Degrees of FMP in extensions of bi-intuitionistic logic

MSc Thesis *(Afstudeerscriptie)*

written by

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MSc in Logic

at the *Universiteit van Amsterdam.*

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Abstract

This thesis contributes to the study of degrees of the finite model property (FMP), initiated by G. Bezhanishvili, N. Bezhanishvili and T. Moraschini (2022). We investigate degrees of FMP in extensions of bi-intuitionistic logic through the lens of universal algebra. Motivated by the characterisation of degrees of FMP for intuitionistic logic, which utilises the Kuznetsov-Gerčiu variety KG, we define its bi-intuitionistic counterpart $\mathsf{bi\text{-}KG}$. We obtain a description of the subdirectly irreducible members of $\mathsf{bi\text{-}KG}$ and, as a result, we prove that they are simple algebras. This enables us to develop a method of comparing subvarieties of $\mathsf{bi\text{-}KG}$ using local embeddability properties of their finitely generated simple members.

As an application of this method, we provide a full description of subvarieties of $\mathsf{bi\text{-}KG}$ with the FMP. Consequently, $\mathsf{bi\text{-}KG}$ turns out to enjoy the FMP, while its least subvariety containing all 1-generated Heyting algebras lacks the FMP. Our main result is a dichotomy-style theorem characterising degrees of FMP of subvarieties of $\mathsf{bi\text{-}KG}$, meaning that the only two degrees of FMP are 1 and $2^{\aleph_0}$. This is in sharp contrast with (intuitionistic) KG, where all cardinals κ with $\kappa \leq \aleph_0$ occur as FMP degrees relative to KG. Finally, we translate the statement into logical terms to obtain a corresponding result about degrees of FMP relative to the extension of bi-intuitionistic logic algebrised by $\mathsf{bi\text{-}KG}$.

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Chapter 1

Introduction

One of the main tools for studying modal logics is Kripke semantics. A normal modal logic is said to be *Kripke complete* if it is sound and complete with respect to a class of Kripke frames. A number of important logics enjoy this property and several powerful methods have been developed for establishing Kripke completeness (see, e.g., [5, Chapter 4]). Initially, it even seemed likely that every normal modal logic is Kripke complete (a historical overview can be found in, e.g., [9, Section 6.8]). However, Fine [15] and Thomason [28] displayed examples of *Kripke incomplete* logics, meaning that no class of Kripke frames provides sound and complete semantics for these logics. Because of this, the question arose whether the gap between completeness and incompleteness could be understood in quantitative terms. In [15] Fine introduced the *degree of incompleteness* of a normal modal logic. Given a logic L, its degree of incompleteness is the cardinality of the set of logics with the same Kripke frames as L. In essence, the degree of incompleteness of L counts the number of logics that cannot be distinguished from L by means of Kripke frames. A striking result due to Blok [6, 7], known as *Blok's dichotomy*, states that every normal modal logic has degree of incompleteness 1 or $2^{\aleph_0}$. Similar results were obtained by Dzobiak [12] and Chagrova [10] with respect to neighbourhood semantics.

This success in measuring incompleteness serves as an invitation to investigate degrees of other properties similar to Kripke completeness. In particular, one important such property is the *finite model property* (FMP for short). A normal modal logic is said to have the FMP if it is the logic of some class of *finite* Kripke frames. Drawing inspiration from Blok's work, G. Bezhanishvili, N. Bezhanishvili and T. Moraschini [3] introduced the term *degree of FMP*. The degree of FMP of a logic L counts the number of logics with the same *finite* Kripke frames as L. By utilising Blok's dichotomy, they showed that the degree of FMP of every normal modal logic is again 1 or $2^{\aleph_0}$. However, one is often interested in what can be said about a particular set of normal modal logics, for instance, extensions of the transitive modal logic **K4** or extensions of the reflexive and transitive modal logic **S4**.

For this purpose, given a normal modal logic L and an extension $L' \supseteq L$, we define the degree of FMP of L' *relative to* L to be the number of extensions of L that have the same finite Kripke frames as L'. This definition can be taken even further, in the setting of intuitionistic logic. Indeed, the intuitionistic propositional calculus (**IPC**) admits its own semantics based on Kripke frames, so the notion of degree of FMP can be analogously defined for extensions of **IPC**.

In this extended terminology, one can ask about the possible degrees of FMP relative to some logic L. In [3], G. Bezhanishvili, N. Bezhanishvili and T. Moraschini gave an answer to this question for **IPC**, **K4** and **S4**. Relative to **IPC**, all cardinals $\{1, 2, \ldots, \aleph_0, 2^{\aleph_0}\}$ occur as the degree of FMP of some logic. This result is referred to as the *antidichotomy theorem*. Consequently, with the help of the celebrated Blok-Esakia isomorphism between extensions of **IPC** and the Grzegorczyk modal system **Grz** (see, e.g., [9, Section 9.6]), they obtained that the same characterisation of degrees of FMP holds relative to **K4** and **S4**.

In order to take the investigation of degrees of FMP even further, [3] suggests as a possible research direction characterising degrees of FMP relative to bi-intuitionistic logic (**bi-IPC**). First introduced and studied by Rauszer [24, 23, 25], **bi-IPC** is a conservative extension of **IPC** with an additional binary connective $\leftarrow$, called *co-implication* (also exclusion, pseudo-difference or subtraction). Similarly to how falsum is dual to verum and conjunction is dual to disjunction, the intuition behind co-implication is that it is dual to implication. This duality can be observed in the definition of *bi-Heyting algebras* – the algebraic models of **bi-IPC**. Bi-Heyting algebras are Heyting algebras with an additional co-implication operation $\leftarrow$, defined by the following property:

$$c \vee b \geq a \iff c \geq a \leftarrow b,$$

for all elements a, b, c in the algebra. This is a dualised version of the implication condition:

$$c \wedge a \leq b \iff c \leq a \rightarrow b.$$

Furthermore, **bi-IPC** also has a Kripke semantics – intuitionistic Kripke models in which the co-implication is interpreted as follows:

$$M, x \Vdash \varphi \leftarrow \psi \iff \exists y \leq x \, (M, y \vDash \varphi \text{ and } M, y \nvDash \psi).$$

This is dual to the interpretation of implication:

$$M, x \Vdash \varphi \rightarrow \psi \iff \forall y \geq x \, (M, y \nvDash \varphi \text{ or } M, y \vDash \psi).$$

Thus, in terms of Kripke frames, **bi-IPC** can be seen as an extension of **IPC** that can express properties dependant not only on the accessibility relation $\leq$, but also on its converse relation $\geq$. In this respect, **bi-IPC** resembles temporal logic, where there are past-looking modalities.

This thesis initiates the study of degrees of FMP relative to **bi-IPC**. In view of the similarity between **IPC** and **bi-IPC**, it appears promising to analyse the approach used in [3] and potentially adapt it to the bi-intuitionistic setting. As already mentioned, relative to **IPC**, for every cardinal κ in $\{1, 2, \ldots, \aleph_0, 2^{\aleph_0}\}$, there exists a superintuitionistic logic with degree of FMP κ. A core step in the proof is constructing a logic with degree of FMP n for every natural number n. This relies on the well-understood properties of the Kuznetsov-Gerčiu logic (**KG**), introduced in [17, 16] for studying the FMP in superintuitionistic logics. For instance, one can obtain continuum many extensions of **KG** with the FMP and continuum many extensions without it (see, e.g., [2, Section 5]). Now the argument for exhibiting logics of finite degrees of FMP runs as follows.

1. For every $n \in \mathbb{N}$, find an extension of **KG** with degree of FMP n relative to **KG**.

2. Prove that for every extension L of **KG**, the degree of FMP of L relative to **KG** coincides with its degree relative to **IPC**.

These proofs rely heavily on the algebraic semantics of **IPC**. In fact, the question of finding degrees of FMP can be entirely reformulated in algebraic terms. Note that all logics of interest correspond to varieties, i.e., equationally defined classes of algebras. Now we define the degree of FMP of a variety V relative to a variety W with $W \supseteq V$ as the number of subvarieties of W with the same finite algebras as V. In this way, the degree of FMP of a logic L', relative to a logic L, coincides with the degree of FMP of the variety algebrising L', relative to the variety algebrising L. We find it more convenient to work purely in terms of varieties, so we take this approach for the majority of the thesis. Nevertheless, we will occasionally provide the most important statements in logical terms.

In this thesis, we take inspiration from the above strategy of utilising **KG** and study a bi-intuitionistic analogue of **KG**, which we name **bi-KG**. Our main contributions can be summarised as follows.

- We introduce the variety bi-KG and examine its subvarieties. We prove that bi-KG is semi-simple and give a description of all finitely generated simple bi-KG algebras. This culminates in a theorem that reduces comparison of subvarieties of bi-KG to local embeddability properties of their finitely generated simple members.

- We present a complete characterisation of subvarieties of bi-KG with the FMP. In particular, bi-KG itself has the FMP, but we can construct continuum many subvarieties of bi-KG that do not have the FMP.

- Finally, we derive a dichotomy-style theorem about degrees of FMP relative to bi-KG, according to which the only possible degrees of FMP relative to bi-KG are 1 and $2^{\aleph_0}$. This is in stark contrast with KG, where for every cardinal κ with $\kappa \leq \aleph_0$ there is a variety with degree of FMP κ relative to KG.

Our work is structured into the following chapters. In chapter 2, we lay out the general preliminaries. Chapter 3 introduces bi-KG and the methods for describing its subvarieties. Chapter 4 deals with the questions regarding finite frames of bi-KG, i.e., the FMP and degrees of FMP in bi-KG. Chapter 5 suggests possible directions for related future studies.

Chapter 2

Preliminaries

In this chapter, we present preliminary notation, definitions and facts that will be used throughout the thesis. Basic mathematical terminology and set-theoretic notation, such as sets, relations, functions and cardinality, will be assumed and used without explanation (see, e.g., [19, Chapter 1]). We also expect familiarity with the language and semantics of first-order logic (see, e.g., [11, 13]) and occasionally, the categorical notion of duality (see, e.g., [1, 20]).

2.1 Ordered sets

We begin with a list of definitions related to ordered sets. These can be considered standard and well-known, but we spell them out to avoid confusion.

Definition 2.1.1 (Poset)**.** Let A be a set and $R \subseteq A \times A$ be a binary relation on A. We call the pair $\langle A, R\rangle$ a *partially ordered set* or a *poset* if the following are satisfied.

- R is reflexive, i.e., $\langle a, a\rangle \in R$ for every $a \in A$.
- R is anti-symmetric, i.e., $\langle a, b\rangle \in R$ and $\langle b, a\rangle \in R$ imply $a = b$, for every $a, b \in A$.
- R is transitive, i.e., $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in R$ imply $\langle a, c\rangle \in R$, for every $a, b, c \in A$.

If the context is clear, we will write A to mean $\langle A, R\rangle$.

Remark 2.1.2. We usually denote the poset relation by $\leq$. In this case, we use the infix notation $a \leq b$. We use $a < b$ to mean $a \leq b$ and $a \neq b$. In addition, $\geq$ denotes the inverse of $\leq$ and $>$ denotes the inverse of $<$.

Definition 2.1.3 (Upper and lower bound)**.** Let $\langle A, \leq \rangle$ be a poset and let $B \subseteq A$. We call a an *upper bound* (resp. a *lower bound*) of B if $x \leq a$ (resp. $a \leq x$) for every $x \in B$.

Definition 2.1.4 (Supremum and infimum)**.** Let $\langle A, \leq \rangle$ be a poset and let $B \subseteq A$. We call e a *supremum* of B and write $e = sup(B)$ if e is an upper bound of B and $e \leq a$ for every upper bound a of B.

Similarly, we call f an *infimum* of B and write $f = inf(B)$ if f is a lower bound of B and $a \leq f$ for every lower bound a of B.

Remark 2.1.5. When suprema and infima exist, they are unique.

Definition 2.1.6 (Upset and downset)**.** Let $\langle A, \leq \rangle$ be a poset and let $B \subseteq A$. We call B *upwards closed* or an *upset* if $x \in B$, $y \in A$ and $x \leq y$ imply $y \in B$. If $C \subseteq A$, we write $\uparrow C$ for the least upset that contains C, namely $\{x \in A \mid \exists y \in C : x \geq y\}$.

Similarly, we call B *downwards closed* or a *downset* if $x \in B$, $y \in A$ and $y \leq x$ imply $y \in B$. If $C \subseteq A$, we write $\downarrow C$ for the least downset that contains C, namely $\{x \in A \mid \exists y \in C : x \leq y\}$.

Definition 2.1.7 (Interval)**.** Let $\langle A, \leq \rangle$ be a poset and let $B \subseteq A$. We call B an *interval* if $x, y \in B$, $z \in A$ and $x \leq z \leq y$ imply $z \in B$. Given $a, b \in A$, we write $[a, b]$ to mean the least interval containing a and b, namely $\{x \in A \mid a \leq x \leq b\}$. We write (a, b) to mean $[a, b] \setminus \{a, b\}$.

The following three notions are common in the context of graphs, but they also make sense for posets.

Definition 2.1.8 (Connected points)**.** Let $\langle A, \leq \rangle$ be a poset and let $a, b \in A$. We say that a and b are *connected* if there exists a finite sequence $a_0, \ldots, a_n$ such that $a_0 = a$, $a_n = b$ and for every $i \in \{0, \ldots, n-1\}$ we have $a_i \leq a_{i+1}$ or $a_i \geq a_{i+1}$.

Definition 2.1.9 (Connected sets)**.** Let $\langle A, \leq \rangle$ be a poset and let $B \subseteq A$. We call B *connected* if every two elements of B are connected.

Definition 2.1.10 (Connected components)**.** Every poset can be uniquely partitioned into subsets such that each subset is connected and has no proper connected supersets. We call the elements of this partition the *connected components* of the poset.

Example 2.1.11. On Figure 2.1 we see two posets. The one on the left is connected, so it has a single connected component. The one on the right has two connected components.

Figure 2.1: Examples of posets

2.2 Universal algebra

Most of our work is done in the context of universal algebra, so we gather here the required terminology and properties. We cover the topics of algebras and operations, universal classes, varieties and generating algebras.

2.2.1 Algebras and operations

The theory on universal algebra is built around the notion of an algebra and some important operations between algebras, which we present here.

Definition 2.2.1 (Language)**.** Let $\mathcal{F}$ be a set and $\tau : \mathcal{F} \to \mathbb{N}$. We call τ an *algebraic signature* or *language* and $\mathcal{F}$ is called the corresponding set of *function symbols*. For $f \in \mathcal{F}$, we call $\tau(f)$ the *arity* of f. If $\tau(f) = 0$, we call f a *constant symbol*.

Definition 2.2.2 (Algebra)**.** Let A be a non-empty set, $\tau : \mathcal{F} \to \mathbb{N}$ be an algebraic language and $F = \{f^A \mid f \in \mathcal{F}, f^A : A^{\tau(f)} \to A\}$, i.e., F contains interpretations of the symbols in $\mathcal{F}$. We call $\langle A, F\rangle$ an *algebra* in the language τ. If the context is clear, we will write A both for the algebra and for the underlying set.

Remark 2.2.3. Algebras can be seen as first-order models containing only functions and constants. Hence we might assume that they come equipped with a satisfaction relation $\vDash$.

Definition 2.2.4 (Similar algebras)**.** We refer to algebras that are in the same language as *similar algebras*.

Definition 2.2.5 (Homomorphism)**.** Let A and B be similar algebras in a language $\tau : \mathcal{F} \to \mathbb{N}$. We call a function $g : A \to B$ a *homomorphism* if for every $f \in \mathcal{F}$ with $n \coloneqq \tau(f)$ and every $a_1, \ldots, a_n \in A$:

$$g(f^A(a_1, \ldots, a_n)) = f^B(g(a_1), \ldots, g(a_n)).$$

We call B a *homomorphic image* of A if there exists a surjective homomorphism from A to B.

