamma \Rightarrow \Pi}\ (\text{EC}) \qquad \frac{H}{H \mid \Gamma \Rightarrow \Pi}\ (\text{EW})$$

Internal Structural rules

$$\frac{H \mid \Gamma_1, \Gamma_2 \Rightarrow \Pi}{H \mid \Gamma_1, \Sigma, \Gamma_2 \Rightarrow \Pi}\ (\text{WL}) \qquad \frac{H \mid \Gamma \Rightarrow}{H \mid \Gamma \Rightarrow u}\ (\text{WR})$$

$$\frac{H \mid \Gamma_1, \Sigma, \Sigma, \Gamma_2 \Rightarrow \Pi}{H \mid \Gamma_1, \Sigma, \Gamma_2 \Rightarrow \Pi}\ (\text{CL}) \qquad \frac{H \mid \Gamma_1, \Sigma_2, \Sigma_1, \Gamma_2 \Rightarrow \Pi}{H \mid \Gamma_1, \Sigma_1, \Sigma_2, \Gamma_2 \Rightarrow \Pi}\ (\text{EL})$$

Left Logical Rules

$$\frac{}{H \mid 0 \Rightarrow \Pi}\ (0\Rightarrow)$$

$$\frac{H \mid \Gamma \Rightarrow s \quad H \mid \Gamma, t \Rightarrow \Pi}{H \mid \Gamma, s \to t \Rightarrow \Pi}\ (\to\Rightarrow)$$

$$\frac{H \mid \Gamma, s \Rightarrow \Pi}{H \mid \Gamma, s \wedge t \Rightarrow \Pi}\ (\wedge\Rightarrow)_1$$

$$\frac{H \mid \Gamma, t \Rightarrow \Pi}{H \mid \Gamma, s \wedge t \Rightarrow \Pi}\ (\wedge\Rightarrow)_2$$

$$\frac{H \mid \Gamma, s \Rightarrow \Pi \quad H \mid \Gamma, t \Rightarrow \Pi}{H \mid \Gamma, s \vee t \Rightarrow \Pi}\ (\vee\Rightarrow)$$

Right Logical Rules

$$\frac{}{H \mid \Gamma \Rightarrow 1}\ (\Rightarrow 1)$$

$$\frac{H \mid \Gamma, s \Rightarrow t}{H \mid \Gamma \Rightarrow s \to t}\ (\Rightarrow\to)$$

$$\frac{H \mid \Gamma \Rightarrow s \quad H \mid \Gamma \Rightarrow t}{H \mid \Gamma \Rightarrow s \wedge t}\ (\Rightarrow\wedge)$$

$$\frac{H \mid \Gamma \Rightarrow s}{H \mid \Gamma \Rightarrow s \vee t}\ (\Rightarrow\vee)_1$$

$$\frac{H \mid \Gamma \Rightarrow t}{H \mid \Gamma \Rightarrow s \vee t}\ (\Rightarrow\vee)_2$$

Figure 2.1: The Hypersequent Calculus HLJ

A *derivation* in a calculus $\mathsf{HLJ} + \mathscr{R}$ is a finite labeled tree such that for any node ν_0, with label l_0 and immediate successors $\nu_1, \ldots, \nu_n$ labeled by $l_1 \ldots, l_n$, respectively, there is a rule in $\mathsf{HLJ}+\mathscr{R}$ having an instance with premises $l_1, \ldots, l_n$ and conclusion l_0. A hypersequent G is *derivable* in a calculus $\mathsf{HLJ}+\mathscr{R}$ from a set of hypersequents $\mathcal{H}$, written $\mathcal{H} \vdash_{\mathsf{HLJ}+\mathscr{R}} G$, if there exists a derivation in $\mathsf{HLJ}+\mathscr{R}$ with leaves labeled by members of $\mathcal{H}$ and root labeled by G. In the case where $\mathcal{H}$ is empty we simply write $\vdash_{\mathsf{HLJ}+\mathscr{R}} G$.

We say that a hypersequent calculus $\mathsf{HLJ}+\mathscr{R}$ is *cut-free* or that the cut-rule is *redundant* if any hypersequent derivable in $\mathsf{HLJ}+\mathscr{R}$, from an empty set of premises, can be derived without using the cut-rule.

2.1.1. Remark. One could have introduced a seemingly more general definition of a hypersequent calculus by allowing extensions of HLJ with arbitrary hypersequent rules. However, in the presence of external contraction (EC) and external weakening (EW), see Figure 2.1, any hypersequent rule is interderivable with a a finite set of reduced hypersequent rules. Thus there is no loss of generality in considering only reduced hypersequent rules.

2.1.2. Remark. It is easy to see that hypersequent calculi are essentially syntactic variants of so-called multi-conclusion calculi [232] and of multi-conclusion rule systems see, e.g., [166, 36, 28, 29].

In addition to the usual internal and external structural rules such as contraction, weakening, and exchange the hypersequent framework allows us to consider a wide range of so-called *structural hypersequent rules* [57, Sec. 3.1], i.e., reduced hypersequent rules not involving any of the logical connectives, which operate on multiple components at once. Thus structural rules have the following form

$$\frac{H \mid \Upsilon_1 \Rightarrow \Psi_1 \quad \ldots \quad H \mid \Upsilon_m \Rightarrow \Psi_m}{H \mid \Upsilon_{m+1} \Rightarrow \Psi_{m+1} \mid \ldots \mid \Upsilon_n \Rightarrow \Psi_n} \ (r)$$

where each Υ_i is a, possibly empty, sequence of sequence-variables and term-variables and each Ψ_i is empty, a stoup-variable, or a term-variable. Hence structural rules differ from arbitrary reduced rules only in that meta-terms are not in general allowed. In particular, this means that structural rules can not serve as new introduction rules for the operations $\wedge, \vee$, and $\to$. If $\mathscr{R}$ is a set of structural hypersequent rules we say that $\mathsf{HLJ}+\mathscr{R}$ is a *structural hypersequent calculus* .

Despite their relative simplicity structural rules have non-trivial expressive power.

2.1.3. Example. The structural hypersequent rule

$$\frac{H \mid \Gamma_1, \Sigma_2 \Rightarrow \Pi_1 \quad H \mid \Gamma_2, \Sigma_1 \Rightarrow \Pi_2}{H \mid \Gamma_1, \Sigma_1 \Rightarrow \Pi_1 \mid \Gamma_2, \Sigma_2 \Rightarrow \Pi_2} \ (\textsc{com})$$

initially due to Avron [11] determines a hypersequent calculus for the intermediate logic LC, defined as IPC $+ (p \to q) \vee (q \to p)$, when added to HLJ. Similarly the rule

$$\frac{H \mid \Gamma_1, \Gamma_2 \Rightarrow}{H \mid \Gamma_1 \Rightarrow \ \mid \Gamma_2 \Rightarrow} \text{(co-wc)}$$

determines a hypersequent calculus for the intermediate logic KC, defined as IPC+ $\neg p \vee \neg\neg p$, when added to HLJ, see, e.g., [56].

2.1.1 Interpreting hypersequents and hypersequent rules

To each sequent of the form $s_1, \ldots, s_n \Rightarrow t$ we associate the equation $s_1 \wedge \ldots \wedge s_n \leq t$. Similarly, to each sequent of the form $s_1, \ldots, s_n \Rightarrow$ we associate the equation $s_1 \wedge \ldots \wedge s_n \leq 0$. In each case we interpret an empty meet as the constant 1.

A hypersequent G is *valid* on a class $\mathcal{K}$ of Heyting algebras, written $\models_{\mathcal{K}} G$, provided that $\mathcal{K} \models \varepsilon_1$ or $\ldots$ or ε_n where ε_i is the equation associated with the i'th component of G. When $\mathcal{K}$ is a singleton, say $\{\mathbf{A}\}$, we may write $\mathbf{A} \models G$ for $\models_{\mathcal{K}} G$. As any sequent may be considered a single component hypersequent, the above also applies to sequents. Moreover, given a finite set of sequents $\mathcal{S} = \{S_1, \ldots, S_m\}$ and a hypersequent G, say $S_{m+1} \mid \ldots \mid S_n$, we say that $\mathcal{S}$ entails G over a class of Heyting algebras $\mathcal{K}$, written $\mathcal{S} \models_{\mathcal{K}} G$, provided that for all $\mathbf{A} \in \mathcal{K}$

$$\mathbf{A} \models \varepsilon_1 \text{ and } \ldots \text{ and } \varepsilon_m \implies \varepsilon_{m+1} \text{ or } \ldots \text{ or } \varepsilon_n,$$

where, for each $i \in \{1, \ldots, n\}$ the equation ε_i is the equation associated with the sequent S_i. In case $\mathcal{K}$ is the singleton $\{\mathbf{A}\}$ we write $\mathcal{S} \models_{\mathbf{A}} G$ in place of $\mathcal{S} \models_{\mathcal{K}} G$.

To interpret reduced hypersequent rules a bit more care is needed. For each term variable t we associate a variable in the language of Heyting algebras $t^\bullet$ in such a way that the map $t \mapsto t^\bullet$ will be injective. In the evident way this extends to a map $^\bullet : \underline{\mathrm{Tm}} \to \mathrm{Tm}$, from the set of meta-terms $\underline{\mathrm{Tm}}$ to the set Tm of terms in the language of Heyting algebras. Similarly, for each sequence variable Γ and each stoup variable Π we associate variables $\Gamma^\bullet$ and $\Pi^\bullet$, respectively, in the language of Heyting algebras in such a way that the functions $\Gamma \mapsto \Gamma^\bullet$ and $\Pi \mapsto \Pi^\bullet$ are both injective.

Given any non-empty sequence Υ consisting of meta-variables for sequences, and meta-terms we extend the realization inductively as follows,

$$(\Upsilon, \Gamma)^\bullet = \Upsilon^\bullet \wedge \Gamma^\bullet \quad \text{and} \quad (\Upsilon, t)^\bullet = \Upsilon^\bullet \wedge t^\bullet.$$

For each meta-sequent $\Upsilon \Rightarrow \Psi$ we then define its *realization* to be the equation $\Upsilon^\bullet \leq \Psi^\bullet$, where we use the convention that $\Upsilon^\bullet$ is the term 1 when Υ is empty and $\Psi^\bullet$ is the term 0 when Ψ is empty. In this way we obtain for each sequent rule

$$\frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \ldots \quad \Upsilon_m \Rightarrow \Psi_m}{\Upsilon_{m+1} \Rightarrow \Psi_{m+1}} \ (r)$$

a quasi-equation

$$\varepsilon_1 \text{ and } \dots \text{ and } \varepsilon_n \implies \varepsilon_{m+1}, \qquad (r^\bullet)$$

where for each $i \in \{1 \dots, m+1\}$ the equation ε_i is the realization of the meta-sequent $\Upsilon_i \Rightarrow \Psi_i$. Similarly, we obtain for each reduced hypersequent rule

$$\frac{H \mid \Upsilon_1 \Rightarrow \Psi_1 \quad \dots \quad H \mid \Upsilon_m \Rightarrow \Psi_m}{H \mid \Upsilon_{m+1} \Rightarrow \Psi_{m+1} \mid \dots \mid \Upsilon_n \Rightarrow \Psi_n} \ (r)$$

a universal clause

$$\varepsilon_1 \text{ and } \dots \text{ and } \varepsilon_m \implies \varepsilon_{m+1} \text{ or } \dots \text{ or } \varepsilon_n, \qquad (r^\bullet)$$

where again for each $i \in \{1, \dots, n\}$ the equation ε_i is the realization of the meta-sequent $\Upsilon_i \Rightarrow \Psi_i$.

2.1.4. Example. The structural hypersequent rules (com) and (co-wc) correspond to the universal clauses

$$x_1 \wedge z_2 \leq y_1 \text{ and } x_2 \wedge z_1 \leq y_2 \implies x_1 \wedge z_1 \leq y_1 \text{ or } x_2 \wedge z_2 \leq y_2$$

and

$$x_1 \wedge x_2 \leq 0 \implies x_1 \leq 0 \text{ or } x_2 \leq 0,$$

respectively.

As we will see, for each quasi-equation or universal clause q one can construct a sequent or hypersequent rule (r), respectively, such that $r^\bullet$ and q are equivalent.

Given a set of (hyper)sequent rules $\mathscr{R}$ let $\mathcal{K}(\mathscr{R})$ denote the class of Heyting algebras validating all the quasi-equations (clauses) $r^\bullet$ with $(r) \in \mathscr{R}$. From the above it immediately follows that $\mathcal{K}(\mathscr{R})$ is a universal class and when $\mathscr{R}$ consists only of sequence rules a quasi-variety. We may then establish the following completeness theorem which is already known in the setting of multi-conclusions rule systems, see [166, Thm. 2.2] and [36, Thm. 2.5].

2.1.5. Theorem. *Let $\mathscr{R}$ be a set of hypersequent rules, $\mathcal{S}$ a finite set of sequents and G a hypersequent. Then the following are equivalent.*

1. $\mathcal{S} \vdash_{\mathsf{HLJ}+\mathscr{R}} G$.
2. $\mathcal{S} \models_{\mathcal{K}(\mathscr{R})} G$.

Proof:
It is easy to verify that each of the rules of the calculus $\mathsf{HLJ}+\mathscr{R}$ preserves validity on members of the class $\mathcal{K}(\mathscr{R})$ which shows that Item 1 implies Item 2.

Conversely, to see that Item 2 implies Item 1 assume that $\mathcal{S} \nvdash_{\mathsf{HLJ}+\mathscr{R}} G$. By Zorn's Lemma there is a maximal set of sequents $\mathcal{T} \supseteq \mathcal{S}$ such that $\mathcal{T} \nvdash_{\mathsf{HLJ}+\mathscr{R}} G$.

We then define a binary relation Θ on the set Tm, of terms in the language of Heyting algebras, as follows

$$s \mathrel{\Theta} t \quad \text{if, and only if,} \quad \mathcal{T} \vdash_{\mathsf{HLJ}+\mathscr{R}} s \Rightarrow t \text{ and } \mathcal{T} \vdash_{\mathsf{HLJ}+\mathscr{R}} t \Rightarrow s.$$

It is straightforward to check that Θ is a congruence on the term algebra $\mathbf{Tm}$ making $\mathbf{A} = \mathbf{Tm}/\Theta$ a Heyting algebra with the property that

$$s/\Theta \leq t/\Theta \quad \text{if, and only if,} \quad \mathcal{T} \vdash_{\mathsf{HLJ}+\mathscr{R}} s \Rightarrow t,$$

for any terms $s, t \in \mathrm{Tm}$.

For each $S \in \mathcal{S}$, say $s_1, \ldots, s_n \Rightarrow t$, we have that $\mathcal{T} \vdash_{\mathsf{HLJ}+\mathscr{R}} s_1, \ldots, s_n \Rightarrow t$ and hence also that $\mathcal{T} \vdash_{\mathsf{HLJ}+\mathscr{R}} s_1 \wedge \ldots \wedge s_n \Rightarrow t$. It follows that $(s_1 \wedge \ldots \wedge s_n)/\Theta \leq t/\Theta$ which shows that the equation associated with the sequent S is true in $\mathbf{A}$ under the natural valuation μ given by $u \mapsto u/\Theta$. In case the right-hand side of the sequent S is empty the argument is completely analogous. Now suppose towards a contradiction that $(\mathbf{A}, \mu) \models G$. Then there must be a component of G, say $s_1, \ldots, s_{n_i} \Rightarrow t_i$, such that $(\mathbf{A}, \mu) \models s_1 \wedge \ldots \wedge s_{n_i} \leq t_i$, in which case $(s_1 \wedge \ldots \wedge s_{n_i})/\Theta \leq t_i/\Theta$. But then $\mathcal{T} \vdash_{\mathsf{HLJ}+\mathscr{R}} s_1 \wedge \ldots \wedge s_{n_i} \Rightarrow t_i$ and, as the sequent $s_1, \ldots, s_{n_i} \Rightarrow s_1 \wedge \ldots \wedge s_{n_i}$ is derivable in HLJ, an application of the rule (CUT) shows that $\mathcal{T} \vdash_{\mathsf{HLJ}+\mathscr{R}} s_1, \ldots, s_{n_i} \Rightarrow t_i$. Applying the rule (EW) we may then conclude that $\mathcal{T} \vdash_{\mathsf{HLJ}+\mathscr{R}} G$, in direct contradiction with the choice of $\mathcal{T}$. This shows that $\mathcal{S} \not\models_{\mathbf{A}} G$.

To see that $\mathbf{A} \in \mathcal{K}(\mathscr{R})$ let $(r) \in \mathscr{R}$ be given, say

$$\frac{H \mid \Upsilon_1 \Rightarrow \Psi_1 \quad \ldots \quad H \mid \Upsilon_m \Rightarrow \Psi_m}{H \mid \Upsilon_{m+1} \Rightarrow \Psi_{m+1} \mid \ldots \mid \Upsilon_n \Rightarrow \Psi_n}\ (r)$$

and consider the corresponding universal clause

$$\varepsilon_1 \text{ and } \ldots \text{ and } \varepsilon_m \implies \varepsilon_{m+1} \text{ or } \ldots \text{ or } \varepsilon_n, \qquad (r^\bullet)$$

with the equation ε_i, say $s_1^i(\vec{x}) \wedge \ldots \wedge s_{k_i}^i(\vec{x}) \leq t_i(\vec{x})$ being the realization of the meta-sequent $\Upsilon_i \Rightarrow \Psi_i$. If for some terms $\vec{u} \in \mathrm{Tm}$ we have that

$$(s_1^i(\vec{u}) \wedge \ldots \wedge s_{k_i}^i(\vec{u}))/\Theta \leq t_i(\vec{u})/\Theta,$$

for all $i \in \{1, \ldots, m\}$, then, as before, $\mathcal{T} \vdash_{\mathsf{HLJ}+\mathscr{R}} s_1^i(\vec{u}), \ldots, s_{k_i}^i(\vec{u}) \Rightarrow t_i(\vec{u})$. Hence by the rule (EW) we obtain that $\mathcal{T} \vdash_{\mathsf{HLJ}+\mathscr{R}} G \mid s_1^i(\vec{u}), \ldots, s_{k_i}^i(\vec{u}) \Rightarrow t_i(\vec{u})$ for all $i \in \{1, \ldots, m\}$. But then applying the rule (r) yields that

$$\mathcal{T} \vdash_{\mathsf{HLJ}+\mathscr{R}} G \mid s_1^{m+1}(\vec{u}), \ldots, s_{k_{m+1}}^{m+1}(\vec{u}) \Rightarrow t_{m+1}(\vec{u}) \mid \ldots \mid s_1^n(\vec{u}), \ldots, s_{k_n}^n(\vec{u}) \Rightarrow t_n(\vec{u}).$$

We claim that this entails that there must be $j \in \{m+1, \ldots, n\}$ such that

$$\mathcal{T} \vdash_{\mathsf{HLJ}+\mathscr{R}} s_1^j(\vec{u}), \ldots, s_{k_j}^j(\vec{u}) \Rightarrow t_j(\vec{u}),$$

from which it will follow that $(s_1^j(\vec{u}) \wedge \ldots \wedge s_{k_j}^j(\vec{u}))/\Theta \leq t_j(\vec{u})/\Theta$ and hence that $\mathbf{A} \models r^\bullet$. To establish this claim suppose for a contradiction that there is no such $j \in \{m+1, \ldots, n\}$. Then since $\mathcal{T}$ is maximal with the property that $\mathcal{T} \nvdash_{\mathsf{HLJ}+\mathscr{R}} G$ we must have that

$$\mathcal{T} \cup \{s_1^j(\vec{u}), \ldots, s_{k_j}^j(\vec{u}) \Rightarrow t_j(\vec{u})\} \vdash_{\mathsf{HLJ}+\mathscr{R}} G$$

for each $j \in \{m+1, \ldots, n\}$. This combined with the fact that

$$\mathcal{T} \vdash_{\mathsf{HLJ}+\mathscr{R}} G \mid s_1^{m+1}(\vec{u}), \ldots, s_{k_{m+1}}^{m+1}(\vec{u}) \Rightarrow t_{m+1}(\vec{u}) \mid \ldots \mid s_1^n(\vec{u}), \ldots, s_{k_n}^n(\vec{u}) \Rightarrow t_n(\vec{u})$$

shows that

$$\mathcal{T} \vdash_{\mathsf{HLJ}+\mathscr{R}} G \mid G \mid \ldots \mid G$$

and hence applying the rule (EC) an appropriate number of times we obtain $\mathcal{T} \vdash_{\mathsf{HLJ}+\mathscr{R}} G$ in direct contradiction with the choice of $\mathcal{T}$. □

The proof of Theorem 2.1.5 makes ample use of the rule (CUT), in particular to ensure that the relation Θ is transitive. For $\mathscr{R}$ a set of so-called analytic structural rules a very different kind of completeness proof can be given which avoids making use of the cut-rule whereby showing that the calculus with and without the cut-rule derives exactly the same hypersequents [57, 60].

2.1.6. REMARK. Note that unlike the standard Lindenbaum-Tarski construction used to establish completeness of sequent calculi the construction in the proof of Theorem 2.1.5 does not produce free algebras for the universal class of Heyting algebras validating the corresponding rules. In fact, in universal classes of algebras free algebras may not exist at all, see, e.g., [135, §53].

2.1.2 Analytic structural clauses

We have seen how any hypersequent rule gives rise to a universal clause. From the definition of structural hypersequent rules it is immediate that if (r) is any such rule then the corresponding universal clause

$$\varepsilon_1 \textsf{ and } \ldots \textsf{ and } \varepsilon_m \implies \varepsilon_{m+1} \textsf{ or } \ldots \textsf{ or } \varepsilon_n, \tag{$r^\bullet$}$$

is such that for $i \in \{1, \ldots, n\}$, the equation ε_i is of the form $x_1 \wedge \ldots \wedge x_{k_i} \leq y$ or $x_1 \wedge \ldots \wedge x_{k_i} \leq 0$ for some, possibly empty, set of variables $\{x_1, \ldots, x_{k_i}, y\}$. We will call such universal clauses *structural*. Given a structural universal clause,

$$\varepsilon_1 \textsf{ and } \ldots \textsf{ and } \varepsilon_m \implies \varepsilon_{m+1} \textsf{ or } \ldots \textsf{ or } \varepsilon_n, \tag{q}$$

with ε_i being the equation $s_i \leq t_i$ we call the variables occurring in one or more of the terms $s_{m+1}, \ldots, s_n$ *left variables* and the variables occurring in one or more of the terms $t_{m+1}, \ldots, t_n$ *right variables*. A structural universal clause q is then called *analytic*, see [60, Def. 4.12], provided that

1. Any left variable occurs in exactly one of the terms $s_{m+1}, \ldots, s_n$, and in the term where it occurs it only occurs once.

2. Any right variable occurs in exactly one of the terms $t_{m+1}, \ldots, t_n$, and in the term where it occurs it only occurs once.

3. For each index $i \in \{1, \ldots, m\}$ the term s_i contains only left variables.

4. For each index $i \in \{1, \ldots, m\}$ the term t_i contains only right variables.

Using the so-called *Ackermann Lemma*, see, e.g., [60, Lem. 4.3], we may show that any universal clauses in the $\{\wedge, 0, 1\}$-reduct of the language of Heyting algebras is equivalent to a finite set of such special clauses. The Ackermann Lemma, which is straightforward to prove, states that any universal clause

$$\varepsilon_1 \text{ and } \ldots \text{ and } \varepsilon_m \implies \varepsilon_{m+1} \text{ or } \ldots \text{ or } s \leq t \text{ or } \ldots \text{ or } \varepsilon_n,$$

is equivalent to each of the following universal clauses

$$\begin{aligned} \varepsilon_1 \text{ and } \ldots \text{ and } \varepsilon_m \text{ and } x \leq s &\implies \varepsilon_{m+1} \text{ or } \ldots \text{ or } x \leq t \text{ or } \ldots \text{ or } \varepsilon_n \\ \varepsilon_1 \text{ and } \ldots \text{ and } \varepsilon_m \text{ and } t \leq y &\implies \varepsilon_{m+1} \text{ or } \ldots \text{ or } s \leq y \text{ or } \ldots \text{ or } \varepsilon_n, \end{aligned}$$

where x and y are variables not occurring in the equation $s \leq t$ nor in any of the equations ε_i for $i \in \{1, \ldots, n\}$.

2.1.7. Lemma. *Any universal clause in the $\{\wedge, 0, 1\}$-reduct of the language of Heyting algebras is equivalent to a structural (analytic) clauses.*

Proof:
Let

$$\varepsilon_1 \text{ and } \ldots \text{ and } \varepsilon_m \implies \varepsilon_{m+1} \text{ or } \ldots \text{ or } \varepsilon_n, \tag{q}$$

with ε_i denoting the equation $s_i \leq t_i$, be a universal clause in the $\{\wedge, 0, 1\}$-reduct of the language of Heyting algebras. We will write P for the right-hand side of the clause q and C for the left-hand side of the clause q.

If for some $i \in \{1, \ldots, m\}$ the term s_i is the constant 0 or the term t_i is the constant 1 then q is equivalent to the universal clause

$$\varepsilon_1 \text{ and } \ldots \text{ and } \varepsilon_{i-1} \text{ and } \varepsilon_{i+1} \text{ and } \ldots \text{ and } \varepsilon_m \implies \varepsilon_{m+1} \text{ or } \ldots \text{ or } \varepsilon_n.$$

If for some $j \in \{m+1, \ldots, n\}$ we have that t_j is neither a single variable or the constant 0 then, letting y_j be a variable not already occurring in q, we see by the Ackermann Lemma that q is equivalent to the universal clause

$$P \text{ and } t_j \leq y_j \implies \varepsilon_{m+1} \text{ or } \ldots \text{ or } \varepsilon_{j-1} \text{ or } s_j \leq y_j \text{ or } \varepsilon_{j+1} \text{ or } \ldots \varepsilon_n.$$

If for some $i \in \{1, \ldots, m\}$ we have that t_i is a term of the form $t_i^1 \wedge t_i^2$ then q is equivalent to the universal clause

$$\varepsilon_1 \text{ and } \ldots \text{ and } \varepsilon_{i-1} \text{ and } s_i \leq t_i^1 \text{ and } s_i \leq t_i^2 \text{ and } \varepsilon_{i+1} \text{ and } \ldots \text{ and } \varepsilon_m \implies C.$$

If for some $j \in \{m+1, \ldots, n\}$ either s_j is the constant 0 or t_j is the constant 1 then q is equivalent to the clause $x \leq x$. Consequently, given any universal clause in the $\{\wedge, 0, 1\}$-reduct of the language of Heyting algebras, applying the above transformations we obtain an equivalent structural universal clause.

Finally, by [60, Thm. 4.15] any structural clause is equivalent, over the class of Heyting algebras, to an analytic structural clause. □

Following [60, Sec. 4.4] we show how to transform any analytic structural clause q, and therefore by Lemma 2.1.7 any universal clause in the $\{\wedge, 0, 1\}$-reduct of the language of Heyting algebras, into a structural hypersequent rule, (q°), such that the clauses $q^{\circ\bullet}$ and q are equivalent and the rules $r^{\bullet\circ}$ and r are interderivable.

Given a structural analytic clause

$$\varepsilon_1 \text{ and } \ldots \text{ and } \varepsilon_m \implies \varepsilon_{m+1} \text{ or } \ldots \text{ or } \varepsilon_n, \tag{q}$$

with ε_i denoting the equation $s_i \leq t_i$, we let $L = \{x_1, \ldots, x_k\}$ and $R = \{y_1, \ldots, y_l\}$ denote the set of left and right variables of q, respectively. We then introduce for each $x \in L$ a sequence variable Γ_x and similarly for each $y \in R$ a stoup variable Π_y. Inductively, we define

$$1^\circ = \emptyset, \quad (s \wedge x)^\circ = s^\circ, \Gamma_x.$$

Furthermore for each equation ε_i we introduce sequence variables Γ_i and Σ_i. Then for each of the equations ε_i of the form $s_i \leq y$ we obtain a meta-sequent

$$\Gamma_i, s_i^\circ, \Sigma_i \Rightarrow \Pi_y.$$

Similarly, for each of the equations ε_i of the form $s_i \leq 0$ we obtain a meta-sequent

$$\Gamma_i, s_i^\circ, \Sigma_i \Rightarrow \quad .$$

In both cases we denote by ε_i° the meta-sequent associated with the equation ε_i. This allows us to define a structural hypersequent rule

$$\frac{H \mid \varepsilon_1^\circ \quad \ldots \quad H \mid \varepsilon_m^\circ}{H \mid \varepsilon_{m+1}^\circ \mid \ldots \mid \varepsilon_n^\circ} \; (q^\circ)$$

We say that a structural rule (r) is *analytic* if it is of the form (q°) for some structural analytic clause q. We say that a hypersequent calculus $\mathsf{HLJ} + \mathscr{R}$ is a *calculus for* a variety of Heyting algebras $\mathcal{V}$ provided,

$$\mathcal{V} \models s \leq t \quad \text{if, and only if,} \quad \vdash_{\mathsf{HLJ}+\mathscr{R}} \; s \Rightarrow t,$$

for any terms s, t in the language of Heyting algebras. We will say that a variety $\mathcal{V}$ *admits* a calculus with a certain property if there is a calculus for $\mathcal{V}$ with that property. Similarly, we say that an intermediate logic L admits a hypersequent calculus with a given property provided that the corresponding variety $\mathcal{V}(\mathsf{L})$ of Heyting algebras admits a hypersequent calculus with that property. From Theorem 2.1.5 it immediately follows that $\mathsf{HLJ}+\mathscr{R}$ is a calculus for a variety of Heyting algebras $\mathcal{V}$ if, and only if, the universal class $\mathcal{K}(\mathscr{R})$, consisting of Heyting algebras validating the set of rules $\mathscr{R}$, generates $\mathcal{V}$.

2.1.3 Syntactic characterization

In the following we will recall the purely syntactic characterization due to Ciabattoni, Galatos, and Terui [57, 60] of the varieties of Heyting algebras admitting structural hypersequent calculi.

2.1.8. Definition ([61]). Let $\mathcal{P}_0 = \mathcal{N}_0$ be a (countable) set of variables and define sets of terms $\mathcal{P}_n, \mathcal{N}_n$ in the language of Heyting algebras by the following grammar

$$\begin{aligned} \mathcal{P}_{n+1} &::= 1 \mid 0 \mid \mathcal{N}_n \mid \mathcal{P}_{n+1} \wedge \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \\ \mathcal{N}_{n+1} &::= 1 \mid 0 \mid \mathcal{P}_n \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid \mathcal{P}_{n+1} \to \mathcal{N}_{n+1} \end{aligned}$$

An equation of the form $1 \approx t$ is called a $\mathcal{P}_n$*-equation* or $\mathcal{N}_n$*-equation* if $t \in \mathcal{P}_n$ or $t \in \mathcal{N}_n$, respectively. A variety of Heyting algebras is called a $\mathcal{P}_n$*-variety* or $\mathcal{N}_n$*-variety* if it can be axiomatized by $\mathcal{P}_n$-equations or $\mathcal{N}_n$-equations, respectively.

A crucial insight is that using the invertible rules of HLJ, i.e., rules the premises of which are derivable whenever the conclusion is, together with the Ackermann Lemma [60, Lem. 4.3], any $\mathcal{P}_3$-equation can be transformed into an analytic structural universal clause and consequently into an analytic structural hypersequent rule. Such rules preserve the redundancy of the cut-rule when added to HLJ [60, Thm. 6.3]. Thus any variety of Heyting algebras axiomatizable by $\mathcal{P}_3$-equations admits a structural hypersequent calculus in which the cut-rule is redundant.

2.1.9. Theorem ([57, 60]). *Let $\mathcal{V}$ be a variety of Heyting algebras. Then the following are equivalent.*

1. *The variety $\mathcal{V}$ admits a structural hypersequent calculus.*
2. *The variety $\mathcal{V}$ admits an analytic structural hypersequent calculus.*
3. *The variety $\mathcal{V}$ is axiomatizable by $\mathcal{P}_3$-terms.*

Proof:
That Items 1 and 2 are equivalent is established in [57], see also [60], just as the fact that Item 3 entails Item 1. That Item 1 entails Item 3 may be seen via an

argument analogous to the one used to prove [59, Prop. 3.10]. We supply the details. Given a structural hypersequent rule (r) let

$$\varepsilon_1 \text{ and } \ldots \text{ and } \varepsilon_m \implies \varepsilon_{m+1} \text{ or } \ldots \text{ or } \varepsilon_n, \qquad (r^\bullet)$$

with ε_i denoting the equation $s_i(\vec{x}) \leq t_i(\vec{y})$, be the corresponding structural universal clause. By Lemma 2.1.7 we may assume that the variables $\vec{x} = \{x_1, \ldots, x_r\}$ and $\vec{y} = \{y_1, \ldots, y_l\}$ are disjoint and that for each $k \in \{1, \ldots, n\}$ the terms s_k are, possible empty, meets of variables, i.e., the constant 1, and the terms t_k are either a single variable or the constant 0. In fact, we may assume that none of the terms s_i are the constant 1 for $i \in \{1, \ldots, m\}$. Let u_r be the term

$$\bigvee_{j=m+1}^{n} \left(\left(\bigwedge_{i=1}^{m} (s_i(\vec{x}) \to t_i(\vec{y})) \right) \to (s_j(\vec{x}) \to t_j(\vec{y})) \right).$$

It is straightforward to verify that u_r belongs to $\mathcal{P}_3$.

We claim that the variety $\mathcal{V}$ of Heyting algebras generated by the universal class $\mathcal{K}(r)$ of Heyting algebra validating $r^\bullet$ coincides with the variety $\mathcal{W}$ of Heyting algebras axiomatized by the equation $1 \approx u_r$.

To see this, consider first any finitely subdirectly irreducible Heyting algebra $\mathbf{A}$ such that $\mathbf{A} \models 1 \approx u_r$. Then for any choice of elements $a_1, \ldots, a_r, b_1, \ldots b_l \in A$

$$1 = \bigvee_{j=m+1}^{n} \left(\left(\bigwedge_{i=1}^{m} \left(s_i^{\mathbf{A}}(\vec{a}) \to t_i^{\mathbf{A}}(\vec{b}) \right) \right) \to \left(s_j^{\mathbf{A}}(\vec{a}) \to t_j^{\mathbf{A}}(\vec{b}) \right) \right).$$

Since $\mathbf{A}$ is finitely subdirectly irreducible it must have a join-irreducible top element, see, e.g., Appendix A.4. We must therefore have that

$$\bigwedge_{i=1}^{m} \left(s_i^{\mathbf{A}}(\vec{a}) \to t_i^{\mathbf{A}}(\vec{b}) \right) \leq s_j^{\mathbf{A}}(\vec{a}) \to t_j^{\mathbf{A}}(\vec{b}),$$

for some $j \in \{m+1, \ldots, n\}$. Consequently, if $s_i^{\mathbf{A}}(\vec{a}) \leq t_i^{\mathbf{A}}(\vec{b})$ for all $i \in \{1, \ldots, m\}$, then also $s_j^{\mathbf{A}}(\vec{a}) \leq t_j^{\mathbf{A}}(\vec{b})$. This shows that $\mathbf{A} \models r^\bullet$ and therefore that any finitely subdirectly irreducible $\mathcal{W}$-algebra belongs to the class $\mathcal{K}(r)$ from which we conclude that $\mathcal{W} \subseteq \mathcal{V}$.

Conversely, let $\mathbf{A}$ be a Heyting algebra such that $\mathbf{A} \models r^\bullet$ and consider elements $a_1, \ldots, a_r, b_1, \ldots b_l \in A$. Define for each $k \in \{1, \ldots, r\}$ an element $c_k \in A$ by

$$c_k = a_k \wedge \bigwedge_{i=1}^{m} \left(s_i^{\mathbf{A}}(\vec{a}) \to t_i^{\mathbf{A}}(\vec{b}) \right),$$

For each term s_l which is a non-empty meet of variables we may conclude that

$$s_l^{\mathbf{A}}(\vec{c}) = s_l^{\mathbf{A}}(\vec{a}) \wedge \bigwedge_{i=1}^{m} \left(s_i^{\mathbf{A}}(\vec{a}) \to t_i^{\mathbf{A}}(\vec{b}) \right).$$

In particular, we have that $s_i^{\mathbf{A}}(\vec{c}) \leq t_i^{\mathbf{A}}(\vec{b})$, for each $i \in \{1, \ldots, m\}$. Hence by the assumption that $\mathbf{A} \models r^\bullet$ there must be $j \in \{m+1, \ldots, n\}$ such that $s_j^{\mathbf{A}}(\vec{c}) \leq t_j^{\mathbf{A}}(\vec{b})$. Either the term s_j is the constant 1 in which case $t_j^{\mathbf{A}}(\vec{b}) = 1$. Otherwise the term s_j is a non-empty meet of variables and hence

$$s_j^{\mathbf{A}}(\vec{a}) \wedge \bigwedge_{i=1}^{m} \left(s_i^{\mathbf{A}}(\vec{a}) \to t_i^{\mathbf{A}}(\vec{b})\right) = s_j^{\mathbf{A}}(\vec{c}) \leq t_j^{\mathbf{A}}(\vec{b}),$$

from which it follows that

$$\bigwedge_{i=1}^{m} \left(s_i^{\mathbf{A}}(\vec{a}) \to t_i^{\mathbf{A}}(\vec{b})\right) \leq s_j^{\mathbf{A}}(\vec{a}) \to t_j^{\mathbf{A}}(\vec{b}).$$

In either case, for any choice of elements $\vec{a}, \vec{b} \in A$, there is $j \in \{m+1, \ldots, n\}$ such that

$$1 = \left(\bigwedge_{i=1}^{m} \left(s_i^{\mathbf{A}}(\vec{a}) \to t_i^{\mathbf{A}}(\vec{b})\right)\right) \to \left(s_j^{\mathbf{A}}(\vec{a}) \to t_j^{\mathbf{A}}(\vec{b})\right),$$

showing that $\mathbf{A} \models 1 \approx u_r$. Therefore $\mathcal{K}(r) \subseteq \mathcal{W}$ and hence $\mathcal{V} = \mathcal{W}$.

It follows that if $\mathsf{HLJ} + \mathscr{R}$ is an analytic structural sequent calculi for the variety $\mathcal{V}$ then the class $\mathcal{K}(\mathscr{R})$ generates $\mathcal{V}$ which must then be axiomatized by the set of $\mathcal{P}_3$-equations $\{1 \approx u_r : r \in \mathscr{R}\}$. $\square$

2.1.10. Remark. Note that the proof of [60, Thm. 6.3] showing the cut-rule is redundant in any analytic structural hypersequent calculus as defined in this chapter is semantic in nature and as such does not directly yield an explicit procedure for transforming a derivation using the cut-rule into a cut-free derivation. However, in concrete cases an explicit cut-elimination procedure may be given, see, e.g., [55, 56]. We also want to emphasize that it is not the case that the cut-rule is redundant in every structural hypersequent calculus but only that any such calculus is effectively equivalent to a structural hypersequent calculus in which the cut-rule is redundant.

2.1.11. Proposition. *Any $\mathcal{P}_3$-variety can be axiomatized by a set of equations of the form $1 \approx u$ where u is a term in the $\vee$-free reduct of the language of Heyting algebras.*

Proof:
Any $\mathcal{P}_3$-equation is equivalent on (finitely) subdirectly irreducible algebras to a set of analytic structural clauses [60, Sec. 4]. As shown in the proof of Theorem 2.1.9 any analytic structural clause is equivalent, on finitely subdirectly irreducible Heyting algebras, to an equation of the form $1 \approx u_1 \vee \ldots \vee u_n$ with the terms $u_1, \ldots, u_n$ belonging to the $\vee$-free reduct of the language of Heyting algebras. It

is not difficult to see that any equation in the language of Heyting algebras of the form $1 \approx s \vee t$ is equivalent to the equation $1 \approx (s \to x) \to ((t \to x) \to x)$, for x a variable not occurring in the term s nor in the term t.

This shows that any variety of Heyting algebras which can be axiomatized by $\mathcal{P}_3$-equation can also be axiomatized by equations of the form $1 \approx u$ where u is a $\vee$-free term. □

2.1.12. Example. For $n \geq 1$ let $\mathcal{BTW}_n, \mathcal{BW}_n$ and $\mathcal{BC}_n$, be the varieties determined by posets of top width at most n, of width at most n, and of cardinality at most n, respectively. See Appendix A.9 for definitions. All of these varieties have axiomatizations given by equations which are ostensibly $\mathcal{P}_3$, and so by Theorem 2.1.9 all of these varieties admit an analytic structural hypersequent calculus. Concretely, the rules (com) and (co-wc) yield cut-free structural hypersequent calculi for $\mathcal{LC} = \mathcal{BW}_1$ and $\mathcal{KC} = \mathcal{BTW}_1$, respectively, when added to HLJ. For more concrete examples of structural hypersequent calculus we refer to [57, 55]. Moreover, the variety of Heyting algebras corresponding to the Kuznetsov-Gerčiu logic [26, 185] is also easily seen to be a $\mathcal{P}_3$-variety.

Theorem 2.1.9 thus gives a very simple syntactic description of the class of varieties which admit analytic, and hence cut-free, structural hypersequent calculi. Our aim is then to supply criteria describing this class of varieties which are syntax independent. Among other things this will allow us to derive negative results showing that certain well-known varieties of Heyting algebras do not admit such calculi. Given the correspondence between structural hypersequent calculi and structural universal clauses we may provide the first algebraic characterization of the class of varieties admitting structural hypersequent calculi.

2.1.13. Proposition. *Let $\mathcal{V}$ be a variety of Heyting algebras. Then the following are equivalent.*

1. *The variety $\mathcal{V}$ admits a structural hypersequent calculus.*
2. *The variety $\mathcal{V}$ is generated by a universal class of Heyting algebras axiomatized by structural universal clauses.*
3. *The variety $\mathcal{V}$ is generated by a universal class of Heyting algebras axiomatized by universal clauses in the $\{\wedge, 0, 1\}$-reduct of the language of Heyting algebras.*

Proof:
The equivalence between Item 1 and Item 2 follows from Theorem 2.1.5 and the correspondence between structural hypersequent calculi and structural universal clauses. That Item 2 is equivalent to Item 3 follows from Lemma 2.1.7. □

This characterization is, however, not very informative and in the following section we shall provide a characterization which we believe to be more enlightening.

2.2 Algebraic characterization

In this section we provide a semantic characterization of the varieties of Heyting algebras, or equivalently intermediate logics, admitting a structural, and therefore also a cut-free, hypersequent calculus in terms of the algebraic semantics. This section builds on the theory of stable classes of Heyting algebras as introduced in [25] and further developed in [29, 34], see also [164] for a thorough treatment of different concepts of stability in general.

Given a set $\tau \subseteq \{\wedge, \vee, \rightarrow, 0, 1\}$ we will let τ_c denote the set $\tau \cap \{0, 1\}$ and τ_o denote the set $\tau \cap \{\wedge, \vee, \rightarrow\}$. Let $\tau \subseteq \{\wedge, \vee, \rightarrow, 0, 1\}$ and let $\mathbf{A}$ and $\mathbf{B}$ be Heyting algebras. We say that a function $h\colon A \to B$ is a *τ-homomorphism* if h is a homomorphism between the τ-reducts of $\mathbf{A}$ and $\mathbf{B}$. If $h\colon A \to B$ is a τ-homomorphism we write $\mathbf{A} \to_\tau \mathbf{B}$. A τ-homomorphism $h\colon \mathbf{A} \to_\tau \mathbf{B}$ is called a *τ-embedding* if the function $h\colon A \to B$ is injective. In this case we write $h\colon \mathbf{A} \hookrightarrow_\tau \mathbf{B}$. We write $\mathbf{A} \hookrightarrow_\tau \mathbf{B}$ to indicate that there is a τ-embedding from $\mathbf{A}$ to $\mathbf{B}$. We say that an algebra $\mathbf{A}$ is a *τ-subalgebra* of an algebra $\mathbf{B}$ provided that the τ-reduct of $\mathbf{A}$ is a subalgebra of the τ-reduct of $\mathbf{B}$. For a class of Heyting algebras $\mathcal{K}$ we write $\mathrm{S}_\tau(\mathcal{K})$ for the class of τ-subalgebras of members of $\mathcal{K}$.

2.2.1. Definition (cf. [164, Def. 3.3.2]). Let $\tau \subseteq \{\wedge, \vee, \rightarrow, 0, 1\}$. A class $\mathcal{K}$ of Heyting algebras is (*finitely*) *τ-stable* provided that whenever $\mathbf{B} \in \mathcal{K}$ and $\mathbf{A} \hookrightarrow_\tau \mathbf{B}$ then $\mathbf{A} \in \mathcal{K}$ for all (finite) Heyting algebras $\mathbf{A}$.

Let $\tau \subseteq \{\wedge, \vee, \rightarrow, 0, 1\}$ and let $\mathcal{K}$ be a class of Heyting algebras. We say that a variety of Heyting algebras $\mathcal{V}$ is *τ-stably generated by* $\mathcal{K}$ provided that it is generated by the class $\mathrm{IS}_\tau(\mathcal{K})$. In particular, if $\mathcal{V}$ is *τ-stably generated by* a class of algebras $\mathcal{K}$ then $\mathrm{IS}_\tau(\mathcal{K}) \subseteq \mathcal{V}$ and so $\mathcal{V}$ is generated by some τ-stable class.

A variety is called *τ-stably generated* provided that it is τ-stably generated by some class of algebras. Thus a variety is τ-stably generated if, and only, it is generated by a τ-stable class.

2.2.2. Definition (cf. [164, Def. 3.3.9]). An intermediate logic L is *τ-stable* if its corresponding variety of Heyting algebras is τ-stably generated.

The τ-stable logics are all well understood in the case when τ is $\{\wedge, \rightarrow\}$, $\{\wedge, \rightarrow, 0\}$, or $\{\wedge, \vee, 0, 1\}$. These logics have all been studied before individually. The $(\wedge, \rightarrow)$-stable logics are known as *subframe logics*, see, e.g., [51, Chap. 11.3], and the $(\wedge, \rightarrow, 0)$-stable logics as *cofinal subframe logics* [254]. The $(\wedge, \vee, 0, 1)$-stable logics are also known as *stable logics* [25]. Expanding the language of

Heyting algebras we also obtain a notion of $(\wedge, \vee, \neg, 1)$-stable logics known as *cofinal stable logics* [29].

Note that if $\tau \subseteq \tau' \subseteq \{\wedge, \vee, \to, 0, 1\}$ then any τ-stable class $\mathcal{K}$ must necessarily also be τ'-stable. In particular, any $(\wedge, 0, 1)$-stably generated variety will be $(\wedge, \vee, 0, 1)$-stably generated. By similar reasoning any $(\wedge, 0, 1)$-stably generated variety will also be $(\wedge, \vee, \neg, 1)$-stably generated. Consequently, $(\wedge, 0, 1)$-stable logics will be both $(\wedge, \vee, 0, 1)$-stable and $(\wedge, \vee, \neg, 1)$-stable.

When τ is either one of the sets $\{\wedge, \to\}$, $\{\wedge, \to, 0\}$ or $\{\wedge, \vee, 0, 1\}$ we have a description of the τ-stably generated varieties using the concept of stable clauses.

2.2.3. Definition. Let $\tau \subseteq \{\wedge, \vee, \to, 0, 1\}$ and let $\mathbf{A}$ be a finite Heyting algebra. For each element $a \in A$ introduce a first-order variable x_a, such that the map $a \mapsto x_a$ becomes injective. By the *τ-stable (universal) clause $q_\tau(\mathbf{A})$ associated with* $\mathbf{A}$ we shall understand the universal clause $\forall \vec{x}\ (P(\vec{x}) \implies C(\vec{x}))$ where

$$P(\vec{x}) = \mathsf{AND}\{x_c \approx c\colon c \in \tau_c\} \text{ and } \mathsf{AND}\{x_a \bullet x_{a'} \approx x_{a \bullet a'}\colon a, a' \in A, \bullet \in \tau_o\}$$
$$C(\vec{x}) = \mathsf{OR}\{x_a \approx x_{a'}\colon a, a' \in A, a \neq a'\}.$$

2.2.4. Remark. Stable universal clauses may be seen as a propositional version of diagrams as known from classic Robinson-style model theory, see, e.g., [157, Chap. 1.4]. Variants of the τ-stable clauses defined above have been studied before under the names stable and canonical multi-conclusion rules [166, 28, 29, 164].

Given a class of Heyting algebras $\mathcal{K}$ we denote by $\mathcal{K}^\omega$ the class of finite members of $\mathcal{K}$. Similarly we denote by $\mathcal{K}_{si}$ and $\mathcal{K}_{fsi}$ the classes of subdirectly irreducible and finitely subdirectly irreducible members of $\mathcal{K}$, respectively. Evidently, $\mathcal{K}^\omega_{si} = \mathcal{K}^\omega_{fsi}$. The following theorem sums up the characterization of being τ-stably generated.

2.2.5. Theorem ([164, 3.3.17]). *Let $\mathcal{V}$ be a variety of Heyting algebras and let L be its corresponding intermediate logic. Then for τ any of the sets $\{\wedge, \to\}$, $\{\wedge, \to, 0\}$ or $\{\wedge, \vee, 0, 1\}$ the following are equivalent.*

1. *The variety $\mathcal{V}$ is τ-stably generated.*
2. *The logic L is τ-stable.*
3. *The variety $\mathcal{V}$ is generated by a τ-stable class of finite algebras.*
4. *The variety $\mathcal{V}$ is generated by a τ-stable universal class.*
5. *The variety $\mathcal{V}$ is generated by a finitely τ-stable universal class.*
6. *The variety $\mathcal{V}$ is generated by a universal class of Heyting algebras axiomatized by τ-stable universal clauses associated with finite subdirectly irreducible Heyting algebras.*

7. *The class* $(\mathrm{IS}_\tau(\mathcal{V}_{fsi}))^\omega \cap \mathcal{V}_{fsi}$ *is contained in* $\mathcal{V}$.

We will prove a version of Theorem 2.2.5 for $\tau = \{\wedge, 0, 1\}$. Together with Proposition 2.1.13 this will provide us with an algebraic characterization of the varieties of Heyting algebras admitting a structural hypersequent calculus.

2.2.1 $(\wedge, 0, 1)$-stably generated varieties

In this subsection we will establish a version of Theorem 2.2.5 for $\tau = \{\wedge, 0, 1\}$. The strategy will be completely similar to the one found in [25, 29, 164]. The following lemma shows that the τ-stable clause associated with a finite algebra $\mathbf{A}$ encodes the property of not containing an isomorphic copy of $\mathbf{A}$ as a τ-subalgebra.

2.2.6. Lemma (cf. [29, Prop. 4.2]). *Let* $\tau \subseteq \{\wedge, \vee, \to, 0, 1\}$ *and let* $\mathbf{A}, \mathbf{B}$ *be Heyting algebras with* $\mathbf{A}$ *finite. Then the following are equivalent.*

1. $\mathbf{B} \not\models q_\tau(\mathbf{A})$.

2. *There exists a* τ*-embedding* $h\colon \mathbf{A} \hookrightarrow_\tau \mathbf{B}$.

Proof:
Given a valuation ν on $\mathbf{B}$ such that $(\mathbf{B}, \nu) \not\models q_\tau(\mathbf{A})$ then we obtain a τ-embedding $h_\nu\colon \mathbf{A} \hookrightarrow_\tau \mathbf{B}$ by letting $h_\nu(a) = \nu(x_a)$. Conversely, given a τ-embedding $h\colon \mathbf{A} \hookrightarrow_\tau \mathbf{B}$ we obtain a valuation ν_h on $\mathbf{B}$ such that $(\mathbf{B}, \nu_h) \not\models q_\tau(\mathbf{A})$ by letting $\nu_h(x_a) = h(a)$. $\square$

We then show that a universal class of Heyting algebras is $(\wedge, 0, 1)$-stable precisely if it is axiomatizable by $(\wedge, 0, 1)$-stable clauses.

2.2.7. Lemma (cf. [29, Prop. 4.5]). *Let* $\tau = \{\wedge, 0, 1\}$ *and let* $\mathcal{U}$ *be a universal class of Heyting algebras. Then the following are equivalent.*

1. *The universal class* $\mathcal{U}$ *is axiomatized by structural universal clauses.*

2. *The universal class* $\mathcal{U}$ *is* τ*-stable.*

3. *The universal class* $\mathcal{U}$ *is finitely* τ*-stable.*

4. *The universal class* $\mathcal{U}$ *is axiomatized by* τ*-stable clauses.*

Proof:
If $\mathcal{U}$ is axiomatized by structural universal clauses then $\mathcal{U}$ must be τ-stable, since universal clauses in the τ-reduct of the language of Heyting algebras are preserved by τ-subalgebras. Moreover, any τ-stable universal class is evidently finitely τ-stable.

We show that if $\mathcal{U}$ is finitely τ-stable then $\mathcal{U}$ is axiomatized by τ-stable clauses. Therefore, assume that $\mathcal{U}$ is finitely τ-stable and let

$$Q = \{q_\tau(\mathbf{A})\colon |\mathbf{A}| < \aleph_0, \mathbf{A} \notin \mathcal{U}\}.$$

We claim that for any Heyting algebra $\mathbf{B}$ we have that $\mathbf{B} \in \mathcal{U}$ if, and only if, $\mathbf{B} \models Q$. To see this let $\mathrm{Th}^{\forall}_{\mathsf{HA}}(\mathcal{U})$ be the universal theory, in the language of Heyting algebras, of $\mathcal{U}$. If $\mathbf{B} \notin \mathcal{U}$ then, by the assumption that $\mathcal{U}$ is a universal class of Heyting algebras, there must be a universal clause $q \in \mathrm{Th}^{\forall}_{\mathsf{HA}}(\mathcal{U})$ such that $\mathbf{B} \not\models q$. Thus, by Lemma A.4.4 we must have a finite $(\wedge, \vee, 0, 1)$-subalgebra, in particular a τ-subalgebra, $\mathbf{C}$ of $\mathbf{B}$ such that $\mathbf{C} \not\models q$, i.e., $\mathbf{C} \notin \mathcal{U}$ whence $q_\tau(\mathbf{C}) \in Q$. By Lemma 2.2.6 we must have that $\mathbf{B} \not\models q_\tau(\mathbf{C})$ and so $\mathbf{B} \not\models Q$.

Conversely, if $\mathbf{B} \not\models Q$ then for some finite Heyting algebra $\mathbf{A} \notin \mathcal{U}$ we have $\mathbf{B} \not\models q_\tau(\mathbf{A})$. By Lemma 2.2.6 it follows that $\mathbf{A}$ is a τ-subalgebra of $\mathbf{B}$. Since $\mathcal{U}$ is assumed to be finitely τ-stable we must conclude that $\mathbf{B} \notin \mathcal{U}$ since otherwise $\mathbf{A} \in \mathcal{U}$.

Finally, by Lemma 2.1.7 any τ-stable clause is equivalent to a finite set of structural universal clauses so if $\mathcal{V}$ is axiomatized by τ-stable clauses it will also be axiomatized by structural universal clauses. $\square$

2.2.8. Definition. Let $\tau \subseteq \{0, \wedge, \vee, \to, 1\}$ and let $\mathbf{A}$ be a finite Heyting algebra. For each element $a \in A$ introduce a variable x_a, such that the map $a \mapsto x_a$ becomes injective. By the *τ-stable equation* $\varepsilon_\tau(\mathbf{A})$ *associated with* $\mathbf{A}$ we shall understand the equation $\bigwedge \Gamma \leq \bigvee \Delta$ where

$$\Gamma = \{x_c \leftrightarrow c\colon c \in \tau_c\} \cup \{x_a \bullet x_{a'} \leftrightarrow x_{a \bullet a'}\colon a, a' \in A, \bullet \in \tau_o\}$$
$$\Delta = \{x_a \to x_{a'}\colon a, a' \in A, a \not\leq a'\}.$$

The τ-stable equations encode information about finite Heyting algebras in almost the same way as the τ-stable clauses. However, a version of Lemma 2.2.6 only holds for so-called *well-connected* Heyting algebras, that is, Heyting algebras validating the universal clause

$$1 \leq x \vee y \implies 1 \leq x \text{ or } 1 \leq y.$$

These are exactly the finitely subdirectly irreducible Heyting algebras, see Appendix A.4. Consequently, any variety of Heyting algebras is generated by its class of well-connected members.

We will need the following lemma showing that homomorphic images of a finite Heyting algebra $\mathbf{A}$ are isomorphic to $(\wedge, 0, 1)$-subalgebras of $\mathbf{A}$.

2.2.9. Lemma. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be finite Heyting algebras. If* $\mathbf{B}$ *is a homomorphic image of* $\mathbf{A}$*, then* $\mathbf{B} \in \mathrm{IS}_{\wedge,0,1}(\mathbf{A})$.

Proof:
If $h\colon \mathbf{A} \twoheadrightarrow \mathbf{B}$ is a surjective Heyting algebra homomorphism, then $\mathbf{B}$ is isomorphic to $\mathbf{A}/\theta$, with the congruence θ the kernel of h. As $\mathbf{A}$ is finite the filter $1/\theta$ must be a principal filter, say $\uparrow a$ for some $a \in A$, whence $\mathbf{B}$ is isomorphic to the interval $[0, a]$. Evidently, we have a $(\wedge, 0, 1)$-embedding g from $[0, a]$ into $\mathbf{A}$, given by,

$$g(b) = \begin{cases} b & \text{if } b < a, \\ 1 & \text{otherwise,} \end{cases}$$

showing that $\mathbf{B} \hookrightarrow_{\wedge,0,1} \mathbf{A}$. □

We may then establish a version of Lemma 2.2.6 for $(\wedge, 0, 1)$-stable equations.

2.2.10. Lemma (cf. [25, Thm. 6.3]). *Let* $\mathbf{A}$, $\mathbf{B}$ *be Heyting algebras with* $\mathbf{A}$ *finite.*

1. *If* $\mathbf{B} \not\models \varepsilon_{\wedge,0,1}(\mathbf{A})$, *then* $\mathbf{A} \hookrightarrow_{\wedge,0,1} \mathbf{B}$.
2. *If* $\mathbf{B}$ *is well-connected and* $\mathbf{A} \hookrightarrow_{\wedge,0,1} \mathbf{B}$, *then* $\mathbf{B} \not\models \varepsilon_{\wedge,0,1}(\mathbf{A})$.

Proof:
If $\mathbf{B} \not\models \varepsilon_{\wedge,0,1}(\mathbf{A})$ then by Lemma A.4.4 we must have a finite Heyting algebra $\mathbf{C}$ which is a $(\wedge, \vee, 0, 1)$-subalgebra of $\mathbf{B}$, and so in particular a $(\wedge, 0, 1)$-subalgebra of $\mathbf{B}$, such that $\mathbf{C} \not\models \varepsilon_{\wedge,0,1}(\mathbf{A})$. This means that there is a valuation ν on $\mathbf{C}$ such that $\nu(\bigwedge \Gamma \to \bigvee \Delta) < 1$, where Γ and Δ are as in Definition 2.2.8. By Wronski's Lemma [253, Lem. 1] there exists a subdirectly irreducible Heyting algebra $\mathbf{D}$ together with a Heyting algebra homomorphism $\pi\colon \mathbf{C} \twoheadrightarrow \mathbf{D}$ such that $\pi(\nu(\bigwedge \Gamma \to \bigvee \Delta)) = c_{\mathbf{D}}$, where $c_{\mathbf{D}}$ denotes the unique co-atom of $\mathbf{D}$. By Lemma 2.2.9 we have that $\mathbf{D}$ is a $(\wedge, 0, 1)$-subalgebra of $\mathbf{C}$ and therefore also a $(\wedge, 0, 1)$-subalgebra of $\mathbf{B}$. We claim that $\mathbf{A}$ is a $(\wedge, 0, 1)$-subalgebra of $\mathbf{D}$ and therefore also a $(\wedge, 0, 1)$-subalgebra of $\mathbf{B}$. We obtain a valuation μ on $\mathbf{D}$ such that $\mu(\bigwedge \Gamma \to \bigvee \Delta) = c_{\mathbf{D}}$ by letting $\mu(x_a) = \pi(\nu(x_a))$. From this it follows that $\mu(\bigwedge \Gamma) = 1$ and $\mu(\bigvee \Delta) = c_{\mathbf{D}}$ and hence we may conclude that $h_\mu\colon \mathbf{A} \to \mathbf{D}$ given by $h_\mu(a) = \mu(x_a)$ is a $(\wedge, 0, 1)$-embedding of $\mathbf{A}$ into $\mathbf{D}$.

Conversely, if there is a $(\wedge, 0, 1)$-embedding $h\colon \mathbf{A} \hookrightarrow_{\wedge,0,1} \mathbf{B}$, then defining a valuation ν_h on $\mathbf{B}$ by $\nu_h(x_a) = h(a)$ we obtain that $\nu_h(\bigwedge \Gamma) = 1$ by the fact that h is a $(\wedge, 0, 1)$-homomorphism. Moreover, by the fact that h is also a $(\wedge, 0, 1)$-embedding we must have that $1 \not\leq \nu_h(x_a \to x_{a'})$ for all $x_a \to x_{a'}$ in Δ. Thus, assuming $\mathbf{B}$ to be well-connected we may conclude that $1 \not\leq \nu_h(\bigvee \Delta)$ and therefore that $\nu(\bigwedge \Gamma) \not\leq \nu(\bigvee \Delta)$. Thus, ν_h witnesses that $\mathbf{B} \not\models \varepsilon_{\wedge,0,1}(\mathbf{A})$. □

The following lemma shows that the varieties of Heyting algebras generated by $(\wedge, 0, 1)$-stable universal classes are in fact axiomatized by $(\wedge, 0, 1)$-stable equations. Thus a variety can be axiomatized by $(\wedge, 0, 1)$-stable equations precisely when it can be axiomatized by $(\wedge, 0, 1)$-stable universal clauses.

2.2.11. Lemma. *Let $\mathcal{V}$ be a variety of Heyting algebras. Then the following are equivalent.*

1. *The variety $\mathcal{V}$ is axiomatized by a collection of $(\wedge, 0, 1)$-stable equations.*

2. *The variety $\mathcal{V}$ is generated by a universal class axiomatized by a collection of $(\wedge, 0, 1)$-stable clauses.*

Proof:
We first observe that as a consequence of Lemma 2.2.6 and Item 1 of Lemma 2.2.10 the clause $q_{\wedge,0,1}(\mathbf{A})$ implies the corresponding equation $\varepsilon_{\wedge,0,1}(\mathbf{A})$ for each finite Heyting algebra $\mathbf{A}$. Similarly by Lemma 2.2.6 and Item 2 of Lemma 2.2.10 we have that for each finite Heyting algebra $\mathbf{A}$ the clause $q_{\wedge,0,1}(\mathbf{A})$ is equivalent to the corresponding equation $\varepsilon_{\wedge,0,1}(\mathbf{A})$ on finitely subdirectly irreducible Heyting algebras.

Assume that $\mathcal{V}$ is axiomatized by a collection of $(\wedge, 0, 1)$-stable equations, say $\{\varepsilon_i\}_{i\in I}$, and let $\mathcal{U}$ be the universal class of Heyting algebras axiomatized by the corresponding $(\wedge, 0, 1)$-stable clauses $\{q_i\}_{i\in I}$. For each $i \in I$ the clause q_i entails the equation ε_i, whence $\mathcal{U} \subseteq \mathcal{V}$. Furthermore, since for each $i \in I$ the clause q_i and the equation ε_i are equivalent on finitely subdirectly irreducible Heyting algebras it follows that $\mathcal{V}_{\mathit{fsi}} \subseteq \mathcal{U} \subseteq \mathcal{V}$ and so the class $\mathcal{U}$ must generate the variety $\mathcal{V}$.

Conversely, assume that $\mathcal{V}$ is generated by a universal class $\mathcal{U} \subseteq \mathcal{V}$ axiomatized by a collection of $(\wedge, 0, 1)$-stable clauses, say $\{q_i\}_{i\in I}$, and let $\mathcal{W}$ be the variety of Heyting algebras axiomatized by the corresponding $(\wedge, 0, 1)$-stable equations $\{\varepsilon_i\}_{i\in I}$. As before we see that $\mathcal{W}_{\mathit{fsi}} \subseteq \mathcal{U} \subseteq \mathcal{W}$ and hence that $\mathcal{W} = \mathcal{V}$. $\square$

2.2.12. Lemma. *Let $\tau \subseteq \{\wedge, \vee, 0, 1\}$ be given. If $\mathcal{K}$ is a τ-stable class, then so is the universal class generated by $\mathcal{K}$.*

Proof:
By [46, Thm. V.2.20] we know that the universal class generated by the class $\mathcal{K}$ is the class $\mathrm{ISP_U}(\mathcal{K})$. Therefore, let $\{\mathbf{B}_i\}_{i\in I}$ be a collection of $\mathcal{K}$-algebras, U an ultrafilter on I, and $\mathbf{A}$ a finite Heyting algebra. If $\mathbf{A} \not\hookrightarrow_\tau \mathbf{B}_i$ for all $i \in I$ then by Lemma 2.2.6 we have that $\mathbf{B}_i \models q_\tau(\mathbf{A})$ for all $i \in I$ and hence by Łoś' Theorem we obtain that $\prod_{i\in I} \mathbf{B}_i/U \models q_\tau(\mathbf{A})$ and so $\mathbf{A} \not\hookrightarrow_\tau \prod_{i\in I} \mathbf{B}_i/U$. Consequently, if $\mathbf{A} \hookrightarrow_\tau \prod_{i\in I} \mathbf{B}_i/U$ then, $\mathbf{A} \hookrightarrow_\tau \mathbf{B}_i$ for some $i \in I$. Moreover, if $\mathbf{B} \in \mathrm{ISP_U}(\mathcal{K})$ and $\mathbf{A}$ is a finite algebra such that $\mathbf{A} \hookrightarrow_\tau \mathbf{B}$, then necessarily $\mathbf{A} \hookrightarrow_\tau \mathbf{B}'$ for some $\mathbf{B}' \in \mathrm{P_U}(\mathcal{K})$ whence by the above we have that $\mathbf{A} \in \mathcal{K}$. We have thus shown that $\mathrm{ISP_U}(\mathcal{K})$ is a finitely τ-stable universal class and as such it must be τ-stable by Lemma 2.2.7. $\square$

2.2.13. Theorem. *Let $\mathcal{V}$ be a variety of Heyting algebras and let L be the corresponding intermediate logic. Then the following are equivalent.*

1. *The logic L is $(\wedge, 0, 1)$-stable.*
2. *The variety $\mathcal{V}$ is $(\wedge, 0, 1)$-stably generated.*
3. *The variety $\mathcal{V}$ is generated by a $(\wedge, 0, 1)$-stable universal class of Heyting algebras.*
4. *The variety $\mathcal{V}$ is generated by a universal class of Heyting algebras axiomatized by $(\wedge, 0, 1)$-stable universal clauses.*
5. *The variety $\mathcal{V}$ is generated by a universal class of Heyting algebras axiomatized by structural universal clauses.*
6. *The variety $\mathcal{V}$ is axiomatized by $(\wedge, 0, 1)$-stable equations.*
7. *The class $\mathrm{IS}_{\wedge,0,1}(\mathcal{V}_{fsi})$ is contained in $\mathcal{V}$.*
8. *The class $(\mathrm{IS}_{\wedge,0,1}(\mathcal{V}_{fsi}))^{\omega}$ is contained in $\mathcal{V}$.*
9. *The variety $\mathcal{V}$ is generated by its finite members and $\mathrm{IS}_{\wedge,0,1}(\mathcal{V}^{\omega}_{fsi}) \subseteq \mathcal{V}$.*

Proof:
Item 1 and Item 2 are equivalent by definition. That Item 2 and Item 3 are equivalent follows from Lemma 2.2.12. The equivalence of Item 3, Item 4 and Item 5 follows from Lemma 2.2.7. That Item 4 and Item 6 are equivalent is the content of Lemma 2.2.11.

To see that Item 6 implies Item 7 we observe that if $\mathcal{V}$ is axiomatized by a collection of stable equations $\{\varepsilon_i\}_{i\in I}$ and $\mathbf{B} \in \mathcal{V}_{fsi}$ then $\mathbf{B} \models q_i$, where q_i is the $(\wedge, 0, 1)$-stable clause corresponding to the equation ε_i. Such clauses are preserved under $(\wedge, 0, 1)$-subalgebras. Consequently, for each $\mathbf{A} \in \mathrm{IS}_{\wedge,0,1}(\mathbf{B})$ we have $\mathbf{A} \models q_i$ and hence $\mathbf{A} \models \varepsilon_i$ showing that $\mathbf{A} \in \mathcal{V}$.

Evidently, Item 7 implies Item 8.

To see that Item 8 implies Item 9 assume that $(\mathrm{IS}_{\wedge,0,1}(\mathcal{V}_{fsi}))^{\omega} \subseteq \mathcal{V}$ and that ε is an equation such that $\mathcal{V} \not\models \varepsilon$. Then there exists $\mathbf{B} \in \mathcal{V}_{fsi}$ such that $\mathbf{B} \not\models \varepsilon$. By Lemma A.4.4 we can find a finite $(\wedge, 0, 1)$-subalgebra $\mathbf{A}$ of $\mathbf{B}$ such that $\mathbf{A} \not\models \varepsilon$. By assumption $\mathbf{A} \in (\mathrm{IS}_{\wedge,0,1}(\mathcal{V}_{fsi}))^{\omega} \subseteq \mathcal{V}$, showing that every equation which is refuted by some member of $\mathcal{V}$ is also refuted by some finite member of $\mathcal{V}$ and hence that $\mathcal{V}$ is generated by its finite members. Evidently, $\mathrm{IS}_{\wedge,0,1}(\mathcal{V}^{\omega}_{fsi}) \subseteq (\mathrm{IS}_{\wedge,0,1}(\mathcal{V}_{fsi}))^{\omega} \subseteq \mathcal{V}$.

Finally, we show that Item 9 implies Item 6. Therefore, assume that $\mathcal{V}$ is generated by its finite members and $\mathrm{IS}_{\wedge,0,1}(\mathcal{V}^{\omega}_{fsi}) \subseteq \mathcal{V}$. Let $\mathcal{K}$ be any set of finite Heyting algebras such that no member of $\mathcal{K}$ is contained in $\mathcal{V}$ and each finite

Heyting algebra which is not a member of $\mathcal{V}$ is isomorphic to some member of $\mathcal{K}$. Consider the set of equations,

$$E = \{\varepsilon_{\wedge,0,1}(\mathbf{A}) : \mathbf{A} \in \mathcal{K}\},$$

and let $\mathcal{W}$ be the variety of Heyting algebras axiomatized by E. We claim that if $\mathbf{B} \in \mathcal{W}^{\omega}_{fsi}$ then $\mathbf{B}$ belongs to $\mathcal{V}$. Otherwise there would be $\mathbf{A} \in \mathcal{K}$ isomorphic to $\mathbf{B}$, whence $\mathbf{B} \models \varepsilon_{\wedge,0,1}(\mathbf{A})$ which by Item 2 of Lemma 2.2.10 is a contradiction. As we have already seen, since the variety $\mathcal{W}$ is axiomatized by $(\wedge, 0, 1)$-stable equations, it must be generated by its finite members. Consequently, $\mathcal{W} \subseteq \mathcal{V}$. On the other hand since by assumption $\mathrm{IS}_{\wedge,0,1}(\mathcal{V}^{\omega}_{fsi}) \subseteq \mathcal{V}$ we obtain from Item 1 of Lemma 2.2.10 that $\mathbf{B} \not\models \varepsilon_{\wedge,0,1}(\mathbf{A})$ entails $\mathbf{A} \in \mathcal{V}$ for each finite $\mathbf{A} \notin \mathcal{V}$ and each $\mathbf{B} \in \mathcal{V}^{\omega}_{fsi}$. Therefore, we may conclude that $\mathcal{V}^{\omega}_{fsi} \models E$ and since $\mathcal{V}$ is generated by its finite members also that $\mathcal{V} \models E$, whence $\mathcal{V} \subseteq \mathcal{W}$. □

We now obtain the following algebraic characterization of the intermediate logics admitting a structural hypersequent calculus and therefore by Theorem 2.1.9 also a cut-free structural hypersequent calculus.

2.2.14. Theorem. *Let $\mathcal{V}$ be a variety of Heyting algebras and let L be its corresponding intermediate logic. Then the following are equivalent.*

1. *The variety $\mathcal{V}$ admits a structural hypersequent calculus.*
2. *The logic L admits a structural hypersequent calculus.*
3. *The class $(\mathrm{IS}_{\wedge,0,1}(\mathcal{V}_{fsi}))^{\omega}$ is contained in $\mathcal{V}$.*
4. *The logic L is $(\wedge, 0, 1)$-stable.*

Proof:
This follows directly from Theorem 2.2.13 and Proposition 2.1.13. □

2.2.15. Remark. Note that if $\mathcal{V}$ is a finitely axiomatizable variety of Heyting algebras such that $\mathrm{IS}_{\wedge,0,1}(\mathcal{V}_{fsi}) \subseteq \mathcal{V}$ then $\mathcal{V}$ admits a structural hypersequent calculus given by only finitely many structural hypersequent rules. To see this simply note that as $\mathrm{IS}_{\wedge,0,1}(\mathcal{V}_{fsi}) \subseteq \mathcal{V}$ the variety $\mathcal{V}$ is axiomatized by a collection of $(\wedge, 0, 1)$-equations. Since $\mathcal{V}$ is finitely axiomatizable we may conclude that only finitely many of the $(\wedge, 0, 1)$-stable equations are required to axiomatize $\mathcal{V}$. Hence by Lemma 2.2.11 $\mathcal{V}$ is determined by a finite number of $(\wedge, 0, 1)$-stable clauses. Thus from the correspondence between $(\wedge, 0, 1)$-stable clauses and structural hypersequent rules we obtain that $\mathcal{V}$ indeed admits a structural hypersequent calculus given by only finitely many structural hypersequent rules.

2.3 Consequences of the characterization

We here present a few consequences of our analysis so far. Recall that a variety of Heyting algebras $\mathcal{V}$ is a $\mathcal{P}_3$-variety provided that it can be axiomatized by $\mathcal{P}_3$-equations. By Theorem 2.2.14 and Theorem 2.1.9 the $\mathcal{P}_3$-varieties are exactly the varieties of Heyting algebras satisfying any, and therefore all, of the equivalent conditions of Theorem 2.2.13.

2.3.1. PROPOSITION. *Any $\mathcal{P}_3$-variety is generated by its finite members, canonical, and elementarily determined.*

Proof:
That any $\mathcal{P}_3$-variety is generated by its finite members is a direct consequence of Theorem 2.2.13. By Proposition 2.1.11 $\mathcal{P}_3$-varieties can be axiomatized by $\vee$-free equations. As such they will be canonical as well as elementarily determined, see, e.g., [254, Thm. 6.8]. □

2.3.2. REMARK. Note that Proposition 2.3.1 entails that every finitely axiomatizable $\mathcal{P}_3$-variety is decidable, i.e., has a decidable equational theory. We do not know if proof search in the corresponding analytic structural calculi yields optimal bounds on the complexity of the decidability problem. However, uniform upper bounds on the complexity of the decidability problem for $\mathcal{P}_3$-varieties can likely be obtained, cf. [192, Sec. 6].

Recall from Appendix A.9 that $\mathcal{BD}_n$ denotes the variety of Heyting algebras corresponding to the intermediate logic BD_n determined by posets of depth at most n.

2.3.3. PROPOSITION. *Let $n \geq 2$ be given. The variety $\mathcal{BD}_n$ does not admit a structural hypersequent calculus.*

Proof:
We know that for $n \geq 2$ the variety $\mathcal{BD}_n$ is not $(\wedge, \vee, 0, 1)$-stably generated [25, Thm. 7.4(2)] and so in particular it cannot be $(\wedge, 0, 1)$-stably generated. Knowing this the proposition is an immediate consequence of Theorem 2.2.14. □

2.3.4. REMARK. That structural hypersequent rules could not capture $\mathcal{BD}_n$, for $n \geq 2$, had been expected, see., e.g., [61, 62]. However, we have not been able to find any proof of this fact in the literature.[2] The variety $\mathcal{BD}_2$ does, however, admit an analytic multi-succedent hypersequent calculus obtained by adding an

[2]Independently Lellmann has shown, using the relational semantics, that the logics BD_n, for $n \geq 2$, cannot be axiomatized by $\mathcal{P}_3$-formulas [193].

additional non-structural hypersequent rule for the introduction of the implication to a multi-succedent version of the calculus HLJ, see [61]. Furthermore, the varieties $\mathcal{BD}_n$, for $n \geq 2$, do admit analytic display calculi [62], analytic labeled sequent calculi [81] as well as so-called path-hypertableau and path-hypersequent calculi [54].

As a final application of Theorem 2.4.11 we show that for certain varieties of Heyting algebras the problem of whether or not they can be axiomatized by $\mathcal{P}_3$-equations, and therefore be given a cut-free structural hypersequent calculus, is decidable.

2.3.5. Proposition. *It is decidable whether a finitely axiomatized variety of Heyting algebras generated by its finite members is a $\mathcal{P}_3$-variety.*

Proof:
We show that there is an effective procedure which given a finite axiomatization of a variety of Heyting algebras $\mathcal{V}$ generated by its finite members decides whether $\mathcal{V}$ is a $\mathcal{P}_3$-variety. Note first that membership of the set $\mathcal{P}_3$ is a decidable property of terms.

Let $\mathcal{V}$ be a variety of Heyting algebra generated by its finite members which is moreover finitely axiomatizable. Without loss of generality we may assume that the given axiomatization of $\mathcal{V}$ consists of a single equation, say $1 \approx s$. We first note that $\mathcal{V}$, being finitely axiomatizable, is a $\mathcal{P}_3$-variety if, and only if, it is axiomatizable by finitely many $\mathcal{P}_3$-equations. Moreover, since the class of $\mathcal{P}_3$-terms is such that $t_1, t_2 \in \mathcal{P}_3$ entails $t_1 \wedge t_2 \in \mathcal{P}_3$ it follows that $\mathcal{V}$ is a $\mathcal{P}_3$-variety if, and only if, it is axiomatizable by a single $\mathcal{P}_3$-equation.

Since $\mathcal{V}$ is both finitely axiomatizable and generated by its finite members the equational theory of $\mathcal{V}$ is decidable. Therefore, we may enumerate all the $\mathcal{P}_3$-equations $1 \approx t$ such that $\mathcal{V} \models 1 \approx t$. Since finitely axiomatizable $\mathcal{P}_3$-varieties are decidable we may check for each $\mathcal{P}_3$-equation $1 \approx t$ such that $\mathcal{V} \models 1 \approx t$ whether $\mathcal{W} \models 1 \approx s$, where $\mathcal{W}$ is the variety axiomatized by the equation $1 \approx t$. It follows that for finitely axiomatizable and decidable varieties the property of being a $\mathcal{P}_3$-variety is semi-decidable.

On the other hand as $\mathcal{V}$ is finitely axiomatizable and since the property of being subdirectly irreducible is a decidable property of finite Heyting algebras, we may enumerate the finite subdirectly irreducible members of $\mathcal{V}$. Again, using the finite axiomatization of $\mathcal{V}$, for each finite subdirectly irreducible members of $\mathcal{V}$ we may check if all of its $(\wedge, 0, 1)$-subalgebras, of which there are only finitely many, belong to $\mathcal{V}$. If $\mathcal{V}$ is not a $\mathcal{P}_3$-variety then, since by assumption $\mathcal{V}$ is generated by its finite members, it follows from Theorem 2.2.13 that there must be some member of $\mathcal{V}^{\omega}_{fsi}$ having a $(\wedge, 0, 1)$-subalgebra which does not belong to $\mathcal{V}$. Consequently, for varieties of Heyting algebras which are both finitely axiomatizable and generated by its finite members the property of not being a $\mathcal{P}_3$-variety is semi-decidable.

Putting these observations together we see that given a finite axiomatization of a variety of Heyting algebras $\mathcal{V}$ generated by its finite members, it is decidable whether $\mathcal{V}$ is a $\mathcal{P}_3$-variety. $\square$

From Proposition 2.3.5 we then immediately obtain the following.

2.3.6. Corollary. *It is decidable whether a finitely axiomatized intermediate logic with the finite model property admits a cut-free structural hypersequent calculus.*

2.3.7. Remark. Note that by [51, Thm. 17.21] neither being decidable, having the finite model property (being generated by its finite members), or being axiomatizable by $\vee$-free formulas (equations) are decidable properties of finitely axiomatizable intermediate logics (varieties of Heyting algebras).

2.4 Poset-based characterization

As we have already seen, by Proposition 2.1.11 any $\mathcal{P}_3$-variety $\mathcal{V}$ is axiomatizable by $\vee$-free equations and hence *elementarily determined*, meaning that there exists a class $\mathcal{F}$ of posets first-order definable in the language of partial orders such that $\mathcal{V}$ is generated by the corresponding class of complex algebras $\mathcal{F}^+ = \{\mathbb{P}^+ : \mathbb{P} \in \mathcal{F}\}$. See Appendix A.6 for the relevant definitions. For an intermediate logic L the corresponding variety $\mathcal{V}(\mathsf{L})$ is elementarily determined if, and only if, the logic L is elementary, i.e., sound and complete with respect to a first-order definable class of posets.

It is known that $\vee$-free formulas have first-order correspondents which are Π_2, i.e., of the form $\forall\vec{x}\exists\vec{y}\Phi(\vec{x},\vec{y})$ with Φ a quantifier-free formula in the language of posets, see [222, §7.4] as well as [52, 53]. In this section we identify the Π_2-sentences which define the elementary classes of posets determining the $\mathcal{P}_3$-varieties, or equivalently the $(\wedge, 0, 1)$-stable logics. We do so using the duality theory for $(\wedge, 0, 1)$-homomorphisms between distributive lattices, see e.g., [32, 127, 110]. However, it will be enough to consider the finite case.

Given a relation $R \subseteq X \times Y$ between sets X and Y for $x \in X$ we write $R[x]$ for the set $\{y \in Y : xRy\}$ and for $Z \subseteq Y$ we write $R^{-1}[Z]$ for the set $\{x \in X : \exists z \in Z\ (xRz)\}$. Finally, given two relations $R_1 \subseteq X \times Y$ and $R_2 \subseteq Y \times Z$ we write $R_1 \circ R_2$ for the relational composition, viz., the relation $R \subseteq X \times Z$ given by

$$xRz \quad \text{if, and only if,} \quad \exists y \in Y\ (xR_1y \text{ and } yR_2z).$$

2.4.1. Definition (cf., e.g., [32, Def. 6.2]). Let $\mathbb{P} = \langle P, \leq_P\rangle$ and $\mathbb{Q} = \langle Q, \leq_Q\rangle$ be posets. A relation $R \subseteq P \times Q$ is called *order-compatible*, or a *generalized Priestley morphism*, provided that $\leq_P \circ R \circ \leq_Q = R$. Moreover, if $R^{-1}[Q] = P$ we say that R is *total* and we say that R is *onto* if for every $q \in Q$ there is $p \in P$ such that $R[p] = {\uparrow} q$.