Definition 2.2.6 (Embedding and isomorphism)**.** Let $f : A \to B$ be an algebra homomorphism. If f is injective, we call f an *embedding* and say that A *embeds* into B. If, in addition, f is surjective, we call f an *isomorphism* and say that A and B are *isomorphic*.

Definition 2.2.7 (Congruence)**.** Let A be an algebra in a language $\tau : \mathcal{F} \to \mathbb{N}$. We call an equivalence relation $\theta \subseteq A \times A$ a *congruence* if for every $f \in \mathcal{F}$ with $n := \tau(f)$:

$$\langle a_1, b_1 \rangle \in \theta \wedge \cdots \wedge \langle a_n, b_n \rangle \in \theta \implies \langle f(a_1, \ldots, f_n), f(b_1, \ldots, b_n) \rangle \in \theta,$$

for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$.

Definition 2.2.8 (Quotient)**.** Let A be an algebra in a language $\tau : \mathcal{F} \to \mathbb{N}$ and θ be a congruence on A. Let B be the algebra in the language τ with an underlying set $A/\theta := \{[a]_\theta \mid a \in A\}$ such that for every $f \in \mathcal{F}$ with $n := \tau(f)$ and every $a_1, \ldots, a_n \in A$:

$$f^{A/\theta}(a_1, \ldots, a_n) = [f^A(a_1, \ldots, a_n)]_\theta.$$

We denote B by A/θ and call it the *quotient algebra* of A by θ.

Remark 2.2.9. The definition of congruence ensures that the operations in the quotient are well-defined.

The following theorem expresses that homomorphic images and quotient algebras are different ways to construct the same structures.

Theorem 2.2.10. *Let A and B be similar algebras. Then B is a homomorphic image of A if and only if B is isomorphic to a quotient of A.*

Proof sketch. Suppose B is a homomorphic image of A, witnessed by the homomorphism $f : A \to B$. Then

$$\mathrm{Ker}(f) := \{\langle a, b \rangle \in A \times A \mid f(a) = f(b)\}$$

is a congruence on A and $A/\mathrm{Ker}(f)$ is isomorphic to B.

Conversely, if B is isomorphic to A/θ, witnessed by an isomorphism $f : A/\theta \to B$, then $g : A \to B$ defined as:

$$g(a) := f([a]_\theta)$$

is a surjective homomorphism. ■

Definition 2.2.11 (Subalgebra)**.** Let A and B be similar algebras such that $A \subseteq B$ as underlying sets. If the inclusion function of A into B is a homomorphism, we write $A \leq B$ and call A a *subalgebra* of B.

Definition 2.2.12 (Direct product)**.** Let $\{A_i \mid i \in I\}$ be a set of similar algebras in a language $\tau : \mathcal{F} \to \mathbb{N}$. Let B be the algebra in the language τ with an underlying set $\Pi_{i\in I} A_i$ such that for every $f \in \mathcal{F}$ with $n \coloneqq \tau(f)$, every $\overline{a}_1, \ldots, \overline{a}_n \in \Pi_{i\in I} A_i$ and $i \in I$:

$$f^{\Pi_{i\in I} A_i}(\overline{a}_1, \ldots, \overline{a}_n)(i) = f^{A_i}(\overline{a}_1(i), \ldots, \overline{a}_n(i)).$$

We denote B by $\Pi_{i\in I} A_i$ and call it the *direct product* of $\{A_i \mid i \in I\}$.

For the next operation on algebras, we need the notion of an ultrafilter.

Definition 2.2.13 (Filter and ultrafilter)**.** Let I be a set and let $F \subseteq \mathcal{P}(I)$ be non-empty. Suppose $A, B \in F$ implies $A \cap B \in F$ and in addition, F is an upset in $\langle \mathcal{P}(I), \subseteq \rangle$. We call F a *filter* on I.

If $F \neq \mathcal{P}(I)$, we call F *proper*. If F is maximal with this property, we call it an *ultrafilter*.

Remark 2.2.14. Equivalently, a proper filter F is an ultrafilter if for every $A \subseteq I$, either $A \in F$ or $I \setminus A \in F$.

Definition 2.2.15 (Ultraproduct)**.** Let $\{A_i \mid i \in I\}$ be a set of similar algebras and U be an ultrafilter on I. Let θ_U be the congruence on $\Pi_{i\in I} A_i$ defined by:

$$\langle \overline{a}, \overline{b} \rangle \in \theta_U \iff \{i \in I \mid \overline{a}(i) = \overline{b}(i)\} \in U.$$

We call $\Pi_{i\in I} A_i/\theta_U$ the *ultraproduct* of $\{A_i \mid i \in I\}$ on U.

Remark 2.2.16. The fact that θ_U is a congruence follows from the properties of filters.

Definition 2.2.17 (Subdirect embedding and product)**.** An embedding $A \to \Pi_{i\in I} A_i$ is called *subdirect* if for every $i \in I$, we have $(\pi_i \circ f)[A] = A_i$. Here π_i denotes the projection onto the i-th coordinate.

If $A \leq \Pi_{i\in I} A_i$ and the inclusion function is a subdirect embedding, we call A a *subdirect product* of $\{A_i \mid i \in I\}$.

Definition 2.2.18 (Subdirect irreducibility)**.** An algebra A is called *subdirectly* irreducible if for every subdirect embedding $f : A \to \Pi_{i\in I} A_i$, there exists $i \in I$ such that $(\pi_i \circ f) : A \to A_i$ is an isomorphism.

Definition 2.2.19 (Class operators)**.** Let $\mathcal{K}$ be a class of similar algebras. We write:

$$\begin{aligned}
\mathbb{I}(\mathcal{K}) &\coloneqq \{A \mid A \text{ is isomorhpic to some } B \in \mathcal{K}\};\\
\mathbb{H}(\mathcal{K}) &\coloneqq \{A \mid A \text{ is a homomorphic image of some } B \in \mathcal{K}\};\\
\mathbb{S}(\mathcal{K}) &\coloneqq \mathbb{I}(\{A \mid A \text{ is a subalgebra of some } B \in \mathcal{K}\});\\
\mathbb{P}(\mathcal{K}) &\coloneqq \mathbb{I}(\{A \mid A \text{ is a direct product of some } \{B_i\}_{i\in I} \subseteq \mathcal{K}\});\\
\mathbb{P}_U(\mathcal{K}) &\coloneqq \mathbb{I}(\{A \mid A \text{ is an ultraproduct of some } \{B_i\}_{i\in I} \subseteq \mathcal{K}\});\\
\mathbb{P}_S(\mathcal{K}) &\coloneqq \mathbb{I}(\{A \mid A \text{ is a subdirect product of some } \{B_i\}_{i\in I} \subseteq \mathcal{K}\}).
\end{aligned}$$

2.2.2 Universal classes

Universal classes are collections of similar algebras that can be defined by universal formulas. Formally, we use the following definitions (see, e.g., [8, Section 5.2]).

Definition 2.2.20 (Universal formula)**.** A first-order formula φ is called universal if it is of the form $\varphi = \forall x_1 \dots \forall x_n \psi$ where ψ is quantifier-free.

Definition 2.2.21 (Universal class)**.** A class of similar algebras is said to be a *universal class* if it is the class of models of some set of universal sentences.

Definition 2.2.22 (Universal class generation)**.** If $\mathcal{K}$ is a class of similar algebras, we write $\mathbb{U}(\mathcal{K})$ for the least universal class that contains $\mathcal{K}$, also known as the universal class *generated* by $\mathcal{K}$.

The following is a useful characterisation of $\mathbb{U}$ in terms of the operators from the previous section. See, e.g., [8, Chapter 5, Theorem 2.20] for a proof.

Theorem 2.2.23. *Universal classes are closed under $\mathbb{S}$ and $\mathbb{P}_u$. Furthermore:*

$$\mathbb{U}(\mathcal{K}) = \mathbb{S}\mathbb{P}_u(\mathcal{K}).$$

Next, we define the auxiliary notion of a local subgraph that will allow for another useful characterisation of universal classes, found in, e.g., [18, Section 1.2]. Intuitively, a local subgraph of an algebra A is a finite subset of A endowed with finitely many partial operations, inherited from A.

Definition 2.2.24 (Local subgraph)**.** Let A be an algebra in a language $\tau : \mathcal{F} \to \mathbb{N}$. We call $\langle X, f_1^X, \dots, f_m^X \rangle$ a *local subgraph* of A, if:

- X is a finite subset of A;

- for every $i \in \{1, \ldots, m\}$, $f_i^X = f_i^A \cap X^{n+1}$ for some $f_i \in \mathcal{F}$ with $n \coloneqq \tau(f_i)$.

If the context allows, we will write X as shorthand for $\langle X, f_1^X, \ldots, f_m^X \rangle$.

Definition 2.2.25 (Local subgraph embeddability)**.** Let $\langle X, f_1^X, \ldots, f_m^X \rangle$ be a local subgraph of A and let B be an algebra similar to A. We call $g : X \to B$ an *embedding* if g is injective and for every $i \in \{1, \ldots, m\}$ and $a_1, \ldots, a_n \in X$ (where n is the arity of f_i^X) such that $f_i^X(a_1, \ldots, a_n) \in X$, we have:

$$g(f_i^X(a_1, \ldots, a_n)) = f_i^B(g(a_1), \ldots, g(a_n)).$$

If such a function exists, we say that X *embeds into* B.

Theorem 2.2.26. *Let $\{A\} \cup \mathcal{K}$ be a class of similar algebras. We have $A \in \mathbb{U}(\mathcal{K})$ if and only if for every local subgraph X of A, there exists $B \in \mathcal{K}$ such that X embeds into B.*

Proof sketch. For the left-to-right direction, we reason as follows. For every local subgraph X of A, there exists a universal sentence φ_X such that for every algebra B, we have $B \vDash \varphi_X$ if and only if X does not embed into B. Now suppose that there exists a local subgraph X of A that does not embed into any member of $\mathcal{K}$, then $\mathcal{K} \vDash \varphi_X$. Since φ_X is universal, $\mathbb{U}(\mathcal{K}) \vDash \varphi_X$. But as $A \nvDash \varphi_X$, this implies $A \notin \mathbb{U}(\mathcal{K})$.

For the right-to-left direction, we use [18, Theorem 1.2.8], which states that if every local subgraph of A embeds into some member of $\mathcal{K}$, then $A \in \mathbb{SP}_U(\mathcal{K})$. By Theorem 2.2.23, this means $A \in \mathbb{U}(\mathcal{K})$. ■

2.2.3 Varieties

We look more closely into a special kind of universal classes, called varieties, that are closed under homomorphic images and direct products. They are very frequently used throughout the text as algebraic models of logics.

The following definition is taken from [8, Section 2.9].

Definition 2.2.27 (Variety)**.** A class of similar algebras V is called a *variety* if it is closed under $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$.

Definition 2.2.28 (Variety generation)**.** Let $\mathcal{K}$ be a class of similar algebras. We write $\mathbb{V}(\mathcal{K})$ for the least variety that contains $\mathcal{K}$ and call it the variety *generated* by $\mathcal{K}$.

Below we state [8, Chapter 2, Theorem 9.5], which is a characterisation of varieties in terms of the operators.

Theorem 2.2.29 (Tarski)**.** *Let $\mathcal{K}$ be a class of similar algebras. Then* $\mathbb{V}(\mathcal{K}) = \mathbb{HSP}(\mathcal{K})$*.*

Similarly to how universal classes are defined by universal formulas, we know that varieties are defined by special formulas called equations. This result is known as Birkhoff's Theorem.

Definition 2.2.30 (Equation)**.** A first-order sentence φ is called an *equation* if it is of the form $\sigma \approx \tau$, where σ and τ are terms and $\approx$ is the formal equality symbol.

The proof of the next theorem can be found in [8, Chapter 2, Theorem 11.9].

Theorem 2.2.31 (Birkhoff)**.** *Let V be a class of similar algebras in a language τ. Then V is a variety if and only if V is equationally definable, i.e., there exists a set of equations Φ such that:*

$$V = \{A \text{ is an algebra in } \tau \mid A \vDash \Phi\}.$$

Next, we see that varieties are always generated by its subdirectly irreducible algebras (see [8, Chapter 2, Corollary 9.7]).

Definition 2.2.32. If $\mathcal{K}$ is a class of algebras, we write $\mathcal{K}_{SI}$ for the collection of subdirectly irreducible members of $\mathcal{K}$.

Theorem 2.2.33. *For every variety V, we have $V = \mathbb{P}_S(V_{SI})$, i.e., V is generated by its subdirectly irreducible members.*

In addition to subdirectly irreducible algebras, there is another important sort of algebras called simple algebras.

Definition 2.2.34 (Simple algebra)**.** An algebra A is called *simple* if it has exactly two congruences (these are $A \times A$ and $\{\langle a, a\rangle \mid a \in A\}$).

Given a class of algebras $\mathcal{K}$, we write $\mathcal{K}_S$ for the collection of simple members of $\mathcal{K}$.

Simple algebras are subdirectly irreducible, but the converse does not hold in general. Nevertheless, in the following chapter we will study a variety where the two notions coincide, i.e., we will have the following.

Definition 2.2.35 (Semi-simple variety)**.** A variety V is called *semi-simple* if $V_{SI} = V_S$.

Finally, we define the finite model property of varieties, which is closely connected to the finite model property of logics.

Definition 2.2.36. Let $\mathcal{K}$ be a class of algebras. We write $Fin(\mathcal{K})$ for the collection of finite members of $\mathcal{K}$.

Definition 2.2.37 (FMP)**.** Let V be a variety. We say that V has the *finite model property* (FMP) if $V = \mathbb{V}(Fin(V))$, i.e., V is generated by its finite members.

2.2.4 Generating algebras

Some important objects that we study are defined in terms of algebra generation, which we cover here.

Definition 2.2.38 (Algebra generation)**.** Let A be an algebra and $X \subseteq A$. We refer to the least subalgebra of A containing X as the subalgebra *generated* by X.

Definition 2.2.39 (n-generated and finitely generated)**.** An algebra A is said to be *n-generated*, for $n \in \mathbb{N}$, if there exist a set $B \subseteq A$ with $|B| \leq n$ that generates A. If A is n-generated for some $n \in \mathbb{N}$, A is said to be *finitely generated*.

Now we have the required terminology to further strengthen Theorem 2.2.33.

Theorem 2.2.40. *Every variety is generated by the set of its finitely generated subdirectly irreducible members.*

Proof sketch. From the fact that each equation contains finitely many variables, one can derive that varieties are generated by their finitely generated members. From here, we can adapt the proof of [8, Chapter 2, Corollary 9.7]. ∎

Definition 2.2.41 (Locally finite)**.** A variety V is said to be *locally finite* if every finitely generated algebra in V is finite.

Definition 2.2.42 (Free algebra)**.** Let $n \in \mathbb{N}$ and let V be a variety. We call $A \in V$ the *free* n-generated algebra in V if A is generated by some $\{a_1, \dots, a_n\} \subseteq A$ and for every $B \in V$ generated by $\{a_1, \dots, a_n\}$, there exists a unique homomorphism $f : A \to B$ that restrict to the identity function on $\{a_1, \dots, a_n\}$, i.e.:

$$f\big|_{\{a_1,\dots,a_n\}} = Id_{\{a_1,\dots,a_n\}}.$$

Note that $a_1, \dots, a_n$ are not necessarily distinct.

2.3 Heyting and bi-Heyting algebras

We presented the general theory of universal algebra so that we can apply it to two concrete classes of algebras – Heyting and bi-Heyting algebras. Here we see what these are. We define them as special cases of simpler structures, which we introduce first.

Definition 2.3.1 (Lattice)**.** Let $\langle A, \leq \rangle$ be a poset where every pair of elements has a supremum and an infimum. Define for every $a, b \in A$:

$$\wedge^A (a, b) = inf\{a, b\}, \qquad \vee^A(a, b) = sup\{a, b\}.$$

We call the algebra $\langle A, \{\wedge^A, \vee^A\}\rangle$ a *lattice*. Thus the language of lattices consists of two binary operations $\wedge$ and $\vee$. We call $\wedge$ a meet and $\vee$ a join.

Remark 2.3.2. Note that the order $\leq$ is not present in $\langle A, \{\wedge^A, \vee^A\}\rangle$, but it can be retrieved by the equivalences:

$$a \leq b \iff a \wedge b = a \iff a \vee b = b.$$

Hence we may assume that lattices come equipped with a partial order.

Remark 2.3.3. With lattices and other classes of algebras that follow, we will drop the superscript (as in $\wedge$ instead of $\wedge^A$) whenever no confusion arises.

Definition 2.3.4 (Dual lattice)**.** Let A be a lattice induced by $\langle A, \leq \rangle$. We define A^∂ to be the lattice induced by $\langle A, \geq \rangle$ and call it the *dual* of A.

Remark 2.3.5. Meets in A^∂ are the joins of A and joins in A^∂ are the meets of A. Intuitively, A^∂ is the upside-down image of A.

Definition 2.3.6 (Bounded lattice)**.** Let A be a lattice with a least element $\bot^A$ and a greatest element $\top^A$. Then A, endowed with the two constants $\bot^A$ and $\top^A$, is called a *bounded lattice*. Thus the language of bounded lattices consists of two constants $\bot$ and $\top$ and two binary operations $\wedge$ and $\vee$.

Definition 2.3.7 (Distributive lattice)**.** Let A be a lattice, such that for every $a, b, c \in A$:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Then A is called a *distributive lattice*.

Definition 2.3.8 (Heyting algebra)**.** Let A be a bounded lattice and let $\rightarrow: A \times A \rightarrow A$ satisfy:

$$c \wedge a \leq b \iff c \leq a \rightarrow b \qquad \text{for all } a, b, c \in A.$$

We call A, endowed with $\rightarrow$, a *Heyting algebra*. Thus the language of Heyting algebras consists of two constants $\bot$ and $\top$ and three binary operations $\wedge$, $\vee$ and $\rightarrow$. We call $\rightarrow$ an *implication*.

Remark 2.3.9. Every Heyting algebra is a distributive lattice.

The following property is useful for determining which ordered structures can be turned into Heyting algebras.

Proposition 2.3.10. Let A be a Heyting algebra. Then for every $a, b \in A$:

$$a \rightarrow b = sup\{c \in A \mid c \wedge a \leq b\}.$$

Moreover, a bounded distributive lattice can be equipped with an implication, i.e., turned into a Heyting algebra, if and only if the above supremum exists for all elements a and b.

Remark 2.3.11. Consequently, a Heyting algebra is uniquely determined by its partial order.

Definition 2.3.12 (bi-Heyting algebra)**.** Let A be a Heyting algebra and let $\leftarrow: A \times A \rightarrow A$ satisfy:

$$a \leq c \vee b \iff a \leftarrow b \leq c \qquad \text{for all } a, b, c \in A.$$

We call A, endowed with $\leftarrow$, a *bi-Heyting algebra*. Thus the language of bi-Heyting algebras consists of two constants $\bot$ and $\top$ and four binary operations $\wedge$, $\vee$, $\rightarrow$ and $\leftarrow$. We call $\leftarrow$ a *co-implication*.

Remark 2.3.13. We will call a bounded lattice with a co-implication only a co-Heyting algebra.

Next, we have the bi-Heyting analogue of Proposition 2.3.10.

Proposition 2.3.14. Let A be a bounded distributive lattice. Then for every $a, b \in A$:

$$a \leftarrow b = inf\{c \in A \mid a \leq c \vee b\}.$$

Moreover, a bounded distributive lattice can be equipped with a co-implication, i.e. turned into a co-Heyting algebra, if and only if the above meet exists for all elements a and b.

Corollary 2.3.15. A bounded distributive lattice A can be turned into a bi-Heyting algebra if and only if A and A^∂ can be equipped with an implication.

We will be interested in the collection of all Heyting algebras and all bi-Heyting algebras.

Definition 2.3.16. Let HA denote the class of all Heyting algebras and $\mathsf{bi\text{-}HA}$ denote the class of all bi-Heyting algebras.

Crucially, we have the following property.

Theorem 2.3.17. *HA and $\mathsf{bi\text{-}HA}$ are varieties.*

Lastly, we recall the so-called Jónsson's Lemma (see, e.g., [8, Chapter 4, Theorem 6.8]). It can be useful for characterising the subdirectly irreducible algebras of a variety. It only applies to congruence-distributive varieties, which we define below.

Definition 2.3.18 (Lattice of congruences)**.** Let A be an algebra. We write $\mathrm{Con}(A)$ for the lattice of congruences of A, where $\theta_1 \wedge \theta_2$ is defined as the intersection of θ_1 and θ_2 and $\theta_1 \vee \theta_2$ is the least congruence containing $\theta_1 \cup \theta_2$, for any pair of congruences θ_1, θ_2 on A.

Definition 2.3.19 (Congruence distributive)**.** An algebra A is said to be *congruence-distributive* if $\mathrm{Con}(A)$ is a distributive lattice. A class of algebras $\mathcal{K}$ is said to be *congruence-distributive* if each of its members is congruence-distributive.

Theorem 2.3.20. *The varieties HA and $\mathsf{bi\text{-}HA}$ (and consequently, all of their subvarieties) are congruence-distributive.*

Theorem 2.3.21 (Jónsson's Lemma)**.** *Let V be a congruence-distributive variety such that $V = \mathbb{V}(\mathcal{K})$ for some class $\mathcal{K}$. Then $V = \mathbb{P}_S\mathbb{H}\mathbb{S}\mathbb{P}_U(\mathcal{K})$. Consequently, $V_{SI} \subseteq \mathbb{H}\mathbb{S}\mathbb{P}_U(\mathcal{K})$.*

Remark 2.3.22. By Theorem 2.3.17, we can freely apply Jónsson's Lemma to subvarieties of HA and $\mathsf{bi\text{-}HA}$.

2.4 Bi-Heyting Duality

When working with varieties, it is sometimes possible to translate the algebraic objects of study into a topological language, which allows us to approach the problem from a different perspective. An example is the so-called Esakia duality between Heyting algebras and Esakia spaces (see, e.g., [14, Chapter 3]). Moreover, bi-Esakia spaces (see, e.g., [27, Chapter 2]), which are a special kind of Esakia spaces, are dual to bi-Heyting algebras. In this section, we outline the Esakia and bi-Esakia dualities and some of their consequences that will be of use.

We assume familiarity with introductory notions from general topology, such as topological spaces, open and closed sets, bases, subbases, continuous functions and compactness. For a text on general topology, refer to, e.g., [21].

Definition 2.4.1 (Esakia space)**.** Let $\mathcal{X} := \langle X, \mathcal{T}, \leq \rangle$ satisfy the following conditions.

- $\langle X, \mathcal{T} \rangle$ is a compact topological space.
- $\langle X, \leq \rangle$ is a poset.
- If $x, y \in X$ and $x \nleq y$, then there exists a clopen set $U \in \mathcal{T}$ such that $x \in U$ and $y \notin U$.
- If $U \in \mathcal{T}$ is clopen, then $\downarrow U$ is clopen.

We call $\mathcal{X}$ an *Esakia space.*

Definition 2.4.2 (bi-Esakia space)**.** If an Esakia space $\langle X, \mathcal{T}, \leq \rangle$ satisfies the additional condition:

- if $U \in \mathcal{T}$ is clopen, then $\uparrow U$ is clopen,

we call it a *bi-Esakia space.*

An Esakia space $\mathcal{X}$ can be seen as an intuitionistic general Kripke frame $\mathfrak{F}$, where the admissible subsets of $\mathfrak{F}$ are the clopen upsets of $\mathcal{X}$.

Definition 2.4.3 (Esakia morphism)**.** Suppose $\mathcal{X} := \langle X, \mathcal{T}, \leq \rangle$ and $\mathcal{X}' := \langle X', \mathcal{T}', \leq' \rangle$ are Esakia spaces. We call $f : X \to X'$ an *Esakia morphism* if:

- f is a continuous function between the topological spaces $\langle X, \mathcal{T} \rangle$ and $\langle X', \mathcal{T}' \rangle$;
- f is monotone, i.e.:

$$\forall x, y \in X : x \leq y \to f(x) \leq f(y);$$

- f satisfies the forth condition, i.e.:

$$\forall x \in X \forall y' \in X'(f(x) \leq' y' \to \exists y \in X(x \leq y \land f(y) = y')).$$

Definition 2.4.4 (bi-Esakia morphism)**.** Let $\mathcal{X} := \langle X, \mathcal{T}, \leq \rangle$ and $\mathcal{X}' := \langle X', \mathcal{T}', \leq' \rangle$ be bi-Esakia spaces. We call $f : X \to X'$ a *bi-Esakia morphism* if it is an Esakia morphism such that:

- f satisfies the back condition, i.e.:

$$\forall x \in X \forall y' \in X'(y' \leq f(x) \to \exists y \in X(y \leq x \land f(y) = y')).$$

Esakia morphisms can be seen as p-morphisms, while bi-Esakia morphisms can be seen as bidirectional p-morphisms.

In the following theorem, we state the Esakia and bi-Esakia dualities together.

Theorem 2.4.5. *The category of (bi-)Heyting algebras with homomorphisms is dually equivalent to the category of (bi-)Esakia spaces with (bi-)Esakia morphisms.*

Below we present the object part of the dualities. As a preliminary, we recall what filters and prime filters on a bounded lattice are.

Definition 2.4.6 (Filter, prime filter)**.** Let A be a bounded lattice and let $\emptyset \neq F \subseteq A$. We call F a *filter* if:

- $a, b \in F$ implies $a \land b \in F$;
- F is an upset.

We call F a *prime filter* if in addition, we have the following:

- $\bot \notin F$;
- $a \lor b \in F$ implies $a \in F$ or $b \in F$.

Definition 2.4.7 (Dual space)**.** Let A be a (bi-)Heyting algebra. We define the *dual (bi-)Esakia space* of A to be $\mathcal{X}(A) := \langle X, \mathcal{T}, \leq \rangle$, where:

- X is the set of prime filters on A;
- $\leq := \subseteq$ is the set inclusion between filters;

- $\mathcal{T}$ is the topology induced by the subbase:

$$\{\varphi(a) \mid a \in A\} \cup \{A \setminus \varphi(a) \mid a \in A\},$$

  where $\varphi(a) \coloneqq \{F \text{ is a prime filter} \mid a \in F\}$ for $a \in A$.

Remark 2.4.8. A (bi-)Heyting algebra is finite precisely when its dual space is finite.

Definition 2.4.9 (Dual algebra)**.** Let $\mathcal{X} \coloneqq \langle X, \mathcal{T}, \leq \rangle$ be a (bi-)Esakia space. Denote by $\mathcal{A}(\mathcal{X})$ the (bi-)Heyting algebra obtained by equipping the set of clopen upsets of $\mathcal{X}$ with the following operations:

- $\bot \coloneqq \emptyset$;
- $\top \coloneqq X$;
- $U \wedge V \coloneqq U \cap V$;
- $U \vee V \coloneqq U \cup V$;
- $U \to V \coloneqq X \setminus {\downarrow}(U \setminus V)$;
- $U \leftarrow V \coloneqq {\uparrow}(U \setminus V)$, in case $\mathcal{X}$ is a bi-Esakia space.

We call $\mathcal{A}(\mathcal{X})$ the *dual (bi-)Heyting algebra* of $\mathcal{X}$.

The object part of the duality is conveyed by the following statement.

Theorem 2.4.10. *If A is a (bi-)Heyting algebra and $\mathcal{Y}$ is a (bi-)Esakia space, then:*

$$A \cong \mathcal{A}(\mathcal{X}(A)),$$
$$\mathcal{Y} \cong \mathcal{X}(\mathcal{A}(\mathcal{Y})).$$

For the rest of the section, we work with the bi-Heyting duality only. Our goal is to state a consequence of this duality that we are going to need in this thesis. For this purpose, we define the following analogue of a generated subframe of a Kripke frame.

Definition 2.4.11 (Generated subspace)**.** Let $\mathcal{X} \coloneqq \langle X, \mathcal{T}, \leq \rangle$ be a bi-Esakia space. We call $\langle X', \mathcal{T}', \leq' \rangle$ a *generated subspace* of $\mathcal{X}$ if:

- $X' \subseteq X$ is closed, an upset and a downset;
- $\langle X', \mathcal{T}' \rangle$ is a topological subspace of $\mathcal{X}$;

- $\leq' = \leq|_{X'}$.

Proposition 2.4.12. Let A and B be bi-Heyting algebras. Then B is a homomorphic image of A if and only if $\mathcal{X}(B)$ is a generated subspace of $\mathcal{X}(A)$.

For our purposes, the main ingredient to take away from this section is the following corollary of Proposition 2.4.12.

Corollary 2.4.13. A bi-Heyting algebra A is simple if and only if $\mathcal{X}(A)$ is non-empty and does not contain non-trivial generated subspaces, i.e., generated subspaces different from $\emptyset$ and $\mathcal{X}(A)$.

2.5 Intuitionistic and bi-intuitionistic logic

Most results in the following chapters are presented in the language of varieties, but in order to appreciate them, we should look at the connection between varieties and logics. In particular, we will focus on the systems **IPC** (intuitionistic propositional calculus) and **bi-IPC** (bi-intuitionistic propositional calculus).

Remark 2.5.1. By a logic, we will refer to a set of formulas, as opposed to a consequence relation.

The logic **IPC** is obtained by removing the law of excluded middle $p \vee \neg p$ from classical propositional logic. Among other purposes, it aims to model constructive reasoning in mathematics. A more thorough presentation can be found in [9, Chapter 2].

Definition 2.5.2 (IPC). Consider the language consisting of the constants $\bot$ and $\top$ and the binary connectives $\wedge$, $\vee$ and $\to$. We use the abbreviation $\neg\varphi \coloneqq \varphi \to \bot$. We work with an infinite set of propositional letters $\{p_0, p_1, \dots\}$. The logic **IPC** is defined as the least set of formulas in this language, containing the axioms:

$$p_0 \to (p_1 \to p_0 \wedge p_1) \qquad p_0 \to (p_1 \to p_0)$$
$$p_0 \wedge p_1 \to p_1 \qquad p_0 \wedge p_1 \to p_0$$
$$p_0 \to p_0 \vee p_1 \qquad p_1 \to p_0 \vee p_1$$
$$(p_0 \to p_2) \to ((p_1 \to p_2) \to (p_0 \vee p_1 \to p_2)) \qquad \bot \to p_0$$
$$(p_0 \to (p_1 \to p_2)) \to ((p_0 \to p_1) \to (p_0 \to p_2))$$

and closed under the rules:

- Modus Ponens: if $\varphi \in \mathbf{IPC}$ and $\varphi \to \psi \in \mathbf{IPC}$, then $\psi \in \mathbf{IPC}$;
- Substitution: if $\varphi \in \mathbf{IPC}$, then $\varphi[\psi/p_i] \in \mathbf{IPC}$ for every formula ψ and $i \in \mathbb{N}$.

Definition 2.5.3 (Superintuitionistic logic)**.** A set of formulas in the language of **IPC** is called a *superintuitionistic logic* if it extends **IPC** and is closed under the rules of Modus Ponens and Substitution.

Note that **IPC** has a Kripke frame semantics. In particular, a finite intuitionistic frame is simply a finite Esakia space. Since we will be interested in the class of finite frames validating a logic, we give the following definition.