The proof of the following lemma is straightforward.

2.4.2. Lemma. *Let $R \subseteq P \times Q$ be an order-compatible relation between posets $\mathbb{P}$ and $\mathbb{Q}$. If $p_1, p_2 \in P$ are such that $p_1 \leq_P p_2$, then $R[p_2] \subseteq R[p_1]$.*

We are interested in total order-compatible relations between posets because they correspond to $(\wedge, 0, 1)$-homomorphisms between the dual Heyting algebras. To be precise we have the following theorem.

2.4.3. Theorem ([32, Sec. 7 and Thm. 8.11], [110, Thm. 12]). *Let $\mathbf{A}$ and $\mathbf{B}$ be finite Heyting algebras with dual posets $\mathbb{P}$ and $\mathbb{Q}$, respectively. There is a one-to-one correspondence between $(\wedge, 0, 1)$-homomorphisms from $\mathbf{A}$ to $\mathbf{B}$ and total order-compatible relations $R \subseteq Q \times P$. Under this correspondence $(\wedge, 0, 1)$-embeddings correspond to onto total order-compatible relations.*

Recall from [203] that a *geometric axiom* is a first-order sentence in the language of partial orders of the form,

$$\forall \vec{w}\ (\varphi(\vec{w}) \implies \exists v\ \mathsf{OR}_{j=1}^{m} \psi_j(\vec{w}, v)),$$

with $\varphi, \psi_1, \ldots, \psi_m$ conjunctions of atomic formulas and the variable v not occurring free in φ. A *geometric implication* is then taken to be a finite conjunction of geometric axioms.[3]

2.4.4. Definition (cf. [188, Def. 2]). We say that a geometric axiom of the form

$$\forall \vec{w}\ (\varphi(\vec{w}) \implies \exists v\ \mathsf{OR}_{j=1}^{m} \psi_j(\vec{w}, v))$$

is *simple* if

(i) There exists $w_0 \in \vec{w}$ such that $\varphi(\vec{w})$ is the conjunction of the atomic formulas $w_0 \leq w$ with $w \in \vec{w}$ such that $w \neq w_0$,

(ii) Every atomic subformula of $\psi_j(\vec{w}, v)$ is of the form $w \leq v$ or $w = v$ for $w \in \vec{w}$ such that $w \neq w_0$.

A *simple geometric implication* is then a conjunction of simple geometric axioms.

2.4.5. Remark. Intermediate logics determined by a class of posets defined by geometric implications have been shown to admit so-called labeled sequent calculi [234, 203, 81]. Thus as a consequence of Proposition 2.4.10 below we obtain that any $(\wedge, 0, 1)$-stable logic admits a cut-free labeled sequent calculus. This is consistent with the existence of a translation of hypersequents into labeled sequents, see, e.g., [224] for an overview.

[3] This name is explained by the fact that any conjunction of geometric axioms is equivalent to a formula of the form $\forall \vec{x}(\varphi(\vec{x}) \implies \psi(\vec{x}))$, where φ and ψ are so-called *geometric formulas*, i.e., first-order formulas containing neither the connective "$\implies$" nor the quantifier "$\forall$". Conversely, any first-order formula of the form $\forall \vec{x}(\varphi(\vec{x}) \implies \psi(\vec{x}))$ with φ and ψ geometric formulas is equivalent to a geometric implication in the sense used here.

In the work of Lahav [188] and Lellmann [192, Sec. 6.1] on constructing analytic hypersequent calculi for modal logics, variants of the simple geometric implications play an important rule. They consider so-called *simple formulas*, viz., first-order formulas, in the language with a single binary relation symbol, of the form $\forall \vec{w} \exists v\ \psi(\vec{w}, v)$ where $\psi(\vec{w}, v)$ is a disjunction of conjunctions of atomic formulas of the form wRv or $w = v$. It is easy to verify that on rooted posets any simple formula is equivalent to a simple geometric implication and vice versa.

2.4.6. Example. The intermediate logics $\mathsf{BTW}_n, \mathsf{BW}_n, \mathsf{BC}_n$, for $n \geq 1$, are all complete with respect to an elementary class of posets determined by simple geometric implications. Furthermore the logics BD_n, for $n \geq 2$, are all complete with respect to an elementary class of posets determined by geometric implications, namely

$$\forall w_1 \ldots w_{n+1}(\mathsf{AND}_{i=1}^{n}(w_i \leq w_{i+1}) \implies \mathsf{OR}_{i \neq j}(w_i = w_j)),$$

which are ostensibly not simple.

2.4.7. Proposition. *Let $R \subseteq P \times Q$ be a total onto order-compatible relation between posets $\mathbb{P} = \langle P, \leq_P \rangle$ and $\mathbb{Q} = \langle Q, \leq_Q \rangle$, with $\mathbb{P}$ rooted. Then for any simple geometric implication γ, we have that $\mathbb{P} \models \gamma$ implies $\mathbb{Q} \models \gamma$.*

Proof:
Without loss of generality we may assume that γ is a simple geometric axiom, say, $\forall \vec{w}\ (\varphi(\vec{w}) \implies \exists v\ \mathsf{OR}_{j=1}^{m} \psi_j(\vec{w}, v))$.

Assume that $\mathbb{P} \models \gamma$. Suppose that $q_0, \ldots, q_n \in Q$ are such that the formula $\varphi(q_0, \ldots, q_n)$ holds in $\mathbb{Q}$, then by the assumption that R is an onto order-compatible relation there are $p_1, \ldots, p_n \in P$ such that $R[p_i] = {\uparrow} q_i$ for each $i \in \{1, \ldots, n\}$. Because $\mathbb{P}$ is rooted there is $p_0 \in P$ such that $\varphi(p_0, p_1, \ldots, p_n)$ holds and so since $\mathbb{P} \models \gamma$ there is $r \in P$ such that $\psi_j(\vec{p}, r)$ holds in $\mathbb{P}$ for some $j \in \{1, \ldots, m\}$.

We claim that there is an element $s \in Q$ such that $\psi_j(\vec{q}, s)$ holds in $\mathbb{Q}$. To establish this claim we consider separately the case where there are no subformulas of $\psi_j(\vec{w}, v)$ of the form $w = v$ and the case where there are subformulas of $\psi_j(\vec{w}, v)$ of this form.

In the first case, since R is total, we have $s \in Q$ such that rRs. We claim that $\psi_j(\vec{q}, s)$ holds in $\mathbb{Q}$. To see this, simply observe that if $p_i \leq_P r$ for some $i \in \{1, \ldots, n\}$ then by Lemma 2.4.2 we have that $s \in R[r] \subseteq R[p_i] = {\uparrow} q_i$, which implies that $q_i \leq_Q s$.

In the second case we have that there is at least one subformula of $\psi_j(\vec{w}, v)$ of the form $w = v$ and consequently at least one index, say i_0, in $\{1, \ldots, n\}$ such that $p_{i_0} = r$. Consequently, $R[r] = R[p_{i_0}] = {\uparrow} q_{i_0}$. We claim that $\psi_j(\vec{q}, q_{i_0})$ holds in $\mathbb{Q}$. To see this, as before, we observe that if $p_i \leq_P r$ for some $i \in \{1, \ldots, n\}$ then ${\uparrow} q_{i_0} = R[r] \subseteq R[p_i] = {\uparrow} q_i$ and hence $q_i \leq_Q q_{i_0}$. Similarly, if for some $i \in \{1, \ldots, n\}$ we have $p_i = r$, then ${\uparrow} q_i = R[p_i] = R[r] = {\uparrow} q_{i_0}$ and hence $q_i = q_{i_0}$.

Since the elements $q_0, \ldots, q_n \in Q$ were arbitrary with the property that $\varphi(q_0, \ldots, q_n)$ was true in $\mathbb{Q}$, this shows that $\mathbb{Q} \models \gamma$. $\square$

2.4.8. PROPOSITION. *Let $\mathcal{F}$ be a class of posets defined by simple geometric implications and let $\mathcal{F}^+ = \{\mathbb{P}^+ : \mathbb{P} \in \mathcal{F}\}$ be the corresponding set of Heyting algebras. Then the variety of Heyting algebras generated by the class $\mathcal{F}^+$ is $(\wedge, 0, 1)$-stably generated.*

Proof:
If $\mathbb{P} \in \mathcal{F}$ and $\mathbb{Q}$ is a subposet of $\mathbb{P}$ with $\uparrow_{\mathbb{P}} Q = Q$, then evidently $\mathbb{Q}$ satisfies any simple geometric implication satisfied by $\mathbb{P}$. Consequently, the variety $\mathcal{V}$ must in fact be determined by the class $\mathcal{F}_r$ of rooted posets belonging to $\mathcal{F}$. Furthermore, because on rooted posets any (simple) geometric implication is equivalent to a positive first-order formula in the language of partial orders we easily see that for any rooted poset $\mathbb{P} \in \mathcal{F}$ and any order-preserving surjection $f : \mathbb{P} \twoheadrightarrow \mathbb{Q}$ we have that $\mathbb{Q} \in \mathcal{F}$. Consequently, by Lemma A.4.4 and discrete duality we have that any equation refuted by some algebra $\mathbb{P}^+$ with $\mathbb{P} \in \mathcal{F}_r$ is in fact also refuted by some algebra $\mathbb{Q}^+$ with $\mathbb{Q}$ a finite member of $\mathcal{F}_r$. It follows that the variety $\mathcal{V}$ is generated by the class of finite members of $\mathcal{F}_r^+$, i.e., the class $(\mathcal{F}_r^+)^\omega$. Finally, letting $\mathcal{K} := \{\mathbf{A} : \exists \mathbf{B} \in (\mathcal{F}_r^+)^\omega (\mathbf{A} \hookrightarrow_{\wedge,0,1} \mathbf{B})\}$ it follows from Proposition 2.4.7 that $\mathcal{K}$ is a $(\wedge, 0, 1)$-stable class of Heyting algebras generating $\mathcal{V}$. $\square$

To establish the converse of Corollary 2.4.8 we need the following lemma which is essentially an exercise in correspondences theory, see, e.g., [228, 222, 68].

2.4.9. LEMMA. *For any analytic structural universal clause q there exists a simple geometric axiom γ_q such that*

$$\mathbb{P} \models \gamma_q \iff \mathbb{P}^+ \models q,$$

for every rooted poset $\mathbb{P}$.

Proof:
Consider an analytic structural universal clause

$$\varepsilon_1 \textsf{ and } \ldots \textsf{ and } \varepsilon_m \implies \varepsilon_{m+1} \textsf{ or } \ldots \textsf{ or } \varepsilon_n, \tag{q}$$

with ε_k denoting the equation $s_k(\vec{x}) \leq t_k(\vec{y})$, for $k \in \{1, \ldots, n\}$. By the assumption that q is analytic we have that each term s_k is a, possibly empty, meet of left variables and each term t_k either 0 or a single right variable. Furthermore, the set $\vec{x} = \{x_1, \ldots, x_l\}$ of left variables and the set $\vec{y} = \{y_1, \ldots, y_r\}$ of right variables are disjoint and each left variable occurs exactly once in exactly one of the terms s_j and similarly each right variable occurs exactly once in exactly one of the terms t_j, with $j \in \{m+1, \ldots, n\}$.

In the following we write $Pr(\vec{x}, \vec{y})$ for the left-hand side of the clause q, and for $k \in \{1, \ldots, n\}$ we let $x_{k_1}, \ldots, x_{k_{r_k}}$ denote the variables occurring in the term $s_k(\vec{x})$, if any, and let y_{k_0} denote the variable occurring in the term $t_k(\vec{y})$, if any.

Let $\mathbb{P}$ be a poset. With the notation above we then have that $\mathbb{P}^+ \not\models q$ if, and only if, there are upsets $U_1, \ldots, U_l$ and $V_1, \ldots, V_r$ of $\mathbb{P}$ such that

$$s_i^{\mathbb{P}^+}(\vec{U}) \subseteq t_i^{\mathbb{P}^+}(\vec{V}), \tag{2.1}$$

for all $i \in \{1, \ldots, m\}$, and

$$s_j^{\mathbb{P}^+}(\vec{U}) \not\subseteq t_j^{\mathbb{P}^+}(\vec{V}), \tag{2.2}$$

for all $j \in \{m+1, \ldots, n\}$. We see that Property 2.2 holds if, and only if, there is $p_j \in P$ such that $p_j \in s_j^{\mathbb{P}^+}(\vec{U})$ and $p_j \notin t_j^{\mathbb{P}^+}(\vec{V})$. Since both $s_j^{\mathbb{P}^+}(\vec{V}) = \bigcap_{d=1}^{r_j} U_{j_d}$ and $t_j^{\mathbb{P}^+}(\vec{U})$ are upsets of $\mathbb{P}$, this in turn is equivalent to the existence of some $p_j \in P$ such that ${\uparrow} p_j \subseteq s_j^{\mathbb{P}^+}(\vec{U})$ and $t_j^{\mathbb{P}^+}(\vec{V}) \subseteq ({\downarrow} p_j)^{\mathrm{c}}$. Unraveling the definitions of the terms s_j and t_j we obtain that Property 2.2 is equivalent to the existence of a $p_j \in P$ such that

$${\uparrow} p_j \subseteq U_{j_1} \quad \text{and} \quad \ldots \quad \text{and} \quad {\uparrow} p_j \subseteq U_{j_{r_j}} \quad \text{and} \quad V_{j_0} \subseteq ({\downarrow} p_{j_0})^{\mathrm{c}}.$$

The special syntactic shape of the clause q ensures that the left variables $\vec{x}$ only occur negatively in $Pr(\vec{x}, \vec{y})$ and that the right variables $\vec{y}$ only occur positively in $Pr(\vec{x}, \vec{y})$. Moreover, every variable among $\vec{x}, \vec{y}$ occurs exactly once somewhere on the right-hand side of the clause q. This implies that, *salva veritate*, we may substitute ${\uparrow} p_j$ for U_{j_d} and $({\downarrow} p_{j_0})^{\mathrm{c}}$ for V_{j_0} everywhere in Equation 2.1. This is essentially an application of the Ackermann Lemma, see, e.g., [68, Sec. 4] for the version used here. From this we may conclude, with some renaming, that $\mathbb{P}^+ \not\models q$ if, and only if, there are $p_{m+1}, \ldots, p_n \in P$ such that

$$\bigcap_{d=1}^{r_i} {\uparrow} p_{i_d} \subseteq ({\downarrow} p_{i_0})^{\mathrm{c}},$$

for each $i \in \{1, \ldots, m\}$. Consequently, $\mathbb{P}^+ \models q$ if, and only if, for all elements $p_{m+1}, \ldots, p_n \in P$ there is some $i \in \{1, \ldots, m\}$ such that

$$\bigcap_{d=1}^{r_i} {\uparrow} p_{i_d} \not\subseteq ({\downarrow} p_{i_0})^{\mathrm{c}}.$$

We easily see that $\bigcap_{d=1}^{r_i} {\uparrow} p_{i_d} \not\subseteq ({\downarrow} p_{i_0})^{\mathrm{c}}$ precisely when there is $q \in P$ such that $p_{i_d} \leq q$ for all $d \in \{1, \ldots, r_i\}$ and $q \leq p_{i_0}$. We therefore obtain that

$$\mathbb{P}^+ \models q \iff \mathbb{P} \models \forall \vec{w} \exists v \ \mathsf{OR}_{i=1}^{m} (\mathsf{AND}_{d=1}^{r_i} (w_{i_d} \leq v) \ \mathsf{and} \ v \leq w_{i_0}).$$

Thus q is equivalent to a formula in the first-order language of posets. In fact, it is easy to see that the formula

$$\forall \vec{w} \exists v \ \mathsf{OR}_{i=1}^{m} (\mathsf{AND}_{d=1}^{r_i} (w_{i_d} \leq v) \ \mathsf{and} \ v \leq w_{i_0}),$$

is equivalent to the formula

$$\forall \vec{w} \exists v \ \mathsf{OR}_{i=1}^{m}(\mathsf{AND}_{d=1}^{r_i}(w_{i_d} \leq v) \text{ and } v = w_{i_0}).$$

Thus we obtain a formula $\psi(\vec{w}, v)$, which is a disjunction of conjunctions of atomic formulas of the form $w \leq v$ and $w = v$, such that

$$\mathbb{P}^+ \models q \iff \mathbb{P} \models \forall \vec{w} \exists v \ \psi(\vec{w}, v).$$

Finally, letting γ_q be the formula $\forall w_0 \forall \vec{w} \ (\mathsf{AND}_{w \in \vec{w}}(w_0 \leq w) \implies \exists v \ \psi(\vec{w}, v))$, for w_0 some fresh first-order variable, we obtain a simple geometric axiom such that γ_q is equivalent to q on rooted posets. □

2.4.10. Proposition. *Any variety of Heyting algebras generated by a $(\wedge, 0, 1)$-stable universal class of Heyting algebras is elementarily determined by a class of posets defined by simple geometric implications.*

Proof:
Let $\mathcal{V}$ be a variety of Heyting algebras generated by a $(\wedge, 0, 1)$-stable universal class, say $\mathcal{U}$. By Lemma 2.2.7 is axiomatized by a collection of structural clauses, say $\{q_i\}_{i \in I}$, which by Lemma 2.1.7 we may assume to be analytic. By an argument completely similar to the one presented in the proof of Proposition 2.4.8, $\mathcal{V}$ will be generated by the class $\mathcal{F}^+ := \{\mathbb{F}^+ \colon \forall i \in I \ (\mathbb{F} \models \gamma_{q_i})\}$, where γ_i is the simply geometric implication corresponding to q_i obtained from Lemma 2.4.9. □

We summarize our findings by amending Theorem 2.1.9 with additional items.

2.4.11. Theorem. *Let $\mathcal{V}$ be a variety of Heyting algebras and let L be its corresponding logic. Then the following are equivalent.*

1. *The variety $\mathcal{V}$ admits a structural hypersequent calculus.*
2. *The variety $\mathcal{V}$ admits an analytic structural hypersequent calculus.*
3. *The logic L admits a structural hypersequent calculus.*
4. *The logic L admits an analytic structural hypersequent calculus.*
5. *The variety $\mathcal{V}$ is axiomatizable by $\mathcal{P}_3$-equations.*
6. *The logic L is axiomatizable by $\mathcal{P}_3$-formulas.*
7. *The class $(\mathrm{IS}_{\wedge,0,1}(\mathcal{V}_{fsi}))^{\omega}$ is contained in $\mathcal{V}$.*
8. *The logic L is $(0, \wedge, 1)$-stable.*
9. *The variety $\mathcal{V}$ is determined by a class of posets defined by simple geometric implications.*
10. *The logic L is sound and complete with respect to a class of posets defined by simple geometric implications.*

2.5 Comparison with $(\wedge, \vee, 0, 1)$-stable logics

What we have done so far shows, perhaps somewhat surprisingly, that the seemingly very restrictive notion of $(\wedge, 0, 1)$-stability yields interesting and, indeed well-known, varieties of Heyting algebra, or equivalently, intermediate logics. In this section we will compare the class of $(\wedge, 0, 1)$-stable intermediate logics to the class of $(\wedge, \vee, 0, 1)$-stable intermediate logics.

2.5.1. PROPOSITION. *The set of* $(\wedge, 0, 1)$*-stable logics is a proper subset of the set of* $(\wedge, \vee, 0, 1)$*-stable logics.*

Proof:
Evidently each $(\wedge, 0, 1)$-stable logic is also a $(\wedge, \vee, 0, 1)$-stable logic. To show that there exist $(\wedge, \vee, 0, 1)$-stable logics which are not $(\wedge, 0, 1)$-stable, consider the following pair of Heyting algebras

A **B**

We easily see that $\mathbf{A}$ is isomorphic to a $(\wedge, 0, 1)$-subalgebra of $\mathbf{B}$ but not a $(\wedge, \vee, 0, 1)$-subalgebra of $\mathbf{B}$. Let $\mathcal{V}$ be the variety axiomatized by the $(\wedge, \vee, 0, 1)$-stable equation $\varepsilon_{\wedge,\vee,0,1}(\mathbf{A})$ associated with $\mathbf{A}$. Then the intermediate logic L corresponding to this variety is $(\wedge, \vee, 0, 1)$-stable [29, Prop. 5.3]. Since $\mathbf{B}$ is well-connected and $\mathbf{A} \not\rightarrowtail_{\wedge,\vee,0,1} \mathbf{B}$, we may conclude that $\mathbf{B}$ belongs to $\mathcal{V}$ [29, Prop. 5.1]. Consequently, assuming that L is $(\wedge, 0, 1)$-stable $\mathbf{A}$ must also belong to $\mathcal{V}$. But then $\mathbf{A} \models \varepsilon_{\wedge,\vee,0,1}(\mathbf{A})$ which, since any finite well-connected Heyting algebra refutes its own $(\wedge, \vee, 0, 1)$-stable equation [29, Prop. 5.1], is a contradiction. □

Despite the fact that there are $(\wedge, \vee, 0, 1)$-stable logics which are not $(\wedge, 0, 1)$-stable, all the examples of $(\wedge, \vee, 0, 1)$-stable logics considered so far [25, Sec. 7] are in fact $(\wedge, 0, 1)$-stable. The following theorem may be seen as explaining why this indeed the case. Furthermore, this also provides us with examples of $(\wedge, \vee, 0, 1)$-stable logics which are not $(\wedge, 0, 1)$-stable.

2.5.2. THEOREM. *For* $\mathbf{A}$ *a finite subdirectly irreducible Heyting algebra the following are equivalent.*

1. *The $(\wedge, \vee, 0, 1)$-stable clause $q_{\wedge,\vee,0,1}(\mathbf{A})$ associated with $\mathbf{A}$ is equivalent to a collection of $(\wedge, 0, 1)$-stable clauses.*

2. *The $(\wedge, \vee, 0, 1)$-stable clause $q_{\wedge,\vee,0,1}(\mathbf{A})$ associated with $\mathbf{A}$ is equivalent to the $(\wedge, 0, 1)$-stable clause $q_{\wedge,0,1}(\mathbf{A})$ associated with $\mathbf{A}$.*

3. *The lattice reduct of $\mathbf{A}$ is projective in the class of distributive lattices.*

Proof:
Evidently Item 2 entails Item 1. Conversely, to see that Item 1 entails Item 2, it suffices, due to Lemma 2.2.6, to show for any Heyting algebra $\mathbf{B}$ that

$$\mathbf{A} \hookrightarrow_{\wedge,0,1} \mathbf{B} \quad \text{if, and only if,} \quad \mathbf{A} \hookrightarrow_{\wedge,\vee,0,1} \mathbf{B}.$$

Since $\{\wedge, 0, 1\} \subseteq \{\wedge, \vee, 0, 1\}$, for each Heyting algebra $\mathbf{B}$, we evidently have

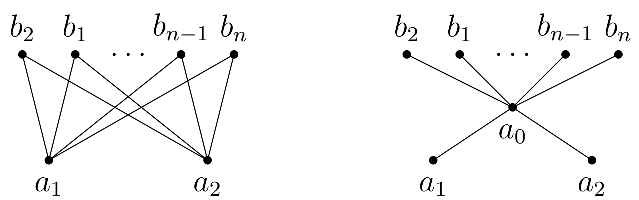
$$\mathbf{A} \hookrightarrow_{\wedge,\vee,0,1} \mathbf{B} \quad \text{implies} \quad \mathbf{A} \hookrightarrow_{\wedge,0,1} \mathbf{B}.$$

To establish the converse let $\mathbf{B}$ be given and suppose that $\mathbf{A} \hookrightarrow_{\wedge,0,1} \mathbf{B}$, say via $h\colon A \hookrightarrow B$. If $\mathbf{A} \not\hookrightarrow_{0,\wedge,\vee 1} \mathbf{B}$ then $\mathbf{B} \models q_{\wedge,\vee,0,1}(\mathbf{A})$ and so since, by assumption, $q_{\wedge,\vee,0,1}(\mathbf{A})$ is equivalent to collection of $(\wedge, 0, 1)$-stable clauses and such clauses are preserved by $(\wedge, 0, 1)$-embeddings we must have that $\mathbf{A} \models q_{\wedge,\vee,0,1}(\mathbf{A})$ which is a contradiction as every Heyting algebra refutes all of the stable clauses associated with it.

To see that Item 2 entails Item 3 suppose that the lattice reduct of $\mathbf{A}$ is not projective in the class of distributive lattices. We exhibit a Heyting algebra $\mathbf{B}$ such that

$$\mathbf{A} \hookrightarrow_{\wedge,0,1} \mathbf{B} \quad \text{and} \quad \mathbf{A} \not\hookrightarrow_{\wedge,\vee,0,1} \mathbf{B},$$

showing that the universal clauses $q_{\wedge,0,1}(\mathbf{A})$ and $q_{\wedge,\vee,0,1}(\mathbf{A})$ are not equivalent. To this effect let $\mathbb{P} := J(\mathbf{A})^{\partial}$, be the order dual of $J(\mathbf{A})$, the poset of join-irreducible elements of $\mathbf{A}$. Note that as $\mathbf{A}$ is finite the Heyting algebra $\mathbb{P}^{+}$ of upsets of $\mathbb{P}$ is isomorphic to $\mathbf{A}$. By Theorem A.7.1, the lattice reduct of $\mathbf{A}$ not being projective entails the existence of $a_1, a_2 \in J(A)$ such that $a_1 \wedge a_2 \notin J_0(A) = J(A) \cup \{0\}$, in particular a_1 and a_2 must be incomparable. Let $b_1, \ldots, b_n$ be an anti-chain of maximal join-irreducible elements below $a_1 \wedge a_2$, such that $\bigvee_{i=1}^{n} b_i = a_1 \wedge a_2$ in $\mathbf{A}$. Necessarily, $n \geq 2$. Given this, let $\mathbb{P}_0$ be the poset obtained from $\mathbb{P}$ by adding a new element a_0 covering a_1, a_2 and covered by $b_1, \ldots, b_n$ as shown below



Thus $|P_0| = |P| + 1$. Evidently there can be no order-preserving surjection from $\mathbb{P}_0$ onto $\mathbb{P}$. Letting $\mathbf{B}$ denote the dual Heyting algebra $\mathbb{P}_0^+$ of $\mathbb{P}_0$, this shows that $\mathbf{A} \not\hookrightarrow_{\wedge,\vee,0,1} \mathbf{B}$. We claim, however, that $\mathbf{A} \hookrightarrow_{\wedge,0,1} \mathbf{B}$. To see this, observe that $\mathbb{P}$ is a subposet of $\mathbb{P}_0$, whence by duality we have a surjective map $h\colon \mathbf{B} \twoheadrightarrow_{\wedge,\vee,0,1} \mathbf{A}$. Moreover, we may easily verify that $h^{-1}(0) = \{0\}$ and $h^{-1}(1) = \{1\}$. Being a finite distributive lattice the meet semi-lattice reduct of $\mathbf{A}$ is projective as a meet semi-lattice by Theorem A.7.4. Therefore, we obtain a meet semi-lattice homomorphism $\overline{h}\colon \mathbf{A} \to \mathbf{B}$ such that $h \circ \overline{h}$ is the identity on $\mathbf{A}$. In particular, $\overline{h}$ must be injective. Moreover, as $h(\overline{h}(0)) = 0$ and $h(\overline{h}(1)) = 1$ it follows that $\overline{h}(0) = 0$ and $\overline{h}(1) = 1$. Thus we have $\overline{h}\colon \mathbf{A} \hookrightarrow_{\wedge,0,1} \mathbf{B}$.

To see that Item 3 entails Item 2 suppose that the lattice reduct of $\mathbf{A}$ is projective in the class of distributive lattices. We claim that the $(\wedge,\vee,0,1)$-stable clause $q_{\wedge,\vee,0,1}(\mathbf{A})$ associated with $\mathbf{A}$ is equivalent to the $(\wedge,0,1)$-stable clause $q_{\wedge,0,1}(\mathbf{A})$ associated with $\mathbf{A}$. As before it suffices to show that $\mathbf{A} \hookrightarrow_{\wedge,0,1} \mathbf{B}$ implies $\mathbf{A} \hookrightarrow_{\wedge,\vee,0,1} \mathbf{B}$. This however follows immediately from Corollary A.7.3. $\square$

2.5.3. Remark. Theorem 2.5.2 can be seen as explaining why all of the examples of $(\wedge,\vee,0,1)$-stable logics considered in [25, Sec. 7] are in fact $(\wedge,0,1)$-stable logics, as all of these logics are axiomatized by $(\wedge,\vee,0,1)$-stable equations associated with finite well-connected Heyting algebras the lattice reducts of which are projective in the class of distributive lattices.

We conclude this section by showing that the $(\wedge,0,1)$-stable logics can be characterized among the $(\wedge,\vee,0,1)$-stable logics as the ones which can be axiomatized by $\vee$-free formulas. For this we will need the following definition.

2.5.4. Definition. A poset $\mathbb{Q}$ is *cofinal in* a poset $\mathbb{P}$ provided that $\mathbb{Q}$ is a subposet of $\mathbb{P}$ and for all $p \in P$ there is $q \in Q$ with $p \leq q$.

We will make use of the following well-known characterization of intermediate logics axiomatizable by $\vee$-free formulas.

2.5.5. Theorem ([254, Thm. 5.7(ii)]). *Let* L *be an intermediate logic. Then the following are equivalent.*

1. *The logic* L *is axiomatizable by* $\vee$*-free formulas.*

2. *For all posets* $\mathbb{P}$ *and* $\mathbb{Q}$ *with* $\mathbb{Q}$ *cofinal in* $\mathbb{P}$*, if* $\mathbb{P} \Vdash \mathsf{L}$*, then* $\mathbb{Q} \Vdash \mathsf{L}$.

Intermediate logics satisfying either of the equivalent conditions of Theorem 2.5.5 are called *cofinal subframe logics* see, e.g., [51, Chap. 11.3]. They enjoy many good properties such as the finite model property, canonicity, and elementarity.

2.5.6. Lemma. *Let $R \subseteq P \times Q$ be an order-compatible relation between finite posets $\mathbb{P} = \langle P, \leq_P \rangle$ and $\mathbb{Q} = \langle Q, \leq_Q \rangle$. Then there is a poset $\mathbb{P}_1$ cofinal in $\mathbb{P}$ and an order-preserving surjection $f : \mathbb{P}_1 \twoheadrightarrow \mathbb{Q}$.*

Proof:
Let $P_0 = \{p \in P : \exists q \in Q \ (R[p] = {\uparrow}q)\}$ and let $\mathbb{P}_0$ be the corresponding subposet of $\mathbb{P}$. Then we see that mapping each $p \in P_0$ to the necessarily unique element $q \in Q$ such that $R[p] = {\uparrow}q$ determines a map $g : P_0 \to Q$ which must be surjective as R is onto. Furthermore, because R is order-compatible we have by Lemma 2.4.2, that $p_1 \leq_P p_2$ implies $R[p_2] \subseteq R[p_1]$ and consequently that g is an order-preserving map from the poset $\mathbb{P}_0$ to the poset $\mathbb{Q}$.

Let $P_1 = P_0 \cup \max(P)$ and let $\mathbb{P}_1$ be the corresponding subposet of $\mathbb{P}$. Evidently $\mathbb{P}_1$ is cofinal in $\mathbb{P}$. We then note that for any $p \in \max(P)$, since R is total, we have $q \in Q$ such that pRq. Moreover, if for some $p_0 \in P_0$ we have $p_0 \leq_P p$ then by Lemma 2.4.2, $R[p] \subseteq R[p_0] = {\uparrow}g(p_0)$. This shows that the map $f : P_0 \cup \max(P) \to Q$ defined by letting $f(p) = g(p)$, if $p \in P_0$ and letting $f(p)$ be some element of $R[p]$ if $p \in \max(P) \backslash P_0$ is a well-defined and order-preserving surjection from $\mathbb{P}_1$ to $\mathbb{Q}$. □

2.5.7. Lemma. *Let $R \subseteq P \times Q$ be an order-compatible relation between finite posets $\mathbb{P} = \langle P, \leq_P \rangle$ and $\mathbb{Q} = \langle Q, \leq_Q \rangle$, with $\mathbb{P}$ rooted. Then there is a rooted poset $\mathbb{P}_1$ cofinal in $\mathbb{P}$ and an order-preserving surjection $f : \mathbb{P}_1 \twoheadrightarrow \mathbb{Q}_1$ such that $\mathbb{Q}$ is cofinal in the poset $\mathbb{Q}_1$.*

Proof:
From Lemma 2.5.6 we know that there is a poset $\mathbb{P}_1$ cofinal in $\mathbb{P}$ together with an order-preserving surjective $f : \mathbb{P}_1 \twoheadrightarrow \mathbb{Q}$. Let r be the root of $\mathbb{P}$. If $r \in P_1$ then the proposition follows. If r is not in P_1, then letting $P_1' := P_1 \cup \{r\}$ we obtain a rooted poset $\mathbb{P}_1'$ cofinal in $\mathbb{P}$. Similarly, by adjoining a new root s to Q we obtain a rooted poset $\mathbb{Q}_1$ in which $\mathbb{Q}$ is cofinal. Evidently, the map f extends to a surjective order-preserving map from $\mathbb{P}_1'$ to $\mathbb{Q}_1$ by mapping r to s. □

2.5.8. Proposition. *Let L be an intermediate logic. Then the following are equivalent.*

1. *L is $(\wedge, 0, 1)$-stable.*
2. *L is a $(\wedge, \vee, 0, 1)$-stable and axiomatizable by $\vee$-free formulas.*

Proof:
Every $(\wedge, 0, 1)$-stable logic is evidently $(\wedge, \vee, 0, 1)$-stable. Furthermore, by Proposition 2.1.11 every $(\wedge, 0, 1)$-stable logic can be axiomatized by $\vee$-free formulas.

Conversely, suppose that L is a $(\wedge, \vee, 0, 1)$-stable logic which can be axiomatized by $\vee$-free formulas. We show that if $\mathbf{A}$ and $\mathbf{B}$ are finite Heyting algebras with $\mathbf{B} \in \mathcal{V}(\mathsf{L})_{fsi}$ and $\mathbf{A} \hookrightarrow_{\wedge,0,1} \mathbf{B}$ then $\mathbf{A} \in \mathcal{V}(\mathsf{L})$. Therefore, let $\mathbb{P}$ be the dual poset of $\mathbf{B}$ and let $\mathbb{Q}$ be the dual poset of $\mathbf{A}$. Since $\mathbf{A} \hookrightarrow_{\wedge,0,1} \mathbf{B}$ we have, by Theorem 2.4.3 a total and onto order-compatible relation $R \subseteq P \times Q$. Moreover, since the algebra $\mathbf{B}$ is subdirectly irreducible we have that the poset $\mathbb{P}$ is rooted and hence by Lemma 2.5.7 that $\mathbb{Q}$ is cofinal in an image $\mathbb{Q}_1$ under an order-preserving map of a rooted poset $\mathbb{P}_1$ cofinal in $\mathbb{P}$. By Theorem 2.5.5 we must have that $\mathbb{P}_1 \Vdash \mathsf{L}$. Moreover, since $\mathbb{P}_1$ is rooted and L is $(\wedge, \vee, 0, 1)$-stable we obtain that $\mathbb{Q}_1 \Vdash \mathsf{L}$, see [25, Thm. 6.7], and so, again using Theorem 2.5.5, we see that $\mathbb{Q} \Vdash \mathsf{L}$. We may therefore conclude that $\mathbf{A} \in \mathcal{V}(\mathsf{L})$, as desired. Since L is $(\wedge, \vee, 0, 1)$-stable we have that $\mathcal{V}(\mathsf{L})$ is generated by its finite members [25, Thm. 6.8]. Thus by Theorem 2.2.13 it follows that the logic L is $(\wedge, 0, 1)$-stable. $\Box$

2.5.9. Remark. It is known that there are continuum-many $(\wedge, \vee, 0, 1)$-stable logics [25, Thm. 6.13] just as it is known that there are continuum-many logics axiomatized by $\vee$-free formulas [51, Thm. 11.19]. However, it is not immediately clear if the techniques used to establish these two results can also be used to construct continuum-many $(\wedge, 0, 1)$-stable logics. Thus we leave as open the problem of determining the cardinality of the set of $(\wedge, 0, 1)$-stable logics.

2.6 Summary and concluding remarks

In this chapter we have looked at intermediate logics admitting structural hypersequent calculi. Using the correspondence between (analytic) structural hypersequent rules and (analytic) structural universal clauses we have shown that the intermediate logics which admit an analytic structural hypersequent calculus are precisely the $(\wedge, 0, 1)$-stable logics. This supplements the previous syntactic characterization of logics admitting structural hypersequent calculi [57, 60] with a purely algebraic characterization. Our semantic characterization has also allowed us to obtain negative results showing that certain logics, such as BD_n, for $n \geq 2$, do not admit a structural hypersequent calculus, let alone an analytic structural hypersequent calculus. We have also provided a characterization of the $(\wedge, 0, 1)$-stable logics in terms of the relational semantics. In particular, we have shown that any $(\wedge, 0, 1)$-stable logic is sound and complete with respect to a class of poset defined by certain Π_2-sentences which we called simple geometric implications. We have also remarked how the simple geometric implications are related to the simple sentences appearing in the work of Lahav [188] and Lellmann [192, Sec. 6.1] concerned with constructing analytic hypersequent calculi for modal logics. Finally, we have compared the $(\wedge, 0, 1)$-stable logics to the $(\wedge, \vee, 0, 1)$-stable logics and shown that the $(\wedge, 0, 1)$-stable logics are exactly the $(\wedge, \vee, 0, 1)$-stable logics which can be axiomatized by $\vee$-free formulas.

Further directions and open problems As mentioned earlier, the work of Ciabattoni, Galatos, and Terui [57, 59, 60], on which the findings in this chapter rely, is carried out in the much more general context of substructural logics. It is therefore natural to ask if a semantic characterization of substructural logics admitting structural (analytic) sequent and hypersequent calculi similar to the ones presented in this chapter can also be found for substructural logics. However, the fact that the comma in the left-hand side of the sequent arrow is interpreted in a (pointed) residuated lattice as the monoidal product and not as the meet raises a number of non-trivial technical issues. Nevertheless, one could hope that some appropriately modified version of $(\cdot, \vee, \mathrm{e})$-stability introduced by Bezhanishvili, Galatos, and Spada [35, Sec. 4] would also play a role in the substructural setting.

It is also natural to ask for semantic characterizations of intermediate logics admitting different types of proof calculi, such as structural display calculi, where good syntactic characterizations already exist [62, 141, 140], or labeled sequent calculi where good descriptions of the first-order theory of the corresponding posets are available [203, 81].

As we have seen there is a close connection between the $(\wedge, 0, 1)$-stable intermediate logics and the modal logics considered by Lahav [188] and Lellmann [192, Sec. 6.1]. It would be interesting to explore this connection further, e.g., by providing a purely semantic characterization of the modal logics they consider. Furthermore, this connections also indicates that the cut-admissibility argument using the relational semantics due to Lahav should have a counterpart in the setting of intermediate logics, and similarly, that the algebraic cut-admissibility arguments used by Ciabattoni, Galatos, Terui, and others should have a counterpart in the setting of modal logics, cf. [21, Sec. 6] for an algebraic proof of cut-admissibility in sequent calculi for a few modal logics.

Finally, we believe that it would be worthwhile to systematically explore the consequences of having a cut-free structural hypersequent calculus. For instance: (i) Conservativity results showing that certain fragments of two logics coincide. (ii) Interpolation results using the calculi to extract procedures for computing interpolants and ideally allowing for further analysis of their complexity, see [186, Sec. 3] and [187] for existing work along these lines. (iii) Complexity results such as obtaining uniform upper, and ideally optimal, bounds on the complexity of proof search as, e.g., already considered in the setting of modal logic [192, Sec. 6].

Chapter 3

MacNeille transferability

As we have outlined in the introduction, there is an interesting link between completions of lattice-based algebras and proof theory. For example, analytic structural hypersequent calculi give rise to universal classes of lattice-based algebras closed under MacNeille completions. In particular, if $\mathcal{V}$ is any variety of Heyting algebras axiomatized by $\mathcal{P}_3$-equations then the universal class $\mathcal{V}_{fsi}$ of its finitely subdirectly irreducible members is closed under MacNeille completions [58, Thm. 4.3]. This is particularly interesting since the only non-trivial varieties of Heyting algebras closed under MacNeille completions are the variety of all Heyting algebras and the variety of all Boolean algebras [151]. Drawing on the connection between the level $\mathcal{P}_3$ and the notion of stability established in Chapter 2, in this chapter, based on [31], we look, from a purely algebraic point of view, at the phenomenon of universal classes of lattices being closed under MacNeille completions.

Our starting point is the notion of (ideal) transferability originally introduced by Grätzer [133, Sec. 10(ii)]. A finite lattice $\mathbf{L}$ is (ideal) transferable if for any lattice $\mathbf{K}$, the lattice $\mathbf{L}$ is a sublattice of the lattice of ideals of $\mathbf{K}$ only if $\mathbf{L}$ is a sublattice of $\mathbf{K}$. We introduce analogous notions of MacNeille and canonical transferability and show how finite transferable lattices give rise to universal classes of lattices which are closed under completions. Thus the problem of finding universal classes of lattice-based algebras closed under MacNeille completions can in some cases be reduced to the problem of finding finite MacNeille transferable lattices.

While we are mainly interested in MacNeille transferability, we also explore the relationships between ideal, MacNeille, and canonical transferability. Concretely, we show that under mild assumptions MacNeille transferability entails canonical transferability which in turn entails ideal transferability.

We provide necessary conditions for a finite lattice to be MacNeille transferable for the class of all lattices. In particular any such lattice must be distributive. This highlights some of the crucial differences between ideal and MacNeille transferability. Nevertheless, we show that, just as in the case of ideal transferability, the concept of (weak) projectivity plays an important role in understanding the concept of MacNeille transferability.

Using the connection between MacNeille transferability and projectivity, we obtain an alternative proof of the fact, first established by purely syntactic methods, that if $\mathcal{V}$ is any variety of Heyting algebras axiomatized by $\mathcal{P}_3$-equations then the universal class $\mathcal{V}_{fsi}$ is closed under MacNeille completions.

We then focus on MacNeille transferability with respect to the class of Heyting algebras and the class of bi-Heyting algebras. In this setting we are able to say much more about necessary and sufficient conditions for different types of MacNeille transferability. In particular, we show that all finite distributive lattices are MacNeille transferable with respect to the class of bi-Heyting algebras.

Finally, we discuss how canonical and MacNeille transferability of finite distributive lattices relate to intermediate logics. In particular we consider the problem of whether all $(\wedge, \vee, 0, 1)$-stable logics are (i) canonical, and (ii) elementary [164, Chap. 3]. In this respect a number of partial results of a positive nature are obtained.

Outline The chapter is structured as follows: In Section 3.1 we give a brief summary of the theory of (ideal) transferability. Then in Section 3.2 we introduce a general notion of transferability and compare different notions of transferability of finite lattices. Section 3.3 is concerned with MacNeille transferability of lattices while Section 3.4 focuses on MacNeille transferability relative to the class of distributive lattices. In Section 3.5 and Section 3.6 we consider MacNeille transferability relative to classes of Heyting algebras and bi-Heyting algebras, respectively. In Section 3.7 we show how transferability relates to questions of canonicity and elementarity of varieties of Heyting algebras and in Section 3.8 we draw some consequence of our results for the $(\wedge, \vee, 0, 1)$-stable logics. Finally, we conclude by discussing some possible directions for further research and listing a number of concrete open problems in Section 3.9.

3.1 Ideal transferability

We shall here recall some basic definitions and results concerning the notion of transferability due to Grätzer. Since we will be considering other analogous notions of transferability later in this chapter we shall include the qualifier "ideal" when talking about what is known in the literature as "transferability" *simpliciter*.

3.1.1. Definition ([133, Sec. 10(ii)]). A lattice $\mathbf{L}$ is *ideal transferable* if whenever there is a lattice embedding $h\colon \mathbf{L} \hookrightarrow \mathrm{Idl}(\mathbf{K})$ of $\mathbf{L}$ into the ideal lattice $\mathrm{Idl}(\mathbf{K})$ of a lattice $\mathbf{K}$, then there is a lattice embedding $h'\colon \mathbf{L} \hookrightarrow \mathbf{K}$. It is *sharply ideal transferable* if the embedding h' can be chosen so that $h'(a) \in h(b)$ if, and only if, $a \leq b$, for all $a, b \in L$. If we restrict $\mathbf{K}$ to belong to some class of lattices $\mathcal{K}$, we say $\mathbf{L}$ is *transferable for*, or *with respect to*, the class $\mathcal{K}$.

Throughout the 1970s and early 1980s an extensive body of work related to ideal transferability was produced, see [133, 105, 106, 205, 14, 134, 107, 189, 138, 20, 137, 214]. Recently the topic has been taken up again by Wehrung [250]. We shall here only mention two of the most striking results which will also be useful later on, referring to [136, pp. 502–503] for a historic account.

3.1.2. Theorem ([136, Thm. 557]). *Let* $\mathbf{L}$ *be a finite lattice. Then the following are equivalent.*

1. *The lattice* $\mathbf{L}$ *is ideal transferable for the class of all lattices.*
2. *The lattice* $\mathbf{L}$ *is sharply ideal transferable for the class of all lattices.*
3. *The lattice* $\mathbf{L}$ *is projective in the class of all lattices.*
4. *The lattice* $\mathbf{L}$ *is a sublattice of a free lattice.*
5. *The lattice* $\mathbf{L}$ *is semi-distributive and satisfies Whitman's condition.*

Recall that a lattice is called *meet semi-distributive* if it satisfies the quasi-equation

$$x \wedge y \approx x \wedge z \implies x \wedge y \approx x \wedge (y \vee z),$$

join semi-distributive if it satisfies the quasi-equation

$$x \vee y \approx x \vee z \implies x \vee y \approx x \vee (y \wedge z),$$

and *semi-distributive* if it satisfies both. Together with the following universal clause, know as *Whitman's condition*, semi-distributivity characterizes the finite sublattices of free lattices [202, 174].

$$x \wedge y \leq u \vee v \implies x \leq u \vee v \text{ or } y \leq u \vee v \text{ or } x \wedge y \leq u \text{ or } x \wedge y \leq v.$$

3.1.3. Remark. Note that being semi-distributive and satisfying Whitman's condition are both decidable properties of finite lattices. Consequently, the property of being ideal transferable, and therefore also sharply ideal transferable, with respect to class of all lattices is a decidable property of finite lattices.

The assumption in Theorem 3.1.2 that the lattice $\mathbf{L}$ is finite is necessary. There exist infinite lattices which are ideal transferable for the class of all lattices but not sharply ideal transferable for the class of all lattices, see [244]. Similarly, as shown by Wehrung [250] even for finite lattices the equivalence between ideal transferability and sharp ideal transferable can also fail when restricting to other classes of lattices such as the variety of modular lattices.

The situation becomes very different when we consider ideal transferability with respect to the class of distributive lattices.

3.1.4. Definition. A finite lattice $\mathbf{L}$ is *faithfully ideal transferable* for some class of lattices $\mathcal{K}$ if for all lattice embeddings $h\colon \mathbf{L} \hookrightarrow \mathrm{Idl}(\mathbf{K})$ with $\mathbf{K} \in \mathcal{K}$ there is a lattice embedding $h'\colon \mathbf{L} \hookrightarrow \mathbf{K}$ such that $h(a) = {\downarrow}x$ implies $h'(a) = x$ for all $a \in L$ and $x \in K$.

We will say that a finite lattice $\mathbf{L}$ is *simultaneously sharply and faithfully ideal transferable* for some class of lattices $\mathcal{K}$, provided that $\mathbf{L}$ is both sharply and faithfully ideal transferable for $\mathcal{K}$ and furthermore for all lattice embeddings $h\colon \mathbf{L} \hookrightarrow \mathrm{Idl}(\mathbf{K})$ with $\mathbf{K} \in \mathcal{K}$ there is a lattice embedding $h'\colon \mathbf{L} \hookrightarrow \mathbf{K}$ witnessing both that $\mathbf{L}$ is sharply and faithfully ideal transferable for $\mathcal{K}$.

3.1.5. Theorem ([105, 205]). *Every finite distributive lattice is simultaneously sharply and faithfully ideal transferable for the class of all distributive lattices.*

Thus any distributive lattice has, up to isomorphism, the same finite sublattices as its ideal lattice. In fact, sharp and faithful transferability allow us to say something about the preservations of bounds. If $\mathbf{K}$ is a distributive lattice with a least element 0, then $\{0\}$ is the least element of $\mathrm{Idl}(\mathbf{K})$. In this case it is easy to see that if $\mathbf{L}$ is a finite distributive lattice and $h\colon \mathbf{L} \hookrightarrow \mathrm{Idl}(\mathbf{K})$ is a lattice embedding which preserves the least element, i.e., $h(0) = \{0\}$, then the lattice embedding $h'\colon \mathbf{L} \hookrightarrow \mathbf{K}$ obtained from Theorem 3.1.5 must map the least element of $\mathbf{L}$ to the least element of $\mathbf{K}$. A similar comment can be made in case $\mathbf{K}$ has a greatest element. In particular, we have that if $\mathbf{K}$ is a bounded distributive lattice then the lattices $\mathbf{K}$ and $\mathrm{Idl}(\mathbf{K})$ have, up to isomorphism, the same finite bounded sublattices.

3.2 General notions of transferability

We here introduce and compare several notions of transferability for different types of completions. By a *completion type* for a class of lattices $\mathcal{K}$ we shall understand a class function associating to each lattice $\mathbf{K} \in \mathcal{K}$ a complete lattice $C(\mathbf{K})$ together with a lattice embedding $e_C\colon \mathbf{K} \hookrightarrow C(\mathbf{K})$. In this chapter we shall only be considering three different completion types: The ideal completion[1], the MacNeille completion, and for bounded lattices the canonical completion. See Appendix A.8 for definitions.

For $\tau \subseteq \{\wedge, \vee, 0, 1\}$, a *$\tau$-lattice* is a lattice, or lattice with one or both bounds, whose basic operations are of type τ. A *τ-homomorphism* is a homomorphism with respect to this type, and a *τ-embedding* is an injective τ-homomorphism. As in Chapter 2 we will sometimes write $h\colon \mathbf{K} \hookrightarrow_\tau \mathbf{K}'$ to indicate that h is a

[1]Note that when $\mathbf{K}$ is a lattice without a least element the ideal lattice $\mathrm{Idl}(\mathbf{K})$ will not be complete for want of a least element. Nevertheless, we will, with some minor abuse of terminology, allow ourself to speak about the ideal completion also in such cases.

τ-embedding of τ-lattice $\mathbf{K}$ and $\mathbf{K}'$. Similarly, we write $\mathbf{K} \hookrightarrow_\tau \mathbf{K}'$ to indicate that there exists some τ-embedding from the τ-lattice $\mathbf{K}$ to the τ-lattice $\mathbf{K}'$.

Throughout this chapter we will assume that if C is a completion type for a class $\mathcal{K}$ of τ-lattices then $C(\mathbf{K})$ is also a τ-lattice and the embeddings $e_C \colon \mathbf{K} \hookrightarrow C(\mathbf{K})$ preserve bounds the type of which belongs to τ, for each $\mathbf{K} \in \mathcal{K}$. This assumption is satisfied by the ideal, the MacNeille, and the canonical completions.

3.2.1. Definition. Let $\tau \subseteq \{\wedge, \vee, 0, 1\}$, $\mathbf{L}$ be a τ-lattice, $\mathcal{K}$ a class of τ-lattices and C be a completion type for $\mathcal{K}$. Then $\mathbf{L}$ is *(τ, C)-transferable for* $\mathcal{K}$ if for any $h \colon \mathbf{L} \hookrightarrow_\tau C(\mathbf{K})$ where $\mathbf{K} \in \mathcal{K}$, there is $h' \colon \mathbf{L} \hookrightarrow_\tau \mathbf{K}$. If the embedding h' can be chosen so that $e_C(h'(a)) \leq h(b)$ if, and only if, $a \leq b$ for all $a, b \in L$ we say that $\mathbf{L}$ is *sharply (τ, C)-transferable* for $\mathcal{K}$. Finally, if h' can be chosen so that $h'(a) = x$ whenever $h(a) = e_C(x)$ for some $a \in L$ and $x \in K$, we say that $\mathbf{L}$ is *faithfully (τ, C)-transferable* for $\mathcal{K}$.

When the completion type C is the ideal completion, MacNeille completion or the canonical completion we will use the terms *τ-ideal transferable*, *τ-MacNeille transferable* and *τ-canonically transferable*, respectively, and when $\tau = \{\wedge, \vee\}$ we will use the terms *ideal transferable*, *MacNeille transferable* and *canonically transferable*, respectively. Similar conventions apply in the presence of the words "sharply" and "faithfully".

As we have seen, the finite lattices which are ideal transferable for the class of all lattices are well understood. There are obvious examples of lattices $\mathbf{L}$ that are MacNeille and canonically transferable for the class of all lattices. Any finite chain, and the 4-element Boolean lattice provide examples. Since the property of being modular is not a property of lattices which is preserved by MacNeille completions [94] nor by canonical completions [147, 150], the pentagon $\mathbf{N}_5$ is neither MacNeille nor canonically transferable for the class of all lattices. Infinite chains can be problematic, as is seen by a simple cardinality argument. For example the chain of real numbers embeds into the MacNeille completion of the chain of rational numbers but of course not into the chain of rational numbers itself. Such difficulties arise also with the traditional study of ideal transferability. Therefore, in this chapter we will only consider transferability of finite lattices.

3.2.1 Transferability and preservation of clauses

Just as in Chapter 2 for every $\tau \subseteq \{\wedge, \vee, 0, 1\}$ and every finite lattice $\mathbf{L}$ we can find a universal clause $q_\tau(\mathbf{L})$ in the language of τ-lattices such that for any τ-lattice $\mathbf{K}$ we have that

$$\mathbf{K} \not\models q_\tau(\mathbf{L}) \quad \text{if, and only if,} \quad \mathbf{L} \hookrightarrow_\tau \mathbf{K}.$$

Given $\tau \subseteq \{\wedge, \vee, 0, 1\}$, a class $\mathcal{J}$ of finite lattices and $\mathcal{K}$ a class of τ-lattices we define,

$$\mathcal{K}_\tau(\mathcal{J}) = \{\mathbf{K} \in \mathcal{K} : \forall \mathbf{L} \in \mathcal{J}\ (\mathbf{K} \models q_\tau(\mathbf{L}))\}.$$

When $\mathcal{J}$ is a finite set, say $\{\mathbf{L}_1, \ldots, \mathbf{L}_n\}$, we write $\mathcal{K}_\tau(\mathbf{L}_1, \ldots, \mathbf{L}_n)$ for the class $\mathcal{K}_\tau(\mathcal{J})$. Thus $\mathcal{K}_\tau(\mathcal{J})$ consist of the members of $\mathcal{K}$ not having any τ-sublattices isomorphic to a member of $\mathcal{J}$. Typically, $\mathcal{K}$ will be a universal class of lattices or a class of τ-lattice reducts of some variety of τ-lattice based algebras. In this case the class $\mathcal{K}_\tau(\mathcal{J})$ is universal for any choice of τ and $\mathcal{J}$. When $\mathcal{K}$ is a class of τ-lattices and C is a completion type for $\mathcal{K}$ such that $C(\mathbf{K})$ belongs to $\mathcal{K}$ for all $\mathbf{K} \in \mathcal{K}$ we say that $\mathcal{K}$ is *closed under C-completions*.

3.2.2. Proposition. *Let $\tau \subseteq \{\wedge, \vee, 0, 1\}$, $\mathcal{K}$ be a class of τ-lattices and C a completion type for $\mathcal{K}$. If $\mathcal{J}$ is any class of finite lattices (τ, C)-transferable for $\mathcal{K}$ then $\mathcal{K}_\tau(\mathcal{J})$ is closed under C-completions.*

Proof:
Let $\mathbf{K} \in \mathcal{K}$ be given. If $C(\mathbf{K}) \notin \mathcal{K}_\tau(\mathcal{J})$, then there is some $\mathbf{L} \in \mathcal{J}$ such that $\mathbf{K} \not\models q_\tau(\mathbf{L})$. Consequently, $\mathbf{L} \hookrightarrow_\tau C(\mathbf{K})$ and so by the assumption that $\mathbf{L}$ is (τ, C)-transferable for $\mathbf{K}$ we have that $\mathbf{L} \hookrightarrow_\tau \mathbf{K}$. But then $\mathbf{K} \not\models q_\tau(\mathbf{L})$ and therefore $\mathbf{K} \notin \mathcal{K}_\tau(\mathcal{J})$. $\square$

Proposition 3.2.2 thus provides a template for constructing (universal) classes of lattices closed under different types of completions.

3.2.2 Relationships between notions of transferability

We first compare the finite ideal and canonically transferable lattices. For a bounded lattice $\mathbf{K}$, let $\mathbf{K}^\delta$ denote the canonical completion of the lattice $\mathbf{K}$, as defined in [111].

3.2.3. Proposition. *Let $\mathcal{K}$ be a class of bounded lattices and let $\tau \subseteq \{\wedge, \vee, 0, 1\}$. If a finite lattice $\mathbf{L}$ is τ-canonically transferable for $\mathcal{K}$, then $\mathbf{L}$ is τ-ideal transferable for $\mathcal{K}$.*

Proof:
Suppose that $\mathbf{L}$ is a finite lattice which is τ-canonically transferable for $\mathcal{K}$ and assume that $\mathbf{L} \hookrightarrow_\tau \mathrm{Idl}(\mathbf{K})$ for some $\mathbf{K} \in \mathcal{K}$. The ideal lattice $\mathrm{Idl}(\mathbf{K})$ is isomorphic to the bounded sublattice of open elements of the canonical extension $\mathbf{K}^\delta$ of $\mathbf{K}$ [111, Lem. 3.3], whence $\mathbf{L} \hookrightarrow_\tau \mathbf{K}^\delta$. Therefore, by the assumption that $\mathbf{L}$ is τ-canonically transferable for the class $\mathcal{K}$, we must have that $\mathbf{L} \hookrightarrow_\tau \mathbf{K}$, showing that $\mathbf{L}$ is τ-ideal transferable for $\mathcal{K}$. $\square$

We now compare the finite canonically and MacNeille transferable lattices. Recall that for a lattice $\mathbf{K}$ we use $\overline{\mathbf{K}}$ to denote the MacNeille completion of $\mathbf{K}$.

3.2.4. Proposition. *Let $\mathcal{K}$ be a class of bounded lattices closed under ultrapowers and let $\tau \subseteq \{\wedge, \vee, 0, 1\}$. If a finite lattice $\mathbf{L}$ is τ-MacNeille transferable for $\mathcal{K}$ then $\mathbf{L}$ is τ-canonically transferable for $\mathcal{K}$.*

Proof:
Suppose that $\mathbf{L}$ is a finite lattice which is τ-MacNeille transferable for $\mathcal{K}$ and assume that $\mathbf{L} \hookrightarrow_\tau \mathbf{K}^\delta$ for some $\mathbf{K} \in \mathcal{K}$. By [112, Thm. 3.2] there is an ultrapower $\mathbf{K}^X/U$ of $\mathbf{K}$ such that $\mathbf{K}^\delta$ is isomorphic to a bounded sublattice of $\overline{\mathbf{K}^X/U}$. Consequently, $\mathbf{L} \hookrightarrow_\tau \overline{\mathbf{K}^X/U}$. By assumption $\mathbf{K}^X/U$ belongs to $\mathcal{K}$ and hence $\mathbf{L} \hookrightarrow_\tau \mathbf{K}^X/U$. Therefore, $\mathbf{K}^X/U$ satisfies the universal clause $q_\tau(\mathbf{L})$ associated with $\mathbf{L}$ and, by Łoś' Theorem, so does $\mathbf{K}$. Thus, $\mathbf{K}$ has a τ-sublattice that is isomorphic to $\mathbf{L}$, showing that $\mathbf{L}$ is transferable for $\mathcal{K}$. □

For a lattice $\mathbf{K}$, we let $\mathbf{K}^+$ be the result of adding a least element to $\mathbf{K}$ if it does not have one, and adding a greatest element to $\mathbf{K}$ if it does not have one. Note that for a finite lattice $\mathbf{L}$, there is a lattice embedding of $\mathbf{L}$ into $\mathbf{K}$ if, and only if, there is a lattice embedding of $\mathbf{L}$ into $\mathbf{K}^+$. Following the convention that the empty set is not an element of the ideal lattice, but is an element of the MacNeille completion when viewed as a lattice of normal ideals, it is easily seen that $\mathrm{Idl}(\mathbf{K})$ is a sublattice of $\mathrm{Idl}(\mathbf{K}^+)$ and $\overline{\mathbf{K}^+} = \overline{\mathbf{K}}$.

3.2.5. Proposition. *Let $\mathcal{K}$ be a class of lattices closed under ultrapowers. If a finite lattice $\mathbf{L}$ is MacNeille transferable for $\mathcal{K}$ then $\mathbf{L}$ is ideal transferable for $\mathcal{K}$.*

Proof:
Let $\mathcal{K}$ be a class of lattices which is closed under ultrapowers, and assume that $\mathbf{L}$ is a finite lattice MacNeille transferable for $\mathcal{K}$. If there is a lattice embedding $\mathbf{L} \hookrightarrow \mathrm{Idl}(\mathbf{K})$ then we also have a lattice embedding $\mathbf{L} \hookrightarrow \mathrm{Idl}(\mathbf{K}^+)$. As in the proof of Proposition 3.2.3 we then obtain an embedding of lattices $\mathbf{L} \hookrightarrow (\mathbf{K}^+)^\delta$. Similarly to the argument in the proof of Proposition 3.2.4 we then obtain that $\mathbf{L}$ is isomorphic to a sublattice of

$$\overline{(\mathbf{K}^+)^X/U} = \overline{(\mathbf{K}^X/U)^+} = \overline{\mathbf{K}^X/U}.$$

By assumption $\mathbf{K}^X/U \in \mathcal{K}$ and consequently we obtain a lattice embedding $\mathbf{L} \hookrightarrow \mathbf{K}^X/U$. Finally, since $\mathbf{L}$ is finite Łoś' Theorem gives an embedding $\mathbf{L} \hookrightarrow \mathbf{K}$ of lattices. □

3.2.3 Transferability and ultrapowers

Recall, e.g., from [46, Def. II.10.14] that an algebra is *locally finite* if all of its finitely generated subalgebras are finite. Evidently, being locally finite as a lattice is independent of whether the type of any existing bounds are taken to be part of the type of the lattice.

3.2.6. Theorem. *Let $\tau \subseteq \{\wedge, \vee, 0, 1\}$, $\mathcal{K}$ be class of τ-lattices which are locally finite as lattices and C some completion type for $\mathcal{K}$. If $\mathcal{K}$ is closed under C-completions then the following are equivalent.*

1. *All finite lattices are (τ, C)-transferable for $\mathcal{K}$.*

2. *For all $\mathbf{K} \in \mathcal{K}$ the completion $C(\mathbf{K})$ is isomorphic to a τ-sublattice of an ultrapower of $\mathbf{K}$.*

Proof:
First assume that all finite lattices are (τ, C)-transferable for $\mathcal{K}$. Let $\mathbf{K} \in \mathcal{K}$ be given and let $\tau' = \tau \cup \{\wedge, \vee\}$. Any algebra is an ultraproduct of its finitely generated subalgebras [46, Thm. V.2.14]. By assumption $C(\mathbf{K}) \in \mathcal{K}$ and as such must be locally finite as a lattice and therefore also as a τ'-lattice. Consequently, $C(\mathbf{K})$ must be isomorphic to a τ'-sublattice of an ultraproduct of its finite τ'-sublattices, say $\prod_{i \in I} \mathbf{L}_i/U$. For each such τ'-sublattice $\mathbf{L}_i$ of $C(\mathbf{K})$ we must have that $\mathbf{L}_i \hookrightarrow_\tau C(\mathbf{K})$ and hence by assumption $\mathbf{L}_i \hookrightarrow_\tau \mathbf{K}$. From this we may conclude that $\prod_{i \in I} \mathbf{L}_i/U \hookrightarrow_\tau \mathbf{K}^I/U$, see, e.g., [179, Prop. 4.2], and hence that $C(\mathbf{K}) \hookrightarrow_\tau \mathbf{K}^I/U$.

Now assume that for all $\mathbf{K} \in \mathcal{K}$ the completion $C(\mathbf{K})$ is isomorphic to a τ-subalgebra of an ultrapower of $\mathbf{K}$. Then consider a finite lattice $\mathbf{L}$ such that $\mathbf{L} \hookrightarrow_\tau C(\mathbf{K})$ for some $\mathbf{K} \in \mathcal{K}$. By assumption there is an ultrapower, say $\mathbf{K}^X/U$, such that $C(\mathbf{K}) \hookrightarrow_\tau \mathbf{K}^X/U$. But then $\mathbf{K}^X/U \not\models q_\tau(\mathbf{L})$ and so by Łoś' Theorem $\mathbf{K} \not\models q_\tau(\mathbf{L})$, showing that $\mathbf{L} \hookrightarrow_\tau \mathbf{K}$. $\square$

Since all distributive lattices are locally finite we obtain the following corollary from Theorem 3.1.5.

3.2.7. Corollary. *Let $\tau \subseteq \{\wedge, \vee, 0, 1\}$ with $\{\wedge, \vee\} \subseteq \tau$. For any distributive τ-lattice $\mathbf{D}$, there is an ultrapower $\mathbf{D}^X/U$ of $\mathbf{D}$ such that* $\mathrm{Idl}(\mathbf{D}) \hookrightarrow_\tau \mathbf{D}^X/U$.

This corollary may be seen as a strong version of the Baker-Hales Theorem [14, Thm. A], see Lemma 3.3.6 below, showing that for any lattice $\mathbf{K}$ the ideal lattice $\mathrm{Idl}(\mathbf{K})$ is a homomorphic image of a sublattice of an ultrapower of $\mathbf{K}$.

3.3 MacNeille transferability for lattices

In the rest of this chapter we will focus on τ-MacNeille transferability with respect to several classes of lattices for different choices of $\tau \subseteq \{\wedge, \vee, 0, 1\}$. In this section we will consider finite lattices which are τ-MacNeille transferable for the class of all τ-lattices.

3.3.1 Transferability with both lattice operations

Our first result in this respect is an immediate consequence of a result due to Harding [145] which states that any lattice can be embedded into the MacNeille completion of a distributive lattice. Consequently, any lattice which is MacNeille transferable for a class of lattices containing the class of all distributive lattices must be a sublattice of a distributive lattice and hence distributive.

3.3.1. THEOREM. *Let $\mathcal{K}$ be a class of lattices containing the class of all distributive lattices. If $\mathbf{L}$ is a finite lattice MacNeille transferable for $\mathcal{K}$, then $\mathbf{L}$ is distributive.*

From Proposition 3.2.5 we obtain that any finite lattice which is MacNeille transferable for the class of all lattices is also ideal transferable. Therefore, by Theorem 3.1.2, it follows that if a finite lattice is MacNeille transferable for the class of all lattices it must be a sublattice of a free lattice. Thus Theorem 3.3.1 entails that any finite lattice MacNeille transferable for the class of all lattices is a finite distributive sublattice of a free lattice. Galvin and Jónsson [103] characterized the finite distributive sublattices of free lattices as exactly those that do not contain a doubly reducible element, viz., an element that is both a non-trivial join and a non-trivial meet. From this we obtain the following result as an immediate consequence of Theorem 3.3.1.

3.3.2. COROLLARY. *If a finite lattice is MacNeille transferable for the class of all lattices, then it is distributive and has no doubly reducible elements.*

3.3.3. REMARK. In fact, as one of the earliest results on ideal transferability Grätzer [134] showed that any lattice which is ideal transferable for the class of all lattices cannot have any doubly reducible elements.

Galvin and Jónsson [103] further characterized all distributive lattices that have no doubly reducible elements. This will be of use in later considerations for us as well. To state their result we will need the following definition.

3.3.4. DEFINITION. Let I be a set with a total order $\leq_I$, and for each $i \in I$ let $\mathbf{K}_i = (K_i, \leq_i)$ be a lattice. Then the *linear sum*, denoted $\bigoplus_I \mathbf{K}_i$, of the family $\{\mathbf{K}_i\}_{i\in I}$ is the disjoint union of the sets K_i with the ordering $\leq$ given by setting $a \leq b$ iff $a \in K_i$ and $b \in K_j$ for some $i <_I j$, or $a, b \in K_i$ for some i and $a \leq_i b$.

For any pair of lattices $\langle \mathbf{K}_1, \mathbf{K}_2 \rangle$ we write $\mathbf{K}_1 \oplus \mathbf{K}_2$ for the linear sum of the family $\{\mathbf{K}_1, \mathbf{K}_2\}$ obtained from the total order $1 < 2$. The operation $\langle \mathbf{K}_1, \mathbf{K}_2 \rangle \mapsto \mathbf{K}_1 \oplus \mathbf{K}_2$ is evidently an associative operation and so we may unambiguously write $\mathbf{K}_1 \oplus \ldots \oplus \mathbf{K}_n$, for lattice $\mathbf{K}_1, \ldots, \mathbf{K}_n$. In particular, for a lattice $\mathbf{K}$, the lattice $\mathbf{1} \oplus \mathbf{K}$ will be the result of adding a new bottom element 0 to $\mathbf{K}$, the lattice $\mathbf{K} \oplus \mathbf{1}$ will be the result of adding a new top element 1 to $\mathbf{K}$, and the lattice $\mathbf{1} \oplus \mathbf{K} \oplus \mathbf{1}$ will be the result of doing both.

3.3.5. THEOREM ([103]). *A distributive lattice has no doubly reducible elements if, and only if, it is a linear sum of lattices each of which is isomorphic to an eight-element Boolean algebra, a one-element lattice, or $\mathbf{2} \times \mathbf{C}$ for a chain $\mathbf{C}$.*

Consequently, the finite lattices MacNeille transferable for the class of all lattices must be of the form $\bigoplus_{i=1}^{n} \mathbf{K}_i$ with each lattice $\mathbf{K}_i$ either the Boolean algebra $\mathbf{2}^3$, the one-element lattice $\mathbf{1}$, or the direct product $\mathbf{2} \times \mathbf{C}$ for some finite chain $\mathbf{C}$.

3.3.2 Transferability without both lattice operations

If instead of considering MacNeille transferability, i.e., τ-MacNeille transferability for $\tau = \{\wedge, \vee\}$, we consider τ-MacNeille transferability for $\tau \subseteq \{\wedge, \vee, 0, 1\}$ such that $\{\wedge, \vee\} \nsubseteq \tau$, the situation changes considerably. For what follows we will need the following slight extension of a result of Baker and Hales [14].

3.3.6. Lemma (cf. [14, Thm. A]). *For a lattice* $\mathbf{K}$*, there is a sublattice* $\mathbf{S}$ *of an ultrapower* $\mathbf{K}^X/U$ *of* $\mathbf{K}$ *and an onto lattice homomorphism* $h\colon \mathbf{S} \twoheadrightarrow \mathrm{Idl}(\mathbf{K})$*. If* $\mathbf{K}$ *has a least (resp. greatest) element, then the least (resp. greatest) element of* $\mathbf{K}^X/U$ *belongs to* $\mathbf{S}$ *and is the only element of* $\mathbf{S}$ *mapped by* h *to the least (resp. greatest) element of* $\mathrm{Idl}(\mathbf{K})$*.*

Proof:
We follow [14]. Let X be the set of all finite subsets of K partially ordered by set inclusion. The collection of principal upsets of X is closed under finite intersections and does not contain the empty set and may therefore be extended to an ultrafilter, say U. Let M be the set of order-preserving maps from X to K. Then $\mathbf{M}$ is a sublattice of $\mathbf{K}^X$. Let $\mathbf{S}$ be the image of $\mathbf{M}$ in $\mathbf{K}^X/U$ under the canonical projection. So the elements of $\mathbf{S}$ are equivalence classes σ/U of order preserving functions $\sigma\colon X \to K$. Define $h\colon \mathbf{S} \twoheadrightarrow \mathrm{Idl}(\mathbf{K})$ by letting $h(\sigma/U)$ be the ideal generated by the image of σ. In [14] it is shown that h is a well defined onto lattice homomorphism.

If $\mathbf{K}$ has a least element 0, then the constant map from X to K taking value 0 belongs to $\mathbf{M}$, and the equivalence class determined by this map is the least element of $\mathbf{K}^X/U$. Since h is onto, it must map least element to least element. If $\sigma\colon X \to K$ is order preserving and $h(\sigma/U)$ is the least element of $\mathrm{Idl}(\mathbf{K})$, then the ideal generated by the image of σ is the zero ideal $\{0\}$, whence σ must be the zero function. So the least element of $\mathbf{S}$ is the only element mapped to the least element of $\mathrm{Idl}(\mathbf{K})$. Similarly, if $\mathbf{K}$ has a greatest element 1 then the greatest element of $\mathbf{K}^X/U$ is the equivalence class of the constant function 1. It belongs to $\mathbf{S}$, and since h is onto it maps greatest element to greatest element. Suppose $h(\sigma/U)$ is the greatest element of $\mathrm{Idl}(\mathbf{K})$. Then the element 1 belongs to the ideal of $\mathbf{K}$ generated by the image of σ, and since σ is order-preserving and the partial order on X is in fact a lattice, 1 must be in the image of σ. But then there is a finite subset $K_0 \subseteq K$ with $\sigma(K_0) = 1$. Since σ is order-preserving, σ takes value 1 on the upset of X generated by K_0, hence σ is in the equivalence class of the constant function 1. $\square$

3.3.7. Theorem. *Let* $\mathbf{L}$ *be a finite distributive lattice and let* $\tau \subseteq \{\wedge, \vee, 0, 1\}$ *be such that* $\{\wedge, \vee\} \not\subseteq \tau$. *Then* $\mathbf{L}$ *is* τ*-MacNeille transferable for the class of all* τ*-lattices.*

Proof:
Assume $\mathbf{K}$ is a τ-lattice and that $h\colon \mathbf{L} \hookrightarrow_\tau \overline{\mathbf{K}}$. By symmetry, we may assume that τ does not contain $\vee$. We consider first the case that τ does not contain $\wedge$, i.e., $\tau \subseteq \{0, 1\}$. If $\mathbf{K}$ is finite, then $\overline{\mathbf{K}} = \mathbf{K}$ and there is nothing to show. Otherwise, since $\mathbf{L}$ is finite, there is a τ-embedding of $\mathbf{L}$ into $\mathbf{K}$.

Assume that τ contains $\wedge$. Viewing the MacNeille completion $\overline{\mathbf{K}}$ as a set of normal ideals we have an embedding $\overline{\mathbf{K}} \hookrightarrow_{\wedge,0,1} \mathrm{Idl}(\mathbf{K}_0)$, where $\mathbf{K}_0$ is taken to be the lattice $\mathbf{1} \oplus \mathbf{K}$ if $\mathbf{K}$ does not have a least element and $\mathbf{K}$ otherwise. We therefore obtain a τ-embedding $j\colon \mathbf{L} \hookrightarrow_\tau \mathrm{Idl}(\mathbf{K}_0)$. By Lemma 3.3.6 we have an ultrapower $(\mathbf{K}_0)^X/U$ of $\mathbf{K}_0$ and a bounded sublattice $\mathbf{S} \hookrightarrow (\mathbf{K}_0)^X/U$ with bounded lattice homomorphism $h\colon \mathbf{S} \twoheadrightarrow \mathrm{Idl}(\mathbf{K}_0)$. Thus we have a diagram

$$\begin{tikzcd}
 & \mathbf{S} \arrow[r, hook] \arrow[d, two heads, "h"] & (\mathbf{K}_0)^X/U \\
\mathbf{L} \arrow[r, hook, "j"'] & \mathrm{Idl}(\mathbf{K}_0) &
\end{tikzcd}$$

of τ-homomorphism. Since $\mathbf{L}$ is distributive, by Theorem A.7.4, the meet semi-lattice reduct of $\mathbf{L}$ is projective in the class of meet semi-lattices. Therefore, there is a meet semi-lattice homomorphism $g\colon \mathbf{L} \to \mathbf{S}$ making the diagram

$$\begin{tikzcd}
 & \mathbf{S} \arrow[r, hook] \arrow[d, two heads, "h"] & (\mathbf{K}_0)^X/U \\
\mathbf{L} \arrow[ur, "g"] \arrow[r, hook, "j"'] & \mathrm{Idl}(\mathbf{K}_0) &
\end{tikzcd}$$

commute. Since j is injective, so is g. Since the composite $h \circ g$ preserves the bounds whose type belongs to τ, and h is injective on those bounds, this shows that g preserves bounds whose type is in τ. Thus, $\mathbf{S}$ has a τ-sublattice isomorphic to $\mathbf{L}$, and hence so does $(\mathbf{K}_0)^X/U$. Therefore, the ultrapower $(\mathbf{K}_0)^X/U$ refutes the clause $q_\tau(\mathbf{L})$ associated with $\mathbf{L}$, and hence by Łoś' Theorem, so does $\mathbf{K}_0$, whence $\mathbf{L} \hookrightarrow_\tau \mathbf{K}_0$. If $\mathbf{K}_0 = \mathbf{K}$, in particular if $0 \in \tau$, then we must have $\mathbf{L} \hookrightarrow_\tau \mathbf{K}$. Otherwise $\mathbf{L} \hookrightarrow_\tau \mathbf{1} \oplus \mathbf{K}$, in which case $\mathbf{K}$ does not have a least element. In particular $0 \notin \tau$ and so we may conclude that also $\mathbf{L} \hookrightarrow_\tau \mathbf{K}$. $\square$

3.3.8. Remark. We note that the proof of Theorem 3.3.7 also shows that any finite distributive lattice is τ-ideal transferable for the class of all τ-lattices when $\tau \subseteq \{\wedge, 0, 1\}$.

From Propositions 3.2.2 and Theorem 3.3.7 it follows that if $\mathcal{K}$ is any class of finite distributive lattice and $\tau = \{\wedge, 0, 1\}$ then the universal class of Heyting algebras $\mathcal{HA}_\tau(\mathcal{K})$ consisting of Heyting algebras not containing any τ-sublattice isomorphic to a member of $\mathcal{K}$ is closed under MacNeille completions. This provides us with an alternative proof of the fact, first established by Ciabattoni, Galatos, and Terui [58, Thm. 4.1], that structural clauses, i.e., universal clauses in the $\{\wedge, 0, 1\}$-reduct of the language of Heyting algebras, are preserved by MacNeille completions of Heyting algebras.

3.3.3 Adding bounds

We conclude this section by discussing the addition of bounds to a given type.

For a lattice $\mathbf{K}$ and $k \in K$, we can consider the principal ideals $({\downarrow}k)_{\mathbf{K}}$ and $({\downarrow}k)_{\overline{\mathbf{K}}}$. Using the abstract characterization of the MacNeille completion of a given lattice as its, up to isomorphism, unique join- and meet-dense completion, it is easily seen that $\overline{({\downarrow}k)_{\mathbf{K}}} = ({\downarrow}k)_{\overline{\mathbf{K}}}$. We say that a class $\mathcal{K}$ of lattices is *closed under principal ideals* if for each $K \in \mathcal{K}$ and each $k \in K$, the lattice $({\downarrow}k)_{\mathbf{K}}$ belongs to $\mathcal{K}$. Similarly, $\mathcal{K}$ is closed under principal filters if each $({\uparrow}k)_{\mathbf{K}}$ belongs to $\mathcal{K}$.

For $\tau \subseteq \{\wedge, \vee, 0, 1\}$ we write τ_0 and τ_1 for the sets $\tau \cup \{0\}$ and $\tau \cup \{1\}$, respectively. Similarly, we write τ_{01} for the set $\tau \cup \{0, 1\}$.

3.3.9. Proposition. *Let $\mathbf{L}$ be a finite lattice, let $\tau \subseteq \{\wedge, \vee, 0\}$, and let $\mathcal{K}$ be a class of τ_1-lattices that is closed under principal ideals. If $\mathbf{L}$ is τ-MacNeille transferable for $\mathcal{K}$, then $\mathbf{L} \oplus \mathbf{1}$ is τ_1-MacNeille transferable for $\mathcal{K}$. Similar results hold for $\tau \subseteq \{\wedge, \vee, 1\}$ and $\mathbf{1} \oplus \mathbf{L}$ when $\mathcal{K}$ is closed under principal filters, and $\tau \subseteq \{\wedge, \vee\}$ and $\mathbf{1} \oplus \mathbf{L} \oplus \mathbf{1}$ when $\mathcal{K}$ is closed under both.*

Proof:
We prove the result for $\tau \subseteq \{\wedge, \vee, 0\}$, the case $\tau \subseteq \{\wedge, \vee, 1\}$ follows by symmetry, and the result for $\tau \subseteq \{\wedge, \vee\}$ follows from these two results combined. Suppose $\mathbf{K} \in \mathcal{K}$ and $h\colon \mathbf{L} \oplus \mathbf{1} \hookrightarrow_{\tau_1} \overline{\mathbf{K}}$ for some $\mathbf{K} \in \mathcal{K}$. Let $\top$ be the top of $\mathbf{L}$ and 1 be the top of $\mathbf{L} \oplus \mathbf{1}$. Then $h(\top) = x$ for some $x < 1$ in $\overline{\mathbf{K}}$. Since the MacNeille completion is meet-dense, there is $k \in K$ with $x \leq k < 1$. Then the restriction $h|L\colon \mathbf{L} \to ({\downarrow}k)_{\overline{\mathbf{K}}} = \overline{({\downarrow}k)_{\mathbf{K}}}$ is a τ-embedding. Since by assumption $({\downarrow}k)_{\mathbf{K}} \in \mathcal{K}$ and $\mathbf{L}$ is τ-MacNeille transferable for $\mathcal{K}$, there is a τ-embedding $\mathbf{L} \hookrightarrow ({\downarrow}k)_{\mathbf{K}}$. Since $k < 1$, there is evidently also a τ_1-embedding of $\mathbf{L} \oplus \mathbf{1}$ into $\mathbf{K}$. □

3.3.10. Remark. We note that $(\wedge, \vee, 0, 1)$-MacNeille transferability is an elusive concept when lattices are not of the form $\mathbf{1} \oplus \mathbf{L}$ or $\mathbf{L} \oplus \mathbf{1}$, i.e., when the bottom element is meet-reducible or the top element join-reducible. For the real unit interval $[0, 1]$, the bounded lattice $\mathbf{K} = ([0, 1] \times [0, 1]) \setminus \{(1, 0), (0, 1)\}$ has no complemented elements other than the bounds, while its MacNeille completion

$\overline{\mathbf{K}} = [0,1] \times [0,1]$ does. Thus, even the 4-element bounded lattice $\mathbf{2} \times \mathbf{2}$ is not $(\wedge, \vee, 0, 1)$-MacNeille transferable for the class of distributive lattices. We return to this matter in the Section 3.5 on MacNeille completions of Heyting algebras where it takes particular significance.