Definition 2.5.4 (Finite frames of a logic)**.** Let L be a superintuitionistic logic. We define:

$$FinFr(L) \coloneqq \{\mathcal{X} \mid \mathcal{X} \text{ is a finite Esakia space such that } \mathcal{A}(\mathcal{X}) \vDash L\}$$

and call it the class of *finite frames* of L.

The importance of Heyting algebras is conveyed by the following theorem, stating that the variety of Heyting algebras serves as semantics for **IPC**.

Theorem 2.5.5. *For every formula φ in the language of **IPC**:*

$$\varphi \in \boldsymbol{IPC} \iff \mathsf{HA} \vDash \varphi \approx \top.$$

In addition to **IPC** itself, superintuitionistic logics are also modelled by varieties of Heyting algebras. In order to present this fact, we define a lattice structure on superintuitionistic logics and on subvarieties of HA.

Proposition 2.5.6. The collection of superintuitionistic logics is a lattice with operations defined as follows:

- $L_1 \wedge L_2 = L_1 \cap L_2$;
- $L_1 \vee L_2 = \mathcal{L}(L_1 \cup L_2)$, where $\mathcal{L}(X)$ is defined to be the least superintuitionistic logic containing X,

for any pair of superintuitionistic logics L_1 and L_2.

Proposition 2.5.7. The set of subvarieties of HA is a lattice with operations defined as follows:

- $V_1 \wedge V_2 = V_1 \cap V_2$;
- $V_1 \vee V_2 = \mathbb{V}(V_1 \cup V_2)$,

for any pair of varieties $V_1, V_2 \subseteq \mathsf{HA}$.

We know that these two lattices are dually isomorphic and the isomorphism is given in the following way.

Definition 2.5.8. Let $V \subseteq \mathsf{HA}$ be a variety. We define the logic:

$$\mathcal{L}_V \coloneqq \{\varphi \mid V \vDash \varphi \approx \top\}.$$

Theorem 2.5.9. *The function $\mathcal{L}_{\_}$ is a dual isomorphism between the lattice of subvarieties of HA and the lattice of superintuitionistic logics.*

Next, we define **bi-IPC**, which is a conservative extension of **IPC** with an additional connective $\leftarrow$, which behaves dually to $\rightarrow$.

Definition 2.5.10 (**bi-IPC**)**.** Consider the language obtained from adding a binary connective $\leftarrow$ to the language of **IPC**. We use an additional abbreviation $\sim\varphi \coloneqq \top \leftarrow \varphi$. The logic **bi-IPC** is the least set of formulas containing the axioms of **IPC** and:

$$p_0 \rightarrow (p_1 \vee (p_1 \leftarrow p_0)) \qquad (p_1 \leftarrow p_0) \rightarrow \sim(p_0 \rightarrow p_1)$$
$$\neg(p_0 \leftarrow p_1) \rightarrow (p_0 \rightarrow p_1) \qquad \neg\sim(p_0 \leftarrow p_0)$$
$$(p_2 \leftarrow (p_1 \leftarrow p_0)) \rightarrow ((p_0 \vee p_1) \leftarrow p_0)$$

which is closed under the rules of **IPC** and:

- Double negation: $\varphi \in$ **bi-IPC**, then $\neg\sim\varphi \in$ **bi-IPC**.

As might be expected, bi-Heyting algebras model bi-intuitionistic logic.

Theorem 2.5.11. *For every formula φ in the language of* **bi-IPC**:

$$\varphi \in \textbf{bi-IPC} \iff \mathsf{bi\text{-}HA} \vDash \varphi \approx \top.$$

Definition 2.5.12 (Bi-superintuitionistic logic)**.** A set of formulas in the language of **IPC** is called a *bi-superintuitionistic logic* if it extends **bi-IPC** and is closed under the rules of Modus Ponens and Substitution and Double negation.

Remark 2.5.13. The finite frames operator $FinFr$ from Definition 2.5.4 can be adapted to bi-superintuitionistic logics. For every bi-intuitionistic logic L, we define:

$$FinFr(L) \coloneqq \{\mathcal{X} \mid \mathcal{X} \text{ is a finite bi-Esakia space such that } \mathcal{A}(\mathcal{X}) \vDash L\}.$$

Remark 2.5.14. Completely analogously to superintuitionistic logics and subvarieties of HA, we can define a lattice structure on bi-superintuitionistic logics and subvarieties of $\mathsf{bi\text{-}HA}$. In the same way as in Definition 2.5.8, we can define $\mathcal{L}$ on subvarieties of $\mathsf{bi\text{-}HA}$.

As a result, we have the following analogous dual isomorphism.

Theorem 2.5.15. *The function $\mathcal{L}_{-}$ is a dual isomorphism between the lattice of subvarieties of $\mathsf{bi\text{-}HA}$ and the lattice of bi-superintuitionistic logics.*

In view of these correspondences, in the following chapters, we freely switch between logics and varieties. In fact, we choose to do the proofs algebraically (and occasionally, order-topologically) and only state the most important results in logical terms.

Chapter 3

Bi-intuitionistic Kuznetsov-Gerčiu variety

The purpose of this chapter is to introduce the bi-intuitionistic variant of the Kuznetsov-Gerčiu variety (resp. the Kuznetsov-Gerčiu logic) and to establish its core properties. We begin by recalling the well-known intuitionistic variety KG. Motivated by it, we proceed to define its bi-Heyting variant $\mathsf{bi\text{-}KG}$. Next, we focus on proving semi-simplicity of $\mathsf{bi\text{-}KG}$, together with a characterisation of its subvarieties. As a result, local subgraphs (see Definition 2.2.24) turn out to be especially useful for comparing subvarieties of $\mathsf{bi\text{-}KG}$ with the help of Theorem 2.2.26. Lastly, we collect a number of useful facts about local embeddability in $\mathsf{bi\text{-}KG}$, which are essential to our further study of this variety.

3.1 The Kuznetsov-Gerčiu variety KG

The variety KG (more precisely, its logic counterpart $\mathbf{KG}$) appeared in the work of Kuznetsov and Gerčiu [17, 16]. It was motivated by studying finite axiomatizability and the FMP of superintuitionistic logics. Moreover, its fine combinatorial properties (see, e.g., [4, Chapter 4]) allow for the construction of subvarieties without the FMP. This makes it a very valuable tool in characterising degrees of FMP of superintuitionistic logics [3]. Recall that the degree of FMP of a variety U relative to a variety V is the number of subvarieties of V with the same finite algebras as U.

In order to define KG, we need to introduce several notions. Recall the notion of an n-generated algebra from Definition 2.2.39. Here we describe the 1-generated Heyting algebras. By Definition 2.2.42, this can be done by first

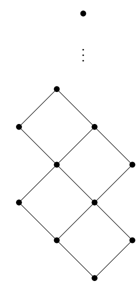

Figure 3.1: Rieger-Nishimura lattice RN

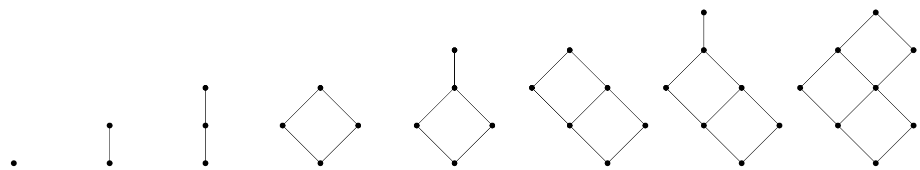

Figure 3.2: Some finite 1-generated Heyting algebras

considering the free Heyting algebra on 1 generator, because the 1-generated Heyting algebras are precisely its homomorphic images. This is the so-called Rieger-Nishimura lattice RN, described in [26, 22], which is depicted in Figure 3.1. Consequently, the other 1-generated Heyting algebras are homomorphic images of RN. Except for RN itself, these images are finite and are, in fact, principal downsets of RN. The first several smallest 1-generated Heyting algebras are depicted in Figure 3.2. Notice that we denote the finite 1-generated algebra with m elements by RN_m. This is convenient, because for every natural number, there exists a unique 1-generated Heyting algebra of that size. Thus we obtain the following definition.

Definition 3.1.1. We denote the free 1-generated Heyting algebra by RN and for every $m \in \mathbb{N}$, we denote the 1-generated Heyting algebra with m elements by RN_m.

Next, we recall the sum operation on Heyting algebras (see, e.g., [14, Appendix A.9]).

Definition 3.1.2 (Sum of Heyting algebras)**.** Let A and B be Heyting algebras. The *sum* $A + B$ of A and B is defined as the algebra obtained by pasting A below B and identifying the top element of A with the bottom element of B.

Example 3.1.3. For instance, $RN_7 = RN_6 + RN_2$.

Remark 3.1.4. Given Heyting algebras A and B, the sum $A + B$ is indeed a Heyting algebra. This can be seen with the help of Proposition 2.3.10.

Remark 3.1.5. By a straightforward argument, the operation $+$ is associative. Therefore, we will write expressions of the form $A_1 + \cdots + A_n$ or $\Sigma_{1 \leq i \leq n} A_i$ without parenthesis.

It is convenient to introduce the following notation:

Definition 3.1.6 (Finite sums)**.** Let $\mathcal{K}$ be a class of Heyting algebras. Define:

$$FinSum(\mathcal{K}) = \{A_1 + \cdots + A_n \mid A_1, \ldots, A_n \in \mathcal{K}, n \in \mathbb{N}^+\}.$$

With these definitions in place, we are now ready to give the notion of KG. It is the variety of Heyting algebras generated by finite sums of 1-generated algebras.

Definition 3.1.7 (Kuznetsov-Gerčiu variety)**.** Let $\mathcal{K}$ denote the class of 1-generated Heyting algebras. Define the *Kuznetsov-Gerčiu variety*:

$$\mathsf{KG} \coloneqq \mathbb{V}(FinSum(\mathcal{K})).$$

As we saw in Section 2.5, subvarieties of HA give rise to superintuitionistic logics via $\mathcal{L}_{-}$ (see Definition 2.5.8). We apply it in the current context.

Definition 3.1.8 (Kuznetsov-Gerčiu logic)**.** Define the *Kuznetsov-Gerčiu logic*:

$$\mathbf{KG} \coloneqq \mathcal{L}_{\mathsf{KG}}.$$

3.2 The bi-intuitionistic variant bi-KG

The similarities between the varieties HA and $\mathsf{bi\text{-}HA}$ motivate a fruitful approach to solving questions of bi-intuitionistic nature by building on the already established methods for HA. Since we are interested in studying degrees of FMP in $\mathsf{bi\text{-}HA}$ and the central tool used for resolving this intuitionistically is KG (see [3]), it is meaningful to look for a bi-intuitionistic counterpart $\mathsf{bi\text{-}KG}$ of KG. In the present section, we introduce a natural such candidate.

In the last section, we defined KG by specifying a class of generators – the class of finite sums of 1-generated Heyting algebras. Now we aim to use the same generators to induce a variety in the bi-Heyting language. But before we are able to do that, we need to make sure that our generators are indeed bi-Heyting algebras. More concretely, we need to turn $RN, RN_1, RN_2, \dots$ into bi-Heyting algebras and verify that the sum operation is well-defined for bi-Heyting algebras.

Proposition 3.2.1. The 1-generated Heyting algebras $RN, RN_1, RN_2, \dots$ can be (uniquely) equipped with a co-implication.

Proof. By referring to Proposition 2.3.14, it suffices to check that for every $A \in \{RN, RN_1, RN_2, \dots\}$, the algebra A possesses the required infima $inf\{c \mid a \leq c \vee b\}$, for every $a, b \in A$. This is a straightforward verification. ∎

As a result of this proposition, we can give the following definition.

Definition 3.2.2. We denote the bi-Heyting algebra obtained by equipping RN with a co-implication by L. For every $m \in \mathbb{N}$, we denote the bi-Heyting algebra obtained by equipping RN_m with a co-implication by L_m.

Next, we ensure the sum operation is well-defined for bi-Heyting algebras.

Proposition 3.2.3. Let A and B be bi-Heyting algebras and let A' and B' be their respective Heyting reducts. Then the sum $A' + B'$ can be equipped with a co-implication in a unique way and hence gives rise to a bi-Heyting algebra.

Proof. Consider the dual lattices $(A')^\partial$ and $(B')^\partial$. Since A and B are bi-Heyting algebras and are equipped with a co-implication, we know that $(A')^\partial$ and $(B')^\partial$ can be equipped with an implication. Thus by viewing them as Heyting algebras, we can write the sum $(B')^\partial + (A')^\partial$. But we know that the sum of two Heyting algebras is a Heyting algebra, therefore $(B')^\partial + (A')^\partial$ can also be equipped with an implication. Now it follows that $A' + B' = \big((B')^\partial + (A')^\partial\big)^\partial$ can be equipped with a co-implication. ∎

Remark 3.2.4. Given $A, B \in \mathsf{bi\text{-}HA}$, we will write $A + B$ to mean the bi-Heyting algebra arising from the sum of the reducts of A and B. Unsurprisingly, $+$ is also associative on bi-Heyting algebras, so we will write expressions of the form $A_1 + \dots + A_n$ or $\Sigma_{1 \leq i \leq n} A_i$ for $A_1, \dots, A_n \in \mathsf{bi\text{-}HA}$.

As a last preparation step, recall the $FinSum$ class operator from last section. We will use it for bi-Heyting algebras as well. In addition, the following shorthand notation will be convenient. Here $\mathcal{G}$ is short for "generators".

Definition 3.2.5. Define $\mathcal{G} \coloneqq FinSum(\{L\} \cup \{L_n \mid n \in \mathbb{N}^+\}$.

Now we are ready to define $\mathsf{bi\text{-}KG}$.

Definition 3.2.6 (Bi-Kuznetsov-Gerčiu variety)**.** We define the *bi-Kuznetsov-Gerčiu variety*:

$$\mathsf{bi\text{-}KG} \coloneqq \mathbb{V}(\mathcal{G}).$$

Again, we have a corresponding bi-superintuitionistic logic.

Definition 3.2.7 (Bi-Kuznetsov-Gerčiu logic)**.** Define the *bi-Kuznetsov-Gerčiu logic*:

$$\mathbf{bi\text{-}KG} \coloneqq \mathcal{L}_{\mathsf{bi\text{-}KG}}.$$

As a consequence of Theorem 2.5.15, the lattice of subvarieties of $\mathsf{bi\text{-}KG}$ is dually isomorphic to the lattice of extensions of **bi-KG**. Thus the results for the variety $\mathsf{bi\text{-}KG}$ that we study in the remainder of this chapter can be translated into results for the logic **bi-KG**.

3.3 Understanding the universal class of the bi-KG generators

Now that we have defined $\mathsf{bi\text{-}KG}$, we are interested in determining its properties. More specifically, we would like to have a good understanding of the lattice of subvarieties of $\mathsf{bi\text{-}KG}$. From the study of universal algebra, we know that every variety is fully characterised by its subdirectly irreducible algebras.

In the current setting, we will see that subdirectly irreducible $\mathsf{bi\text{-}KG}$ algebras have a particular form, which will allow us to work with them in a very concrete way. The core step towards finding this form is to study the universal class $\mathbb{U}(\mathcal{G})$, which turns out to almost match the class of simple $\mathsf{bi\text{-}KG}$ algebras. In order to characterise $\mathbb{U}(\mathcal{G})$, we need to introduce a few new notions, which arise in relation to $\mathsf{bi\text{-}KG}$.

Firstly, we present a slight generalisation of the sum operation. So far we have worked with sums of Heyting and bi-Heyting algebras, which are bounded lattices. But it is also possible to glue together a lattice without a top element and a lattice without a bottom element, resulting in a bi-Heyting algebra. In order to allow this, we extend the definition of the sum operation to possibly unbounded lattices.