3.4 MacNeille transferability for distributive lattices

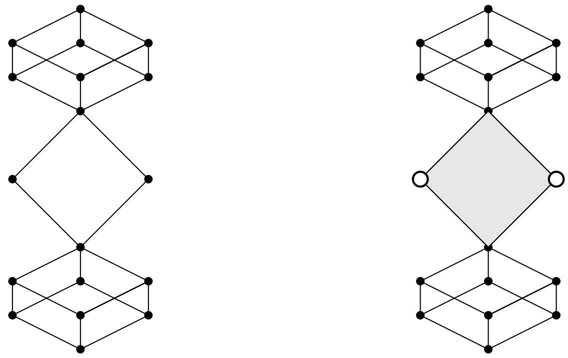
One can consider MacNeille transferability for the class of distributive lattices. However, since distributive lattices are not generally closed under MacNeille completions [94], to do so one would also have to consider non-distributive lattices. Therefore, when establishing negative results we consider MacNeille transferability for the class $\mathcal{DM}$ of distributive lattices whose MacNeille completions are distributive. This class includes many well-known classes of lattice such as (the lattice reducts of) Heyting algebras, co-Heyting algebras, and bi-Heyting algebras. Since MacNeille transferability for $\mathcal{DM}$ holds vacuously for any non-distributive lattice, we consider only the case when a finite distributive lattice is MacNeille transferable for $\mathcal{DM}$.

In contrast to Theorem 3.1.5, which says that every finite distributive lattice is ideal transferable for the class of all distributive lattices, we have the following.

3.4.1. THEOREM. *There is a finite distributive lattice* $\mathbf{L}$ *that is not MacNeille transferable for the class* $\mathcal{DM}$.

Proof:
Consider the lattices $\mathbf{L}$, in Figure 3.1 at the left, and $\mathbf{K}$, in Figure 3.1 at the right. Here the shaded middle portion of K is $([0,1] \times [0,1]) \setminus \{(0,1),(1,0)\}$, the product of two copies of the real unit interval with the "corners" removed. The MacNeille completion $\overline{\mathbf{K}}$ simply reinserts the missing "corners", and there is a lattice embedding of $\mathbf{L}$ into the distributive $\overline{\mathbf{K}}$. Using the fact that $[0,1] \times [0,1]$ does not contain a sublattice isomorphic to an eight-element Boolean algebra and $([0,1] \times [0,1]) \setminus \{(0,1),(1,0)\}$ does not have any complemented elements, it is easily seen that $\mathbf{L}$ is not isomorphic to a sublattice of $\mathbf{K}$. $\square$

Before moving to some positive results, we note that a lattice being projective in the class of distributive lattices is not the same as it being distributive and projective in the class of all lattices. The finite lattices that are projective in the class of distributive lattices are characterized in [17] as exactly those where the meet of two join-irreducible elements is either join-irreducible or the least element 0, see Theorem A.7.1.



L **K**

Figure 3.1: The distributive lattices **L** and **K**.

3.4.2. THEOREM. *Let $\tau \subseteq \{\wedge, \vee, 0, 1\}$ with $\{0,1\} \not\subseteq \tau$. Every finite lattice that is projective in the class of distributive lattices is τ-MacNeille transferable for the class of all distributive τ-lattices.*

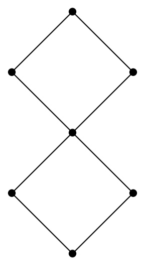
Proof:
Assume that $\tau \subseteq \{\wedge, \vee, 0\}$. Since the order dual of a projective distributive lattice is again a projective distributive lattice the case where $\tau \subseteq \{\wedge, \vee, 1\}$ follows by symmetry.

Let $\mathbf{P}$ be a finite distributive lattice that is projective in the class of distributive lattices. Let $\mathbf{K}$ be a distributive τ-lattice with $\mathbf{P} \hookrightarrow_\tau \overline{\mathbf{K}}$. By Theorem 3.3.7, we have that $\mathbf{P} \hookrightarrow_{\tau'} \mathbf{K}$, where $\tau' = \tau \backslash \{\vee\}$. Since $\mathbf{P}$ is projective among the distributive lattices, applying Proposition A.7.2 we obtain a τ-embedding $\mathbf{P} \hookrightarrow_\tau \mathbf{K}$. $\square$

Our next theorem will show that there are finite non-projective distributive lattices that are MacNeille transferable for the class of distributive lattices. Let $\mathbf{D}_4$ be the seven-element distributive lattice that has a doubly reducible element shown in Figure 3.2. Since the lattice $\mathbf{D}_4$ contains join-irreducible elements whose meet is a non-zero join-reducible element it is not projective as a distributive lattice.

3.4.3. THEOREM. *The lattice $\mathbf{D}_4$ is MacNeille transferable for the class of distributive lattices.*

Proof:
Suppose that $\mathbf{K}$ is a distributive lattice that does not contain $\mathbf{D}_4$ as a sublattice. We must show that $\overline{\mathbf{K}}$ does not contain $\mathbf{D}_4$ as a sublattice. By Theorem 3.3.5, $\mathbf{K}$ is a linear sum of lattices $\bigoplus_I \mathbf{K}_i$ with each $\mathbf{K}_i$ isomorphic to either an eight-element Boolean algebra, a one-element lattice, or $\mathbf{2} \times \mathbf{C}$ for a chain $\mathbf{C}$, for some totally order set I. We describe the MacNeille completion of $\mathbf{K}$.



$\mathbf{D}_4$

Figure 3.2: A seven-element non-projective distributive lattice.

Let $\overline{I}$ denote the MacNeille completion of the totally order set I. For $x \in \overline{I}$, let $\mathbf{M}_x = \overline{\mathbf{K}_x}$ if $x \in I$, and let $\mathbf{M}_x$ be a one-element lattice otherwise. Let 0_x be the least element of $\mathbf{M}_x$ and 1_x be the greatest element of $\mathbf{M}_x$. Set $\mathbf{M} = \bigoplus_{\overline{I}} \mathbf{M}_x$.

If $S \subseteq M$, let $S' = \{x \in \overline{I} : S \cap M_x \neq \emptyset\}$, and let $z = \bigvee_{\overline{I}} S'$. Then it is not difficult to see that $\bigvee_{\mathbf{M}_z}(S \cap M_z)$, where it is understood that the join of the empty set in $\mathbf{M}_z$ is 0_z, is the least upper bound of S in $\mathbf{M}$. Similar remarks hold for the greatest lower bound of S in $\mathbf{M}$. Thus, $\mathbf{M}$ is complete and evidently $\mathbf{K}$ is a sublattice of $\mathbf{M}$.

We show that $\mathbf{K}$ must be a join- and meet-dense subalgebra of a quotient of $\mathbf{M}$. To this end define a special covering pair in M to be an ordered pair $(1_x, 0_y)$ where $x, y \in I$ with x covered by y and either $1_x \notin K_x$ or $0_y \notin K_y$. This implies that 1_x is join-irreducible in $\mathbf{M}$, or 0_y is meet-irreducible in $\mathbf{M}$, or both. Let θ be the set of all special covering pairs together with the diagonal of M^2. It is easy to see that special covering pairs cannot overlap, and hence the relation θ is a lattice congruence on $\mathbf{M}$. Evidently, $\mathbf{M}/\theta$ is complete and $\mathbf{K}$ may be identified with a sublattice of $\mathbf{M}/\theta$. We claim that the quotient $\mathbf{M}/\theta$ is the MacNeille completion of $\mathbf{K}$. To see that $\mathbf{K}$ is join-dense in $\mathbf{M}/\theta$ consider a special covering pair $(1_x, 0_y)$ such that $p = 0_y/\theta = 1_x/\theta$ does not belong to the image of the embedding $a \mapsto a/\theta$ of $\mathbf{K}$ into $\mathbf{M}/\theta$. Then $1_x \notin K_x$ and so 1_x must be the join in $\mathbf{M}_x$ of the non-zero elements of the set K_x, showing that $p = \bigvee\{a/\theta : a \in K_x\}$. Similarly, $0_x \notin K_y$ whence $p = \bigwedge\{a/\theta : a \in K_y\}$. If $a \in M_z$ for some $z \in I$ and a is not part of some special covering pair then it is easy to see that $a/\theta = \bigvee\{b/\theta : b \in {\downarrow}a \cap K_z\}$ and $a/\theta = \bigwedge\{b/\theta : b \in {\uparrow}a \cap K_z\}$. It remains to consider elements of $\overline{I} \backslash I$. For $z \in \overline{I} \backslash I$ we have that z is the join of the elements in I strictly below it and the meet of the elements of I strictly above it. Therefore, letting $S = \bigcup\{K_x : z > x \in I\}$ and $S' = \bigcup\{K_y : z < y \in I\}$, for $a \in M_z$ we must have that $a/\theta = \bigvee\{b/\theta : b \in S\}$ and $a/\theta = \bigwedge\{b/\theta : b \in S'\}$. This shows that $a \mapsto a/\theta$ is indeed a join- and meet-dense embedding of $\mathbf{K}$ into $\mathbf{M}/\theta$.

It remains to show that $\overline{\mathbf{K}} = \mathbf{M}/\theta$ has no doubly reducible elements, and hence does not have a sublattice that is isomorphic to $\mathbf{D}_4$. Along the way, we will show that $\mathbf{M}/\theta$ is in fact distributive. Let $z \in \overline{I}$. Then $\mathbf{M}_z$ is either a one-element lattice, the MacNeille completion of an eight-element Boolean algebra, which is an eight-element Boolean algebra, or $\overline{\mathbf{2} \times \mathbf{C}}$ for some chain $\mathbf{C}$. The MacNeille completion $\overline{\mathbf{2} \times \mathbf{C}}$ depends on whether $\mathbf{C}$ is bounded. If it is, then $\overline{\mathbf{2} \times \mathbf{C}} = \mathbf{2} \times \overline{\mathbf{C}}$. If $\mathbf{C}$ has a greatest, but no least element, then $\overline{\mathbf{2} \times \mathbf{C}} = (\mathbf{2} \times \overline{\mathbf{C}}) \setminus \{(1, 0_{\overline{\mathbf{C}}})\}$, and similarly if $\mathbf{C}$ has a least, but no greatest element. In any case $\overline{\mathbf{2} \times \mathbf{C}}$ is a sublattice of $\mathbf{2} \times \overline{\mathbf{C}}$. Therefore, each summand $\mathbf{M}_z$ is distributive and has no doubly reducible elements. Thus, $\mathbf{M}$ is distributive and has no doubly reducible elements. In forming the quotient $\mathbf{M}/\theta$ we only collapse covering pairs $(1_x, 0_y)$ where either 1_x is not a proper join, or 0_y is not a proper meet, and hence introduce no doubly reducible elements into the quotient. □

3.4.4. Remark. Note that even though the lattice $\mathbf{D}_4$ is not projective in the class of distributive lattices it is projective in the class of relatively complemented lattices [250, Thm. 4.2].

Theorems 3.4.1 and 3.4.3 shows that MacNeille transferability for the class of distributive lattices is rather different than ideal transferability for the class of distributive lattices. However, it is not clear how robust Theorem 3.4.3 is, in the sense that the argument provided cannot easily be adapted to show that the lattice $\mathbf{D}_4$ is τ-MacNeille transferable for the class of all bounded distributive lattices when τ contains one or more bounds.

3.5 MacNeille transferability for Heyting algebras

In this section we consider τ-MacNeille transferability for the class $\mathcal{HA}$ of Heyting algebras for $\tau = \{\wedge, \vee, 0, 1\}$. Here we treat members of $\mathcal{HA}$ as bounded lattices and note that the notion of MacNeille transferability does not involve the Heyting implication. Since the MacNeille completion of a Heyting algebra is distributive, and in fact is a Heyting algebra, see [16, p. 238] or [151, Thm 2.3], results of the previous section also apply to the class of Heyting algebras as well.

Using Theorem 3.4.2, Proposition 3.3.9, and the fact that for a Heyting algebra $\mathbf{K}$ each principle ideal $(\downarrow k)_{\mathbf{K}}$ and filter $(\uparrow k)_{\mathbf{K}}$ are Heyting algebras, gives the following result.

3.5.1. Theorem. *For a finite distributive lattice $\mathbf{P}$ that is projective in the class of distributive lattices, the lattices $\mathbf{1} \oplus \mathbf{P}$ and $\mathbf{P} \oplus \mathbf{1}$ are $(\wedge, \vee, 0, 1)$-MacNeille transferable for $\mathcal{HA}$, and for the seven-element distributive lattice $\mathbf{D}_4$ of Figure 3.2, the lattice $\mathbf{1} \oplus \mathbf{D}_4 \oplus \mathbf{1}$ is $(\wedge, \vee, 0, 1)$-MacNeille transferable for $\mathcal{HA}$.*

We consider a closely related result, but one that to the best of our knowledge requires a completely different approach.

3.5.2. THEOREM. *For the seven-element distributive lattice* $\mathbf{D}_4$ *shown in Figure 3.2, the lattice* $\mathbf{D}_4 \oplus \mathbf{1}$ *is* $(\wedge, \vee, 0, 1)$*-MacNeille transferable for* $\mathcal{HA}$.

Proof:
Let $\mathbf{K}$ be a Heyting algebra and suppose there is a bounded sublattice of $\overline{\mathbf{K}}$ isomorphic to $\mathbf{D}_4 \oplus \mathbf{1}$. Then there are normal ideals P, Q, R, S of $\mathbf{K}$ situated as in Figure 3.3. We will show that $\mathbf{K}$ has a bounded sublattice isomorphic to $\mathbf{D}_4 \oplus \mathbf{1}$. If there are $0 < p_1 < p \in P$ and $0 < q_1 < q \in Q$, then since $P \wedge Q = 0$ we

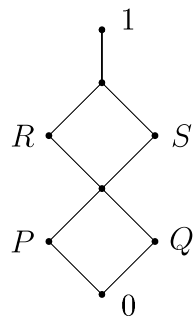


Figure 3.3: The lattice $\mathbf{D}_4 \oplus \mathbf{1}$ as a bounded sublattice of $\overline{\mathbf{K}}$

have that $p \wedge q = 0$, hence also that $p_1 \wedge q_1 = 0$. Furthermore, it follows from applications of the distributive law that

$$\begin{aligned}(p \vee q_1) \wedge (q \vee p_1) &= ((p \vee q_1) \wedge q) \vee ((p \vee q_1) \wedge p_1)\\ &= ((p \wedge q) \vee (q_1 \wedge q)) \vee ((p \wedge p_1) \vee (q_1 \wedge p_1))\\ &= (0 \vee q_1) \vee (p_1 \vee 0)\\ &= p_1 \vee q_1\end{aligned}$$

Consequently, $\{0, p_1, q_1, p_1 \vee q_1, p \vee q_1, q \vee p_1, p \vee q, 1\}$ is a bounded sublattice of $\mathbf{K}$ isomorphic to $\mathbf{D}_4 \oplus \mathbf{1}$.

Thus it remains to consider the case where one of the normal ideals P and Q, say without loss of generality P, is an atom of $\overline{\mathbf{K}}$. In this case $P = \downarrow p$ for some atom p of $\mathbf{K}$. Choose $r \in R \setminus S$ and $s \in S \setminus R$ with $p \leq r, s$. Since $\mathbf{K}$ is a Heyting subalgebra of $\overline{\mathbf{K}}$, see, e.g., [151, Sec. 2], the pseudo-complement $\neg p$ of p in $\mathbf{K}$ is the pseudo-complement of p in $\overline{\mathbf{K}}$. Therefore, $\neg p \geq Q$, giving $p \vee \neg p \geq P \vee Q = R \wedge S \geq r \wedge s$. It follows that

$$p \vee (r \wedge s \wedge \neg p) = (p \vee (r \wedge s)) \wedge (p \vee \neg p) = (r \wedge s) \wedge (p \vee \neg p) = r \wedge s.$$

Thus, as $p \wedge (r \wedge s \wedge \neg p) = 0$ we see that $\{0, p, r \wedge s \wedge \neg p, r \wedge s, r, s, r \vee s, 1\}$ is a bounded sublattice of $\mathbf{K}$ isomorphic to $\mathbf{D}_4 \oplus \mathbf{1}$. □

3.5.3. Remark. Note that in the proof of Theorem 3.5.2 we only made use of the existence of pseudo-complements and the fact that the embedding a Heyting algebra into its MacNeille completion preserves pseudo-complements. Since the MacNeille completion of any pseudo-complemented lattice is again pseudo-complemented and the embedding into the MacNeille completion preserves pseudo-complements we may in fact conclude that the lattice $\mathbf{D}_4 \oplus \mathbf{1}$ is $(\wedge, \vee, 0, 1)$-transferable for the class of all pseudo-complemented lattices.

To make further progress with positive results, we consider more restrictive classes of Heyting algebras. We will make use of Esakia duality for Heyting algebras, see, e.g., [87, 24] or Appendix A.6. For the Esakia space X of a Heyting algebra $\mathbf{K}$, the maximum of X, denoted $\max(X)$, is the set of maximal elements of the underlying poset of X.

3.5.4. Definition. Let $\mathbf{K}$ be a Heyting algebra with dual Esakia space X and let $n \geq 1$ be a natural number. We say that $\mathbf{K}$ and X

(i) have *width* n if n is the maximal cardinality of an anti-chain in X,

(ii) have *top width* n if n is the cardinality of the maximum $\max(X)$ of X.

We say that $\mathbf{K}$ and X have *finite* (*top*) *width* if they have (top) width n for some natural number n. We let $\mathcal{HA}_w$ denote the class of Heyting algebras of finite width, and $\mathcal{HA}_t$ the class of Heyting algebras of finite top width.

Let $\mathbf{K}$ be a Heyting algebra. We call $S \subseteq K$ *orthogonal* if $a \wedge b = 0$ for any distinct $a, b \in S$. An element a of K is *regular* if $a = \neg\neg a$. Since the regular elements form a Boolean algebra $\mathrm{Rg}(\mathbf{K})$ that is a $(\wedge, 0)$-subalgebra of $\mathbf{K}$, see Appendix A.5, an orthogonal set in $\mathrm{Rg}(\mathbf{K})$ is an orthogonal set in $\mathbf{K}$. The following two lemmas are easily proved, see, e.g., [25, Thm. 7.5(1) and Thm. 7.5(3)].

3.5.5. Lemma. *Let* $\mathbf{K}$ *be a Heyting algebra and* $n \geq 1$ *a natural number. Then* $\mathbf{K}$ *has width at most* n *if, and only if, the algebra* $\mathbf{2}^{n+1}$ *is not isomorphic to a sublattice of* $\mathbf{K}$.

3.5.6. Lemma. *Let* $\mathbf{K}$ *be a Heyting algebra and* $n \geq 1$ *a natural number. The following are equivalent.*

1. *The algebra* $\mathbf{K}$ *has top width at most* n.
2. *The algebra* $\mathbf{2}^{n+1} \oplus \mathbf{1}$ *is not isomorphic to a bounded sublattice of* $\mathbf{K}$.
3. *The maximal cardinality of an orthogonal set in* $\mathbf{K}$ *is at most* n.
4. *The cardinality of the algebra* $\mathrm{Rg}(\mathbf{K})$ *is at most* 2^n.

If a finite distributive lattice $\mathbf{D}$ is MacNeille transferable for $\mathcal{HA}$, i.e., τ-MacNeille transferable for $\tau = \{\wedge, \vee\}$, then by definition it is also MacNeille transferable for $\mathcal{HA}_w$. The converse holds as well, since $\mathbf{D}$ is a sublattice of a finite Boolean algebra, hence of any Heyting algebra that is not of finite width. Similar reasoning shows that $\mathbf{D} \oplus \mathbf{1}$ is MacNeille transferable for $\mathcal{HA}_t$ if, and only if, $\mathbf{D} \oplus \mathbf{1}$ is MacNeille transferable for $\mathcal{HA}$. However, the notion of $(\wedge, \vee, 0, 1)$-MacNeille transferability for $\mathcal{HA}_w$ and $\mathcal{HA}_t$ differs from that of $\mathcal{HA}$. We begin with the following, first recalling that an element a in a Heyting algebra $\mathbf{A}$ is *complemented* if $a \wedge \neg a = 0$, or equivalently $\neg\neg a = a$.

3.5.7. Lemma. *If $\mathbf{K}$ is a Heyting algebra of finite top width, then the complemented elements of the MacNeille completion $\overline{\mathbf{K}}$ belong to $\mathbf{K}$.*

Proof:
Let x be a complemented element of $\overline{\mathbf{K}}$. Since $\mathbf{K}$ is a Heyting subalgebra of $\overline{\mathbf{K}}$, for $a \in K$, we have that $\neg a$ is the pseudo-complement of a in both $\mathbf{K}$ and $\overline{\mathbf{K}}$. Suppose $a \leq x$. Then $a \leq \neg\neg a \leq \neg\neg x = x$ and hence the normal ideal $N = \{a \in K : a \leq x\}$ must be generated by the regular elements of $\mathbf{K}$ below x. Since $\mathbf{K}$ has finite top width, by Lemma 3.5.6, there are only finitely many regular elements of $\mathbf{K}$. The regular elements of $\mathbf{K}$ form a join semi-lattice with $\neg\neg(a \vee b)$ the least upper bound of a, b in $\mathrm{Rg}(\mathbf{K})$. Consequently, there must be a largest regular element in N, showing that N is a principal ideal of $\mathbf{K}$. Since $\mathbf{K}$ is join-dense in $\overline{\mathbf{K}}$, we have $x = \bigvee N$, hence $x \in K$. □

3.5.8. Remark. Lemma 3.5.7 may be false without the assumption of finite top width as is seen by considering the MacNeille completion of an incomplete Boolean algebra.

3.5.9. Theorem.

1. *The class of finite lattices that are $(\wedge, \vee, 0, 1)$-MacNeille transferable for the class $\mathcal{HA}_t$ is closed under finite products.*

2. *The class of finite lattices that are $(\wedge, \vee, 0, 1)$-MacNeille transferable for the class $\mathcal{HA}_w$ is closed under finite products.*

Proof:
Let $\mathbf{D}_1$ and $\mathbf{D}_2$ be finite lattices that are $(\wedge, \vee, 0, 1)$-MacNeille transferable for $\mathcal{HA}_t$. Suppose that $\mathbf{K} \in \mathcal{HA}_t$ and $h\colon \mathbf{D}_1 \times \mathbf{D}_2 \hookrightarrow \overline{\mathbf{K}}$ is a bounded lattice embedding. Let $h(1,0) = x$ and $h(0,1) = y$. Then x, y are complemented elements of $\overline{\mathbf{K}}$, and since $\mathbf{K}$ is of finite top width, Lemma 3.5.7 implies $x, y \in K$.

The restrictions $h \,|\, {\downarrow}(1,0)\colon {\downarrow}(1,0) \hookrightarrow ({\downarrow}x)_{\overline{\mathbf{K}}}$ and $h \,|\, {\downarrow}(0,1)\colon {\downarrow}(0,1) \hookrightarrow ({\downarrow}y)_{\overline{\mathbf{K}}}$ are bounded lattice embeddings. But ${\downarrow}(1,0)$ and ${\downarrow}(0,1)$ are isomorphic to $\mathbf{D}_1$

and $\mathbf{D}_2$ respectively, while $({\downarrow}x)_{\overline{\mathbf{K}}}$ and $({\downarrow}y)_{\overline{\mathbf{K}}}$ are isomorphic to $\overline{({\downarrow}x)_{\mathbf{K}}}$ and $\overline{({\downarrow}y)_{\mathbf{K}}}$ respectively. Since $\mathbf{D}_1$ and $\mathbf{D}_2$ are $(\wedge, \vee, 0, 1)$-MacNeille transferable for the class $\mathcal{HA}_t$ and the Heyting algebras $({\downarrow}x)_{\mathbf{K}}$ and $({\downarrow}y)_{\mathbf{K}}$ belong to $\mathcal{HA}_t$, there are bounded sublattices of $({\downarrow}x)_{\mathbf{K}}$ and $({\downarrow}y)_{\mathbf{K}}$ isomorphic to $\mathbf{D}_1$ and $\mathbf{D}_2$ respectively. Since $\mathbf{K}$ is isomorphic to $({\downarrow}x)_{\mathbf{K}} \times ({\downarrow}y)_{\mathbf{K}}$, we may conclude that $\mathbf{K}$ has a bounded sublattice isomorphic to $\mathbf{D}_1 \times \mathbf{D}_2$. The argument for $\mathcal{HA}_w$ is identical, using that $\mathcal{HA}_w \subseteq \mathcal{HA}_t$. $\square$

As we will see below, the class of finite lattices that are $(\wedge, \vee, 0, 1)$-MacNeille transferable for $\mathcal{HA}$ is not closed under binary products. In fact, Theorem 3.5.12 shows that the product of any two non-trivial finite distributive lattices is not $(\wedge, \vee, 0, 1)$-MacNeille transferable for the class $\mathcal{HA}$. Thus, the results of Theorem 3.5.9 fail if we replace $\mathcal{HA}_t$ or $\mathcal{HA}_w$ by $\mathcal{HA}$. To establish this we require a preliminary definition and lemma.

Let $\mathbb{T}_1, \dots, \mathbb{T}_k$ be trees with respective roots $r_1, \dots, r_k$. Suppose for each $i \in \{2, \dots, k-1\}$ there are two distinct maximal nodes t_i^r and t_i^l of the tree $\mathbb{T}_i$ such that $t_{i-1}^r = t_i^l$ and $T_{i-1} \cap T_i = \{t_{i-1}^r\}$, and $T_i \cap T_j = \emptyset$ for $j \notin \{i-1, i, i+1\}$, as shown in Figure 3.4.

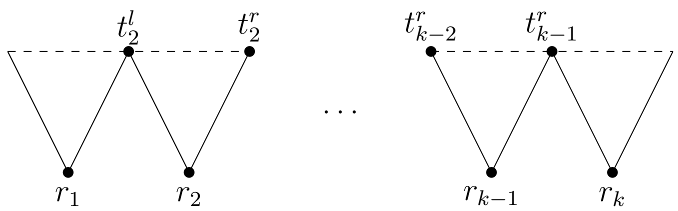


Figure 3.4: A tree sum of posets.

Let $\mathbb{T}$ be the union of the trees $\mathbb{T}_i$. Any poset of this form is called a *tree sum*. The following very simple tree sums will play an important role.

3.5.10. Definition. For a natural number $n \geq 1$ let $\mathbb{S}_n$ be the finite connected poset shown in Figure 3.5.

3.5.11. Lemma. *For each finite connected poset $\mathbb{P}$, there is a natural number $n \geq 1$ and an order-preserving map from $\mathbb{S}_n$ onto $\mathbb{P}$.*

Proof:
By [30, Lem. 16], for every finite connected poset $\mathbb{P}$ there is a tree sum $\mathbb{T}$ such that $\mathbb{P}$ is the image of $\mathbb{T}$ under an order-preserving map. Therefore, without loss

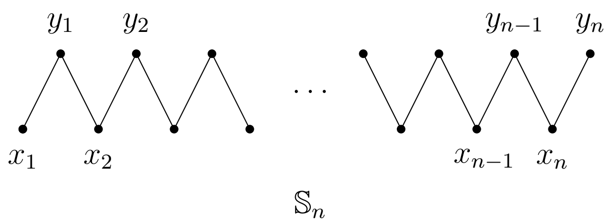


Figure 3.5: The poset $\mathbb{S}_n$.

of generality, we may assume that $\mathbb{P}$ is a tree sum, say $\bigcup_{i=1}^{k} \mathbb{T}_i$, with nodes labeled as in Figure 3.4. We then define recursively

$$\begin{aligned} n_1 &= |T_1| - 1 \\ n_i &= n_{i-1} + |T_i| - 2 \quad \text{for } 2 \leq i \leq k. \end{aligned}$$

Let $n = n_k$ and define an order-preserving map $f\colon \mathbb{S}_n \to \mathbb{P}$ as follows. Let f map $y_1, \ldots, y_{n_1 - 1}$ bijectively onto $T_1 \setminus \{r_1, t_2^l\}$. The exact nature of this bijection is irrelevant. Let f map $x_1, \ldots, x_{n_1}$ to r_1 and y_{n_1} to t_2^l. For each $2 \leq i < k$ let h map $y_{n_{i-1}+1}, \ldots, y_{n_i - 1}$ bijectively onto $T_i \setminus \{r_i, t_i^l, t_i^r\}$, map $x_{n_{i-1}+1}, \ldots, x_{n_i}$ to r_i, and map y_{n_i} to t_i^r. Finally, let f map $y_{n_{k-1}+1}, \ldots, y_n$ bijectively onto $T_k \setminus \{r_k, t_{k-1}^r\}$ and map $x_{n_{k-1}+1}, \ldots, x_n$ to r_k. Then f is the desired map. □

3.5.12. THEOREM. *Any finite distributive lattice* $(\wedge, \vee, 0, 1)$*-MacNeille transferable for the class* $\mathcal{HA}$ *is directly indecomposable.*

Proof:
We first consider the case of a finite directly decomposable distributive lattice $\mathbf{D}$ of the form $\mathbf{D}_1 \times \mathbf{D}_2$ with $\mathbf{D}_1$ and $\mathbf{D}_2$ directly indecomposable. We will construct a Heyting algebra $\mathbf{K}$ such that $\mathbf{D}$ is isomorphic to a bounded sublattice of $\overline{\mathbf{K}}$ but not to a bounded sublattice of $\mathbf{K}$. This will show that $\mathbf{D}$ is not $(\wedge, \vee, 0, 1)$-MacNeille transferable for $\mathcal{HA}$.

To this effect let X be the Esakia space whose domain is the countable set $\{x_i, y_i, w_i, z_i : i \geq 1\} \cup \{\infty\}$ topologized as the one-point compactification of the discrete topology on $\{x_i, y_i, w_i, z_i : i \geq 1\}$ with compactification point ∞ and whose ordering is as shown in Figure 3.6.

Let $\mathbf{K}$ be the Heyting algebra of clopen upsets of X. Since $\varnothing$ and X are the only clopen upsets of X that are also downsets, the only complemented elements of $\mathbf{K}$ are the trivial elements $0, 1$. Thus $\mathbf{K}$ has no non-trivial complemented elements and consequently $\mathbf{D}$ is not isomorphic to a bounded sublattice of $\mathbf{K}$

It is easy to see that $\mathbf{K}$ is in fact a bi-Heyting algebra. Therefore, the elements of the MacNeille completion of $\mathbf{K}$ are the regular open upsets of X, see [151,

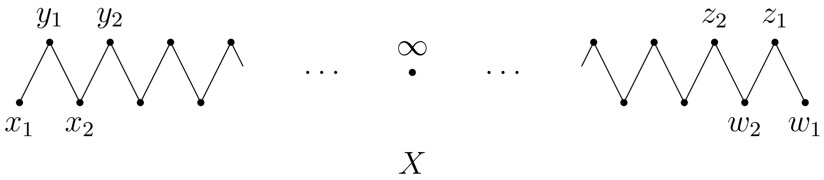


Figure 3.6: The Easkia space X.

Thm. 3.8]. It follows that $\{x_i, y_i : i \geq 1\}$ and $\{w_i, z_i : i \geq 1\}$ are the new complemented elements of $\overline{\mathbf{K}}$. Thus, letting $\overline{X}$ denote the dual Esakia space of the algebra $\overline{\mathbf{K}}$ we see that this space is isomorphic to a space with domain $\{x_i, y_i, w_i, z_i : i \geq 1\} \cup \{\infty_1, \infty_2\}$ topologized as the two-point compactification of $\{x_i, y_i, w_i, z_i : i \geq 1\}$. Consequently, $\overline{X}$ is the disjoint union of two Esakia spaces X_1 and X_2, each of which carries the topology of the one-point compactification of the discrete topology on $\{x_i, y_i : i \geq 1\}$ and $\{w_i, z_i : i \geq 1\}$, respectively; see Figure 3.7.

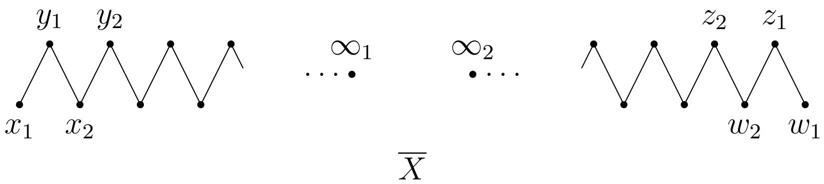


Figure 3.7: The Esakia space $\overline{X}$.

To show that $\mathbf{D}$ is isomorphic to a bounded sublattice of $\overline{\mathbf{K}}$, by duality, it is suffices to show that the Esakia dual Y of $\mathbf{D}$ is a continuous order-preserving image of $\overline{X}$. Since $\mathbf{D}$ is a product of two finite directly indecomposable distributive lattice its dual Esakia space must be discrete and order-disconnected. Let $Y = Y_1 \cup Y_2$ be the decomposition of the dual space Y of $\mathbf{D}$ into its two order-connected components Y_1 and Y_2.

We show that there is a continuous order-preserving map from $\overline{X}$ onto Y. By Lemma 3.5.11, there are natural numbers n_1 and n_2 together with order-preserving onto maps $f_1 \colon S_{n_1} \twoheadrightarrow Y_1$ and $f_2 \colon S_{n_2} \twoheadrightarrow Y_2$. We have that $\overline{X}$ is the disjoint union of X_1 and X_2 and we can regard $\mathbb{S}_{n_1}$ as a subposet of the underlying poset of X_1 and $\mathbb{S}_{n_2}$ as a subposet of the underlying poset of X_2 in the obvious way. Let m_1 be a minimal element of Y_1 that is below $f_1(y_{n_1})$ and let m_2 be a minimal element of Y_2 that is below $f_2(y_{n_2})$. Define

$$f(s) = \begin{cases} f_1(s) & \text{if } s \in S_{n_1}, \\ f_2(s) & \text{if } s \in S_{n_2}, \\ m_1 & \text{if } s \in X_1 \setminus S_{n_1}, \\ m_2 & \text{if } s \in X_2 \setminus S_{n_2}. \end{cases}$$

Then f is the desired continuous order-preserving map from $\overline{X}$ onto Y, showing that $\mathbf{D}$ is a bounded sublattice of $\overline{\mathbf{K}}$.

The general case may then be established as follows. Consider a finite directly decomposable distributive lattice $\mathbf{D}$ of the form $\prod_{i=1}^{n} \mathbf{D}_i$, with $n \geq 3$ and each factor $\mathbf{D}_i$ directly indecomposable. By the above $\mathbf{D}_1 \times \mathbf{D}_2$ is a bounded sublattice of $\overline{\mathbf{K}}$. Therefore, letting $\mathbf{D}'$ be the direct product $\prod_{i=3}^{n} \mathbf{D}_i$ we must have that $\mathbf{D}$ is isomorphic to a bounded sublattice of the Heyting algebra $\overline{\mathbf{K}} \times \mathbf{D}' = \overline{\mathbf{K} \times \mathbf{D}'}$, but not to a bounded sublattice of $\mathbf{K} \times \mathbf{D}'$. This shows that the lattice $\mathbf{D}$ is not $(\wedge, \vee, 0, 1)$-MacNeille transferable for the class $\mathcal{HA}$. □

To conclude this section, we give an example which shows that even in the setting of Heyting algebras, sharp MacNeille transferability and MacNeille transferability is not the same.

3.5.13. PROPOSITION. *The lattice $\mathbf{D}_4$ of Figure 3.2 is not sharply MacNeille transferable for the class $\mathcal{HA}$.*

Proof:
Let $\mathbf{K}'$ be the sublattice of the Euclidean plane $\mathbb{R}^2$ with domain

$$\{(x, y) \mid 0 \leq x < 1, 0 \leq y \leq x\} \cup \{(x, y) \mid 1 < x \leq 2, 0 \leq y \leq 1\}.$$

and let $\mathbf{K}$ be the quotient of the lattice $\mathbf{K}' \oplus \mathbf{2}^2$, displayed at the left in Figure 3.8, obtained by identifying the greatest element of lattice $\mathbf{K}'$ with the least element of lattice $\mathbf{2}^2$. It is routine to verify that $\mathbf{K}$ is a Heyting algebra. The MacNeille completion $\overline{\mathbf{K}}$ of $\mathbf{K}$ inserts the "missing" line $\{(1, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\}$. The black circles in the picture of $\overline{\mathbf{K}}$ at right in Figure 3.8 define a lattice embedding h from $\mathbf{D}_4$ into $\overline{\mathbf{K}}$. Suppose there is a lattice embedding $h' \colon \mathbf{D}_4 \hookrightarrow \mathbf{K}$ that is sharp, meaning that $h'(a) \leq h(b)$ if, and only if, $a \leq b$, for all $a, b \in D_4$. Recall that $\mathbf{D}_4$ is a quotient of the linear sum of two four-element Boolean algebras. Because h' is sharp, the maps h and h' must agree on the top element and the two co-atoms of $\mathbf{D}_4$. Consequently, h' must map the doubly reducible element of $\mathbf{D}_4$ to the element $(2, 1)$ of $\mathbf{K}$. Since $h \colon \mathbf{D}_4 \hookrightarrow \overline{\mathbf{K}}$ maps the bottom element of $\mathbf{D}_4$ to the element $(1, 0)$, it follows that $h' \colon \mathbf{D}_4 \hookrightarrow \mathbf{K}$ must map the bottom element of $\mathbf{D}_4$ to some element $(a, 0)$ of $\mathbf{K}$ with $a < 1$. However, there do not exist elements in $\mathbf{K}$ whose meet is $(a, 0)$ for some $a < 1$ and whose join is $(2, 1)$. □

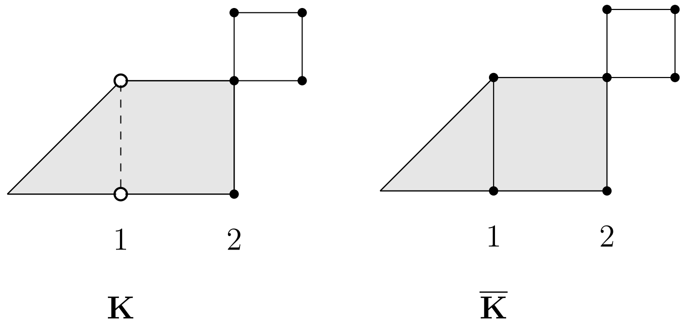


Figure 3.8: The Heyting algebras $\mathbf{K}$ and $\overline{\mathbf{K}}$ as sublattices of the Euclidean plane.

3.5.14. REMARK. The lattice $\mathbf{D}_4$ also plays a central role in the work of Wehrung [250] where it is shown that $\mathbf{D}_4$ is sharply ideal transferable for a variety of lattice $\mathcal{V}$ if, and only if, $\mathcal{V}$ is contained in the variety of lattice generated by the class of all lattices of length 2.

3.6 MacNeille transferability for bi-Heyting algebras

Restricting attention to the class $bi\mathcal{HA}$ of bi-Heyting algebras, stronger results of a positive nature can be obtained. Recall that a Heyting algebra $\mathbf{K}$ is a bi-Heyting algebra if the order dual of $\mathbf{K}$ is also a Heyting algebra. We note that the property of being a bi-Heyting algebra is preserved under MacNeille completions, see, e.g., [151]. Recall further that a subset U of a topological space X is *regular open* provided that $\mathsf{IC}(U) = U$, where I and C denote the interior and closure operator on X, respectively.

3.6.1. LEMMA. *Let X be an Esakia space of finite width. If U, V are regular open upsets of X, then so is the union $U \cup V$.*

Proof:
Suppose the width of X is n. Then, by [72, p. 3], X can be covered by n maximal chains $C_1, \ldots, C_n$. By [87, Lem. III.2.8], each of these maximal chains are closed in X. To show that $U \cup V$ is regular open, it is enough to show that $\mathsf{IC}(U \cup V) \subseteq U \cup V$.

Let $x \in \mathsf{IC}(U \cup V)$. Then there is a clopen set F with $x \in F \subseteq \mathsf{C}(U \cup V) = \mathsf{C}(U) \cup \mathsf{C}(V)$. Let $S = \{i \leq n : x \in C_i\}$ and $T = \{i \leq n : x \notin C_i\}$. Consider the following statement

$$U \cap C_i \subseteq \mathsf{C}(V) \quad \text{for each } i \in S. \tag{$*$}$$

Suppose $(*)$ holds. Then since $D := \bigcup\{C_i : i \in T\}$ is a closed set and $x \notin D$, there is a clopen F' with $x \in F' \subseteq F \subseteq \mathsf{C}(U) \cup \mathsf{C}(V)$ and F' disjoint from

D. In particular, $F' \subseteq (F' \cap \mathsf{C}(U)) \cup \mathsf{C}(V)$. Since $C_1 \cup \cdots \cup C_n$ covers X and F' is disjoint from D, and hence disjoint from each $C_i = \mathsf{C}(C_i)$ for $i \in T$, we have $F' \cap \mathsf{C}(U) = \bigcup\{F' \cap \mathsf{C}(U \cap C_i) : i \in S\}$. Then condition $(*)$ gives that $F' \cap \mathsf{C}(U) \subseteq \mathsf{C}(V)$, hence $F' \subseteq \mathsf{C}(V)$. Thus, $x \in \mathsf{IC}(V) = V$.

Suppose that the condition $(*)$ does not hold. Then there is i with $x \in C_i$ and $U \cap C_i \nsubseteq \mathsf{C}(V)$. Let $y \in U \cap C_i$ with $y \notin \mathsf{C}(V)$. Note that if $x \in U$, then *a fortiori* $x \in U \cup V$. On the other hand if $x \notin U$, then since x, y belong to the chain C_i and U is an upset, we must have $x < y$. Since $y \notin \mathsf{C}(V)$, there is a clopen G with $y \in G$ and G disjoint from V. So G is disjoint from $\mathsf{C}(V)$. Since X is an Esakia space and V is an upset, $\mathsf{C}(V)$ is an upset, see, e.g., [151, Lem. 3.6(3)]. Therefore, the fact that G is disjoint from $\mathsf{C}(V)$ implies that the clopen set ${\downarrow}G$ is disjoint from the closed up $\mathsf{C}(V)$. Thus, $F'' = F \cap {\downarrow}G$ is a clopen neighborhood of x disjoint from $\mathsf{C}(V)$. Since $F'' \subseteq F \subseteq \mathsf{C}(U) \cup \mathsf{C}(V)$, we have $F'' \subseteq \mathsf{C}(U)$. Consequently, $x \in \mathsf{IC}(U) = U$. □

3.6.2. Proposition. *If* $\mathbf{K}$ *is a bi-Heyting algebra of finite width, then* $\overline{\mathbf{K}}$ *is a bounded sublattice of* $\mathrm{Idl}(\mathbf{K})$.

Proof:
Let X be the dual Esakia space of $\mathbf{K}$. It is well known that $\mathrm{Idl}(\mathbf{K})$ is isomorphic to the lattice of open upsets of X, and since $\mathbf{K}$ is a bi-Heyting algebra, by [151, Thm 3.8], $\overline{\mathbf{K}}$ is isomorphic to the lattice of regular open upsets of X. By Lemma 3.6.1, the lattice of regular open upsets of X is a bounded sublattice of the lattice of open upsets of X. □

3.6.3. Remark. Note that the assumption of finite width cannot be dropped from Proposition 3.6.2 as for example considering the dual Stone spaces of incomplete infinite Boolean algebras shows.

As a consequence, we obtain the following, letting $bi\mathcal{HA}_w$ denote the class of bi-Heyting algebras of finite width.

3.6.4. Theorem. *Let* $\mathbf{D}$ *be a finite distributive lattice.*

1. *The lattice* $\mathbf{D}$ *is simultaneously sharply and faithfully MacNeille transferable for the class* $bi\mathcal{HA}_w$.

2. *The lattice* $\mathbf{D}$ *is MacNeille transferable for the class* $bi\mathcal{HA}$.

Proof:
If $\mathbf{K}$ is a bi-Heyting algebra of finite width and $h\colon \mathbf{D} \hookrightarrow \overline{\mathbf{K}}$ is a lattice embedding, then since $\overline{\mathbf{K}}$ is a sublattice of $\mathrm{Idl}(\mathbf{K})$ by Proposition 3.6.2, we have a lattice

embedding $h\colon \mathbf{D} \hookrightarrow \mathrm{Idl}(\mathbf{K})$. Then simultaneous sharp and faithful MacNeille transferability follows directly from Theorem 3.1.5.

That $\mathbf{D}$ is MacNeille transferable for the class of all bi-Heyting algebras then follows from the fact that $\mathbf{D}$ is a sublattice of a finite Boolean algebra, and hence by Lemma 3.5.5, of any bi-Heyting algebra of infinite width. □

For any finite lattice $\mathbf{L}$, being simultaneously sharply and faithfully MacNeille transferable with respect to a class of bounded lattices $\mathcal{K}$ entails that $\mathbf{L}$ is $(\wedge, \vee, 0, 1)$-transferable with respect to $\mathcal{K}$. Thus, we obtain the following corollary.

3.6.5. Corollary. *Any finite distributive lattice is $(\wedge, \vee, 0, 1)$-transferable with respect to the class of bi-Heyting algebras of finite width.*

This together with Theorem 3.2.6 yields the following result.

3.6.6. Corollary. *If $\mathbf{K}$ is a bi-Heyting algebra of finite width then $\overline{\mathbf{K}}$ is isomorphic to a bounded sublattice of an ultrapower of $\mathbf{K}$.*

3.7 Canonicity and elementarity

In this section we will show how classes of finite distributive τ-canonically and τ-MacNeille transferable lattices give rise to varieties of Heyting algebras which are canonical and elementarily determined, respectively. This will allow us to obtain results about τ-stable logics in Section 3.8. As we are concerned with applications to $(\wedge, 0, 1)$- and $(\wedge, \vee, 0, 1)$-stable intermediate logics we will here mainly focus on the cases $\tau = \{\wedge, 0, 1\}$ and $\tau = \{\wedge, \vee, 0, 1\}$.

3.7.1 Transferability and canonicity

We show how τ-canonical transferability can be used to establish canonicity of varieties of Heyting algebras. If $\mathbf{A}$ is a Heyting algebra then the canonical completion $\mathbf{A}^\delta$ is also a Heyting algebra and $\mathbf{A}$ is not only a sublattice, but in fact a Heyting subalgebra, of $\mathbf{A}^\delta$, see, e.g., [109, Prop. 2].

3.7.1. Proposition. *Let $\tau \subseteq \{\wedge, \vee, 0, 1\}$ and let $\mathcal{J}$ be a class of finite distributive lattices which are all τ-canonically transferable for the class of Heyting algebras. Then the variety of Heyting algebras generated by the class $\mathcal{HA}_\tau(\mathcal{J})$ is canonical.*

Proof:
Since all members of $\mathcal{J}$ are τ-canonically transferable for the class of Heyting algebras, we obtain that the class $\mathcal{HA}_\tau(\mathcal{J})$ is closed under canonical completions by Proposition 3.2.2. Evidently, $\mathcal{HA}_\tau(\mathcal{J})$ is a universal class of Heyting algebras

and by [111, Thm. 6.8] any universal class closed under canonical completions generates a canonical variety. $\square$

By Proposition 3.7.1 any finite lattice which is τ-MacNeille transferable for the class of all Heyting algebras is also τ-canonically transferable for the class of all Heyting algebras, whence we obtain the following corollary.

3.7.2. Corollary. *Let $\tau \subseteq \{\wedge, \vee, 0, 1\}$ and let $\mathcal{J}$ be a class of finite distributive lattices which are all τ-MacNeille transferable for the class of Heyting algebras. Then the variety of Heyting algebras generated by the class $\mathcal{HA}_\tau(\mathcal{J})$ is canonical.*

We will show that the assumption in Proposition 3.7.1 that $\mathcal{J}$ is a class of finite distributive lattices which are all τ-canonically transferable for the class of all Heyting algebras can be replaced with a weaker assumption.

3.7.3. Definition. We say that a Heyting algebra is *pseudo-finite* if it is a model of the first-order theory, in the language of Heyting algebras, of the class of all finite Heyting algebras. Given a class of Heyting algebras $\mathcal{K}$ we let $\mathcal{K}^{pf}$ denote the class of pseudo-finite members of $\mathcal{K}$.

3.7.4. Remark. It is not difficult to see that the property of being the reduct of a co-Heyting algebra is expressible in the first-order language of lattices. Thus as all finite Heyting algebras are (reducts of) bi-Heyting algebras so are all pseudo-finite Heyting algebras. Similarly, it can be shown that any element in a pseudo-finite Heyting algebra must be the join of the join-irreducible elements below it and the meet of the meet-irreducible elements above it.

The following lemma is essentially a variation of [111, Thm. 6.8]. Recall that for a class of algebras $\mathcal{K}$ we write $\mathcal{K}_{si}$ and $\mathcal{K}_{fsi}$ for the classes of subdirectly irreducible and finitely subdirectly irreducible members of $\mathcal{K}$ respectively.

3.7.5. Lemma. *Let $\mathcal{V}$ be a variety of Heyting algebras generated by its finite members. If $\mathbf{A}^\delta \in \mathcal{V}$ for all $\mathbf{A} \in \mathcal{V}_{si}^{pf}$, then $\mathcal{V}$ is canonical.*

Proof:
Consider $\mathbf{A} \in \mathcal{V}$ and let $\kappa = |A|$. Then $\mathbf{A}$ is a homomorphic image of the freely κ-generated $\mathcal{V}$-algebra $\mathbf{F}_\mathcal{V}(\kappa)$. Since by assumption $\mathcal{V}$ is generated by its finite members the algebra $\mathbf{F}_\mathcal{V}(\kappa)$ embeds into a direct product of finite subdirectly irreducible members of $\mathcal{V}$, say $\prod_{i\in I} \mathbf{A}_i$. Since canonical completions of Heyting algebras preserve surjective and injective homomorphisms [111, Thm. 5.4] the algebra $\mathbf{A}^\delta$ is a homomorphic image of a subalgebra of the algebra $(\prod_{i\in I} \mathbf{A}_i)^\delta$.

Let $\beta(I)$ be the set of all ultrafilters over the set I. As the dual Stone space of the Boolean algebra $\wp(I)$ the set $\beta(I)$ carries a Stone topology. In fact, the direct product $\prod_{i\in I} \mathbf{A}_i$ has a Boolean product representation over $\beta(I)$, with stalks $\mathbf{A}_U$ given by the ultraproduct $\prod_{i\in I}\mathbf{A}_i/U$, for each $U\in\beta(I)$, see, e.g., [115, Thm. 3.17].

The canonical completion of a Boolean product is the direct product of the canonical completions of the stalks, see, e.g., [111, Lem. 6.7], whence,

$$\left(\prod_{i\in I}\mathbf{A}_i\right)^{\delta}=\prod_{U\in\beta(I)}\mathbf{A}_U^{\delta}.$$

By Łoś' Theorem the algebra $\mathbf{A}_U$ is evidently pseudo-finite for each $U\in\beta(I)$. Similarly, since the property of being a subdirectly irreducible Heyting algebra is first-order definable in the language of lattices $\mathbf{A}_U\in\mathcal{V}_{si}^{pf}$, for all $U\in\beta(I)$. Consequently, by assumption $\mathbf{A}_U^{\delta}\in\mathcal{V}$ for all $U\in\beta(I)$, showing that $(\prod_{i\in I}\mathbf{A}_i)^{\delta}$ and therefore also $\mathbf{A}^{\delta}$ belongs to $\mathcal{V}$. $\Box$

In the rest of this section we will focus on the case $\tau=\{\wedge,0,1\}$ or $\tau=\{\wedge,\vee,0,1\}$.

3.7.6. Lemma. *Let τ be either $\{\wedge,0,1\}$ or $\{\wedge,\vee,0,1\}$. If $\mathcal{V}$ is a variety of Heyting algebras generated by the universal class $\mathcal{HA}_\tau(\mathcal{J})$, for some class of finite distributive lattices $\mathcal{J}$, then $\mathcal{V}_{fsi}\subseteq\mathcal{HA}_\tau(\mathcal{J})$.*

Proof:
In case $\tau=\{\wedge,0,1\}$ this follows from Lemmas 2.2.6 and 2.2.10 and in case $\tau=\{\wedge,\vee,0,1\}$ from [29, Prop. 5.1, Claim. 5.2]. $\Box$

3.7.7. Proposition. *Let τ be either $\{\wedge,0,1\}$ or $\{\wedge,\vee,0,1\}$, let $\mathcal{J}$ be a class of finite distributive lattices, and let $\mathcal{V}$ the variety of Heyting algebras generated by the universal class $\mathcal{HA}_\tau(\mathcal{J})$. If all members of $\mathcal{J}$ are τ-canonically transferable for the class $\mathcal{V}_{si}^{pf}$, then the variety $\mathcal{V}$ is canonical.*

Proof:
That $\mathcal{V}$ is generated by its finite members follows from Theorem 2.2.5. Thus by Lemma 3.7.5 it suffices to show that $\mathbf{A}^{\delta}\in\mathcal{V}$ for all $\mathbf{A}\in\mathcal{V}_{si}^{pf}$.

Let $\mathbf{A}\in\mathcal{V}_{si}^{pf}$ be given. If $\mathbf{A}^{\delta}\notin\mathcal{V}$ then $\mathbf{A}^{\delta}\notin\mathcal{HA}_\tau(\mathcal{J})\subseteq\mathcal{V}$. But then there will be $\mathbf{D}\in\mathcal{J}$ such $\mathbf{D}\hookrightarrow_\tau\mathbf{A}^{\delta}$. Since $\mathbf{A}$ is subdirectly irreducible, by Lemma 3.7.6, we must have that $\mathbf{A}\in\mathcal{HA}_\tau(\mathcal{J})$, whence $\mathbf{D}\not\hookrightarrow_\tau\mathbf{A}$. This is in direct contradiction with the assumption that all members of $\mathcal{J}$ are τ-canonically transferable for the class $\mathcal{V}_{si}^{pf}$. $\Box$

Since for any variety of Heyting algebras $\mathcal{V}$ the class $\mathcal{V}_{si}^{pf}$ is closed under ultrapowers combining Propositions 3.7.7 and 3.7.1 we obtain the following corollary.

3.7.8. Corollary. *Let τ be either $\{\wedge, 0, 1\}$ or $\{\wedge, \vee, 0, 1\}$, let $\mathcal{J}$ be a class of finite distributive lattices, and let $\mathcal{V}$ the variety of Heyting algebras generated by the class $\mathcal{HA}_\tau(\mathcal{J})$. If all members of $\mathcal{J}$ are τ-MacNeille transferable for the class $\mathcal{V}^{pf}_{si}$, then the variety $\mathcal{V}$ is canonical.*

3.7.2 Transferability and elementarity

We show how classes of finite distributive lattices τ-MacNeille transferable for the class of all Heyting algebras give rise to elementarily determined varieties of Heyting algebras. Recall that a variety of Heyting algebras $\mathcal{V}$ is elementarily determined if there exists a first-order definable class of posets $\mathcal{F}$ such that $\mathcal{V}$ is generated by the class $\mathcal{F}^+ = \{\mathbb{P}^+ : \mathbb{P} \in \mathcal{F}\}$, where $\mathbb{P}^+$ denotes the dual Heyting algebra of upsets of the poset $\mathbb{P}$. It is well known that being elementarily determined implies being canonical, see, e.g., [51, Thm. 10.22].

A version of the following theorem for Boolean algebras with operators can be found in Givant [123, Thm. 1.35]. We supply a proof of the analogous result for Heyting algebras.

3.7.9. Theorem (cf. [127, Lem. 3.6.5]). *Let $\{\mathbb{P}_i \colon i \in I\}$ be a set of posets and let U be an ultrafilter on I. Then,*

$$\overline{\prod_{i\in I} \mathbb{P}_i^+/U} \cong \left(\prod_{i\in I} \mathbb{P}_i/U\right)^+.$$

Proof:
The algebra $(\prod_{i\in I} \mathbb{P}_i/U)^+$ is evidently complete and so it suffices to show that $\prod_{i\in I} \mathbb{P}_i^+/U$ is join- and meet-dense in $(\prod_{i\in I} \mathbb{P}_i/U)^+$. To this end we define a map $h\colon \prod_{i\in I} \mathbb{P}_i^+/U \to (\prod_{i\in I} \mathbb{P}_i/U)^+$ as follows,

$$h(\lambda/U) = \{p/U \in \prod_{i\in I} \mathbb{P}_i/U : [\![p \in \lambda]\!] \in U\},$$

where $[\![p \in \lambda]\!]$ denotes the set $\{i \in I : p(i) \in \lambda(i)\}$. Note that if $p/U = q/U$ and $\lambda/U = \sigma/U$ then we have that $[\![p \in \lambda]\!] \in U$ if, and only if, $[\![q \in \sigma]\!] \in U$, hence h does in fact define a function from $\prod_{i\in I} \mathbb{P}_i^+/U$ to the powerset of $\prod_{i\in I} \mathbb{P}_i/U$. To see that $h(\lambda/U)$ is indeed an upset for all $\lambda \in \prod_{i\in I} \mathbb{P}_i^+$ let $p/U \geq q/U \in h(\lambda/U)$. Then letting $[\![q \leq p]\!]$ denote the set $\{i \in I : q(i) \leq p(i)\}$ we see that

$$[\![p \in \lambda]\!] \supseteq [\![q \leq p]\!] \cap [\![q \in \lambda]\!] \in U,$$

and hence $p/U \in h(\lambda/U)$. This shows that h is a well-defined function from $\prod_{i\in I} \mathbb{P}_i^+/U$ to $(\prod_{i\in I} \mathbb{P}_i/U)^+$. Moreover, h is easily seen to be an order embedding.

Given $V \in (\prod_{i\in I} \mathbb{P}_i/U)^+$ we have that

$$V = \bigvee\{{\uparrow}(p/U) : p/U \in V\} \quad \text{and} \quad V = \bigwedge\{({\downarrow}(p/U))^{\mathrm{c}} : p/U \notin V\}.$$

Thus it suffices to show that the upsets of $(\prod_{i\in I}\mathbb{P}_i/U)^+$ of the form $\uparrow(p/U)$ and $(\downarrow(p/U))^c$ are in the image of the map h.

Given $p \in \prod_{i\in I}\mathbb{P}_i$, we define $\lambda,\sigma \in \prod_{i\in I}\mathbb{P}_i^+$ as follows

$$\lambda(i) = \uparrow p(i) \quad \text{and} \quad \sigma(i) = (\downarrow p(i))^c,$$

and hence we obtain elements $\lambda/U, \sigma/U \in \prod_{i\in I}\mathbb{P}_i^+/U$. We then observe that

$$\begin{aligned} h(\lambda/U) &= \{q/U \in \prod_{i\in I}\mathbb{P}_i/U : [\![q \in \lambda]\!] \in U\} \\ &= \{q/U \in \prod_{i\in I}\mathbb{P}_i/U : [\![p \leq q]\!] \in U\} \\ &= \{q/U \in \prod_{i\in I}\mathbb{P}_i/U : p/U \leq q/U\} = \uparrow(p/U), \end{aligned}$$

and similarly, letting $[\![q \nleq p]\!]$ denote the complement of the set $[\![q \leq p]\!]$ we see that

$$\begin{aligned} h(\sigma/U) &= \{q/U \in \prod_{i\in I}\mathbb{P}_i/U : [\![q \in \sigma]\!] \in U\} \\ &= \{q/U \in \prod_{i\in I}\mathbb{P}_i/U : [\![q \nleq p]\!] \in U\} \\ &= \{q/U \in \prod_{i\in I}\mathbb{P}_i/U : p/U \nleq q/U\} = (\downarrow(p/U))^c, \end{aligned}$$

which concludes the argument. □

3.7.10. Proposition (cf. [129, Cor. 3.2] and [112, Thm. 4.10]). *Let $\mathcal{U}$ be a universal class of Heyting algebras such that $\overline{\mathbf{A}} \in \mathcal{U}$ for all bi-Heyting algebras $\mathbf{A} \in \mathcal{U}$, then*

$$\mathcal{U}^- := \{\mathbb{P} : \mathbb{P}^+ \in \mathcal{U}\}$$

is an elementary class of posets.

Proof:
The class $\mathcal{U}^-$ is evidently closed under isomorphisms. For any poset $\mathbb{P}$ the algebra $\mathbb{P}^+$ is in fact a bi-Heyting algebra. Moreover the property of being a bi-Heyting algebra is preserved by the formation of ultraproducts. Thus by Theorem 3.7.9 the class $\mathcal{U}^-$ is closed under the formation of ultraproducts and reflects ultrapowers, i.e., $\mathbb{P}$ belongs to $\mathcal{U}^-$ whenever some ultrapower of $\mathbb{P}$ does. It is a classic result from model theory, see, e.g., [157, Cor. 8.5.13], that these three properties characterize the first-order definable classes of structures in a give language. □

We conclude this section with a theorem which will play an important role in the following section.

3.7.11. Theorem (cf. [112, Thm. 3.8]). *Let τ be either $\{\wedge, 0, 1\}$ or $\{\wedge, \vee, 0, 1\}$, let $\mathcal{J}$ be a class of finite distributive lattices, and let $\mathcal{V}$ the variety of Heyting algebras generated by the universal class $\mathcal{HA}_\tau(\mathcal{J})$. If all members of $\mathcal{K}$ are τ-MacNeille transferable for the class of bi-Heyting algebras belonging to $\mathcal{V}_{fsi}$, then the variety $\mathcal{V}$ is elementarily determined and hence canonical.*

Proof:
By Theorem 2.2.5 the variety $\mathcal{V}$ will be generated by its finite members. In particular, letting $\mathcal{U}$ be the universal class $\mathcal{V}_{fsi}$ we must have that $\mathcal{V}$ is generated by the class $\{\mathbb{P}^+ : \mathbb{P} \in \mathcal{U}^-\}$, where $\mathcal{U}^- = \{\mathbb{P} : \mathbb{P}^+ \in \mathcal{U}\}$. Thus it suffices to show that $\mathcal{U}^-$ is elementary and hence by Proposition 3.7.10 that $\overline{\mathbf{A}} \in \mathcal{U}$ for all bi-Heyting algebras $\mathbf{A} \in \mathcal{U}$. Therefore, let $\mathbf{A} \in \mathcal{U}$ be a bi-Heyting algebra. If $\overline{\mathbf{A}} \notin \mathcal{U} = \mathcal{V}_{fsi}$ then, since the property of being finitely subdirectly irreducible is preserved by passing to the MacNeille completion, $\overline{\mathbf{A}} \notin \mathcal{V}$ and therefore $\overline{\mathbf{A}} \notin \mathcal{HA}_\tau(\mathcal{J})$. Consequently, we must have $\mathbf{D} \in \mathcal{J}$ such that $\mathbf{D} \hookrightarrow_\tau \overline{\mathbf{A}}$. Since by assumption $\mathbf{D}$ is τ-MacNeille transferable for the class of bi-Heyting algebras belonging to $\mathcal{V}_{fsi}$ it follows that $\mathbf{D} \hookrightarrow_\tau \mathbf{A}$. By Lemma 3.7.6 we have that $\mathcal{V}_{fsi} \subseteq \overline{\mathcal{HA}_\tau(\mathcal{J})}$ and hence $\mathbf{D} \not\hookrightarrow_\tau \mathbf{A}$, which is a contradiction. We may therefore conclude that $\overline{\mathbf{A}} \in \mathcal{U}$. $\square$

3.8 Connections to logic

In this section we will show how the results obtained so far relate to intermediate logics. Recall from Chapter 2 that an intermediate logic L is τ-stable for some $\tau \subseteq \{\wedge, \vee, 0, 1\}$ provided that the corresponding variety of Heyting algebras $\mathcal{V}(\mathsf{L})$ is generated by a τ-stable class of Heyting algebras.

We have already seen, Proposition 2.3.1, that all $(\wedge, 0, 1)$-stable logics are canonical and elementarily determined. Using the method developed in this chapter we will establish analogous results for $(\wedge, \vee, 0, 1)$-stable logics [25, 29, 164]. Unlike $(\wedge, 0, 1)$-stable logics it is still not known whether all $(\wedge, \vee, 0, 1)$-stable logics are elementarily determined [164, Prob. 2] or even canonical [164, Prob. 1]. We will establish some partial results related to these problems and point to some strategies for a positive solution.

For a natural number $n \geq 1$ we let $\mathcal{BTW}_n$ and $\mathcal{BW}_n$ denote the varieties of Heyting algebras generated by the class of Heyting algebras of top width at most n and width at most n, respectively. We let BTW_n and BW_n denote the intermediate logics corresponding to the varieties $\mathcal{BTW}_n$ and $\mathcal{BW}_n$, respectively. It is not difficult to see that all the finitely subdirectly irreducible members of $\mathcal{BTW}_n$ and $\mathcal{BW}_n$ have top width and width at most n, respectively. Moreover, these logics are all $(\wedge, \vee, 0, 1)$-stable [25, Thm. 7.3] and in fact $(\wedge, 0, 1)$-stable.

Recall that an intermediate logic L is elementarily determined provided that its corresponding variety of Heyting algebras $\mathcal{V}(\mathsf{L})$ is generated by a class of Heyting algebras $\{\mathbb{P}^+ : \mathbb{P} \in \mathcal{F}\}$ with $\mathcal{F}$ a an elementary class of posets.

Theorems 3.7.11 and 3.3.7 provide an alternative argument to the effect that all $(\wedge, 0, 1)$-stable intermediate logics are elementarily determined and therefore also canonical. Therefore for the rest of this section we will focus on the case $\tau = \{\wedge, \vee, 0, 1\}$.

3.8.1. PROPOSITION. *Any $(\wedge, \vee, 0, 1)$-stable logic containing the logic BW_n, for some natural number $n \geq 1$, is elementarily determined and hence canonical.*

Proof:
Let L be a $(\wedge, \vee, 0, 1)$-stable logic with $\mathsf{BW}_n \subseteq \mathsf{L}$, for some $n \geq 1$. By Theorem 2.2.5 the variety $\mathcal{V}(\mathsf{L})$ is generated by a universal class of Heyting algebras of the form $\mathcal{HA}_{\wedge,\vee,0,1}(\mathcal{J})$, for some class of finite distributive lattices $\mathcal{J}$. Moreover, since $\mathsf{BW}_n \subseteq \mathsf{L}$ we have that $\mathcal{V}(\mathsf{L}) \subseteq \mathcal{BW}_n$ and so in particular any finitely subdirectly irreducible member of $\mathcal{V}(\mathsf{L})$ has width at most n.

By Corollary 3.6.5 we have that all members of $\mathcal{J}$ are $(\wedge, \vee, 0, 1)$-MacNeille transferable for the class of bi-Heyting algebras belonging to $\mathcal{V}_{fsi}$ whence the proposition is a direct consequence of Theorem 3.7.11. $\square$

To show that Proposition 3.8.1 includes logics which are not covered by Proposition 2.3.1 we construct a family of $(\wedge, \vee, 0, 1)$-stable logics none of which are $(\wedge, 0, 1)$-stable but all of which contain BW_n for some $n \geq 1$. For each natural number $n \geq 1$ let $\mathcal{J}_n$ be the set

$$\{\mathbf{D}_4 \oplus \mathbf{1},\ \mathbf{2}^{n+1},\ \mathbf{1} \oplus \mathbf{2}^{n+1},\ \mathbf{2}^{n+1} \oplus \mathbf{1},\ \mathbf{1} \oplus \mathbf{2}^{n+1} \oplus \mathbf{1}\}.$$

Let $\mathcal{V}_n$ be the variety generated by the $(\wedge, \vee, 0, 1)$-stable class $\mathcal{HA}_{\wedge,\vee,0,1}(\mathcal{J}_n)$ and let L_n be the corresponding intermediate logic.

3.8.2. PROPOSITION. *For each $n \geq 1$ the logic L_n is $(\wedge, \vee, 0, 1)$-stable but not $(\wedge, 0, 1)$-stable and contains the logic BW_n. Furthermore, the logics L_m and L_n are different for natural numbers $m, n \geq 1$ with $m \neq n$.*

Proof:
Let $n \geq 1$ be given. By construction $\mathcal{V}(\mathsf{L}_n) = \mathcal{V}_n$ is generated by the $(\wedge, \vee, 0, 1)$-stable class $\mathcal{HA}_{\wedge,\vee,0,1}(\mathcal{J}_n)$, whence L_n is $(\wedge, \vee, 0, 1)$-stable.

By [29, Prop. 5.1] a subdirectly irreducible Heyting algebra $\mathbf{A}$ validates L_n if, and only if, none of the algebras in $\mathcal{J}_n$ are isomorphic to a bounded sublattice of $\mathbf{A}$. From this we may conclude that the algebra $\mathbf{2}^2 \oplus \mathbf{2}^2 \oplus \mathbf{1}$ is a subdirectly irreducible algebra belonging to $\mathcal{V}_n$. Since $\mathbf{D}_4 \oplus \mathbf{1}$ is isomorphic to a $(\wedge, 0, 1)$-sublattice of this algebra, we deduce from Theorem 2.2.13 that L_n is not $(\wedge, 0, 1)$-stable.

Similarly, since non of the subdirectly irreducible members of $\mathcal{V}_n$ has a bounded sublattice of the form $\mathbf{2}^{n+1}$, $\mathbf{1} \oplus \mathbf{2}^{n+1}$, $\mathbf{2}^{n+1} \oplus \mathbf{1}$, or $\mathbf{1} \oplus \mathbf{2}^{n+1} \oplus \mathbf{1}$ none of them has a sublattice isomorphic to $\mathbf{2}^{n+1}$. By Lemma 3.5.5 it follows that each subdirectly irreducible member of $\mathcal{V}_n$ is of width at most n, whence $\mathcal{V}_n \subseteq \mathcal{BW}_n$ and therefore $\mathsf{BW}_n \subseteq \mathsf{L}$.

Finally, if $m > n$, then $\mathbf{1} \oplus \mathbf{2}^{n+1} \oplus \mathbf{1}$ is a subdirectly irreducible member of $\mathcal{V}_m$ but not of $\mathcal{V}_n$, showing that $\mathcal{V}_m \neq \mathcal{V}_n$ and therefore that $\mathsf{L}_m \neq \mathsf{L}_n$. $\square$

We conclude this section by making a few observations showing how finding a positive answer to the question of whether every $(\wedge, \vee, 0, 1)$-stable logic is elementarily determined can be reduced to a question about $(\wedge, \vee, 0, 1)$-MacNeille transferability.

By a *proper* intermediate logic we shall understand an intermediate logic not equal to IPC.

3.8.3. Proposition. *Any proper $(\wedge, \vee, 0, 1)$-stable intermediate logic contains the logic BTW_n for some natural number $n \geq 1$.*

Proof:
Let L be a proper $(\wedge, \vee, 0, 1)$-stable logic. We show that there exist a natural number $n \geq 1$ such that $\mathcal{V}(\mathsf{L}) \subseteq \mathcal{BTW}_n$, and therefore that $\mathsf{BTW}_n \subseteq \mathsf{L}$. Suppose not, then for each natural number $n \geq 1$ there is $\mathbf{A} \in \mathcal{V}(\mathsf{L})$ such that $\mathbf{A} \notin \mathcal{BTW}_n$. Without loss of generality we may assume that $\mathbf{A}$ is subdirectly irreducible. It follows that $\mathbf{A}$ has top width strictly greater than n. By Lemma 3.5.6 we have that $\mathbf{2}^{n+1} \oplus \mathbf{1}$ is isomorphic to a bounded sublattice of $\mathbf{A}$. Since by assumption L is a $(\wedge, \vee, 0, 1)$-stable logic the class of finite subdirectly irreducible members of $\mathcal{V}(\mathsf{L})$ is closed under taking bounded sublattices, see Theorem 2.2.5. We may therefore conclude that $\mathbf{2}^{n+1} \oplus \mathbf{1}$ belongs to $\mathcal{V}(\mathsf{L})$. Moreover, since each finite subdirectly irreducible Heyting algebra is a bounded sublattice of $\mathbf{2}^m \oplus \mathbf{1}$, for some natural number m, we obtain that $\mathcal{V}(\mathsf{L})$ contains all finite subdirectly irreducible Heyting algebras. As these generate the variety of all Heyting algebras, it follows that $\mathcal{V}(\mathsf{L}) = \mathcal{HA}$, in direct contradiction with the assumption that the logic L is proper. $\square$

Two τ-stable universal classes may generate the same variety and hence give rise to the same τ-stable intermediate logic while one of them is closed under MacNeille completions and the other is not. For instance, for $\tau = \{\wedge, \vee, 0, 1\}$ the two classes $\mathcal{HA}_\tau(\mathbf{2}^2)$ and $\mathcal{HA}_\tau(\mathbf{1} \oplus \mathbf{2}^2)$ both generate the variety of all Heyting algebras $\mathcal{HA}$, but by Theorems 3.5.2 and 3.5.12, respectively, $\mathcal{HA}_\tau(\mathbf{1} \oplus \mathbf{2}^2)$ is closed under MacNeille completions while $\mathcal{HA}_\tau(\mathbf{2}^2)$ is not.

However, from Theorem 2.2.5 we know that if L is a $(\wedge, \vee, 0, 1)$-stable intermediate logic, then the corresponding variety $\mathcal{V}(\mathsf{L})$ of Heyting algebras is generated by a $(\wedge, \vee, 0, 1)$-stable universal class $\mathcal{HA}_{\wedge,\vee,0,1}(\mathcal{J})$ with $\mathcal{J}$ a class of finite distributive lattice of the form $\mathbf{D} \oplus \mathbf{1}$. Thus Proposition 3.8.3 together with Theorem 3.7.11 entail that all $(\wedge, \vee, 0, 1)$-stable intermediate logics are elementarily determined provided that all finite distributive lattice of the form $\mathbf{D} \oplus \mathbf{1}$ are

$(\wedge, \vee, 0, 1)$-MacNeille transferable for the class of bi-Heyting algebras of finite top width. Similarly, invoking Corollary 3.7.8, we obtain that any $(\wedge, \vee, 0, 1)$-stable intermediate logic is canonical if all finite distributive lattice of the form $\mathbf{D} \oplus \mathbf{1}$ are τ-canonically transferable for the class of pseudo-finite Heyting algebras of finite top width.

3.9 Summary and concluding remarks

In this chapter we have considered three different notions of transferability for finite lattices, namely ideal, MacNeille and canonical transferability. Our main motivation for this was to find universal classes of lattices closed under completions and we have shown how such classes can indeed be obtained from families of finite transferable lattices. We have compared the three notions of ideal, MacNeille and canonical transferability, showing that, under mild assumptions, MacNeille transferability implies canonical transferability which in turn implies ideal transferability.

Our main focus has been on MacNeille transferability. We have provided necessary conditions for a finite lattice to be MacNeille transferable for the class of all lattices just as we have given necessary conditions for a finite distributive lattice to be $(\wedge, \vee, 0, 1)$-MacNeille transferable with respect to the class of all Heyting algebras.

In terms of sufficient conditions for transferability we have shown that every finite projective distributive lattice is $(\wedge, 0, 1)$-MacNeille transferable with respect to the class of all lattices. Moreover, we have shown that every finite projective distributive lattice of the form $\mathbf{D} \oplus \mathbf{1}$ or $\mathbf{1} \oplus \mathbf{D}$ is $(\wedge, \vee, 0, 1)$-MacNeille transferable with respect to the class of all distributive lattices. However, we have also seen examples of finite non-projective distributive lattices which are MacNeille transferable and $(\wedge, \vee, 0, 1)$-MacNeille transferable for the class of all distributive lattices and the class of all Heyting algebras, respectively. Furthermore, we have shown that every finite distributive lattice is MacNeille transferable with respect to the class of all bi-Heyting algebras.

Finally, we have discussed how considerations on MacNeille and canonical transferability can yield results about canonicity and elementarity of certain classes of intermediate logics.

Further directions and open problems Although we have mainly considered notions of MacNeille transferability we believe that the concept of canonical transferability for various classes of lattices is well worth looking into. We would also like to look at the notions of transferability in the context of lattices with operators. In particular, τ-MacNeille and τ-canonical transferability for Heyting algebras when $\tau \subseteq \{\wedge, \vee, \rightarrow, \neg, 0, 1\}$. Using [152, Prop. 4.2] the results concerning MacNeille transferability with respect to the class of Heyting algebras obtained in this chapter can be transfered to the setting of S4-algebras for appropriately defined notions of MacNeille transferability of modal algebras. Other than that, the concept of transferability of modal algebras is still completely unexplored.

Having shown how syntax independent methods can be used to find universal classes of lattice-based algebras closed under various types of completions, we would like to see if syntactic methods like the ones found in [58] can be of help for establishing new results about transferability. In particular, adapting the ALBA-framework, see, e.g., [141, 68, 67], to find purely syntactic descriptions of universal clauses preserved by MacNeille completions of bi-Heyting algebras seems to be promising, cf. [124, 245]. For an early discussion on syntactic issues related to ideal transferability see [14, §2.3, §5.2].

We end this chapter with a selection of concrete open problems.

1. Is there a variety of lattices $\mathcal{V}$ such that the property of being ideal, or MacNeille, transferable for $\mathcal{V}$ is not a decidable property of finite lattices?

2. Is every $(\wedge, \vee, 0, 1)$-stable intermediate logic determined by a $(\wedge, \vee, 0, 1)$-stable universal class closed under MacNeille, or canonical, completions?

3. Give a complete characterization of the finite distributive lattices which are τ-MacNeille transferable for $\mathcal{K}$ in the following cases:

   (i) $\tau = \{\wedge, \vee\}$ and $\mathcal{K}$ is the class of all lattices,

   (ii) $\tau = \{\wedge, \vee\}$ and $\mathcal{K}$ is the class of all Heyting algebras,

   (iii) $\tau = \{\wedge, \vee, 0, 1\}$ and $\mathcal{K}$ is the class of all Heyting algebras,

   (iv) $\tau = \{\wedge, \vee, 0, 1\}$ and $\mathcal{K}$ is the class of all pseudo-finite Heyting algebras,

   (v) $\tau = \{\wedge, \vee, 0, 1\}$ and $\mathcal{K}$ is the class of all bi-Heyting algebras,

   (vi) $\tau = \{\wedge, \vee, 0, 1\}$ and $\mathcal{K}$ is the class of all bi-Heyting algebras of finite top width.

Chapter 4

Hyper-MacNeille completions of Heyting algebras

In connection with their proof of the admissibility of the cut-rule in certain types of structural hypersequent calculi, Ciabattoni, Galatos, and Terui [60] introduced a new type of completion of (pointed) residuated lattices which they called the hyper-MacNeille completion. Among other things they established that any variety of Heyting algebras axiomatized by $\mathcal{P}_3$-equations must be closed under this type of completion. Thus closure under hyper-MacNeille completions is a necessary condition for a variety of Heyting algebras to be determined by $\mathcal{P}_3$-equations and consequently for the corresponding intermediate logic to admit an analytic structural hypersequent calculus.

In this chapter, based on [153], we consider more closely this type of completion in the context of Heyting algebras. We first identify the concept of a De Morgan supplemented Heyting algebra as being helpful for understanding the hyper-MacNeille completions of Heyting algebras. These algebras may be viewed as Heyting algebras equipped with a "co-negation" satisfying both of the De Morgan laws. We prove that for De Morgan supplemented Heyting algebras the MacNeille and hyper-MacNeille completions coincide. This generalizes the fact that the MacNeille and hyper-MacNeille completions coincide for subdirectly irreducible algebras [60, Prop. 6.6], at least in the context of Heyting algebras.

We also show that the De Morgan supplemented Heyting algebras are precisely the Heyting algebras which are isomorphic to Boolean products of finitely subdirectly irreducible Heyting algebras. This connection allows us to draw inspiration from previous work on MacNeille completions of Boolean products of lattices [146, 73]. Concretely, we establish the following close connection between MacNeille and hyper-MacNeille completions of Heyting algebras.

1. The hyper-MacNeille completion of a Heyting algebra $\mathbf{A}$ is the MacNeille completion of some De Morgan supplemented Heyting algebra $Q(\mathbf{A})$.

2. A variety $\mathcal{V}$ of Heyting algebras is closed under hyper-MacNeille completions

if, and only if, the class of De Morgan supplemented members of $\mathcal{V}$ is closed under MacNeille completions.

Specifically, the last item allows us to turn the question of which varieties of Heyting algebras are closed under hyper-MacNeille completions into the question of which varieties of De Morgan supplemented Heyting algebras are closed under MacNeille completions.

Our analysis also allows us to show that any finitely generated variety of Heyting algebras is closed under hyper-MacNeille completions. From this and the results of Chapter 2 it follows that being axiomatizable by $\mathcal{P}_3$-equations is not a necessary condition for being closed under hyper-MacNeille completions. In fact, there are varieties of Heyting algebras, such as the variety $\mathcal{BD}_2$ corresponding to the logic of posets having depth at most 2, which are closed under hyper-MacNeille completions but which are neither axiomatizable by $\mathcal{P}_3$-equations nor finitely generated.

Finally, we show that the sufficient conditions for the hyper-MacNeille completion to be regular, identified by Ciabattoni, Galatos, and Terui [60, Thm. 6.11], are in fact also necessary, at least in the context of Heyting algebras.

Outline The chapter is structured as follows: Section 4.1 contains some basic results concerning supplemented distributive lattices. Section 4.2 introduces the hyper-MacNeille completion of a Heyting algebra. Section 4.3 describes how the hyper-MacNeille completion of any Heyting algebra can be obtained as the MacNeille completion of its algebra of dense open sections. In Section 4.4 a number of examples of varieties closed under hyper-MacNeille completions are considered. Section 4.5 determines central and supplemented elements of the hyper-MacNeille completion and it is shown that the hyper-MacNeille completion is always Hausdorff with finitely subdirectly irreducible stalks. Section 4.6 gives necessary and sufficient conditions for the hyper-MacNeille completion to be regular. Finally, Section 4.7 contains a few concluding remarks.

4.1 Supplemented distributive lattices

We here cover some basic facts about supplemented lattices which will be used throughout this chapter. Since we will only consider lattices which are bounded we will simply leave out this qualifier.