Definition 3.3.1 (Sum of possibly unbounded lattices)**.** Let A and B be lattices. If A has a top element and B has a bottom element, the *sum* $A+B$ is defined to be the algebra obtained by pasting A below B and identifying the top element of A with the bottom element of B. Otherwise, $A+B$ is defined to be the algebra obtained by pasting A below B, without any identification.

Example 3.3.2. Let RN^u denote the Rieger-Nishimura lattice without its top element. Then $RN = RN^u + RN_1$. In this case, we make no identifications, because the first summand RN^u is unbounded from above.

This new sum operation is again associative, so we can write finite sums without parentheses. In fact, we can write arbitrary linear sums.

Definition 3.3.3. Let I be a linear order and let $\{A_i\}_{i\in I}$ be a set of lattices indexed by I. We write $\Sigma_{i\in I}A_i$ for the lattice obtained by pasting all A_i on top of each other according to the order of I and making the following identifications. For every $i \in I$, if i has a direct successor $j \in I$ (meaning $i < j$ and there are no elements between them) and in addition, A_i has a top element and A_j has a bottom element, we identify the top of A_i with the bottom of A_j. If $\Sigma_{i\in I}A_i$ has a least and a greatest element, we call the sum *bounded*.

Remark 3.3.4. From now on, if we have a sum of lattices A that can be equipped with an implication and a co-implication, we will consider it as a bi-Heyting algebra (unless explicitly stated otherwise). So for example, saying that B is a subalgebra of A will tacitly imply that B is closed under $\to$ and $\leftarrow$.

Secondly, we give names to some particular algebras that will play an essential role in understanding bi-KG. By L^u, L_u and L^u_u we will refer to the corresponding unbounded lattices depicted in Figure 3.3 (the index u indicates the directions in which they are unbounded).

Definition 3.3.5. We call L^u the lattice L without its top element; we call L_u the dual of L^u; we call L^u_u the lattice whose every principle upset is isomorphic to L^u and whose every principal downset is isomorphic to L_u.

Amongst the many algebras we have introduced in relation to bi-KG, we are interested in the ones that cannot be represented as a sum of *strictly* smaller summands. For example, if we look at the algebras $L_1, L_2, L_3, \dots,$ the ones that satisfy this property are $L_1, L_2, L_4, L_6, L_8, \dots$ (see Figure 3.4). Notice that, for instance, L_3, L_5 and L does not satisfy this property, since $L_3 = L_2 + L_2$, $L_5 = L_4 + L_2$ and $L = L^u + L_1$.

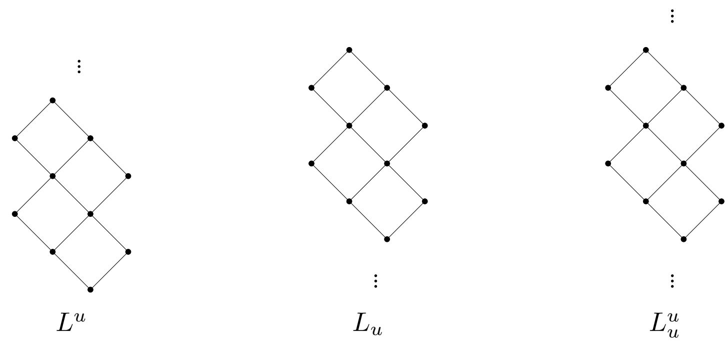

Figure 3.3: Infinite prime algebras

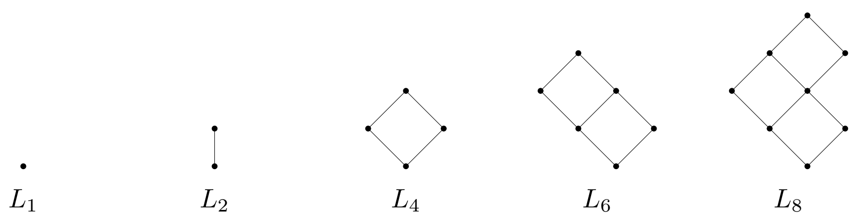

Figure 3.4: Some finite prime algebras

Definition 3.3.6 (Prime algebras)**.** Let L be a lattice. If for every pair of lattices K and M we have that $L = K + M$ implies $K = L$ or $K = M$, we call L *prime*. Moreover, we define the following set of prime algebras:

$$\mathcal{P} \coloneqq \{L^u, L_u, L_u^u\} \cup \{L_n \mid L_n \text{ is prime}, n \in \mathbb{N}^+\}.$$

Remark 3.3.7*.* The set $\mathcal{P}$ coincides with $\{L^u, L_u, L_u^u, L_1\} \cup \{L_{2n} \mid n \in \mathbb{N}^+\}$.

Remark 3.3.8*.* It is straightforward to see that any bounded sum of elements of $\mathcal{P}$ can be equipped with an implication and a co-implication.

We are ready to prove the main result of this section – a characterisation of $\mathbb{U}(\mathcal{G})$. As we will see in the next section, this result encapsulates most of the technicalities behind the characterisation of subvarieties of bi-KG. It states that:

$$\mathbb{U}(\mathcal{G}) = \{\Sigma_{i\in I} P_i \mid I \text{ is a linear order}, \{P_i\}_{i\in I} \in \mathcal{P}^I, \Sigma_{i\in I} P_i \text{ is bounded }\}.$$

We prove each of the two inclusions separately. We begin by introducing terminology that will be used in the proof of the first inclusion (Theorem 3.3.11).

Definition 3.3.9. Let A be a lattice and $a, b \in A$. We call the pair $\langle a, b \rangle$:

- *boundary*, if $a = b$ and $A = \uparrow\{a\} \cup \downarrow\{a\}$.
- *neighbouring*, if $\{a, b\}$ is an anti-chain, $A = \uparrow\{a, b\} \cup \downarrow\{a, b\}$ and all of the intervals $(a, a \vee b), (b, a \vee b), (a \wedge b, a), (a \wedge b, b)$ are empty (recall the notation (a, b) from Definition 2.1.7).
- *distant*, if it is a non-neighbouring anti-chain.

We call two neighbouring pairs $\langle a, b \rangle$ and $\langle c, d \rangle$ *adjacent*, if $\{a, b\} \cap \{c, d\} = \emptyset$ and for every $e \in A \setminus \{a, b, c, d\}$, we have:

$$e \in \uparrow\{a, b\} \cap \uparrow\{c, d\} \text{ or } e \in \downarrow\{a, b\} \cap \downarrow\{c, d\}.$$

Example 3.3.10. Consider the Rieger-Nishimura lattice (Figure 3.1). The only boundary pairs are $\langle \bot, \bot \rangle$ and $\langle \top, \top \rangle$. The neighbouring pairs are exactly the pairs of distinct points lying on the same Y-axis. The distant pairs are all pairs of incomparable points that are 1 unit apart on the Y-axis. Two neighbouring pairs are adjacent precisely when they are 1 unit apart on the Y-axis.

Theorem 3.3.11. *The following inclusion holds:*

$$\mathbb{U}(\mathcal{G}) \subseteq \{\Sigma_{i \in I} P_i \mid I \textit{ is a linear order}, \{P_i\}_{i \in I} \in \mathcal{P}^I, \Sigma_{i \in I} P_i \textit{ is bounded } \}.$$

Proof. The universal class $\mathbb{U}(\mathcal{G})$ is defined by the universal sentences that hold in $\mathcal{G}$. Our proof strategy is to find some concrete universal sentences that are specific enough to ensure that every algebra that validates them is a bounded sum of elements of $\mathcal{P}$. The choice of formulas is guided by the goal of describing the local structure of the members of $\mathcal{G}$. We do this through the introduction of local patterns that can be described by universal formulas.

Below we work with first-order formulas in the language of bi-Heyting algebras, so we need to be careful with notation. For the remainder of the proof, we use the logical symbols $\wedge$ (conjunction), $\vee$ (disjunction), $\Rightarrow$ (implication) and the algebraic symbols $\times$ (meet), $+$ (join), $\rightarrow$ (Heyting implication), $\leftarrow$ (co-Heyting implication).

Before we start with the local patterns, let us see how to define a two-element anti-chain. The property of being an anti-chain is expressed as follows.

$$\varphi_{ac}(x, y) \coloneqq x \nleq y \wedge y \nleq x$$

Our first local patterns are the *Base patterns*, which will be used as building blocks for other patterns.

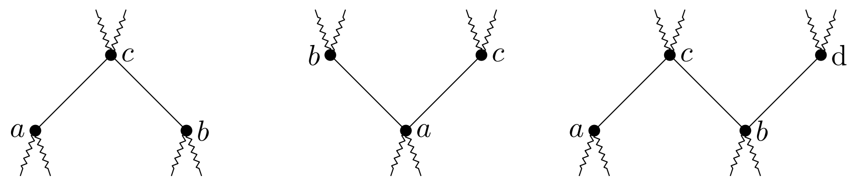

Figure 3.5: Base patterns

Intuitively, these patterns describe the existence of distinct a, b, c (and possibly d) with the displayed relations between them. Moreover, every other element in the algebra is inside one of the curly segments. A curly segment above a point means that there could be points greater than it; a curly segment below a point means that there could be points less than it.

The Base patterns are defined by the following formulas:

$$\begin{aligned}
\varphi_{bp1}(x_a, x_b, x_c) \coloneqq\ & \varphi_{ac}(x_a, x_b) \wedge x_a < x_c \wedge x_b < x_c \wedge \\
& \forall y(y \neq x_a \wedge y \neq x_b \wedge y \neq x_c \Rightarrow \\
& \qquad y > x_c \vee y < x_a \vee y < x_b) \\
\varphi_{bp2}(x_a, x_b, x_c) \coloneqq\ & \varphi_{ac}(x_b, x_c) \wedge x_a < x_b \wedge x_a < x_c \wedge \\
& \forall y(y \neq x_a \wedge y \neq x_b \wedge y \neq x_c \Rightarrow \\
& \qquad y > x_b \vee y > x_c \vee y < x_a) \\
\varphi_{bp3}(x_a, x_b, x_c, x_d) \coloneqq\ & \varphi_{ac}(x_a, x_b) \wedge \varphi_{ac}(x_a, x_d) \wedge \varphi_{ac}(x_c, x_d) \wedge \\
& x_a < x_c \wedge x_b < x_c \wedge x_b < x_d \wedge \\
& \forall y(y \neq x_a \wedge y \neq x_b \wedge y \neq x_c \wedge y \neq x_d \Rightarrow \\
& \qquad y < x_a \vee y < x_b \vee y > x_c \vee y > x_d).
\end{aligned}$$

With the Base patterns in place, we can now present the main patterns of interest – the *Neighbouring patterns*. Suppose we are given an anti-chain $\{a, b\}$ in a member A of $\mathcal{G}$. The Neighbouring patterns capture the possible cases of $\langle a, b\rangle$ being a neighbouring pair. Below we present the patterns pictorially, immediately followed by the formula that defines them. So the pair of incomparable points $\langle a, b\rangle$ in A satisfies Neighbouring pattern 1 if and only if $A \vDash \varphi_{np1}(a, b)$ (and same for the other Neighbouring patterns).

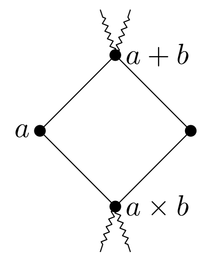

Figure 3.6: Neighbouring pattern 1

$$\varphi_{np1}(x_a, x_b) \coloneqq \varphi_{bp1}(x_a, x_b, x_a + x_b) \land \varphi_{bp2}(x_a \times x_b, x_a, x_b)$$

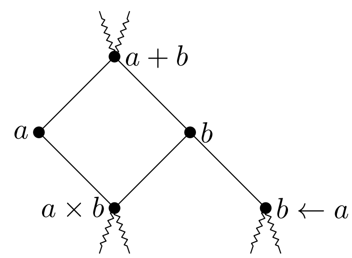

Figure 3.7: Neighbouring pattern 2

$$\varphi_{np2}(x_a, x_b) \coloneqq \varphi_{bp1}(x_a, x_b, x_a + x_b) \land \varphi_{bp3}(x_b \leftarrow x_a, x_a \times x_b, x_b, x_a)$$

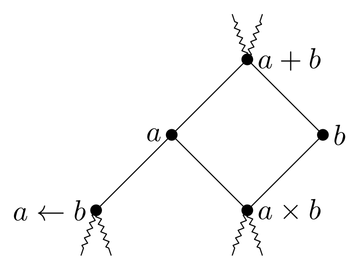

Figure 3.8: Neighbouring pattern 3

$$\varphi_{np3}(x_a, x_b) \coloneqq \varphi_{bp1}(x_a, x_b, x_a + x_b) \land \varphi_{bp3}(x_a \leftarrow x_b, x_a \times x_b, x_a, x_b)$$

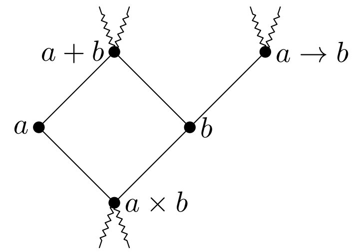

Figure 3.9: Neighbouring pattern 4

$$\varphi_{np4}(x_a, x_b) \coloneqq \varphi_{bp3}(x_a, x_b, x_a + x_b, x_a \to x_b) \wedge \varphi_{bp2}(x_a \times x_b, x_a, x_b)$$

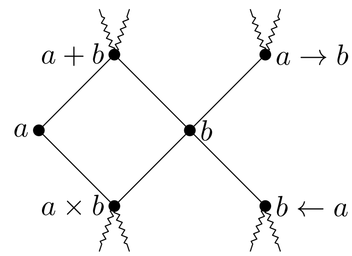

Figure 3.10: Neighbouring pattern 5

$$\varphi_{np5}(x_a, x_b) \coloneqq \varphi_{bp3}(x_a, x_b, x_a + x_b, x_a \to x_b) \wedge \varphi_{bp3}(x_b \leftarrow x_a, x_a \times x_b, x_b, x_a)$$

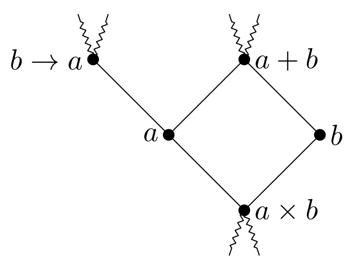

Figure 3.11: Neighbouring pattern 6

$$\varphi_{np6}(x_a, x_b) \coloneqq \varphi_{bp3}(x_b, x_a, x_a + x_b, x_b \to x_a) \wedge \varphi_{bp2}(x_a \times x_b, x_a, x_b)$$

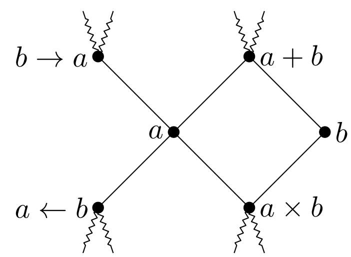

Figure 3.12: Neighbouring pattern 7

$$\varphi_{np7}(x_a, x_b) \coloneqq \varphi_{bp3}(x_b, x_a, x_a + x_b, x_b \to x_a) \wedge \\ \varphi_{bp3}(x_a \leftarrow x_b, x_a \times x_b, x_a, x_b)$$

Again, all depicted points are understood to be distinct and all other points in the algebra are inside the curly segments. In addition, some points have labels indicating the algebraic operations.

But in a member A of $\mathcal{G}$, an anti-chain $\{a, b\}$ need not satisfy any of these patterns. What we claim is that if none of the Neighbouring patterns are satisfied, the pair $\langle a, b\rangle$ is distant and it satisfies one of the following *Distant patterns*.

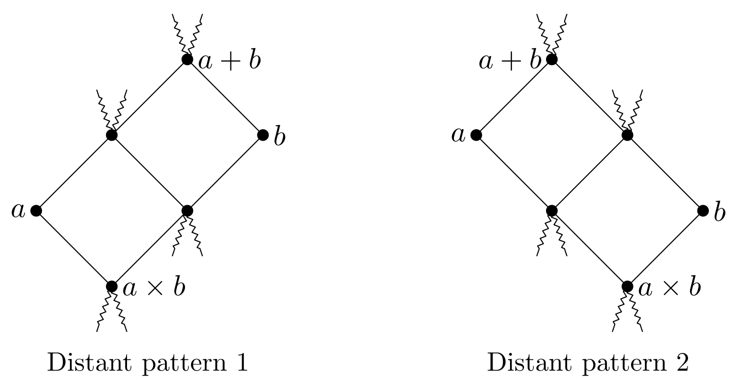

Figure 3.13: Distant patterns

While we cannot directly define these patterns with universal formulas, there is a list of *Distant pattern properties* that are restrictive enough for our purposes, as we will see later.

$$\varphi_{dpp1}(x_a, x_b) \coloneqq \forall y(y > x_a \wedge y \ngeq x_a + x_b \Rightarrow x_a < y \times (x_a + x_b) < x_a + x_b)$$
$$\varphi_{dpp2}(x_a, x_b) \coloneqq \forall y(y > x_b \wedge y \ngeq x_a + x_b \Rightarrow x_b < y \times (x_a + x_b) < x_a + x_b)$$
$$\varphi_{dpp3}(x_a, x_b) \coloneqq \forall y(y < x_a \wedge y \nleq x_a \times x_b \Rightarrow x_a \times x_b < y + (x_a \times x_b) < x_a)$$
$$\varphi_{dpp4}(x_a, x_b) \coloneqq \forall y(y < x_b \wedge y \nleq x_a \times x_b \Rightarrow x_a \times x_b < y + (x_a \times x_b) < x_b)$$
$$\varphi_{dpp5}(x_a, x_b) \coloneqq \forall y(x_a < y < x_a + x_b \Rightarrow \varphi_{bp3}(x_a, y \times x_b, y, x_b))$$
$$\varphi_{dpp6}(x_a, x_b) \coloneqq \forall y(x_b < y < x_a + x_b \Rightarrow \varphi_{bp3}(x_b, y \times x_a, y, x_a))$$

So if an anti-chain $\{a, b\}$ in a member A of $\mathcal{G}$ satisfies Distant pattern 1 or 2, then it satisfies all Distant pattern properties, i.e., $A \vDash \bigwedge_{1 \le i \le 6} \varphi_{dppi}(a, b)$.

By combining the Neighbouring pattern formulas and the Distant pattern property formulas, we obtain the *Anti-chain pattern formula*.

$$\varphi_{acp}(x, y) = \bigvee_{1 \le i \le 7} \varphi_{npi}(x, y) \vee \left(\bigwedge_{1 \le i \le 6} \varphi_{dppi}(x, y) \right)$$

We express the fact that any two-element anti-chain in a member of $\mathcal{G}$ satisfies the *Anti-chain pattern* as:

$$\varphi \coloneqq \forall x \forall y(\varphi_{ac}(x, y) \Rightarrow \varphi_{acp}(x, y)).$$

We are now ready to proceed with the main part of the proof. Let $A \in \mathbb{U}(\mathcal{G})$, it suffices to show that A is a sum of elements of $\mathcal{P}$, because A is assumed to be bounded. By a straightforward verification we can see that φ is equivalent to a universal formula and φ holds in $\mathcal{G}$. Therefore φ holds in A. We will show that we can partition A into neighbouring and boundary pairs.

We can directly see that if an anti-chain $\{a, b\}$ in A satisfies a Neighbouring pattern, then $\langle a, b \rangle$ is a neighbouring pair. But the converse also holds, as will follow from the lemma below.

Lemma. *Let $\{a, b\}$ be an anti-chain in A such that $\langle a, b \rangle$ does not satisfy any Neighbouring pattern. Then $(a, a+b) \cup (b, a+b) \neq \emptyset$ and $(a \times b, a) \cup (a \times b, b) \neq \emptyset$.*

Proof of Lemma. Observe that in members of $\mathcal{G}$, there does not exist three-element anti-chains and moreover, this property can be expressed with the universal formula:

$$\forall x \forall y \forall z(\varphi_{ac}(x, y) \wedge \varphi_{ac}(y, z) \wedge \varphi_{ac}(x, z) \Rightarrow x = y \vee y = z \vee x = z).$$

Hence there are no three-element anti-chains in A either. Since $\{a,b\}$ is an anti-chain, this implies $A = \uparrow\{a,b\} \cup \downarrow\{a,b\}$.

Be begin by proving that at least one of the intervals $(a, a+b)$, $(b, a+b)$, $(a \times b, a)$ and $(a \times b, b)$ is non-empty. For suppose the contrary, with a view to contradiction. Since by assumption $\langle a,b \rangle$ does not satisfy any Neighbouring pattern, it satisfies all Distant pattern properties. By $\varphi_{dpp1}(a,b)$ and $\varphi_{dpp2}(a,b)$ we have $\uparrow\{a,b\} = \{a,b\} \cup \uparrow\{a+b\}$. Similarly, by $\varphi_{dpp3}(a,b)$ and $\varphi_{dpp4}(a,b)$ we have $\downarrow\{a,b\} = \{a,b\} \cup \downarrow\{a \times b\}$. But now these two properties, together with $A = \uparrow\{a,b\} \cup \downarrow\{a,b\}$, imply that $\langle a,b \rangle$ satisfies Neighbouring Pattern 1, which is a contradiction. Therefore at least one of the intervals is non-empty.

Consider the case where $(a, a+b)$ is non-empty, i.e there exists $x \in A$ with $(a < x < a+b)$, then $a \times b < x \times b < b$ (by basic properties of lattices). Therefore $(a, a+b) \cup (b, a+b) \neq \emptyset$ and $(a \times b, a) \cup (a \times b, b) \neq \emptyset$. The other cases are analogous. This concludes the proof of the Lemma.

Consequently, an anti-chain $\{a,b\}$ in A satisfies a Neighbouring pattern if and only if $\langle a,b \rangle$ is neighbouring.

Observation 1. Let $\{a,b\}$ be an anti-chain in A such that $\langle a,b \rangle$ is distant, i.e., not neighbouring. We show that a and b belong to two adjacent neighbouring pairs. By the above Lemma, we know $(a, a+b) \cup (b, a+b) \neq \emptyset$. Consider the case where $(a, a+b) \neq \emptyset$ (the other case $(b, a+b) \neq \emptyset$ is similar). Then there exists $x \in (a, a+b)$ with $x \times b \in (a \times b, b)$. By $\varphi_{dpp5}(a,b)$ we have that $\langle a, x \times b, x, a \rangle$ satisfy Base Pattern 3. From the definition of Base Pattern 3 it follows that the intervals $(x \times b, x)$ and $(x \times b, b)$ are empty. Therefore by the above Lemma, the pair $\langle x, b \rangle$ is neighbouring. Similarly, the pair $\langle a, x \times b \rangle$ is neighbouring. Now we see that $\langle x, b \rangle$ and $\langle a, x \times b \rangle$ are adjacent.

Observation 2. Notice that if $\langle a,b \rangle$ is an arbitrary neighbouring pair in A, then it has adjacent neighbouring pairs $\langle a', b' \rangle$ and $\langle a'', b'' \rangle$ above and below respectively (meaning $a', b' \in \uparrow\{a,b\}$ and $a'', b'' \in \downarrow\{a,b\}$). Take, for example, Neighbouring pattern 5. There $\{a+b, a \to b\}$ is an anti-chain and the intervals $(b, a+b)$ and $(b, a \to b)$ are empty, hence by the Lemma $\langle a+b, a \to b \rangle$ is neighbouring. This gives us the desired adjacent pair above $\langle a,b \rangle$. Similarly, $\langle a \times b, b \leftarrow a \rangle$ is neighbouring, so we obtain an adjacent pair below $\langle a,b \rangle$.

We are ready to see how A can be written as a sum of element of $\mathcal{P}$. Let $\{a,b\}$ be an anti-chain in A. Either $\langle a,b \rangle$ is neighbouring or by Observation 1, a and b are part of two adjacent neighbouring pairs. Traverse A by repeatedly applying Observation 2 upwards and downwards, at most ω times in each direction, until reaching a boundary pair. If we reach boundary pairs above and below, this means we have traversed a finite element of $\mathcal{P}$ inside A. If we only reach a boundary pair below, this means we have traversed L^u. If we only reach a boundary pair above, we have traversed L_u. If neither direction reaches a boundary pair, we have traversed L_u^u.

By inductively running this procedure for all two-element anti-chains (skipping the ones already traversed), we obtain a set of linearly ordered elements of $\mathcal{P}$. The rest of the points in A are boundaries and can be written as sums of L_1 and L_2, again elements of $\mathcal{P}$. Taking the sum of all these parts gives the whole A. ■

In preparation of the next theorem, we point out the following.

Remark 3.3.12. Notice that when writing sums of elements of $\mathcal{P}$, technically, we can obtain the same algebra in different ways. For instance $L_6 = L_1 + L_6 = L_1 + L_6 + L_1 = \cdots$. However, when writing $\Sigma_{i \in I} P_i$, we will always tacitly assume that this sum does not contain any redundant L_1 summands, i.e., L_1 summands that can be contracted. With this in place, every algebra has a canonical decomposition into prime summands. We will often refer to these prime summands.

Theorem 3.3.13. *The following inclusion holds:*

$$\{\Sigma_{i \in I} P_i \mid I \text{ is a linear order}, \{P_i\}_{i \in I} \in \mathcal{P}^I, \Sigma_{i \in I} P_i \text{ is bounded }\} \subseteq \mathbb{U}(\mathcal{G}).$$

Proof. We prove this using Theorem 2.2.26. Let A be a bounded sum of elements of $\mathcal{P}$ and let X be an arbitrary local subgraph of A. Our strategy is to find a local subgraph Y of A that extends X and embeds into a member of $\mathcal{G}$. We construct Y in the following steps, where every step adds finitely many new points to Y.

1. Take $Y \coloneqq X$.

2. Close Y under meets and joins. Since distributive lattices are locally finite, this adds finitely many points.

3. For all $x \in Y$, if x is contained in a finite prime summand P and x is neither the least, nor the greatest element in P, add the whole P to Y.

4. For all $x \in Y$, if x is contained in an L^u summand P, add ${\downarrow}\{x\} \cap P$ to Y. In this way, the part of P that lies in Y will be an initial segment of P.

5. For all $x \in Y$, if x is contained in an L_u summand P, add $\uparrow\{x\} \cap P$ to Y. In this way, the part of P that lies in Y will be a final segment of P.
6. For all $x, y \in Y$, if x and y are contained in the same L_u^u summand P, add (x, y) to Y. In this way, since Y was closed under meets and joins in step (2), the part of P that lies in Y will be a closed interval.

These properties guarantee that Y can be constructed by stacking finitely many finite elements of $\mathcal{P}$, say $P_1, \ldots, P_n$. Now take the algebra:

$$\Sigma_{1 \leq i < n}(P_i + S_i) + P_n,$$

where for every $1 \leq i < n$, $S_i = L_1$ if $P_i \cap P_{i+1} \neq \emptyset$ and $S_i = L_2$ otherwise. One can directly check that Y embeds into a member of $FinSum(\mathcal{G})$. ■

Corollary 3.3.14. The following identity holds:

$$\mathbb{U}(\mathcal{G}) = \{\Sigma_{i \in I} P_i \mid I \text{ is a linear order}, \{P_i\}_{i \in I} \in \mathcal{P}^I, \Sigma_{i \in I} P_i \text{ is bounded }\}.$$

3.4 Characterising subvarieties of bi-KG

We have now developed sufficient machinery to prove a promised characterisation of subvarieties of bi-KG. As mentioned in the previous section, this is achieved by understanding the subdirectly irreducible members of bi-KG and in fact, we need only consider the finitely generated ones. And since we will show that bi-KG is semi-simple (see Definition 2.2.35), these are precisely the finitely generated simple members of bi-KG. Furthermore, we will describe a convenient way to compare varieties generated by a given class of finitely generated simple algebras.

We start off with the description of simple bi-KG algebras.

Lemma 3.4.1. *All members of* $\mathbb{U}(\mathcal{G})$ *except the ones isomorphic to* L_1, L_4 *and* L_6, *are simple.*

Proof. We make use of Corollary 2.4.13, i.e., we work with the dual spaces of members of $\mathbb{U}(\mathcal{G})$.

Let $A \in \mathbb{U}(\mathcal{G})$. Firstly, consider the case where A is prime. By Corollary 3.3.14 and boundedness of A, we know A is a finite member of $\mathcal{P}$. In Figure 3.14 we see the dual spaces of the first smallest finite members of $\mathcal{P}$ (except $\mathcal{X}(L_1)$, which is empty). It is straightforward to conclude that the only spaces that are either empty or contain a non-trivial upwards- and downwards-closed subset are the duals of L_1, L_4 and L_6.

Secondly, consider the case where A is not prime. There are two options.

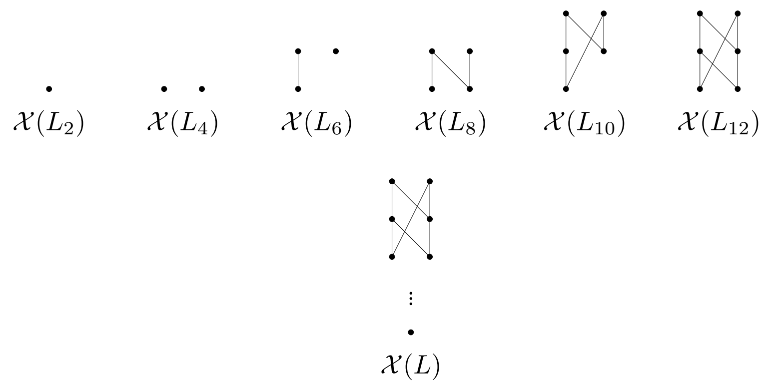

Figure 3.14: Dual spaces of some prime algebras and L

- In the decomposition of A into prime summands, all but one summand is L_1. This means that the only non-L_1 summand is L^u, L_u or L^u_u, hence A is $L^u + L_1 = L$ or $L_1 + L_u = L^\partial$, or $L_1 + L^u_u + L_1$. By examining the Rieger-Nishimura ladder $\mathcal{X}(L)$, which is the dual of the Rieger-Nishimura lattice (see Figure 3.14), we see that it has no non-trivial upwards- and downwards-closed subsets. The other cases L^∂ and $L_1 + L^u_u + L_1$ admit a similar argument.

- In the decomposition of A into prime summands, there exist two non-L_1 prime summands B and C. Let F_B and F_C be prime filters in A and B respectively. By spelling out the definition of a prime filter and a sum of algebras, we see that $\uparrow F_B$ and $\uparrow F_C$ are prime filters in A and every prime filter in A is comparable with F_B or F_C.

  Now suppose $Y \subseteq \mathcal{X}_A$ is upwards- and downwards-closed. Suppose Y is non-empty, i.e., there exists $x \in Y$ and let G be the prime filter in A that corresponds to x. Now notice that G is comparable to F_B or F_C, i.e., Y contains the point corresponding to F_B or the point corresponding to F_C. Moreover, F_B and F_C are comparable, so Y contains both of these points. Finally, every prime filter is comparable to F_B or F_C, thus Y contains every point in X, i.e., Y = X.

■

Theorem 3.4.2. *bi-KG is semi-simple and* bi-KG$_S = \mathbb{U}(\mathcal{G}) \setminus \mathbb{I}(\{L_1, L_4, L_6\})$.