4.1.1. Definition. A *supplemented* lattice is a distributive lattice $\mathbf{D}$ such that every $a \in D$ has a *supplement*, i.e., a, necessarily unique, element ${\sim} a \in D$ satisfying $1 = a \vee b$ if, and only if, ${\sim} a \leq b$, for all $b \in D$.

The following examples of supplemented lattices will play an important role.

4.1.2. EXAMPLE. The following types of lattice are always supplemented:

(i) Boolean algebras,

(ii) Finite distributive lattices,

(iii) Distributive lattices with a join-irreducible top element.

Evidently, the supplement of an element a in a distributive lattice $\mathbf{D}$ is the least element $b \in D$ such that $a \vee b = 1$. Thus it is easy to see that a distributive lattice $\mathbf{D}$ is supplemented if, and only if, its order dual is pseudo-complemented, see Appendix A.3. In particular, any result about pseudo-complemented lattices holds for supplemented lattices in an order dual version.

4.1.3. LEMMA. *Let* $\mathbf{D}$ *be a supplemented lattice, then for all* $a, b \in D$ *we have*

1. $\sim(a \wedge b) = \sim a \vee \sim b$.

2. $\sim(a \vee b) \leq \sim a \wedge \sim b$.

Proof:
As the supplement operation is order-reversing we must have $\sim a \vee \sim b \leq \sim(a \wedge b)$ for all $a, b \in D$. Conversely, since both $\sim a \vee \sim b \vee a = 1$ and $\sim a \vee \sim b \vee b = 1$, by distributivity we have that $1 = \sim a \vee \sim b \vee (a \wedge b)$, whence $\sim(a \wedge b) \leq \sim a \vee \sim b$.
To establish Item 2 we simply note that by distributivity $1 = (a \vee b) \vee (\sim a \wedge \sim b)$ whence $\sim(a \vee b) \leq \sim a \wedge \sim b$. □

However, the equation $\sim x \wedge \sim y \leq \sim(x \vee y)$ will in general not be satisfied, as is easily seen, e.g., by considering the five-element lattice $\mathbf{1} \oplus (\mathbf{2} \times \mathbf{2})$. Supplemented lattices which do satisfy this equation will play an important role.

4.1.4. DEFINITION. A supplemented lattice $\mathbf{D}$ is *De Morgan supplemented* provided that it validates the equation $\sim(x \vee y) \approx \sim x \wedge \sim y$.

Again we will list some important examples of such lattices.

4.1.5. EXAMPLE. The following lattices are supplemented with a De Morgan supplement:

(i) Boolean algebras,

(ii) Distributive lattices with a join-irreducible top element,

(iii) Any direct or Boolean product of distributive lattices with join-irreducible top elements.

4.1.6. Definition. Let $\mathbf{D}$ be a supplemented distributive lattice. An element $a \in D$ is *co-regular* provided that $\sim\sim a = a$. We denote by $\mathrm{CoRg}(\mathbf{D})$ the set of co-regular elements of $\mathbf{D}$.

Since the co-regular elements are simply the regular elements of the order dual the following is well known, see also Appendix A.5.

4.1.7. Lemma (cf. [16, Thm. VIII.2.1(xi), Thm. VIII.4.3]). *Let $\mathbf{D}$ be a supplemented lattice.*

1. $\mathrm{CoRg}(\mathbf{D}) = \{\sim a \in D : a \in D\}$.
2. $\mathrm{CoRg}(\mathbf{D})$ *is a Boolean algebra which is a* $(\vee, 0, 1)$*-subalgebra of* $\mathbf{D}$.
3. *The assignment* $a \mapsto \sim\sim a$ *is a supplemented lattice homomorphism from* $\mathbf{D}$ *onto* $\mathrm{CoRg}(\mathbf{D})$.

4.1.8. Definition. An element c of a distributive lattice $\mathbf{D}$ is *central* if there is, a necessarily unique, element $c' \in D$ such that $0 = c \wedge c'$ and $1 = c \vee c'$. We denote by $Z(\mathbf{D})$ the set of central elements of $\mathbf{D}$.

It is easy to see that if $\mathbf{D}$ is pseudo-complemented then the central elements of $\mathbf{D}$ are precisely the elements with $c \vee \neg c = 1$. Similarly, in a supplemented lattice the central elements are precisely the elements with $c \wedge \sim c = 0$. If $\mathbf{D}$ is a distributive lattice which is both pseudo-complemented and supplemented then $\neg a \leq \sim a$, for all $a \in D$. Consequently, in any such lattice the central elements must be those for which the pseudo-complement and supplement coincide.

4.1.9. Proposition (cf. [16, Thm.VIII.7.1]). *Let $\mathbf{D}$ be a supplemented lattice then the following are equivalent.*

1. $Z(\mathbf{D}) = \mathrm{CoRg}(\mathbf{D})$.
2. $\mathbf{D} \models \sim x \wedge \sim\sim x \approx 0$.
3. *The supplement on $\mathbf{D}$ is a De Morgan supplement.*

Proof:
That Items 1 and 2 are equivalent is easy to verify just as the fact that Item 3 implies Item 2.

To see that Item 2 entails Item 3 let $a, b \in D$ be given. It suffices to show that $\sim a \wedge \sim b \leq \sim(a \vee b)$. We first observe that $\sim a \leq \sim(a \vee b) \vee b$ since $1 = \sim(a \vee b) \vee a \vee b$. But then

$$\sim(a \vee b) \vee b \vee \sim\sim a \geq \sim a \vee \sim\sim a = 1,$$

and hence also $\sim b \leq \sim(a \vee b) \vee \sim\sim a$. From this we then see that

$$\begin{aligned}\sim(a \vee b) &= \sim(a \vee b) \vee 0 \\ &= \sim(a \vee b) \vee (\sim a \wedge \sim\sim a) \\ &= (\sim(a \vee b) \vee \sim a) \wedge (\sim(a \vee b) \vee \sim\sim a) \\ &\geq \sim a \wedge \sim b,\end{aligned}$$

which concludes the proof. □

4.1.10. Remark. Proposition 4.1.9 shows that the distributive lattices with a De Morgan supplement are precisely the distributive lattices with order duals being so-called *Stone lattices*, see [139] or [16, Chap. VIII.7].

When a distributive lattice is both pseudo-complemented and supplemented computing infinitary joins and meets of central elements is particularly easy.

4.1.11. Proposition. *Let* $\mathbf{D}$ *be a distributive lattice which is both pseudo-complemented and supplemented. If* $\mathbf{D}$ *is complete then so is* $Z(\mathbf{D})$*.*

Proof:
To see that $Z(\mathbf{D})$ is complete it suffices to show that any set of central elements in $\mathbf{D}$ has a greatest lower bound in $Z(\mathbf{D})$. Therefore, let $\{c_i\}_{i\in I}$ be a collection of central elements. As $\mathbf{D}$ is complete a greatest lower bound $c := \bigwedge_{i\in I} c_i$ of the family $\{c_i\}_{i\in I}$ exists in $\mathbf{D}$. We claim that c is central with complement $c' := \bigvee_{i\in I} \neg c_i$. For all $i \in I$ we have $\neg c_i \wedge c \leq \neg c_i \wedge c_i = 0$ and hence $\neg c_i \leq \neg c$. Therefore, $c' \leq \neg c$ which implies that $c' \wedge c = 0$. Furthermore, since each c_i is central we have $c_i \vee c' \geq c_i \vee \neg c_i = 1$ and hence $\sim c' \leq c_i$. Therefore, $\sim c' \leq c$ which implies that $c' \vee c = 1$. □

4.1.12. Remark. Note that Proposition 4.1.11 also proves that if $\{c_i\}_{i\in I}$ is a collection of central elements in a complete distributive lattice $\mathbf{D}$ which is both pseudo-complemented and supplemented then $\neg \bigwedge_{i\in I} \neg c_i = \bigvee_{i\in I} \neg\neg c_i = \bigvee_{i\in I} c_i$. Consequently, also the join, taken in $\mathbf{D}$, of any family of central elements in $\mathbf{D}$ must be central. This shows that $Z(\mathbf{D})$ is in fact a *complete sublattice* of $\mathbf{D}$.

4.1.1 Minimal prime filters

As we will see, the minimum of the dual Esakia space X of a Heyting algebra $\mathbf{A}$ will play an important role in describing the hyper-MacNeille completion of $\mathbf{A}$. The set of minimal prime filters will in general not be a closed subset of X and therefore not a Stone space when equipped with the subspace topology determined by X. We first characterize the distributive lattices with the property that the minimum of their dual Priestley space is closed. We refer to Appendix A.6 for basic concepts from duality theory.

4.1.13. Definition. Let $\mathbf{D}$ be a distributive lattice. An element $a \in D$ satisfying

$$\forall b \in D \ (1 = a \vee b \implies b = 1)$$

is called *co-dense*. We denote by $\mathrm{CoDn}(\mathbf{D})$ the set of co-dense elements of $\mathbf{D}$.

4.1.14. Remark. It is easy to see that in a supplemented lattice $\mathbf{D}$ the co-dense elements are precisely the elements $a \in D$ such that $\sim a = 1$.

4.1.15. Definition (cf. [237]). A bounded distributive lattice $\mathbf{D}$ is called a ∇_*-*lattice* if for all $a \in D$ there is an element $b \in D$ such that $a \vee b = 1$ and $a \wedge b \in \mathrm{CoDn}(\mathbf{D})$.

Since in any supplemented lattice elements of the form $a \wedge \sim a$ are always co-dense it is easy to see that such lattices will be ∇_*-lattices. However, ∇_*-lattices need not be supplemented, as is easily seen, e.g., by considering the order dual of the bounded distributive lattice $(\omega \times \mathbf{2}) \oplus \mathbf{1}$, obtained by adding a top element to the lattice $\omega \times \mathbf{2}$, cf. [237, Sec. 4].

As the next proposition shows, the ∇_*-lattices pick out exactly the distributive lattices, and so in particular the Heyting algebras, for which the set of minimal prime filters is closed.

4.1.16. Proposition (cf. [237, Thm. 1]). *Let $\mathbf{D}$ be a distributive lattice and let X be its dual Priestley space. Then the following are equivalent.*

1. *The set $\min(X)$ is a closed subset of X.*

2. *The lattice $\mathbf{D}$ is a ∇_*-lattice.*

Proof:
First, assume that $\min(X)$ is closed in X. Let U be a clopen upset of X. Then $C_1 := (U \cap \min(X))$ and $C_2 := {\uparrow}(X \backslash U)$ are two disjoint closed down- and upsets, respectively. Consequently, we have a clopen upset V such that $C_1 \cap V = \emptyset$ and $C_2 \subseteq V$, see, e.g., [74, Lem. 11.21(ii)(b)]. It follows that $U \cup V = X$ and that $U \cap V$ is disjoint from $\min(X)$. Therefore any clopen upset W such that $W \cup (U \cap V) = X$ must contain $\min(X)$ and hence be equal to X. This means that $U \cap V$ is co-dense, showing that the lattice of clopen upsets of X is a ∇_*-lattice.

Second, assuming that $\mathbf{D}$ is a ∇_*-lattice we show that

$$\min(X) = \bigcap \{X \backslash \widehat{a} : a \in \mathrm{CoDn}(\mathbf{D})\},$$

and so being the intersection of closed sets, the set $\min(X)$ must be closed. Assume first that $x \in \min(X)$ and that $a \in \mathrm{CoDn}(\mathbf{D})$. Hence for all $b \in D$ we have that $b \vee a = 1$ entails $b = 1$. It follows that the ideal I generated by $(D \backslash x) \cup \{a\}$ must be proper and as such it may be extended to a prime ideal, say $y \supseteq I \supseteq D \backslash x$.

But then we have that $D \backslash y \subseteq x$ is a prime filter and so, by the minimality of x, that $D \backslash y = x$ whence $a \notin x$. This shows that $\min(X) \subseteq \bigcap\{X \backslash \widehat{a} : a \in \mathrm{CoDn}(\mathbf{D})\}$. Conversely, if $x \in \bigcap\{X \backslash \widehat{a} : a \in \mathrm{CoDn}(\mathbf{D})\}$ and $y \subseteq x$ then, since $\mathbf{D}$ is a ∇_*-lattice, for any $a \in x$ we have $b \in D$ such that $a \vee b = 1$ and $a \wedge b \in \mathrm{CoDn}(\mathbf{D})$. So, as y is a prime filter, we obtain that either $a \in y$ or $b \in y$. Furthermore, since $a \wedge b \in \mathrm{CoDn}(\mathbf{D})$ we have that $a \wedge b \notin x$, whence $b \notin x$ as $a \in x$ by assumption. It follows that $b \notin y$ whence $a \in y$. As $a \in x$ was arbitrary, we obtain that $y = x$ and therefore that $x \in \min(X)$, showing that $\bigcap\{X \backslash \widehat{a} : a \in \mathrm{CoDn}(\mathbf{D})\} \subseteq \min(X)$. $\square$

4.1.17. Remark. It is worth noting that the proof of Proposition 4.1.16 shows that $\min(X) \subseteq \bigcap\{X_{\mathbf{D}} \backslash \widehat{a} : a \in \mathrm{CoDn}(\mathbf{D})\}$ holds for any distributive lattice $\mathbf{D}$. In particular, this means that if $a \in D$ is co-dense and $x \in \min(X)$ then $x \notin \widehat{a}$, which is to say, $a \notin x$.

4.1.18. Corollary. *Let $\mathbf{D}$ be a supplemented lattice with dual Priestley space X. Then $\min(X)$ is a closed subspace of X.*

The following is most likely already known in the order dual formulation. However, as we have not be able to find a direct reference we include a proof.

4.1.19. Proposition. *Let $\mathbf{D}$ be a supplemented lattice with dual Priestley space X. Then $\min(X)$ is a Stone space the dual Boolean algebra of which is isomorphic to $\mathrm{CoRg}(\mathbf{D})$.*

Proof:
Since $\mathbf{D}$ is a supplemented lattice $\min(X)$ is a closed subset of X by Corollary 4.1.18. Consequently, being a closed subset of a Stone space, the set $\min(X)$ equipped with the subspace topology is again a Stone space. By [217, Lem. 12] the dual space of the Stone space $\min(X)$ is the homomorphic image of $\mathbf{D}$ determined by the congruence θ defined as

$$a\theta b \quad \text{if, and only if,} \quad \forall x \in \min(X)\ (a \in x \iff b \in x).$$

We show that $a\theta b$ if, and only if, ${\sim} a = {\sim} b$, for all $a, b \in D$. If ${\sim} a = {\sim} b$ and $x \in \min(X)$ is such that $a \in x$ then ${\sim} b = {\sim} a \notin x$, as $a \wedge {\sim} a$ is co-dense, and so $b \in x$ as $b \vee {\sim} b = 1$. Conversely, if ${\sim} a \neq {\sim} b$, then ${\sim} a \leftrightarrow {\sim} b < 1$ and so there is $x \in X$ such that ${\sim} a \leftrightarrow {\sim} b \notin x$. Since X is a Priestley space, by Proposition A.6.6 there is $y \subseteq x$, with $y \in \min(X)$ and consequently ${\sim} a \leftrightarrow {\sim} b \notin y$. Therefore, y separates ${\sim} a$ and ${\sim} b$, say, without loss of generality, ${\sim} a \in y$ and ${\sim} b \notin y$. Since y is minimal and ${\sim} a \wedge a$ is co-dense we have that $a \notin y$. On the other hand as ${\sim} b \notin y$ we must have $b \in y$, implying that y separates a and b. This shows that $a\theta b$ if, and only if, ${\sim} a = {\sim} b$.

Recall from Lemma 4.1.7 that $a \mapsto {\sim}{\sim} a$ is a surjective homomorphism $\mathbf{D} \twoheadrightarrow \mathrm{CoRg}(\mathbf{D})$ of supplemented lattice. Since the equation ${\sim}{\sim}{\sim} x \approx {\sim} x$ is always satisfied in a supplemented lattice, we see that the kernel of this map coincides with the congruence θ. From this we may conclude that $\mathbf{D}/\theta$ is isomorphic to $\mathrm{CoRg}(\mathbf{D})$, as desired. $\square$

4.1.2 Hausdorff lattices

Mimicking the construction of the Pierce sheaf of a commutative regular ring [212] one can represent any lattice as a weak Boolean product over its center, see Appendix A.2 for definitions. We describe the construction for distributive lattices noting that something very similar can be done for arbitrary lattices, see [73]. Given a distributive lattice $\mathbf{D}$ let X be the dual Stone space of its center $Z(\mathbf{D})$. Then for each $x \in X$ we obtain a congruence $\equiv_x$ on $\mathbf{D}$ by letting

$$a \equiv_x b \quad \text{if, and only if,} \quad \exists c \in x \ (a \wedge c = b \wedge c).$$

Then, with $\mathbf{D}_x$ denoting the quotient $\mathbf{D}/{\equiv_x}$, it may be shown that $\mathbf{D} \hookrightarrow \prod_{x \in X} \mathbf{D}_x$ is always a subdirect representation of $\mathbf{D}$ as a weak Boolean product of the family $\{\mathbf{D}_x : x \in X\}$. Following [73] we refer to this as the *usual representation of* $\mathbf{D}$ *over its center*.

4.1.20. Definition ([73]). A distributive lattice is *Hausdorff* if the usual representation over its center is a Boolean product.

The terminology comes from the fact that when viewing the usual presentation of a distributive lattice $\mathbf{D}$ over its center as a sheaf presentation, then the induced sheaf space is Hausdorff if, and only if, the lattice $\mathbf{D}$ is Hausdorff [195, 146].

4.1.21. Remark. It is easy to verify that when $\mathbf{D}$ is a Heyting algebra then the congruence $\equiv_x$ is in fact a Heyting algebra congruence and so the embedding obtained from the usual representation of $\mathbf{D}$ over its center is an embedding of Heyting algebras. A similar remark applies, *mutatis mutandis*, when $\mathbf{D}$ is a supplemented lattice.

Knowing that a lattice $\mathbf{D}$ is Hausdorff will usually not be very useful if the stalks of $\mathbf{D}$ are not in some way simpler than $\mathbf{D}$ itself. The case where the stalks are directly indecomposable or in some other sense irreducible is often of interest. In the following we will determine the Heyting algebras which are Boolean products of a family of finitely subdirectly irreducible (fsi) Heyting algebras. The finitely subdirectly irreducible Heyting algebras have a particularly simple description, namely, they are the well-connected Heyting algebras, viz., Heyting algebras satisfying the universal clause

$$1 \approx x \vee y \implies x \approx 1 \text{ or } y \approx 1.$$

See Appendix A.4 for details.

4.1.22. Proposition. *Let* $\mathbf{A}$ *be a Heyting algebra. If* $\mathbf{A}$ *is a Boolean product with fsi stalks then* $\mathbf{A}$ *is supplemented with a De Morgan supplement.*

Proof:
Suppose that $\mathbf{A}$ is a Boolean product of a family of fsi Heyting algebras $\{\mathbf{A}_x : x \in X\}$. Then for all $a \in A$ we have that the set $[\![a = 1]\!] = \{x \in X : a(x) = 1\}$ is clopen. Hence by the patchwork property there is an element $a' \in A$ such that

$$a'(x) = \begin{cases} 1 & \text{if } a(x) < 1, \\ 0 & \text{if } a(x) = 1. \end{cases}$$

As each of the stalks $\mathbf{A}_x$ of $\mathbf{A}$ are well-connected it is easy to see that a' is the supplement of a in $\mathbf{A}$. Moreover, because the stalks are well-connected, for $a, b \in A$ and $x \in X$ we must have that

$$\begin{aligned} ({\sim}(a \vee b))(x) = 0 &\iff (a \vee b)(x) = 1 \\ &\iff a(x) = 1 \text{ or } b(x) = 1 \\ &\iff ({\sim}\, a)(x) = 0 \text{ or } ({\sim}\, b)(x) = 0 \\ &\iff ({\sim}\, a \wedge {\sim}\, b)(x) = 0. \end{aligned}$$

Similarly, again using the fact that the stalks are well-connected, we see that

$$\begin{aligned} ({\sim}(a \vee b))(x) = 1 &\iff (a \vee b)(x) < 1 \\ &\iff a(x) < 1 \text{ and } b(x) < 1 \\ &\iff ({\sim}\, a)(x) = 1 \text{ and } ({\sim}\, b)(x) = 1 \\ &\iff ({\sim}\, a \wedge {\sim}\, b)(x) = 1. \end{aligned}$$

Consequently, since for each $c \in A$ and all $x \in X$ we have that $({\sim}\, c)(x) = 0$ or $({\sim}\, c)(x) = 1$, this shows that ${\sim}(a \vee b) = {\sim}\, a \wedge {\sim}\, b$. □

Let $\mathbf{A}$ be a Heyting algebra with dual Esakia space X. Since each $x \in X$ is a filter on $\mathbf{A}$ we obtain a Heyting algebra congruence θ_x on $\mathbf{A}$ by

$$a \theta_x b \quad \text{if, and only if,} \quad a \leftrightarrow b \in x.$$

Moreover, since each $x \in X$ is prime, the quotient $\mathbf{A}/\theta_x$ is well-connected or equivalently finitely subdirectly irreducible.

4.1.23. Lemma. *Let* $\mathbf{A}$ *be Heyting algebra with dual Esakia space* X. *Then* $\mathbf{A}$ *is a subdirect product of the family* $\{\mathbf{A}/\theta_x : x \in \min(X)\}$.

Proof:
It suffices to show that for all $a, b \in A$, if $a \neq b$ then there exists $x \in \min(X)$ such that $(a, b) \notin \theta_x$. If $a \neq b$ then $a \leftrightarrow b < 1$ and so there must be a prime filter x'

of $\mathbf{A}$ not containing $a \leftrightarrow b$. Since X is a Priestley space we have that x' extends some minimal prime filter, say $x \subseteq x'$. Evidently, $a \leftrightarrow b \notin x$, whence $(a, b) \notin \theta_x$, as desired. □

The subdirect presentation of $\mathbf{A}$ over the minimum of its dual Esakia space determined by Lemma 4.1.23 can be seen as a generalized version of the usual weak Boolean product representation of $\mathbf{A}$ over its center. Furthermore, as we will see in Section 4.3 given a Heyting algebra $\mathbf{A}$ with dual Esakia space X, the space $\min(X)$ and the family of quotients $\{\mathbf{A}/\theta_x : x \in \min(X)\}$ completely determine the hyper-MacNeille completion of $\mathbf{A}$. We first show that when $\mathbf{A}$ is supplemented with a De Morgan supplement then the subdirect representation of $\mathbf{A}$ given by Lemma 4.1.23 is in fact (isomorphic to) the usual representation of $\mathbf{A}$ over the center of $\mathbf{A}$.

4.1.24. Theorem (cf. [247, Thm. 9.5]). *Let $\mathbf{A}$ be a Heyting algebra. Then the following are equivalent.*

1. *The algebra $\mathbf{A}$ is supplemented with a De Morgan supplement.*
2. *The algebra $\mathbf{A}$ is a Hausdorff lattice with finitely subdirectly irreducible stalks.*

Proof:
That Item 2 entails Item 1 has already been established in Proposition 4.1.22. Thus it only remains to establish the converse implication.

Suppose that $\mathbf{A}$ is supplemented with a De Morgan supplement, and let X be the dual Stone space of the center of $\mathbf{A}$. We must show that $[\![a = b]\!]$ is a clopen subset of X for all $a, b \in A$. Therefore, let $a, b \in A$ be given. As the relation $\equiv_x$ is in fact a congruence of Heyting algebras we see that

$$[\![a = b]\!] = \{x \in X : a \leftrightarrow b \equiv_x 1\}.$$

Let d be the element $a \leftrightarrow b$. Then for $x \in X$ we see that $d \equiv_x 1$ if, and only if, there is $c \in x$ such $c \leq d$. As any central element is evidently co-regular and $\sim\sim d \leq d$ we see that $c \leq d$, if and only if, $c \leq \sim\sim d$, for any central element c of A. From Proposition 4.1.9 we know that co-regular elements of $\mathbf{A}$ are all central. In particular $\sim\sim d$ must be central. It follows that $d \equiv_x 1$ precisely when $\sim\sim d$ belongs to x. This shows that $[\![a = b]\!]$ is the clopen subset $\widehat{\sim\sim d}$ of X.

To see that the stalks are $\mathbf{A}_x$ are finitely subdirectly irreducible let $x \in X$ be given and consider $a, b \in A$ such that $a \vee b \equiv_x 1$. Then there is $c \in x$ such that $c \wedge (a \vee b) = c$, implying that $c \leq a \vee b$. But then $\sim\sim c \leq \sim\sim(a \vee b)$. Since the central and the supplemented elements of $\mathbf{A}$ coincide we obtain that $c \leq \sim\sim(a \vee b)$. Moreover, using the De Morgan laws we see that $c \leq \sim(\sim a \wedge \sim b)$. As $\sim(\sim a \wedge \sim b)$ is co-regular and therefore central we obtain that $\sim(\sim a \wedge \sim b) \in x$.

Another use of the De Morgan laws shows that $\sim\sim a \vee \sim\sim b \in x$ and hence that either $\sim\sim a \in x$ or $\sim\sim b \in x$, as x is a prime filter. In the first case we must have that $a \equiv_x 1$ and in the second that $b \equiv_x 1$. This shows that $\mathbf{A}_x$ is well-connected and hence finitely subdirectly irreducible. □

4.1.25. Remark. We note that $(\sim(x \leftrightarrow y) \wedge x) \vee (\neg\sim(x \leftrightarrow y) \wedge z)$ is a discriminator term on any fsi Heyting algebra, viewed as a supplemented Heyting algebra, cf. [229, Sec. 5] as well as [241]. Consequently, Theorem 4.1.24 shows that the class of De Morgan supplemented Heyting algebras forms a discriminator variety of supplemented Heyting algebras, see, e.g., [46, Chap. IV.9].

When $\mathbf{A}$ is supplemented with a De Morgan supplement not only will $\mathbf{A}$ be Hausdorff but, as the following proposition will show, the stalks can be give a more familiar description.

4.1.26. Proposition. *Let $\mathbf{A}$ be a supplemented Heyting algebra with dual Esakia space X. Then for any $x \in \min(X)$ and any $a, b \in A$ we have that $a\theta_x b$ if, and only if, there is $c \in x \cap Z(\mathbf{A})$ such that $c \wedge a = c \wedge b$.*

Proof:
If $a\theta_x b$ then we have that $d := a \leftrightarrow b \in x$. Then $c := \sim\sim d$ is a co-regular, and hence by Proposition 4.1.9 central, element below d. Consequently,

$$c \wedge a = (c \wedge d) \wedge a = c \wedge (d \wedge a) = c \wedge (d \wedge b) = c \wedge b.$$

Thus it suffices to show that $c \in x$. If not, then $\sim c \in x$ but $\sim c = \sim d$ and so $d \wedge \sim d \in x$ which is a contradiction since x is assumed to be minimal and as such does not contain any co-dense elements. Conversely, if we have $c \in x \cap Z(\mathbf{A})$ such that $c \wedge a = c \wedge b$, then $c \leq a \leftrightarrow b$ and hence $a \leftrightarrow b \in x$ showing that $a\theta_x b$, as desired. □

By Corollary 4.1.18 we know that when $\mathbf{A}$ is a supplemented Heyting algebra with dual Esakia space X then $\min(X)$ is closed in X and hence is a Stone space in the subspace topology. Moreover, by Propositions 4.1.9 and 4.1.19, when $\mathbf{A}$ is De Morgan supplemented the dual Stone space of the center of $\mathbf{A}$ may be identified with $\min(X)$. This, together with Proposition 4.1.26, yields the following corollary.

4.1.27. Corollary. *A Heyting algebra $\mathbf{A}$ with dual Esakia space X is supplemented with a De Morgan supplement if, and only if, the subdirect embedding $\mathbf{A} \hookrightarrow \prod_{x \in \min(X)} \mathbf{A}/\theta_x$ is a Boolean product representation of $\mathbf{A}$.*

As mentioned before when computing the hyper-MacNeille completion of a Heyting algebra $\mathbf{A}$ the subdirect presentation $\mathbf{A} \hookrightarrow \prod_{x \in \min(X)} \mathbf{A}/\theta_x$ will play an essential role also when $\mathbf{A}$ is not (De Morgan) supplemented.

4.2 Hyper-MacNeille completions

In this section we will provide the definition of the hyper-MacNeille completion of a Heyting algebra. We first give a presentation of the hyper-MacNeille completion of Heyting algebras which is slightly simpler than the original. We then recall the original definition from [60, Sec. 6.2] and show that the two presentations indeed determine the same Heyting algebras.

4.2.1 Heyting frames and complete Heyting algebras

To define the hyper-MacNeille completion of a Heyting algebra it will not be necessary to introduce the notion of a residuated hyper-frame as done in [60, Sec. 5.1]. Instead, following [242], we will make use of *Heyting frames* [243, Sec. 6.1] which are essentially the residuated frames [97] giving rise to complete Heyting algebras.

4.2.1. Definition ([243]). A structure $\mathbf{W} = \langle W_0, W_1, N, \circ, \varepsilon, \rightsquigarrow\rangle$ is called a *Heyting frame* provided that

(i) (W_0, W_1, N) is a polarity,

(ii) $\langle W_0, \circ, \varepsilon\rangle$ is a monoid,

(iii) $\rightsquigarrow\colon W_0 \times W_1 \to W_1$ is a function satisfying,

$$\begin{aligned}
&\forall w_1, w_2, \in W_0 \forall u \in W_1\ ((w_1 \circ w_2)Nu \iff w_2 N w_1 \rightsquigarrow u),\\
&\forall w \in W_0 \forall u \in W_1\ ((w \circ w)Nu \implies wNu),\\
&\forall w \in W_0 \forall u \in W_1\ (\varepsilon N u \implies wNu),\\
&\forall w_1, w_2 \in W_0 \forall u \in W_1\ ((w_1 \circ w_2)Nu \implies (w_2 \circ w_1)Nu).
\end{aligned}$$

Recall from Appendix A.8 that any polarity (W_0, W_1, N) induces a pair of functions

$$L\colon \wp(W_1) \to \wp(W_0) \quad \text{and} \quad U\colon \wp(W_0) \to \wp(W_1),$$

defined by

$$L(X) := \{w \in W_0 : \forall u \in X\ (wNu)\} \quad \text{and} \quad U(Y) := \{u \in W_1 : \forall w \in Y\ (wNu)\},$$

for $X \subseteq W_1$ and $Y \subseteq W_0$. The composition of these functions determines a closure operator $\gamma_N := LU\colon \wp(W_0) \to \wp(W_0)$, the closed elements of which form a complete lattice with meets and joins defined as

$$\bigwedge_{i\in I} Z_i = \bigcap_{i\in I} Z_i \quad \text{and} \quad \bigvee_{i\in I} Z_i = \gamma_N\left(\bigcup_{i\in I} Z_i\right),$$

respectively.

The point of introducing the additional structure of a Heyting frame is to insure that lattice induced by the polarity (W_0, W_1, N) will be a Heyting algebra.

4.2.2. Theorem ([243, Lem. 13]). *Let* $\mathbf{W} = \langle W_0, W_1, N, \circ, \varepsilon, \rightsquigarrow\rangle$ *be a Heyting frame. Then the lattice* $\mathbf{W}^+$ *induced by the polarity* (W_0, W_1, N) *is a complete Heyting algebra, with Heyting implication defined as*

$$X \to Y := \{w \in W_0 : \forall v \in X\ (v \circ w \in Y)\},$$

and with least element $\gamma_N(\emptyset)$ *and greatest element* W_0*.*

It is not difficult to verify that any Heyting algebra $\mathbf{A}$ gives rise to a Heyting frame $\mathbf{W_A} := \langle \mathbf{A}_\wedge, \mathbf{A}_\wedge, \leq, \wedge, 1, \to\rangle$, such that the induced complete Heyting algebra $\mathbf{W}^+_\mathbf{A}$ is in fact isomorphic to the MacNeille completion $\overline{\mathbf{A}}$ of $\mathbf{A}$. One can also construct "Lindenbaum-Tarski" Heyting frames on the basis of sequent and hypersequent calculi. However, the exact definitions are somewhat more intricate, and so for details we refer to [243, 60, 97, 59].

4.2.2 Definition and basic properties

Let $\mathbf{A} = \langle A, \wedge, \vee, \to, 0, 1\rangle$ be a Heyting algebra. The algebra $\mathbf{A}$ induces two monoids, namely the monoids $\mathbf{A}_\wedge = \langle A, \wedge, 1\rangle$ and $\mathbf{A}_\vee = \langle A, \vee, 0\rangle$. Consequently, forming the direct product $\mathbf{M_A} := \mathbf{A}_\vee \times \mathbf{A}_\wedge$ we obtain a monoid with monoidal operation $\circ$ given by

$$(s, a) \circ (t, b) := (s \vee t, a \wedge b),$$

and unit $(0, 1)$. We then define a relation $N \subseteq A^2 \times A^2$ by letting

$$(s, a)N(t, b) \quad \text{if, and only if,} \quad s \vee t \vee (a \to b) = 1.$$

Furthermore, letting

$$(s, a) \rightsquigarrow (t, b) = (s \vee t, a \to b),$$

we see that

$$\begin{aligned} ((s_1, a_1) \circ (s_2, a_2))N(t, b) &\iff s_1 \vee s_2 \vee t \vee ((a_1 \wedge a_2) \to b) = 1 \\ &\iff s_2 \vee (s_1 \vee t) \vee (a_2 \to (a_1 \to b)) = 1 \\ &\iff (s_2, a_2)N(s_1 \vee t, (a_1 \to b)) \\ &\iff (s_2, a_2)N(s_1, a_1) \rightsquigarrow (t, b). \end{aligned}$$

The monoid $\mathbf{M_A}$ is evidently commutative and idempotent. Furthermore, for all $(t, b) \in A^2$ we have that $(0, 1)N(t, b)$ if, and only if, $t \vee b = 1$. Therefore, since the Heyting implication is order-reversing in its first argument, for all $(s, a), (t, b) \in A^2$ we see that $(0, 1)N(t, b)$ entails $(s, a)N(t, b)$. Thus the structure $\mathbf{J_A} := (\mathbf{M_A}, \mathbf{M_A}, N, \circ, (0, 1), \rightsquigarrow)$ is a Heyting frame. Consequently, by Theorem 4.2.2, we obtain a complete Heyting algebra $\mathbf{J}^+_\mathbf{A}$. It turns out that this is in fact a completion of $\mathbf{A}$.

4.2.3. Proposition (cf. [60, Thm. 5.20]). *There is an embedding of Heyting algebras* $e\colon \mathbf{A} \hookrightarrow \mathbf{J}^+_{\mathbf{A}}$ *given by* $b \mapsto L(0,b) = \{(s,a) \in A^2 : s \vee (a \to b) = 1\}$.

Proof:
As a general fact about closure operators induced by polarities we have that the elements of the algebra $\mathbf{J}^+_{\mathbf{A}}$ are precisely the elements of the form $L(X)$ for $X \subseteq A^2$. In particular this shows that the map e is well defined. For each $b \in A$ we have that $(0,b) \in L(0,b)$. Consequently, if $L(0,b_1) \leq L(0,b_2)$ then $(0,b_1) \in L(0,b_2)$ whence $b_1 \to b_2 = 1$ and hence $b_1 \leq b_2$. This shows that the map e is injective.

Evidently, we have that $e(1) = L(0,1) = A^2$. Moreover, $e(0) = \{(s,a) \in A^2 : s \vee \neg a = 1\}$ and so since the Heyting implication is order-preserving in its second argument we see that $(s,a)N(t,b)$ for all $(s,a) \in e(0)$ and all $(t,b) \in A^2$. Thus $e(0) \subseteq L(X)$ for any $X \subseteq A^2$ and hence $e(0)$ must be the least element of the algebra $\mathbf{J}^+_{\mathbf{A}}$. This shows that the map e preserves the bounds.

From the fact that the Heyting implication preserves binary meets in its second argument, for $b_1, b_2 \in A$ we see that

$$\begin{aligned}
e(b_1 \wedge b_2) &= \{(s,a) \in A^2 : s \vee (a \to (b_1 \wedge b_2)) = 1\} \\
&= \{(s,a) \in A^2 : s \vee ((a \to b_1) \wedge (a \to b_2)) = 1\} \\
&= \{(s,a) \in A^2 : (s \vee (a \to b_1)) \wedge (s \vee (a \to b_2)) = 1\} \\
&= \{(s,a) \in A^2 : (s \vee (a \to b_1)) = 1 \text{ and } (s \vee (a \to b_2)) = 1\} \\
&= e(b_1) \cap e(b_2) \\
&= e(b_1) \wedge e(b_2).
\end{aligned}$$

Thus the map e preserves binary meets. In particular e must be order-preserving. Form this we may conclude that $e(b_1) \vee e(b_1) \leq e(b_1 \vee b_2)$ and $e(b_1 \to b_2) \leq e(b_1) \to e(b_2)$, for all $b_1, b_2 \in A$.

In order to show that e preserves binary joins it will be sufficient to show that $e(b_1 \vee b_2) \leq e(b_1) \vee e(b_2)$, for all $b_1, b_2 \in A$. Therefore, let $b_1, b_2 \in A$ be given, and let $(s,a) \in e(b_1 \vee b_2)$. By definition we have that $e(b_1) \vee e(b_2) = LU(e(b_1) \cup e(b_2))$ and so to show that $(s,a) \in e(b_1) \vee e(b_2)$ we must show that $(s,a)N(t,b)$ for all $(t,b) \in U(e(b_1) \cup e(b_2))$. Therefore, let $(t,b) \in U(e(b_1) \cup e(b_2))$ be given. Then $(t,b) \in U(e(b_i))$ for each $i \in \{1,2\}$. As $(0,b_i) \in U(e(b_i))$ it follows that $t \vee (b_i \to b) = 1$ for $i \in \{1,2\}$. Consequently, we must have

$$\begin{aligned}
1 &= (t \vee (b_1 \to b)) \wedge (t \vee (b_2 \to b)) \\
&= t \vee ((b_1 \to b) \wedge (b_2 \to b)) \\
&= t \vee ((b_1 \vee b_2) \to b).
\end{aligned}$$

By the assumption that $(s,a) \in e(b_1 \vee b_2)$ we also have that $s \vee (a \to (b_1 \vee b_2)) = 1$.

Using distributivity we then see that

$$\begin{aligned}1 &= (s \vee (a \to (b_1 \vee b_2))) \wedge (t \vee ((b_1 \vee b_2) \to b)) \\ &\leq ((s \vee t) \vee (a \to (b_1 \vee b_2))) \wedge ((s \vee t) \vee ((b_1 \vee b_2) \to b)) \\ &= (s \vee t) \vee ((a \to (b_1 \vee b_2)) \wedge ((b_1 \vee b_2) \to b)) \\ &\leq s \vee t \vee (a \to b).\end{aligned}$$

This shows that $(s, a)N(t, b)$, as desired.

Finally, to show that e preserves the Heyting implication it suffices to show that $e(b_1) \to e(b_2) \leq e(b_1 \to b_2)$, for all $b_1, b_2 \in A$. Therefore, let $b_1, b_2 \in A$ be given. By definition the set $e(b_1) \to e(b_2)$ consist of the elements $(s, a) \in A^2$ such that $(s, a) \circ (s_1, a_1) \in e(b_2)$ for all $(s_1, a_1) \in e(b_1)$. Since $(0, b_1) \in e(b_1)$ it follows that $(s, a) \circ (0, b_1) \in e(b_2)$, for all $(s, a) \in e(b_1)$. For each $(s, a) \in A^2$ we see that $(s, a) \circ (0, b_1) = (s, a \wedge b_1)$. Therefore, if $(s, a) \in e(b_1)$ then

$$s \vee ((a \wedge b_1) \to b_2) = s \vee (a \to (b_1 \to b_2)) = 1,$$

showing that $(s, a) \in e(b_1 \to b_2)$. Therefore, $e(b_1) \to e(b_2) \leq e(b_1 \to b_2)$, as desired. $\square$

4.2.4. Definition (cf. [60, Sec. 6.2]). For a Heyting algebra $\mathbf{A}$ we define the *hyper-MacNeille completion* of $\mathbf{A}$ to be the algebra $\mathbf{J}^+_{\mathbf{A}}$, and denote it by $\mathbf{A}^+$.

The original definition of the hyper-MacNeille completion due to Ciabattoni, Galatos, and Terui [60, Sec. 6.2] was based on a slightly different Heyting frame. Namely, given a Heyting algebra $\mathbf{A}$ let $\mathbf{F}(A^2)$ be the free commutative monoid on the set A^2. For $h, g \in F(A^2)$ we write $h \mid g$ for their product in $\mathbf{F}(A^2)$. The universal property of $\mathbf{F}(A^2)$ then induces a monoid homomorphism $(-)^* \colon \mathbf{F}(A^2) \to \mathbf{A}_\vee$ by letting $(a, b)^* = a \to b$, for each $(a, b) \in A^2$. This map has a section $(-)_* \colon A \to F(A^2)$ given by $a \mapsto (1, a)$, which is however not a monoid homomorphism. We then obtain a monoid $\mathbf{M}^*_{\mathbf{A}} \coloneqq \mathbf{F}(A^2) \times \mathbf{A}_\wedge$, with monoidal operation $\bullet$ given by

$$(h, a) \bullet (g, b) \coloneqq (h \,|\, g, a \wedge b),$$

and unit $(\varepsilon, 1)$, where ε denotes the unit in $\mathbf{F}(A^2)$. Just as before it is easy to verify that the structure $\mathbf{H}_{\mathbf{A}} \coloneqq (\mathbf{M}^*_{\mathbf{A}}, \mathbf{M}^*_{\mathbf{A}}, N^*, \bullet, (\varepsilon, 1), \rightsquigarrow)$ is a Heyting frame with

$$(h, a)N^*(g, b) \quad \text{if, and only if,} \quad h^* \vee g^* \vee (a \to b) = 1,$$

and

$$(h, a) \rightsquigarrow (g, b) \coloneqq (h \,|\, g, a \to b).$$

4.2.5. Theorem. *If* $\mathbf{A}$ *is a Heyting algebra, then the induced Heyting algebras* $\mathbf{J}^+_{\mathbf{A}}$ *and* $\mathbf{H}^+_{\mathbf{A}}$ *are isomorphic.*

Proof:
To avoid confusion we will here write and U^* and L^* for the functions induced by the relation N^*.

For $Z \in \mathbf{H}^+_{\mathbf{A}}$ let $J(Z) \coloneqq \{(h^*, a) : (h, a) \in Z\}$. We claim that this assignment is an isomorphism between $\mathbf{H}^+_{\mathbf{A}}$ and $\mathbf{J}^+_{\mathbf{A}}$. To see that this assignment indeed defines a map from $\mathbf{H}^+_{\mathbf{A}}$ to $\mathbf{J}^+_{\mathbf{A}}$ let $Z \in \mathbf{H}^+_{\mathbf{A}}$ and $(s, a) \in LU(J(Z))$ be given. Observe that if $(g, b) \in U^*(Z)$ then $h^* \vee g^* \vee (a \to b) = 1$ for all $(h, a) \in Z$. But then $(g^*, b) \in U(J(Z))$ and so $(s, a)N(g^*, b)$. Consequently, $1 = s \vee g^* \vee (a \to b) = (s_*)^* \vee g^* \vee (a \to b)$, and hence $(s_*, a)N^*(g, b)$. This shows that $(s_*, a) \in L^*U^*(Z) = Z$, and therefore $(s, a) = ((s_*)^*, a) \in J(Z)$. We have thus established that $LU(J(Z)) \subseteq J(Z)$ and hence that $J(Z) \in \mathbf{J}^+_{\mathbf{A}}$, showing that the map $Z \mapsto J(Z)$ is well defined.

To see that this map is in fact an isomorphism we first observe that since the partial orders on $\mathbf{H}^+_{\mathbf{A}}$ and $\mathbf{J}^+_{\mathbf{A}}$ are given by set-theoretic inclusion we have that $Z_1 \leq Z_2$ implies $J(Z_1) \leq J(Z_2)$ for all $Z_1, Z_2 \in \mathbf{H}^+_{\mathbf{A}}$. Conversely, if $J(Z_1) \leq J(Z_2)$ for some $Z_1, Z_2 \in \mathbf{H}^+_{\mathbf{A}}$ then for $(h, a) \in Z_1$ we have that $(h^*, a) \in J(Z_2)$, implying that $h_0^* = h$ for some h_0 such that $(h_0, a) \in Z_2$. Given $(g, b) \in U^*(Z_2)$ we then have that $(h_0, a)N^*(g, b)$ and hence that $(h^*, a) = (h_0^*, a)N(g^*, b)$, whence $(h, a)N^*(g, b)$ yielding $(h, a) \in L^*U^*(Z_2) = Z_2$. This shows that $Z \mapsto J(Z)$ is an order embedding. Thus it suffices to prove that it is also surjective. Therefore, given $Z \in \mathbf{J}^+_{\mathbf{A}}$ we let

$$H(Z) = L^*U^*(\{(s_*, a) : (s, a) \in Z\}).$$

Evidently, $H(Z) \in \mathbf{H}^+_{\mathbf{A}}$. We claim that $J(H(Z)) = Z$. It is easily seen that $Z \subseteq J(H(Z))$. On the other hand, for each $(t, b) \in U(Z)$ we have $(t_*, b) \in U^*(\{(s_*, a) \colon (s, a) \in Z\})$. Consequently, if $(h, a) \in H(Z)$, then for all $(t, b) \in U(Z)$ we have $(h, a)N^*(t_*, b)$ and therefore also $(h^*, a)N((t_*)^*, b) = (t, b)$. This implies that $(h^*, a) \in LU(Z) = Z$ for all $(h, a) \in H(Z)$. From this we may conclude that $J(H(Z)) \subseteq Z$, as desired. □

The following are completely standard facts, see, e.g., [60, Lem. 5.3], but included here for easy reference.

4.2.6. Lemma. *For all* $Z \in \mathbf{A}^+$ *and all* $b \in A$ *we have that*

1. $Z = \bigwedge\{L(t, b) \in A^+ : Z \subseteq L(t, b)\}$,
2. $Z = \bigvee\{LU(s, a) \in A^+ : (s, a) \in Z\}$,
3. $LU(0, b) = L(0, b)$.

Proof:
We first observe that since $LU(-)$ is an order-preserving function on $\wp(A^2)$ such that $LU(Z) = Z$ for all $Z \in A^+$, we must have

$$Z = LU(Z) = \bigwedge\{L(t,b) \in A^+ \colon (t,b) \in U(Z)\},$$

for each $Z \in A^+$. Now, Item 1 follows from the easy observation that $(t,b) \in U(Z)$ if, and only if, $Z \subseteq L(t,b)$.

To see that Item 2 holds note that if $(s,a) \in Z$ then $LU(s,a) \subseteq LU(Z) = Z$, hence $\bigvee\{LU(s,a) \in A^+ \colon (s,a) \in Z\} \subseteq Z$. Conversely, since $(s,a) \in LU(s,a)$ we have that $Z \subseteq \bigcup\{LU(s,a) \in A^+ \colon (s,a) \in Z\}$ yielding,

$$\begin{aligned} Z = LU(Z) &\subseteq LU\Big(\bigcup\{LU(s,a) \in A^+ \colon (s,a) \in Z\}\Big) \\ &= \bigvee\{LU(s,a) \in A^+ \colon (s,a) \subseteq Z\}. \end{aligned}$$

Finally, to see that Item 3 holds we observe that since $(0,b) \in U(0,b)$ then $LU(0,b) \subseteq L(0,b)$. Conversely, if $(u,c) \in U(0,b)$ then $1 = u \vee (c \to b)$ and so if $(s,a) \in L(0,b)$ then $1 = s \vee (a \to b)$ yielding

$$1 = u \vee s \vee ((a \to b) \wedge (b \to c)) \leq u \vee s \vee (a \to c).$$

This shows that $(s,a)N(u,c)$ and so since $(u,c) \in U(0,b)$ was arbitrary that $L(0,b) \subseteq LU(0,b)$. □

Recall that an embedding of algebras $e \colon \mathbf{A} \hookrightarrow \mathbf{B}$ is *essential* if for any homomorphism $h \colon \mathbf{B} \to \mathbf{C}$ we have that h is a monomorphism whenever $h \circ e \colon \mathbf{A} \to \mathbf{C}$ is a monomorphism. The MacNeille completion $\mathbf{A} \hookrightarrow \overline{\mathbf{A}}$ of a Heyting algebra $\mathbf{A}$ is always an essential extension. We show that the same is the case for the hyper-MacNeille completion.

4.2.7. Lemma. *An embedding of Heyting algebras $e \colon \mathbf{A} \hookrightarrow \mathbf{B}$ is essential if for all $b \in B$ with $b < 1$ there is $a \in A$ with $a < 1$ such that $b \leq e(a)$.*

Proof:
Suppose that $e \colon \mathbf{A} \hookrightarrow \mathbf{B}$ is an embedding of Heyting algebras satisfying the conditions of the lemma. Let $h \colon \mathbf{B} \to \mathbf{C}$ be a Heyting algebra homomorphism such that $h \circ e \colon \mathbf{A} \hookrightarrow \mathbf{C}$ is an embedding. As congruences of Heyting algebras are determined by the equivalence class of the top element 1, to see that h is an embedding, it suffices to show that $h^{-1}(1) = 1$. Therefore, let $b \in B$ be such that $b < 1$. Then by assumption we have $a \in A$ with $a < 1$ such that $b \leq e(a)$. But then $h(b) \leq h(e(a)) < 1$, showing that $b \notin h^{-1}(1)$. □

4.2.8. Lemma. *Let* $\mathbf{A}$ *be a Heyting algebra and let* $(t,b) \in A^2$ *then* $L(t,b) \leq L(0, t \vee b)$. *Moreover,* $L(0, t \vee b) \neq 1$ *when* $L(t,b) \neq 1$.

Proof:
We claim that $t \vee (a \to b) \leq (a \to (t \vee b))$ and therefore that $s \vee t \vee (a \to b) = 1$ implies $s \vee (a \to (t \vee b)) = 1$ for all $(s,a) \in A^2$, showing that $L(t,b) \leq L(0, t \vee b)$. We have that

$$\begin{aligned} t \vee (a \to b) \leq (a \to (t \vee b)) &\iff a \wedge (t \vee (a \to b)) \leq t \vee b \\ &\iff (a \wedge t) \vee (a \wedge (a \to b)) \leq t \vee b \\ &\iff (a \wedge t) \vee (a \wedge b) \leq t \vee b \\ &\iff a \wedge (t \vee b) \leq t \vee b, \end{aligned}$$

establishing the desired result. Finally, it is easily seen that if $t \vee b = 1$ then $L(t,b) = 1$ thus if $L(t,b) \neq 1$ then, as $a \mapsto L(0,a)$ is an injection, we must have $L(0, t \vee b) \neq 1$. ☐

4.2.9. Proposition. *If* $\mathbf{A}$ *is a Heyting algebra, then* $e \colon \mathbf{A} \hookrightarrow \mathbf{A}^+$ *is an essential extension.*

Proof:
Since the set $\{L(t,b) \in A^+ : (t,b) \in A^2\}$ is meet-dense in $\mathbf{A}^+$ this is now a direct consequence of Lemma 4.2.7 and Lemma 4.2.8. ☐

4.2.3 Hyper-MacNeille completions of De Morgan supplemented lattices

As established in [60, Prop. 6.6] the hyper-MacNeille completion coincides with the MacNeille completion for subdirectly irreducible algebras. We will show that in fact the hyper-MacNeille completion coincides with the MacNeille completion for an even larger class of algebras, namely the class of supplemented Heyting algebras with a De Morgan supplement or equivalently the class of Boolean products of fsi Heyting algebras.

4.2.10. Lemma. *If* $\mathbf{A}$ *is supplemented with a De Morgan supplement, then*

$$L(t,b) = L(0, {\sim} t \to b) \quad \text{and} \quad U(s,a) = U(0, {\sim} s \wedge b),$$

for all $(s,a),(t,b) \in A^2$.

Proof:
Since $\mathbf{A}$ is De Morgan supplemented we have that $\sim(s \vee t) = \sim s \wedge \sim t$ for all $(s, a), (t, b) \in A^2$. Consequently, for $(t, b) \in A^2$ we have that

$$\begin{aligned} L(t, b) &= \{(s, a) \in A^2 : 1 = s \vee t \vee (a \to b)\} \\ &= \{(s, a) \in A^2 : \sim(s \vee t) \leq a \to b\} \\ &= \{(s, a) \in A^2 : \sim(s \vee t) \wedge a \leq b\} \\ &= \{(s, a) \in A^2 : \sim s \wedge \sim t \wedge a \leq b\} \\ &= \{(s, a) \in A^2 : \sim s \wedge a \leq \sim t \to b\} \\ &= \{(s, a) \in A^2 : \sim s \leq a \to (\sim t \to b)\} \\ &= \{(s, a) \in A^2 : 1 = s \vee (a \to (\sim t \to b))\} \\ &= L(0, \sim t \to b) \end{aligned}$$

Similarly, for $(s, a) \in A^2$ we have that

$$\begin{aligned} U(s, a) &= \{(t, b) \in A^2 : 1 = s \vee t \vee (a \to b)\} \\ &= \{(t, b) \in A^2 : \sim s \wedge \sim t \wedge a \leq b\} \\ &= \{(t, b) \in A^2 : \sim t \leq (\sim s \wedge a) \to b\} \\ &= \{(t, b) \in A^2 : 1 = t \vee ((\sim s \wedge a) \to b)\} \\ &= U(0, \sim s \wedge a), \end{aligned}$$

which concludes the proof. □

4.2.11. Proposition. *If $\mathbf{A}$ is supplemented with a De Morgan supplement, then $\mathbf{A}^+$ is isomorphic to $\overline{\mathbf{A}}$.*

Proof:
By Lemmas 4.2.10 and 4.2.6 we have that $\mathbf{A}$ is both join- and meet-densely embedded into $\mathbf{A}^+$. As $\mathbf{A}^+$ is complete we may conclude that $\mathbf{A}^+$ is isomorphic to the MacNeille completion $\overline{\mathbf{A}}$ of $\mathbf{A}$. □

It is worth noting that Proposition 4.2.11 applies to Boolean algebras.

4.2.12. Corollary. *If $\mathbf{A}$ is a Boolean algebra, then $\mathbf{A}^+$ is isomorphic to the MacNeille completion of $\mathbf{A}$.*

Furthermore, since the Heyting algebras with a De Morgan supplement are precisely the Boolean products with fsi stalks, we obtain the following corollary.

4.2.13. Corollary. *Let $\mathbf{A}$ be a Heyting algebra. If $\mathbf{A}$ is a Boolean product of fsi Heyting algebras, then $\mathbf{A}^+$ is isomorphic to $\overline{\mathbf{A}}$.*

Thus to determine the hyper-MacNeille completion of an arbitrary Heyting algebra we must as a special case be able to determine the MacNeille completion of Boolean products of fsi Heyting algebras. Unfortunately, not much is known about how to compute MacNeille completions of Boolean products of lattices. However, what we do know is the following.

Let $\mathbf{D}$ be any distributive lattice[1] and let $\mathbf{D} \hookrightarrow \prod_{x \in X} \mathbf{D}_x$ be the usual weak Boolean product representation of $\mathbf{D}$ over its center as defined in Subsection 4.1.2. It can be shown, see [146, 73], that there is a homomorphic image of a subalgebra of the direct product $\prod_{x \in X} \mathbf{D}_x$ called, *the algebra of dense open sections of* $\mathbf{D}$ and denote by $R(\mathbf{A})$, with the property that $\mathbf{D}$ is join- and meet-densely embedded in $R(\mathbf{D})$ whenever $\mathbf{D}$ is Hausdorff[2]. However, the algebra $R(\mathbf{D})$ need not be complete. Nevertheless, there are natural criteria for $R(\mathbf{D})$ to be complete and therefore the MacNeille completion of $\mathbf{D}$. Namely, as established by Harding [146], when the size of the stalks of $\mathbf{D}$ are uniformly bounded on a dense open set, then $R(\mathbf{D})$ will be complete.

4.2.14. Proposition ([146, Prop. 4]). *Let $\mathbf{D}$ be any Hausdorff lattice and let X be the dual Stone space of the center of $\mathbf{D}$. If there is a natural number n and a dense open $U \subseteq X$ such that $\mathbf{D}_x$ has size at most n for all $x \in U$, then $\overline{\mathbf{D}} = R(\mathbf{D})$.*

From this, and the fact that De Morgan supplemented Heyting algebras are Hausdorff, we immediately obtain the following corollary.

4.2.15. Corollary. *Let $\mathbf{A}$ be a De Morgan supplemented Heyting algebra with dual Esakia space X. If there is a natural number n and a dense open $U \subseteq \min(X)$ such that $|\mathbf{A}/\theta_x| \leq n$ for all $x \in U$, then $\mathbf{A}^+$ is isomorphic to $R(\mathbf{A})$.*

This discussion also shows that in general there is little hope of giving a better description of the hyper-MacNeille completion of an arbitrary Heyting algebra without first being able to determine the MacNeille completions of Boolean products of (fsi) Heyting algebras. In fact, in Section 4.3 we will obtain a description of the hyper-MacNeille completion of an arbitrary Heyting algebra $\mathbf{A}$ as the MacNeille completion of some Boolean product of fsi Heyting algebras.

4.3 Algebras of dense open sections

In this section we will show that the hyper-MacNeille completion of Heyting algebra $\mathbf{A}$ is the MacNeille completion of some Hausdorff Heyting algebra $Q(\mathbf{A})$ determined by $\mathbf{A}$. This makes the hyper-MacNeille completion of $\mathbf{A}$ much like the

[1] For what follows the assumption that the lattice in question is distributive is not essential, see [73].

[2] In fact, a slightly weaker condition suffices, see [73, Lem. 6.8].

MacNeille completion of a Boolean product, where the subdirect representation $\mathbf{A} \hookrightarrow \prod_{x\in\min(X)} \mathbf{A}/\theta_x$ plays the role of a Boolean product representation.

Given a Heyting algebra $\mathbf{A}$ with dual Esakia space X we will always consider $\mathbf{A}$ as a subalgebra of $\prod_{x\in\min(X)} \mathbf{A}/\theta_x$. Consequently, for $a \in A$ and $x \in \min(X)$, we have that $a(x)$ denotes the image of a under the canonical projection $\prod_{x\in\min(X)} \mathbf{A}/\theta_x \twoheadrightarrow \mathbf{A}/\theta_x$. It is easy to verify that for $a \in A$ and $x \in \min(X)$,

$$a(x) = 1 \quad \text{if, and only if,} \quad a \in x.$$

For any $f, g \in \prod_{x\in\min(X)} \mathbf{A}/\theta_x$, we let

$$[\![f = g]\!] \coloneqq \{x \in \min(X) : f(x) = g(x)\},$$

and likewise, *mutatis mutandis*, for $[\![f < g]\!]$ and $[\![f \leq g]\!]$.

As we have seen, the algebra of dense open sections plays a crucial role when considering MacNeille completions of Boolean products of lattices. This is also the case when considering hyper-MacNeille completions. However, in this case we will be working with a sheaf of algebras the base space of which is not necessarily compact and so in particular not a Stone space.

4.3.1. Lemma. *Let $\mathbf{A}$ be a Heyting algebra with dual Esakia space X. For each $a \in A$ and each $x \in \min(X) \cap \widehat{a}$ there is $s < 1$ such that $a \vee s = 1$ and $x \notin \widehat{s}$.*

Proof:
Since x is minimal we have that $\{x\}$ and $\uparrow(X\backslash\widehat{a})$ are disjoint closed down- and upsets, respectively. As X is a Priestley space we have clopen upset $\widehat{s}$ such that $\{x\} \cap \widehat{s} = \emptyset$ and $\uparrow(X\backslash\widehat{a}) \subseteq \widehat{s}$. It follows that $x \notin \widehat{s}$. Furthermore, we see that

$$\widehat{a \vee s} = \widehat{a} \cup \widehat{s} \supseteq \widehat{a} \cup \uparrow(X\backslash\widehat{a}) = X,$$

from which it follows that $a \vee s = 1$. $\square$

Recall from Appendix A.6 that any Priestley space $(X, \leq, \tau)$ gives rise to two spectral spaces $(X, \tau^{\uparrow})$ and $(X, \tau^{\downarrow})$ with $\tau^{\uparrow}$ the collection of open upsets of $(X, \leq, \tau)$ and $\tau^{\downarrow}$ the collection of open downsets of $(X, \leq, \tau)$.

4.3.2. Proposition. *Let $\mathbf{A}$ be a Heyting algebra with $(X, \leq, \tau)$ its dual Esakia space. The subspace topology on $\min(X)$ inherited from the Priestley topology τ on X coincides with the subspace topology on $\min(X)$ inherited form the spectral topology $\tau^{\downarrow}$ on X.*

Proof:
Since $\tau^{\downarrow} \subseteq \tau$ it suffices to show that for each basic open $U \in \tau$ there is $V \in \tau^{\downarrow}$ such that $U \cap \min(X) = V \cap \min(X)$. The sets of the form $\widehat{a}$, and $\widetilde{a} = X\backslash\widehat{a}$, with $a \in A$, form a basis for the topology τ. Consequently, as $\widetilde{a} \in \tau^{\downarrow}$ for all $a \in A$

we only need to show that the elements of the form $\widehat{a} \cap \min(X)$ are open in the subspace topology on $\min(X)$ inherited from $\tau^{\downarrow}$. To see this let $x \in \widehat{a} \cap \min(X)$ be given. By Lemma 4.3.1 we have $s_x < 1$ such that $a \vee s_x = 1$ and $x \in \widetilde{s_x}$. Furthermore, if $y \in \widetilde{s_x} \cap \min(X)$ then $y \notin \widehat{s_x}$ and therefore $s_x(y) < 1$. As $\mathbf{A}/\theta_y$ is well-connected this entails that $a(y) = 1$ and hence $y \in \widehat{a}$. This shows that $x \in \widetilde{s_x} \cap \min(X) \subseteq \widehat{a} \cap \min(X)$ from which it follows that the set $\widehat{a} \cap \min(X)$ is open the subspace topology on $\min(X)$ inherited from $\tau^{\downarrow}$. □

In this section we will always consider $\min(X)$ as a topological space equipped with the subspace topology inherited from the spectral space $(X, \tau^{\downarrow})$. That is, the topology determined by basic opens of the form $\widehat{s} \cap \min(X)$ for $s \in A$. Note that by Proposition 4.3.2 the sets $[\![s = 1]\!]$ for $s \in A$ are open in this topology.

Given a Heyting algebra $\mathbf{A}$ with dual Esakia space X we would like to equip the disjoint union of the quotients $\{A/\theta_x : x \in \min(X)\}$ with a topology allowing us to talk about continuity of elements of the direct product $\prod_{x \in \min(X)} \mathbf{A}/\theta_x$. In order to avoid cumbersome notation we will always assume that the underlying sets of the algebras $\mathbf{A}/\theta_x$ are disjoint, knowing that we can always replace them with isomorphic copies to satisfy this assumption.

4.3.3. Proposition. *Let* $\mathbf{A}$ *be a Heyting algebra with dual Esakia space* X*. The collection of sets*

$$\mathcal{O}(a, s) := \{a(x) : x \in [\![s < 1]\!]\},$$

with $(s, a) \in A^2$*, forms a basis for a topology on* $\bigcup_{x \in \min(X)} A/\theta_x$*.*

Proof:
We have that each element of $\bigcup_{x \in \min(X)} A/\theta_x$ is of the form $c(x)$ for some $c \in A$ and some $x \in \min(X)$. Because x is a prime filter and as such proper, we must have $s \in A$ such that $s \notin x$, implying that $s(x) < 1$. This shows that the sets of the form $\mathcal{O}(a, s)$ cover $\bigcup_{x \in \min(X)} A/\theta_x$.

If $c \in A$ and $x \in \min(X)$ are such that $c(x)$ belongs to $\mathcal{O}(a_1, s_1) \cap \mathcal{O}(a_2, s_2)$ then we have that $c(x) = a_1(x) = a_2(x)$ and $s_1(x) < 1$ and $s_2(x) < 1$. In particular, for $a := a_1 \leftrightarrow a_2$ we have $a(x) = 1$, implying that $x \in \widehat{a}$. Consequently, by Lemma 4.3.1 we must have $s \in A$ with $s < 1$ such that $s \vee a = 1$ and $x \notin \widehat{s}$. This means that $s(x) < 1$ and that $s(y) < 1$ entails $a(y) = 1$, which is to say that $a_1(y) = a_2(y)$ whenever $s(y) < 1$, for all $y \in \min(X)$. From this we may conclude that,

$$c(x) \in \mathcal{O}(a_1 \wedge a_2, s \vee s_1 \vee s_2) \subseteq \mathcal{O}(a_1, s_2) \cap \mathcal{O}(a_2, s_2),$$

by using the fact that each of the factors $\mathbf{A}/\theta_y$ are fsi Heyting algebras and as such well-connected, implying that $(s \vee s_1 \vee s_2)(y) < 1$ if, and only if, $s(y) < 1$, $s_1(y) < 1$, and $s_2(y) < 1$. □

Given a Heyting algebra $\mathbf{A}$ with dual Esakia space X we will equip the (disjoint) union $\bigcup_{x\in\min(X)} A/\theta_x$ with the topology determined by the family $\{\mathcal{O}(a,s) : (s,a) \in A^2\}$.

Recall that a function $f\colon X \to Y$ between topological spaces is *continuous at a point* $x \in X$ provided that for any open neighborhood V of $f(x)$ there is an open neighborhood U of x such that $f[U] \subseteq V$. Furthermore, a function $f\colon X \to Y$ between topological spaces is *continuous on a set* $U \subseteq X$ if f is continuous at each $x \in U$.

4.3.4. Lemma. *Let* $\mathbf{A}$ *be a Heyting algebra with dual Esakia space* X, $f \in \prod_{x\in\min(X)} \mathbf{A}/\theta_x$ *and* $x \in \min(X)$. *Then* f *is continuous at* x *if, and only if, there is* $(s_x, a_x) \in A^2$ *such that* $s_x(x) < 1$ *and*

$$\forall y \in \min(X)\ (s_x(y) < 1 \implies a_x(y) = f(y)).$$

Proof:
Suppose first that there is $(s_x, a_x) \in A^2$ such that $s_x(x) < 1$, and $s_x(y) < 1$ entails $a_x(y) = f(y)$ for all $y \in \min(X)$. Then given any open neighborhood V of $f(x)$, since the sets $\mathcal{O}(a,s)$, with $(a,s) \in A^2$, form a basis for the topology on the co-domain, we have $(s,a) \in A^2$ such that $f(x) \in \mathcal{O}(a,s) \subseteq V$. This entails that $f(x) = a(x)$ and $s(x) < 1$. But then $[\![s \vee s_x < 1]\!]$ is an open neighborhood of x such that $f[[\![s \vee s_x < 1]\!]] \subseteq \mathcal{O}(a,s) \subseteq V$, showing that f is continuous at x.

Conversely, suppose that f is continuous at x. Evidently there must be an element $a_x \in A$ with $a_x(x) = f(x)$ and an element $s_x \in A$ such that $s(x) < 1$. We then have that $\mathcal{O}(a_x, s_x)$ is an open neighborhood of $f(x)$. Consequently, since f is continuous at x there must be an open neighborhood of x, say U, such that $f[U] \subseteq \mathcal{O}(a_x, s)$. As the sets $[\![t < 1]\!]$, with $t \in A$, form a basis for the topology on the domain we must have $s_x \in A$ with $[\![s_x < 1]\!] \subseteq U$ and hence $f[[\![s_x < 1]\!]] \subseteq \mathcal{O}(a_x, s)$, showing that $s_x(y) < 1$ entails $a_x(y) = f(y)$ for all $y \in \min(X)$. $\square$

4.3.5. Definition. For a Heyting algebra $\mathbf{A}$ with dual Esakia space X let $\mathcal{D}(\mathbf{A})$ be the set of elements of $\prod_{x\in\min(X)} \mathbf{A}/\theta_x$ which are continuous on some dense open subset of $\min(X)$.

4.3.6. Remark. Note that $\mathbf{A} \hookrightarrow \prod_{x\in\min(X)} \mathbf{A}/\theta_x$ is not in general a Boolean product; the base space $\min(X)$ might not be compact and the patchwork property is not guaranteed to hold. Therefore, the subalgebra of $\prod_{x\in\min(X)} \mathbf{A}/\theta_x$ consisting of elements which are continuous at every point will not necessarily coincide with the algebra $\mathbf{A}$, but might be larger.

For $f \in \prod_{x\in\min(X)} \mathbf{A}/\theta_x$ we let U_f be the interior of the set of points at which f is continuous. Thus $f \in \mathcal{D}(\mathbf{A})$ if, and only if, U_f is a dense open set.

4.3.7. PROPOSITION. *Let* $\mathbf{A}$ *be a Heyting algebra with dual Esakia space* X*. Then* $\mathcal{D}(\mathbf{A})$ *is a Heyting subalgebra of the direct product* $\prod_{x\in\min(X)} \mathbf{A}/\theta_x$*.*

Proof:
It is easy to see that the least and the greatest elements of $\prod_{x\in\min(X)} \mathbf{A}/\theta_x$ are continuous everywhere and so belong to $\mathcal{D}(\mathbf{A})$. Let $f_1, f_2 \in \mathcal{D}(\mathbf{A})$ and $* \in \{\wedge, \vee, \rightarrow\}$ be given. We claim that $f_1 * f_2$ is continuous on $U_{f_1} \cap U_{f_2}$. To see this let $x \in U_{f_1} \cap U_{f_2}$ be given. By Lemma 4.3.4 we have pairs $(s^1_x, a^1_x), (s^2_x, a^2_x) \in A^2$ witnessing that f_1 and f_2 are continuous on x. If $(s^1_x \vee s^2_x)(y) < 1$ for some $y \in \min(X)$, since the algebra $\mathbf{A}/\theta_y$ is well-connected, $s^1_x(y) < 1$ and $s^2_x(y) < 1$ whence $(a^1_x * a^2_x)(y) = (f_1 * f_2)(y)$. Similarly, we have that $(s^1_x \vee s^2_x)(x) < 1$. Thus by, again by Lemma 4.3.4, the pair $(s^1_x \vee s^2_x, a^1_x * a^2_x)$ witnesses that $f_1 * f_2$ is continuous at x. Since the intersection of two dense open sets is again dense open this concludes the proof. ☐

Given a Heyting algebra $\mathbf{A}$ and $f \in \mathcal{D}(\mathbf{A})$ we define

$$\Delta(f) := \{(s, a) \in A^2 : \forall x \in U_f\ (s(x) < 1 \implies a(x) \leq f(x))\}$$

4.3.8. PROPOSITION. *Let* $\mathbf{A}$ *be a Heyting algebra. Then* $\Delta(f)$ *belongs to* A^+ *for all* $f \in \mathcal{D}(\mathbf{A})$*.*

Proof:
We need to show that $\Delta(f) = LU(\Delta(f))$. Therefore, as we always have $\Delta(f) \subseteq LU(\Delta(f))$, it suffices to argue that $LU(\Delta(f)) \subseteq \Delta(f)$. We first observe that for all $x \in U_f$, since f is continuous at x we have by Lemma 4.3.4 a pair $(s_x, a_x) \in A^2$ such that $s_x(x) < 1$, and $a_x(y) = f(y)$ whenever $s_x(y) < 1$. Suppose that a choice of such pairs (s_x, a_x) has been made for each $x \in U_f$. Then for $x \in U_f$ and $(s, a) \in \Delta(f)$ we must have that $U_f \subseteq [\![s_x \vee s \vee (a \rightarrow a_x) = 1]\!]$. As the set $[\![s_x \vee s \vee (a \rightarrow a_x) = 1]\!]$ is closed in $\min(X)$ and U_f is dense, we may conclude that $s_x \vee s \vee (a \rightarrow a_x) = 1$, for all $(s, a) \in \Delta(f)$. But then $(s_x, a_x) \in U(\Delta(f))$, for all $x \in U_f$. Consequently, if $(s, a) \in LU(\Delta(f))$ it follows that $s \vee s_x \vee (a \rightarrow a_x) = 1$ for all $x \in U_f$. As $s_x(x) < 1$ it follows that $s(x) < 1$ implies $a(x) \leq a_x(x) = f(x)$, for each $x \in U_f$, and hence $(s, a) \in \Delta(f)$. ☐

4.3.9. LEMMA. *For any Heyting algebra* $\mathbf{A}$ *and* $f_1, f_2 \in \mathcal{D}(\mathbf{A})$ *we have that* $\Delta(f_1) \leq \Delta(f_2)$ *if, and only if,* $[\![f_1 \leq f_2]\!]$ *contains a dense open set.*

Proof:
We first observe that for each $x \in U_{f_1}$ we have $(s_x, a_x) \in \Delta(f_1)$ for any pair $(s_x, a_x) \in A^2$ witnessing the fact that f_1 is continuous at x. Thus, if $\Delta(f_1) \leq \Delta(f_2)$, then for $x \in U_{f_1}$ we have that $(s_x, a_x) \in \Delta(f_2)$, implying that $s_x(y) < 1$

entails $a_x(y) \leq f_2(y)$. As $s_x(x) < 1$ and $a_x(x) = f_1(x)$, we see that the dense open set U_{f_1} is a subset of $[\![f_1 \leq f_2]\!]$.

Conversely, if $[\![f_1 \leq f_2]\!]$ contains a dense open set, say U, then for all $y \in U \cap U_{f_1}$ and all $(s, a) \in \Delta(f_1)$ we have that $s(y) < 1$ implies $a(y) \leq f_2(y)$. Given, $x \in U_{f_2}$ we let $(s_x, a_x) \in A^2$ be a pair witnessing that f_2 is continuous at x. Then we have that $U \cap U_{f_1} \subseteq [\![s_x \vee s \vee (a \to a_x) = 1]\!]$. Since the intersection of two dense open sets is again a dense open set and since $[\![s_x \vee s \vee (a \to a_x) = 1]\!]$ is closed we must have that $s_x \vee s \vee (a \to a_x) = 1$, for all $x \in U_{f_2}$. Since $s_x(x) < 1$ for each $x \in U_{f_2}$, we obtain that $s(x) < 1$ implies $a(x) \leq a_x(x) = f_2(x)$, for all such $x \in U_{f_2}$. This proves that $(s, a) \in \Delta(f_2)$, as desired. □

Let Θ be the relation on $\mathcal{D}(\mathbf{A})$ defined by letting

$$f_1 \Theta f_2 \quad \text{if, and only if,} \quad [\![f_1 = f_2]\!] \text{ contains a dense open set.}$$

Since the collection of dense open sets is closed under binary intersections, we see that Θ is in fact a Heyting algebra congruence. Thus by Proposition 4.3.7 we have that $\mathcal{D}(\mathbf{A})/\Theta$ is a homomorphic image of a subalgebra of the direct product $\prod_{x \in \min(X)} \mathbf{A}/\theta_x$. In particular, for any Heyting algebra $\mathbf{A}$, the algebra $\mathcal{D}(\mathbf{A})/\Theta$ belongs to any variety of Heyting algebras containing $\mathbf{A}$.

4.3.10. Definition (cf. [146]). By *the algebra of dense open sections of* a Heyting algebra $\mathbf{A}$ we shall understand the algebra $\mathcal{D}(\mathbf{A})/\Theta$, which we will denote by $Q(\mathbf{A})$.

4.3.11. Remark. The construction of the algebra $Q(\mathbf{A})$ from the algebra $\mathbf{A}$ is akin to the construction of the maximal ring of quotients of a commutative semi-simple ring [18], in particular of the ring of all real-valued functions on some completely regular space, see, e.g., [89, Chap. 2]. Of course, for each Boolean algebra $\mathbf{B}$ the MacNeille completion $\overline{\mathbf{B}}$ considered as a Boolean ring is exactly the maximal ring of quotients of $\mathbf{B}$ considered as a Boolean ring [44].

4.3.12. Proposition. *Let $\mathbf{A}$ be a Heyting algebra. Then there is an order-embedding $Q(\mathbf{A}) \hookrightarrow \mathbf{A}^+$ of posets.*

Proof:
We claim that the map $f/\Theta \mapsto \Delta(f)$ is an order-embedding. That this maps is well defined follows directly from Lemma 4.3.9. To see that it is order-preserving we observe that if $f_1/\Theta \leq f_2/\Theta$ then $f_1/\Theta = f_1/\Theta \wedge f_2/\Theta = (f_1 \wedge f_2)/\Theta$ whence $[\![f_1 = f_1 \wedge f_2]\!]$ contains a dense open set. Since $[\![f_1 \leq f_2]\!] = [\![f_1 = f_1 \wedge f_2]\!]$ we obtain from Lemma 4.3.9 that $\Delta(f_1) \leq \Delta(f_2)$.

Similarly, to see that the map is also order-reflecting we observe that if $\Delta(f_1) \leq \Delta(f_2)$ then the set $[\![f_1 \leq f_2]\!] = [\![f_1 = f_1 \wedge f_2]\!]$ contains a dense open set and therefore $f_1/\Theta = f_1/\Theta \wedge f_2/\Theta$ which shows that $f_1/\Theta \leq f_2/\Theta$. $\square$

In the following we will show that this embedding is both join- and meet-dense and therefore that $\overline{Q(\mathbf{A})} \cong \mathbf{A}^+$.

4.3.13. Definition. Let $\mathbf{A}$ be a Heyting algebra with dual Esakia space X. For $(s,a),(t,b) \in A^2$ we define functions $f(t,b), g(s,a) \in \prod_{x\in\min(X)} \mathbf{A}/\theta_x$ as follows

$$f(t,b)(x) = \begin{cases} 1 & \text{if } t(x) = 1 \\ b(x) & \text{otherwise} \end{cases} \quad \text{and} \quad g(s,a)(x) = \begin{cases} 0 & \text{if } s(x) = 1 \\ a(x) & \text{otherwise} \end{cases}$$

4.3.14. Lemma. *For any Heyting algebra $\mathbf{A}$ and any $(s,a),(t,b) \in A^2$ the functions $g(s,a)$ and $f(t,b)$ are everywhere continuous.*

Proof:
Let $(t,b) \in A^2$ be given. By Lemma 4.3.4 we have that $f(t,b)$ is continuous at $x \in X$ if, and only if, there is $(s,a) \in A^2$ such that $s(x) < 1$, and $s(y) < 1$ entails $a(y) = f(t,b)(y)$ for all $y \in \min(X)$. From the definition of $f(t,b)$ we see that $a(y) = f(t,b)(y)$ precisely when $a(y) = t(y) = 1$ or when $t(y) < 1$ and $a(y) = b(y)$. Given this it is easy to see that if $t(x) < 1$, then (t,b) witnesses that $f(t,b)$ is continuous at x. Thus the set of points at which $f(t,b)$ is continuous contains the open set $[\![t < 1]\!]$. Moreover, if $s \in A$ is such that

$$[\![t < 1]\!] \cap [\![s < 1]\!] = \emptyset,$$

then $s(y) < 1$ entails $t(y) = 1$ for all $y \in \min(X)$, and so for any $x \in [\![s < 1]\!]$ we have that $(s,1)$ will witness that $f(t,b)$ is continuous at x. Consequently, we obtain that $f(t,b)$ is continuous on the open set

$$[\![t < 1]\!] \cup \bigcup\{[\![s < 1]\!] : [\![t < 1]\!] \cap [\![s < 1]\!] = \emptyset\} = [\![t < 1]\!] \cup \mathsf{I}([\![t = 1]\!]),$$

where $\mathsf{I}(-)$ denotes the interior operator on $\min(X)$. By Proposition 4.3.2 we have that $\mathsf{I}([\![t = 1]\!]) = [\![t = 1]\!]$ and so $f(t,b)$ is continuous everywhere.

A similar argument shows that for each $(s,a) \in A^2$ the corresponding function $g(s,a)$ is continuous everywhere. $\square$

4.3.15. Proposition. *For any Heyting algebra $\mathbf{A}$ and any $(s,a),(t,b) \in A^2$ we have that*

$$\Delta(f(t,b)) = L(t,b) \quad \textit{and} \quad \Delta(g(s,a)) = LU(s,a).$$

Proof:
Let X denote the dual Esakia space of $\mathbf{A}$. Using the fact that $f(t,b)$ is continuous everywhere we see that

$$\begin{aligned}\Delta(f(t,b)) &= \{(s,a) : \forall x \in U_{f(t,b)}\ (s(x) < 1 \implies a(x) \leq f(t,b)(x))\}\\ &= \{(s,a) : \forall x \in \min(X)\ (s(x) < 1 \implies (t(x) < 1 \implies a(x) \leq b(x)))\}\\ &= \{(s,a) : \min(X) \subseteq [\![s \vee t \vee (a \to b) = 1]\!]\}\\ &= \{(s,a) : s \vee t \vee (a \to b) = 1\}\\ &= L(t,b).\end{aligned}$$

To see that $\Delta(g(s,a)) = LU(s,a)$, we first show that $\Delta(g(s,a)) \subseteq LU(s,a)$. Therefore, let $(s_0,a_0) \in \Delta(g(s,a))$ be given and let $(t,b) \in U(s,a)$. Then $s \vee t \vee (a \to b) = 1$, showing that $s(x) < 1$ implies $t(x) < 1$ or $a(x) \leq b(x)$. Since $g(s,a)$ is continuous everywhere we have that $s_0(x) < 1$ implies $a_0(x) \leq g(s,a)(x)$ for all $x \in \min(X)$. If $t(x) < 1$ and $s_0(x) < 1$ then either $s(x) = 1$ or $a(x) \leq b(x)$. In the former case $g(s,a)(x) = 0$ and so $a_0(x) \leq b(x)$. In the latter case, $a_0(x) \leq g(s,a)(x) = a(x) \leq b(x)$. In either case $a_0(x) \leq b(x)$. This shows that $s_0 \vee t \vee (a_0 \to b) = 1$, i.e., $(s_0,a_0) \in L(t,b)$. Since $(t,b) \in U(s,a)$ was arbitrary we may then conclude that $(s_0,a_0) \in LU(s,a)$.