Proof. We prove $\mathbb{V}(\mathcal{G})_{SI} \subseteq \mathbb{U}(\mathcal{G}) \setminus \mathbb{I}(\{L_1, L_4, L_6\}) \subseteq \mathbb{V}(\mathcal{G})_S$.

- First inclusion. By Jónsson's Lemma (Theorem 2.3.21), $\mathbb{V}(\mathcal{G})_{SI} \subseteq \mathbb{HSP}_U(\mathcal{G})$. We claim that the class $\mathbb{SP}_U(\mathcal{G}) = \mathbb{U}(\mathcal{G})$ is closed under homomorphic images. Indeed, by Lemma 3.4.1, every $A \in \mathbb{U}(\mathcal{G}) \setminus \mathbb{I}(\{L_1, L_4, L_6\})$ is simple and has no non-trivial homomorphic images, while $\mathbb{H}(\{L_1, L_4, L_6\}) = \mathbb{I}(\{L_1, L_2, L_4, L_6\}) \subseteq \mathbb{U}(\mathcal{G})$. Therefore we have $\mathbb{V}(\mathcal{G})_{SI} \subseteq \mathbb{U}(\mathcal{G})$. Finally, we can directly check that L_1, L_4 and L_6 are not subdirectly irreducible, so $\mathbb{V}(\mathcal{G})_{SI} \subseteq \mathbb{U}(\mathcal{G}) \setminus \mathbb{I}(\{L_1, L_4, L_6\})$.

- Second inclusion. Again by Lemma 3.4.1, $\mathbb{U}(\mathcal{G}) \setminus \mathbb{I}(\{L_1, L_4, L_6\})$ is a class of simple algebras. Now:

$$\mathbb{U}(\mathcal{G}) \setminus \mathbb{I}(\{L_1, L_4, L_6\}) = \big(\mathbb{U}(\mathcal{G}) \setminus \mathbb{I}(\{L_1, L_4, L_6\})\big)_S \subseteq \mathbb{V}(\mathcal{G})_S.$$

Finally, since simple algebras are always subdirectly irreducible, we get $\mathbb{V}(\mathcal{G})_S \subseteq \mathbb{V}(\mathcal{G})_{SI}$ and thus $\mathbb{V}(\mathcal{G})_{SI} = \mathbb{U}(\mathcal{G}) \setminus \mathbb{I}(\{L_1, L_4, L_6\}) = \mathbb{V}(\mathcal{G})_S$. This proves both properties stated in the theorem. ■

As a consequence, every subvariety of bi-KG is generated by its simple members. In addition, this property holds even if we consider only finitely generated simple algebras. This motivates us to give the following description of finitely generated members of $\mathbb{U}(\mathcal{G})$.

Proposition 3.4.3. The finitely generated members of $\mathbb{U}(\mathcal{G})$ are the finite bounded sums of members of $\mathcal{P}$.

Proof. Firstly, we show that if $A \in \mathbb{U}(\mathcal{G})$ is finitely generated, then it is a finite bounded sum of members of $\mathcal{P}$. By Corollary 3.3.14, A is an arbitrarily large bounded sum of members of $\mathcal{P}$. Assume towards a contradiction that A contains infinitely many prime summands. Let $G \subseteq A$ be a finite set of generators of A. Notice that if $a \in A$ lies inside a prime summand disjoint from G, then a cannot belong to the subset generated by G. But since every element of G can belong to at most two prime summands and there are infinitely many prime summands, there exists an element not generated by G. This contradicts our assumption.

Secondly, by Corollary 3.3.14, finite bounded sums of members of $\mathcal{P}$ are in $\mathbb{U}(\mathcal{G})$, so it suffices to show they are finitely generated. It is straightforward to see that every member of $\mathcal{P}$ is finitely generated and that finite sums of finitely generated algebras are finitely generated. This gives us the desired result. ■

Remark 3.4.4. Theorem 3.4.2 and Proposition 3.4.3 together give us that the finitely generated simple bi-KG algebras coincide with the finite bounded sums of elements of $\mathcal{P}$, with the exception of L_1, L_4 and L_6.

In our setting, finitely generated simple algebras are important enough to have a special notation.

Definition 3.4.5 (Finitely generated simple members)**.** Let $V \subseteq \mathsf{bi\text{-}KG}$ be a variety. We write $FGS(V)$ to refer to the class of finitely generated simple members of V. Alternatively, by Remark 3.4.4, $FGS(V)$ is the class of those finite bounded sums of elements of $\mathcal{P}$ that lie in V, except L_1, L_4 and L_6.

The following two results summarise the work of this section and together are useful for comparing subvarieties of bi-KG.

Theorem 3.4.6. *Let $V_1, V_2 \subseteq$ bi-KG be varieties. Then $V_1 = V_2$ if and only if $FGS(V_1) = FGS(V_2)$.*

Proof. This follows from Proposition 2.2.40 and the semi-simplicity of bi-KG (Theorem 3.4.2). ■

Theorem 3.4.7. *Let V be a variety given by $V = \mathbb{V}(\mathcal{K})$, where $\mathcal{K} \subseteq FGS(\textit{bi-KG})$ and let $A \in FGS(\textit{bi-KG})$. Then $A \in V$ if and only if for every local subgraph X of A, there exists $B \in \mathcal{K}$ such that X embeds into B.*

Proof. ($\Longrightarrow$) Suppose $A \in V$. By Jónsson's Lemma (Theorem 2.3.21) and simplicity of A, we get $A \in \mathbb{HSP}_U(\mathcal{K})$. Since all members of $\mathcal{K}$ are simple, by Theorem 3.4.2 we know $\mathcal{K} \subseteq \mathbb{U}(\mathcal{G})$, hence $\mathbb{U}(\mathcal{K}) \subseteq \mathbb{U}(\mathcal{G})$. So the only non-simple members of $\mathbb{U}(\mathcal{K})$ are potentially L_1, L_4 and L_6 and for all $B \in \{L_1, L_4, L_6\}$, we have $\mathbb{H}(B) = \mathbb{S}(B)$. Therefore $\mathbb{U}(\mathcal{K})$ is closed under $\mathbb{H}$, which allows us to deduce $A \in \mathbb{HSP}_U(\mathcal{K}) = \mathbb{HU}(\mathcal{K}) = \mathbb{U}(\mathcal{K})$. Now the desired conclusion follows from Theorem 2.2.26.

($\Longleftarrow$) If every local subgraph of A embeds into a member of $\mathcal{K}$, then by Theorem 2.2.26, $A \in \mathbb{U}(\mathcal{K}) \subseteq \mathbb{V}(\mathcal{K}) = V$. ■

With these properties at hand, we have a very strong grasp of subvarieties of bi-KG. Theorem 3.4.6 reduces comparing subvarieties to determining the membership of finitely generated simple algebras, Theorem 3.4.7 prescribes how to determine this membership and Remark 3.4.4 describes the structure of these algebras in terms of finite bounded sums of prime components.

3.5 Local and weak embeddability for simple bi-KG algebras

In view of Theorem 3.4.7, it is useful to understand when all local subgraphs of a given finitely generated simple bi-KG algebra embed into other finitely generated simple bi-KG algebras. The present section discusses some terminology and properties that will simplify working with local subgraphs and their embeddability properties. Our aim is to gather the more technical aspects here, so that later on we can use them freely and keep the exposition lighter and more conceptual.

Definition 3.5.1 (Local embeddability)**.**

1. Let A and B be algebras. If every local subgraph of A embeds into B, we write $A \overset{loc}{\hookrightarrow} B$ and say that A *locally embeds* into B.

2. Let $\{A\} \cup \mathcal{K}$ be a class of algebras. If every local subgraph of A embeds into a member of $\mathcal{K}$, we write $A \overset{loc}{\hookrightarrow} K$ and say that A *locally embeds* into K.

Suppose $A, B \in FGS(\mathsf{bi\text{-}KG})$ and A has a prime summand P. Sometimes it will be useful to determine whether every local subgraph of P embeds into B, because if the answer is negative, we can immediately deduce $A \overset{loc}{\not\hookrightarrow} B$. But in general, we cannot assume that P contains the least and greatest element of A. For this reason, we introduce the following notion.

Definition 3.5.2 (Weak embeddability)**.**

1. Let A, B be bi-Heyting algebras. If every local subgraph of A whose language does not contain the constants $\bot$ and $\top$ embeds into B, we write $A \overset{w}{\hookrightarrow} B$ and say that A *weakly embeds* into B.

2. Let $\{A\} \cup \mathcal{K}$ be a class of bi-Heyting algebras. If every local subgraph of A whose language does not contain the constants $\bot$ and $\top$ embeds into a member of $\mathcal{K}$, we write $A \overset{w}{\hookrightarrow} \mathcal{K}$ and say that A *weakly embeds* into $\mathcal{K}$.

Remark 3.5.3. Note the difference between not containing the constants $\bot$ and $\top$ and not containing their interpretations, i.e., the least and greatest element. The above definition does allow including the least and greatest element in the local subgraph, but does not require that they are sent to minima and maxima, respectively.

In the proof of Theorem 3.3.13, we constructed a local subgraph Y with some "good" properties that turned out to be convenient. Moreover, we saw that any local subgraph of a member of $FinSum(\mathcal{P})$ can be extended to a graph that satisfies these "good" properties. So without loss of generality, we may disregard completely local subgraphs which are not "good". Below we make this precise.

Definition 3.5.4 ((Weakly) normal local subgraph)**.** Let $A \in FinSum(\mathcal{P})$ and let X be a local subgraph of A. We call X *normal* if it satisfies the following properties:

- X contains the constants $\bot$ and $\top$ (assuming A has $\bot$ and $\top$ in its signature in the first place).
- X is closed under meets and joins.
- If $x \in X$, x is contained in a finite prime summand P and x is neither the least, nor the greatest element in P, then the whole P is in X.
- If $x \in X$ and x is contained in an L^u summand P, then $\downarrow\{x\} \cap P$ is in X.
- If $x \in X$ and x is contained in an L_u summand P, then $\uparrow\{x\} \cap P$ is in X.
- If $x, y \in X$ are contained in the same L_u^u summand, then (x, y) is in X.

If X satisfies all of the above, except its language does not comprise the constant symbols $\bot$ and $\top$, we call X *weakly normal*.

Proposition 3.5.5. Let $\{A\} \cup \mathcal{K}$ be a class of algebras. Then $A \overset{loc}{\hookrightarrow} \mathcal{K}$ if and only if every normal local subgraph of A embeds into a member of $\mathcal{K}$.

Proof. Follows directly from the idea of extending X to Y in the proof of Theorem 3.3.13. ■

Proposition 3.5.6. Let $\{A\} \cup \mathcal{K}$ be a class of algebras. Then $A \overset{w}{\hookrightarrow} \mathcal{K}$ if and only if every weakly normal local subgraph of A embeds into a member of $\mathcal{K}$.

Proof. An adaptation of the above argument. ■

In what follows, we collect several facts about local and weak embeddability in $FGS(\mathsf{bi\text{-}KG})$. They will be used in the following chapter, when we want to prove that specific bi-KG algebras do or do not embed into other bi-KG algebras.

Proposition 3.5.7. Any finite sum of L_2 and L_4 weakly embeds into L^u, L_u and L_u^u.

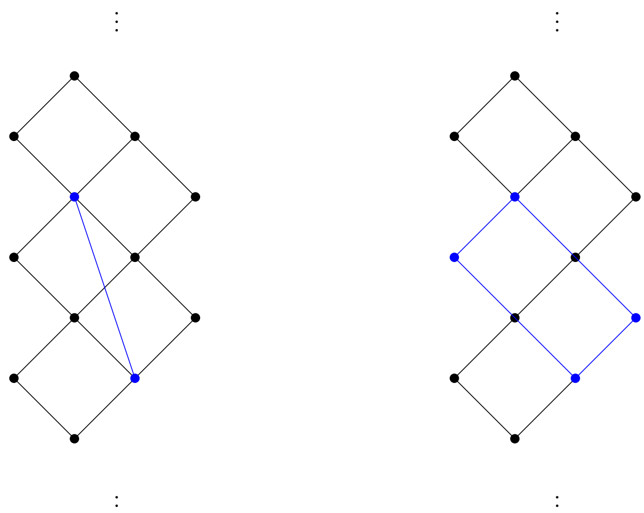

Figure 3.15: Weakly embedding L_2 and L_4 into L^u, L_u or L_u^u

Proof. We see in Figure 3.15 that L_2 and L_4 embed weakly into a middle section of L^u, L_u and L_u^u. And since all three of them contain infinitely many consecutive such sections, we deduce that any finite sum of L_2 and L_4 weakly embeds into L^u, L_u and L_u^u. ∎

Recall Definition 3.3.9, where we introduced the terms *boundary*, *neighbouring* and *distant* pairs of lattice elements. We also introduced the term *adjacent* pairs of lattice elements. These will be used again in the present section. We will also use the following slightly modified terminology.

Definition 3.5.8. Let A be a lattice and $a, b \in A$.

- We call a a *boundary point* if $\langle a, a\rangle$ is a boundary pair.
- We call a and b *neighbouring points* if the pair $\langle a, b\rangle$ is neighbouring.
- We call a and b *distant points* if $\langle a, b\rangle$ is a distant pair.

The following Proposition has a relatively long proof, but most of it is dealing with technicalities.

Proposition 3.5.9. Let $P \in \mathcal{P} \setminus \{L_1, L_2\}$ and suppose a weakly normal local subgraph X of P with $|X| > 2$ embeds into some $A \in FGS(\mathsf{bi\text{-}KG})$ via f. Then $f[X]$ lies inside a single prime summand of A.

Proof. Assume towards a contradiction that there exist $a, b \in f[X]$ that do not lie inside a single prime summand of A. Then a and b must be comparable, so without loss of generality, assume $a \leq b$. Our contradiction will be finding three distinct points in X, such that each of them is comparable with every other point in X. Consider the following cases.

- Neither a, nor b is boundary. Let Q, resp. R, be the prime summand of A that contains a, resp b. Take $c := sup(Q \cap f[X])$ and $d := inf(R \cap f[X])$. These supremum and infimum exist, because $f[X]$ is finite. By the basic properties of embeddings and the fact that X is a lattice (by normality), we have $c \in Q \cap f[X]$ and $d \in R \cap f[X]$. Notice that c and d are comparable with every other point in $f[P]$, thus $f^{-1}(c)$ and $f^{-1}(d)$ are comparable with every other point in X. But since, by normality, X is isomorhpic as a lattice to a member of $\mathcal{P}$, this is only possible if $f^{-1}(c) = minX$ and $f^{-1}(d) = maxX$. Therefore $a = c$ and $b = d$.

  Since $|X| > 2$, there exists some point $p \in (minX, maxX)$. Let S be a prime summand of A containing $f(p)$. Let $e := sup(S \cap f[X]) \in S \cap f[X]$. Again, e is comparable to every point in $f[X]$. Now $S \neq Q$, $S \neq R$ and a, b not boundary imply $a < e < b$. But this means $f^{-1}(a) < f^{-1}(e) < f^{-1}(b)$ are all comparable with every other point in X, which is a contradiction.

- a is boundary, b is not. Take $c := a$ and take d as in the previous case. By the same argument as above, $b = d$. Again, there exists $f^{-1}(a) < p < f^{-1}(b)$. If $f(p)$ is boundary, take $e := f(p)$, otherwise take S to be the prime summand of A containing $f(p)$ and take $e := sup(S \cap f[X])$. Again, $a < e < b$ and $f^{-1}(a) < f^{-1}(e) < f^{-1}(b)$, which is a contradiction.

- a is not boundary, b is. This case is symmetric to the previous one.