Finally, we show that $LU(s,a) \subseteq \Delta(g(s,a))$. For each $x \in \min(X)$ we have $(t_x,b_x) \in A^2$ witnessing that $g(s,a)$ is continuous at x. That is to say $t_x(x) < 1$ and $t_x(y) < 1$ implies $b_x(y) = g(s,a)(y)$ for all $y \in \min(X)$. It is not difficult to see that this entails that $s \vee t_x \vee (a \to b_x) = 1$ and hence that $(t_x,b_x) \in U(s,a)$. Consequently, we have that if $(s_0,a_0) \in LU(s,a)$ then in particular, $s_0 \vee t_x \vee (a_0 \to b_x) = 1$ for all $x \in \min(X)$. But then, as $t_x(x) < 1$ for each $x \in \min(X)$, we may conclude that $s_0(x) < 1$ entails $a_0(x) \leq b_x(x) = g(s,a)(x)$. This shows that $(s_0,a_0) \in \Delta(g(s,a))$. □

Since by Lemma 4.2.6 the sets $\{L(t,b) \in A^+ : (t,b) \in A^2\}$ and $\{LU(s,a) \in A^+ : (s,a) \in A^2\}$ are meet- and join-dense in A^+, respectively, we obtain that the order-embedding $Q(\mathbf{A}) \hookrightarrow \mathbf{A}^+$ given by Proposition 4.3.12 is both meet- and join-dense. Consequently, since $\mathbf{A}^+$ is complete, the MacNeille completion of $Q(\mathbf{A})$ must be isomorphic, as a poset and therefore also as a Heyting algebra, to $\mathbf{A}^+$. We have thus established the following theorem.

4.3.16. Theorem. *Let* $\mathbf{A}$ *be a Heyting algebra. Then* $\mathbf{A}^+ \cong \overline{Q(\mathbf{A})}$.

4.3.17. Remark. Constructing different types of completions of lattice-based algebras using sheaf representations is by no means a new idea as references [44, 146, 73] show. We point to two more examples of this phenomenon. Given a completely regular Baire space X, the Dedekind completion of the Riesz space $C(X)$ of real-valued continuous functions with domain X may be obtained as the Riesz space consisting of certain bounded real-valued functions from X which are continuous on a dense set and identified by equality on a dense set [77, Thm. 6.1], see also [89, Chap. 4]. Similarly, lateral completions of ℓ-groups may be obtained from sheaf representations in a way resembling the construction of $Q(\mathbf{A})$, see [226].

On the face of it Theorem 4.3.16 does not appear to say much about the hyper-MacNeille completion of $\mathbf{A}$. Nevertheless, since the algebra $Q(\mathbf{A})$ may have a richer structure than $\mathbf{A}$, some useful information can still be gained. For example, the algebra $Q(\mathbf{A})$ will always belong to any variety containing $\mathbf{A}$. Moreover, as the following proposition shows, this algebra will always be supplemented with a De Morgan supplement.

4.3.18. Proposition. *For any Heyting algebra $\mathbf{A}$ the algebra $Q(\mathbf{A})$ is supplemented with a De Morgan supplement.*

Proof:
We show that $\mathcal{D}(\mathbf{A})$ is supplemented with a De Morgan supplement. It is then easy to see that the congruence Θ will also be a congruence with respect to the supplementation operation.

Let X denote the dual Esakia space of $\mathbf{A}$. Then for $f \in \mathcal{D}(\mathbf{A})$ we define $\widetilde{f}$ by

$$\widetilde{f}(x) = \begin{cases} 0 & \text{if } f(x) = 1 \\ 1 & \text{otherwise.} \end{cases}$$

It is not hard to see that $\widetilde{f}$ is a supplement of f in $\prod_{x \in \min(X)} \mathbf{A}/\theta_x$ and that it satisfies the De Morgan laws. Thus, it remains to be shown that $\widetilde{f}$ is continuous on a dense open set. Therefore, let U_f be the interior of the set of points at which f is continuous. By assumption U_f is a dense open set. We show that the set of points at which $\widetilde{f}$ is continuous contains the open set $(\llbracket f < 1 \rrbracket \cap U_f) \cup \mathsf{I}(\llbracket f = 1 \rrbracket)$.

If $x \in \llbracket f < 1 \rrbracket \cap U_f$ then we have that f is continuous at x with $f(x) < 1$. Consequently, we must have $(s, a) \in A^2$ such that $s(x) < 1$, and $s(y) < 1$ implies $a(y) = f(y)$, for all $y \in \min(X)$. It is then easy to see that $(s \vee a, 1)$ witnesses that $\widetilde{f}$ is continuous at x. If $x \in \mathsf{I}(\llbracket f = 1 \rrbracket)$ then we have $s \in A$ such that $x \in \llbracket s < 1 \rrbracket \subseteq \llbracket f = 1 \rrbracket$ and hence $(s, 0)$ witnesses that $\widetilde{f}$ is continuous at x.

Finally, we show that $(\llbracket f < 1 \rrbracket \cap U_f) \cup \mathsf{I}(\llbracket f = 1 \rrbracket)$ is dense. Therefore, let $\llbracket s < 1 \rrbracket$ be a non-empty basic open subset of $\min(X)$. If $\llbracket s < 1 \rrbracket \cap U_f \cap \llbracket f < 1 \rrbracket = \emptyset$ then, as U_f is dense, we have that $\llbracket s < 1 \rrbracket \cap \llbracket f < 1 \rrbracket = \emptyset$ and hence that $\llbracket s < 1 \rrbracket \subseteq \llbracket f = 1 \rrbracket$. But since $\llbracket s < 1 \rrbracket$ is a non-empty open set we must have that $\llbracket s < 1 \rrbracket \cap \mathsf{I}(\llbracket f = 1 \rrbracket) \neq \emptyset$. □

4.3.19. Remark. From Proposition 4.3.18 and Theorem 4.1.24 we obtain that the algebra $Q(\mathbf{A})$ is Hausdorff with fsi stalks for all Heyting algebras $\mathbf{A}$.

4.3.1 MacNeille completions of De Morgan supplemented Heyting algebras

From Proposition 4.3.18 and Theorem 4.3.16 it follows that one can understand the hyper-MacNeille completions of Heyting algebras by looking at MacNeille completions of De Morgan supplemented Heyting algebras.

4.3.20. PROPOSITION. *The MacNeille completion of a De Morgan supplemented Heyting algebra is again a De Morgan supplemented Heyting algebra.*

Proof:
Let $\mathbf{A}$ be a De Morgan supplemented Heyting algebra. It is well known that the MacNeille completion of a Heyting algebra is also a Heyting algebra, see, e.g., [151, Thm. 2.3]. We show that $\overline{\mathbf{A}}$ is also supplemented.

For $x \in \overline{\mathbf{A}}$ we claim that $z := \bigvee\{\sim a \in A : x \leq a \in A\}$ is the supplement of x in $\overline{\mathbf{A}}$. To see this, we first note that if $y \in \overline{A}$ is such that $y \vee x = 1$ then for each $a \in S$ with $a \geq x$ we have that $y \vee a = 1$. Consequently, for each $b \in A$ with $b \geq y$ we have $\sim a \leq b$. Since A is meet-dense in $\overline{\mathbf{A}}$ this shows that $\sim a \leq y$ and hence that $z \leq y$. To see that $x \vee z = 1$, consider $a \in A$ such that $x \vee z \leq a$ then $x \leq a$ and $z \leq a$ from which we may conclude that $\sim a \leq a$ and therefore that $a = 1$. Since A is meet-dense in $\overline{\mathbf{A}}$ we then obtain that $x \vee z = 1$. This shows that $\bigvee\{\sim a \in A : x \leq a \in A\}$ is indeed the supplement of x in $\overline{\mathbf{A}}$.

It remains to show that the supplement on $\overline{\mathbf{A}}$ is a De Morgan supplement. By Proposition 4.1.9 it suffices to show that $\sim x \wedge \sim\sim x = 0$ for all $x \in \overline{A}$. Therefore, let $x \in \overline{A}$ be given. By the above we have that $\sim x = \bigvee\{\sim a \in A : x \leq a \in A\}$. Since $\mathbf{A}$ is a Heyting algebra it follows that

$$\sim x \wedge \sim\sim x = \bigvee\{\sim a \wedge \sim b : x \leq a \in A,\ \sim x \leq b \in A\}.$$

If $a, b \in A$ are such that $x \leq a$ and $\sim x \leq b$, then $\sim b \leq \sim\sim x \leq x \leq a$ and therefore $\sim a \leq \sim\sim b$. It then follows that $\sim a \wedge \sim b \leq \sim\sim b \wedge \sim b$. But as $\mathbf{A}$ was assumed to be supplemented with a De Morgan supplement, by Proposition 4.1.9, we obtain that $\sim\sim b \wedge \sim b = 0$. We may then conclude that $\sim x \wedge \sim\sim x = 0$, as desired. $\square$

Given any class $\mathcal{K}$ of Heyting algebras we denote by $\mathcal{K}_{dms}$ the class of De Morgan supplemented members of $\mathcal{K}$.

4.3.21. THEOREM. *A variety of Heyting algebras $\mathcal{V}$ is closed under hyper-MacNeille completions if, and only if, the class $\mathcal{V}_{dms}$ is closed under MacNeille completions.*

Proof:
Let $\mathcal{V}$ be a variety of Heyting algebras. For each $\mathbf{A} \in \mathcal{V}$ we have that $Q(\mathbf{A})$ belongs to $\mathcal{V}$ and hence to $\mathcal{V}_{dms}$ by Proposition 4.3.18. Thus if $\mathcal{V}$ is such that $\mathcal{V}_{dms}$

is closed under MacNeille completions, then $\overline{Q(\mathbf{A})}$ belongs to $\mathcal{V}$ for all $\mathbf{A} \in \mathcal{V}$. Therefore, by Theorem 4.3.16 the variety $\mathcal{V}$ is closed under hyper-MacNeille completions. Conversely, if $\mathcal{V}$ is closed under hyper-MacNeille completions, then in particular $\mathbf{A}^+ \in \mathcal{V}$ for each $\mathbf{A} \in \mathcal{V}_{dms}$. However, by Proposition 4.2.11 we have that $\mathbf{A}^+$ is isomorphic to $\overline{\mathbf{A}}$ and hence $\overline{\mathbf{A}} \in \mathcal{V}$. Finally, by Proposition 4.3.20 we may conclude that $\overline{\mathbf{A}} \in \mathcal{V}_{dms}$, showing that this class is indeed closed under MacNeille completions. □

As we will see in the following section, MacNeille completions of supplemented Heyting algebras are already fairly easy to work with compared to the general case.

4.4 Varieties closed under hyper-MacNeille completions

In this section we discuss two methods of finding varieties of Heyting algebras closed under hyper-MacNeille completions based on the results obtained so far. We also give more direct descriptions of the hyper-MacNeille completion of special classes of Heyting algebras.

4.4.1 ALBA-type arguments

The existence of a supplement operation makes it possible to use syntactic methods analogous to the ones developed in [124, 245], or, alternatively, ALBA-type methods as in, e.g., [68, 236, 141, 255], to establish preservation of equations under MacNeille completions. In particular, this allows us to prove that there are varieties of Heyting algebras closed under hyper-MacNeille completions which are not determined by $\mathcal{P}_3$-equations. Consider, for example, the variety $\mathcal{BD}_2$ of Heyting algebras satisfying the equation

$$1 \approx x_2 \vee (x_2 \to (x_1 \vee \neg x_1)). \tag{bd_2}$$

This variety is not determined by $\mathcal{P}_3$-equations, see Proposition 2.3.3. Nevertheless, we claim that the variety $\mathcal{BD}_2$ is closed under hyper-MacNeille completions. By Theorem 4.3.21 it suffices to show that the class of supplemented members of $\mathcal{BD}_2$ is closed under MacNeille completions.

We first observe that on supplemented Heyting algebras the defining equation for $\mathcal{BD}_2$ is equivalent to the equation

$$\sim x_2 \wedge x_2 \leq x_1 \vee \neg x_1. \tag{4.1}$$

Let $\mathbf{A}$ be a supplemented Heyting algebra. Since the set A is both join- and meet-dense in $\overline{A}$ we have that $\overline{\mathbf{A}}$ validates Equation (4.1) if, and only if, the following two-sorted quasi-equation holds

$$(a \leq {\sim} x_2 \wedge x_2 \text{ and } x_1 \vee \neg x_1 \leq b) \implies a \leq b, \tag{4.2}$$

where a, b are understood to be universally quantified over A and x_1, x_2 universally quantified over $\overline{A}$. This is essentially mimicking the "approximation" step of the ALBA algorithm. The crucial observation is that the polarities have been reversed in the antecedent and so the operations now appear in a position where they are invertible. This shows that the quasi-equation (4.2) is equivalent to the quasi-equation

$$(a \leq {\sim} x_2 \text{ and } a \leq x_2 \text{ and } x_1 \leq b \text{ and } \neg x_1 \leq b) \implies a \leq b. \tag{4.3}$$

Finally, it is not difficult to see that the quasi-equation (4.3) is equivalent to the quasi-equation

$$(a \leq {\sim} a \text{ and } \neg b \leq b) \implies a \leq b. \tag{4.4}$$

This last step is essentially a version of what is known as the Ackermann Lemma or the *minimal valuation argument* in the context of correspondence theory, see, e.g., [68]. Now, the quasi-equation (4.4) only depends on $\mathbf{A}$, it is expressing that every co-dense element is below every dense element. Moreover, the above reasoning also shows that the quasi-equation (4.4) is equivalent to the equation (4.1) on all supplemented Heyting algebras. Hence since by assumption $\mathbf{A}$ belongs to $\mathcal{BD}_2$, the quasi-equation (4.4) is valid on $\mathbf{A}$ and hence $\overline{\mathbf{A}}$ belongs to $\mathcal{BD}_2$.

It is not difficult to show that a similar kind of argument works for other varieties of Heyting algebras such that $\mathcal{LC}$ and $\mathcal{KC}$.

4.4.2 Hyper-MacNeille completions of finitely generated varieties

There is another way to provide examples of varieties of Heyting algebras closed under hyper-MacNeille completions but not necessarily determined by $\mathcal{P}_3$-equations. Namely, by finding conditions on a variety of Heyting algebras $\mathcal{V}$ ensuring that the algebra of dense open sections $Q(\mathbf{A})$ is complete for all $\mathbf{A} \in \mathcal{V}$.

4.4.1. Lemma. *Let $\mathbf{A}$ be a Heyting algebra with dual Esakia space X and let $f, g \in \mathcal{D}(\mathbf{A})$. If $f/\Theta = g/\Theta$ then $f(x) = g(x)$ for all $x \in \min(X)$ at which both f and g are continuous.*

Proof:
Suppose that f and g are both continuous at $x \in \min(X)$. By Lemma 4.3.4 there are $(s, a), (t, b) \in A^2$ such that $s(x), t(x) < 1$ and,

$$[\![s < 1]\!] \subseteq [\![f = a]\!] \quad \text{and} \quad [\![t < 1]\!] \subseteq [\![g = b]\!].$$

Consider the open set $U = [\![s \vee t \vee (a \leftrightarrow b) < 1]\!]$. It is easy to see that $U \cap [\![f = g]\!] = \emptyset$. As $f/\Theta = g/\Theta$ the set $[\![f = g]\!]$ contains a dense open set and consequently has a non-empty intersection with every non-empty open set. We may therefore conclude that $U = \emptyset$ and hence that $s \vee t \vee (a \leftrightarrow b) = 1$. Because the factor $\mathbf{A}/\theta_x$ is well-connected and both $s(x) < 1$ and $t(x) < 1$ we must have that $(a \leftrightarrow b)(x) = 1$, proving that $f(x) = a(x) = b(x) = g(x)$. □

The proof of the following theorem follows that of [146, Prop. 4] closely.

4.4.2. Theorem (cf., [146, Prop. 4]). *Let $\mathbf{A}$ be a Heyting algebra with dual Esakia space X. If the cardinalities of the quotients $\{\mathbf{A}/\theta_x : x \in \min(X)\}$ are uniformly bounded on a dense open set of $\min(X)$, then $Q(\mathbf{A})$ is complete.*

Proof:
Assume that there exists $n \in \omega$ and dense open set $U \subseteq \min(X)$ such that $|\mathbf{A}/\theta_x| \leq n$ for all $x \in U$. We show that for any non-empty set $T \subseteq \mathcal{D}(\mathbf{A})$ the set $\{f/\Theta : f \in T\}$ has a least upper bound in $Q(\mathbf{A})$, from which it follows that $Q(\mathbf{A})$ is complete. Therefore, let a non-empty set $T \subseteq \mathcal{D}(\mathbf{A})$ be given. For each $f \in T$ fix a dense open set U_f on which f is continuous, and let $U_T := \bigcup_{f \in T} U_f$. Now pick $g \in \prod_{x \in \min(X)} \mathbf{A}/\theta_x$ such that $g(x) = \bigvee\{f(x) : f \in T \text{ and } x \in U_f\}$ for each $x \in U_T \cap U$. We show that g is continuous on a dense open set. To this end we prove that each non-empty open set $V \subseteq \min(X)$ contains a non-empty open set on which g is continuous. Given a non-empty open set $V \subseteq \min(X)$ let S_V be the set of finite sequences of triples $(s_i, a_i, f_i)_{i=1}^k$, with $(s_i, a_i) \in A^2$ and $f_i \in \mathcal{D}(\mathbf{A})$, such that,

(i) $V \supseteq [\![s_1 < 1]\!] \supseteq [\![s_2 < 1]\!] \supseteq \ldots \supseteq [\![s_k < 1]\!] \neq \emptyset$,

(ii) $[\![s_i < 1]\!] \subseteq [\![f_i = a_i]\!]$ for each $i \in \{1, \ldots, k\}$,

(iii) the chain $f_1(x) \leq (f_1 \vee f_2)(x) \leq \ldots \leq \left(\bigvee_{i=1}^k f_i\right)(x)$ is strictly increasing, for each $x \in [\![s_k < 1]\!]$.

Since V is non-empty open and T is non-empty we have that there exists at least one sequence in S_V of length 1. Since the sets of the form $[\![s < 1]\!]$ are open and as such have non-empty intersection with the set U, we have that the maximal length of a sequence in S_V is at most n. Therefore, let $(s_i, a_i, f_i)_{i=1}^q$ be a sequence in S_V of maximal length. We claim that g is continuous at any point of $[\![s_q < 1]\!]$.

Let $a := \bigvee_{i=1}^q a_i$, by Lemma 4.3.4 it suffices to show that g agrees with a on $[\![s_q < 1]\!]$. If this is not the case then we have $x \in [\![s_q < 1]\!]$ such that $a(x) \neq g(x)$ By construction f_i is continuous at x for all $i \in \{1, \ldots, q\}$ and so we must have that $a(x) \leq g(x)$. So since $a(x) \neq g(x)$ we must have $f_{q+1} \in T$ with $x \in U_{f_{q+1}}$ and $f_{q+1}(x) \not\leq a(x)$. Because f_{q+1} is continuous at x we have by Lemma 4.3.4 a pair $(s, a_{q+1}) \in A^2$ such that $x \in [\![s < 1]\!] \subseteq [\![f_{q+1} = a_{q+1}]\!]$. In particular, we must have that $a_{q+1}(x) \not\leq a(x)$. Thus letting $s_{q+1} = s_q \vee s \vee (a_{q+1} \to a)$, we obtain that $[\![s_q < 1]\!] \supseteq [\![s_{q+1} < 1]\!] \ni x$. Moreover, $[\![s_{q+1} < 1]\!] \subseteq [\![s < 1]\!] \subseteq [\![f_{q+1} = a_{q+1}]\!]$. Finally, if $y \in [\![s_{q+1} < 1]\!]$ then $(\bigvee_{i=1}^{q} f_i)(y) = a(y) < (a \vee a_{q+1})(y) = (\bigvee_{i=1}^{q+1} f_i)(y)$. Consequently, the chain $f_1(y) \leq (f_1 \vee f_2)(y) \leq \ldots \leq (\bigvee_{i=1}^{q+1} f_i)(y)$ must be strictly increasing. This shows that $(s_i, a_i, f_i)_{i=1}^{q+1} \in S_V$, contradicting the choice of q as the maximal length of a sequence in S_V.

We now show that g/Θ is the least upper bound of the set $\{f/\Theta : f \in T\}$. Given $f \in T$ we have that $[\![f \leq g]\!]$ contains the dense open set $U \cap U_f$ and consequently, that $f/\Theta \leq g/\Theta$. Thus g/Θ is an upper bound of $\{f/\Theta : f \in T\}$. Suppose that $h/\Theta \in Q(\mathbf{A})$ is also an upper bound of $\{f/\Theta : f \in T\}$. Then for each $f \in T$ we have that $(f \vee h)/\Theta = h/\Theta$ and so by Lemma 4.4.1 we obtain that $f(x) \leq h(x)$ for all x at which both f and h are continuous. Thus if h is continuous at x, then $f(x) \leq h(x)$ for all $f \in T$ such that $x \in U_f$. Consequently, letting U_h be a dense open set of points at which h is continuous we have that $g(x) \leq h(x)$ for all points in the dense open set $U_T \cap U \cap U_h$, proving that $g/\Theta \leq h/\Theta$, as desired. □

Thus in case the cardinalities of the algebras $\{\mathbf{A}/\theta_x : x \in \min(X)\}$ are uniformly bounded on a dense open subset of $\min(X)$ we have that $\mathbf{A}^+ = Q(\mathbf{A})$. By Jónsson's Lemma, see [172, Sec. 6] or Appendix A.2, if $\mathcal{V}$ is a congruence distributive variety generated by a finite set of finite algebras $\mathcal{K}$, then the finitely subdirectly irreducible members of $\mathcal{V}$ must all be a homomorphic image of a subalgebra of some member of $\mathcal{K}$. Since all varieties of Heyting algebras are congruence distributive, we obtain that if $\mathbf{A}$ belongs to a finitely generated variety of Heyting algebras $\mathcal{V}$, then the cardinalities of the algebras $\{\mathbf{A}/\theta_x : x \in \min(X)\}$ are uniformly bounded by the cardinality of any algebra generating $\mathcal{V}$. Since $Q(\mathbf{A})$ always belongs to any variety containing $\mathbf{A}$ we obtain the following corollary.

4.4.3. Corollary. *If $\mathcal{V}$ is a finitely generated variety of Heyting algebras, then $\mathcal{V}$ is closed under hyper-MacNeille completions.*

Again, using the characterization from Chapter 2 of varieties of Heyting algebras determined by $\mathcal{P}_3$-equations it is not hard to come up with examples of finitely generated varieties of Heyting algebras which are not axiomatizable by such equations, e.g., the variety generated by the Heyting algebra $(\mathbf{2} \times \mathbf{2}) \oplus \mathbf{1}$, obtained by adding a new top element to the four element Boolean algebra. Note, however, that not all varieties determined by $\mathcal{P}_3$-equations are finitely generated.

4.4.3 Isolated minimal points

We conclude this section by considering Heyting algebras $\mathbf{A}$ for which the hyper-MacNeille completion is easily determined in terms of the quotients $\mathbf{A}/\theta_x$, with $x \in \min(X)$, where X is the dual Esakia space of $\mathbf{A}$.

When the isolated points of $\min(X)$ are dense then the algebra of dense open sections, and therefore also the hyper-MacNeille completion, of $\mathbf{A}$ is particularly easy to describe. For a topological space X we denote by $\mathrm{iso}(X)$ the set of isolated points of X.

4.4.4. Proposition. *If* $\mathbf{A}$ *is a Heyting algebra with dual Esakia space* X *and* $\mathrm{iso}(\min(X))$ *a dense subset of* $\min(X)$*, then* $\mathbf{A}^+ \cong \prod_{x\in\mathrm{iso}(\min(X))} \overline{\mathbf{A}/\theta_x}$.

Proof:
As the MacNeille completion of a direct product is the direct product of the MacNeille completions of the factors, by Theorem 4.3.16 it suffices to show that $Q(\mathbf{A}) \cong \prod_{x\in\mathrm{iso}(\min(X))} \mathbf{A}/\theta_x$. As the set of isolated points of $\min(X)$ is assumed to be dense, it follows that for $f, g \in \prod_{x\in\min(X)} \mathbf{A}/\theta$ we have $f/\Theta = g/\Theta$ if, and only if, f and g agree on all the points of $\mathrm{iso}(\min(X))$. Consequently, we have an order-embedding $Q(\mathbf{A}) \hookrightarrow \prod_{x\in\mathrm{iso}(\min(X))} \mathbf{A}/\theta_x$ given by $f/\Theta \mapsto f \mid \mathrm{iso}(\min(X))$, i.e., restricting a representative to the set of isolated points. Moreover, since any function is continuous at any isolated point this map must be surjective and therefore $Q(\mathbf{A}) \cong \prod_{x\in\mathrm{iso}(\min(X))} \mathbf{A}/\theta_x$, as desired. □

4.4.5. Remark. The fact that the hyper-MacNeille completion coincides with the MacNeille completion for Heyting algebras with a De Morgan supplement, in particular for Boolean algebras, shows that Proposition 4.4.4 is not true unconditionally. In the Boolean case the only non-trivial finitely subdirectly irreducible Boolean algebra is the two element chain $\mathbf{2}$. Consequently, if Proposition 4.4.4 were to hold in general then the MacNeille completion of any Boolean algebra would be isomorphic to a power of $\mathbf{2}$. This, however, is not the case as the latter is always atomic but the former not. In fact, for any Boolean algebra $\mathbf{B}$ with dual Stone space X we have that $\min(X) = X$. Since for Boolean algebras the MacNeille and the hyper-MacNeille completions coincide, Proposition 4.4.4 implies that if $\mathbf{B}$ is a Boolean algebra with $\mathrm{iso}(X)$ dense in X then $\overline{\mathbf{B}} \cong \mathbf{2}^{\mathrm{iso}(X)}$. It is well known, see, e.g., [180, Prop 7.18], that the isolated points of X are in one-to-one correspondence with the atoms of $\mathbf{B}$ and that $\mathrm{iso}(X)$ is dense in X if, and only if, $\mathbf{B}$ is atomic. Thus as a special case of Proposition 4.4.4 we obtain the easily verifiable fact that $\overline{\mathbf{B}} \cong \wp(\mathrm{At}(\mathbf{B}))$, for atomic Boolean algebras $\mathbf{B}$ with $\mathrm{At}(\mathbf{B})$ the set of atoms of $\mathbf{B}$.

For any Heyting algebra $\mathbf{A}$ with dual Esakia space X, given distinct elements $x, y \in \min(X)$ we can always find $a_{x,y} \in A$ with $a_{x,y}(x) = 1$ and $a_{x,y}(y) \neq 1$. Suppose one choice of such $a_{x,y}$ has been made for each $x \neq y$. If $\min(X)$ is finite we may then define elements $a_x = \bigwedge\{a_{x,y} : y \neq x\}$ and $a'_x = \bigvee\{a_y : y \neq x\}$ of A. It is clear that $a_x(y) = 1$ if, and only if, $y = x$. That $a'_x(y) = 1$ if, and only if, $y \neq x$ follows from the assumption that $\mathbf{A}/\theta_x$ is finitely subdirectly irreducible and hence well-connected. As $[\![a'_x < 1]\!] = \{x\}$ we may then conclude that $\min(X)$ is a discrete space. Thus as a special case of Proposition 4.4.4 we obtain the following corollary.

4.4.6. Corollary. *Let* $\mathbf{A}$ *be a Heyting algebra with dual Esakia space* X*. If* $\min(X)$ *is finite then* $\mathbf{A}^+ \cong \prod_{x\in\min(X)} \overline{\mathbf{A}/\theta_x}$.

4.5 Central and supplemented elements

Even though in the general case a complete and transparent description of the hyper-MacNeille completion $\mathbf{A}^+$ of $\mathbf{A}$ might be out of reach, we will still be able to determine a number of properties of $\mathbf{A}^+$ both absolutely as well as in terms of data given by $\mathbf{A}$. Most importantly, we will show that $\mathbf{A}^+$ is always supplemented with a De Morgan supplement or equivalently by Theorem 4.1.24 a Hausdorff lattice with finitely subdirectly irreducible stalks. Furthermore, we will be able to determine the center of $\mathbf{A}^+$.

4.5.1. Theorem. *If is* $\mathbf{A}$ *a Heyting algebra then* $\mathbf{A}^+$ *is a Hausdorff lattice with fsi stalks.*

Proof:
By Theorem 4.1.24 it suffices to show that $\mathbf{A}^+$ is supplemented with a De Morgan supplement. From Theorem 4.3.16 we know that $\mathbf{A}^+$ is the MacNeille completion of the Heyting algebra $Q(\mathbf{A})$. As this algebra is supplemented with a De Morgan supplement it follows from Proposition 4.3.20 that so is $\mathbf{A}^+$. □

Thus $\mathbf{A}^+$ will always have a somewhat well-behaved Boolean product representation. In this section we will make some initial attempts at understanding the local structure of $\mathbf{A}^+$ for an arbitrary Heyting algebra $\mathbf{A}$. We will give a description of the center $Z(\mathbf{A}^+)$ of $\mathbf{A}$, but the stalks of $\mathbf{A}^+$ do not appear to admit a simple description. In fact, let $\mathbf{A}$ be a complete Heyting algebra which is also a Hausdorff lattice with fsi stalks at least one of which is incomplete, e.g., the lattice described in [73, Prop. 7.9]. Then $\mathbf{A}^+ = \overline{\mathbf{A}} = \mathbf{A}$. This shows that the stalks of the hyper-MacNeille completion of an arbitrary Heyting algebra will not necessarily be complete.

Recall that for any Heyting algebra $\mathbf{A}$ we have an embedding of Heyting algebras $e\colon \mathbf{A} \hookrightarrow \mathbf{A}^+$ given by $e(a) = L(0, a)$, where

$$L(t, b) = \{(s, a) \in A^2 : s \vee t \vee (a \to b) = 1\},$$

for all $(t, b) \in A^2$.

4.5.2. Proposition. *Let* $\mathbf{A}$ *be a Heyting algebra and let* $c \in A$*. Then* $L(c, 0)$ *is a central element of* $\mathbf{A}^+$*. Moreover,* $L(c, 0) = 1$ *precisely when* $c = 1$*, and* $L(c, 0) = 0$ *precisely when* $c \in \mathrm{CoDn}(\mathbf{A})$*.*

Proof:
We claim that $L(c,0) \vee \neg L(c,0) = 1$. To see this we make two observations. Firstly, (i) by definition of the Heyting implication in $\mathbf{A}^+$

$$\neg L(c,0) = \{(t,b) \in A^2 \colon \forall (s,a) \in L(c,0)\ ((s \vee t, a \wedge b) \in L(0,0))\}.$$

Consequently, $(c,1) \in \neg L(c,0)$, since if $(s,a) \in L(c,0)$, then $1 = s \vee c \vee \neg a = (s \vee c) \vee \neg(a \wedge 1)$, and hence $(s \vee c, a \wedge 1) \in L(0,0)$. Secondly, (ii) $(d,1) \in L(c,0)$ for all $d \in A$ such that $1 = c \vee d$.

Then let $L(t,b) \in A^+$ be such that $L(c,0) \vee \neg L(c,0) \subseteq L(t,b)$. Then $L(c,0) \cup \neg L(c,0) \subseteq L(t,b)$, hence by (i) we must have $(c,1) \in L(t,b)$ and therefore also that $1 = c \vee t \vee (1 \to b) = c \vee (t \vee b)$. Consequently, $(t \vee b, 1) \in L(c,0)$ by (ii). But then $(t \vee b, 1) \in L(t,b)$ and hence $1 = t \vee (t \vee b) \vee (1 \to b) = t \vee b$. This, however, entails that $L(t,b) = A^2$ since for $(s,a) \in A^2$ we always have $t \vee b \leq s \vee t \vee (a \to b)$. By Lemma 4.2.6(1) we must have that $L(c,0) \vee \neg L(c,0) = A^2$, showing that $L(c,0)$ is indeed central.

Finally, we see that $L(c,0) = 1$ if, and only if, $s \vee c \vee \neg a = 1$ for all $(s,a) \in A^2$, which is easily seen to happen only when $c = 1$. Similarly, $L(c,0) = L(0,0)$ if, and only if, $s \vee c \vee \neg a = 1$ implies $s \vee \neg a = 1$ for all $(s,a) \in A^2$. As all elements of A are of the form $s \vee \neg a$ we obtain that $L(c,0) = L(0,0)$ if, and only if, $c \vee d = 1$ implies $d = 1$, which is to say that c is co-dense. $\square$

4.5.3. Proposition. *The map $\Phi \colon \mathbf{A} \to Z(\mathbf{A}^+)$ given by $c \mapsto L(c,0)$ is a bounded lattice homomorphism.*

Proof:
We have already seen that $\Phi(0) = 0$ and that $\Phi(1) = 1$. Furthermore, for $c, d \in A$ we have

$$\begin{aligned}
\Phi(c \wedge d) &= \{(s,a) \in A^2 \colon 1 = s \vee (c \wedge d) \vee \neg a\} \\
&= \{(s,a) \in A^2 \colon 1 = (s \vee c \vee \neg a) \wedge (s \vee d \vee \neg a)\} \\
&= \{(s,a) \in A^2 \colon 1 = s \vee c \vee \neg a \text{ and } 1 = s \vee d \vee \neg a\} \\
&= \Phi(d) \wedge \Phi(d).
\end{aligned}$$

To see that $\Phi(c \vee d) = \Phi(c) \vee \Phi(d)$ we first note that $\Phi(c \vee d) \geq \Phi(c) \vee \Phi(d)$ since Φ is order-preserving. For the converse, let $(s,a) \in \Phi(c \vee d)$ be given. To show $(s,a) \in \Phi(c) \vee \Phi(d)$ we must show that $(s,a) \in LU(\Phi(c) \cup \Phi(d))$. Therefore, let $(t,b) \in U(\Phi(c) \cup \Phi(d))$ be given. From $(s,a) \in \Phi(c \vee d)$ we may deduce $(s \vee c, a) \in \Phi(d)$ and hence that $s \vee c \vee t \vee (a \to b) = 1$, as $(t,b) \in U(\Phi(d))$. Therefore, letting $u := s \vee t \vee (a \to b)$ we see that $(u,1) \in \Phi(c)$. As $(t,b) \in U(\Phi(c))$ it follows that

$$1 = u \vee t \vee (1 \to b) = u \vee t \vee b = s \vee t \vee (a \to b) \vee t \vee b = s \vee t \vee (a \to b).$$

Because $(t,b) \in U(\Phi(c) \cup \Phi(d))$ was arbitrary, this shows $(s,a) \in LU(\Phi(c) \cup \Phi(d))$, from which we may conclude that $\Phi(c \vee d) \leq \Phi(c) \vee \Phi(d)$. □

By Proposition 4.1.9 we know that in a distributive lattice which is De Morgan supplemented the central and the co-regular elements coincide. To obtain a description of the co-regular elements of $\mathbf{A}^+$ requires that we first compute the supplements of elements of the form $L(t,b)$.

4.5.4. LEMMA. *Let* $\mathbf{A}$ *be a Heyting algebra and let* $(t,b) \in A^2$. *Then* $\neg\Phi(t \vee b)$ *is the supplement of* $L(t,b)$ *in* $\mathbf{A}^+$.

Proof:
By the definition of the Heyting implication in $\mathbf{A}^+$ we have that

$$Z_1 \to Z_2 = \{(s,a) \in A^2 : \forall (t,b) \in Z_1 \ ((s \vee t, a \wedge b) \in Z_2)\},$$

for $Z_1, Z_2 \in A^+$. In particular,

$$\neg L(t \vee b, 0) = \{(s,a) \in A^2 : \forall (s',a') \in L(t \vee b, 0) \ (s \vee s' \vee \neg(a \wedge a') = 1)\}.$$

We first show that $L(t,b) \vee Z = 1$ entails $\neg L(t \vee b, 0) \leq Z$, for all $Z \in A^+$. Since $Z = \bigwedge\{L(t',b') : Z \leq L(t',b')\}$ it suffices to establish the claim for elements of the form $L(t',b')$. Therefore, assume $L(t,b) \vee L(t',b') = 1$, then since $L(t,b), L(t',b') \leq L(t \vee t', b \vee b')$ we must have that $L(t \vee t', b \vee b') = 1$ and hence $t \vee t' \vee b \vee b' = 1$. From this it follows that $(t' \vee b', 1) \in L(b \vee t, 0)$. Consequently, if $(s,a) \in \neg L(t \vee b, 0)$ then in particular we must have $(s \vee (t' \vee b'), a \wedge 1) \in L(0,0)$, i.e., $s \vee t' \vee b' \vee \neg a = 1$. But then,

$$1 = s \vee t' \vee b' \vee \neg a \leq s \vee t' \vee b' \vee (a \to b') = s \vee t' \vee (a \to b'),$$

showing that $(s,a) \in L(t',b')$. Thus, we have shown $\neg L(t \vee b, 0) \leq L(t',b')$.

To see that $L(t,b) \vee \neg L(t \vee b, 0) = 1$ we first observe that since $(t \vee b, 1) \in \neg L(t \vee b, 0)$ we have that if $\neg L(t \vee b, 0) \leq L(t',b')$ then $t \vee b \vee t' \vee b' = 1$ meaning that $(t' \vee b', 1) \in L(t,b)$. Consequently, if also $L(t,b) \leq L(t',b')$ then $(t' \vee b', 1) \in L(t',b')$ and hence $t' \vee b' = 1$ from which we may conclude $L(t',b') = 1$. So $L(t,b) \vee \neg L(t \vee b, 0) \leq L(t',b')$ entails $L(t',b') = 1$. Since the elements of the form $L(t',b')$ are meet-dense in A^+ we obtain that $L(t,b) \vee \neg L(t \vee b, 0) = 1$, as desired. □

Since $\Phi\colon \mathbf{A} \to Z(\mathbf{A}^+)$ is a homomorphism of bounded lattices, the quotient $\mathbf{A}/\ker\Phi$ is a bounded sublattice of $Z(\mathbf{A}^+)$.

4.5.5. PROPOSITION. *Let* $\mathbf{A}$ *be a Heyting algebra. Then* $Z(\mathbf{A}^+)$ *is the MacNeille completion of the free Boolean extension of the distributive lattice* $\mathbf{A}/\ker\Phi$.

Proof:
Since $\mathbf{A}/\ker\Phi$ embeds into $Z(\mathbf{A}^+)$ as a bounded distributive lattice we have that the free Boolean extension $B(\mathbf{A}/\ker\Phi)$ of $\mathbf{A}/\ker\Phi$ is a Boolean subalgebra of $Z(\mathbf{A}^+)$. This follows from the fact that the category of Boolean algebras and Boolean algebra homomorphisms is a reflective subcategory of the category of bounded distributive lattices and bounded lattice homomorphism with the free Boolean extension a reflector preserving monomorphisms [16, Cor. V.4.3]. Since $\mathbf{A}^+$ is complete and supplemented we have that $Z(\mathbf{A}^+)$ is complete by Proposition 4.1.11. Moreover, all meets and joins in $Z(\mathbf{A}^+)$ are computed as in $\mathbf{A}^+$. As both $Z(\mathbf{A}^+)$ and $B(\mathbf{A}/\ker\Phi)$ are Boolean algebras it suffices to show that $B(\mathbf{A}/\ker\Phi)$ is meet-dense in $Z(\mathbf{A}^+)$.

Therefore let $Z \in \mathbf{A}^+$ be a central element. Then ${\sim}Z = \neg Z$. By Lemma 4.2.6 the elements of the form $L(t,b)$ are meet-dense in A^+. Consequently, since each element in the image of Φ is central we may conclude that

$$\begin{aligned}
Z &= \neg\sim Z \\
&= \neg\sim\Big(\bigwedge\big\{L(t,b) \in A^+ : Z \leq L(t,b)\big\}\Big) \\
&= \neg\Big(\bigvee\big\{{\sim}L(t,b) \in A^+ : Z \leq L(t,b)\big\}\Big) \\
&= \neg\Big(\bigvee\big\{\neg\Phi(t\vee b) \in A^+ : Z \leq L(t,b)\big\}\Big) \\
&= \bigwedge\big\{\neg\neg\Phi(t\vee b) \in A^+ : Z \leq L(t,b)\big\} \\
&= \bigwedge\big\{\Phi(t\vee b) \in A^+ : Z \leq L(t,b)\big\},
\end{aligned}$$

where we used that by Lemma 4.5.4 ${\sim}L(t,b) = \neg\Phi(t\vee b)$, for all $(t,b) \in A^2$. This shows that the image of Φ is meet-dense in $Z(\mathbf{A}^+)$. ◻

Thus one way to understand the structure of $Z(\mathbf{A}^+)$ is to understand the structure of the distributive lattice $\mathbf{A}/\ker\Phi$. We will do so by showing how the dual Priestley space of $\mathbf{A}/\ker\Phi$ can be obtained from the dual space of $\mathbf{A}$ in a natural way.

4.5.6. PROPOSITION. *The kernel of the map $\Phi\colon \mathbf{A} \to Z(\mathbf{A}^+)$ consists of the pairs $(c,d) \in A^2$ such that*

$$\forall a \in A\ (1 = a \vee c \iff 1 = a \vee d).$$

Proof:
We have that $\Phi(c) = \Phi(d)$ precisely when

$$\forall (s,a) \in A^2\ (1 = s\vee c\vee\neg a \iff 1 = s\vee d\vee\neg a).$$

Thus as any element of A is of the form $s\vee\neg a$, for some $(s,a) \in A^2$ the statement of the proposition follows. ◻

4.5.7. REMARK. The distributive lattice congruence $\ker\Phi$ has also been studied in its order dual version by Speed, see, [238, Sec. 5] and [237].

In order to characterise the dual Priestley space of the distributive lattice $\mathbf{A}/\ker\Phi$ the following lemma will be useful.

4.5.8. LEMMA. *Let* $\mathbf{D}$ *be a distributive lattice with dual Priestley space* X. *Then* $\mathrm{CoDn}(\mathbf{D}) = \{a \in D : \min(X) \cap \widehat{a} = \emptyset\}$.

Proof:
By Remark 4.1.17 $\min(X) \subseteq \bigcap\{X\backslash\widehat{a} : a \in \mathrm{CoDn}(\mathbf{D})\}$. Thus for each $a \in \mathrm{CoDn}(\mathbf{D})$ we have that $\min(X) \subseteq X\backslash\widehat{a}$, which implies that $\min(X) \cap \widehat{a} = \emptyset$. Conversely, if $a \notin \mathrm{CoDn}(\mathbf{D})$ then we have $c < 1$ such that $a \vee c = 1$. Consequently, for any $x \in X$ either $a \in x$ or $c \in x$. Since $c < 1$ we have a prime filter $x \in X$ such that $c \notin x$. Moreover, since X is a Priestley space we must have $y \subseteq x$ for some $y \in \min(X)$. But then $c \notin y$ whence $a \in y$, i.e., $y \in \widehat{a}$, showing that $\widehat{a} \cap \min(X) \neq \emptyset$. □

4.5.9. PROPOSITION. *Let* $\mathbf{A}$ *be a Heyting algebra with dual Esakia space* X. *Then the dual Priestley space of* $\mathbf{A}/\ker\Phi$ *is the closure in* X *of the set* $\min(X)$.

Proof:
Under the correspondence between homomorphic images of a distributive lattice and closed subspaces of its dual space, see, e.g., [217, Lem. 12], we have that the dual Priestley space of $\mathbf{A}/\ker\Phi$ is determined by the closed set

$$Y = \{x \in X : \forall (a,b) \in \ker\Phi\ (a \in x \iff b \in x)\}.$$

We show that this set coincides with the closure of $\min(X)$.

First we determine the closure of $\min(X)$. As the sets $\widehat{a} = \{x \in X : a \in x\}$, for $a \in A$, and their complements form a basis for the topology on X, the closure $\mathsf{C}(S)$ of any set $S \subseteq A$ consist of points x satisfying

$$\forall a \in A\ ((a \in x \implies \exists y \in S(a \in y)) \,\text{and}\, (a \notin x \implies \exists y \in S(a \notin y))).$$

By Proposition A.6.6 any prime filter on $\mathbf{A}$ contains a minimal prime filter, and so for all $a \in A$, if $a \notin x$ for some $x \in X$ then also $a \notin y$ for some minimal prime filter $y \in X$. Therefore, we may conclude that

$$\mathsf{C}(\min(X)) = \{x \in X : \forall a \in A\ (a \in x \implies \widehat{a} \cap \min(X) \neq \emptyset)\} = X\backslash \bigcup_{a\in\mathrm{CoDn}(A)} \widehat{a}.$$

where the last equality follows from Lemma 4.5.8. We then observe that if $x \in Y$, then since x is proper we must have that $0 \notin x$ and hence $a \notin x$, for all $a \in A$

such that $(a, 0) \in \ker \Phi$. By Proposition 4.5.6 it follows that $(a, 0) \in \ker \Phi$ if, and only if, $a \in \mathrm{CoDn}(A)$, whence

$$Y \subseteq X \backslash \bigcup \{\widehat{a} : a \in \mathrm{CoDn}(\mathbf{A})\} = \mathsf{C}(\min(X)).$$

Lastly, we observe that $\min(X) \subseteq Y$. To see this, let $x \in \min(X)$ be given and suppose that $(a, b) \in \ker \Phi$ with $a \in x$. By Lemma 4.3.1 we have $s \in A$ with $s < 1$ such that $a \vee s = 1$ and $s \notin x$. Since $(a, b) \in \ker \Phi$ we have that $b \vee s = 1$ and hence $b \in x$, from which we may conclude that $x \in Y$. This shows that

$$\min(X) \subseteq Y \subseteq \mathsf{C}(\min(X)),$$

and since Y is closed this entails $Y = \mathsf{C}(\min(X))$, as desired. □

The Stone space of the free Boolean extension of a distributive lattice $\mathbf{D}$ is the underlying Stone space of the dual Priestley space of $\mathbf{D}$, see, e.g., [130, Prop. 1.1.14]. Moreover, the MacNeille completion of a Boolean algebra $\mathbf{B}$ is isomorphic to $\mathrm{RO}(X)$, the algebra of regular open subsets of dual Stone space X of $\mathbf{B}$, see, e.g., [151, Thm. 3.8]. Putting these two facts together we obtain that for any Heyting algebra with dual Esakia space X the center of $\mathbf{A}^+$ is isomorphic to the algebra of regular opens of the Stone space $\mathsf{C}(\min(X))$. It is not difficult to show that if $X \subseteq Y$ is a dense subspace of a topological space Y, then the map $U \mapsto U \cap X$ is an isomorphism of Boolean algebras $\mathrm{RO}(Y)$ and $\mathrm{RO}(X)$. Consequently, since $\min(X)$ is dense in $\mathsf{C}(\min(X))$, when both are considered as subspaces of X, we obtain the following description of the center of $\mathbf{A}^+$.

4.5.10. Corollary. *If $\mathbf{A}$ is a Heyting algebra with dual Esakia space X, then $Z(\mathbf{A}^+)$ is isomorphic to $\mathrm{RO}(\min(X))$.*

Finally, we compare the center of $\mathbf{A}^+$ to that of $Q(\mathbf{A})$, the algebra of dense open sections of $\mathbf{A}$.

4.5.11. Proposition. *For any Heyting algebra $\mathbf{A}$, the center of $Q(\mathbf{A})$ is isomorphic to the center of $\mathbf{A}^+$.*

Proof:
Let X denote the dual Esakia space of $\mathbf{A}$. For each open U of $\min(X)$ let χ_U denote the characteristic function for U. We show that $U \mapsto \chi_U/\Theta$ is an isomorphism from $\mathrm{RO}(\min(X))$ to $Z(Q(\mathbf{A}))$.

To see that this map is indeed well defined let U be an open subset of $\min(X)$. If $x \in U$, then, since U is open, we have $s \in A$ such that $x \in [\![s < 1]\!] \subseteq U$. Thus by Lemma 4.3.4, the pair $(s, 1)$ will witness that χ_U is continuous at x. If $x \in \mathsf{I}(\min(X) \backslash U)$, then we have $s \in A$ such that $x \in [\![s < 1]\!] \subseteq \min(X) \backslash U$. Thus the pair $(s, 0)$ will witness that χ_U is continuous at x. We have thus shown that

χ_U is continuous on the dense open set $U \cup \mathsf{I}(\min(X)\backslash U)$. Therefore, $\chi_U \in \mathcal{D}(\mathbf{A})$ for all open subsets U of $\min(X)$. Furthermore, for each open $U \subseteq \min(X)$, letting $U' = \mathsf{I}(\min(X)\backslash U)$, we see that $\chi_U \wedge \chi_{U'} = 0$ and that $\chi_U \vee \chi_{U'}$ agrees with 1 on the dense open set $U \cup U'$. Consequently, χ_U will be central in $Q(\mathbf{A})$ with complement $\chi_{U'}$.

Evidently, $U \mapsto \chi_U/\Theta$ is an order-preserving map. If $U_1, U_2 \in \mathrm{RO}(\min(X))$ are such that $\chi_{U_1}/\Theta \leq \chi_{U_2}/\Theta$ then we have a dense open subset U contained in $[\![\chi_{U_1} \leq \chi_{U_2}]\!]$. But then $U \cap U_1 \subseteq U_2$ and hence $\mathsf{IC}(U \cap U_1) \subseteq \mathsf{IC}(U_2) = U_2$. Because U is dense we have that $\mathsf{IC}(U \cap U_1) = \mathsf{IC}(U_1) = U_1$ and therefore $U_1 \subseteq U_2$. This show that $U \mapsto \chi_U/\Theta$ is also order-reflecting and hence an order-embedding.

For each open subset U of $\min(X)$ we have that χ_U and $\chi_{\mathsf{IC}(U)}$ agree on a dense open set, namely $U \cup (\min(X) \setminus \mathsf{C}(U))$. Therefore, since $\mathsf{IC}(U)$ is always a regular open subset of $\min(X)$, to establish that the map $U \mapsto \chi_U/\Theta$ is surjective it suffices to show that for f/Θ central in $Q(\mathbf{A})$ we have $f/\Theta = \chi_U/\Theta$ for some open subset U of $\min(X)$.

Suppose that $f \in \mathcal{D}(\mathbf{A})$ is such that f/Θ is central in $Q(\mathbf{A})$. Then there is f' in $\mathcal{D}(\mathbf{A})$ with f'/Θ the complement of f/Θ in $Q(\mathbf{A})$. Let U be a dense open set on which both f and f' are continuous. Let $V_1 = [\![f = 1]\!] \cap U$ and $V_0 = [\![f = 0]\!] \cap U$. If $x \in V_1$ then by Lemma 4.3.4 we have $(s_x, a_x) \in A^2$ such that $s_x(x) < 1$, and $s_x(y) < 1$ implies $a_x(y) = f(y)$, for all $y \in \min(X)$. Consequently, the set $[\![s_x < 1]\!] \cap [\![a_x = 1]\!] \cap U$ is an open neighborhood of x contained in V_1. This shows that V_1 is open. Similarly, we see that V_0 is open. We claim that $V_1 \cup V_0$ is dense. By assumption the sets $[\![f \vee f' = 1]\!]$ and $[\![f \wedge f' = 0]\!]$ both contain dense open sets, say U_1 and U_0. As each of the quotients $\mathbf{A}/\theta_x$ is well-connected we obtain that if $x \in U_1 \cap U_2$ then $f(x) = 1$ or $f(x) = 0$. Consequently, the dense open $U \cap U_1 \cap U_2$ is contained in $V_1 \cup V_0$, which must therefore be dense. This shows that $f/\Theta = \chi_U/\Theta$. □

Since $Z(\mathbf{A}^+)$ is complete we obtain that the center of $Q(\mathbf{A})$ is a complete Boolean algebra for each Heyting algebra $\mathbf{A}$. Even if the algebra $Q(\mathbf{A})$ is itself not always complete, some suprema are guaranteed to exist. Recall from [73] that in a lattice $\mathbf{L}$ with a least element 0 a family of elements $\{a_i \in L : i \in I\}$ is *pairwise disjoint* if $a_i \wedge a_j = 0$ for $i \neq j$. Recall further that a lattice $\mathbf{L}$ with a least element is *orthogonally complete* if for any family of pairwise disjoint central elements $\{c_i \in Z(\mathbf{L}) : i \in I\}$ and every I-indexed family $\{a_i \in L : i \in I\}$, the family $\{a_i \wedge c_i \in L : i \in I\}$ has a least upper bound in $\mathbf{L}$.

4.5.12. Proposition. *The algebra $Q(\mathbf{A})$ is orthogonally complete for any Heyting algebra $\mathbf{A}$.*

Proof:
Let X denote the dual Esakia space of $\mathbf{A}$ and let $\{c_i \in Z(Q(\mathbf{A})) : i \in I\}$ be a family of pairwise disjoint central elements of $Q(\mathbf{A})$. By Proposition 4.5.11

we have a family $\{U_i \in \mathrm{RO}(\min(X)) : i \in I\}$ of pairwise disjoint regular open subsets of $\min(X)$ such that $c_i = \chi_{U_i}/\Theta$. Let $U := \bigcup_{i\in I} U_i$. Given a family $\{f_i/\Theta \in Q(\mathbf{A}) : i \in I\}$ we define $f \in \prod_{x\in\min(X)} \mathbf{A}/\theta_x$ by letting $f(x) = f_i(x)$ if $x \in U_i$ and $f(x) = 0$ if $x \notin U$. Evidently, f is continuous on the dense open set $U \cup \mathsf{I}(\min(X)\backslash U)$ and therefore a member of $\mathcal{D}(\mathbf{A})$. We claim that f/Θ is the least upper bound of the family $\{f_i/\Theta \wedge \chi_{U_i}/\Theta \in Q(\mathbf{A}) : i \in I\}$. By construction of f we have that $f_i \wedge \chi_{U_i} \leq f$ and hence f/Θ must be an upper bound of the family $\{f_i/\Theta \wedge \chi_{U_i}/\Theta \in Q(\mathbf{A}) : i \in I\}$. If g/Θ is another upper bound of this family, then for each $i \in I$ we have a dense open V_i with $V_i \subseteq [\![f_i \wedge \chi_{U_i} \leq g]\!]$. Consequently, letting $W := \bigcup_{i\in I}(V_i \cap U_i)$, we see that $W \subseteq [\![f \leq g]\!]$. Given open V with $W \cap V = \emptyset$ we must have that $V_i \cap U_i \cap V = \emptyset$ for all $i \in I$. As each V_i is dense this entails that $U_i \cap V = \emptyset$ for each $i \in I$ and hence that $V \subseteq \min(X)\backslash U$. It follows that $W \cup \mathsf{I}(\min(X)\backslash U)$ is a dense open set. Since f agrees with 0 on $\mathsf{I}(\min(X)\backslash U)$ we obtain that $W \cup \mathsf{I}(\min(X)\backslash U) \subseteq [\![f \leq g]\!]$ and therefore that $f/\Theta \leq g/\Theta$. □

4.5.13. Remark. Let $\mathbf{D}$ be a Hausdorff lattice with X the dual Stone space of its center. If $\mathbf{D}$ has complete stalks on a dense subset of X, then by [73, Thm. 7.8], the lattice $\mathbf{D}$ is complete if, and only if, it is orthogonally complete. This together with Proposition 4.5.12 and Jónsson's Lemma can be used to give an alternative proof of the fact that finitely generated varieties of Heyting algebras are closed under hyper-MacNeille completions.

4.6 Regular hyper-MacNeille completions

Recall that a completion $e\colon \mathbf{D} \hookrightarrow \mathbf{C}$ of a lattice $\mathbf{D}$ is *regular* if the embedding preserves all infima and suprema which exist in $\mathbf{D}$. The MacNeille completion of a lattice is always regular while for instance the canonical completions is never regular except in trivial cases, see, e.g., [148, Sec. 3]. There is at least one variety of Heyting algebras which is not closed under MacNeille completions but in which every algebra regularly embeds into some complete algebra belonging to the variety, namely the variety $\mathcal{V}(\mathbf{3})$ generated by the three element chain, see, [149]. It is currently not known whether there are other varieties of Heyting algebras with this property. In this section we will give necessary and sufficient conditions for the completion $e\colon \mathbf{A} \hookrightarrow \mathbf{A}^+$ to be regular. In particular, this will show that the hyper-MacNeille completion will not provide examples of varieties of Heyting algebras which are not closed under MacNeille completions but nevertheless admit regular completions.

4.6.1. Definition (cf. [60, Def. 6.10]). A distributive lattice $\mathbf{D}$ is called *externally distributive* if

$$(\forall s \in S\ (a \vee s = 1)) \implies a \vee \bigwedge S = 1,$$

for all $S \cup \{a\} \subseteq D$ such that S has a greatest lower bound in $\mathbf{D}$.

4.6.2. Remark. Any supplemented Heyting algebra is externally distributive. However, not every externally distributive Heyting algebra is supplemented. To see this, consider any non-supplemented algebra with only essentially finite meets such as the linear sum $\omega \oplus (\mathbf{2} \times \omega^\partial)$. Of course, for complete distributive lattices being externally distributive is equivalent to being supplemented.

4.6.3. Lemma. *Let $\mathbf{A}$ be a Heyting algebra. Then for all $(t, b) \in A^2$ and all $c \in A$ we have that $(t, b) \in U(0, c)$ if, and only if, $L(0, c) \subseteq L(t, b)$.*

Proof:
First let $(t, b) \in A^2$ and $c \in A$ be such that $(t, b) \in U(0, c)$. Then $t \vee (c \to b) = 1$. If $(s, a) \in L(0, c)$ we have $s \vee (a \to c) = 1$. From this it follows that

$$\begin{aligned} 1 &= (t \vee (c \to b)) \wedge (s \vee (a \to c)) \\ &\leq ((s \vee t) \vee (c \to b)) \wedge ((s \vee t) \vee (a \to c)) \\ &= (s \vee t) \vee ((a \to c) \wedge (c \to b)) \\ &\leq s \vee t \vee (a \to c). \end{aligned}$$

Consequently, $(s, a) \in L(t, b)$ and hence $L(0, c) \subseteq L(t, b)$. Conversely, let $(t, b) \in A^2$ and $c \in A$ be such that $L(0, c) \subseteq L(t, b)$. Since $(0, c) \in L(0, c)$ we must have that $(0, c) \in L(t, b)$ and hence that $(t, b) \in U(0, c)$. □

4.6.4. Theorem (cf. [60, Prop. 6.11]). *If $\mathbf{A}$ is an externally distributive Heyting algebra then the embedding $e \colon \mathbf{A} \hookrightarrow \mathbf{A}^+$ is regular.*

Proof:
Let $C \subseteq A$ be a subset with a greatest lower bound in $\mathbf{A}$. Then we have that

$$\begin{aligned} e\Big(\bigwedge C\Big) &= \Big\{(s, a) \in A^2 : 1 = s \vee (a \to \bigwedge C)\Big\} \\ &= \Big\{(s, a) \in A^2 : 1 = s \vee \bigwedge \{a \to c \in A : c \in C\}\Big\} \\ &= \big\{(s, a) \in A^2 : \forall c \in C \ (1 = s \vee (a \to c))\big\} \\ &= \bigcap_{c \in C} L(0, c) \\ &= \bigwedge_{c \in C} e(c). \end{aligned}$$

where the third equality from the top follows from the assumption that $\mathbf{A}$ is externally distributive. Similarly let $C \subseteq A$ be a subset with a least upper bound in $\mathbf{A}$. Then we have that

$$\begin{aligned}
U\Big(0, \bigvee C\Big) &= \Big\{(t,b) \in A^2 : 1 = t \vee (\bigvee C \to b)\Big\} \\
&= \Big\{(t,b) \in A^2 : 1 = t \vee \bigwedge\{c \to b \in A : c \in C\}\Big\} \\
&= \big\{(t,b) \in A^2 : \forall c \in C\ (1 = t \vee (c \to b))\big\} \\
&= \bigcap_{c \in C} U(0,c).
\end{aligned}$$

where again the third equality from the top follows from the assumption that $\mathbf{A}$ is externally distributive. By Lemma 4.6.3 we then obtain that

$$\begin{aligned}
e\left(\bigvee C\right) &= L\Big(0, \bigvee C\Big) \\
&= LU\Big(0, \bigvee C\Big) \\
&= L\Big(\bigcap\{U(0,c) : c \in C\}\Big) \\
&= L\big(\big\{(t,b) \in A^2 : \forall c \in C\ (L(0,c) \subseteq L(t,b))\big\}\big) \\
&= \bigcap \{L(t,b) : \forall c \in C\ (L(0,c) \subseteq L(t,b))\} \\
&= \bigcap \left\{L(t,b) : \bigcup_{c \in C} L(0,c) \subseteq L(t,b)\right\} \\
&= \bigvee_{c \in C} L(0,c) \\
&= \bigvee_{c \in C} e(c).
\end{aligned}$$

Thus e preserves all existing meets and joins in $\mathbf{A}$. $\square$

Using the fact that the hyper-MacNeille completion of a Heyting algebra is always supplemented we also obtain that being externally distributive is in fact a necessary condition for the hyper-MacNeille completion to be regular. More precisely we have the following.

4.6.5. THEOREM. *Let $\mathbf{A}$ be a Heyting algebra and let $e\colon \mathbf{A} \hookrightarrow \mathbf{A}^+$ be the embedding $a \mapsto L(0,a)$, then the following are equivalent.*

1. *The embedding $e\colon \mathbf{A} \hookrightarrow \mathbf{A}^+$ is regular.*

2. *The embedding $e\colon \mathbf{A} \hookrightarrow \mathbf{A}^+$ is meet-regular.*

3. *The algebra $\mathbf{A}$ is externally distributive.*

Proof:
We have already seen, Theorem 4.6.4, that $\mathbf{A}$ being externally distributive is a sufficient condition for the embedding $e\colon \mathbf{A} \to \mathbf{A}^+$ to be regular. Thus it suffices to show that $\mathbf{A}$ is externally distributive if the embedding e is meet-regular. Therefore, assume that $S \subseteq A$ is a subset having a greatest lower bound in $\mathbf{A}$ and that $a \in A$ is such that $s \vee a = 1$ for all $s \in S$. Using the fact that every element in the image of e is supplemented we see that $\sim e(a) \leq e(s)$ for each $s \in S$. From this it follows that $\sim e(a) \leq \bigwedge_{s \in S} e(s)$ and so by meet-regularity we obtain that $\sim e(a) \leq e(\bigwedge S)$ and therefore that $1 = e(a) \vee e(\bigwedge S) = e(a \vee \bigwedge S)$, showing that $a \vee \bigwedge S = 1$. $\square$

4.6.6. Remark. Let $\mathbf{A}_1$ be the Heyting algebra of empty and co-finite subsets of the natural numbers. It is easy to see that $\mathbf{A}_1$ belongs to the variety of Heyting algebras $\mathcal{V}(\mathbf{3})$ generated by the chain $\mathbf{3}$ and hence to any variety of Heyting algebras properly containing the variety of Boolean algebras. The algebra $\mathbf{A}_1$ is evidently not externally distributive. Let $\mathcal{V}$ be a variety of Heyting algebras such that $\mathcal{BA} \subsetneq \mathcal{V} \subsetneq \mathcal{HA}$. Since $\mathcal{V}$ is not closed under MacNeille completions [151] we must have an algebra $\mathbf{A}_2 \in \mathcal{V}$ such that $\overline{\mathbf{A}_2} \notin \mathcal{V}$. In particular $\mathbf{A}_2$ must be incomplete. It follows that the direct product $\mathbf{A}_3 \coloneqq \mathbf{A}_1 \times \mathbf{A}_2$ is an incomplete member of the variety $\mathcal{V}$ which is not externally distributive. Furthermore we must have that $\overline{\mathbf{A}_3}$ does not belong to $\mathcal{V}$ and that the embedding $e\colon \mathbf{A}_3 \hookrightarrow \mathbf{A}_3^+$ is not regular. This shows that hyper-MacNeille completions will not in any immediate way yield non-trivial varieties of Heyting algebras, other than the variety of Boolean algebras, admitting regular completions.

4.7 Summary and concluding remarks

In this chapter we have looked at the hyper-MacNeille completion, first introduced by Ciabattoni, Galatos, and Terui [60], from a more algebraic perspective. In particular we have shown how in the context of Heyting algebras tools from universal algebra and duality theory can be used to obtain both new results, as well as new proofs of some already known facts, about the hyper-MacNeille completion. Concretely, we have identified the notion of a De Morgan supplemented Heyting algebra as central for understanding the hyper-MacNeille completion of Heyting algebras. We have shown that the MacNeille and hyper-MacNeille completions coincide for De Morgan supplemented Heyting algebras. Moreover, we have established that the hyper-MacNeille completion of a Heyting algebra $\mathbf{A}$ is the MacNeille completion of some De Morgan supplemented Heyting algebra $Q(\mathbf{A})$ belonging to the variety generated by $\mathbf{A}$. As a consequence of this, we obtained that a variety of Heyting algebras is closed under hyper-MacNeille completions if, and only if, the class of its De Morgan supplemented members is closed under MacNeille completions. Using this characterization we have provided examples of varieties of Heyting algebras closed under hyper-MacNeille completions but not axiomatizable by $\mathcal{P}_3$-equations. In particular, we have shown that all finitely generated varieties of Heyting algebras must be closed under hyper-MacNeille completions. We have also provided a description of the center of the algebra $Q(\mathbf{A})$ for an arbitrary Heyting algebra $\mathbf{A}$ and shown that $Q(\mathbf{A})$ is always orthogonally complete. Finally, we have identified necessary and sufficient conditions for the hyper-MacNeille completion to be regular.

Further directions and open problems One obvious question which remains to be answered is how to use our new perspective on the hyper-MacNeille completion to show that all varieties of Heyting algebras axiomatized by $\mathcal{P}_3$-equations are closed under hyper-MacNeille completions [60, Thm. 6.8]. One would hope that the results about varieties of Heyting algebras axiomatized by $\mathcal{P}_3$-equations obtained in Chapter 2 should turn out to be helpful in relation to this problem.

As we have seen the hyper-MacNeille completion is always a Hausdorff Heyting algebra with finitely subdirectly irreducible stalks. Thus one possible strategy for determining which varieties of Heyting algebras are closed under hyper-MacNeille completions would be to obtain a better description of the stalks of $\mathbf{A}^+$. It is tempting to conjecture that for any Heyting algebra $\mathbf{A}$ with dual Esakia space X the stalks of $\mathbf{A}^+$, and therefore $\mathbf{A}^+$ itself, belong to the variety of Heyting algebras generated by the algebras $\overline{\mathbf{A}/\theta_x}$ for $x \in \min(X)$.

Another line of further investigation would be to establish a relationship between the hyper-MacNeille completion of a Heyting algebra $\mathbf{A}$ and the canonical completion of $\mathbf{A}$. We also believe that it will be worthwhile to systematically develop syntactic methods for showing that equations are preserved under MacNeille completions of (De Morgan) supplemented Heyting algebras. As mentioned in Section 4.4 this can either be done pursuing topological methods presented by Givant and Venema [124] and Theunissen and Venema [245] or by emulating ALBA-like arguments, see, e.g., [68, 236, 141, 255], with the MacNeille completion taking the place of the canonical completion.

Finally, we would like to have a better understanding of the class operation which takes a Heyting algebra $\mathbf{A}$ to its algebra of dense open sections $Q(\mathbf{A})$. In particular, we would like to understand exactly how it relates to seemingly similar constructions in other areas of algebra.

Chapter 5

Integrally closed residuated lattices

In this chapter, based on [122], we change perspective in two respects. First, we switch from considering Heyting algebras to considering residuated lattices and various closely related types of algebras. Second, instead of being concerned with properties related to hypersequent calculi and $\mathcal{P}_3$-equations we look at a specific $\mathcal{N}_2$-equation and an equivalent non-standard sequent calculus for the equational theory of residuated lattices determined by this equation. As we will see, even though this equation belongs to the level $\mathcal{N}_2$, the approach of Ciabattoni, Galatos, and Terui [59] cannot be applied to obtain an equivalent cut-free structural sequent calculus. Nevertheless, we show that algebraic methods can still yield some proof-theoretical insights, although of a different type than those found in [59].

Concretely, we look at residuated lattices satisfying the equation $x\backslash x \approx \mathrm{e}$, or equivalently the equation $x/x \approx \mathrm{e}$, viz., the so-called integrally closed residuated lattices [92, Chap. XII.3]. These structures encompass a large number of well-known residuated lattices, such as integral residuated lattices, ℓ-groups [6], cancellative residuated lattices [13], and GBL-algebras [102]. Moreover, as we will show, integrally closed residuated lattices are also connected to Dubreil-Jacotin semi-groups [78, 93, 218, 41], pseudo BCI-algebra [165, 178, 181, 79], sirmonoids [219, 83] and algebras for Casari's comparative logic [48, 49, 50, 210, 198].

We show that any integrally closed residuated lattice satisfies the equation $x\backslash\mathrm{e} \approx \mathrm{e}/x$. Consequently, we may expand the type of integrally closed residuated lattices with an additional unary operation, interpreted in each integrally closed residuated lattice $\mathbf{A}$ as the function mapping $a \in A$ to the element $a\backslash\mathrm{e} = \mathrm{e}/a$, which we denote by $-a$. Composing this operation with itself gives rise to a nucleus on any integrally closed residuated lattice analogous to the well-known double negation nucleus on Heyting algebras. Using this nucleus we show that the variety $\mathcal{IcRL}$ of integrally closed residuated lattices admits a Glivenko theorem [126, 100] with respect to the variety $\mathcal{LG}$ of ℓ-groups. That is, for any residuated lattice terms s, t we have

$$\mathcal{LG} \models s \leq t \quad \text{if, and only if,} \quad \mathcal{IcRL} \models --s \leq --t.$$

This relationship between the varieties $\mathcal{IcRL}$ and $\mathcal{LG}$ allows us to establish the soundness, with respect to the variety $\mathcal{IcRL}$, of the following non-standard version of the weakening rule

$$\frac{\Gamma, \Pi \Rightarrow u \quad \models_{\mathcal{LG}} \Delta \Rightarrow \mathrm{e}}{\Gamma, \Delta, \Pi \Rightarrow u} \ (\mathcal{LG}\text{-w})$$

where the premise $\models_{\mathcal{LG}} \Delta \Rightarrow \mathrm{e}$ may be understood as a side-condition for weakening. Variants of this rule have already been considered before in the context of BCI-algebras [178] and Casari's comparative logic [198]. Adding this rule to the ordinary sequent calculus for the equational theory of residuated lattices we obtain a sound and complete calculus for the equational theory of integrally closed residuated lattices. Furthermore, using a standard argument we show that the cut-rule is eliminable in this calculus. From this the decidability of the equational theory of integrally closed residuated lattices follows.

Finally, we use the cut-free calculus for the equational theory of integrally closed residuated lattices to obtain conservativity results concerning the equational theories of pseudo BCI-algebra, sirmonoids, and the algebras for Casari's comparative logic.

Outline The chapter is structured as follows: In Section 5.1 we establish some basic facts about the structure of integrally closed residuated lattices and in Section 5.2 we use these to prove that the variety of integrally closed residuated lattices enjoys the Glivenko property with respect to the variety of ℓ-groups. Then in Section 5.3 we construct a sequent calculus for the equational theory of the variety of integrally closed residuated lattices which we show admits cut-elimination. Section 5.4 contains a discussion of the relationship between integrally closed residuated lattices and sirmonoids, while in Section 5.5 the relationship between integrally closed residuated lattices and the algebras for Casari's comparative logic is discussed. Finally, Section 5.6 contains a few concluding remarks.

5.1 The structure of integrally closed residuated lattices

In this section we establish some basic facts about the structure of integrally closed residuated lattices. We refer to Appendix A.3 for the definition of residuated lattices. By an *integrally closed residuated lattice* we shall understand a residuated lattice satisfying the equations

$$x\backslash x \approx \mathrm{e} \quad \text{and} \quad x/x \approx \mathrm{e}.$$

We denote by $\mathcal{IcRL}$ the variety of integrally closed residuated lattices. As we shall see the equations $x\backslash x \approx \mathrm{e}$ and $x/x \approx \mathrm{e}$ are in fact equivalent and so either one of them suffices to define the variety $\mathcal{IcRL}$ relative to the variety of all residuated lattices.

Recall that an element a of a residuated lattice $\mathbf{A}$ is *idempotent* provided that $a^2 = a$. The following proposition shows that the property of being integrally closed is completely determined by the structure of the idempotent elements.

5.1.1. PROPOSITION. *Let* $\mathbf{A}$ *be a residuated lattice. Then the following are equivalent.*

1. *The residuated lattice* $\mathbf{A}$ *is integrally closed.*
2. *The residuated lattice* $\mathbf{A}$ *satisfies the quasi-equations*
$$xy \leq x \implies y \leq \mathrm{e} \quad \textit{and} \quad yx \leq x \implies y \leq \mathrm{e}.$$
3. *The monoidal unit* e *is the largest idempotent element of* $\mathbf{A}$.

Proof:
That Item 1 implies Item 2 follows immediately by applying residuation. That Item 2 implies Item 3 is likewise easy to see. We show that all elements of $\mathbf{A}$ of the form $a\backslash a$ and a/a are idempotent from which it follows that Item 3 implies Item 1. Therefore, let $a \in A$ be given. Since $a(a\backslash a) \leq a$ we must have $a(a\backslash a)(a\backslash a) \leq a(a\backslash a) \leq a$ and hence $(a\backslash a)^2 \leq (a\backslash a)$. On the other hand, since $\mathrm{e} \leq a\backslash a$ we also have $a\backslash a \leq (a\backslash a)^2$, which shows that $a\backslash a$ is indeed idempotent. The fact that the element a/a is idempotent for all $a \in A$ follows from a completely analogous argument. □

5.1.2. REMARK. This shows that the variety $\mathcal{IcRL}$ of integrally closed residuated lattices is $(\cdot, \vee, \mathrm{e})$-stable in the sense of definition [35, Def. 4.6], even in the stronger sense that any residuated lattice which is a $(\cdot, \vee, \mathrm{e})$-subalgebra of an integrally closed residuated lattice must also be integrally closed.

Thus Proposition 5.1.1 shows that whether or not a residuated lattice is integrally closed is completely determined by the structure of its underlying partially ordered monoid. We note that partially ordered monoids with a largest idempotent element are special instances of what is known as *Dubreil-Jacotin semigroups* which have been studied extensively, see, e.g., [41, Chap. 12–13] and [42, Chap. 3.25] for an overview. Proposition 5.1.1 also allows us to easily identify many examples of integrally closed residuated lattices. For example any *integral residuated lattice* is integrally closed. Similarly any *cancellative residuated lattice* [13] must be integrally closed. In particular, any *ℓ-group* will be integrally closed. Consequently, any direct product of an integral residuated lattice and an ℓ-group must be integrally closed, whence, by [102, Cor. 5.3], all *GBL-algebras* will be integrally closed. Of these four types of algebras only integral residuated lattices and ℓ-groups will play a role in this chapter. We refer to Appendix A.3 for definitions.

5.1.3. LEMMA. *Any upper or lower bounded integrally closed residuated lattice is integral.*

Proof:
Suppose that $\top$ is the greatest element of an integrally closed residuated lattice $\mathbf{A}$. Then $\top \cdot a \leq \top$ and hence $a \leq \top\backslash\top = \mathrm{e}$ for all $a \in A$. So $\mathbf{A}$ is integral. Moreover, any residuated lattice with a least element $\bot$ has a greatest element, namely $\bot\backslash\bot$, and so any lower bounded integrally closed residuated lattice must also be integral. $\square$

Since every finite residuated lattice is bounded, we obtain the following description of finite integrally closed residuated lattices.

5.1.4. COROLLARY. *A finite residuated lattice is integrally closed if, and only if, it is integral.*

Also, since there are integrally closed residuated lattices that are not integral, e.g., any non-trivial ℓ-group, the variety $\mathcal{IcRL}$ of integrally closed residuated lattices does not have the finite model property.

5.1.5. COROLLARY. *$\mathcal{IcRL}$ is not generated by its finite members.*

A residuated lattice $\mathbf{A}$ is called e-*cyclic* if the two unary operations

$$a \mapsto a\backslash\mathrm{e} \quad \text{and} \quad a \mapsto \mathrm{e}/a$$

on A coincide. The next result shows that integrally closed residuated lattices are e-cyclic and that either one of the defining equations for this variety, relative to the variety $\mathcal{RL}$ of all residuated lattices, suffices to imply the other.

5.1.6. PROPOSITION. *Any residuated lattice satisfying either $x\backslash x \approx \mathrm{e}$ or $x/x \approx \mathrm{e}$ is* e-*cyclic and integrally closed.*

Proof:
Let $\mathbf{A}$ be a residuated lattice satisfying $x\backslash x \approx \mathrm{e}$, noting that the case where $\mathbf{A}$ satisfies $x/x \approx \mathrm{e}$ is symmetrical. Consider any $a \in A$. By residuation, $a(a\backslash\mathrm{e}) \leq \mathrm{e}$, so $a(a\backslash\mathrm{e})a \leq a$, giving $(a\backslash\mathrm{e})a \leq a\backslash a = \mathrm{e}$ and hence $a\backslash\mathrm{e} \leq \mathrm{e}/a$. But also $(\mathrm{e}/a)a \leq \mathrm{e}$, so $a \leq (\mathrm{e}/a)\backslash\mathrm{e} \leq \mathrm{e}/(\mathrm{e}/a)$, giving $a(\mathrm{e}/a) \leq \mathrm{e}$ and hence $\mathrm{e}/a \leq a\backslash\mathrm{e}$. So $\mathbf{A}$ satisfies $x\backslash\mathrm{e} \approx \mathrm{e}/x$, i.e., $\mathbf{A}$ is e-cyclic.

Consider now again any $a \in A$. Since, as shown above, $(a\backslash\mathrm{e})a \leq \mathrm{e}$, we also have $a \leq (a\backslash\mathrm{e})\backslash\mathrm{e}$. But then, using the fact that the equation $x\backslash(y/z) \approx (x\backslash y)/z$ is valid in all residuated lattices,

$$a/a \leq ((a\backslash\mathrm{e})\backslash\mathrm{e})/a = (a\backslash\mathrm{e})\backslash(\mathrm{e}/a) = (a\backslash\mathrm{e})\backslash(a\backslash\mathrm{e}) = \mathrm{e}.$$

Hence $\mathbf{A}$ satisfies $x/x \approx \mathrm{e}$ and is therefore integrally closed. $\square$

For any e-cyclic residuated lattice $\mathbf{A}$ and $a \in A$, we write $-a$ to denote the element $a\backslash\mathrm{e} = \mathrm{e}/a$ of A. We will consider the type of e-cyclic residuated lattices to be $\langle 2,2,2,2,2,1,0\rangle$, with the additional unary operation interpreted as $a \mapsto -a$ on any e-cyclic residuated lattice. The next lemma collects some useful properties of this operation.

5.1.7. LEMMA. *Each of the following equations and quasi-equations axiomatizes the variety $\mathcal{IcRL}$ relative to the variety of* e-*cyclic residuated lattices:*

(i) $-(x\backslash y) \approx -y/-x$,

(ii) $-(y/x) \approx -x\backslash -y$,

(iii) $x(-x)y \leq \mathrm{e} \implies y \leq \mathrm{e}$,

(iv) $y(-x)x \leq \mathrm{e} \implies y \leq \mathrm{e}$.

Proof:
To see that the equation (i) axiomatizes the variety $\mathcal{IcRL}$ relative to the variety of e-cyclic residuated lattices, let $\mathbf{A}$ be any e-cyclic residuated lattice and consider $a, b \in A$. Since $a(-a)b \leq b$, it follows that $(-a)b \leq a\backslash b \leq --(a\backslash b)$ and hence $-(a\backslash b)(-a)b \leq \mathrm{e}$, yielding $-(a\backslash b) \leq -b/-a$. Note also that

$$a(a\backslash b)(-b/-a)(-a) \leq b(-b) \leq \mathrm{e}.$$

Hence if $\mathbf{A}$ is integrally closed, it follows that $(a\backslash b)(-b/-a) \leq (-a)/(-a) = \mathrm{e}$, implying $-b/-a \leq -(a\backslash b)$, which shows that $\mathbf{A}$ satisfies $-(x\backslash y) \approx -y/-x$. Conversely, if $\mathbf{A}$ satisfies $-(x\backslash y) \approx -y/-x$, then we have that

$$a\backslash a \leq (-a/-a)(a\backslash a) = -(a\backslash a)(a\backslash a) \leq \mathrm{e},$$

implying that $\mathbf{A}$ satisfies $x\backslash x \approx \mathrm{e}$ and so is integrally closed. The proof showing that the equation (ii) axiomatizes the variety $\mathcal{IcRL}$ relative to the variety of e-cyclic residuated lattices is symmetrical.

To see that the quasi-equation (iii) axiomatizes the variety $\mathcal{IcRL}$ relative to the variety of e-cyclic residuated lattices, consider first any integrally closed residuated lattice $\mathbf{A}$ and $a, b \in A$. If $a(-a)b \leq \mathrm{e}$, then $(-a)b \leq -a$ and hence $b \leq -a\backslash -a = \mathrm{e}$, implying that $\mathbf{A}$ satisfies $x(-x)y \leq \mathrm{e} \implies y \leq \mathrm{e}$. Suppose next that $\mathbf{A}$ is an e-cyclic residuated lattice that satisfies $x(-x)y \leq \mathrm{e} \implies y \leq \mathrm{e}$ and consider $a \in A$. Then $a(-a)(-a\backslash -a) \leq \mathrm{e}$ yields $-a\backslash -a \leq \mathrm{e}$. But also $-a(a/a)a \leq \mathrm{e}$, implying $a/a \leq -a\backslash -a \leq \mathrm{e}$. Therefore, $\mathbf{A}$ satisfies $x/x \approx \mathrm{e}$ and so is integrally closed. The proof showing that the quasi-equation (iv) axiomatizes the variety $\mathcal{IcRL}$ relative to the variety of e-cyclic residuated lattices is symmetrical. □

For any e-cyclic residuated lattice $\mathbf{A}$, the map $\alpha\colon A \to A$ given by

$$a \mapsto --a$$

is a nucleus on the induced partially ordered monoid $\langle A, \leq, \cdot, \mathrm{e}\rangle$, i.e., an increasing, order-preserving, idempotent map satisfying $\alpha(a)\alpha(b) \leq \alpha(ab)$ for all $a, b \in A$, see, e.g., [100, Lem. 5.2]. Consequently, the image of A under α can be equipped with the structure of a residuated lattice

$$\mathbf{A}_\alpha = \langle \alpha[A], \wedge, \vee_\alpha, \cdot_\alpha, \backslash, /, \mathrm{e}\rangle,$$

where $a \vee_\alpha b := \alpha(a \vee b)$ and $a \cdot_\alpha b := \alpha(a \cdot b)$, for all $a, b \in \alpha[A]$. See Section A.5 of the appendix for details.

Suppose now that $\mathbf{A}$ is an integrally closed residuated lattice satisfying the equation $--x \approx x$. Then for any $a \in A$ we have,

$$a(-a) = --(a(-a)) = -(a\backslash(--a)) = -(a\backslash a) = -\mathrm{e} = \mathrm{e}.$$

That is, $\mathbf{A}$ satisfies the equation $x(x\backslash \mathrm{e}) \approx \mathrm{e}$ and is therefore an ℓ-group. In this case, the operation $a \mapsto -a$ is the group inverse operation and α is therefore the identity map on $\mathbf{A}$, whence $\mathbf{A} = \mathbf{A}_\alpha$. On the other hand, if $\mathbf{A}$ is an integral residuated lattice, then $-a = \mathrm{e}$ for all $a \in A$ and α maps every element to the unit e, so $\mathbf{A}_\alpha$ is trivial. More generally, if $\mathbf{A}$ is integrally closed, then α and its image enjoy the following properties. Call a homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ between residuated lattices $\mathbf{A}$ and $\mathbf{B}$ e-*principal* provided that $h^{-1}(\mathrm{e}^{\mathbf{B}}) \subseteq {\downarrow}\mathrm{e}^{\mathbf{A}}$, cf. [41, Chap. 12.2].

5.1.8. PROPOSITION. *Let $\mathbf{A}$ be an integrally closed residuated lattice.*

1. *The map $\alpha\colon \mathbf{A} \to \mathbf{A}_\alpha$ is a surjective homomorphism of residuated lattices.*
2. *The residuated lattice $\mathbf{A}_\alpha$ is an ℓ-group.*
3. *Any homomorphism $h\colon \mathbf{A} \to \mathbf{G}$ with $\mathbf{G}$ an ℓ-group factors through the ℓ-group $\mathbf{A}_\alpha$.*
4. *The residuated lattice $\mathbf{A}_\alpha$ is, up to isomorphism, the unique* e-*principal homomorphic image of $\mathbf{A}$ which is an ℓ-group.*

Proof:
To establish Item 1 we first note that any nucleus on the induced partially ordered monoid of a residuated lattice preserves the monoidal structure and joins, see, e.g., [98, Thm. 3.34(2)]. By parts (i) and (ii) of Lemma 5.1.7, this nucleus also preserves the residual operations. Moreover, $\alpha(\mathrm{e}) = \mathrm{e}$. It therefore suffices to

show that α preserves binary meets. First note that since $(-a)(--a) \leq \mathrm{e}$, also $a(-a)(--a) \leq a$, and, since $b(-b) \leq \mathrm{e}$, it follows that $a(-a)b(-b)(--a) \leq a$. Similarly, $a(-a)b(-b)(--b) \leq b$, and hence

$$a(-a)b(-b)(--a \wedge --b) \leq a \wedge b \leq --(a \wedge b).$$

By residuation, $a(-a)b(-b)(--a \wedge --b)(-(a \wedge b)) \leq \mathrm{e}$, and hence, applying part (iii) of Lemma 5.1.7 twice, $(--a \wedge --b)(-(a \wedge b)) \leq \mathrm{e}$. By residuation again,

$$--a \wedge --b \leq --(a \wedge b).$$

Since α is order-preserving, $--(a \wedge b) = --a \wedge --b$, as desired.

To establish Item 2 we observe that by Item 1 $\mathbf{A}_\alpha$ is an integrally closed residuated lattice. But also for any $a \in A$, we have $--\alpha(a) = \alpha(\alpha(a)) = \alpha(a)$, so $\mathbf{A}_\alpha$ satisfies the equation $--x \approx x$ and is therefore an ℓ-group.

For Item 3 suppose that $h\colon \mathbf{A} \to \mathbf{G}$ is a homomorphism of residuated lattices with $\mathbf{G}$ an ℓ-group. We claim that $\ker \alpha \subseteq \ker h$. To see this let $a, b \in A$ be given such that $(a, b) \in \ker \alpha$, meaning that $--a = --b$. But then from the assumption that h is a homomorphism we obtain $--h(a) = --h(b)$. Since $\mathbf{G}$ is assumed to be an ℓ-group this entails that $h(a) = h(b)$, which implies $(a, b) \in \ker h$. By the universal property of quotients we obtain a homomorphism $\overline{h}\colon \mathbf{A}_\alpha \to \mathbf{G}$, such that $h = \overline{h} \circ \alpha$.

Finally, to establish Item 4 we first observe that by Items 1 and 2 the algebra $\mathbf{A}_\alpha$ is a homomorphic image of $\mathbf{A}$ which is an ℓ-group. Furthermore, we see that if $a \in A$ is such that $\alpha(a) = \mathrm{e}$ then $a \leq \mathrm{e}$, as α, being a nucleus on $\mathbf{A}$, is increasing. This shows that $\alpha\colon \mathbf{A} \twoheadrightarrow \mathbf{A}_\alpha$ is indeed e-principal. Now assume that $\mathbf{G}$ an ℓ-group and $h\colon \mathbf{A} \twoheadrightarrow \mathbf{G}$ is a surjective e-principal homomorphism of residuated lattices. Then by Item 3 we must have a, necessarily surjective, homomorphism $\overline{h}\colon \mathbf{A}_\alpha \twoheadrightarrow \mathbf{G}$. Now if $(a, b) \in A^2$ belongs to $\ker h$ then since $\mathbf{G}$ is an ℓ-group, and so in particular integrally closed, we have $h(a\backslash b) = h(a)\backslash h(b) = \mathrm{e}$. Consequently, as h is assumed to be e-principal we obtain that $a\backslash b \leq \mathrm{e}$. By Lemma 5.1.7(i) it follows that $\mathrm{e} \leq -(a\backslash b) = -b/-a$, whence $-a \leq -b$. A completely analogous argument shows that $-b \leq -a$, and hence that $-a = -b$. But then $--a = --b$, which shows that $(a, b) \in \ker \alpha$. Consequently, $\ker h \subseteq \ker \alpha$ from which we may conclude that $\overline{h}$ is injective and hence an isomorphism. $\square$

The following theorem may be seen as a version of an analogous theorem for ordered semi-groups see, e.g., [78], [92, Thm. XII.3.1] and [41, Thm. 12.5].

5.1.9. THEOREM. *Let* $\mathbf{A}$ *be a residuated lattice. Then the following are equivalent.*

1. *The residuated lattice* $\mathbf{A}$ *is integrally closed.*

2. *The residuated lattice* $\mathbf{A}$ *admits an* e*-principal homomorphic image which is an* ℓ*-group.*

Proof:
That Item 1 entails Item 2 follows from Proposition 5.1.8. To see that Item 2 entails Item 1, assume that $\mathbf{G}$ is an ℓ-group, and so in particular integrally closed, with an e-principal homomorphism $h\colon \mathbf{A} \twoheadrightarrow \mathbf{G}$. Then since h preservers the residuals, we immediately see that $h(a\backslash a) = h(a)\backslash h(a) = \mathrm{e}$, for all $a \in A$. Since h is e-principal this entails that $a\backslash a \leq \mathrm{e}$ for each $a \in A$, showing that $\mathbf{A}$ is integrally closed. □

The connection between integrally closed residuated lattices and ℓ-groups allows us to deduce that certain properties are enjoyed by the former by knowing that they are enjoyed by the latter. The following proposition serves as an example of this.

5.1.10. PROPOSITION. *Every integrally closed residuated lattice is torsion-free, i.e., satisfies the quasi-equation*

$$x^n \approx \mathrm{e} \implies x \approx \mathrm{e},$$

for every natural number $n \geq 1$.

Proof:
Let $\mathbf{A}$ be an integrally closed residuated lattice. We prove by induction on natural numbers $n \geq 1$ that $\mathbf{A}$ satisfies the quasi-equation $x^n \approx \mathrm{e} \implies x \approx \mathrm{e}$. The case $n = 1$ is trivial. For the inductive step, suppose that $n > 1$ and $a^n = \mathrm{e}$ for some $a \in A$. Then, since $\alpha\colon \mathbf{A} \to \mathbf{A}_\alpha$ is a homomorphism, $\alpha(a)^n = \alpha(a^n) = \alpha(\mathrm{e}) = \mathrm{e}$. But ℓ-groups are torsion-free, see, e.g., [6, Prop. 1.1.6(b)], so $\alpha(a) = \mathrm{e}$ and therefore $-a = ---a = -\alpha(a) = -\mathrm{e} = \mathrm{e}$. Moreover, since by assumption $a^n = \mathrm{e}$ we also have $a^{n-1} \leq -a$ by residuation. From this we see that

$$\mathrm{e} = (-a)a^n = (-a)aa^{n-1} \leq a^{n-1} \leq -a = \mathrm{e}.$$

Hence $a^{n-1} = \mathrm{e}$ and, by the induction hypothesis, $a = \mathrm{e}$. □

We turn our attention now to varieties of integrally closed residuated lattices. Given any class $\mathcal{K} \subseteq \mathcal{IcRL}$, we denote by $\mathcal{K}_\alpha$ the class $\{\mathbf{A}_\alpha \mid \mathbf{A} \in \mathcal{K}\} \subseteq \mathcal{LG}$, recalling that $\mathcal{LG}$ denotes the variety of ℓ-groups.

5.1.11. PROPOSITION. *Let $\mathcal{V}$ be any variety of integrally closed residuated lattices.*

1. *The class $\mathcal{V}_\alpha$ forms a variety of ℓ-groups.*

2. *If $\mathcal{V}$ is defined relative to $\mathcal{IcRL}$ by a set of equations E, then $\mathcal{V}_\alpha$ is defined relative to $\mathcal{LG}$ by E.*

Hence the map $\mathcal{V} \mapsto \mathcal{V}_\alpha$ is an interior operator on the lattice of subvarieties of $\mathcal{IcRL}$ whose image is the lattice of subvarieties of $\mathcal{LG}$.

Proof:
Let $\mathcal{V}$ be a variety of integrally closed residuated lattices defined relative to $\mathcal{IcRL}$ by a set of equations E and let $\mathcal{W}$ be the variety of ℓ-groups defined relative to $\mathcal{LG}$ by E. Clearly $\mathcal{W} = \mathcal{W}_\alpha \subseteq \mathcal{V}_\alpha$. But also each $\mathbf{A}_\alpha \in \mathcal{V}_\alpha$ is, by Proposition 5.1.8, an ℓ-group and a homomorphic image of $\mathbf{A} \in \mathcal{V}$. So $\mathcal{V}_\alpha \subseteq \mathcal{W}$.

The second item of the proposition then follows from the observation that $\mathcal{V}_\alpha = \mathcal{V}$ if, and only if, $\mathcal{V} \subseteq \mathcal{LG}$. □

5.2 A Glivenko theorem for ℓ-groups

In this section we will establish a correspondence between the validity of equations in an integrally closed residuated lattice $\mathbf{A}$ and the corresponding ℓ-group $\mathbf{A}_\alpha$, analogous to the one between Heyting algebras and Boolean algebras first established by Glivenko [126], see also [16, Thm. IX.5.3]. In addition, we will consider the relationship between the equational theories of integrally closed and integral residuated lattices as well as the quasi-equational theories of integrally closed residuated lattices and ℓ-groups.

We will here denote by $\mathbf{Tm}$ the term algebra for residuated lattices over a fixed countably infinite set of variables.

5.2.1. Lemma. *For any integrally closed residuated lattice $\mathbf{A}$ and $s, t \in \mathrm{Tm}$,*

$$\mathbf{A}_\alpha \models s \le t \quad \textit{if, and only if,} \quad \mathbf{A} \models --s \le --t.$$

Proof:
Suppose first that $\mathbf{A} \models --s \le --t$. Since $\mathbf{A}_\alpha$ is a homomorphic image of $\mathbf{A}$, also $\mathbf{A}_\alpha \models --s \le --t$. But $\mathbf{A}_\alpha$ is an ℓ-group, and so $\mathbf{A}_\alpha \models s \le t$.

Now suppose that $\mathbf{A} \not\models --s \le --t$. Then there exists a homomorphism $\nu\colon \mathbf{Tm} \to \mathbf{A}$ such that $\nu(--s) \not\le \nu(--t)$. Since α is a homomorphism from $\mathbf{A}$ to $\mathbf{A}_\alpha$, we obtain a homomorphism $\alpha \circ \nu\colon \mathbf{Tm} \to \mathbf{A}_\alpha$ such that

$$(\alpha \circ \nu)(s) = \alpha(\nu(s)) = --\nu(s) = \nu(--s) \not\le \nu(--t) = \alpha(\nu(t)) = (\alpha \circ \nu)(t).$$

Hence $\mathbf{A}_\alpha \not\models s \le t$ as required. □

Following [100], we will say that a variety $\mathcal{V}$ of residuated lattices admits the (*equational*) *Glivenko property* with respect to a variety $\mathcal{W}$ of residuated lattices if for all $s, t \in \mathrm{Tm}$, both of the equivalences

$$\mathcal{V} \models \mathrm{e}/(s\backslash\mathrm{e}) \le \mathrm{e}/(t\backslash\mathrm{e}) \quad \text{if, and only if,} \quad \mathcal{W} \models s \le t,$$

and

$$\mathcal{V} \models (\mathrm{e}/s)\backslash \mathrm{e} \leq (\mathrm{e}/t)\backslash \mathrm{e} \quad \text{if, and only if,} \quad \mathcal{W} \models s \leq t,$$

are satisfied. It may then easily be observed that if $\mathcal{V}$ is a variety of e-cyclic residuated lattices, this simplifies to the condition,

$$\mathcal{V} \models --s \leq --t \quad \text{if, and only if,} \quad \mathcal{W} \models s \leq t.$$

Let us also note the following useful consequence of this property.

5.2.2. Proposition. *If $\mathcal{V}$ is a variety of residuated lattices admitting the Glivenko property with respect to a variety of residuated lattices $\mathcal{W}$, then for all $s \in \mathrm{Tm}$,*

$$\mathcal{V} \models s \leq \mathrm{e} \quad \textit{if, and only if,} \quad \mathcal{W} \models s \leq \mathrm{e}.$$

Proof:
The equation $x \leq \mathrm{e}/(x\backslash \mathrm{e})$ and quasi-equation $x \leq \mathrm{e} \implies \mathrm{e}/(x\backslash \mathrm{e}) \leq \mathrm{e}$ are valid in all residuated lattices. Hence for all $s \in \mathrm{Tm}$,

$$\begin{aligned} \mathcal{W} \models s \leq \mathrm{e} &\iff \mathcal{V} \models \mathrm{e}/(s\backslash \mathrm{e}) \leq \mathrm{e}/(\mathrm{e}\backslash \mathrm{e}) \\ &\iff \mathcal{V} \models \mathrm{e}/(s\backslash \mathrm{e}) \leq \mathrm{e} \\ &\iff \mathcal{V} \models s \leq \mathrm{e}, \end{aligned}$$

establishing the proposition. □

For integrally closed residuated lattices, we obtain the following pivotal result.

5.2.3. Theorem. *Any variety $\mathcal{V}$ of integrally closed residuated lattices admits the Glivenko property with respect to the variety of ℓ-groups $\mathcal{V}_\alpha$.*

Proof:
Suppose that $\mathcal{V}_\alpha \models s \leq t$. For any $\mathbf{A} \in \mathcal{V}$, it follows that $\mathbf{A}_\alpha \models s \leq t$, and hence $\mathbf{A} \models --s \leq --t$, by Lemma 5.2.1. So $\mathcal{V} \models --s \leq --t$. The other implication follows from the fact that $\mathcal{V}_\alpha \subseteq \mathcal{V}$ and $\mathcal{V}_\alpha \models --x \approx x$. □

In particular, the decision problem for the equational theory of a variety of integrally closed residuated lattices $\mathcal{V}$ is at least as difficult as the decision problem for the equational theory of the corresponding variety of ℓ-groups $\mathcal{V}_\alpha$.

Applying Theorem 5.2.3 to the variety of all integrally closed residuated lattices we obtain the following.

5.2.4. Corollary. *The variety of integrally closed residuated lattices admits the Glivenko property with respect to the variety of ℓ-groups, and hence for all $s \in \mathrm{Tm}$,*

$$\mathcal{LG} \models s \leq \mathrm{e} \quad \textit{if, and only if,} \quad \mathcal{I}c\mathcal{RL} \models s \leq \mathrm{e}.$$

As the next result demonstrates, $\mathcal{IcRL}$ is in fact the largest variety of residuated lattices which admits the Glivenko property with respect to the variety of all ℓ-groups.

5.2.5. THEOREM. *Let $\mathcal{V}$ be a variety of integrally closed residuated lattices that is axiomatized relative to $\mathcal{IcRL}$ by equations of the form $s \leq \mathrm{e}$. Then $\mathcal{V}$ is the largest variety of residuated lattices admitting the Glivenko property with respect to the variety $\mathcal{V}_\alpha$.*

Proof:
By Theorem 5.2.3, $\mathcal{V}$ admits the Glivenko property with respect to $\mathcal{V}_\alpha$. Now suppose that $\mathcal{W}$ is any variety of residuated lattices admitting the Glivenko property with respect to $\mathcal{V}_\alpha$. By assumption, $\mathcal{V}$ is axiomatized relative to $\mathcal{IcRL}$ by a set of equations E of the form $s \leq \mathrm{e}$, so, by Proposition 5.1.11, the variety $\mathcal{V}_\alpha$ is axiomatized relative to $\mathcal{LG}$ by E. But also by Proposition 5.2.2, all members of the variety $\mathcal{W}$ must satisfy all the equations in E as well as $x\backslash x \leq \mathrm{e}$. So $\mathcal{W} \subseteq \mathcal{V}$. $\square$

Applying Theorem 5.2.5 to the varieties $\mathcal{IRL}$ and $\mathcal{Triv}$ of all integral and all trivial residuated lattices, respectively, we obtain the following.

5.2.6. COROLLARY.

1. *The variety $\mathcal{IcRL}$ is the largest variety of residuated lattices that admits the Glivenko property with respect to the variety $\mathcal{LG}$.*

2. *The variety $\mathcal{IRL}$ is the largest variety of residuated lattices that admits the Glivenko property with respect to the variety $\mathcal{Triv}$.*

It is not the case that every variety $\mathcal{V}$ of integrally closed residuated lattices is the largest variety of residuated lattices admitting the Glivenko property with respect to the corresponding variety $\mathcal{V}_\alpha$ of ℓ-groups. For example, if $\mathcal{V}$ is the variety of commutative integrally closed residuated lattices then $\mathcal{V}_\alpha$ is the variety of Abelian ℓ-groups. However, for any integral residuated lattice $\mathbf{A}$, the ℓ-group $\mathbf{A}_\alpha$ is trivial, so the largest variety admitting the Glivenko property with respect to the variety of Abelian ℓ-groups must contain all integral residuated lattices.

5.2.1 Two additional translations

We conclude this section by describing further syntactic relationships existing between the variety $\mathcal{IcRL}$ and the varieties $\mathcal{IRL}$ and $\mathcal{LG}$. Recall, e.g., from [170, Sec. 5] that for any residuated lattice $\mathbf{A}$, the *negative cone* of $\mathbf{A}$ is the residuated lattice $\mathbf{A}^-$ with universe $A^- = \{a \in A \mid a \leq \mathrm{e}\}$, monoid and lattice operations inherited from $\mathbf{A}$, and residuals defined by

$$a\backslash_- b := (a\backslash b) \wedge \mathrm{e} \quad \text{and} \quad b/_- a := (b/a) \wedge \mathrm{e},$$

for $a, b \in A^-$. For each term $s \in \mathrm{Tm}$ we define a corresponding term s^- by the following recursion:

$$\mathrm{e}^- = \mathrm{e} \quad \text{and} \quad x^- = x \wedge \mathrm{e} \quad \text{for each variable } x,$$

$$(s * t)^- = s^- * t^- \quad \text{for } * \in \{\wedge, \vee, \cdot\},$$
$$(s \backslash t)^- = (s^- \backslash t^-) \wedge \mathrm{e} \quad \text{and} \quad (s/t)^- = (s^-/t^-) \wedge \mathrm{e}.$$

It is then straightforward, see, e.g., [170, Lem. 5.10], to prove that for any residuated lattice $\mathbf{A}$ and $s, t \in \mathrm{Tm}$,

$$\mathbf{A}^- \models s \approx t \quad \text{if, and only if,} \quad \mathbf{A} \models s^- \approx t^-.$$

Since the negative cone of an integrally closed residuated lattice is integral and an integral residuated lattice is integrally closed, we obtain the following result.

5.2.7. Proposition. *For any $s, t \in \mathrm{Tm}$,*

$$\mathcal{IRL} \models s \approx t \quad \textit{if, and only if,} \quad \mathcal{I}c\mathcal{RL} \models s^- \approx t^-.$$

Thus the decision problem for the equational theory of integrally closed residuated lattices is at least as difficult as the decision problem for integral residuated lattices.

Corollary 5.2.4 shows how the equational theory of ℓ-groups may be interpreted in the equational theory of integrally closed residuated lattices. We now show that also the quasi-equational theory of ℓ-groups can be interpreted in the quasi-equational theory of integrally closed residuated lattices. To this end we define for each term $s \in \mathrm{Tm}$ a corresponding term s^α by the following recursion:

$$\mathrm{e}^\alpha = \mathrm{e} \quad \text{and} \quad x^\alpha = --x \quad \text{for each variable } x,$$

$$(s * t)^\alpha = s^\alpha * t^\alpha \quad \text{for } * \in \{\wedge, \backslash, /\},$$
$$(s \cdot t)^\alpha = --(s^\alpha \cdot t^\alpha) \quad \text{and} \quad (s \vee t)^\alpha = --(s^\alpha \vee t^\alpha).$$

5.2.8. Proposition (cf. [168, Prop. 28]). *For each quasi-equation q in the language of residuated lattices there is a quasi-equation q^α effectively computable from q such that*

$$\mathcal{LG} \models q \quad \textit{if, and only if,} \quad \mathcal{I}c\mathcal{RL} \models q^\alpha.$$

Proof:
Suppose that q is the quasi-equation

$$s_1 \approx t_1 \text{ and } \ldots \text{ and } s_m \approx t_m \implies s_{m+1} \approx t_{m+1},$$

with $\{x_1, \dots, x_k\}$ the set of variables occurring in q. Let ρ be the following conjunction of equations,

$$--x_1 \approx x_1 \text{ and } \dots \text{ and } --x_k \approx x_k.$$

We then let q^α be the quasi-equation,

$$\rho \text{ and } s_1^\alpha \approx t_1^\alpha \text{ and } \dots \text{ and } s_m^\alpha \approx t_m^\alpha \implies s_{m+1}^\alpha \approx t_{m+1}^\alpha.$$

A straightforward induction shows that for any $s \in \mathrm{Tm}$ the term function $(s^\alpha)^{\mathbf{A}}$ on $\mathbf{A}$ induced by the term s^α and the term function $s^{\mathbf{A}_\alpha}$ on $\mathbf{A}_\alpha$ induced by the term s agree on their common domain, viz., $\{a \in A : --a = a\}$.

Now assume that $\mathcal{LG} \models q$. To see that $\mathcal{IcRL} \models q^\alpha$ let $\mathbf{A}$ be an integrally closed residuated lattice and let $a_1, \dots, a_k \in A$ be such that $--a_i = a_i$, for all $i \in \{1, \dots, k\}$, and $(s_j^\alpha)^{\mathbf{A}}(\vec{a}) = (t_j^\alpha)^{\mathbf{A}}(\vec{a})$ for all $j \in \{1, \dots, m\}$. Then by the above observation we have that $s_j^{\mathbf{A}_\alpha}(\vec{a}) = t_j^{\mathbf{A}_\alpha}(\vec{a})$, for all $j \in \{1, \dots, m\}$. Consequently, as $\mathcal{LG} \models q$ we have that $\mathbf{A}_\alpha \models q$ from which we may conclude that

$$(s_{m+1}^\alpha)^{\mathbf{A}}(\vec{a}) = s_{m+1}^{\mathbf{A}_\alpha}(\vec{a}) = t_{m+1}^{\mathbf{A}_\alpha}(\vec{a}) = (t_{m+1}^\alpha)^{\mathbf{A}}(\vec{a}).$$

This shows that $\mathbf{A} \models q^\alpha$ and therefore also that $\mathcal{IcRL} \models q^\alpha$. To establish the converse implication we observe that since all ℓ-groups satisfy the equation $x \approx --x$ it follows that $\mathcal{LG} \models s \approx s^\alpha$ for each $s \in \mathrm{Tm}$. Consequently, the quasi-equations q and q^α are equivalent on any ℓ-group and so as $\mathcal{LG} \subseteq \mathcal{IcRL}$ we obtain that

$$\mathcal{IcRL} \models q^\alpha \implies \mathcal{LG} \models q,$$

as desired. $\square$

This shows that the word problem for integrally closed residuated lattices is at least as difficult as the word problem for ℓ-groups.

5.3 Proof theory and decidability

In this section we will construct a sequent calculus for the equational theory of integrally closed residuated lattices by adding a *non-standard weakening rule* to the standard sequent calculus RL, presented in Figure 5.1, for the equational theory of residuated lattices. We prove that this calculus admits cut-elimination and obtain as a consequence a proof of the decidability, indeed PSPACE-completeness, of the equational theory of integrally closed residuated lattices.

In the following we shall by a (*single-succedent*) *sequent* understand an expression of the form $\Gamma \Rightarrow t$ where Γ is a finite, possible empty, sequence of terms $s_1, \dots, s_n \in \mathrm{Tm}$ and $t \in \mathrm{Tm}$. Sequent rules, calculi, and derivations are defined in the usual way, see, e.g., [98, 199] or Chapter 2. We say that

Identity Axioms

$$\frac{}{s \Rightarrow s}\ (\textsc{id})$$

Cut Rule

$$\frac{\Gamma_2 \Rightarrow s \quad \Gamma_1, s, \Gamma_3 \Rightarrow u}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow u}\ (\textsc{cut})$$

Left Operation Rules

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow u}{\Gamma_1, \mathrm{e}, \Gamma_2 \Rightarrow u}\ (\mathrm{e}{\Rightarrow})$$

$$\frac{\Gamma_2 \Rightarrow s \quad \Gamma_1, t, \Gamma_3 \Rightarrow u}{\Gamma_1, t/s, \Gamma_2, \Gamma_3 \Rightarrow u}\ ({/}{\Rightarrow})$$

$$\frac{\Gamma_2 \Rightarrow s \quad \Gamma_1, t, \Gamma_3 \Rightarrow u}{\Gamma_1, \Gamma_2, s\backslash t, \Gamma_3 \Rightarrow u}\ ({\backslash}{\Rightarrow})$$

$$\frac{\Gamma_1, s, t, \Gamma_2 \Rightarrow u}{\Gamma_1, s \cdot t, \Gamma_2 \Rightarrow u}\ ({\cdot}{\Rightarrow})$$

$$\frac{\Gamma_1, s, \Gamma_2 \Rightarrow u}{\Gamma_1, s \wedge t, \Gamma_2 \Rightarrow u}\ ({\wedge}{\Rightarrow})_1$$

$$\frac{\Gamma_1, t, \Gamma_2 \Rightarrow u}{\Gamma_1, s \wedge t, \Gamma_2 \Rightarrow u}\ ({\wedge}{\Rightarrow})_2$$

$$\frac{\Gamma_1, s, \Gamma_2 \Rightarrow u \quad \Gamma_1, t, \Gamma_2 \Rightarrow u}{\Gamma_1, s \vee t, \Gamma_2 \Rightarrow u}\ ({\vee}{\Rightarrow})$$

Right Operation Rules

$$\frac{}{\Rightarrow \mathrm{e}}\ ({\Rightarrow}\mathrm{e})$$

$$\frac{\Gamma, s \Rightarrow t}{\Gamma \Rightarrow t/s}\ ({\Rightarrow}{/})$$

$$\frac{s, \Gamma \Rightarrow t}{\Gamma \Rightarrow s\backslash t}\ ({\Rightarrow}{\backslash})$$

$$\frac{\Gamma_1 \Rightarrow s \quad \Gamma_2 \Rightarrow t}{\Gamma_1, \Gamma_2 \Rightarrow s \cdot t}\ ({\Rightarrow}{\cdot})$$

$$\frac{\Gamma \Rightarrow s}{\Gamma \Rightarrow s \vee t}\ ({\Rightarrow}{\vee})_1$$

$$\frac{\Gamma \Rightarrow t}{\Gamma \Rightarrow s \vee t}\ ({\Rightarrow}{\vee})_2$$

$$\frac{\Gamma \Rightarrow s \quad \Gamma \Rightarrow t}{\Gamma \Rightarrow s \wedge t}\ ({\Rightarrow}{\wedge})$$

Figure 5.1: The Sequent Calculus RL

a sequent $s_1, \ldots, s_n \Rightarrow t$ is *valid* in a class $\mathcal{K}$ of residuated lattices, written as $\models_{\mathcal{K}} s_1, \ldots, s_n \Rightarrow t$, if $\mathcal{K} \models s_1 \cdots s_n \leq t$, where the empty product in a residuated lattice is understood to be the monoidal unit e. A sequent is derivable in the calculus RL if, and only if, it is valid in the variety $\mathcal{RL}$ of all residuated lattices, see, e.g., [98, 199], and RL admits cut-elimination, i.e., there is an algorithm that transforms any derivation of a sequent in RL into a derivation of the sequent that does not use the cut-rule.

We define IcRL to be the sequent calculus consisting of the rules of RL together with the non-standard rule

$$\frac{\Gamma, \Pi \Rightarrow u \quad \models_{\mathcal{LG}} \Delta \Rightarrow \mathrm{e}}{\Gamma, \Delta, \Pi \Rightarrow u}\ (\mathcal{LG}\text{-w})$$

where the premise $\models_{\mathcal{LG}} \Delta \Rightarrow \mathrm{e}$ may be understood as a side-condition for weakening that is decidable [158], indeed co-NP-complete [99]. In fact, the condition $\models_{\mathcal{LG}} \Delta \Rightarrow \mathrm{e}$ can be understood proof-theoretically as requiring a derivation in some calculus for ℓ-groups, such as the hypersequent calculus admitting cut-elimination provided in [99]. Thus the rule ($\mathcal{LG}$-W) may be viewed as a special case of the standard weakening rule

$$\frac{\Gamma, \Pi \Rightarrow u}{\Gamma, \Delta, \Pi \Rightarrow u}\ \text{(W)}$$

in which the sequence Δ is restricted by some (decidable) side-condition.

5.3.1. PROPOSITION. *A sequent is derivable in the calculus* IcRL *if, and only if, it is valid in all integrally closed residuated lattices.*

Proof:
Assume first that $s_1, \ldots, s_n \Rightarrow t$ is a sequent valid in all integrally closed residuated lattices. We show that $s_1, \ldots, s_n \Rightarrow t$ is derivable in the calculus IcRL via a Lindenbaum-Tarski algebra construction. Namely, as usual, it is easy to verify that the binary relation Θ on Tm defined by

$$u \,\Theta\, v \quad \text{if, and only if,} \quad \text{the sequents } u \Rightarrow v \text{ and } v \Rightarrow u \text{ are derivable in } \mathsf{IcRL},$$

is a congruence on the term algebra $\mathbf{Tm}$. Moreover, since the sequent $x\backslash x \Rightarrow \mathrm{e}$ is derivable in IcRL, the quotient $\mathbf{Tm}/\Theta$ will be an integrally closed residuated lattice which must satisfy

$$u/\Theta \leq v/\Theta \quad \text{if, and only if,} \quad \text{the sequents } u \Rightarrow v \text{ is derivable in } \mathsf{IcRL}.$$

Consider the homomorphism from $\mathbf{Tm}$ to $\mathbf{Tm}/\Theta$ mapping each term u to the equivalence class u/Θ. Since by assumption the equation $s_1 \cdots s_n \leq t$ is true in the residuated lattice $\mathbf{Tm}/\Theta$, it follows that $s_1 \cdots s_n/\Theta \leq t/\Theta$ and hence that the sequent $s_1 \cdots s_n \Rightarrow t$ is derivable in IcRL. An application of (CUT) with the derivable sequent $s_1, \ldots, s_n \Rightarrow s_1 \cdots s_n$ shows that also $s_1, \ldots, s_n \Rightarrow t$ is derivable in IcRL.

Conversely, to show that any sequent derivable in IcRL is valid in all integrally closed residuated lattices, we recall, e.g., from [98, 199], that the rules of RL preserve validity of sequents in $\mathcal{RL}$ and it therefore suffices to show that the rule ($\mathcal{LG}$-W) preserves validity in $\mathcal{IcRL}$. Suppose that $\models_{\mathcal{IcRL}} \Gamma, \Pi \Rightarrow u$ and $\models_{\mathcal{LG}} \Delta \Rightarrow \mathrm{e}$. Writing s_1, s_2, and t for the products of the terms in Γ, Π, and Δ, respectively, we have that $\mathcal{IcRL} \models s_1 s_2 \leq u$ and $\mathcal{LG} \models t \leq \mathrm{e}$. By Corollary 5.2.4, we obtain that $\mathcal{IcRL} \models t \leq \mathrm{e}$ and hence that $\mathcal{IcRL} \models s_1 t s_2 \leq u$, yielding that, $\models_{\mathcal{IcRL}} \Gamma, \Delta, \Pi \Rightarrow u$, as desired. $\square$

5.3.2. PROPOSITION. *The calculus* IcRL *admits cut-elimination.*

Proof:
It suffices, as usual, to prove that if there are cut-free derivations d_1 of the sequent $\Gamma_2 \Rightarrow s$ and d_2 of the sequent $\Gamma_1, s, \Gamma_3 \Rightarrow u$ in IcRL, i.e.,

$$\dfrac{\dfrac{\vdots\, d_1}{\Gamma_2 \Rightarrow s} \quad \dfrac{\vdots\, d_2}{\Gamma_1, s, \Gamma_3 \Rightarrow u}}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow u}\ (\textsc{cut})$$

then there is a cut-free derivation of the sequent $\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow u$ in IcRL. We proceed by induction on the lexicographically ordered pair $\langle c, h\rangle$ where c is the term complexity of s and h is the sum of the heights of the derivations d_1 and d_2. The cases where the last steps in the derivations d_1 and d_2 are applications of rules of RL are standard, see, e.g., [98, Chap. 4.1]. We therefore just consider the cases where the last step is an application of the rule ($\mathcal{LG}$-W). Suppose first that $\Gamma_2 = \Pi_1, \Delta, \Pi_2$ and d_1 ends with

$$\dfrac{\dfrac{\vdots\, d_1'}{\Pi_1, \Pi_2 \Rightarrow s} \quad \models_{\mathcal{LG}} \Delta \Rightarrow \mathrm{e}}{\Pi_1, \Delta, \Pi_2 \Rightarrow s}\ (\mathcal{LG}\text{-W})$$

By the induction hypothesis, we obtain a cut-free derivation d_3 of the sequent $\Gamma_1, \Pi_1, \Pi_2, \Gamma_3 \Rightarrow u$ in IcRL, and hence a cut-free derivation in IcRL ending with

$$\dfrac{\dfrac{\vdots\, d_3}{\Gamma_1, \Pi_1, \Pi_2, \Gamma_3 \Rightarrow u} \quad \models_{\mathcal{LG}} \Delta \Rightarrow \mathrm{e}}{\Gamma_1, \Pi_1, \Delta, \Pi_2, \Gamma_3 \Rightarrow u}\ (\mathcal{LG}\text{-W})$$

Suppose next that $\Gamma_3 = \Pi_1, \Delta, \Pi_2$ and d_2 ends with

$$\dfrac{\dfrac{\vdots\, d_2'}{\Gamma_1, s, \Pi_1, \Pi_2 \Rightarrow u} \quad \models_{\mathcal{LG}} \Delta \Rightarrow \mathrm{e}}{\Gamma_1, s, \Pi_1, \Delta, \Pi_2 \Rightarrow u}\ (\mathcal{LG}\text{-W})$$

By the induction hypothesis, we obtain a cut-free derivation d_3 of the sequent $\Gamma_1, \Gamma_2, \Pi_1, \Pi_2 \Rightarrow u$ in IcRL, and hence a cut-free derivation in IcRL ending with

$$\dfrac{\dfrac{\vdots\, d_3}{\Gamma_1, \Gamma_2, \Pi_1, \Pi_2 \Rightarrow u} \quad \models_{\mathcal{LG}} \Delta \Rightarrow \mathrm{e}}{\Gamma_1, \Gamma_2, \Pi_1, \Delta, \Pi_2 \Rightarrow u}\ (\mathcal{LG}\text{-W})$$

The analogous case where $\Gamma_1 = \Pi_1, \Delta, \Pi_2$ is very similar.

Suppose finally that $\Gamma_1, s, \Gamma_3 = \Pi_1, \Delta_1, s, \Delta_2, \Pi_2$ and d_2 ends with

$$\dfrac{\dfrac{\vdots\, d_2'}{\Pi_1, \Pi_2 \Rightarrow u} \quad \models_{\mathcal{LG}} \Delta_1, s, \Delta_2 \Rightarrow \mathrm{e}}{\Pi_1, \Delta_1, s, \Delta_2, \Pi_2 \Rightarrow u}\ (\mathcal{LG}\text{-W})$$

By Proposition 5.3.1, we have that $\models_{\mathcal{IcRL}} \Gamma_2 \Rightarrow s$ and hence that $\models_{\mathcal{LG}} \Gamma_2 \Rightarrow s$. But then also $\models_{\mathcal{LG}} \Delta_1, \Gamma_2, \Delta_2 \Rightarrow \mathrm{e}$ and so we obtain a cut-free derivation in IcRL ending with

$$\dfrac{\dfrac{\vdots\, d_2'}{\Pi_1, \Pi_2 \Rightarrow u} \quad \models_{\mathcal{LG}} \Delta_1, \Gamma_2, \Delta_2 \Rightarrow \mathrm{e}}{\Pi_1, \Delta_1, \Gamma_2, \Delta_2, \Pi_2 \Rightarrow u}\ (\mathcal{LG}\text{-w})$$

This takes care of all the possible cases and so concludes the proof. □

The cut-elimination argument of Proposition 5.3.2 applies also to sequent calculi for other varieties of integrally closed residuated lattices. First, let $\mathcal{V}$ be any variety of residuated lattices axiomatized relative to $\mathcal{RL}$ by a set of equations in the $\{\vee, \cdot, \mathrm{e}\}$-reduct of the language of residuated lattices. It is shown in [97, Sec. 3] that $\mathcal{V}$ can then be axiomatized by so-called *simple* equations, viz., equations of the form $s \leq t_1 \vee \ldots \vee t_n$ where each of $s, t_1, \ldots, t_n$ is either e or a product of variables and s contains at most one occurrence of any variable. Moreover, a sequent calculus for the equational theory of $\mathcal{V}$ that admits cut-elimination is obtained by adding to the calculus RL for each such equation $s \leq t_1 \vee \ldots \vee t_n$, a so-called *simple* rule

$$\dfrac{\Gamma, \Psi(t_1), \Pi \Rightarrow u \quad \ldots \quad \Gamma, \Psi(t_n), \Pi \Rightarrow u}{\Gamma, \Psi(s), \Pi \Rightarrow u}$$

where $\Psi(\mathrm{e})$ is the empty sequence and $\Psi(x_1 \cdots x_m)$, for not necessarily distinct variables $x_1, \ldots, x_m$, is the sequence of meta-variables $\Gamma_{x_1}, \ldots, \Gamma_{x_m}$. We then obtain a sequent calculus for the equational theory of the variety $\mathcal{W}$ of integrally closed members of $\mathcal{V}$ that also admits cut-elimination by adding the rule

$$\dfrac{\Gamma, \Pi \Rightarrow u \quad \models_{\mathcal{W}_\alpha} \Delta \Rightarrow \mathrm{e}}{\Gamma, \Delta, \Pi \Rightarrow u}\ (\mathcal{W}_\alpha\text{-w})$$

In particular, a sequent calculus for the equational theory of commutative integrally closed residuated lattices is obtained by adding to IcRL the (left) exchange rule

$$\dfrac{\Gamma_1, \Pi_2, \Pi_1, \Gamma_2 \Rightarrow u}{\Gamma_1, \Pi_1, \Pi_2, \Gamma_2 \Rightarrow u}\ (\textsc{el})$$

and replacing $\mathcal{LG}$ with the variety $\mathcal{AbLG}$ of Abelian ℓ-groups in the rule ($\mathcal{LG}$-w).

We use the cut-elimination of Proposition 5.3.2 to establish the decidability of the equational theory of $\mathcal{IcRL}$.

5.3.3. Theorem. *The equational theory of integrally closed residuated lattices is decidable, indeed PSPACE-complete.*

Proof:
For PSPACE-hardness, it suffices to recall that the equational theory of integral residuated lattices is PSPACE-complete [160, Thm. 5.2] and consider the translation described in Proposition 5.2.7. For inclusion, it suffices by Savitch's theorem [230] to observe that a non-deterministic PSPACE algorithm for deciding validity of sequents is obtained by guessing and checking a cut-free derivation in IcRL, recording only the branch of the derivation from the root to the current point. Note that for the application of the rule ($\mathcal{LG}$-W), we use the fact that the equational theory of $\mathcal{LG}$ is co-NP complete [99] and therefore in PSPACE. □

The decidability of the equational theory of integrally closed residuated lattices should be contrasted with the following fact.

5.3.4. PROPOSITION. *The quasi-equational theory of integrally closed residuated lattice is undecidable.*

Proof:
Since the word problem for ℓ-groups is undecidable [125] so is the quasi-equational theory of ℓ-groups. Consequently, we obtain from Proposition 5.2.8 that the quasi-equational theory of $\mathcal{IcRL}$ is likewise undecidable. □

We continue with a few more consequences of Propositions 5.3.1 and 5.3.2.

5.3.5. PROPOSITION. *Any freely generated integrally closed residuated lattice satisfies Whitman's condition*

$$x_1 \wedge x_2 \leq y_1 \vee y_2 \implies x_1 \wedge x_2 \leq y_1 \text{ or } x_1 \wedge x_2 \leq y_2 \text{ or } x_1 \leq y_1 \vee y_2 \text{ or } x_2 \leq y_1 \vee y_2.$$

Proof:
Let $\mathbf{F}(\kappa)$ be the free integrally closed residuated lattice on κ-many generators and let s_1, s_2, t_1, t_2 be residuated lattice terms in at most κ-many variables. If in $\mathbf{F}(\kappa)$ we have $s_1^{\mathbf{F}(\kappa)} \wedge s_2^{\mathbf{F}(\kappa)} \leq t_1^{\mathbf{F}(\kappa)} \vee t_2^{\mathbf{F}(\kappa)}$, then $\mathcal{IcRL} \models s_1 \wedge s_2 \leq t_1 \vee t_2$, and so by Proposition 5.3.1 the sequent $s_1 \wedge s_2 \Rightarrow t_1 \vee t_2$ must be derivable in the calculus IcRL. Consequently, by Proposition 5.3.2 there must be a cut-free derivation of the sequent $s_1 \wedge s_2 \Rightarrow t_1 \vee t_2$ in the calculus IcRL. By inspecting the rules of IcRL we see that the last rule applied in a cut-free derivation of the sequent $s_1 \wedge s_2 \Rightarrow t_1 \vee t_2$ must either be one of the rules $(\wedge\Rightarrow)_1$ or $(\wedge\Rightarrow)_2$, or one of the rules $(\Rightarrow\vee)_1$ or $(\Rightarrow\vee)_2$. In the former case we obtain that either the sequent $s_1 \Rightarrow t_1 \vee t_2$ or the sequent $s_2 \Rightarrow t_1 \vee t_2$ is derivable in IcRL. In the latter case we obtain that either the sequent $s_1 \wedge s_2 \Rightarrow t_1$ or the sequent $s_1 \wedge s_2 \Rightarrow t_2$ is derivable in IcRL. Consequently, by Proposition 5.3.1 we have that there must be $i \in \{1, 2\}$ such that $\mathcal{IcRL} \models s_1 \wedge s_2 \leq t_i$ or $\mathcal{IcRL} \models s_i \leq t_1 \vee t_2$. But then we may conclude that $s_1^{\mathbf{F}(\kappa)} \wedge s_2^{\mathbf{F}(\kappa)} \leq t_i^{\mathbf{F}(\kappa)}$ or $s_i^{\mathbf{F}(\kappa)} \leq t_1^{\mathbf{F}(\kappa)} \vee t_2^{\mathbf{F}(\kappa)}$ for some $i \in \{1, 2\}$,

showing that the algebra $\mathbf{F}(\kappa)$ satisfies Whitman's condition. $\Box$

Following [98, Chap. 5.1.1] we say that a variety of residuated lattices $\mathcal{V}$ enjoys *the disjunction property* provided that

$$\mathcal{V} \models \mathrm{e} \leq s \vee t \quad \text{implies} \quad \mathcal{V} \models \mathrm{e} \leq s \text{ or } \mathcal{V} \models \mathrm{e} \leq t,$$

for all residuated lattice terms s, t. By an argument very similar to the proof of Proposition 5.3.5 we obtain the following result first established, in a much more general context, by Horčík and Terui via an algebraic argument.

5.3.6. Proposition ([160, Sec. 3]). *The variety $\mathcal{IcRL}$ enjoys the disjunction property.*

Proof:
Let s, t be residuated lattice terms such that $\mathcal{IcRL} \models \mathrm{e} \leq s \vee t$. Then by Proposition 5.3.1 the sequent $\mathrm{e} \Rightarrow s \vee t$ is derivable in the calculus IcRL. Consequently, by Proposition 5.3.2 there must be a cut-free derivation of the sequent $\mathrm{e} \Rightarrow s \vee t$ in the calculus IcRL. By inspecting the rules of IcRL we see that the last rule applied in a cut-free derivation of the sequent $\mathrm{e} \Rightarrow s \vee t$ must be either the rule $(\mathrm{e} \Rightarrow)$, or one of the rules $(\Rightarrow \vee)_1$ or $(\Rightarrow \vee)_2$. In the latter case we may immediately conclude that either the sequent $\mathrm{e} \Rightarrow s$ or the sequent $\mathrm{e} \Rightarrow t$ is derivable in IcRL. In the former case we obtain that the sequent $\Rightarrow s \vee t$ has a cut-free derivation in the calculus IcRL, which by the same reasoning as above can only be the case if the sequent $\Rightarrow s$ or the sequent $\Rightarrow t$ is derivable in IcRL. In all cases Proposition 5.3.1 then yields that $\mathcal{IcRL} \models \mathrm{e} \leq s$ or $\mathcal{IcRL} \models \mathrm{e} \leq t$. $\Box$

We conclude this section by discussing how some simpler proof formalisms fail to capture the equational theory of $\mathcal{IcRL}$ in an analytic manner. Evidently, the equations $x \backslash x \approx \mathrm{e}$ and $x/x \approx \mathrm{e}$ belong to the class $\mathcal{N}_2$ described in [59], but are not acyclic in the sense defined there. Therefore, the method for constructing equivalent analytic sequent calculi in that paper does not apply. Indeed, the defining equations for $\mathcal{IcRL}$ cannot be equivalent to any set of analytic structural sequent rules as defined in [59] and including the simple rules of [97]. If this were the case, then, by [59, Thm. 6.3], the variety $\mathcal{IcRL}$ would be closed under MacNeille completions. However, by Lemma 5.1.3, any bounded, hence in particular complete, integrally closed residuated lattice is integral. Consequently, the completion of an integrally closed residuated lattice $\mathbf{A}$ will be integrally closed only if $\mathbf{A}$ is already integral. This shows that the variety $\mathcal{IcRL}$ is not closed under MacNeille completions. Furthermore, since $\mathcal{IcRL}$ is not a subvariety of the variety of integral residuated lattices we must have a subdirectly irreducible integrally closed residuated lattice which is not integral whence its MacNeille completion will not be integrally closed. Thus the class of subdirectly irreducible integrally closed residuated lattices is not closed under MacNeille completions and so by

[60, Prop. 6.6] it follows that the variety $\mathcal{I}c\mathcal{RL}$ is also not closed under hyper-MacNeille completions. Consequently, by [60, Thm. 6.8], the equations $x\backslash x \approx \mathrm{e}$ and $x/x \approx \mathrm{e}$ are also not equivalent to any collection of analytic structural hypersequent rules.

5.4 Sirmonoids and pseudo BCI-algebras

In this section we relate suitable reducts of integrally closed lattices to semi-integral residuated partially ordered monoids, studied in [219, 83], and pseudo BCI-algebras, defined in [79] as non-commutative versions of BCI-algebras [165].

A *residuated partially ordered monoid* is a structure $\mathbf{M} = \langle M, \preceq, \cdot, \backslash, /, \mathrm{e}\rangle$ such that

(i) The structure $\langle M, \cdot, \mathrm{e}\rangle$ is a monoid,

(ii) The structure $\langle M, \preceq\rangle$ is a partial order,

(iii) The structure $\langle M, \cdot, \backslash, /\rangle$ is an algebra of type $\langle 2, 2, 2\rangle$ satisfying

$$y \preceq x\backslash z \iff x \cdot y \preceq x \iff x \preceq z/y.$$

Such a structure is called *semi-integral* if the monoidal unit e is a maximal element of $\langle M, \preceq\rangle$. We will refer to semi-integral residuated partially ordered monoids as *sirmonoids*. We call an element a in a residuated partially ordered monoid $\mathbf{M}$ *square-increasing* provided that $a \preceq a^2$.

5.4.1. Proposition. *Let $\mathbf{M} = \langle M, \preceq, \cdot, \backslash, /, \mathrm{e}\rangle$ be a residuated partially ordered monoid. Then the following are equivalent.*

1. *The residuated partially ordered monoid $\mathbf{M}$ is semi-integral.*

2. *The residuated partially ordered monoid $\mathbf{M}$ satisfies*

$$x \preceq y \iff x\backslash y = \mathrm{e} \iff y/x = \mathrm{e}.$$

3. *The monoidal unit* e *is the largest square-increasing element of* $\mathbf{M}$.

Proof:
Assume first that $\mathbf{M}$ is semi-integral. Let $a, b \in M$ be given. If $a \preceq b$, then by residuation $\mathrm{e} \preceq a\backslash b$ and $\mathrm{e} \preceq b/a$, whence $a\backslash b = \mathrm{e}$ and $b/a = \mathrm{e}$ by semi-integrality. Of course, if $a\backslash b = \mathrm{e}$ then in particular $\mathrm{e} \preceq a\backslash b$ and hence by residuation $a \preceq b$. Similarly, if $b/a = \mathrm{e}$ then $a \preceq b$.

Next assume that $\mathbf{M}$ is such that $a \preceq b \iff a\backslash b = \mathrm{e} \iff b/a = \mathrm{e}$, for all $a, b \in M$. Then if $a \in M$ is a square-increasing element, i.e., $a \preceq a^2$ then

$\mathrm{e} = a\backslash a^2$. However, by residuation we have that $a \preceq a\backslash a^2$, showing that the monoidal unit e is the largest square-increasing element of **M**.

Finally, assume that the monoidal unit e is the largest square-increasing element of **M**. If $a \in M$ is such that $\mathrm{e} \preceq a$ then, since the monoidal operation preserves the order, we must have $a \preceq a^2$, whence a is square-increasing and consequently $a \preceq \mathrm{e}$, showing that **M** is semi-integral. □

5.4.2. Remark. By Proposition 5.4.1 every semi-integral residuated partial ordered monoid **M** must be such that the monoidal unit e is the largest idempotent element of **M**. Thus every such structure is a so-called *integral Dubreil-Jacotin semi-group*, see [41, Chap. 13.3]. However the converse is not true. To see this, consider any residuated lattice $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, \mathrm{e}\rangle$. Letting $\leq$ be the lattice order on **A** we obtain a residuated partially ordered monoid $\mathbf{M}_{\mathbf{A}}^{\leq} = \langle A, \leq, \cdot, \backslash, /, \mathrm{e}\rangle$. Since $\leq$ is a lattice order it is easy to see that $\mathbf{M}_{\mathbf{A}}^{\leq}$ is semi-integral if, and only if, **A** is integral. Thus, if **A** is any non-integral integrally closed residuated lattice, e.g., a non-trivial ℓ-group, then the corresponding residuated partially ordered monoid $\mathbf{M}_{\mathbf{A}}^{\leq}$ will have the monoidal unit as its largest idempotent element but will not be semi-integral.

It is not difficult to verify that any sirmonoid satisfies the following equations and quasi-equation:

(i) $((x\backslash z)/(y\backslash z))/(x\backslash y) \approx \mathrm{e}$,

(ii) $(y/x)\backslash((z/y)\backslash(z/x)) \approx \mathrm{e}$,

(iii) $\mathrm{e}\backslash x \approx x$,

(iv) $x/\mathrm{e} \approx x$,

(v) $(x \cdot y)\backslash z \approx y\backslash(x\backslash z)$,

(vi) $x\backslash y \approx \mathrm{e} \;\;\&\;\; y\backslash x \approx \mathrm{e} \implies x \approx y$.

Conversely, as the following proposition shows, any structure $\langle S, \cdot, \backslash, /, \mathrm{e}\rangle$ satisfying (i)–(vi) gives rise to a sirmonoid.

5.4.3. Proposition. *Let* $\mathbf{S} = \langle S, \cdot, \backslash, /, \mathrm{e}\rangle$ *a structure satisfying (i)–(vi) above. Then* $\langle S, \preceq, \cdot, \backslash, /, \mathrm{e}\rangle$ *is a sirmonoid with the relation* $\preceq$ *on* S *defined by*

$$a \preceq b \quad \text{if, and only if,} \quad a\backslash b = \mathrm{e}.$$

Moreover, any sirmonoid arises in this way.

Proof:
We first observe that if $a, b \in S$ are such that $a\backslash b = \mathrm{e}$ then

$$b/a = ((\mathrm{e}\backslash b)/\mathrm{e})/(\mathrm{e}\backslash a) = ((\mathrm{e}\backslash b)/(a\backslash b))/(\mathrm{e}\backslash a) = \mathrm{e}.$$

And similarly, if $b/a = \mathrm{e}$ then $a\backslash b = \mathrm{e}$. We may therefore conclude that

$$a \preceq b \iff a\backslash b = \mathrm{e} \iff b/a = \mathrm{e},$$

for all $a, b \in S$.

We then show that the relation $\preceq$ is indeed a partial order. To see this, we first observe that by Items (ii), (iii), and (iv),

$$a\backslash a = \mathrm{e}\backslash(a\backslash a) = (\mathrm{e}/\mathrm{e})\backslash(a\backslash a) = (\mathrm{e}/\mathrm{e})\backslash((a/\mathrm{e})\backslash(a/\mathrm{e})) = \mathrm{e},$$

for each $a \in A$. Hence the relation $\preceq$ is reflexive. To see that $\preceq$ is transitive, let $a, b, c \in S$ be such that $a \preceq b$ and $b \preceq c$. Then $b/a = c/b = \mathrm{e}$ whence by Items (iii) and (ii) we see that

$$c/a = \mathrm{e}\backslash(\mathrm{e}\backslash(c/a)) = (b/a)\backslash((c/b)\backslash(c/a) = \mathrm{e},$$

showing that $a \preceq c$. Hence the relation $\preceq$ is transitive. Since by assumption $\mathbf{S}$ satisfies the quasi-equation (vi) the relation $\preceq$ is antisymmetric and therefore a partial order on S.

It remains to establish the residuation property. For this we first show that the equation $x\backslash(z/y) \approx (x\backslash z)/y$ is satisfied by the structure $\mathbf{S}$. To see this, let $a, b, c \in S$ be given. Applying Items (i) and (iii) we first observe that

$$(c/(a\backslash c))/a = ((\mathrm{e}\backslash c)/(a\backslash c))/(\mathrm{e}\backslash a) = \mathrm{e},$$

whence $a \preceq c/(a\backslash c)$. An application of Item (ii) then shows that

$$((a\backslash c)/b)\backslash((c/(a\backslash c))\backslash(c/b)) = \mathrm{e},$$

and hence that $(a\backslash c)/b \preceq (c/(a\backslash c))\backslash(c/b)$. Lastly, applying Items (i) and (iv) shows that the operation $x \mapsto x\backslash d$ is antitone for each $d \in S$ and so we may conclude that

$$(a\backslash c)/b \preceq (c/(a\backslash c))\backslash(c/b) \preceq a\backslash(c/b).$$

Therefore, by transitivity of the relation $\preceq$, we obtain that $(a\backslash c)/b \preceq a\backslash(c/b)$. That $a\backslash(c/b) \preceq (a\backslash c)/b$ is shown by a completely symmetric argument, so that we have $a\backslash(c/b) = (a\backslash c)/b$

Applying Item (v) we then see that,

$$\begin{aligned}(a \cdot b) \preceq c &\iff (a \cdot b)\backslash c = \mathrm{e}\\ &\iff b\backslash(a\backslash c) = \mathrm{e}\\ &\iff b \preceq a\backslash c\\ &\iff (a\backslash c)/b = \mathrm{e}\\ &\iff a\backslash(c/b) = \mathrm{e}\\ &\iff a \preceq c/b,\end{aligned}$$

for all $a, b, c \in S$. Thus $\langle S, \preceq, \cdot, \backslash, /, \mathrm{e}\rangle$ is indeed a residuated partially ordered monoid which is evidently semi-integral.

Finally, as any sirmonoid $\langle S, \preceq, \cdot, \backslash, /, \mathrm{e}\rangle$ is such that the structure $\langle S, \cdot, \backslash, /, \mathrm{e}\rangle$ satisfies (i)–(vi) the last part of the proposition follows from Proposition 5.4.1. □

It follows that a sirmonoid may be identified with an algebraic structure $\mathbf{S} = \langle S, \cdot, \backslash, /, \mathrm{e}\rangle$ of type $\langle 2, 2, 2, 0\rangle$ satisfying (i)–(vi). We let $\mathcal{S}i\mathcal{RM}$ denote the quasi-variety of sirmonoids.

5.4.4. Proposition. *A residuated lattice is integrally closed if, and only if, its* $\{\cdot, \backslash, /, \mathrm{e}\}$*-reduct is a sirmonoid.*

Proof:
Let $\mathbf{A}$ be a residuated lattice. If its $\{\cdot, \backslash, /, \mathrm{e}\}$-reduct is a sirmonoid, then, by Proposition 5.4.3 the relation $\preceq$ on A given by

$$a \preceq b \iff a\backslash b = \mathrm{e} \iff b/a = \mathrm{e},$$

is a partial order and so must in particular be reflexive. We must therefore have that $a\backslash a = a/a = \mathrm{e}$ for all $a \in A$, i.e., $\mathbf{A}$ is integrally closed.

Conversely, suppose that $\mathbf{A}$ is integrally closed. It is easy to see that the equation $((x\backslash z)/(y\backslash z))/(x\backslash y) \approx \mathrm{e}$ is satisfied in any ℓ-group whence by Corollary 5.2.4 we have that $\mathcal{I}c\mathcal{RL} \models ((x\backslash z)/(y\backslash z))/(x\backslash y) \leq \mathrm{e}$. Since the equation $\mathrm{e} \leq ((x\backslash z)/(y\backslash z))/(x\backslash y)$ holds in any residuated lattice we may conclude that indeed $\mathbf{A} \models ((x\backslash z)/(y\backslash z))/(x\backslash y) \approx \mathrm{e}$. A similar argument shows that $\mathbf{A} \models (y/x)\backslash((z/y)\backslash(z/x)) \approx \mathrm{e}$. As any residuated lattice satisfies the remaining items (iii)–(vi) of the quasi-equational definition of a sirmonoid we obtain that the $\{\cdot, \backslash, /, \mathrm{e}\}$-reduct of $\mathbf{A}$ is indeed a sirmonoid. □

5.4.5. Remark. By Proposition 5.4.4 any integrally closed residuated lattice $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, \mathrm{e}\rangle$ gives rise to two residuated partially ordered monoids, namely $\mathbf{M}^{\leq}_{\mathbf{A}} = \langle A, \leq, \cdot, \backslash, /, \mathrm{e}\rangle$, where $\leq$ is the lattice order determined by $\mathbf{A}$, and $\mathbf{M}^{\preceq}_{\mathbf{A}} = \langle A, \preceq, \cdot, \backslash, /, \mathrm{e}\rangle$, where $\preceq$ is the partial order determined by the residuals of $\mathbf{A}$, i.e., $a \preceq b$, if, and only if, $\mathrm{e} = a\backslash b$. We always have that the order $\preceq$ is stronger then $\leq$ and that they coincide if, and only if, $\mathbf{A}$ is integral.

An algebraic structure $\mathbf{B} = \langle B, \backslash, /, \mathrm{e}\rangle$ of type $\langle 2, 2, 0\rangle$ satisfying the equations (i)-(iv) and quasi-equation (vi) is called a *pseudo BCI-algebra*, see, [79, 83]. The $\{\backslash, /, \mathrm{e}\}$-reduct of any sirmonoid is clearly a pseudo BCI-algebra. More notably, we have the following result first established by Raftery and van Alten [219, Thm. 2] in the commutative setting.

5.4.6. Theorem ([83, Thm. 3.3]). *Every pseudo BCI-algebra is a subreduct of a sirmonoid.*

It follows from Theorem 5.4.6 that the quasi-equational theory of sirmonoids is a conservative extension of the quasi-equational theory of pseudo BCI-algebras.

We first observe that the analogue of Theorem 5.4.6 fails in the context of sirmonoids and integrally closed residuated lattices. That is, not every sirmonoid is a subreduct of an integrally closed residuated lattice. By Proposition 5.1.10, $\{\cdot, \backslash, /, \mathrm{e}\}$-subreducts of integrally closed residuated lattices must satisfy the quasi-equation $x^n \approx \mathrm{e} \implies x \approx \mathrm{e}$ for any natural number $n \geq 1$. However, it follows from Proposition 5.4.8 below that there are sirmonoids that do not satisfy all of these quasi-equations. On the other hand, it is known that any sirmonoid satisfying $x \preceq \mathrm{e}$ is a subreduct of an integral, and hence integrally closed, residuated lattice [184].

The quasi-equational theory of integrally closed residuated lattices is, as we have just seen, not a conservative extension of the quasi-equational theory of sirmonoids. However, as we will show in Theorem 5.4.14, such a conservative extension result does hold if we restrict to equational theories. To establish this we follow the strategy developed in Sections 5.1 and 5.3.

5.4.1 A Glivenko theorem for groups

In order to obtain a version of Theorem 5.2.3 in the setting of sirmonoids we need to show how groups may be identified with a certain class of sirmonoids. For this we need a few basic facts about sirmonoids. We first note that since any sirmonoid satisfies the equations $x\backslash x \approx \mathrm{e}$ and $x/x \approx \mathrm{e}$, an argument completely similar to the proof of the first part of Proposition 5.1.6 shows that any sirmonoid satisfies the equation $x\backslash \mathrm{e} \approx \mathrm{e}/x$. Consequently, as before, we denote by $-a$ the element $a\backslash \mathrm{e} = \mathrm{e}/a$ for $a \in S$. Thus we may consider the type of sirmonds to be $\langle 2, 2, 2, 1, 0\rangle$ with the unary operation interpreted on all sirmonoids as $a \mapsto -a$.

5.4.7. Lemma. *The following equations are satisfied by any sirmonoid.*

(i) $-(x\backslash y) \approx (-y)/(-x)$,

(ii) $-(y/x) \approx (-x)\backslash(-y)$,

(iii) $-(x \cdot y) \approx y\backslash(-x)$,

(iv) $-(x \cdot y) \approx (-y)/x$.

Proof:
Let $\mathbf{S}$ be a sirmonoid. Notice that for $a, b \in S$ we have $a(-b)b \preceq a$ and therefore $-b \preceq a\backslash(a/b)$. That is, $(-b)\backslash(a\backslash(a/b)) = \mathrm{e}$. But also $(-b)b(a\backslash(a/b)) \preceq a\backslash(a/b)$ and hence $b(a\backslash(a/b)) \preceq (-b)\backslash(a\backslash(a/b)) = \mathrm{e}$, yielding $a\backslash(a/b) \preceq b\backslash\mathrm{e} = -b$. So $\mathbf{S}$ satisfies $x\backslash(x/y) \approx -y$. Analogously, $\mathbf{S}$ satisfies $(x\backslash y)/y \approx -x$ and hence for all $a, b \in S$,

$$(-b)/(-a) = \big((a\backslash b)\backslash((a\backslash b)/b)\big)/(-a) = \big((a\backslash b)\backslash(-a)\big)/(-a) = -(a\backslash b).$$

That is, $\mathbf{S}$ satisfies the equation $-(x\backslash y) \approx (-y)/(-x)$. A symmetric argument shows that $\mathbf{S}$ also satisfies the equation $-(y/x) \approx (-x)\backslash(-y)$.

To see that the equation $-(xy) \approx y\backslash(-x)$ is valid let $a, b \in S$ be given. Then

$$-(a \cdot b) = (a \cdot b)\backslash \mathrm{e} = b\backslash(a\backslash \mathrm{e}) = b\backslash(-a),$$

which implies that this equation is indeed valid.

Finally, recall from the proof of Proposition 5.4.3 that, being a sirmonoid, $\mathbf{S}$ satisfies the equation $x\backslash(z/y) \approx (x\backslash z)/y$. From this it follows that

$$-(a \cdot b) = b\backslash(-a) = b\backslash(\mathrm{e}/a) = (b\backslash \mathrm{e})/a = (-b)/a,$$

for all $a, b \in S$. Consequently, $\mathbf{S}$ satisfies the equation $-(x \cdot y) \approx (-y)/x$. □

5.4.8. PROPOSITION ([82],[83, Sec. 2]). *The class of sirmonoids satisfying the equation $--x \approx x$ is term equivalent to the variety of groups.*

Proof:
Given a group $\mathbf{G} = \langle G, \cdot, ^{-1}, \mathrm{e}\rangle$, let $S(\mathbf{G})$ be the structure $\langle G, \cdot, \backslash, /, \mathrm{e}\rangle$ defined by letting

$$a\backslash b := a^{-1} \cdot b \quad \text{and} \quad b/a := b \cdot a^{-1},$$

for $a, b \in G$. It is straightforward to show that this defines a sirmonoid in which the equation $--x \approx x$ is satisfied. Conversely, given a sirmonoid $\mathbf{S} = \langle S, \cdot, \backslash, /, \mathrm{e}\rangle$ satisfying the equation $--x \approx x$ we observe, by Lemma 5.4.7(iii), that

$$a(-a) = --(a(-a)) = -(-a\backslash -a) = -\mathrm{e} = \mathrm{e},$$

for each $a \in S$. Similarly,

$$(-a)a = --((-a)a) = -(a\backslash --a) = -(a\backslash a) = -\mathrm{e} = \mathrm{e},$$

for each $a \in S$. Thus the element $-a$ is the inverse of $a \in S$ in the monoid $\langle S, \cdot, \mathrm{e}\rangle$ which is therefore a group which we shall denote by $G(\mathbf{S})$.

Evidently, $G(S(\mathbf{G})) = \mathbf{G}$ for all groups $\mathbf{G}$. Conversely, given any sirmonoid $\mathbf{S}$ satisfying the equation $--x \approx x$, Lemma 5.4.7 shows that for $a, b \in S$, we have

$$(-a)b = --((-a)b) = -(-b/-a) = (--a)\backslash(--b) = a\backslash b.$$

Similarly, $b(-a) = b/a$, showing that $S(G(\mathbf{S})) = \mathbf{S}$, which concludes the proof. □

In light of Proposition 5.4.8 we will call a sirmonoid a *group* provided that it satisfies the equation $--x \approx x$. We denote by $\mathcal{G}rp$ the class of such sirmonoids.

As before, given any sirmonoid $\mathbf{S} = \langle S, \cdot, \backslash, /, \mathrm{e}\rangle$ we obtain a nucleus $\alpha\colon S \to S$ on the partially ordered monoid $\langle S, \preceq, \cdot, \mathrm{e}\rangle$ given by $a \mapsto --a$. Consequently, for each sirmonoid $\mathbf{S} = \langle S, \cdot, \backslash, /, \mathrm{e}\rangle$ we have a corresponding residuated partially ordered monoid

$$\mathbf{S}_\alpha = \langle \alpha[S], \preceq, \cdot_\alpha, \backslash, /, \mathrm{e}\rangle, \quad \text{where } a \cdot_\alpha b \coloneqq \alpha(a \cdot b).$$

Following the terminology of Section 5.1 we call a homomorphism $h\colon \mathbf{S} \to \mathbf{T}$ of sirmonoids $\mathbf{S}$ and $\mathbf{T}$ e-principal provided that $h^{-1}(\mathrm{e}^{\mathbf{T}}) \subseteq {\downarrow}\mathrm{e}^{\mathbf{S}}$. We then obtain the following analogue of Proposition 5.1.8.

5.4.9. Proposition. *Let* $\mathbf{S}$ *be a sirmonoid.*

1. *The map* $\alpha\colon \mathbf{S} \to \mathbf{S}_\alpha$ *is a surjective homomorphism of sirmonoids.*
2. *The sirmonoid* $\mathbf{S}_\alpha$ *is a group.*
3. *Any homomorphism* $h\colon \mathbf{S} \to \mathbf{G}$ *of sirmonoids with* $\mathbf{G}$ *a group factors through the group* $\mathbf{S}_\alpha$.
4. *The sirmonoid* $\mathbf{S}_\alpha$ *is, up to isomorphism, the unique* e*-principal homomorphic image of* $\mathbf{S}$ *which is group.*

Proof:
To establish Item 1 we observe that since $\alpha\colon S \to S$ is a nucleus on the partially ordered monoid $\langle S, \preceq, \cdot, \mathrm{e}\rangle$ and $\alpha(\mathrm{e}) = \mathrm{e}$, it follows that α is a surjective monoid homomorphism between $\langle S, \cdot, \mathrm{e}\rangle$ and $\langle \alpha[S], \cdot_\alpha, \mathrm{e}\rangle$. Moreover, by Lemma 5.4.7 we see that for all $a, b \in S$,

$$\alpha(a\backslash b) = --(a\backslash b) = -((-b)/(-a)) = (--a)\backslash(--b) = \alpha(a)\backslash\alpha(b).$$

Analogously, $\alpha(b/a) = \alpha(b)/\alpha(a)$, so α is a sirmonoid homomorphism.

To see that Item 2 holds we note that since α is a homomorphism, any equation satisfied by $\mathbf{S}$ is also satisfied by $\mathbf{S}_\alpha$. In particular $\mathbf{S}_\alpha$ satisfies Items (i)–(v) of the definition of a sirmonoid. Moreover, since $\mathbf{S}_\alpha$ is a $\{\backslash, /, \mathrm{e}\}$-subreduct of $\mathbf{S}$ it follows that $\mathbf{S}_\alpha$ also satisfies Item (vi) of the definition of a sirmonoid. To prove that $\mathbf{S}_\alpha$ is a group, it suffices to show that it satisfies the equation $--x \approx x$. But α is idempotent and hence $--\alpha(a) = \alpha(\alpha(a)) = \alpha(a)$ for every $a \in S$, as required.

For Item 3 assume that $h\colon \mathbf{S} \to \mathbf{G}$ is a homomorphism of sirmonoids with $\mathbf{G}$ a group and let $a, b \in \ker\alpha$ be given. Then $--a = --b$ whence $--h(a) = --h(b)$ and consequently, by the assumption that $\mathbf{G}$ is a group also $h(a) = h(b)$. This shows that $\ker\alpha \subseteq \ker h$ and therefore we obtain an induced homomorphism of sirmonoids $\overline{h}\colon \mathbf{S}_\alpha \to \mathbf{G}$ such that $h = \overline{h} \circ \alpha$.

Finally, to establish Item 4 we first note that the homomorphism $\alpha\colon \mathbf{S} \to \mathbf{S}_\alpha$ is an e-principal surjection. Consider a surjective e-principal homomorphism of sirmonoids $h\colon \mathbf{S} \twoheadrightarrow \mathbf{G}$ with $\mathbf{G}$ a group. If $(a, b) \in \ker h$ then since $\mathbf{G}$ is a group we must have that $a \backslash b \in h^{-1}(\mathrm{e})$. By e-principality it follows that $a \backslash b \preceq \mathrm{e}$. But then $\mathrm{e} \preceq -(a \backslash b) = -b / -a$ whence $-a \preceq -b$. Similarly $-b \preceq -a$, showing that $-a = -b$ and hence that $(a, b) \in \ker \alpha$. Consequently, $\ker h \subseteq \ker \alpha$ whence the induced map $\overline{h} \colon \mathbf{S}_\alpha \twoheadrightarrow \mathbf{G}$ must be an isomorphism. $\square$

In every sirmonoid $\mathbf{S} = \langle S, \preceq, \cdot, \backslash, /, \mathrm{e} \rangle$ the map $a \mapsto -a$ is both antitone, by residuation, and monotone, because $\mathbf{S} \models -(x \backslash y) \approx -y/-x$. Therefore, as $\mathbf{S} \models ---x \approx -x$, we must have that

$$\mathbf{S} \models -s \preceq -t \quad \text{if, and only if,} \quad \mathbf{S} \models -s \approx -t,$$

for all terms s, t in the language of sirmonoids. Consequently, for any sirmonoid $\mathbf{S}$ the nucleus image $\alpha[S]$ consists precisely of the set of maximal elements of the poset $\langle M, \preceq \rangle$. In particular, for any group $\mathbf{G}$ the order $\preceq$ must be discrete, whence

$$\mathbf{G} \models s \preceq t \quad \text{if, and only if,} \quad \mathbf{G} \models s \approx t,$$

for all sirmonoid terms s, t.

The proof of the following result now mirrors the proof of Lemma 5.2.1 and is therefore omitted.

5.4.10. Lemma. *For any sirmonoid* $\mathbf{S}$ *and residuated monoid terms* s, t*,*

$$\mathbf{S}_\alpha \models s \approx t \quad \textit{if, and only if,} \quad \mathbf{S} \models --s \approx --t.$$

Given any class $\mathcal{K}$ of sirmonoids, we let $\mathcal{K}_\alpha$ denote the corresponding class of groups $\{\mathbf{S}_\alpha \mid \mathbf{S} \in \mathcal{K}\}$. The proof of the following Glivenko-style result proceeds very similarly to the proof of Proposition 5.1.11 and is therefore omitted.

5.4.11. Proposition. *Let* $\mathcal{Q}$ *be any quasi-variety of sirmonoids defined relative to* $\mathcal{SiRM}$ *by a set of equations* E*. Then* $\mathcal{Q}_\alpha$ *is a variety of groups defined relative to* $\mathcal{G}rp$ *by* E*, and for any residuated monoid terms* s, t*,*

$$\mathcal{Q}_\alpha \models s \approx t \quad \textit{if, and only if,} \quad \mathcal{Q} \models --s \approx --t.$$

In particular, we obtain the following Glivenko-style property for $\mathcal{SiRM}$ with respect to the variety of groups.

5.4.12. Corollary. *For any residuated monoid terms* s, t*,*

$$\mathcal{G}rp \models s \approx t \quad \textit{if, and only if,} \quad \mathcal{SiRM} \models --s \approx --t.$$

We use this result to prove that the equational theory of $\mathcal{IcRL}$ is a conservative extension of the equational theory of $\mathcal{SiRM}$. We call a sequent $s_1, \dots, s_n \Rightarrow t$ an *m-sequent* if $s_1, \dots, s_n, t$ are residuated monoid terms, and say that it is valid in a class $\mathcal{K}$ of sirmonoids, denoted $\models_{\mathcal{K}} s_1, \dots, s_n \Rightarrow t$, if $\mathcal{K} \models s_1 \cdots s_n \preceq t$, recalling that the empty product is understood to be the monoidal unit e.

5.4.13. Proposition. *An m-sequent is derivable in the calculus* IcRL *if, and only if, it is valid in all sirmonoids.*

Proof:
Suppose first that $\Gamma \Rightarrow t$ is an m-sequent which is valid in all sirmonoids. By Proposition 5.4.4, it is also valid in all integrally closed residuated lattices, and hence, by Proposition 5.3.1, derivable in the calculus IcRL.

To establish the converse implication it suffices to show that all the rules of IcRL apart from (cut) preserve validity in $\mathcal{SiRM}$. For the key case of ($\mathcal{LG}$-w), suppose that $\models_{\mathcal{SiRM}} \Gamma, \Pi \Rightarrow u$ and $\models_{\mathcal{LG}} \Delta \Rightarrow \mathrm{e}$. Letting s_1, s_2, and t denote the products of the terms in Γ, Π, and Δ, respectively, we obtain $\mathcal{SiRM} \models s_1 s_2 \preceq u$ and $\mathcal{LG} \models t \leq \mathrm{e}$. We claim that $\mathcal{G}rp \models t \approx \mathrm{e}$. Otherwise, the equation $t \approx \mathrm{e}$ would fail in the free group on countably infinitely many generators. Since this group can be totally ordered, see, e.g., [62, Thm. 3.4], we would then have an ℓ-group in which $\mathrm{e} < t$, contradicting $\mathcal{LG} \models t \leq \mathrm{e}$. Hence, by Corollary 5.4.12, we obtain $\mathcal{SiRM} \models t \preceq \mathrm{e}$. So $\mathcal{SiRM} \models s_1 t s_2 \preceq u$; that is, $\models_{\mathcal{SiRM}} \Gamma, \Delta, \Pi \Rightarrow u$. □

5.4.14. Theorem. *The equational theory of integrally closed residuated lattices is a conservative extension of the equational theories of sirmonoids and pseudo BCI-algebras.*

Proof:
Since the equational theory of sirmonoids is a conservative extension of the equational theory of pseudo BCI-algebras, see Theorem 5.4.6, it suffices to show that the equational theory of integrally closed residuated lattices is a conservative extension of the equational theory of sirmonoids. Therefore, let s, t be terms in the language of sirmonoids. By Proposition 5.4.4 we have that

$$\mathcal{SiRM} \models s \approx t \quad \text{implies} \quad \mathcal{IcRL} \models s \approx t.$$

Conversely, if $\mathcal{IcRL} \models s \approx t$, then both of the m-sequents $s \Rightarrow t$ and $t \Rightarrow s$ are valid in $\mathcal{IcRL}$ and hence, by Proposition 5.3.1, derivable in the calculus IcRL. But then by Propositions 5.4.13 we must have $\mathcal{SiRM} \models s \approx t$. We have thus shown that

$$\mathcal{SiRM} \models s \approx t \quad \text{if, and only if,} \quad \mathcal{IcRL} \models s \approx t,$$

for all terms s, t in the language of sirmonids. □

By the previous result and the fact that the equational theory for $\mathcal{IcRL}$ is sound and complete with respect to the cut-free version of the calculus IcRL, we obtain that the sequent calculus consisting of the rules of IcRL restricted to m-sequents and omitting the rules for $\wedge$ and $\vee$ is sound and complete for the variety of sirmonoids and admits cut-elimination. Similarly, if we further remove the rules for $\cdot$, we obtain a sound and complete calculus for the variety of pseudo BCI-algebras that admits cut-elimination.

5.4.15. Corollary. *The equational theories of sirmonoids and pseudo BCI-algebras are decidable.*

Similar results hold for *BCI-algebras* [165], axiomatized relative to pseudo BCI-algebras by the equation $x\backslash y \approx y/x$, and *sircomonoids* [219], axiomatized relative to sirmonoids by the equation $x\backslash y \approx y/x$ or the equation $x \cdot y \approx y \cdot x$. In particular, the equational theory of commutative integrally closed residuated lattices is a conservative extension of the equational theories of sircomonoids and BCI-algebras. Let us remark also that the decidability of the equational theory of BCI-algebras was first establish in [178] using a sequent calculus with a restricted version of the rule ($\mathcal{LG}$-w).

We conclude this section by noting that by an argument completely analogous to the proof of Proposition 5.2.8 we obtain the following.

5.4.16. Proposition. *For each quasi-equation q in the language of sirmonoids there is a quasi-equation q^α effectively computable from q such that*

$$\mathcal{G}rp \models q \quad \textit{if, and only if,} \quad \mathcal{S}i\mathcal{RM} \models q^\alpha.$$

Since the word problem for groups is undecidable, see, e.g., [225, Chap. 12.8], so is the quasi-equational theory of groups, leading to the following corollary.

5.4.17. Corollary. *The quasi-equational theory of sirmonoids is undecidable.*

5.5 Casari's comparative logic

The results of the previous sections extend with only minor modifications to the setting of *pointed residuated lattices*, or *FL-algebras*, viz., residuated lattices with an extra constant operation f. As before, we call such an algebra *integrally closed* if it satisfies the equation $x\backslash x \approx \mathrm{e}$, or equivalently, the equation $x/x \approx \mathrm{e}$. It is then straightforward to prove analogues of Lemma 5.1.3 and Proposition 5.1.6, simply adding the word "pointed" before every occurrence of the word "residuated lattice".

An ℓ-group can be identified with an integrally closed pointed residuated lattice satisfying $(x\backslash \mathrm{e})\backslash \mathrm{e} \approx x$ and $\mathrm{f} \approx \mathrm{e}$. However, to show that the map $\alpha\colon \mathbf{A} \to \mathbf{A}$ given by $a \mapsto --a$ on an integrally closed pointed residuated lattice $\mathbf{A}$ defines a homomorphism onto an ℓ-group $\mathbf{A}_\alpha = \langle \alpha[A], \wedge, \vee_\alpha, \cdot_\alpha, \backslash, /, \mathrm{e}, \alpha(\mathrm{f})\rangle$, we need also that $\alpha(\mathrm{f}) = \mathrm{e}$. Assuming this condition, we obtain analogues of Propositions 5.1.8 and 5.1.11, and Theorem 5.2.3 for integrally closed pointed residuated lattices satisfying the equation $\mathrm{f}\backslash \mathrm{e} \approx \mathrm{e}$. Note, however, that our definition of the Glivenko

Identity Axioms

$$\frac{}{s \Rightarrow s}\ (\textsc{id})$$

Cut Rule

$$\frac{\Gamma_2 \Rightarrow s, \Delta_1 \quad \Gamma_1, s, \Gamma_3 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta_1, \Delta_2}\ (\textsc{cut})$$

Structural Rules

$$\frac{\Gamma_1, \Pi_2, \Pi_1, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Pi_1, \Pi_2, \Gamma_2 \Rightarrow \Delta}\ (\textsc{el}) \qquad \frac{\Gamma \Rightarrow \Delta_1, \Sigma_2, \Sigma_1, \Delta_2}{\Gamma \Rightarrow \Delta_1, \Sigma_1, \Sigma_2, \Delta_2}\ (\textsc{er})$$

Left Operation Rules

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \mathrm{e}, \Gamma_2 \Rightarrow \Delta}\ (\mathrm{e}\Rightarrow)$$

$$\frac{}{\mathrm{f} \Rightarrow}\ (\mathrm{f}\Rightarrow)$$

$$\frac{\Gamma_2 \Rightarrow s, \Delta_2 \quad \Gamma_1, t, \Gamma_3 \Rightarrow \Delta_1}{\Gamma_1, s \to t, \Gamma_2, \Gamma_3 \Rightarrow \Delta_1, \Delta_2}\ (\to\Rightarrow)$$

$$\frac{\Gamma_1, s, t, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, s \cdot t, \Gamma_2 \Rightarrow \Delta}\ (\cdot\Rightarrow)$$

$$\frac{\Gamma_1, s, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, s \wedge t, \Gamma_2 \Rightarrow \Delta}\ (\wedge\Rightarrow)_1$$

$$\frac{\Gamma_1, t, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, s \wedge t, \Gamma_2 \Rightarrow \Delta}\ (\wedge\Rightarrow)_2$$

$$\frac{\Gamma_1, s, \Gamma_2 \Rightarrow u \quad \Gamma_1, t, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, s \vee t, \Gamma_2 \Rightarrow \Delta}\ (\vee\Rightarrow)$$

Right Operation Rules

$$\frac{}{\Rightarrow \mathrm{e}}\ (\Rightarrow\mathrm{e})$$

$$\frac{\Gamma \Rightarrow \Delta_1, \Delta_2}{\Gamma \Rightarrow \Delta_1, \mathrm{f}, \Delta_2}\ (\Rightarrow\mathrm{f})$$

$$\frac{\Gamma, s \Rightarrow t, \Delta}{\Gamma \Rightarrow s \to t, \Delta}\ (\Rightarrow\to)$$

$$\frac{\Gamma_1 \Rightarrow s, \Delta_1 \quad \Gamma_2 \Rightarrow t, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow s \cdot t, \Delta_1, \Delta_2}\ (\Rightarrow\cdot)$$

$$\frac{\Gamma \Rightarrow \Delta_1, s, \Delta_2}{\Gamma \Rightarrow \Delta_1, s \vee t, \Delta_2}\ (\Rightarrow\vee)_1$$

$$\frac{\Gamma \Rightarrow \Delta_1, t, \Delta_2}{\Gamma \Rightarrow \Delta_1, s \vee t, \Delta_2}\ (\Rightarrow\vee)_2$$

$$\frac{\Gamma \Rightarrow \Delta_1, s, \Delta_2 \quad \Gamma \Rightarrow \Delta_1, t, \Delta_2}{\Gamma \Rightarrow \Delta_1, s \wedge t, \Delta_2}\ (\Rightarrow\wedge)$$

Figure 5.2: The Sequent Calculus InCPRL

property for pointed residuated lattices now diverges from the definition of [100], which considers the operations $a \mapsto \mathrm{f}/(a\backslash\mathrm{f})$ and $a \mapsto (\mathrm{f}/a)\backslash\mathrm{f}$.

We now turn our attention to a particular class of algebras introduced by Casari in [49], see also [48, 50, 210, 198], to model comparative reasoning in natural language. For any commutative pointed residuated lattice $\mathbf{A}$, we write $a \to b$ for the common result of $a\backslash b$ and b/a; we also define $\neg a := a \to \mathrm{f}$ and $a + b := \neg a \to b$ and say that $\mathbf{A}$ is *involutive* if it satisfies $\neg\neg x \approx x$. We call an involutive commutative integrally closed pointed residuated lattice satisfying the equation $\mathrm{f} \to \mathrm{e} \approx \mathrm{e}$, or equivalently the equation $\mathrm{f} \cdot \mathrm{f} \approx \mathrm{f}$, a *Casari algebra*, also called a *lattice-ordered pregroup* in [49]. We denote the variety of Casari algebras by $\mathcal{CA}$ and the variety of Abelian ℓ-groups (Casari algebras satisfying $\mathrm{f} \approx \mathrm{e}$)

by $\mathcal{A}b\mathcal{LG}$. The reasoning described above yields the following Glivenko-style property for Casari algebras, first established in [198].

5.5.1. PROPOSITION ([198, Prop. 1]). *For any pointed residuated lattice terms* s,t,

$$\mathcal{A}b\mathcal{LG} \models s \leq t \quad \textit{if, and only if,} \quad \mathcal{CA} \models --s \leq --t.$$

A sequent calculus for the equational theory of Casari algebras was defined by Metcalfe in [198]. We consider here *multi-succedent sequents* defined as expressions of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite, possibly empty, sequences of pointed residuated lattice terms. Generalizing our definition for single-succedent sequents, we say that a multi-succedent sequent $s_1, \ldots, s_n \Rightarrow t_1, \ldots, t_m$ is *valid* in a class $\mathcal{K}$ of pointed residuated lattices, denoted by $\models_{\mathcal{K}} s_1, \ldots, s_n \Rightarrow t_1, \ldots, t_m$, if $\mathcal{K} \models s_1 \cdots s_n \leq t_1 + \cdots + t_m$, where the empty product is understood as being the monoidal unit e and the empty sum as being the constant f.

The multi-succedent sequent calculus CA consists of the calculus InCPRL for involutive commutative pointed residuated lattices defined in Figure 5.2 extended with the rule

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \quad \models_{\mathcal{A}b\mathcal{LG}} \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\mathcal{A}b\mathcal{LG}\text{-w})$$

The next proposition collects some results from [198], noting that these can also be easily established using the methods of the previous sections.

5.5.2. PROPOSITION ([198, Thms. 3, 4, and 7]).

1. *A multi-succedent sequent is derivable in the calculus* CA *if, and only if, it is valid in* $\mathcal{CA}$.
2. *The calculus* CA *admits cut-elimination.*
3. *The equational theory of Casari algebras is decidable.*

We are now able to establish the main result of this section.

5.5.3. THEOREM. *The equational theory of Casari algebras is a conservative extension of the equational theories of commutative integrally closed residuated lattices, sircomonoids, and BCI-algebras.*

Proof:
The equational theory of commutative integrally closed residuated lattices is a conservative extension of the equational theories of sircomonoids and BCI-algebras by Theorem 5.4.14. Hence it suffices to show that the equational theory of Casari algebras is a conservative extension of the equational theory of commutative integrally closed residuated lattices.

Let ClcRL be the sequent calculus CA restricted to single-succedent sequents, i.e., sequents of the form $\Gamma \Rightarrow t$ where the constant f does not occur in Γ or t. Then a single-succedent sequent is derivable in the calculus ClcRL if, and only if, it is valid in all commutative integrally closed residuated lattices. It therefore suffices to show that if a single-succedent sequent is derivable in the calculus CA, then it is also derivable in the calculus ClcRL. To this end, a simple induction on the height of a cut-free derivation shows that whenever a sequent $\Gamma \Rightarrow \Delta$ not containing any occurrence of the constant f is derivable in CA, the sequence Δ must be non-empty. In particular, no sequent of the form $\Gamma \Rightarrow$, where the constant f does not occur in Γ, is derivable in CA. But then a straightforward induction on the height of a cut-free derivation shows that any single-succedent sequent derivable in CA, must also be derivable in ClcRL. □

5.6 Summary and concluding remarks

In this chapter we have investigated the structure of integrally closed residuated lattices. In particular, we have shown that the variety of integrally closed residuated lattices enjoys the Glivenko property with respect to the variety of ℓ-groups. This fact was then used to establish the soundness of a certain non-standard weakening rule giving rise to a cut-free sequent calculus for the equational theory of integrally closed residuated lattices. As a direct consequence we obtained the decidability of the equational theory of integrally closed residuated lattices. Finally, we showed how the integrally closed residuated lattices enjoys a close connection to many other well-known structures, such as Dubreil-Jacotin semigroups, (pseudo) BCI-algebras, sir(co)monoids and algebras for Casari's comparative logic.

Further directions and open problems As we have seen the $\{\wedge, \vee\}$-free subreducts of a integrally closed residuated lattices are always sirmonoids. However, as we have also seen not all sirmonoids are subreducts of integrally closed residuated lattices. We leave the characterization of the quasi-variety of sirmonoids which are subreducts of integrally closed residuated lattice as an open problem. In fact, the analogous problem for groups, see, e.g., [6, Thm. 5.9] and [66, Thm. 2.2], appears to be open [88, Rem. 3.9].

Finally, we believe that it would be worthwhile to investigate the concept of non-standard structural rules in a more systematic way, by (i) identifying side-conditions on different kinds of structural rules which are compatible with the standard cut-elimination procedure, and (ii) determining more varieties of residuated lattices with an equational theory admitting an analytic sequent calculus consisting of the rules from RL together with a set of non-standard structural rules.

Appendix A

Technical preliminaries

This appendix contains a brief account of some of the central technical background for this thesis. Almost everything presented here is already known and in most cases well known. We did not aim at providing an extensive historical account of the material but simply to define concepts and explain results used in the thesis. Often the references we give to a definition or result will not necessarily be to the original sources.

A.1 Partially ordered sets

For the material covered in this section see, [74, 85] and [98, Chap. 3.1]

Posets A *partially ordered set*, or *poset*, is a structure $\mathbb{P} = \langle P, \leq \rangle$ with $\leq$ a binary relation on a set P which is (i) *reflexive*, (ii) *transitive*, and (iii) *anti-symmetric*, i.e., for all $p, q, r, \in P$,

(i) $p \leq p$,

(ii) $p \leq q$ and $q \leq r$ implies $p \leq r$,

(iii) $p \leq q$ and $q \leq p$ implies $p = q$.

If $p \leq q$ we say that p is *less than or equal to* q which in turn is said to be *greater than or equal to* p. In case $p \leq q$ and $p \neq q$ we write $p < q$ and say that p is *strictly less than* q which in turn is said to be *strictly greater than* p.

Infima and suprema Let $\mathbb{P} = \langle P, \leq \rangle$ be a poset and let $S \subseteq P$. We call $p \in P$ a *lower*, respectively *upper bound* of S provided that $p \leq s$, respectively $p \geq s$, for all $s \in S$. We denote by $L(S)$ and $U(S)$ the collection of lower and upper bounds of S, respectively. A poset $\mathbb{P} = \langle P, \leq \rangle$ is called a *meet semi-lattice* if every two elements p, q of P have a greatest lower bound in $\mathbb{P}$, denoted by $p \wedge q$, and called the *meet* or *infimum of* p *and* q. Similarly, a poset $\mathbb{P} = \langle P, \leq \rangle$ is called a *join semi-lattice* if every two elements p, q of P have a least upper bound in $\mathbb{P}$, denoted by $p \vee q$, and called the *join* or *supremum of* p *and* q. A poset which is both a meet and a join semi-lattice is called a *lattice*. A lattice $\mathbb{P} = \langle P, \leq \rangle$ is called *distributive* provided that

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r) \quad \text{and} \quad p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r),$$

for all $p, q, r \in P$.

Bounds A poset with a least, or greatest, element is called *lower bounded*, or *upper bounded*, respectively. A *bounded* poset is a poset which is at the same time lower and upper bounded. Least and greatest elements are also referred to as *bottom* and *top* elements, respectively. The least element of a lower bounded poset is sometimes referred to as the *root* of the poset. Hence lower bounded posets may also be referred to as being *rooted*.

Complete posets A poset $\mathbb{P}$ is *complete* provided that every subset $S \subseteq P$ has a greatest lower bound in $\mathbb{P}$, denoted $\bigwedge S$, and called the *meet* or *infimum of* S. As is easily seen this is equivalent to requiring that every subset $S \subseteq P$ has a least upper bound in $\mathbb{P}$, denoted $\bigvee S$, and called the *join* or *supremum of* S. Any complete poset is necessarily a bounded lattice.

Order duals Given any partial order $\mathbb{P} = \langle P, \leq \rangle$ we obtain another partial order $\leq^\partial$ on P by letting

$$p \leq^\partial q \quad \text{if, and only if,} \quad q \leq p,$$

for all $p, q \in P$. We will refer to the poset $\mathbb{P}^\partial := \langle P, \leq^\partial \rangle$ as the *order dual* of the poset $\mathbb{P}$. It is easy to see that if $\mathbb{P}$ is a lattice then so is $\mathbb{P}^\partial$. Moreover, if $\mathbb{P}$ is a complete or distributive lattice, then so is $\mathbb{P}^\partial$.

Irreducible elements An element p in a lattice $\mathbb{P} = \langle P, \leq \rangle$ is called *meet-irreducible* if $p = q \wedge r$ entails $p = q$ or $p = r$ for all $q, r \in P$. An element p in a complete lattice $\mathbb{P} = \langle P, \leq \rangle$ is called *completely meet-irreducible* if $p = \bigwedge S$ entails $p \in S$ for all subsets $S \subseteq P$. In case $\mathbb{P}$ is upper bounded we will in addition require (completely) meet-irreducible elements to be different from the top element of $\mathbb{P}$. An element p in a (complete) lattice $\mathbb{P}$ is called (*completely*) *join-irreducible* provided p is (completely) meet-irreducible in the order dual $\mathbb{P}^\partial$.

Filters A non-empty subset $F \subseteq P$ of a poset $\mathbb{P} = \langle P, \leq \rangle$ is called a *filter* if it is (i) *upward closed* and (ii) *downward directed*, i.e., for all $p, q \in P$,

(i) if $p \in F$ and $p \leq q$, then $q \in F$,

(ii) if $p, q \in F$ there is $r \in F$ such that $r \leq p$ and $r \leq q$.

We note that when $\mathbb{P}$ is a meet semi-lattice, condition (ii) may be replaced with the requirement that F is closed under binary meets, i.e., that $p \wedge q \in F$ for all $p, q \in F$. A filter is *proper* if its set-theoretic complement is non-empty. Evidently, in a poset $\mathbb{P} = \langle P, \leq \rangle$, any set of the form $\{q \in P : p \leq q\}$, for $p \in P$, is a filter. Filters of this form are called *principal*. A subset $F \subseteq P$ of a poset $\mathbb{P}$ is a *normal filter* provided that $UL(F) = F$. It is not hard to see that any non-empty normal filter is indeed a filter. An *ideal* of $\mathbb{P}$ is then defined to be a filter of the order dual $\mathbb{P}^\partial$. Similarly, a *normal ideal* of $\mathbb{P}$ is a normal filter of the order dual $\mathbb{P}^\partial$.

Maps between posets A function $f : P \to Q$ between posets $\mathbb{P} = \langle P, \leq_P \rangle$ and $\mathbb{Q} = \langle Q, \leq_Q \rangle$ is called *order-preserving* provided that

$$p \leq_P q \quad \text{implies} \quad f(p) \leq_Q f(q), \qquad \text{for all } p, q \in P.$$

Similarly, a function $f : P \to Q$ between posets $\mathbb{P}$ and $\mathbb{Q}$ is called *order-reversing*, or *antitone*, if f is an order-preserving map from the poset $\mathbb{P}$ to the poset $\mathbb{Q}^\partial$. An order-preserving function $f : \mathbb{P} \to \mathbb{Q}$ between posets $\mathbb{P} = \langle P, \leq_P \rangle$ and $\mathbb{Q} = \langle Q, \leq_Q \rangle$ is called an *order-reflecting* provided that

$$f(p) \leq_Q f(q) \quad \text{implies} \quad p \leq_P q, \qquad \text{for all } p, q \in P.$$

A function between posets which is both order-preserving and order-reflecting is called an *order-embedding*. A *subposet* of a poset $\mathbb{P} = \langle P, \leq_P \rangle$ is a poset $\mathbb{Q} = \langle Q, \leq_Q \rangle$ such that $Q \subseteq P$ and the inclusion map from Q to P is an order-embedding. An *isomorphism* is a surjective order-embedding and two posets are said to be *isomorphic* if there exists an isomorphism between them.

Closure and interior operators An order-preserving function $f : \mathbb{P} \to \mathbb{P}$ on a poset $\mathbb{P} = \langle P, \leq \rangle$ is called a *closure operator* provided that

$$p \leq f(p) \quad \text{and} \quad ff(p) \leq f(p) \qquad \text{for all } p \in P.$$

An *interior operator* on a poset $\mathbb{P}$ is then a closure operator on the order dual $\mathbb{P}^\partial$. If $f : \mathbb{P} \to \mathbb{P}$ is a closure (interior) operator on a poset $\mathbb{P}$ we call the elements in the image $f[P] \coloneqq \{f(p) \in P : p \in P\}$ *f-closed* (*f-open*) or simply *closed* (*open*) if the map f is clear from the context. One may readily verify that the closed (open) elements are precisely the *fixed points* of f, i.e., the elements $p \in P$ such that $f(p) = p$. We denote by $\mathbb{P}_f$ the poset obtained by restricting the order on P to the set of closed (open) elements.

A.1.1. Proposition. *Let $f : \mathbb{P} \to \mathbb{P}$ be a closure operator on a poset $\mathbb{P}$.*

1. *If $S \subseteq f[P]$ has a greatest lower bound $\bigwedge S$ in $\mathbb{P}$ then $\bigwedge S \in f[P]$.*

2. *If $T \subseteq f[P]$ has a least upper bound $\bigvee T$ in $\mathbb{P}$ then $f(\bigvee T)$ is the least upper bound of T in $\mathbb{P}_f$.*

In particular, if $\mathbb{P}$ is a (complete) lattice then $\mathbb{P}_f$ will also be a (complete) lattice.

A corresponding proposition holds, *mutatis mutandis*, of the open elements of any interior operator.

Adjoints An *adjunction* from a poset $\mathbb{P} = \langle P, \leq_P \rangle$ to a poset $\mathbb{Q} = \langle Q, \leq_Q \rangle$ is a pair (f, g) of order-preserving maps $f : P \to Q$ and $g : Q \to P$ such that

$$f(p) \leq_Q q \quad \text{if, and only if,} \quad p \leq_P g(q), \qquad \text{for all } p \in P \text{ and } q \in Q.$$

The pair (f, g) is called an *adjoint pair* with the map f called the *left* or *lower adjoint* of the map g which in turn is called the *right* or *upper adjoint* of f. We say that an order-preserving map $f : \mathbb{P} \to \mathbb{Q}$ *has an upper adjoint* if there is an order-preserving map $g : \mathbb{Q} \to \mathbb{P}$ making (f, g) an adjoint pair. The notion of *having a lower adjoint* is defined analogously. A *Galois connection*[1] from a poset $\mathbb{P}$ to a poset $\mathbb{Q}$ is an adjunction from the poset $\mathbb{P}$ to the poset $\mathbb{Q}^\partial$.

A.1.2. Proposition. *Let $f : \mathbb{P} \to \mathbb{Q}$ be an order-preserving map between complete posets.*

1. *The map f has an upper adjoint if, and only if, f preserves all suprema.*

2. *The map f has a lower adjoint if, and only if, f preserves all infima.*

A.1.3. Proposition. *Let (f, g) be a Galois connection from a poset $\mathbb{P}$ to poset $\mathbb{Q}$.*

1. *The maps $g \circ f : \mathbb{P} \to \mathbb{P}$ and $f \circ g : \mathbb{Q} \to \mathbb{Q}$ are both closure operators.*

2. *The $(g \circ f)$-closed elements of $\mathbb{P}$ are precisely the element in $g[Q]$ and the $(f \circ g)$-closed elements of $\mathbb{Q}$ are precisely the element in $f[P]$.*

3. *The posets $\mathbb{P}_{g \circ f}$ and $\mathbb{Q}^\partial_{f \circ g}$ are isomorphic.*

A.2 Universal algebra

Unless stated otherwise all the material in this section can be found in [46, 16, 65]. We assume that the reader is familiar with the most basic concepts of model theory, from, e.g., [157].

[1] Also in places referred to as an *antitone Galois connection*.

A.2.1 Types, algebras, terms, and equations

Operations Given a natural number $n \in \omega$ and a set A, an *operation of arity* n on A is a function $f\colon A^n \to A$. An *operation* on a set A is then an operation on A of arity n for some $n \in \omega$. We take the empty product A^0 to be a singleton set and hence operations of arity 0 may be identified with elements of A.

Types A *finitary algebraic type*, or simply *type*, is a function $\Omega\colon \kappa \to \omega$, for some cardinal κ. The cardinal κ is called the *order* of Ω and is denoted by $o(\Omega)$. A *subtype* of a type Ω is a type Ω_0 together with an injective function $i\colon o(\Omega_0) \hookrightarrow o(\Omega)$ such that $\Omega_0(\alpha) = \Omega(i(\alpha))$ for all $\alpha < o(\Omega_0)$

Algebras An Ω-*algebra* or *algebra of type* Ω, or simply *algebra* if the type is understood from the context, is a pair $\mathbf{A} = \langle A, O\rangle$ with A a non-empty set, called the *carrier* or *universe* of $\mathbf{A}$, and O an $o(\Omega)$-indexed list $\langle f_\alpha\rangle_{\alpha<o(\Omega)}$ of operations on A such that the operation f_α has arity $\Omega(\alpha)$. As we will only be concerned with algebras with finitely many operations a type may simply be indicated by a finite list of natural numbers. By a class of *similar algebras* we will understand a class of Ω-algebras $\mathcal{K}$ for some fixed but unspecified type Ω. We will generally use the same symbol to denote the operations with index α on all algebras of a fixed type Ω. Furthermore, in the presence of multiple algebras of the same type it can be useful to indicate that an operation f_α, for $\alpha < o(\Omega)$, belongs to an algebra $\mathbf{A}$ by writing $f_\alpha^{\mathbf{A}}$.

Reducts Let Ω be a type and Ω_0 a subtype. An Ω_0-algebra $\mathbf{B} = \langle B, O'\rangle$ is called an Ω_0-*reduct* of an Ω-algebra $\mathbf{A} = \langle A, O\rangle$ provided that $A = B$ and $f_\alpha^{\mathbf{B}} = f_{i(\alpha)}^{\mathbf{A}}$, for all $\alpha < o(\Omega_0)$.

Languages To each type Ω and each cardinal λ corresponds a language L_Ω^λ consisting of a function symbol F_α of arity $\Omega(\alpha)$ for each $\alpha < o(\Omega)$, together with a set of variables $\mathrm{X}_\lambda = \{x_\xi : \xi < \lambda\}$ of cardinality λ. We will also use x, y, z, u, v, w to denote members of X_λ. Each Ω-algebra is a model of the language L_Ω^λ in the standard sense, interpreting each function symbol F_α with index α as the operation f_α with the same index. In practice, given a class of Ω-algebras we will not explicitly distinguish between an operation and its corresponding function symbol.

Terms Let $\Omega\colon \kappa \to \omega$ be a type and let λ be a cardinal. For each $X \subseteq X_\lambda$ We define the set $\mathrm{Tm}(\Omega, X)$ of Ω-*terms* with variables in X by the following recursion:

(i) any variable $x_\xi \in X$ belongs to $\mathrm{Tm}(\Omega, X)$,

(ii) if $\alpha < o(\Omega)$ and $t_1, \dots, t_{\Omega(\alpha)}$ belong to $\mathrm{Tm}(\Omega, X)$, then $F_\alpha(t_1, \dots, t_{\Omega(\alpha)})$ belongs to $\mathrm{Tm}(\Omega, X)$.

In case $X = X_\lambda$ we write $\mathrm{Tm}(\Omega, \lambda)$ for the set $\mathrm{Tm}(\Omega, X)$. Furthermore, when the type is clear from the context we will simply talk about *terms* and denote the set of terms in at most λ-many variables by $\mathrm{Tm}(\lambda)$, or simply Tm in case $\lambda = \aleph_0$. Given a term t and set of variables $X = \{x_{\xi_1}, \dots, x_{\xi_n}\}$ we may write $t(x_{\xi_1}, \dots, x_{\xi_n})$ to indicate that $t \in \mathrm{Tm}(\Omega, X)$ and say that the *variables occurring in t are among X* . Note that every term belongs to $t \in \mathrm{Tm}(\Omega, X)$ for some finite set X.

Term functions Given a type Ω along with an Ω-algebra $\mathbf{A}$ for any Ω-term $t(x_{\xi_1}, \dots, x_{\xi_n})$ we define a function $t^{\mathbf{A}} \colon A^n \to A$, called the *term function of arity n* determined by t, by the following recursion:

(i) if t is the variable $x_{\xi_i} \in$ X, then for all $a_1, \dots, a_n \in A$,

$$t^{\mathbf{A}}(a_1, \dots, a_n) = a_i,$$

(ii) if t is the term $F_\alpha(t_1, \dots, t_{\Omega(\alpha)})$ for $\alpha < o(\Omega)$ and $t_1, \dots, t_{\Omega(\alpha)}$ are terms in $\mathrm{Tm}(\Omega, X)$, then for all $a_1, \dots, a_n \in A$,

$$t^{\mathbf{A}}(a_1, \dots, a_n) = f_\alpha^{\mathbf{A}}(t_1^{\mathbf{A}}(a_1, \dots, a_n), \dots, t_{\Omega(\alpha)}^{\mathbf{A}}(a_1, \dots, a_n)).$$

A function $f \colon A^n \to A$ is called a *term function* if $f = t^{\mathbf{A}}$ for some Ω-term $t(x_{\xi_1}, \dots, x_{\xi_n})$.

Term equivalence Given types Ω_1 and Ω_2, an Ω_1-algebra $\mathbf{A}_1$ is said to be *term equivalent* to an Ω_2-algebra $\mathbf{A}_2$ provided that the algebras $\mathbf{A}_1$ and $\mathbf{A}_2$ have the same universe and for every $\alpha < o(\Omega_1)$ there exists an Ω_2-term t such that $f_\alpha^{\mathbf{A}_1} = t^{\mathbf{A}_2}$ and conversely for every $\alpha < o(\Omega_2)$ there exists an Ω_1-term s such that $f_\alpha^{\mathbf{A}_2} = s^{\mathbf{A}_1}$. We say that a class $\mathcal{K}_1$ of Ω_1-algebras is *term equivalent* to a class $\mathcal{K}_2$ of Ω_2-algebras provided that any member of $\mathcal{K}_1$ is term equivalent to some member of $\mathcal{K}_2$ and vice versa.

Equations Let $X \subseteq X_\lambda$ be given. An Ω-*equation in X* is a pair of terms $s, t \in \mathrm{Tm}(\Omega, X)$, written $s \approx t$. A *universal Ω-clause in X* is an expression of the form

$$\varepsilon_1 \textsf{ and } \dots \textsf{ and } \varepsilon_m \implies \varepsilon_{m+1} \textsf{ or } \dots \textsf{ or } \varepsilon_n,$$

where $m \in \omega$ and ε_i is an Ω-equation for all $i \in \{1, \dots, n\}$. A universal Ω-clause of the form

$$\varepsilon_1 \textsf{ and } \dots \textsf{ and } \varepsilon_m \implies \varepsilon_{m+1},$$

is called a *quasi Ω-equation in X*. Again when the type Ω is clear from the context we will simply refer to *equations*, *quasi-equations* and *universal clauses in X*. Similarly, when the set of variables X is clear from the context we simply speak of *equations*, *quasi-equations* and *universal clauses*. Given equations ε_1 and ε_2 we will write

$$\varepsilon_1 \iff \varepsilon_2$$

to denote the conjunction of the quasi-equations

$$\varepsilon_1 \implies \varepsilon_2 \quad \text{and} \quad \varepsilon_2 \implies \varepsilon_1.$$

Validity We say that an Ω-algebra $\mathbf{A}$ *validates*, or *satisfies*, an equation $s \approx t$ in some finite set $X = \{x_{\xi_1}, \dots, x_{\xi_k}\}$ provided that the corresponding term functions $s^{\mathbf{A}}, t^{\mathbf{A}} \colon A^k \to A$ coincide. In this case we write $\mathbf{A} \models s \approx t$. Note that this notion does not depend of the choice of finite X such that $s, t \in \mathrm{Tm}(\Omega, X)$. Similarly, an Ω-algebra $\mathbf{A}$ *validates*, or *satisfies*, a universal Ω-clause

$$s_1 \approx t_1 \text{ and } \dots \text{ and } s_m \approx t_m \implies s_{m+1} \approx t_{m+1} \text{ or } \dots \text{ or } s_n \approx t_n, \qquad (q)$$

in $X = \{x_{\xi_1}, \dots, x_{\xi_k}\}$, written $\mathbf{A} \models q$, provided that for all $a_1, \dots, a_k \in A$ if

$$s_i^{\mathbf{A}}(a_1, \dots, a_k) = t_i^{\mathbf{A}}(a_1, \dots, a_k)$$

for each $i \in \{1, \dots, m\}$, then there is $j \in \{m+1, \dots, n\}$ such that

$$s_j^{\mathbf{A}}(a_1, \dots, a_k) = t_j^{\mathbf{A}}(a_1, \dots, a_k).$$

Again this notion is independent of the choice of finite X such that all the terms s_i, t_i with $i \in \{1, \dots, n\}$ belong to $\mathrm{Tm}(\Omega, X)$. If Q is a set of clauses we write $\mathbf{A} \models Q$ to indicate that $\mathbf{A} \models q$ for all $q \in Q$.

Valuations A *valuation* on an Ω-algebra $\mathbf{A}$ is a function $\nu \colon X \to A$ for some $X \subseteq \mathrm{X}_\lambda$. We say that an Ω-equation $s \approx t$ in $X = \{x_{\xi_1}, \dots, x_{\xi_k}\}$ is *true* under a valuation $\nu \colon X \to A$ on an Ω-algebra $\mathbf{A}$, written $(\mathbf{A}, \nu) \models s \approx t$, provided that

$$s^{\mathbf{A}}(\nu(x_{\xi_1}), \dots, \nu(x_{\xi_k})) = t^{\mathbf{A}}(\nu(x_{\xi_1}), \dots, \nu(x_{\xi_k})).$$

Similarly, a universal Ω-clause

$$s_1 \approx t_1 \text{ and } \dots \text{ and } s_m \approx t_m \implies s_{m+1} \approx t_{m+1} \text{ or } \dots \text{ or } s_n \approx t_n, \qquad (q)$$

in $X = \{x_{\xi_1}, \dots, x_{\xi_k}\}$ is *true* under a valuation $\nu \colon X \to A$ on an Ω-algebra $\mathbf{A}$, written $(\mathbf{A}, \nu) \models q$, provided that if

$$s_i^{\mathbf{A}}(\nu(x_{\xi_1}), \dots, \nu(x_{\xi_k})) = t_i^{\mathbf{A}}(\nu(x_{\xi_1}), \dots, \nu(x_{\xi_k}))$$

for each $i \in \{1, \dots, m\}$, then there is $j \in \{m+1, \dots, n\}$ such that

$$s_j^{\mathbf{A}}(\nu(x_{\xi_1}), \dots, \nu(x_{\xi_k})) = t_j^{\mathbf{A}}(\nu(x_{\xi_1}), \dots, \nu(x_{\xi_k})).$$

It is not difficult to see that an equation or universal clause is satisfied on an algebra $\mathbf{A}$ if, and only if, it is true under all valuations.

A.2.2 Operations on class of similar algebras

Homomorphisms A *homomorphism* from an Ω-algebras $\mathbf{A}$ to an Ω-algebra $\mathbf{B}$ is a function $h\colon A \to B$ such that

$$h(f_\alpha^{\mathbf{A}}(a_1, \dots, a_{\Omega(\alpha)})) = f_\alpha^{\mathbf{B}}(h(a_1), \dots, h(a_{\Omega(\alpha)})),$$

for all $\alpha < o(\Omega)$ and all $a_1, \dots, a_{\Omega(\alpha)} \in A$. We write $h\colon \mathbf{A} \to \mathbf{B}$ to indicate that h is a homomorphism from an algebra $\mathbf{A}$ to an algebra $\mathbf{B}$ of the same type. Similarly, we write $\mathbf{A} \to \mathbf{B}$ to indicate that there is a homomorphism from the algebra $\mathbf{A}$ to the algebra $\mathbf{B}$.

Subalgebras An injective homomorphism between two algebras of the same type is called an *embedding*. We write $h : \mathbf{A} \hookrightarrow \mathbf{B}$ to denote that h is an embedding from $\mathbf{A}$ to $\mathbf{B}$ and $\mathbf{A} \hookrightarrow \mathbf{B}$ to denote that there is an embedding from $\mathbf{A}$ to $\mathbf{B}$. An *isomorphism* is a surjective embedding and two algebras of the same type are *isomorphic* if there is an isomorphism between them. A *subalgebra* of an algebra $\mathbf{A}$ is an algebra $\mathbf{B}$ of the same type such that $B \subseteq A$ and the identity function on A restricted to B is an embedding from $\mathbf{B}$ to $\mathbf{A}$. When Ω_0 is a subtype of a type Ω we call an Ω_0-subalgebra of the Ω_0-reduct of an Ω-algebra $\mathbf{A}$ an Ω_0-*subreduct* of $\mathbf{A}$. Given any Ω-algebra $\mathbf{A}$ the collection of its subalgebras is closed under arbitrary intersections. Consequently, for every subset $S \subseteq A$ there is a least, with respect to the partial order of set-theoretic inclusion, subalgebra of $\mathbf{A}$ containing the set S. We denote this algebra by $\langle S \rangle$ and call it *the subalgebra generated by* S. We say that an algebra $\mathbf{A}$ is *generated by* a subset $S \subseteq A$ if $\mathbf{A} = \langle S \rangle$. An algebra is *finitely generated* if it is generated by one of its finite subsets. An algebra $\mathbf{A}$ is *locally finite* provided that all of its finitely generated subalgebras are finite. A class of algebras $\mathcal{K}$ is *locally finite* if all of its members are.

Homomorphic images A *homomorphic image* of an algebra $\mathbf{A}$ is an algebra $\mathbf{B}$ of the same type such that there is a surjective homomorphism from $\mathbf{A}$ onto $\mathbf{B}$. We write $h : \mathbf{A} \twoheadrightarrow \mathbf{B}$ to denote that h is a surjective homomorphism from $\mathbf{A}$ to $\mathbf{B}$ and $\mathbf{A} \twoheadrightarrow \mathbf{B}$ to denote that $\mathbf{B}$ is a homomorphic image of $\mathbf{A}$.

Congruences A binary relation $R \subseteq A^2$ on a set A is *compatible with* an operation $f\colon A^n \to A$ if

$$f(a_1, \dots, a_n) R f(b_1, \dots, b_n),$$

for all $a_1, b_1, \dots a_n, b_n \in A$ such that $a_i R b_i$ for all $i \in \{1, \dots, n\}$. A *congruence* on an Ω-algebra $\mathbf{A}$ is an equivalence relation $\theta \subseteq A^2$ compatible with the operation f_α for all $\alpha < o(\Omega)$. The set of congruences on an algebra $\mathbf{A}$ is closed under arbitrary intersections and hence forms a complete lattice under the partial order of set-theoretic inclusion. We denote this lattice by $\mathrm{Con}(\mathbf{A})$. We let Δ, or $\Delta_{\mathbf{A}}$, denote the least element and ∇, or $\nabla_{\mathbf{A}}$, the greatest element of the lattice $\mathrm{Con}(\mathbf{A})$. Evidently, $\Delta_{\mathbf{A}}$ is the identity relation on any algebra $\mathbf{A}$ and $\nabla_{\mathbf{A}}$ is the relation A^2. We call an algebra $\mathbf{A}$ *congruence distributive* when the lattice $\mathrm{Con}(\mathbf{A})$ is distributive. Similarly, we call a class of similar algebras $\mathcal{K}$ congruence distributive provided all members of $\mathcal{K}$ are congruence distributive. The compatibility condition insures that the set A/θ of equivalence classes under a congruence relation θ on an Ω-algebra $\mathbf{A}$ may be equipped with the structure of an Ω-algebra which we denote by $\mathbf{A}/\theta$. Namely, letting a/θ denote the equivalence class of $a \in A$ under θ, we define for all $\alpha < o(\Omega)$ an operation f_α on A/θ by

$$f_\alpha^{\mathbf{A}/\theta}(a_1/\theta, \dots, a_{\Omega(\alpha)}/\theta) = f_\alpha^{\mathbf{A}}(a_1, \dots, a_{\Omega(\alpha)})/\theta,$$

for all $a_1, \dots, a_{\Omega(\alpha)} \in A$. The projection map π_θ given by $a \mapsto a/\theta$ is then a surjective homomorphism of Ω-algebras and so any congruence of $\mathbf{A}$ determines a homomorphic image of $\mathbf{A}$. The algebra $\mathbf{A}/\theta$ is often referred to as *the quotient of* $\mathbf{A}$ *under* θ or simply a *quotient of* $\mathbf{A}$.

Kernels Conversely, any homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ of Ω-algebras determines a congruence

$$\ker h = \{(a, b) \in A^2 : h(a) = h(b)\}$$

on $\mathbf{A}$, called the *kernel* of h. Given any congruence θ on $\mathbf{A}$ such that $\theta \subseteq \ker h$ we obtain a homomorphism $\overline{h}\colon \mathbf{A}/\theta \to \mathbf{B}$ of Ω-algebras with the property that $h = \overline{h} \circ \pi_\theta$ by letting $\overline{h}(a/\theta) = h(a)$, for all $a \in A$. In diagrammatic form:

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "h"] \arrow[d, two heads, "\pi_\theta"'] & \mathbf{B} \\
\mathbf{A}/\theta \arrow[ur, "\overline{h}"'] &
\end{tikzcd}$$

The map $\overline{h}$ is injective, if, and only if, $\ker h \subseteq \theta$. Consequently, if $h\colon \mathbf{A} \twoheadrightarrow \mathbf{B}$ is a surjective homomorphism of Ω-algebras, then the quotient $\mathbf{A}/\ker h$ of $\mathbf{A}$ is isomorphic to $\mathbf{B}$. This establishes a bijective correspondence between congruences on $\mathbf{A}$ and homomorphic images of $\mathbf{A}$.

Reduced and direct products A *direct product* or simply *product* of a family of Ω-algebras $\{\mathbf{A}_i : i \in I\}$ is an Ω-algebra, denoted by $\prod_{i \in I} \mathbf{A}_i$, with carrier set the Cartesian product $\prod_{i \in I} A_i$ of the family $\{A_i : i \in I\}$ of the underlying universes and operations given by

$$f_\alpha^{\prod_{i \in I} \mathbf{A}_i}(a_1, \dots, a_{\Omega(\alpha)})(i) = f_\alpha^{\mathbf{A}_i}(a_1(i), \dots, a_{\Omega(\alpha)}(i)),$$

for all $\alpha < o(\Omega)$, all $a_1, \dots, a_{\Omega(\alpha)} \in \prod_{i \in I} A_i$ and all $i \in I$. The algebras $\mathbf{A}_i$ are often called the *factors* of the direct product $\prod_{i \in I} \mathbf{A}_i$. For each $i \in I$ the map given by $a \mapsto a(i)$ is a surjective homomorphism from $\prod_{i\in I} \mathbf{A}_i$ to $\mathbf{A}_i$. We denote this map by π_i and refer to it as the *canonical projection.* In case $I = \emptyset$ we will use the convention that the direct product $\prod_{i\in I} \mathbf{A}_i$ is a *trivial algebra* , i.e., a one element algebra. Given a family of Ω-algebras $\{\mathbf{A}_i : i \in I\}$, any two elements a_1, a_2 of $\prod_{i\in I} A_i$ give rise to a subset of I

$$[\![a_1 = a_2]\!] = \{i \in I : a_1(i) = a_2(i)\},$$

called the *equalizer* of a_1, a_2. A *filter over* a set I is a filter of the partially ordered set $\langle \wp(I), \subseteq\rangle$ of subsets of I. Given any family $\{\mathbf{A}_i : i \in I\}$ of similar algebras each filter F over I determines a congruence θ_F on the direct product $\prod_{i\in I} \mathbf{A}_i$ given by

$$a_1 \; \theta_F \; a_2 \iff [\![a_1 = a_2]\!] \in F.$$

The resulting algebra $\prod_{i\in I} \mathbf{A}_i/\theta_F$, denoted by $\prod_{i\in I} \mathbf{A}_i/F$, is called a *reduced product* and in case F is an ultrafilter over I, i.e., a proper maximal filter of the poset $\langle \wp(I), \subseteq\rangle$, an *ultraproduct*. For an ultrafilter U over a set I and an algebra $\mathbf{A}$ we call the ultraproduct of the family $\{\mathbf{A}_i : i \in I\}$, obtained by letting $\mathbf{A}_i = \mathbf{A}$ for all $i \in I$, an *ultrapower* of $\mathbf{A}$ and denote it by $\mathbf{A}^I/U$. We will be using the convention that a reduced product on the empty family is a trivial algebra. The central result about ultraproducts is due to Łos, see, e.g., [157, Thm. 8.5.3] for a proof.

A.2.1. THEOREM (Łos). *Let Ω be a type, $\{\mathbf{A}_i : i \in I\}$ a non-empty family of Ω-algebras, and χ a sentence in the first-order language of Ω-algebras. Then for any ultrafilter U on I we have*

$$\prod_{i\in I} \mathbf{A}_i/U \models \chi \quad \textit{if, and only if,} \quad \{i \in I : \mathbf{A}_i \models \chi\} \in U.$$

In particular, any Ω-algebra is elementarily equivalent to any of its ultrapowers.

Subdirect products Let $\{\mathbf{A}_i : i \in I\}$ be a family of Ω-algebras. An Ω-algebra $\mathbf{A}$ is a *subdirect product* of the family $\{\mathbf{A}_i : i \in I\}$ provided that

(i) the algebra $\mathbf{A}$ is a subalgebra of the direct product $\prod_{i\in I} \mathbf{A}_i$,

(ii) for each $i \in I$ the canonical projection $\pi_i \colon \prod_{i\in I} \mathbf{A}_i \twoheadrightarrow \mathbf{A}_i$ is surjective when restricted to A, i.e., $\pi_i[A] = A_i$.

A *subdirect embedding* is an embedding $e \colon \mathbf{A} \hookrightarrow \prod_{i\in I} \mathbf{A}_i$ of similar algebras such that the image of A under e is a subdirect product of the family $\{\mathbf{A}_i : i \in I\}$.

Boolean products Let $\{\mathbf{A}_x : x \in X\}$ be a family of similar algebras indexed by a set X carrying a *Stone* or *Boolean topology*, i.e., topology which is compact, Hausdorff, and zero-dimensional. We call a subdirect product $\mathbf{A}$ of the family $\{\mathbf{A}_x : x \in X\}$ a *weak Boolean product* if,

(i) the equalizer $[\![a_1 = a_2]\!]$ is open for all $a_1, a_2 \in A$,

(ii) for each $a_1, a_2 \in A$ and each *clopen*[2] subset U of X there is $a_3 \in A$ such that

$$[\![a_1 = a_3]\!] = U \quad \text{and} \quad [\![a_2 = a_3]\!] = X \backslash U.$$

The last condition is often referred to as the *patchwork property*. A *Boolean product* $\mathbf{A}$ of the family $\{\mathbf{A}_x : x \in X\}$ is a weak Boolean product where for all a_1, a_2 the equalizer $[\![a_1 = a_2]\!]$ is also closed and hence clopen. If $\mathbf{A}$ is isomorphic to a (weak) Boolean product of $\{\mathbf{A}_x : x \in X\}$ we may refer to X as the *base space*, the factors $\mathbf{A}_x$ as the *stalks* of $\mathbf{A}$, and the elements of $\mathbf{A}$ as *sections*. This terminology comes from the fact that a Boolean product is essentially the same as an *algebra of global sections* of a *sheaf of algebras* over a Boolean space, see, e.g., [47].

Class operations Given a class of similar algebras $\mathcal{K}$ we denote by

(i) $\mathrm{I}(\mathcal{K})$ the class of all algebras isomorphic to some member of $\mathcal{K}$,

(ii) $\mathrm{S}(\mathcal{K})$ the class of all subalgebras of some member of $\mathcal{K}$,

(iii) $\mathrm{H}(\mathcal{K})$ the class of all homomorphic images of some member of $\mathcal{K}$,

(iv) $\mathrm{P}(\mathcal{K})$ the class of all products of families of members of $\mathcal{K}$,

(v) $\mathrm{P_R}(\mathcal{K})$ the class of all reduced products of families of members of $\mathcal{K}$,

(vi) $\mathrm{P_U}(\mathcal{K})$ the class of all ultraproducts of families of members of $\mathcal{K}$,

(vii) $\mathrm{P_S}(\mathcal{K})$ the class of all subdirect products of families of members of $\mathcal{K}$,

(viii) $\mathrm{P_B}(\mathcal{K})$ the class of all Boolean products of families of members of $\mathcal{K}$.

A.2.3 Subdirectly irreducible algebras

An algebra $\mathbf{A}$ is (*finitely*) *subdirectly irreducible* if for each subdirect embedding $e\colon \mathbf{A} \hookrightarrow \prod_{i\in I} \mathbf{A}_i$ of similar algebras (with I finite) there is an index $i \in I$ such that the homomorphism $\pi_i \circ e\colon \mathbf{A} \to \mathbf{A}_i$ is an isomorphism. Given a class $\mathcal{K}$ of similar algebras we denote by

[2]That is, simultaneously closed and open.

(i) $\mathcal{K}_{si}$ the class of subdirectly irreducible members of $\mathcal{K}$.

(ii) $\mathcal{K}_{fsi}$ the class of finitely subdirectly irreducible members of $\mathcal{K}$.

Evidently, we always have $\mathcal{K}_{si} \subseteq \mathcal{K}_{fsi}$. The property of being (finitely) subdirectly irreducible is completely determined by the structure of the congruence lattice.

A.2.2. THEOREM ([46, Thm. II.8.4]). *Let* $\mathbf{A}$ *be a non-trivial algebra.*

1. *The algebra* $\mathbf{A}$ *is subdirectly irreducible if, and only if, the least congruence* $\Delta_{\mathbf{A}}$ *is completely meet-irreducible in the lattice* $\mathrm{Con}(\mathbf{A})$.
2. *The algebra* $\mathbf{A}$ *is finitely subdirectly irreducible if, and only if, the least congruence* $\Delta_{\mathbf{A}}$ *is meet-irreducible in the lattice* $\mathrm{Con}(\mathbf{A})$.

It follows that a non-trivial algebra $\mathbf{A}$ is subdirectly irreducible precisely when $\mathrm{Con}(\mathbf{A})$ has a second least element, namely $\bigcap\{\theta \in \mathrm{Con}(\mathbf{A}) : \theta \neq \Delta_{\mathbf{A}}\}$. A central fact is the following result, originally due to Birkhoff, showing that any algebra of any type always has "enough" subdirectly irreducible quotients, see, e.g., [46, Thm. II.8.6] for a proof.

A.2.3. THEOREM (Birkhoff). *Any algebra is isomorphic to a subdirect product of its subdirectly irreducible homomorphic images.*

A.2.4 Varieties, quasi-varieties, and universal classes

Let Ω be a type. By an *equational* Ω*-theory* we understand a set of Ω-equations in the countably infinite set of variables $X_{\aleph_0}$. The notions of *quasi-equational* and *universal* Ω*-theories* are defined completely analogously. A class $\mathcal{K}$ of Ω-algebras is called

(i) a *variety* if it is the class of models of some equational Ω-theory,

(ii) a *quasi-variety* if it is the class of models of some quasi-equational Ω-theory,

(iii) a *universal class* if it is the class of models of some universal Ω-theory.

These types of classes are completely determined by the closure properties with respect to certain class operations. The first item of the following theorem is due to Birkhoff, the second item is due to Mal'cev, and the last item appears to be folklore. For proofs, see, e.g., [46, Thm. II.11.9, Thm. V.2.25, Thm. V.2.20].

A.2.4. THEOREM (Birkhoff, Mal'cev). *Let* $\mathcal{K}$ *be a non-empty class of similar algebras.*

1. *The class* $\mathcal{K}$ *is a variety if, and only if, its is closed under the formation of direct products, homomorphic images, and subalgebras.*

2. *The class $\mathcal{K}$ is a quasi-variety if, and only if, it is closed under the formation of reduced products, subalgebras, and isomorphic copies,*

3. *The class $\mathcal{K}$ is a universal class if, and only if, its is closed under the formation of ultraproducts, subalgebras, and isomorphic copies.*

It turns out that these closure conditions may be expressed more succinctly. The first item of the following theorem is due to Tarski, the second item is due to Mal'cev, and the last item appears to be folklore. For proofs, see, e.g., [46, Thm. II.9.5, Thm. V.2.20, Thm. V.2.25].

A.2.5. Theorem (Tarski, Mal'cev). *Let $\mathcal{K}$ be a non-empty class of similar algebras.*

1. *The class $\mathcal{K}$ is a variety if, and only if, $\mathcal{K} = \mathrm{HSP}(\mathcal{K})$,*

2. *The class $\mathcal{K}$ is a quasi-variety if, and only if, $\mathcal{K} = \mathrm{ISP_R}(\mathcal{K})$,*

3. *The class $\mathcal{K}$ is a universal class if, and only if, $\mathcal{K} = \mathrm{ISP_U}(\mathcal{K})$.*

We say that a variety $\mathcal{V}$ of similar algebras is *generated* by a class $\mathcal{K} \subseteq \mathcal{V}$ if $\mathcal{V}$ is the least variety containing the class $\mathcal{K}$, or equivalently, if $\mathcal{V} = \mathrm{HSP}(\mathcal{K})$. Thus a class $\mathcal{K}$ generates a variety $\mathcal{V}$ if, and only if, $\mathcal{K} \subseteq \mathcal{V}$ and each equation which is refuted by some algebra in $\mathcal{V}$ is also refuted by some algebra in $\mathcal{K}$. We say that $\mathcal{V}$ is *finitely generated* if it is generated by a finite class of finite algebras. Finally, we say that a class is *finitely approximable* or *has the finite model property* if it is generated by the class of its finite members. Similar definitions apply, *mutatis mutandis*, to quasi-varieties and universal classes. The following is then an immediate consequence of Theorems A.2.3 and A.2.5(1).

A.2.6. Theorem (Birkhoff). *Any variety of similar algebras is generated by its (finitely) subdirectly irreducible members.*

A.2.5 Jónsson's Lemma

The following useful property of congruence distributive varieties is sometimes referred to as Jónsson's Lemma [172, Sec. 3]. See also [46, Thm. IV.6.8] or [169, Cor. 1.5] for proofs.

A.2.7. Theorem (Jónsson). *Let $\mathcal{V}$ be a congruence distributive variety of similar algebras. If $\mathcal{V}$ is generated by $\mathcal{K} \subseteq \mathcal{V}$, then*

$$\mathcal{V}_{fsi} \subseteq \mathrm{HSP_U}(\mathcal{K}),$$

and consequently,

$$\mathcal{V} = \mathrm{IP_S HSP_U}(\mathcal{K}).$$

In particular, since $\mathrm{P_U}(\mathcal{K}) \subseteq \mathrm{I}(\mathcal{K})$ whenever $\mathcal{K}$ is a finite set of finite similar algebras, see, e.g., [46, Lem. IV.6.5], we must have that if $\mathcal{V}$ is a finitely generated congruence distributive variety, then

$$\mathcal{V}_{fsi} \subseteq \mathrm{HS}(\mathcal{K}),$$

for any finite set of finite algebras $\mathcal{K}$ generating $\mathcal{V}$. From this it follows that in a finitely generated congruence distributive variety all finitely subdirectly irreducible algebras are finite.

A.2.6 Free algebras

Let Ω be a type and let $\mathcal{K}$ be a class of Ω-algebras, $\mathbf{A} \in \mathcal{K}$ and $S \subseteq A$. For a cardinal λ we say that $\mathbf{A}$ is *freely λ-generated over $\mathcal{K}$* provided that

(i) the cardinality of S is λ,

(ii) the set S generates the algebra $\mathbf{A}$,

(iii) for any $\mathbf{B} \in \mathcal{K}$ and function $f\colon S \to B$ there is a homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ of Ω-algebras making the following diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d, hook] & \mathbf{B} \\ \mathbf{A} \arrow[ur] & \end{tikzcd}$$

commute.

An algebra $\mathbf{A}$ is *freely generated over $\mathcal{K}$*, for $\mathcal{K}$ some class of similar algebras if it is freely λ-generated over $\mathcal{K}$ for some cardinal λ. It is easy to show that if λ is a cardinal and $\mathbf{A}$, $\mathbf{B}$ are both freely λ-generated over a class of similar algebras $\mathcal{K}$, then $\mathbf{A}$ and $\mathbf{B}$ must be isomorphic as Ω-algebras. Consequently, for each cardinal λ there is (up to isomorphism) at most one member of $\mathcal{K}$ which is freely λ-generated over $\mathcal{K}$. In case there exists a member of $\mathcal{K}$ which is freely λ-generated over $\mathcal{K}$ we denote it, or rather some (canonically chosen) representative of it, by $\mathbf{F}_{\mathcal{K}}(\lambda)$ and call it *the free λ-generated $\mathcal{K}$ algebra* or *the free $\mathcal{K}$ algebra on λ-many generators.* A *free* $\mathcal{K}$-algebra is then a member of $\mathcal{K}$ which is a freely λ-generated over $\mathcal{K}$ for some cardinal λ. The following result is originally due to Birkhoff, see, e.g., [46, Thm. II.10.12] for a proof.

A.2.8. Theorem (Birkhoff). *Let Ω be a type and let $\mathcal{K}$ be any class of Ω-algebras such that* $\mathrm{ISP}(\mathcal{K}) \subseteq \mathcal{K}$*. Then free λ-generated $\mathcal{K}$-algebras exist for all cardinals* $\lambda > 0$*. Moreover, if $\Omega(\alpha) = 0$ for some $\alpha < o(\Omega)$ then free λ-generated $\mathcal{K}$-algebras exist for all cardinals λ.*

In particular any variety of similar algebras has free λ-algebras for all cardinals $\lambda > 0$. We conclude by noting that if $\mathcal{V}$ is a variety of Ω-algebras and $s(x_{\xi_1}, \ldots, x_{\xi_n}), t(\xi_1, \ldots, x_{\xi_n})$ are Ω-terms, with $n \geq 1$, then

$$\mathcal{V} \models s \approx t \quad \text{if, and only if,} \quad \mathbf{F}_{\mathcal{K}}(\lambda) \models s \approx t,$$

for all $\lambda \geq n$, see, e.g., [46, Cor. II.11.15] for a proof. In particular, any variety is generated by its free algebras.

A.3 Algebras

We here recall the definitions of the various algebraic structures which play a role in this thesis.

Semi-groups A *semi-group* is an algebra $\mathbf{A} = \langle A, \cdot \rangle$ of type $\langle 2 \rangle$ satisfying the equation

$$x \cdot (y \cdot z) \approx (x \cdot y) \cdot z.$$

In any semi-group $\langle A, \cdot \rangle$ we may simply write ab for the product $a \cdot b$. Similarly, we may also leave out parentheses writing $a \cdot b \cdot c$ for the element $a \cdot (b \cdot c) = (a \cdot b) \cdot c$. Finally, for $a \in A$ and natural number $n \geq 1$ we write a^n for the n-fold product of a with itself, i.e., $a^1 = a$ and $a^{n+1} = a^n a$. A *monoid* is an algebra $\mathbf{A} = \langle A, \cdot, \mathrm{e} \rangle$ of type $\langle 2, 0 \rangle$, with $\langle A, \cdot \rangle$ a semi-group, satisfying the equations

$$x \cdot \mathrm{e} \approx \mathrm{e} \quad \text{and} \quad \mathrm{e} \cdot x \approx \mathrm{e}.$$

We refer to the element e as the (*monoidal*) *unit* of $\mathbf{A}$. If $a, b \in A$ are elements of a monoid $\mathbf{A} = \langle A, \cdot, \mathrm{e} \rangle$ such that

$$a \cdot b = \mathrm{e} \quad \text{and} \quad b \cdot a = \mathrm{e},$$

we call b the *inverse* of a and vice verse. A *group* is a monoid in which every element has an inverse. It is not difficult to show that any element of a monoid can have at most one inverse. Thus, a group may be identified with an algebra $\mathbf{A} = \langle A, \cdot, ^{-1}, \mathrm{e} \rangle$ of type $\langle 2, 1, 0 \rangle$, with $\langle A, \cdot, \mathrm{e} \rangle$ a monoid, satisfying the equations

$$x \cdot x^{-1} \approx \mathrm{e} \quad \text{and} \quad x^{-1} \cdot x \approx \mathrm{e}.$$

An algebra $\mathbf{A}$ with a semi-group reduct $\langle A, \cdot \rangle$ is *commutative* provided that the equation

$$x \cdot y \approx y \cdot x,$$

is satisfied. Similarly, an algebra $\mathbf{A}$ with a semi-group reduct $\langle A, \cdot \rangle$ is *idempotent* if the equation

$$x \cdot x \approx x,$$

is satisfied.

Lattices A *semi-lattice*, respectively *bounded semi-lattice*, is an idempotent, commutative semi-group, respectively, monoid. A *lattice* is an algebra $\langle L, \wedge, \vee \rangle$ of type $\langle 2, 2 \rangle$ with $\langle L, \wedge \rangle$ and $\langle L, \vee \rangle$ a pair of semi-lattices satisfying the equations

$$x \approx x \vee (x \wedge y) \quad \text{and} \quad x \approx x \wedge (x \vee y).$$

A *bounded lattice* is an algebra $\langle L, \wedge, \vee, 0, 1 \rangle$ of type $\langle 2, 2, 0, 0 \rangle$ such that $\langle L, \wedge, \vee \rangle$ is a lattice and $\langle L, \wedge, 1 \rangle$ and $\langle L, \vee, 0 \rangle$ are bounded semi-lattices. A lattice $\mathbf{L} = \langle L, \wedge, \vee \rangle$ is *distributive* if either one of the following two equivalent equations are satisfied

$$x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z) \qquad x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z).$$

We denote by $\mathcal{DL}$ the class of distributive lattices.

Lattices as algebras and as posets Given a lattice $\mathbf{L} = \langle L, \wedge, \vee \rangle$ we obtain a partial ordered set $\langle L, \leq \rangle$ by letting $a \leq b$ if, and only if, $a \wedge b = a$, or equivalently $a \vee b = b$. This partial order is such that every pair of elements $a, b \in L$ has a least upper bound, namely $a \vee b$, and a greatest lower bound, namely $a \wedge b$. Thus the poset $\langle L, \leq \rangle$ is a lattice in the sense of Section A.1. Conversely, given any poset $\mathbb{P} = \langle P, \leq \rangle$ which is a lattice in the sense of Section A.1 we obtain an algebra $\langle P, \wedge, \vee \rangle$ by letting $a \wedge b$ be the greatest lower bound of a and b in $\mathbb{P}$ and similarly letting $a \vee b$ be the least upper bound of a and b in $\mathbb{P}$. Given any type Ω with the property that all Ω-algebra have a lattice reduct, for Ω-terms s and t we will let the expression $s \leq t$ be an abbreviation for the equation $s \wedge t \approx s$ and also refer to it as an equation.

Pseudo-complemented distributive lattice A *pseudo-complemented distributive lattice* is an algebra $\langle D, \wedge, \vee, \neg, 0, 1 \rangle$ of type $\langle 2, 2, 1, 0, 0 \rangle$ such that the structure $\langle D, \wedge, \vee, 0, 1 \rangle$ is a bounded distributive lattice satisfying the quasi-equations

$$x \wedge y \leq 0 \quad \Longleftrightarrow \quad y \leq \neg x.$$

Similarly, a *supplemented distributive lattice* is an algebra $\langle D, \wedge, \vee, \sim, 0, 1 \rangle$ of type $\langle 2, 2, 1, 0, 0 \rangle$ such that $\langle D, \wedge, \vee, 0, 1 \rangle$ is a bounded distributive lattice satisfying the quasi-equations

$$1 \leq x \vee y \quad \Longleftrightarrow \quad {\sim} x \leq y.$$

Thus a bounded distributive lattice is the reduct of a supplemented distributive lattice if, and only if, its order dual is the reduct of a pseudo-complemented distributive lattice and vice versa. The class of pseudo-complemented (supplemented) distributive lattices forms a variety, see, e.g., [16, Thm. VIII.3.1]. An element $a \in A$ of a pseudo-complement distributive lattice $\langle D, \wedge, \vee, \neg, 0, 1 \rangle$ is called *central* or *complemented* if $a \vee \neg a = 1$. A *Boolean algebra* is a pseudo-complemented distributive lattice in which every element is central. Consequently, Boolean algebras may be identified with pseudo-complemented distributive lattices satisfying the equation

$$x \vee \neg x \approx 1$$

or supplemented distributive lattices satisfying the equation

$$x \wedge \sim x \approx 0.$$

Thus the class of Boolean algebras forms a variety which we denote by $\mathcal{BA}$.

Residuated lattices A *lattice-ordered monoid* is an algebra $\langle A, \wedge, \vee, \cdot, \mathrm{e}\rangle$ of type $\langle 2, 2, 2, 0\rangle$ with $\langle A, \wedge, \vee\rangle$ a lattice and $\langle A, \cdot, \mathrm{e}\rangle$ a monoid satisfying the equation

$$x \cdot (y \vee z) \cdot w \approx (x \cdot y \cdot w) \vee (x \cdot z \cdot w).$$

A *residuated lattice-ordered monoid* or simply *residuated lattice* is an algebra $\langle A, \wedge, \vee, \cdot, \backslash, /, \mathrm{e}\rangle$ of type $\langle 2, 2, 2, 2, 2, 0\rangle$ such that $\langle A, \wedge, \vee, \cdot, \mathrm{e}\rangle$ is a lattice-ordered monoid satisfying the following quasi-equations

$$x \cdot y \leq z \iff y \leq x\backslash z \iff x \leq z/y.$$

The class of residuated lattices is in fact a variety, see, e.g., [98, Thm. 2.7], which we denote by $\mathcal{RL}$. A residuated lattice $\langle A, \wedge, \vee, \cdot, \backslash, /, \mathrm{e}\rangle$ is *integral* if it satisfies the equation

$$x \leq \mathrm{e}.$$

We denote by $\mathcal{IRL}$ the variety of integral residuated lattices. A *pointed residuated lattice*, or *FL-algebra*, is an algebra $\langle A, \wedge, \vee, \cdot, \backslash, /, \mathrm{e}, \mathrm{f}\rangle$ of type $\langle 2, 2, 2, 2, 2, 0, 0\rangle$ such that $\langle A, \wedge, \vee, \cdot, \backslash, /, \mathrm{e}\rangle$ is a residuated lattice. We will use the convention that a property defined for residuated lattices holds for an FL-algebra provided it holds for its residuated lattice reduct.

ℓ-groups A *lattice-ordered group*, or simply *ℓ-group*, is a residuated lattice $\langle A, \wedge, \vee, \cdot, \backslash, /, \mathrm{e}\rangle$ satisfying the equation

$$(\mathrm{e}/x) \cdot x \approx \mathrm{e}.$$

It is not difficult to see that the class of ℓ-groups is term equivalent to the class of algebras $\mathbf{A} = \langle A, \wedge, \vee, \cdot, ^{-1}, \mathrm{e}\rangle$ of type $\langle 2, 2, 2, 1, 0\rangle$ such that

(i) the structure $\langle A, \wedge, \vee, \cdot, \mathrm{e}\rangle$ is a lattice-ordered monoid,

(ii) the structure $\langle A, \cdot, ^{-1}, \mathrm{e}\rangle$ is a group.

We denote the variety of all ℓ-groups by $\mathcal{LG}$.

Heyting algebras A *Heyting algebra* is an algebra $\langle A, \wedge, \vee, \to, 0, 1\rangle$ of type $\langle 2, 2, 2, 0, 0\rangle$, with $\langle A, \wedge, \vee, 0, 1\rangle$ a bounded lattice, satisfying the quasi-equations

$$x \wedge y \leq z \iff y \leq x \to z$$

or equivalently, see, e.g., [16, Thm. IX.4.1], the equations

(i) $x \wedge (x \to y) \approx x \wedge y$,

(ii) $x \wedge (y \to z) \approx x \wedge ((x \wedge y) \to (x \wedge z))$,

(iii) $z \wedge ((x \wedge y) \to x) \approx z$.

Consequently, the class of Heyting algebras is a variety which we denote by $\mathcal{HA}$. The Boolean algebras are up to term equivalence the Heyting algebras satisfying the equation

$$x \vee \neg x \approx 1,$$

where $\neg x$ is an abbreviation for the term $x \to 0$. Thus the variety of Boolean algebras may be identified with a subvariety of Heyting algebras. Furthermore, we note that the class of Heyting algebras is term equivalent to the class of commutative, integral and idempotent FL-algebras satisfying the equation

$$\mathrm{f} \leq x.$$

Consequently, the variety of Heyting algebras may be identified with a subvariety of FL-algebras.

Bi-Heyting algebras A *co-Heyting algebra* is an algebra $\langle A, \wedge, \vee, \leftarrow, 0, 1\rangle$ of type $\langle 2, 2, 2, 0, 0\rangle$, with $\langle A, \wedge, \vee, 0, 1\rangle$ a bounded lattice, satisfying the quasi-equations,

$$z \leq x \vee y \iff z \leftarrow x \leq y.$$

A *bi-Heyting algebra*[3] is then an algebra $\langle A, \wedge, \vee, \to, \leftarrow, 0, 1\rangle$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$ such that $\langle A, \wedge, \vee, \to, 0, 1\rangle$ is a Heyting algebra and $\langle A, \wedge, \vee, \leftarrow, 0, 1\rangle$ is a co-Heyting algebra. Since any bounded distributive lattice can be the reduct of at most one Heyting algebra and at most one co-Heyting algebra, we will say that a Heyting algebra is a bi-Heyting algebra when its bounded lattice reduct is also the reduct of a co-Heyting algebra.

A.4 Heyting algebras

We here collected a number of useful facts about Heyting algebras, see, e.g, [16, Chap. IX] or [221].

[3] Sometimes also called a double Heyting algebra, see, e.g., [229].

Pseudo-complements For any Heyting algebra $\mathbf{A} = \langle A, \wedge, \vee, \to, 0, 1\rangle$ we define a unary operation $\neg\colon A \to A$ by $\neg a := a \to 0$, for all $a \in A$. One may then easily verify that the algebra $\langle A, \wedge, \vee, \neg, 0, 1\rangle$ is a pseudo-complemented distributive lattice.

Infima and suprema If $\mathbf{A}$ is a Heyting algebra and $\{a\} \cup S \cup T \subseteq A$ is such that S has a greatest lower bound in $\mathbf{A}$ and T has a least upper bound in $\mathbf{A}$, then

$$a \wedge \bigvee T = \bigvee_{t\in T}(a \wedge t), \quad \bigvee T \to a = \bigwedge_{t\in t}(t \to a), \quad \text{and} \quad a \to \bigwedge S = \bigwedge_{s\in S}(a \to s).$$

In particularly, $\neg \bigvee T = \bigwedge_{t\in T} \neg t$. In fact, this holds true of any pseudo-complemented distributive lattice.

Filters and congruences It is easy to verify that a non-empty intersection of a family of filters is again a filter. Since any Heyting algebra $\mathbf{A}$ has a greatest element, namely 1, any filter on $\mathbf{A}$ must contain 1 whence it follows that the intersection of any non-empty family of filters on $\mathbf{A}$ is again non-empty and hence a filter. Consequently, the collection of filters on $\mathbf{A}$ forms a complete lattice, with partial order given by set-theoretic inclusion, which we denote by $\mathrm{Fil}(\mathbf{A})$. Given any filter F on a Heyting algebra $\mathbf{A}$ one may show that the relation θ_F on A defined by

$$a\ \theta_F\ b \quad \text{if, and only if,} \quad (a \to b) \wedge (b \to a) \in F,$$

is a Heyting algebra congruence on $\mathbf{A}$. Conversely, given any Heyting algebra congruence θ on $\mathbf{A}$ the equivalence class $1/\theta$ of the top element 1 is filter on $\mathbf{A}$. This establishes a one-to-one correspondence between Heyting algebra congruences and filters on $\mathbf{A}$.

A.4.1. Proposition ([16, Lem.IX.4.4]). *For any Heyting algebra* $\mathbf{A}$ *there is an isomorphism of lattices* $\mathrm{Con}(\mathbf{A})$ *and* $\mathrm{Fil}(\mathbf{A})$.

In particular, since $\mathrm{Fil}(\mathbf{A})$ is readily seen to be a distributive lattice for any Heyting algebra $\mathbf{A}$, it follows that the variety $\mathcal{HA}$ of Heyting algebras is congruence distributive.

Subdirectly irreducible Heyting algebras As another application of Proposition A.4.1 we obtain an intrinsic description of the subdirectly irreducible and finitely subdirectly irreducible Heyting algebras.

A.4.2. Theorem ([16, Thm.IX.4.5]). *A Heyting algebra is subdirectly irreducible, if and only if, it has a second greatest element.*

It is easy to see that any upper bounded lattice with a second greatest element must satisfy the universal clause

$$x \vee y \approx 1 \implies x \approx 1 \text{ or } y \approx 1.$$

Upper bounded distributive lattices satisfying this clause are called *well-connected*. Thus all subdirectly irreducible Heyting algebras are well-connected but not vice versa. The following is already known but since we have been unable to find a direct reference we provide an argument here.

A.4.3. PROPOSITION. *A Heyting is finitely subdirectly irreducible, if, and only if, it is well-connected.*

Proof:
Let $\mathbf{A}$ be a finitely subdirectly irreducible Heyting algebra and suppose that $a_1, a_2 \in A$ are such that $a_1 \vee a_2 = 1$, then the principal filters $\uparrow a_i = \{b \in A : a_i \leq b\}$, for $i \in \{1, 2\}$ are such that $\uparrow a_1 \cap \uparrow a_2 = \{1\}$. Letting θ_i be the corresponding congruences $\theta_{\uparrow a_i}$, for $i \in \{1, 2\}$, we have, by Proposition A.4.1, that $\theta_1 \wedge \theta_2 = \Delta_{\mathbf{A}}$. Consequently, by Theorem A.2.2(2), $\theta_i = \Delta_{\mathbf{A}}$ for some $i \in \{1, 2\}$. But then $\uparrow a_i = \{1\}$, and hence $a_i = 1$, which shows that $\mathbf{A}$ is well-connected.

Conversely, let $\mathbf{A}$ be a well-connected Heyting algebra and let θ_1, θ_2 be congruences on $\mathbf{A}$ such that $\theta_1 \wedge \theta_2 = \Delta_{\mathbf{A}}$ and $\theta_2 \neq \Delta_{\mathbf{A}}$. Then by Proposition A.4.1 we must have $a_2 \in 1/\theta_2$ with $a_2 \neq 1$. Now, for each $a_1 \in 1/\theta_1$ we must have that $a_1 \vee a_2 (\theta_1 \wedge \theta_2) 1$ and hence that $a_1 \vee a_2 = 1$. By the assumption that $\mathbf{A}$ is well-connected we obtain that $a_1 = 1$ for each $a_1 \in 1/\theta_1$ showing that $1/\theta_1 = \{1\}$ and hence by Proposition A.4.1 that $\theta_1 = \Delta_{\mathbf{A}}$. Thus $\Delta_{\mathbf{A}}$ is meet-irreducible in the lattice $\mathrm{Con}(\mathbf{A})$, whence $\mathbf{A}$ is finitely subdirectly irreducible by Theorem A.2.2(2). □

Finite model property We conclude this section by establishing that the variety of Heyting algebras enjoys the finite model property. In fact, a stronger statement holds which is useful on its own. Given Heyting algebras $\mathbf{A}$ and $\mathbf{B}$ we write $\mathbf{B} \hookrightarrow_{\wedge,\vee,0,1} \mathbf{A}$ to indicate that the bounded lattice reduct of $\mathbf{B}$ is a subreduct of $\mathbf{A}$.

A.4.4. LEMMA ([29, Lem. 4.3]). *Let $\mathbf{A}$ be a Heyting algebra and q a universal clause in the language of Heyting algebras. If $\mathbf{A} \not\models q$ then there is a finite Heyting algebra $\mathbf{B} \hookrightarrow_{\wedge,\vee,0,1} \mathbf{A}$ such that $\mathbf{B} \not\models q$.*

From this it follows that every equation which is refuted on some Heyting algebra is also refuted on some finite Heyting algebra. Therefore, the equational theory of the class of finite Heyting algebras coincides with the equational theory of all Heyting algebras. From this the following theorem is an immediate consequence.

A.4.5. THEOREM. *The variety of Heyting algebras is generated by its finite members.*

A.5 Nuclei

We here introduce the concept of nuclei on partially ordered monoids and discuss some of their most fundamental properties. Nuclei were first considered by Schmidt and Tsinakis [231], Simmons [233], and Rosenthal [223] in the context of implicative semi-lattices, locales, and quantales, respectively. Our exposition follows closely the one in [98, Chap. 3.4.11] and [97, Sec. 2].

Nuclei A *partially ordered monoid* is a structure $\langle P, \leq, \cdot, \mathrm{e}\rangle$ such that $\langle P, \leq\rangle$ is a poset and $\langle P, \cdot, \mathrm{e}\rangle$ is a monoid with the property that

$$p \leq q \quad \text{implies} \quad r \cdot p \leq r \cdot q \quad \text{and} \quad p \leq q \quad \text{implies} \quad p \cdot r \leq q \cdot r,$$

for all $p, q, r \in P$. One may readily verify that if $\langle P, \leq, \cdot\rangle$ is a partially ordered monoid then so is the order dual $\langle P, \leq^\partial, \cdot\rangle$. A *residuated partially ordered monoid* is a structure $\langle P, \leq, \backslash, /, \cdot, \mathrm{e}\rangle$ with $\langle P, \leq, \cdot, \mathrm{e}\rangle$ a partially ordered monoid such that

$$p \cdot r \leq q \quad \text{if, and only if,} \quad r \leq p\backslash q \quad \text{if, and only if,} \quad p \leq q/r,$$

for all $p, q, r \in P$.

A.5.1. Proposition (See, e.g., [98, Lem.3.33]). *Let $\mathbf{P} = \langle P, \leq, \cdot, \mathrm{e}\rangle$ be a partially ordered monoid and let $\gamma\colon P \to P$ be a closure operator on $\langle P, \leq\rangle$. Then the following are equivalent.*

1. *$\gamma(p) \cdot \gamma(q) \leq \gamma(p \cdot q)$, for all $p, q \in P$.*
2. *$\gamma(\gamma(p) \cdot \gamma(q)) = \gamma(p \cdot q)$, for all $p, q \in P$.*

In case $\mathbf{P}$ is a residuated partially ordered monoid, say $\langle P, \leq, \cdot, \backslash, /, \mathrm{e}\rangle$, the above conditions are equivalent to the condition that the elements $p\backslash\gamma(q)$ and $\gamma(q)/p$ are closed for all $p, q \in P$.

A *nucleus* on a partially ordered monoid $\langle P, \leq, \cdot, \mathrm{e}\rangle$ is a closure operator $\gamma\colon P \to P$ on the poset $\langle P, \leq\rangle$ satisfying either of the equivalent conditions of Proposition A.5.1. A *co-nucleus* on a partially ordered monoid $\langle P, \leq, \cdot, \mathrm{e}\rangle$ is then a nucleus on the order dual $\langle P, \leq^\partial, \cdot, \mathrm{e}\rangle$.

Retractions Given a partially ordered monoid $\mathbf{P} = \langle P, \leq, \cdot, \mathrm{e}\rangle$ with a nucleus $\gamma\colon P \to P$ we let $\mathbf{P}_\gamma$ be the structure $\langle \gamma[P], \leq, \cdot_\gamma, \gamma(\mathrm{e})\rangle$ with $p \cdot_\gamma q = \gamma(p \cdot q)$ for all $p, q \in \gamma[P]$. Similarly, given a residuated partially ordered monoid $\mathbf{P} = \langle P, \leq, \cdot, \backslash, /, \mathrm{e}\rangle$ and a nucleus $\gamma\colon P \to P$ we let $\mathbf{P}_\gamma$ be the structure $\langle \gamma[P], \leq, \cdot_\gamma, \backslash, /, \gamma(\mathrm{e})\rangle$. By Proposition A.5.1, $p\backslash q, q/p \in \gamma[P]$ for each $p, q \in \gamma[P]$, and so this is indeed well defined. In both cases we refer to the structure $\mathbf{P}_\gamma$ as the *γ-retraction* of $\mathbf{P}$.

A.5.2. Proposition (See, e.g., [98, Thm.3.34(1)(2)]). *If* $\mathbf{P}$ *is a (residuated) partially ordered monoid and* $\gamma\colon P \to P$ *is a nucleus, then* $\mathbf{P}_\gamma$ *is also a (residuated) partially ordered monoid. Furthermore, the map* $\gamma\colon \mathbf{P} \twoheadrightarrow \mathbf{P}_\gamma$ *is an order-preserving monoid homomorphism.*

Given a lattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \mathrm{e}\rangle$ and a nucleus $\gamma\colon A \to A$ on the induced partially ordered monoid $\langle A, \leq, \cdot, \mathrm{e}\rangle$ we let $\mathbf{A}_\gamma$ be the structure $\langle \gamma[A], \wedge, \vee_\gamma, \cdot_\gamma, \gamma(\mathrm{e})\rangle$ with the operation $\vee_\gamma$ defined as in Proposition A.1.1, i.e., $a \vee_\gamma b = \gamma(a \vee b)$, for all $a, b \in A$. Similarly, given a residuated lattice $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, \mathrm{e}\rangle$ with a nucleus $\gamma\colon A \to A$ we let $\mathbf{A}_\gamma$ be the structure $\langle \gamma[A], \wedge, \vee_\gamma, \cdot_\gamma, \backslash, /, \gamma(\mathrm{e})\rangle$. In both cases we refer to the structure $\mathbf{A}_\gamma$ as the γ*-retraction* of $\mathbf{A}$.

A.5.3. Proposition (See, e.g., [98, Thm.3.34(1)(2)]). *If* $\mathbf{A}$ *is a (residuated) lattice-ordered monoid and* $\gamma\colon A \to A$ *is a nucleus on the induced partially ordered monoid* $\langle A, \leq, \cdot, \mathrm{e}\rangle$*, then* $\mathbf{A}_\gamma$ *is also a (residuated) lattice-ordered monoid. Furthermore, the monoid homomorphism* $\gamma\colon \mathbf{A} \twoheadrightarrow \mathbf{A}_\gamma$ *is join-preserving.*

Regular elements If $\mathbf{A} = \langle A, \wedge, \vee, \neg, 0, 1\rangle$ is a pseudo-complemented distributive lattice then the map $a \mapsto \neg\neg a$ is a nucleus on the induced partially ordered monoid $\langle A, \leq, \wedge, 1\rangle$ see, e.g., [16, Thm. VIII.2.1]. The closed elements determined by this nucleus are usually called *regular* and the set of all such elements is denoted by $\mathrm{Rg}(A)$. Since both the least element 0 and the greatest element 1 of $\mathbf{A}$ are closed with respect to this nucleus we obtain as a special case of Proposition A.5.3 that the set of regular elements carries the structure of a bounded lattice $\mathrm{Rg}(\mathbf{A}) := \langle \mathrm{Rg}(A), \wedge, \vee_{\neg\neg}, 0, 1\rangle$.

A.5.4. Proposition (See, e.g., [16, Thm. VIII.4.3]). *Let* $\mathbf{A} = \langle A, \wedge, \vee, \neg, 0, 1\rangle$ *be a pseudo-complemented distributive lattice. Then* $\mathrm{Rg}(\mathbf{A})$ *is a Boolean algebra and the map* $a \mapsto \neg\neg a$ *is a homomorphism of pseudo-complemented lattices.*

By order duality we then obtain that if $\mathbf{A} = \langle A, \wedge, \vee, \sim, 0, 1\rangle$ is a supplemented distributive lattice then the map $a \mapsto {\sim}{\sim} a$ is a co-nucleus on the induced partially ordered monoid $\langle A, \leq, \vee, 0\rangle$. The open elements determined by the co-nucleus $a \mapsto {\sim}{\sim} a$ are called co-regular. We denote by $\mathrm{CoRg}(A)$ the set of co-regular elements of $\mathbf{A}$. Completely analogously to Proposition A.5.4 we obtain that the set $\mathrm{CoRg}(A)$ of co-regular elements of any supplemented distributive lattice $\mathbf{A}$ carries the structure of a Boolean algebra which we denote by $\mathrm{CoRg}(\mathbf{A})$. Finally, since any Heyting algebra $\mathbf{A} = \langle A, \wedge, \vee, \to, 0, 1\rangle$ determines a pseudo-complemented distributive lattice, namely $\langle A, \wedge, \vee, \neg, 0, 1\rangle$ we obtain a nucleus $a \mapsto \neg\neg a$ on the induced partially ordered monoid $\langle A, \leq, \wedge, 1\rangle$. As before we refer to closed elements of this nucleus as the regular elements of $\mathbf{A}$ and denote the corresponding Boolean algebra by $\mathrm{Rg}(\mathbf{A})$.

A.6 Dualities

We here review some different types of dualities for various classes of bounded distributive lattices and Heyting algebras. We will assume that the reader is familiar with the rudiments of category theory, see, e.g., [194, 191].

A.6.1 Discrete duality

We first consider certain subcategories of the category of bounded distributive lattices and bounded lattice homomorphisms for which the opposite categories admit a fairly natural description. The material presented here can be found in [23, Sec. 7], [119, Sec. 4–5], and [252].

Up- and downsets Given a poset $\mathbb{P} = \langle P, \leq \rangle$ and $U \subseteq P$ we define

$$\downarrow U := \{p \in P : \exists q \in U \ (p \leq q)\} \quad \text{and} \quad \uparrow U := \{p \in P : \exists q \in U \ (q \leq p)\}.$$

A subset U of a poset $\mathbb{P} = \langle P, \leq \rangle$ is *upward closed*, or simply an *upset*, if $\uparrow U = U$. A *downward closed*, or simply *downset*, of a poset $\mathbb{P}$ is then an upset of its order dual $\mathbb{P}^\partial$. The collection of all upsets of a poset $\mathbb{P}$, ordered by set-theoretic inclusion, is a lattice which we denote by $\mathbb{P}^+$.

Perfect lattices For any poset $\mathbb{P}$ the lattice $\mathbb{P}^+$ will be complete and *completely distributive* in the sense that arbitrary meets distribute over arbitrary joins and vice versa, see, e.g., [220, Def. 3]. We call a complete and completely distributive lattice *perfect*[4] provided that every element is the join of the set of completely join-irreducible elements below it, see, e.g., [115, Thm. 2.1] or [74, Thm. 10.29] for an alternative characterization of such lattices. Conversely, any perfect distributive lattice $\mathbf{D}$ will be isomorphic to $\mathbb{P}^+$ for some poset, which we denote by $\mathbf{D}_+$, namely the order dual of the poset $J^\infty(\mathbf{D}) = \langle J^\infty(D), \leq \rangle$ of completely join-irreducible elements with the partial order determined by $\mathbf{D}$.

Birkhoff-Raney duality Any order-preserving function $f \colon \mathbb{P} \to \mathbb{Q}$ between posets $\mathbb{P}$ and $\mathbb{Q}$ induces a *complete lattice homomorphism*, i.e., a function preserving arbitrary meets and joins, $f^+ \colon \mathbb{Q}^+ \to \mathbb{P}^+$ by letting $f^+(U) := f^{-1}(U)$ for all upsets U of $\mathbb{Q}$. By Proposition A.1.2 any complete bounded lattice homomorphism $h \colon \mathbf{D} \to \mathbf{E}$ between perfect lattices has a lower adjoint $h_+ \colon \mathbf{E} \to \mathbf{D}$. It is easy to verify that h_+ maps completely join-irreducible elements to completely join-irreducible elements and hence restricts to an order-preserving map from the poset $J^\infty(\mathbf{E})$ to the poset $J^\infty(\mathbf{D})$. Consequently, we may consider h_+ as an order-preserving map from the poset $\mathbf{E}_+$ to the poset $\mathbf{D}_+$. This gives rise

[4] These lattices are also known as: bi-algebraic, doubly algebraic [115], completely join-generated [119] or completely prime-algebraic [252].

to a dual equivalence of categories originally due to Raney [220] in the form of a representation theorem.

A.6.1. THEOREM (Raney). *The category of perfect distributive lattices and complete lattice homomorphisms is dually equivalent to the category of posets and order-preserving functions.*

As the functors involved map finite objects to finite objects we obtain the following corollary originally due to Birkhoff [37] in the form of a representation theorem.

A.6.2. COROLLARY (Birkhoff). *The category of finite distributive lattices and bounded lattice homomorphisms is dually equivalent to the category of finite posets and order-preserving maps.*

We note the following useful properties of this duality which is straightforward to establish.

A.6.3. PROPOSITION. *Let $f\colon \mathbb{P} \to \mathbb{Q}$ be an order-preserving map between posets.*

1. *The map $f\colon \mathbb{P} \to \mathbb{Q}$ is surjective if, and only if, the corresponding homomorphism $f^+\colon \mathbb{Q}^+ \to \mathbb{P}^+$ is injective.*
2. *The map $f\colon \mathbb{P} \to \mathbb{Q}$ is injective if, and only if, the corresponding homomorphism $f^+\colon \mathbb{Q}^+ \to \mathbb{P}^+$ is surjective.*

de Jongh-Troelstra duality It it not difficult to see that for any poset $\mathbb{P}$ the lattice $\mathbb{P}^+$ is a Heyting algebra with Heyting implication defined by

$$U \to V = P \backslash {\downarrow}(U \backslash V),$$

for all upsets U, V of $\mathbb{P}$. As observed by de Jongh and Troelstra [171], for any order-preserving map of posets $f\colon \mathbb{P} \to \mathbb{Q}$, the induced lattice homomorphism $f^+\colon \mathbb{Q}^+ \to \mathbb{P}^+$ will be a homomorphism of Heyting algebras if, and only if, the map $f\colon \mathbb{P} \to \mathbb{Q}$ is a so-called *p-morphism*[5], meaning that for all $p \in P$ and $q \in Q$,

$$f(p) \leq_Q q \quad \text{implies} \quad f(r) = q \text{ for some } r \in P \text{ such that } p \leq_P r.$$

This, in combination with Theorem A.6.1, yields the following duality.

A.6.4. THEOREM. *The category of perfect Heyting algebras and complete Heyting algebra homomorphisms is dually equivalent to the category of posets and p-morphisms.*

Since the functors involved map finite objects to finite objects the duality restricts to finite objects.

A.6.5. COROLLARY. *The category of finite Heyting algebras and Heyting algebra homomorphisms is dually equivalent to the category of finite posets and p-morphisms.*

[5] Sometimes also called *strongly isotone* [171], *bounded* [39], or *open* [119].

A.6.2 Topological duality

We now describe the opposite categories of both the category of bounded distributive lattices and bounded lattice homomorphism and the category of Heyting algebras and Heyting algebra homomorphisms. We will assume that the reader is familiar with basic concepts from topology, see, e.g., [84] or [74, App. A]. A thorough treatment of the material covered here can be found in [74, 201, 27, 109].

Ordered topological spaces A *Priestley space* is a, possibly empty, partially ordered, compact, and totally ordered disconnected topological space, i.e., a tuple $\langle X, \leq, \tau \rangle$ such that

(i) The pair $\langle X, \leq \rangle$ is a partial order,

(ii) The pair $\langle X, \tau \rangle$ is a compact topological space,

(iii) For every $x, y \in X$ such that $x \not\leq y$ there is a clopen upset U of X such that $x \in U$ and $y \notin U$.

We will often refer to a Priestley space $\langle X, \leq, \tau \rangle$ by the set X. We note that not every poset can be equipped with a topology resulting in a Priestley space. Given a Priestley space $\langle X, \leq, \tau \rangle$ we write $\min(X)$ and $\max(X)$ for the sets of minimal and maximal elements of X, respectively. The following useful property was first observed by Esakia [87], see also [33, Thm. 2.3.24] for a proof.

A.6.6. Proposition. *Let $\langle X, \leq, \tau \rangle$ be a Priestley space. Then for each $x \in X$ there are $x_0 \in \min(X)$ and $x_1 \in \max(X)$ such that $x_0 \leq x \leq x_1$.*

Priestley duality The collection $\mathsf{ClpUp}(X)$ of all clopen upsets of a Priestley space X evidently forms a bounded distributive lattice, with meets and joins being set-theoretic intersection and union, respectively. We denote this lattice by X^*. Conversely, let $\mathbf{D} = \langle D, \wedge, \vee, 0, 1 \rangle$ be a bounded distributive lattice. A proper filter F of D is *prime* if

$$a \vee b \in F \quad \text{implies} \quad a \in F \text{ or } b \in F \qquad \text{for all } a, b \in D.$$

We denote the set of prime filters of $\mathbf{D}$ by $\mathbf{D}_*$. Given any $a \in D$ we obtain two sets of prime filters

$$\widehat{a} \coloneqq \{x \in \mathbf{D}_* : a \in x\} \quad \text{and} \quad \widetilde{a} \coloneqq \{x \in \mathbf{D}_* : a \notin x\}.$$

The family $\{\widehat{a}, \widetilde{a} : a \in D\}$ generates a topology τ on the set $\mathbf{D}_*$ such that the resulting partially ordered topological space $\langle \mathbf{D}_*, \subseteq, \tau \rangle$ is a Priestley space. We refer to this space as the *dual Priestley space of* $\mathbf{D}$. If $f \colon X \to Y$ is a function between Priestley spaces X and Y, which is both continuous and order-preserving, then we obtain a homomorphism of bounded distributive lattices $f^*\colon Y^* \to X^*$ by letting $f^*(U) := f^{-1}(U)$ for all $Y \in \mathsf{ClpUp}(Y)$. Conversely, for any homomorphism $h\colon \mathbf{D} \to \mathbf{E}$ of bounded distributive lattices we obtain a function $h_*\colon \mathbf{E}_* \to \mathbf{D}_*$ which is both continuous and order-preserving by letting $h_*(x) := h^{-1}(x)$ for all $x \in \mathbf{E}_*$. This determines the following dual equivalence of categories first established by Priestley [216].

A.6.7. Theorem (Priestley). *The category of bounded distributive lattices and bounded lattice homomorphisms is dually equivalent to the category of Priestley spaces and continuous order-preserving maps.*

Spectral spaces For any bounded distributive lattice $\langle D, \wedge, \vee, 0, 1\rangle$ each of the families $\{\widehat{a} : a \in D\}$ and $\{\widetilde{a} : a \in D\}$ also generates a topology on the set of prime filters of $\mathbf{D}$, denoted by $\tau^{\uparrow}$ and $\tau^{\downarrow}$ respectively. The resulting topological spaces $\langle D_*, \tau^{\uparrow}\rangle$ and $\langle D_*, \tau^{\downarrow}\rangle$ are what is known as *spectral spaces* and all spectral spaces arise in this way [156]. The lattice $\mathbf{D}$ may be recovered as the lattice of compact open subsets of the space $\langle D_*, \tau^{\uparrow}\rangle$ while its order dual may be recovered as the lattice of compact open subsets of the space $\langle D_*, \tau^{\downarrow}\rangle$. This forms the basis of a purely topological description of the opposite category of the category of bounded distributive lattices and bounded lattice homomorphisms originally due to Stone in the form of a representation theorem [240].

Stone duality It is easy to see that the topology on any Priestley space is both Hausdorff and zero-dimensional, i.e., has a basis of clopen sets. Zero-dimensional compact Hausdorff spaces are known as *Stone spaces*[6]. Thus any Stone space may be identified with a Priestley space in which the partial order is trivial, i.e., is the equality relation. It is not hard to show that a bounded distributive lattice is a Boolean algebra if, and only if, its dual Priestley space is a Stone space. Furthermore, since the category of Boolean algebras and Boolean algebra homomorphisms is a full subcategory of the category of bounded distributive lattices and bounded lattice homomorphisms, we obtain, as a special case of Theorem A.6.7, the following theorem originally due to Stone [239] in the form of a representation theorem.

A.6.8. Theorem (Stone). *The category of Boolean algebras and Boolean algebra homomorphisms is dually equivalent to the category of Stone spaces and continuous maps.*

Esakia duality An *Esakia space* is a Priestley space in which ${\downarrow}U$ is open for any open set U. A bounded distributive lattice is a (reduct of a) Heyting algebra precisely when its dual Priestley space is an Esakia space, in which case we refer

[6] Often also referred to as *Boolean spaces*.

to it as its *dual Esakia space*. Just as in the discrete case, a bounded lattice homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ between Heyting algebras is a Heyting algebra homomorphism if, and only if, the dual continuous order-preserving map $h_*\colon \mathbf{B}_* \to \mathbf{A}_*$ is a p-morphism. Consequently, we obtain the following dual equivalence of categories originally established by Esakia [86].

A.6.9. Theorem (Esakia). *The category of Heyting algebras and Heyting algebra homomorphisms is dually equivalent to the category of Esakia spaces and continuous p-morphisms.*

Finally, we note that a completely analogous version of Proposition A.6.3 holds for all the topological dualities discussed above.

A.7 Projective lattices

We here collect a few useful results concerning (weakly) projective distributive lattices and semi-lattices.

Projectivity Let $\mathcal{K}$ be a class of similar algebras. An algebra $\mathbf{P} \in \mathcal{K}$ is said to be *weakly projective in*, or *relative to* $\mathcal{K}$ see, e.g., [16, Chap. I.20.13], if any diagram

$$\begin{tikzcd} & \mathbf{A} \arrow[d, two heads] \\ \mathbf{P} \arrow[r] & \mathbf{B} \end{tikzcd}$$

of homomorphisms between $\mathcal{K}$-algebras can be completed to a commutative diagram

$$\begin{tikzcd} & \mathbf{A} \arrow[d, two heads] \\ \mathbf{P} \arrow[r] \arrow[ur] & \mathbf{B} \end{tikzcd}$$

of homomorphism between $\mathcal{K}$-algebras. Since we are only considering weakly projective algebras in this thesis we will simply use the word *projective* to mean weakly projective.

Finite projective distributive lattices Given a bounded distributive lattice $\mathbf{D} = \langle D, \wedge, \vee, 0, 1, \rangle$ we let $J_0(\mathbf{D})$ denote the poset of join-irreducible elements of $\mathbf{D}$ including the least element 0, and let $J_0(D)$ denote the underlying set of this poset. Recall that $\mathcal{DL}$ denotes the class of distributive lattices.

A.7.1. Theorem ([17]). *For any finite distributive lattice* $\mathbf{D}$ *the following are equivalent.*

1. *The lattice* $\mathbf{D}$ *is projective in the class* $\mathcal{DL}$.

2. *For all* $a, b \in J_0(D)$ *the meet* $a \wedge b$ *belongs to* $J_0(D)$.

From this it may be deduced that a finite distributive lattice is projective if, and only if, its order dual is, see also [15, Thm. 7.1]. Given any two algebras $\mathbf{A}, \mathbf{B}$ with (semi-)lattice reducts and appropriate $\tau \subseteq \{\wedge, \vee\}$ we write $h\colon \mathbf{A} \to_\tau \mathbf{B}$ to indicate that $h\colon A \to B$ is a function preserving all the operations from τ and $h\colon \mathbf{A} \hookrightarrow_\tau \mathbf{B}$ ($h\colon \mathbf{A} \twoheadrightarrow_\tau \mathbf{B}$) to indicate that h is moreover injective (surjective). In case the reducts have one or more bounds we use similar notation for appropriate $\tau \subseteq \{\wedge, \vee, 0, 1\}$. The following useful property of finite projective distributive lattices is a straightforward consequence of [17, Thm. 4]. However, as we have been unable to find a direct reference we supply a proof.

A.7.2. Proposition. *Let* $\mathbf{P} = \langle P, \wedge, \vee, 0, 1\rangle$ *be a finite distributive lattice which is projective in the class* $\mathcal{DL}$ *and let* $\mathbf{D}$ *be a (lower bounded) distributive lattice. Then for any* $h\colon \mathbf{P} \to_\wedge \mathbf{D}$ *(*$h\colon \mathbf{P} \to_{\wedge,0} \mathbf{D}$*) there is* $g\colon \mathbf{P} \to_{\wedge,\vee} \mathbf{D}$ *(*$g\colon \mathbf{P} \to_{\wedge,\vee,0} \mathbf{D}$*). Furthermore, the function* g *is injective whenever* h *is.*

Proof:
Let $h\colon \mathbf{P} \to_\wedge \mathbf{D}$ be given. Since $\mathbf{P}$ is assumed to be projective in the class $\mathcal{DL}$ it follows from Theorem A.7.1 that the poset $J_0(\mathbf{P})$ is a meet semi-lattice which is a subalgebra of the meet semi-lattice reduct of $\mathbf{P}$. Consequently, restricting h to $J_0(\mathbf{P})$ we obtain a function $h_0\colon J_0(\mathbf{P}) \hookrightarrow_\wedge \mathbf{D}$. Because the lattice reduct of $\mathbf{P}$ is projective in the class $\mathcal{DL}$, it follows from [17, Thm. 4] that by letting

$$g(a) := \bigvee\{h_0(b) \in D : J_0(P) \ni b \leq a\},$$

we obtain a function $g\colon \mathbf{P} \to_{\wedge,\vee} \mathbf{D}$. If $\mathbf{D}$ is lower bounded with least element 0 then $g(0) = h_0(0) = h(0)$ and so $g(0) = 0$, provided that $h(0) = 0$.

For the last part of the statement assume that the function h is injective. Since the function h preserves meets, it follows that h must be order-reflecting. It is easy to see that $g(a) \leq h(a)$ for all $a \in P$. It follows that if $a, b \in P$ are such that $g(a) \leq g(b)$, then for all $c \in J_0(P)$ with $c \leq a$ we have

$$h(c) = g(c) \leq g(a) \leq g(b) \leq h(b),$$

and consequently $c \leq b$. Because $\mathbf{P}$ is finite a must be a join of the join-irreducible elements below it and hence $a \leq b$. This shows that g is injective. □

Recall that an upper bounded distributive lattice is well-connected precisely when its top element is join-irreducible. Knowing this the following is an immediate consequence of Proposition A.7.2 above.

A.7.3. Corollary. *Let* $\mathbf{P} = \langle P, \wedge, \vee, 0, 1\rangle$ *be a well-connected finite distributive lattice which is projective in the class* $\mathcal{DL}$ *and let* $\mathbf{D}$ *be a bounded distributive lattice. Then for any* $h\colon \mathbf{P} \to_{\wedge,0,1} \mathbf{D}$ *there is* $g\colon \mathbf{P} \to_{\wedge,\vee,0,1,} \mathbf{D}$ *which is injective whenever* h *is.*

Finite projective semi-lattices We conclude this section by recalling a useful description of a large class of projective meet semi-lattices.

A.7.4. Theorem ([159, Cor. 5.4]). *The meet semi-lattice reduct of any finite distributive lattice is projective in the class of meet semi-lattices.*

As an immediate consequence we obtain that for any surjection $h\colon \mathbf{D} \twoheadrightarrow_{\wedge} \mathbf{E}$ between finite distributive lattices there must be a function $g\colon \mathbf{E} \to_{\wedge} \mathbf{D}$ such that $h \circ g$ is the identity on $\mathbf{E}$. In particular, the function g must be injective.

A.8 Lattice completions

We here collect some basic facts and definitions concerning different types of completions of lattices.

Completions A lattice $\mathbf{L} = \langle L, \wedge, \vee\rangle$ is *complete* provided that every subset S of L has a least upper bound, denoted by $\bigvee S$ and referred to as the *join of* S, or equivalently every subset S of L has a greatest lower bound, denoted by $\bigwedge S$ and referred to as the *meet of* S. A *completion* of a lattice $\mathbf{L}$ is a complete lattice $\mathbf{C}$ together with an embedding of lattices $\mathbf{L} \hookrightarrow \mathbf{C}$. A lattice completion $\mathbf{L} \to \mathbf{C}$ which preserves all existing meets (joins) in $\mathbf{L}$ is called *meet-regular* (*join-regular*). A lattice completion which is both meet-regular and join-regular is called *regular*.

Polarities We now describe a general template for constructing completions of lattices going back to Birkhoff [38, Chap. IV], see, e.g., also [104, 108, 148, 113]. A *polarity* is a tuple (W_0, W_1, N) where $N \subseteq W_0 \times W_1$ is a binary relation. Any polarity (W_0, W_1, N) induces a pair of functions

$$L_N\colon \wp(W_1) \to \wp(W_0) \quad \text{and} \quad U_N\colon \wp(W_0) \to \wp(W_1)$$

given by

$$L_N(X) = \{w \in W_0 \colon \forall u \in X\ (wNu)\} \text{ and } U_N(Y) = \{w \in W_1 \colon \forall v \in Y\ (vNw)\}.$$

In cases where no confusion can arise we may simply omit the subscripts. It is easy to see that, considered as maps between posets $\mathbb{W}_0 = \langle \wp(W_0), \subseteq\rangle$ and $\mathbb{W}_1 = \langle \wp(W_1), \subseteq\rangle$, both L_N and U_N are antitone and satisfy

$$Y \subseteq L_N(X) \quad \text{if, and only if,} \quad X \subseteq U_N(Y),$$

for all $X \in \wp(W_1)$ and $Y \in \wp(W_0)$. Consequently, these maps form a Galois connection from the poset $\mathbb{W}_0$ to the poset $\mathbb{W}_1$. Therefore, by Proposition A.1.3 we obtain two closure operators $L_N U_N\colon \mathbb{W}_0 \to \mathbb{W}_0$ and $U_N L_N\colon \mathbb{W}_1 \to \mathbb{W}_1$, which we denote by γ_N and δ_N respectively. Since the poset $\mathbb{W}_0$ is complete, it follows from Proposition A.1.1 that the set of γ_N-closed elements forms a complete lattice under set-theoretic inclusion with meets and joins given by

$$\bigwedge_{i\in I} Z_i \coloneqq \bigcap_{i\in I} Z_i \quad \text{and} \quad \bigvee_{i\in I} Z_i \coloneqq \gamma_N\left(\bigcup_{i\in I} Z_i\right),$$

for all sets $\{Z_i : i \in I\}$ of γ_N-closed elements. A similar construction is given by considering the δ_N-closed sets resulting in a complete lattice which is the order dual of the lattice of γ_N-closed sets.

Ideal completions Given a lattice $\mathbf{L}$ with a least element, the set of ideals is closed under arbitrary intersections and hence by Proposition A.1.1 we obtain a closure operator on the poset $\langle \wp(L), \subseteq\rangle$ by mapping every subset to the least ideal containing it. In case $\mathbf{L}$ does not contain a least element we obtain a closure operation on the poset $\langle \wp(L)\backslash\{\emptyset\}, \subseteq\rangle$ by mapping any non-empty set to the least ideal containing it. In either case we obtain by Proposition A.1.1, a lattice consisting of all ideals of $\mathbf{L}$ which we denote by $\mathrm{Idl}(\mathbf{L})$. The map $a \mapsto {\downarrow}a$ is evidently a lattice embedding from $\mathbf{L}$ into $\mathrm{Idl}(\mathbf{L})$. In case $\mathbf{L}$ has a least element the lattice $\mathrm{Idl}(\mathbf{L})$ will be complete and hence a completion of $\mathbf{L}$ called the *ideal completion*. If $\mathbf{L}$ does not have a least element then neither does $\mathrm{Idl}(\mathbf{L})$. In particular, this lattice will not be complete. However, the lattice $\mathrm{Idl}_0(\mathbf{L})$ obtained by adding a least element to $\mathrm{Idl}(\mathbf{L})$ will be complete and hence a completion of $\mathbf{L}$. Nevertheless, we may in some cases still refer to $\mathrm{Idl}(\mathbf{L})$ as the *ideal completion* of $\mathbf{L}$ even when $\mathbf{L}$ lacks a least element. For any lattice $\mathbf{L}$ the ideal completion $\mathrm{Idl}(\mathbf{L})$, or $\mathrm{Idl}_0(\mathbf{L})$ in case $\mathbf{L}$ lacks a least element, of $\mathbf{L}$ may alternatively be characterized as the, up to isomorphism, unique completion $e\colon \mathbf{L} \hookrightarrow \mathbf{C}$ of $\mathbf{L}$ satisfying the following two properties

(i) every element of C is a join of elements of the set $e[L]$,

(ii) all subsets $\{a\} \cup T \subseteq L$ satisfy $e(a) \leq \bigvee e[T]$, if, and only if, there is a finite subset $T_0 \subseteq T$ such that $a \leq \bigvee T_0$.

In case $\mathbf{D}$ is a bounded distributive lattice the ideal completion $\mathrm{Idl}(\mathbf{D})$ may be identified with the lattice of all open upsets of the dual Priestley space of $\mathbf{D}$, see, e.g., [27, Cor. 6.3].

MacNeille completions Since the set of normal ideals of any lattice $\mathbf{L}$ is closed under arbitrary intersections we obtain a closure operator on the poset $\langle \wp(L), \subseteq\rangle$ by mapping every subset to the least normal ideal containing it. Recall that, unlike ideals, normal ideals are allowed to be empty. Thus by Proposition A.1.1 we obtain a complete lattice which we denote by $\overline{\mathbf{L}}$. As before we may readily verify that the map $a \mapsto {\downarrow}a$ is a lattice embedding. Thus $\overline{\mathbf{L}}$ is a completion of $\mathbf{L}$ which we call the *MacNeille completion*. The MacNeille completion $\overline{\mathbf{L}}$ of a lattice $\mathbf{L}$ may alternatively be characterized as the, up to isomorphism, unique completion $e\colon \mathbf{L} \hookrightarrow \mathbf{C}$ of $\mathbf{L}$ with the following two properties

(i) every element of C is a meet of elements of the set $e[L]$,

(ii) every element of C is a join of elements of the set $e[L]$.

A completion satisfying (i) is often referred to as being *meet-dense*, while a completion satisfying (ii) is often referred to as being *join-dense*. From these two properties it can be deduced that the MacNeille completion is always regular. In case $\mathbf{D}$ is a bounded distributive lattice the MacNeille completion of $\mathbf{D}$ may be identified with the lattice of so-called *regular* open upsets of the dual Priestley space of $\mathbf{D}$, i.e., open upsets U of D_* such that $\mathsf{JD}(\mathsf{U}) = \mathsf{U}$, where for $S \subseteq D_*$, we denote by $\mathsf{J}(\mathsf{S})$ and $\mathsf{D}(\mathsf{S})$ the largest open upset contained in S and the least closed upset containing S, respectively. Note that this lattice will not necessarily be distributive. When $\mathbf{D}$ is the (reduct of a) bi-Heyting algebra we have that $\mathsf{JD}(\mathsf{U}) = \mathsf{IC}(\mathsf{U})$, for all open upsets U of D_*, with $\mathsf{I}(-)$ and $\mathsf{C}(-)$ denoting the interior and closure operator respectively. We refer to [151, Sec. 3] for details.

Canonical completions Let $\mathbf{L}$ be a bounded lattice. As shown by Gehrke and Harding [111] there is, up to isomorphism, a unique completion $e\colon \mathbf{L} \hookrightarrow \mathbf{C}$ of $\mathbf{L}$ with the following three properties

(i) every element of C is a meet of joins of elements of the set $e[L]$,

(ii) every element of C is a join of meets of elements of the set $e[L]$,

(iii) all subsets $S, T \subseteq L$ satisfy that $\bigwedge e[S] \leq \bigvee e[T]$, if, and only if, there are finite subsets $S_0 \subseteq T$ and $T_0 \subseteq T$ such that $\bigwedge S_0 \leq \bigvee T_0$.

This completion is called the *canonical completion*[7] of $\mathbf{L}$ and is denoted by $\mathbf{L}^\delta$. Various special cases of this type of completion have been studied independently see, e.g., [175, 176, 114, 120]. The canonical completion of any bounded distributive lattice is again a bounded distributive lattice [114] and similarly the canonical completion of any Heyting algebra is again a Heyting algebra [109, Sec. 2]. In case $\mathbf{D}$ is a bounded distributive lattice, the lattice $\mathbf{D}^\delta$ may be identified with the lattice of all upsets of the dual Priestley space of $\mathbf{D}$, see, e.g., [114, Sec. 2].

A.9 Intermediate logics

We here review the basic definitions concerning intermediate logics. For an in depth treatment of intermediate logics see, e.g., [51].

[7] Often also referred to as the canonical *extension*.

The language of intuitionistic propositional logic Fix a countable set of propositional letters $\mathsf{P}_{\aleph_0} \coloneqq \{p_n : n \in \omega\}$. For each $\mathsf{P} \subseteq \mathsf{P}_{\aleph_0}$ let $\mathrm{Fm}(\mathsf{P})$ be the set of formulas produced by the following grammar

$$\varphi ::= \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi, \quad p \in \mathsf{P}.$$

We write Fm for $\mathrm{Fm}(\mathsf{P}_{\aleph_0})$ and call the elements of this set *formulas in the language of intuitionistic propositional logic*. In this language both the connectives $\leftrightarrow$ and $\neg$ as well as the constant $\top$ are definable, e.g., as $(\varphi \to \psi) \wedge (\psi \to \varphi)$, $\varphi \to \bot$, and $\bot \to \bot$, respectively. If $\mathsf{P} = \{p_{n_1}, \ldots, p_{n_k}\}$ we may write $\varphi(p_{n_1}, \ldots, p_{n_k})$ to indicate that φ belongs to $\mathrm{Fm}(\mathsf{P})$.

Intermediate logics By an *intermediate* (*propositional*) *logic* we shall understand any proper subset of Fm containing the formulas

(i) $p_0 \to (p_1 \to p_1)$,

(ii) $(p_0 \to (p_1 \to p_2)) \to ((p_0 \to p_1) \to (p_0 \to p_2))$,

(iii) $(p_0 \wedge p_1) \to p_0$,

(iv) $(p_0 \wedge p_1) \to p_1$,

(v) $p_0 \to (p_1 \to (p_0 \wedge p_1))$,

(vi) $p_0 \to (p_0 \vee p_1)$,

(vii) $p_1 \to (p_0 \vee p_1)$,

(viii) $(p_0 \to p_2) \to ((p_1 \to p_2) \to ((p_0 \vee p_1) \to p_2))$,

(ix) $\bot \to p_0$,

and closed under the rules

$$\frac{\varphi \to \psi \quad \varphi}{\psi}\ (\textsc{mp}) \qquad \text{and} \qquad \frac{\varphi(p_{n_1}, \ldots, p_{n_k})}{\varphi(\psi_1, \ldots, \psi_k)}\ (\textsc{Sub})$$

where $\varphi(\psi_1, \ldots, \psi_k)$ denotes the result of uniformly substituting, for all $i \in \{1, \ldots, k\}$, the formula ψ_i for the propositional letter p_{n_i}. Since the collection of intermediate logics is evidently closed under arbitrary intersections there will be a least, with respect to set-theoretic inclusion, intermediate logic which we will call (*the*) *intuitionistic propositional logic*[8] and denote by IPC.

[8] Also called (*the*) *intuitionistic propositional calculus*.

Algebraic semantics To any formula $\varphi \in \mathrm{Fm}$ we may associate a Heyting algebra term $\mathrm{tm}(\varphi)$ in the evident way. Consequently, to any non-trivial variety of Heyting algebras $\mathcal{V}$ we may associate a collection of formulas

$$\mathsf{L}_{\mathcal{V}} := \{\varphi \in \mathrm{Fm} : \mathcal{V} \models \mathrm{tm}(\varphi) \approx 1\},$$

which is easily seen to be an intermediate logic. Conversely, to any intermediate logic L we may associate a class of Heyting algebra equations,

$$E(\mathsf{L}) := \{\mathrm{tm}(\varphi) \approx \mathrm{tm}(\psi) \colon \varphi \leftrightarrow \psi \in \mathsf{L}\}$$

and hence also a variety of Heyting algebras

$$\mathcal{V}(\mathsf{L}) := \{\mathbf{A} \in \mathcal{HA} \colon \mathbf{A} \models E(\mathsf{L})\},$$

which must necessarily be non-trivial. This establishes a one-to-one correspondence between the class of non-trivial varieties of Heyting algebras and intermediate logics. One part of this correspondence may be viewed as an algebraic completeness theorem.

A.9.1. Theorem. *Let* L *be an intermediate logic and let* $\varphi \in \mathrm{Fm}$. *Then,*

$$\varphi \in \mathsf{L} \quad \textit{if, and only if,} \quad \mathcal{V}(\mathsf{L}) \models \mathrm{tm}(\varphi) \approx 1.$$

Thus any intermediate logic is completely determined by its corresponding variety of Heyting algebras.

Relational semantics The algebraic completeness of IPC together with the discrete duality for Heyting algebras A.6.1 leads to an alternative semantics for IPC based on posets. This is the so-called *Kripke semantics* [183], see also [142]. A *valuation* on a poset $\mathbb{Q} = \langle Q, \leq \rangle$ is a function $V \colon \mathsf{P} \to \mathbb{Q}^+$. We may then define the relation $\Vdash$ between tuples $\langle \mathbb{Q}, V, q \rangle$, with $q \in Q$, and formulas $\varphi \in \mathrm{Fm}$ by the following recursion:

$$\begin{array}{rll}
\mathbb{Q}, V, q \Vdash \bot & \text{never,} & \\
\mathbb{Q}, V, q \Vdash p_i & \text{if, and only if,} & q \in V(p_i), \\
\mathbb{Q}, V, q \Vdash \varphi \wedge \psi & \text{if, and only if,} & \mathbb{Q}, V, q \Vdash \varphi \;\text{ and }\; \mathbb{Q}, V, q \Vdash \psi, \\
\mathbb{Q}, V, q \Vdash \varphi \vee \psi & \text{if, and only if,} & \mathbb{Q}, V, q \Vdash \varphi \;\text{ or }\; \mathbb{Q}, V, q \Vdash \psi, \\
\mathbb{Q}, V, q \Vdash \varphi \to \psi & \text{if, and only if,} & \text{for all } r \geq q,\; \mathbb{Q}, V, r \Vdash \varphi \;\text{ implies } \mathbb{Q}, V, r \Vdash \psi.
\end{array}$$

If $\mathbb{Q}, V, q \Vdash \varphi$ for all $q \in Q$ we write $\mathbb{Q}, V \Vdash \varphi$. Similarly, if $\mathbb{Q}, V \Vdash \varphi$ for all valuations V on $\mathbb{Q}$ we simply write $\mathbb{Q} \Vdash \varphi$. Finally if $\mathcal{F}$ is a class of posets we write $\mathcal{F} \Vdash \varphi$ if $\mathbb{Q} \Vdash \varphi$ for all $\mathbb{Q} \in \mathcal{F}$. A straightforward induction on the complexity of $\varphi \in \mathrm{Fm}$ shows that

$$\mathbb{Q} \Vdash \varphi \quad \text{if, and only if,} \quad \mathbb{Q}^+ \models \mathrm{tm}(\varphi) \approx 1,$$

where $\mathrm{tm}(\varphi)$ is the Heyting algebra term associated with the formula φ.

Basic properties of intermediate logics Let L be an intermediate logic and let $\mathcal{F}$ be a class of posets. We say that

(i) the logic L is *sound* with respect to the class $\mathcal{F}$ if $\varphi \in \mathsf{L}$ implies $\mathcal{F} \Vdash \varphi$ for all $\varphi \in \mathrm{Fm}$,

(ii) the logic L is *complete* with respect to the class $\mathcal{F}$ if $\mathcal{F} \Vdash \varphi$ implies $\varphi \in \mathsf{L}$ for all $\varphi \in \mathrm{Fm}$,

(iii) the logic L is *determined*, or *characterized*, by the class $\mathcal{F}$ if L is both sound and complete with respect to $\mathcal{F}$,

(iv) the logic L is *(Kripke) complete* if is determined by some class of posets,

(v) the logic L has the *finite model property* if it is determined by some class of finite posets,

(vi) the logic L is *elementary* if it is determined by a first-order definable class of posets,

(vii) the logic L is *canonical* if the corresponding variety $\mathcal{V}(\mathsf{L})$ is closed under canonical completions.

It is not hard to see that an intermediate logic L is complete if, and only if, the corresponding variety of Heyting algebras $\mathcal{V}(\mathsf{L})$ is generated by $\{\mathbb{P}^+ : \mathbb{P} \in \mathcal{F}\}$ for some class of posets $\mathcal{F}$. In particular, since all finite Heyting algebras are of the form $\mathbb{P}^+$ for some poset $\mathbb{P}$ we have that L enjoys the finite model property if, and only if, the variety $\mathcal{V}(\mathsf{L})$ is generated by its finite members. Finally, since the canonical completion of any Heyting algebra is a perfect lattice we have that any canonical logic is necessarily complete.

Some intermediate logics Given a set of formula Φ in the language of intuitionistic propositional logic we write $\mathsf{IPC}+\Phi$ to denote the least set of formulas in the language of intuitionistic propositional logic which contains the set $\mathsf{IPC}\cup\Phi$ and is closed under the rules (MP) and (Sub). In case Φ is finite, say $\Phi = \{\varphi_1, \dots, \varphi_n\}$, we write $\mathsf{IPC} + \varphi_1 + \ldots + \varphi_n$ in place of $\mathsf{IPC} \cup \Phi$. We here give a list of some of the intermediate logics which we often consider in this thesis. For any natural number $n \geq 1$ we define

$$\begin{aligned}
\mathsf{BW}_n &:= \mathsf{IPC} + \bigvee_{i=0}^{n}\left(p_i \to \bigvee_{j\neq i} p_j\right),\\
\mathsf{BTW}_n &:= \mathsf{IPC} + \bigvee_{i=0}^{n}\left(\bigwedge_{j<i} p_j \to \neg\neg p_i\right),\\
\mathsf{BC}_n &:= \mathsf{IPC} + \bigvee_{i=0}^{n}\left(\bigwedge_{j<i} p_j \to p_i\right),\\
\mathsf{BD}_n &:= \mathsf{IPC} + \beta_n,
\end{aligned}$$

where the formula β_n is defined by the following recursion:

$$\beta_1 := p_0 \vee \neg p_0 \qquad \text{and} \qquad \beta_{n+1} := p_{n+1} \vee (p_{n+1} \to \beta_n).$$

The logic BW_1 is often called the *Gödel-Dummett logic* and is denoted by LC while the logic BTW_1 is often called the *Jankov logic*, or the *logic of weak excluded middle*, and is denoted by KC. The *width* of a poset $\mathbb{P}$ is the least cardinal κ such that any anti-chain in $\mathbb{P}$ is of size at most κ. Similarly, the *top width* of a poset $\mathbb{P}$ is the cardinality of the set of maximal element of $\mathbb{P}$. Finally, the *depth* of a poset $\mathbb{P}$ is the least cardinal κ such that any chain in $\mathbb{P}$ is of size at most κ. With these definitions the following characterizations may be established for any natural number $n \geq 1$, see, e.g., [51, Chap. 2.5].

1. The logic BW_n is the logic determined by the class of posets of width at most n.

2. The logic BTW_n is the logic determined by the class of posets of bounded top width at most n.

3. The logic BC_n is the logic determined by the class of posets of cardinality at most n.

4. The logic BD_n is the logic determined by the class of posets of bounded depth at most n.

Furthermore, all these logics are canonical and enjoy the finite model property, see, e.g., [51, Chap. 10–11].

Bibliography

[1] M. D'Agostino and M. Mondadori. The taming of the cut. Classical refutations with analytic cut. *J. Logic Comput.*, 4(3):285–319, 1994.

[2] A. Akbar Tabatabai and R. Jalali. Universal proof theory: Semi-analytic rules and uniform interpolation. arXiv:1808.06258, 2018.

[3] M. Alizadeh, F. Derakhshan, and H. Ono. Uniform interpolation in substructural logics. *Rev. Symb. Log.*, 7(3):455–483, 2014.

[4] A. R. Anderson and N. D. Belnap, Jr. *Entailment.* Princeton Univ. Press, Princeton, N.J., 1975.

[5] A. R. Anderson, N. D. Belnap, Jr., and J. M. Dunn. *Entailment. The logic of relevance and necessity. Vol. II.* Princeton Univ. Press, Princeton, N.J., 1992.

[6] M. Anderson and T. Feil. *Lattice-Ordered Groups.* D. Reidel Publishing Co., Dordrecht, 1988.

[7] F. Aschieri, A. Ciabattoni, and F. A. Genco. Gödel logic: from natural deduction to parallel computation. In *2017 32nd Annual ACM/IEEE Sympos. on Logic in Computer Science (LICS)*, pages 1–12. IEEE, NJ, 2017.

[8] F. Aschieri, A. Ciabattoni, and F. A. Genco. Classical proofs as parallel programs. In *Proc. Ninth Int. Sympos. on Games, Automata, Logics, and Formal Verification*, volume 277 of *Electron. Proc. Theor. Comput. Sci. (EPTCS)*, pages 43–57. EPTCS, 2018.

[9] A. Avellone, M. Ferrari, and P. Miglioli. Duplication-free tableau calculi and related cut-free sequent calculi for the interpolable propositional intermediate logics. *Log. J. IGPL*, 7(4):447–480, 1999.

[10] A. Avron. A constructive analysis of RM. *J. Symbolic Logic*, 52(4):939–951, 1987.

[11] A. Avron. Hypersequents, logical consequence and intermediate logics for concurrency. *Ann. Math. Artificial Intelligence*, 4(3-4):225–248, 1991.

[12] A. Avron. The method of hypersequents in the proof theory of propositional non-classical logics. In *Logic: From foundations to applications (Staffordshire, 1993)*, Oxford Sci. Publ., pages 1–32. Oxford Univ. Press, New York, 1996.

[13] P. Bahls, J. Cole, N. Galatos, P. Jipsen, and C. Tsinakis. Cancellative residuated lattices. *Algebr. Univ.*, 50(1):83–106, 2003.

[14] K. A. Baker and A. W. Hales. From a lattice to its ideal lattice. *Algebr. Univ.*, 4:250–258, 1974.

[15] R. Balbes. Projective and injective distributive lattices. *Pacific J. Math.*, 21:405–420, 1967.

[16] R. Balbes and Ph. Dwinger. *Distributive Lattices*. Univ. of Missouri Press, Columbia, 1974.

[17] R. Balbes and A. Horn. Projective distributive lattices. *Pacific J. Math.*, 33:273–279, 1970.

[18] B. Banaschewski. Maximal rings of quotients of semi-simple commutative rings. *Arch. Math.*, 16:414–420, 1965.

[19] B. Banaschewski and G. Bruns. Categorical characterization of the MacNeille completion. *Arch. Math.*, 18:369–377, 1967.

[20] B. Banaschewski and G. Bruns. A proof that $\mathfrak{J}(L)$ belongs to $\mathbf{HSP}\{L\}$. *Algebr. Univ.*, 9(3):395, 1979.

[21] F. Belardinelli, P. Jipsen, and H. Ono. Algebraic aspects of cut elimination. *Studia Logica*, 77(2):209–240, 2004.

[22] N. D. Belnap, Jr. Display logic. *J. Philos. Logic*, 11(4):375–417, 1982.

[23] G. Bezhanishvili. Varieties of monadic Heyting algebras. Part II: Duality theory. *Studia Logica*, 62(1):21–48, 1999.

[24] G. Bezhanishvili, editor. *Leo Esakia on Duality in Modal and Intuitionistic Logics*, volume 4 of *Outstanding Contributions to Logic*. Dordrecht: Springer, 2014.

[25] G. Bezhanishvili and N. Bezhanishvili. Locally finite reducts of Heyting algebras and canonical formulas. *Notre Dame J. Form. Log.*, 58(1):21–45, 2017.

[26] G. Bezhanishvili, N. Bezhanishvili, and D. de Jongh. The Kuznetsov-Gerčiu and Rieger-Nishimura logics. The boundaries of the finite model property. *Logic Log. Philos.*, 17(1-2):73–110, 2008.

[27] G. Bezhanishvili, N. Bezhanishvili, D. Gabelaia, and A. Kurz. Bitopological duality for distributive lattices and Heyting algebras. *Math. Structures Comput. Sci.*, 20(3):359–393, 2010.

[28] G. Bezhanishvili, N. Bezhanishvili, and R. Iemhoff. Stable canonical rules. *J. Symbolic Logic*, 81(1):284–315, 2016.

[29] G. Bezhanishvili, N. Bezhanishvili, and J. Ilin. Cofinal stable logics. *Studia Logica*, 104(6):1287–1317, 2016.

[30] G. Bezhanishvili and M. Gehrke. Completeness of S4 with respect to the real line: Revisited. *Ann. Pure Appl. Logic*, 131(1-3):287–301, 2005.

[31] G. Bezhanishvili, J. Harding, J. Ilin, and F. M. Lauridsen. MacNeille transferability and stable classes of Heyting algebras. *Algebr. Univ.*, 79(3):Art. 55, 21, 2018.

[32] G. Bezhanishvili and R. Jansana. Priestley style duality for distributive meet-semilattices. *Studia Logica*, 98(1-2):83–122, 2011.

[33] N. Bezhanishvili. *Lattices of Intermediate and Cylindric Modal Logics*. PhD thesis, University of Amsterdam, 2006.

[34] N. Bezhanishvili and D. de Jongh. Stable formulas in intuitionistic logic. *Notre Dame J. Form. Log.*, 59(3):307–324, 2018.

[35] N. Bezhanishvili, N. Galatos, and L. Spada. Canonical formulas for k-potent commutative, integral residuated lattices. *Algebr. Univ.*, 77(3):321–343, 2017.

[36] N. Bezhanishvili and S. Ghilardi. Multiple-conclusion rules, hypersequents syntax and step frames. In R. Goré et al., editors, *Advances in Modal Logic*, volume 10, pages 54–73. College Publications, 2014.

[37] G. Birkhoff. Rings of sets. *Duke Math. J.*, 3(3):443–454, 1937.

[38] G. Birkhoff. *Lattice Theory*. American Mathematical Society, New York, N. Y., second edition, 1948.

[39] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge Univ. Press, Cambridge, 2001.

[40] W. J. Blok and D. Pigozzi. Algebraizable logics. *Mem. Amer. Math. Soc.*, 77(396), 1989.

[41] T. S. Blyth. *Lattices and Ordered Algebraic Structures*. Springer-Verlag London, Ltd., London, 2005.

[42] T. S. Blyth and M. F. Janowitz. *Residuation Theory*. Pergamon Press, Oxford-New York-Toronto, Ont., 1972.

[43] G. Boolos. Don't eliminate cut. *J. Philos. Logic*, 13(4):373–378, 1984.

[44] B. Brainerd and J. Lambek. On the ring of quotients of a Boolean ring. *Canad. Math. Bull.*, 2:25–29, 1959.

[45] J. P. Burgess. *Philosophical Logic*. Princeton Univ. Press, Princeton, N.J., 2009.

[46] S. Burris and H. P. Sankappanavar. *A Course in Universal Algebra: The Millennium Edition*. Springer, 2012. Available online: http://www.math.uwaterloo.ca/~snburris/.

[47] S. Burris and H. Werner. Sheaf constructions and their elementary properties. *Trans. Amer. Math. Soc.*, 248(2):269–309, 1979.

[48] E. Casari. Comparative logics. *Synthese*, 73(3):421–449, 1987.

[49] E. Casari. Comparative logics and abelian ℓ-groups. In C. Bonotto, R. Ferro, S. Valentini, and A. Zanardo, editors, *Logic Colloquium '88*, pages 161–190. Elsevier, 1989.

[50] E. Casari. Conjoining and disjoining on different levels. In M. L. D. Chiara, editor, *Logic and Scientific Methods*, pages 261–288. Kluwer, Dordrecht, 1997.

[51] A. V. Chagrov and M. Zakharyaschev. *Modal Logic*. Clarendon Press, 1997.

[52] L. A. Chagrova. On the first order definability of intuitionistic formulas with restrictions on occurrences of the connectives. In M. L. Kanovich, editor, *Logical Methods for Constructing Effective Algorithms*, pages 135–136. Kalinin State Univ., Kalinin, 1986. (Russian).

[53] L. A. Chagrova. An algorithm for constructing first order equivalents for disjunction free formulas. In Y. M. Gorchakov, editor, *Logical and Algebraic Constructions*, pages 96–100. Kalinin State Univ., Kalinin, 1987. (Russian).

[54] A. Ciabattoni and M. Ferrari. Hypertableau and path-hypertableau calculi for some families of intermediate logics. In R. Dyckhoff, editor, *TABLEAUX 2000, Proc.*, volume 1847 of *LNCS*, pages 160–174. Springer, 2000.

[55] A. Ciabattoni and M. Ferrari. Hypersequent calculi for some intermediate logics with bounded Kripke models. *J. Logic Comput.*, 11(2):283–294, 2001.

[56] A. Ciabattoni, D. M. Gabbay, and N. Olivetti. Cut-free proof systems for logics of weak excluded middle. *Soft Comput.*, 2(4):147–156, 1998.

[57] A. Ciabattoni, N. Galatos, and K. Terui. From axioms to analytic rules in nonclassical logics. In *Proc. 23th Annual IEEE Sympos. on Logic in Computer Science, LICS 2008*, pages 229–240. IEEE Computer Society, 2008.

[58] A. Ciabattoni, N. Galatos, and K. Terui. MacNeille completions of FL-algebras. *Algebr. Univ.*, 66(4):405–420, 2011.

[59] A. Ciabattoni, N. Galatos, and K. Terui. Algebraic proof theory for substructural logics: Cut-elimination and completions. *Ann. Pure Appl. Logic*, 163(3):266–290, 2012.

[60] A. Ciabattoni, N. Galatos, and K. Terui. Algebraic proof theory: Hypersequents and hypercompletions. *Ann. Pure Appl. Logic*, 168(3):693–737, 2017.

[61] A. Ciabattoni, P. Maffezioli, and L. Spendier. Hypersequent and labelled calculi for intermediate logics. In D. Galmiche et al., editors, *TABLEAUX 2013, Proc.*, volume 8123 of *LNCS*, pages 81–96. Springer, 2013.

[62] A. Ciabattoni and R. Ramanayake. Power and limits of structural display rules. *ACM Trans. Comput. Log.*, 17(3):17, 2016.

[63] A. Ciabattoni, R. Ramanayake, and H. Wansing. Hypersequent and display calculi—a unified perspective. *Studia Logica*, 102(6):1245–1294, 2014.

[64] A. Ciabattoni and K. Terui. Towards a semantic characterization of cut-elimination. *Studia Logica*, 82(1):95–119, 2006.

[65] P. M. Cohn. *Universal Algebra.* D. Reidel Publishing Co., Dordrecht-Boston, Mass., second edition, 1981.

[66] P. Conrad. Right-ordered groups. *Michigan Math. J.*, 6:267–275, 1959.

[67] W. Conradie, Y. Fomatati, A. Palmigiano, and S. Sourabh. Algorithmic correspondence for intuitionistic modal mu-calculus. *Theoret. Comput. Sci.*, 564:30–62, 2015.

[68] W. Conradie and A. Palmigiano. Algorithmic correspondence and canonicity for distributive modal logic. *Ann. Pure Appl. Logic*, 163(3):338–376, 2012.

[69] G. Corsi. A cut-free calculus for Dummett's LC quantified. *Z. Math. Logik Grundlag. Math.*, 35:289–301, 1989.

[70] D. Coumans, M. Gehrke, and L. van Rooijen. Relational semantics for full linear logic. *J. Appl. Log.*, 12(1):50–66, 2014.

[71] P. Crawley. Regular embeddings which preserve lattice structure. *Proc. Amer. Math. Soc.*, 13:748–752, 1962.

[72] P. Crawley and R. P. Dilworth. *Algebraic Theory of Lattices*. Prentice-Hall, 1973.

[73] G. D. Crown, J. Harding, and M. F. Janowitz. Boolean products of lattices. *Order*, 13(2):175–205, 1996.

[74] B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order*. Cambridge Univ. Press, New York, second edition, 2002.

[75] R. Dedekind. *Essays on the theory of numbers. I: Continuity and irrational numbers. II: The nature and meaning of numbers*. Dover Publications, Inc., New York, 1963.

[76] H. van Ditmarsch, W. van der Hoek, and B. Kooi. *Dynamic Epistemic Logic*. Springer Publishing Company, 2007.

[77] N. Dăneţ. The Dedekind completion of $C(X)$ with pointwise discontinuous functions. In *Ordered Structures and Applications*, Trends Math., pages 111–126. Birkhäuser/Springer, Cham, 2016.

[78] M.-L. Dubreil-Jacotin. Quelques propriétés des équivalences régulières par rapport à la multiplication et à l'union, dans un treillis à multiplication commutative avec élément unitée. *C. R. Acad. Sci. Paris*, 232:287–289, 1951.

[79] W. A. Dudek and Y. B. Jun. Pseudo-BCI algebras. *East Asian Math. J.*, 24(2):187–190, 2008.

[80] J. M. Dunn, M. Gehrke, and A. Palmigiano. Canonical extensions and relational completeness of some substructural logics. *J. Symbolic Logic*, 70(3):713–740, 2005.

[81] R. Dyckhoff and S. Negri. Proof analysis in intermediate logics. *Arch. Math. Logic*, 51(1-2):71–92, 2012.

[82] G. Dymek. p-semisimple pseudo-BCI-algebras. *J. Mult.-Valued Logic Soft Comput.*, 19(5-6):461–474, 2012.

[83] P. Emanovský and J. Kühr. Some properties of pseudo-BCK- and pseudo-BCI-algebras. *Fuzzy Sets and Systems*, 339:1–16, 2018.

[84] R. Engelking. *General Topology.* Heldermann Verlag, Berlin, second edition, 1989.

[85] M. Erné, J. Koslowski, A. Melton, and G. E. Strecker. A primer on Galois connections. In *Papers on general topology and applications (Madison, WI, 1991)*, volume 704 of *Ann. New York Acad. Sci.*, pages 103–125. New York Acad. Sci., New York, 1993.

[86] L. L. Esakia. Topological Kripke models. *Dokl. Akad. Nauk SSSR*, 214:298–301, 1974. (Russian).

[87] L. L. Esakia. *Heyting Algebras.* I *Duality theory.* "Metsniereba", Tbilisi, 1985. (Russian) with an English translation forthcoming.

[88] B. Fine, A. Gaglione, G. Rosenberger, and D. Spellman. Orderable groups, elementary theory, and the Kaplansky conjecture. *Groups Complex. Cryptol.*, 10(1):43–52, 2018.

[89] N. J. Fine, L. Gillman, and J. Lambek. *Rings of Quotients of Rings of Functions.* McGill Univ. Press, Montreal, Que., 1966.

[90] M. Fitting. Prefixed tableaus and nested sequents. *Ann. Pure Appl. Logic*, 163(3):291–313, 2012.

[91] J. M. Font. *Abstract Algebraic Logic.* College Publications, London, 2016.

[92] L. Fuchs. *Partially Ordered Algebraic Systems.* Pergamon Press, Oxford-London-New York-Paris; Addison-Wesley Publishing Co., Inc., Reading, Mass.-Palo Alto, Calif.-London, 1963.

[93] L. Fuchs. On group homomorphic images of partially ordered semigroups. *Acta Sci. Math. (Szeged)*, 25:139–142, 1964.

[94] N. Funayama. On the completion by cuts of distributive lattices. *Proc. Imp. Acad. Tokyo*, 20:1–2, 1944.

[95] D. M. Gabbay. *Labelled Deductive Systems, Volume 1.* The Clarendon Press, Oxford Univ. Press, New York, 1996.

[96] D. M. Gabbay and L. Maksimova. *Interpolation and Definability.* The Clarendon Press, Oxford Univ. Press, Oxford, 2005.

[97] N. Galatos and P. Jipsen. Residuated frames with applications to decidability. *Trans. Amer. Math. Soc.*, 365(3):1219–1249, 2013.

[98] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*. Elsevier, 2007.

[99] N. Galatos and G. Metcalfe. Proof theory for lattice-ordered groups. *Ann. Pure Appl. Logic*, 8(167):707–724, 2016.

[100] N. Galatos and H. Ono. Glivenko theorems for substructural logics over FL. *J. Symbolic Logic*, 71(4):1353–1384, 2006.

[101] N. Galatos and H. Ono. Cut elimination and strong separation for substructural logics: An algebraic approach. *Ann. Pure Appl. Logic*, 161(9):1097–1133, 2010.

[102] N. Galatos and C. Tsinakis. Generalized MV-algebras. *J. Algebra*, 283(1):254–291, 2005.

[103] F. Galvin and B. Jónsson. Distributive sublattices of a free lattice. *Canad. J. Math.*, 13:265–272, 1961.

[104] B. Ganter and R. Wille. *Formale Begriffsanalyse*. Springer-Verlag, Berlin, 1996.

[105] H. S. Gaskill. On the relation of a distributive lattice to its lattice of ideals. *Bull. Austral. Math. Soc.*, 7:377–385, 1972. Corrigendum: ibid. 8 (1973), 317–318.

[106] H. S. Gaskill. On transferable semilattices. *Algebr. Univ.*, 2:303–316, 1972.

[107] H. S. Gaskill and C. R. Platt. Sharp transferability and finite sublattices of free lattices. *Canad. J. Math.*, 27(5):1036–1041, 1975.

[108] M. Gehrke. Generalized Kripke frames. *Studia Logica*, 84(2):241–275, 2006.

[109] M. Gehrke. Canonical extensions, Esakia spaces, and universal models. In G. Bezhanishvili, editor, *Leo Esakia on Duality in Modal and Intuitionistic Logics*, pages 9–41. Springer, Dordrecht, 2014.

[110] M. Gehrke. Duality in computer science. In *Proc. of the 31st Annual ACM-IEEE Sympos. on Logic in Computer Science (LICS 2016)*, page 15. ACM, New York, 2016.

[111] M. Gehrke and J. Harding. Bounded lattice expansions. *J. Algebra*, 238:345–371, 2001.

[112] M. Gehrke, J. Harding, and Y. Venema. MacNeille completions and canonical extensions. *Trans. Amer. Math. Soc.*, 358(2):573–590, 2006.

[113] M. Gehrke, R. Jansana, and A. Palmigiano. Δ_1-completions of a poset. *Order*, 30(1):39–64, 2013.

[114] M. Gehrke and B. Jónsson. Bounded distributive lattices with operators. *Math. Japon.*, 40(2):207–215, 1994.

[115] M. Gehrke and B. Jónsson. Bounded distributive lattice expansions. *Math. Scand.*, 94(1):13–45, 2004.

[116] G. Gentzen. Untersuchungen über das logische Schließen. I. *Math. Z.*, 39(1):176–210, 1935.

[117] G. Gentzen. Untersuchungen über das logische Schließen. II. *Math. Z.*, 39(1):405–431, 1935.

[118] S. Ghilardi. Unification through projectivity. *J. Logic Comput.*, 7(6):733–752, 1997.

[119] S. Ghilardi. A First Introduction to the Algebra of Sentences. Available from the authors website https://homes.di.unimi.it/ghilardi/, 2000.

[120] S. Ghilardi and G. Meloni. Constructive canonicity in non-classical logics. *Ann. Pure Appl. Logic*, 86(1):1–32, 1997.

[121] S. Ghilardi and M. Zawadowski. *Sheaves, Games, and Model Completions.* Kluwer Academic Publishers, Dordrecht, 2002.

[122] J. Gil-Férez, F. M. Lauridsen, and G. Metcalfe. Self-cancellative residuated lattices. arXiv:1902.08144, 2019.

[123] S. Givant. *Duality Theories for Boolean Algebras with Operators.* Springer, Cham, 2014.

[124] S. Givant and Y. Venema. The preservation of Sahlqvist equations in completions of Boolean algebras with operators. *Algebr. Univ.*, 41(1):47–84, 1999.

[125] A. M. W. Glass and Y. Gurevich. The word problem for lattice-ordered groups. *Trans. Amer. Math. Soc.*, 280(1):127–138, 1983.

[126] V. Glivenko. Sur quelques points de la logique de M. Brouwer. *Bull. Cl. Sci., Acad. R. Belg., Ser. 5*, 15:183–188, 1929.

[127] R. Goldblatt. Varieties of complex algebras. *Ann. Pure Appl. Logic*, 44(3):173–242, 1989.

[128] R. Goldblatt. *Logics of Time and Computation*, volume 7. Stanford Univ., Center for the Study of Language and Information, Stanford, CA, second edition, 1992.

[129] R. Goldblatt. Persistence and atomic generation for varieties of Boolean algebras with operators. *Studia Logica*, 68(2):155–171, 2001.

[130] S. J. van Gool. *On Sheaves and Duality*. PhD thesis, Radboud Universiteit Nijmegen, 2014.

[131] S. J. van Gool, G. Metcalfe, and C. Tsinakis. Uniform interpolation and compact congruences. *Ann. Pure Appl. Logic*, 168(10):1927–1948, 2017.

[132] V. Goranko and D. Vakarelov. Elementary canonical formulae: Extending Sahlqvist's theorem. *Ann. Pure Appl. Logic*, 141(1-2):180–217, 2006.

[133] G. Grätzer. Universal algebra. In *Trends in Lattice Theory (Sympos., U.S. Naval Academy, Annapolis, Md., 1966)*, pages 173–210. Van Nostrand Reinhold, New York, 1970.

[134] G. Grätzer. A property of transferable lattices. *Proc. Amer. Math. Soc.*, 43:269–271, 1974.

[135] G. Grätzer. *Universal Algebra*. Springer, New York, second edition, 2008.

[136] G. Grätzer. *Lattice Theory: Foundation*. Birkhäuser/Springer Basel AG, Basel, 2011.

[137] G. Grätzer and C. R. Platt. A characterization of sharply transferable lattices. *Canad. J. Math.*, 32(1):145–154, 1980.

[138] G. Grätzer, C. R. Platt, and B. Sands. Embedding lattices into lattices of ideals. *Pacific J. Math.*, 85(1):65–75, 1979.

[139] G. Grätzer and E. T. Schmidt. On a problem of M. H. Stone. *Acta Math. Acad. Sci. Hungar.*, 8:455–460, 1957.

[140] G. Greco, P. Jipsen, F. Liang, A. Palmigiano, and A. Tzimoulis. Algebraic proof theory for LE-logics. arXiv:1808.04642v1, 2018.

[141] G. Greco, M. Ma, A. Palmigiano, A. Tzimoulis, and Z. Zhao. Unified correspondence as a proof-theoretic tool. *J. Logic Comput.*, 28(7):1367–1442, 2018.

[142] A. Grzegorczyk. A philosophically plausible formal interpretation of intuitionistic logic. *Nederl. Akad. Wetensch. Proc. Ser. A 67*, 26:596–601, 1964.

[143] S. Haack. *Deviant Logic.* Cambridge Univ. Press, Cambridge, 1974.

[144] P. Hájek. *Metamathematics of Fuzzy Logic.* Kluwer Academic Publishers, Dordrecht, 1998.

[145] J. Harding. Any lattice can be regularly embedded into the MacNeille completion of a distributive lattice. *Houston J. Math.*, 19(1):39–44, 1993.

[146] J. Harding. Completions of orthomodular lattices. II. *Order*, 10(3):283–294, 1993.

[147] J. Harding. Canonical completions of lattices and ortholattices. *Tatra Mt. Math. Publ.*, 15:85–96, 1998.

[148] J. Harding. Completions of ordered algebraic structures: A survey. In *Interval/Probabilistic Uncertainty and Non-Classical Logics*, volume 46 of *Advances in Soft Computing*, pages 231–244. Springer, 2008.

[149] J. Harding. A regular completion for the variety generated by the three-element Heyting algebra. *Houston J. Math.*, 34(3):649–660, 2008.

[150] J. Harding. Modularity is not canonical. *Algebr. Univ.*, 80(1):80:8, 2019.

[151] J. Harding and G. Bezhanishvili. MacNeille completions of Heyting algebras. *Houston J. Math.*, 30(4):937–952, 2004.

[152] J. Harding and G. Bezhanishvili. MacNeille completions of modal algebras. *Houston J. Math.*, 33(2):355–384, 2007.

[153] J. Harding and F. M. Lauridsen. Hyper-MacNeille completions of Heyting algebras. manuscript, 2019.

[154] J. Harding and Q. Yang. Regular completions of lattices. *Houston J. Math.*, 38(3):685–691, 2012.

[155] J. Hintikka. *Knowledge and Belief: An Introduction to the Logic of the Two Notions.* Cornell Univ. Press, Cornell, 1962.

[156] M. Hochster. Prime ideal structure in commutative rings. *Trans. Amer. Math. Soc.*, 142:43–60, 1969.

[157] W. Hodges. *A Shorter Model Theory.* Cambridge Univ. Press, Cambridge, 1997.

[158] W. C. Holland and S. H. McCleary. Solvability of the word problem in free lattice-ordered groups. *Houston J. Math.*, 5(1):99–105, 1979.

[159] A. Horn and N. Kimura. The category of semilattices. *Algebr. Univ.*, 1(1):26–38, 1971.

[160] R. Horčík and K. Terui. Disjunction property and complexity of substructural logics. *Theoret. Comput. Sci.*, 412(31):3992–4006, 2011.

[161] T. Hosoi. Gentzen-type formulation of the propositional logic LQ. *Studia Logica*, 47(1):41–48, 1988.

[162] R. Iemhoff. On the admissible rules of intuitionistic propositional logic. *J. Symbolic Logic*, 66(1):281–294, 2001.

[163] R. Iemhoff. Uniform interpolation and the existence of sequent calculi. *Arch. Math. Logic*, 58(1–2):155–181, 2018.

[164] J. Ilin. *Filtrations Revisited: Lattices of Stable Non-Classical Logics.* PhD thesis, University of Amsterdam, 2018.

[165] K. Iséki. An algebra related with a propositional calculus. *Proc. Japan Acad.*, 42:26–29, 1966.

[166] E. Jeřábek. Canonical rules. *J. Symbolic Logic*, 74(4):1171–1205, 2009.

[167] E. Jeřábek. A note on the substructural hierarchy. *Math. Log. Q.*, 62(1-2):102–110, 2016.

[168] P. Jipsen and F. Montagna. The Blok-Ferreirim theorem for normal GBL-algebras and its application. *Algebr. Univ.*, 60(4):381–404, 2009.

[169] P. Jipsen and H. Rose. *Varieties of lattices*, volume 1533 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1992.

[170] P. Jipsen and C. Tsinakis. A survey of residuated lattices. In J. Martinez, editor, *Ordered Algebraic Structures*, pages 19–56. Kluwer, 2002.

[171] D. H. J. de Jongh and A. S. Troelstra. On the connection of partially ordered sets with some pseudo-Boolean algebras. *Nederl. Akad. Wetensch. Proc. Ser. A 69*, 28:317–329, 1966.

[172] B. Jónsson. Algebras whose congruence lattices are distributive. *Math. Scand.*, 21:110–121 (1968), 1967.

[173] B. Jónsson. On the canonicity of Sahlqvist identities. *Studia Logica*, 53(4):473–491, 1994.

[174] B. Jónsson and J. E. Kiefer. Finite sublattices of a free lattice. *Canad. J. Math.*, 14:487–497, 1962.

[175] B. Jónsson and A. Tarski. Boolean algebras with operators. I. *Amer. J. Math.*, 73:891–939, 1951.

[176] B. Jónsson and A. Tarski. Boolean algebras with operators. II. *Amer. J. Math.*, 74:127–162, 1952.

[177] R. Kashima. Cut-free sequent calculi for some tense logics. *Studia Logica*, 53(1):119–135, 1994.

[178] R. Kashima and Y. Komori. The word problem for free BCI-algebras is decidable. *Math. Japon.*, 37(6):1025–1029, 1992.

[179] H. J. Keisler. The ultraproduct construction. In *Ultrafilters across Mathematics*, volume 530 of *Contemp. Math.*, pages 163–179. Amer. Math. Soc., Providence, R.I., 2010.

[180] S. Koppelberg. *Handbook of Boolean Algebras, Volume 1.* North-Holland Publishing Co., Amsterdam, 1989. Edited by J. D. Monk and Robert Bonnet.

[181] T. Kowalski and S. Butchart. A note on monothetic BCI. *Notre Dame J. Form. Log.*, 47(4):541–544, 2006.

[182] M. Kracht. Power and weakness of the modal display calculus. In *Proof Theory of Modal Logic (Hamburg, 1993)*, volume 2 of *Appl. Log. Ser.*, pages 93–121. Kluwer Acad. Publ., Dordrecht, 1996.

[183] S. A. Kripke. Semantical analysis of intuitionistic logic. I. In *Formal Systems and Recursive Functions (Proc. Eighth Logic Colloq., Oxford, 1963)*, pages 92–130. North-Holland, Amsterdam, 1965.

[184] J. Kühr. Pseudo BCK-algebras and residuated lattices. In *Contributions to General Algebra 16*, pages 139–144. Heyn, Klagenfurt, 2005.

[185] A. V. Kuznecov and V. J. Gerčiu. The superintuitionistic logics and finitary approximability. *Dokl. Akad. Nauk SSSR*, 195:1029–1032, 1970.

[186] R. Kuznets. Multicomponent proof-theoretic method for proving interpolation properties. *Ann. Pure Appl. Logic*, 169(12):1369–1418, 2018.

[187] R. Kuznets and B. Lellmann. Interpolation for intermediate logics via hyper- and linear nested sequents. In *Advances in Modal Logic*, volume 12, pages 473–492. College Publications, London, 2018.

[188] O. Lahav. From frame properties to hypersequent rules in modal logics. In *2013 28th Annual ACM/IEEE Sympos. on Logic in Computer Science (LICS 2013)*, pages 408–417. IEEE Computer Soc., Los Alamitos, CA, 2013.

[189] H. Lakser. Finite projective lattices are sharply transferable. *Algebr. Univ.*, 5(3):407–409, 1975.

[190] F. M. Lauridsen. Intermediate logics admitting a structural hypersequent calculus. *Studia Logica*, 107(2):247–282, 2019.

[191] T. Leinster. *Basic Category Theory*. Cambridge Univ. Press, Cambridge, 2014.

[192] B. Lellmann. Hypersequent rules with restricted contexts for propositional modal logics. *Theoret. Comput. Sci.*, 656:76–105, 2016.

[193] B. Lellmann. Private Communication, 2017.

[194] S. Mac Lane. *Categories for the Working Mathematician*. Springer-Verlag, New York, second edition, 1998.

[195] A. Macintyre. Model-completeness for sheaves of structures. *Fund. Math.*, 81(1):73–89, 1973.

[196] H. M. MacNeille. Partially ordered sets. *Trans. Amer. Math. Soc.*, 42(3):416–460, 1937.

[197] S. Maehara. Lattice-valued representation of the cut-elimination theorem. *Tsukuba J. Math.*, 15(2):509–521, 1991.

[198] G. Metcalfe. Proof calculi for Casari's comparative logics. *J. Logic Comput.*, 16(4):405–422, 2006.

[199] G. Metcalfe, F. Paoli, and C. Tsinakis. Ordered algebras and logic. In H. Hosni and F. Montagna, editors, *Uncertainty and Rationality*, pages 1–85. Publications of the Scuola Normale Superiore di Pisa, Vol. 10, 2010.

[200] G. E. Minc. On some calculi of modal logic. In V. Orevkov, editor, *The Calculi of Symbolic Logic I*, volume 98 of *Proc. of the Steklov Inst. of Math.*, pages 97–124, 1971.

[201] P. J. Morandi. Dualities in Lattice Theory. Available from the authors website https://wordpress.nmsu.edu/pamorand/, 2005.

[202] J. B. Nation. Finite sublattices of a free lattice. *Trans. Amer. Math. Soc.*, 269(1):311–337, 1982.

[203] S. Negri. Contraction-free sequent calculi for geometric theories with an application to Barr's theorem. *Arch. Math. Logic*, 42(4):389–401, 2003.

[204] S. Negri and J. von Plato. *Structural Proof Theory*. Cambridge Univ. Press, Cambridge, 2001.

[205] E. Nelson. The embedding of a distributive lattice into its ideal lattice is pure. *Algebr. Univ.*, 4:135–140, 1974.

[206] M. Okada. Phase semantic cut-elimination and normalization proofs of first- and higher-order linear logic. *Theoret. Comput. Sci.*, 227(1-2):333–396, 1999.

[207] M. Okada. A uniform semantic proof for cut-elimination and completeness of various first and higher order logics. *Theoret. Comput. Sci.*, 281(1-2):471–498, 2002.

[208] M. Okada and K. Terui. The finite model property for various fragments of intuitionistic linear logic. *J. Symbolic Logic*, 64(2):790–802, 1999.

[209] H. Ono. Closure operators and complete embeddings of residuated lattices. *Studia Logica*, 74(3):427–440, 2003.

[210] F. Paoli. The proof theory of comparative logic. *Logique et Analyse*, 171:357–370, 2000.

[211] F. Paoli. *Substructural Logics: A Primer.* Kluwer Academic Publishers, Dordrecht, 2002.

[212] R. S. Pierce. *Modules over Commutative Regular Rings*, volume 70 of *Memoirs of the Amer. Math. Soc.* American Mathematical Society, Providence, R.I., 1967.

[213] A. M. Pitts. On an interpretation of second-order quantification in first-order intuitionistic propositional logic. *J. Symbolic Logic*, 57(1):33–52, 1992.

[214] C. R. Platt. Finite transferable lattices are sharply transferable. *Proc. Amer. Math. Soc.*, 81(3):355–358, 1981.

[215] G. Pottinger. Uniform, cut-free formulations of T, S4, S5, (abstract). *J. Symbolic Logic*, 48:900, 1983.

[216] H. A. Priestley. Representation of distributive lattices by means of ordered Stone spaces. *Bull. London Math. Soc.*, 2:186–190, 1970.

[217] H. A. Priestley. Ordered topological spaces and the representation of distributive lattices. *Proc. London Math. Soc.*, 24(3):507–530, 1972.

[218] J. Querré. Plus grand groupe image homomorphe et isotone d'un monoïde ordonné: Caracteŕrisations des anneaux "clos". *Acta Math. Acad. Sci. Hungar.*, 19:129–146, 1968.

[219] J. G. Raftery and C. J. van Alten. Residuation in commutative ordered monoids with minimal zero. *Rep. Math. Logic*, 34:23–57, 2000. Corrigendum: ibid. 39 (2005), 133–135.

[220] G. N. Raney. Completely distributive complete lattices. *Proc. Amer. Math. Soc.*, 3:677–680, 1952.

[221] H. Rasiowa and R. Sikorski. *The Mathematics of Metamathematics*. Państwowe Wydawnictwo Naukowe, Warsaw, 1963.

[222] P. H. Rodenburg. *Intuitionistic Correspondence Theory*. PhD thesis, University of Amsterdam, 1986.

[223] K. I. Rosenthal. *Quantales and their Applications*. Longman Scientific & Technical, Harlow, 1990.

[224] R. Rothenberg. *On the Relationship Between Hypersequent Calculi and Labelled Sequent Calculi for Intermediate Logics with Geometric Kripke Semantics*. PhD thesis, University of St Andrews, 2010.

[225] J. J. Rotman. *An Introduction to the Theory of Groups*. Springer-Verlag, New York, fourth edition, 1995.

[226] W. Rump and Y. C. Yang. Lateral completion and structure sheaf of an Archimedean l-group. *J. Pure Appl. Algebra*, 213(1):136–143, 2009.

[227] V. V. Rybakov. *Admissibility of Logical Inference Rules*. North-Holland Publishing Co., Amsterdam, 1997.

[228] H. Sahlqvist. Completeness and correspondence in the first and second order semantics for modal logic. In *Proc.of the Third Scand. Logic Sympos. (Univ. Uppsala, Uppsala, 1973)*, pages 110–143. Stud. Logic Found. Math., Vol. 82. North-Holland, Amsterdam, 1975.

[229] H. P. Sankappanavar. Heyting algebras with dual pseudocomplementation. *Pacific J. Math.*, 117(2):405–415, 1985.

[230] W. J. Savitch. Relationships between nondeterministic and deterministic tape complexities. *J. Comput. Syst. Sci.*, 4(2):177–192, 1970.

[231] J. Schmidt and C. Tsinakis. Relative pseudo-complements, join-extensions, and meet-retractions. *Math. Z.*, 157(3):271–284, 1977.

[232] D. J. Shoesmith and T. J. Smiley. *Multiple-Conclusion Logic*. Cambridge Univ. Press, Cambridge-New York, 1978.

[233] H. Simmons. A framework for topology. In *Logic Colloquium '77 (Proc. Conf., Wrocław, 1977)*, volume 96 of *Stud. Logic Foundations Math.*, pages 239–251. North-Holland, Amsterdam-New York, 1978.

[234] A. K. Simpson. *The Proof Theory and Semantics of Intuitionistic Modal Logic*. PhD thesis, University of Edinburgh, 1994.

[235] O. Sonobe. A Gentzen-type formulation of some intermediate logics. *J. Tsuda College*, 7:7–14, 1975.

[236] S. Sourabh. *Correspondence and Canonicity in Non-Classical Logic*. PhD thesis, University of Amsterdam, 2015.

[237] T. P. Speed. Some remarks on a class of distributive lattices. *J. Austral. Math. Soc.*, 9:289–296, 1969.

[238] T. P. Speed. Two congruences on distributive lattices. *Bull. Soc. Roy. Sci. Liège*, 38:86–95, 1969.

[239] M. H. Stone. The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc.*, 40(1):37–111, 1936.

[240] M. H. Stone. Topological representations of distributive lattices and Brouwerian logics. *Čas. Mat. Fys.*, 67(1):1–25, 1938.

[241] C. J. Taylor. Expansions of dually pseudocomplemented Heyting algebras. *Studia Logica*, 105(4):817–841, 2017.

[242] K. Terui. Herbrand's Theorem via Hypercanonical Extensions, 2013. Presentation delivered at TbiLLC 2013 in Gadauri on September 24.

[243] K. Terui. MacNeille completion and Buchholz' omega rule for parameter-free second order logics. In *Computer Science Logic 2018*, volume 119 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 37, 19. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2018.

[244] T. Tey. On transferable distributive lattices. *Nanta Math.*, 8(2):50–60, 1975.

[245] M. Theunissen and Y. Venema. MacNeille completions of lattice expansions. *Algebr. Univ.*, 57(2):143–193, 2007.

[246] A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory*, volume 43. Cambridge Univ. Press, Cambridge, second edition, 2000.

[247] D. J. Vaggione. Locally Boolean spectra. *Algebr. Univ.*, 33(3):319–354, 1995.

[248] L. Viganò. *Labelled Non-Classical Logics.* Kluwer Academic Publishers, Dordrecht, 2000.

[249] H. Wansing. *Displaying Modal Logic.* Kluwer Academic Publishers, Dordrecht, 1998.

[250] F. Wehrung. Relative projectivity and transferability for partial lattices. *Order*, 35(1):111–132, 2018.

[251] A. M. Wille. A Gentzen system for involutive residuated lattices. *Algebr. Univ.*, 54(4):449–463, 2005.

[252] G. Winskel. Prime algebraicity. *Theoret. Comput. Sci.*, 410(41):4160–4168, 2009.

[253] A. Wronski. Intermediate logics and the disjunction property. *Rep. Math. Logic*, 1:39–51, 1973.

[254] M. Zakharyaschev. Canonical formulas for K4, part II: Cofinal subframe logics. *J. Symbolic Logic*, 61(2):421–449, 1996.

[255] Z. Zhao. *Unified Correspondence and Canonicity.* PhD thesis, Delft University of Technology, 2018.

Index

Samenvatting

In dit proefschrift, met de titel *Snede en Completeringen: Algebraïsche aspecten van structurele bewijstheorie*, kijken we naar verschillende aspecten van het samenspel tussen structurele bewijstheorie en algebraïsche semantiek voor verschillende niet-klassieke propositionele logica's. We verkennen verbanden tussen bewijstheorie en algebra zoals ze betrekking hebben op structurele sequenten en hypersequenten calculi voor intermediaire and substructurele logica's. Deze verdanden zijn in het bijzonder sterk voor logica's die worden geassocieerd met de niveaus $\mathcal{P}_3$ en $\mathcal{N}_2$ van de substructurele hiërarchie van Ciabattoni, Galatos en Terui. Daarom onderzoeken we verschillende algebraïsche aspecten van deze twee niveaus, waarbij completeringen van tralies and tralie-algebra's een prominente rol spelen.

Hoofdstuk 2 gaat over de kwestie welke intermediaire logica's een structurele hypersequenten calculus toelaten. Deze vraag wordt beantwoord door de notie van $(\wedge, 0, 1)$-stabiele logica's te introduceren, welke een versterking is van de notie van $(\wedge, \vee, 0, 1)$-stabiele logica's, geïntroduceerd door Guram en Nick Bezhanishvili. We laten zien dat de $(\wedge, 0, 1)$-stabiele logica's precies de intermediaire logica's zijn die een structurele hypersequenten calculus toelaten. We onderzoeken deze logica's verder op zichzelf en laten zien, in het bijzonder, dat ze correct en volledig zijn ten opzichte van een eerste-orde definieerbare klasse van partieel geordende verzamelingen.

In Hoofdstuk 3 introduceren we verschillende noties van MacNeille en canonieke overdraagbaarheid voor eindige tralies, analoog aan Grätzers notie van ideaal-overdraagbaarheid. We demonstreren hoe eindige overdraagbare tralies leiden tot universele klasses van tralies gesloten onder completeringen. We concentreren ons hoofdzakelijk op het leveren van de noodzakelijke of voldoende voorwaarden voor een eindig distributief tralie om MacNeille overdraagbaar tralie voor verschillende klasses van Heytingalgebra's te zijn. Als laatste bespreken we hoe MacNeille en canonieke overdraagbaarheid van eindige distributieve tralies gerelateerd zijn aan het probleem van het bepalen van de elementairheid en canonicteit van $(\wedge, \vee, 0, 1)$-stabiele logica's.

Hoofdstuk 4 bevat een verkenning van het concept hyper-MacNeille completering, geïntroduceerd door Ciabattoni, Galatos en Terui, zoals het van toepassing is in de context van Heytingalgebra's. We isoleren de notie van een De Morgan gesupplementeerde Heytingalgebra als centraal voor het begrip van de hyper-MacNeille completeringen van Heytingalgebra's. We laten zien dat de MacNeille en hyper-MacNeille completeringen overeenkomen voor De Morgan gesupplementeerde Heytingalgebra's. Verder laten we zien dat de hyper-MacNeille completering van een Heytingalgebra de MacNeille completering van een De Morgan gesupplementeerde Heytingalgebra is. Als laatste geven we noodzakelijke en voldoende voorwaarden voor de hyper-MacNeille completering van een Heytingalgebra om een reguliere completering te zijn.

Uiteindelijk stellen we, in Hoofdstuk 5, een eigenschap vast, in de stijl van Glivenko, voor de variëteit van integraal gesloten geresidueerde tralies in relatie tot de variëteit van tralie-groepen. Dit wordt gebruikt voor een niet-standaard sequenten calculus voor de equationele theorie van integraal gesloten geresidueerde tralies. Met deze calculus bewijzen we de beslisbaarheid van de equationele theorie van integraal gesloten geresidueerde tralies. Ten slotte vergelijken we de equationele theorie van integraal gesloten geresidueerde tralies met de equationele theorieën van pseudo BCI-algebras, semi-integrale geresidueerde partieel geordende monoïden en algebras voor Casaris comparatieve logica.

Abstract

In this thesis, entitled *Cuts and Completions: Algebraic aspects of structural proof theory*, we look at different aspects of the interplay between structural proof theory and algebraic semantics for several non-classical propositional logics. Concretely, we explore connections between proof theory and algebra as they relate to structural sequent and hypersequent calculi for intermediate and substructural logics. Such connections are particularly strong for logics associated with the levels $\mathcal{P}_3$ and $\mathcal{N}_2$ of the substructural hierarchy introduced by Ciabattoni, Galatos, and Terui. Therefore, we investigate different algebraic aspects of these two levels. Among the algebraic aspects considered, completions of lattices and lattice-based algebras take on a prominent role.

In Chapter 2 we consider the question of which intermediate logics admit a structural hypersequent calculus. This question is answered by introducing the notion of $(\wedge, 0, 1)$-stable logics which is a strengthening of the notion of $(\wedge, \vee, 0, 1)$-stable logics introduced by Guram and Nick Bezhanishvili. We show that the $(\wedge, 0, 1)$-stable logics are precisely the intermediate logics which admit a structural hypersequent calculus. We further investigate these logics in their own right, showing in particular that they are all sound and complete with respect to a first-order definable class of partially ordered sets.

In Chapter 3 we introduce various notions of MacNeille and canonical transferability for finite lattices analogous to Grätzer's notion of (ideal) transferability. We show how finite transferable lattices give rise to universal classes of lattices closed under completions. We focus mainly on providing necessary or sufficient conditions for a finite distributive lattice to be MacNeille transferable for different classes of Heyting algebras. Lastly, we discuss how MacNeille and canonical transferability of finite distributive lattices is related to the problem of establishing the elementarity and the canonicity of $(\wedge, \vee, 0, 1)$-stable logics.

Chapter 4 contains an investigation of the concept of hyper-MacNeille completions introduced by Ciabattoni, Galatos, and Terui as it applies in the setting of Heyting algebras. We single out the notion of a De Morgan supplemented Heyting algebra as being central for understanding the hyper-MacNeille completions of Heyting algebras. We show that the MacNeille and hyper-MacNeille completions coincide for De Morgan supplemented Heyting algebras. Furthermore, we show that the hyper-MacNeille completion of any Heyting algebra is the MacNeille completion of some De Morgan supplemented Heyting algebra. Lastly, we provide necessary and sufficient conditions for the hyper-MacNeille completion of a Heyting algebra to be a regular completion.

Finally, in Chapter 5 we establish a Glivenko-style property for the variety of integrally closed residuated lattices with respect to the variety of lattice-ordered groups. This is used to construct a non-standard sequent calculus for the equational theory of integrally closed residuated lattices. Using this calculus we prove the decidability of the equational theory of integrally closed residuated lattices. Lastly, we compare the equational theory of integrally closed residuated lattices with the equational theories of pseudo BCI-algebras, semi-integral residuated partially ordered monoids, and algebras for Casari's comparative logic.

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