- Both a and b are boundary. Here a and b directly turn out to be comparable to all other points in $f[X]$. Take $p \in X$ in the same way as before. Take S to be a prime summand of A containing $f(p)$. Since a and b are not in the same prime summand, we know that the least element of S is not a or the greatest element of S is not b. These cases are symmetric, so assume a is not the least element of S. Now take $e := inf(S \cap f[P]) \in S \cap f[X]$. Again, we have $a < e < b$ and $f^{-1}(a) < f^{-1}(e) < f^{-1}(b)$, which is a contradiction.

■

Definition 3.5.10 ((Weakly) total local subgraph)**.** Let A be a finite algebra. We call the local subgraph $X = A$ the *total* local subgraph of A. We call the local subgraph obtained by removing the constants $\bot$ and $\top$ from the language of the total local subgraph of A the *weakly total* local subgraph of A.

Proposition 3.5.11. Let $P = L_n$ for $n > 4$ and let X be the weakly total local subgraph of P. Suppose X embeds into $A \in FGS(\textsf{bi-KG})$ via f. Then $f[X]$ is a prime summand of A. Consequently, A contains a prime summand L_n.

Proof. By Proposition 3.5.9, let Q be the prime summand of A containing $f[X]$.

Observation 1. Any two neighbouring points in X map to neighbouring points in A.

Proof of Observation 1. Assume towards a contradiction that there exist two neighbouring points $a, b \in X$ that do not map to neighbouring points in A. Since a and b are not comparable, we know $f(a)$ and $f(b)$ are also not comparable. The only way to have two distinct non-neighbouring incomparable points in a member of $FGS(\textsf{bi-KG})$ is for them to be distant. Then notice that $f(a) \to f(b) = f(b)$, $f(b) \to f(a) = f(a)$, $f(a) \leftarrow f(b) = f(a)$ and $f(b) \leftarrow f(a) = f(b)$. By properties of embeddings, $a \to b = b$, $b \to a = a$, $a \leftarrow b = a$, $b \leftarrow a = b$. But given that a and b are neighbouring and $P \in \mathcal{P}$, this can only happen if $P = L_4$, which is a contradiction with $n > 4$.

Observation 2. Adjacent pairs of neighbouring points map to adjacent pairs of neighbouring points.

Proof of Observation 2. Suppose $\langle a, b\rangle$ and $\langle c, d\rangle$ are adjacent neighbouring pairs. By Observation 1, they map to neighbouring pairs. By adjacency, we have $\{a \wedge b, a \vee b\} \cap \{c, d\} \neq \emptyset$, so the same holds for their images, i.e., $\{f(a) \wedge f(b), f(a) \vee f(b)\} \cap \{f(c), f(d)\} \neq \emptyset$. This is only possible if $\langle f(a), f(b)\rangle$ and $\langle f(c), f(d)\rangle$ are adjacent.

We prove the main statement using Observation 2. By injectivity of f, we have $|Q| \geq n$. Now assume towards a contradiction that $|Q| > n$. By Observation 2, $f[X]$ is an interval in A, because there are no "gaps" between adjacent pairs. Since $|Q| > n$, this means $f(minX) \neq minQ$ or $f(maxX) \neq maxQ$. These two cases are symmetric, so assume $f(minX) \neq minQ$. Let $\langle a, b\rangle$ be the lowest neighbouring pair in X and notice that $a \leftarrow b = a$ and $b \leftarrow a = b$. But since $f(a) \wedge f(b) \neq minQ$, we know that $\langle f(a), f(b)\rangle$ is not the lowest neighbouring pair in Q, thus $f(a) \leftarrow f(b) \neq f(a)$ or $f(b) \leftarrow f(a) \neq f(b)$. This is a contradiction, which confirms $|Q| = n$ and therefore $Q = L_n$. ■

Proposition 3.5.12. Let $P = L^u$ and $\mathcal{K} \subseteq FGS(\mathsf{bi\text{-}KG})$. Let $\pi(\mathcal{K})$ be the following property: there exists $A \in \mathcal{K}$ that contains a prime summand L^u or for every $n \in \mathbb{N}$ there exists $m > n$ and $A \in \mathcal{K}$ such that A contains a prime summand L_m. Then $P \overset{w}{\hookrightarrow} \mathcal{K}$ if and only if $\pi(\mathcal{K})$.

Proof. ($\Longrightarrow$). We prove this direction by contraposition. Assume $\pi(\mathcal{K})$ does not hold and let $n \in \mathbb{N}$ be such that $n > 2$ and n is a strict upper bound on the size of finite prime summands occurring in $\mathcal{K}$. Let X be a weakly normal subgraph of P of size at least n. Note that, by normality, X is a finite initial segment of L^u. We will derive a contradiction from the assumption that X embeds into some $A \in \mathcal{K}$ via some f. By Proposition 3.5.9, $f[X]$ lies inside a single prime summand of A. Now the choice of n and injectivity of embeddings imply that Q is infinite and moreover, $Q = L_u$ or $Q = L_u^u$ ($Q = L^u$ is not possible by the assumed failure of $\pi(\mathcal{K})$). Let $\langle a, b\rangle$ be the lowest neighbouring pair in X. Using Observation 1 from Proposition 3.5.11, we know that $f(a)$ and $f(b)$ are neighbouring. Now notice that $a \leftarrow b = a$ and $b \leftarrow a = b$, but since Q has no least element, we have $f(a) \leftarrow f(b) \neq f(a)$ or $f(b) \leftarrow f(a) \neq f(b)$. Hence we have a contradiction.

($\Longleftarrow$) Assume $\pi(\mathcal{K})$ holds. We choose an arbitrary normal subgraph X of P and embed it into a member of $\mathcal{K}$. If some $A \in \mathcal{K}$ contains an L^u summand, we can directly embed X into A. Otherwise, choose $A \in \mathcal{K}$ such that A contains an L_n summand for $n \geq |X|$. Once again, we can see that X embeds into A. ■

An interesting observation is that in the last two propositions, it was essential for our proof that we are working in the signature with a co-implication. Indeed, in KG weak embeddability is notably less rigid (see [4, Chapter 4]).

Proposition 3.5.13. Let $P = L_u$ and $\mathcal{K} \subseteq FGS(\mathsf{bi\text{-}KG})$. Let $\rho(\mathcal{K})$ be the following property: there exists $A \in \mathcal{K}$ that contains a prime summand L_u or for every $n \in \mathbb{N}$ there exists $m > n$ and $A \in \mathcal{K}$ such that A contains a prime summand L_m. Then $P \overset{w}{\hookrightarrow} \mathcal{K}$ if and only if $\rho(\mathcal{K})$.

Proof. Notice how this proposition is symmetric to Proposition 3.5.12. And since the signature of bi-Heyting algebras allows for symmetric arguments, a symmetric version of the above proof establishes the current result. ■

Proposition 3.5.14. Let $P = L_u^u$ and $\mathcal{K} \subseteq FGS(\mathsf{bi\text{-}KG})$. Let $\sigma(\mathcal{K})$ be the following property: there exists $A \in \mathcal{K}$ that contains a prime summand $Q \in \{L^u, L_u, L_u^u\}$ or for every $n \in \mathbb{N}$ there exists $m > n$ and $A \in \mathcal{K}$ such that A contains a prime summand L_m. Then $P \overset{w}{\hookrightarrow} \mathcal{K}$ if and only if $\sigma(\mathcal{K})$.

Proof. The only difference between the current proposition and the previous two propositions is that, in contrast with L^u and L_u, we have that L_u^u weakly embeds into L^u, L_u and L_u^u. This is due to the fact that any weakly normal subgraph of L_u^u is a closed interval and it is straightforward to verify that it embeds into L^u, L_u and L_u^u. ■

Chapter 4

FMP and degrees of FMP in bi-KG

After the introduction of bi-KG and some general properties about its subvarieties, we proceed with questions related to its finite members. In particular, in the present chapter we study the finite model property in subvarieties of bi-KG and degrees of FMP relative to bi-KG.

Both of these problems have been studied for KG, see [4, Chapter 4] and [3] respectively. Degrees of FMP in KG are especially interesting, because they are an essential step towards finding degrees of FMP relative to HA (see [3]). In analogy with the case of Heyting algebras, our hope is that understanding degrees of FMP relative to bi-KG will aid the efforts of characterising degrees of FMP relative to the whole bi-HA.

4.1 Characterising FMP in bi-KG

In this section we present several results about the FMP of subvarieties of bi-KG. We start with a general characterisation of subvarieties of bi-KG with the FMP and later on, we draw as corollaries facts about the FMP in particular subvarieties.

Before we proceed with the main results, we make a convenient observation about varieties with the same finite members, which will be useful throughout the whole chapter. Recall that by Theorem 3.4.6, two subvarieties of bi-KG coincide if and only if they have the same finitely generated simple members. Similarly, what we will see now is that when comparing the classes of finite algebras of two subvarieties of bi-KG, it suffices to compare only their finite simple algebras.

Definition 4.1.1 (Finite simple members)**.** Let $V \subseteq \mathsf{bi\text{-}HA}$ be a variety. By $FinS(V)$ we denote the class of finite simple members of V.

Proposition 4.1.2. Let $V_1, V_2 \subseteq \mathsf{bi\text{-}HA}$ be varieties. Then $Fin(V_1) = Fin(V_2)$ if and only if $FinS(V_1) = FinS(V_2)$.

Proof. The left-to-right implication follows directly, so we focus on the converse. Suppose $FinS(V_1) = FinS(V_2)$ and let $A \in Fin(V_1)$. We show that $A \in V_2$. By Remark 2.4.8, the space $\mathcal{X}(A)$ is finite. Let $X_1, \ldots, X_n$ be the subspaces of $\mathcal{X}(A)$, corresponding to the connected components (recall Definition 2.1.10) of the poset of $\mathcal{X}(A)$. Denote by $A_1, \ldots, A_n$ the dual algebras of $X_1, \ldots, X_n$ respectively. Since $X_1, \ldots, X_n$ are generated subspaces of $\mathcal{X}(A)$, it follows that $A_1, \ldots, A_n \in V$. Moreover, connectedness of the duals implies that $A_1, \ldots, A_n$ are simple by Proposition 2.4.13. Now, by assumption, it follows that $A_1, \ldots, A_n \in V_2$. But notice that $\mathcal{X}(A)$ is the sum of $X_1, \ldots, X_n$, so A is the product of $A_1, \ldots, A_n$. This allows us to deduce $A \in V_2$ and since A was an arbitrary member of V_1, we conclude $V_1 \subseteq V_2$.

The other inclusion $V_2 \subseteq V_1$ follows by a completely symmetric argument. ∎

Remark 4.1.3. For a variety $V \subseteq \mathsf{bi\text{-}KG}$, we have $FinS(V) \subseteq FGS(V)$.

Recall that a variety V has the finite model property if it is generated by its finite members (Definition 2.2.37). In the case where $V \subseteq \mathsf{bi\text{-}KG}$, we can say more.

Proposition 4.1.4. A variety $V \subseteq \mathsf{bi\text{-}KG}$ has the FMP if and only if $V = \mathbb{V}(FinS(V))$.

Proof. ($\Longrightarrow$) Suppose $V = \mathbb{V}(Fin(V))$. Let $W := \mathbb{V}(FinS(V))$. We have $FinS(V) = FinS(W)$, so by Proposition 4.1.2, $Fin(V) = Fin(W)$. Now $V = \mathbb{V}(Fin(V))$ and $W \subseteq V$ imply $V = W$.

($\Longleftarrow$) Suppose $V = \mathbb{V}(FinS(V))$. We have $FinS(V) \subseteq Fin(V)$, thus $Fin(V) = V$. ∎

Remark 4.1.5. Now, by Theorems 3.4.6 and 3.4.7, determining whether a subvariety of $\mathsf{bi\text{-}KG}$ has the FMP is equivalent to checking whether each of its finitely generated simple algebras locally embeds into its class of finite simple algebras. In this way, we can rely on the extensive embeddability properties described in Section 3.5.

For the general characterisation of the FMP, we will show that a variety V has the FMP if and only if for each $A \in V$, there exists a class of special finite algebras $\mathcal{B} \subseteq V$ such that $A \overset{loc}{\hookrightarrow} \mathcal{B}$. We call these special finite algebras *m-compressions* of A, where $m \in \mathbb{N}$ is a parameter. The role of m is to accommodate for the size of the local subgraph that we want to embed.

Before defining what an m-compression is formally, we will give an intuitive explanation. Suppose we are given $A \in FGS(\textsf{bi-KG})$ with its prime decomposition. The idea is to collapse some of the prime summands of A in such a way that the resulting algebra is finite. More specifically, we are allowed to collapse consecutive summands that are of type L_1, L_2, L_4, L^u, L_u or L_u^u. An intuitive justification for this choice is that, by Proposition 3.5.11, large finite prime summands are inflexible with respect to embedding, i.e., we do not want to collapse them. While we can freely merge L_1, L_2, L_4 and L_u^u summands, L^u summands can only appear at the beginning of a collapse block and L_u summands can only appear at the end of a collapse block. This is due to the embeddability restrictions observed in the proofs of Proposition 3.5.12 and 3.5.13. The result of this collapse is a finite L_p summand for $p \geq m$ (this is where the parameter m comes into the picture). In addition, we are sometimes allowed to "disconnect" collapsed summands from their adjacent summands. This is done by adding an L_2 before or after a collapse. The purpose of this is technical and only comes up at the end of the proof of Theorem 4.1.9, left to right direction.

Definition 4.1.6 (m-compression)**.** Let $A \in FGS(\textsf{bi-KG})$ with a prime decomposition $A = P_1 + \cdots + P_n$ and let $m \in \mathbb{N}$. Let $\mathcal{T}_1, \ldots, \mathcal{T}_k$ be a partition of $\{P_1, \ldots, P_n\}$ such that:

$$\begin{aligned}
&\mathcal{T}_1 = \langle P_1, P_2, \ldots, P_{i_1} \rangle,\\
&\mathcal{T}_2 = \langle P_{i_1+1}, P_{i_1+2}, \ldots, P_{i_2} \rangle,\\
&\ldots\\
&\mathcal{T}_k = \langle P_{i_{k-1}+1}, P_{i_{k-1}+2}, \ldots, P_n \rangle.
\end{aligned}$$

Moreover, suppose that for each $j \in \{1, \ldots, k\}$, one of the following conditions holds:

- $\mathcal{T}_j = \langle P_{i_j} \rangle$ for a finite P_{i_j}.
- $\mathcal{T}_j = \langle P_{i_{j-1}}, \ldots, P_{i_j} \rangle$ where:
  - each of $P_{i_{j-1}}, \ldots, P_{i_j}$ is isomorphic to L_1, L_2, L_4, L^u, L_u or L_u^u;
  - at least one of $P_{i_{j-1}}, \ldots, P_{i_j}$ is infinite;

– only $P_{i_{j-1}}$ could be an L^u summand;

– only P_{i_j} could be an L_u summand.

An algebra C of the form:

$$C = C_1 + \Sigma_{j\in\{2,\dots,k\}}(D_j + C_j)^{1}$$

is called an *m-compression* of A, if for each $j \in \{1, \dots, k\}$:

- $C_j = P_{i_j}$, if $\mathcal{T}_j$ contains a single finite element P_{i_j} or
- $C_j = L_p \in \mathcal{P}$ for $p \geq m$, if $\mathcal{T}_j$ contains an infinite element;

and for each $j \in \{2, \dots, k\}$:

- $D_j \in \{L_1, L_2\}$, if the last member of $\mathcal{T}_{j-1}$ is unbounded above (i.e., L^u or L_u^u) or the first member of $\mathcal{T}_j$ is unbounded below (i.e., L_u or L_u^u) or
- $D_j = L_1$, otherwise.

Remark 4.1.7. All m-compression algebras are finite.

Example 4.1.8. Let $A \coloneqq L_6 + L^u + L_u + L_2 + L_{10} + L_u^u + L_4$.

- The algebra B:

$$\begin{aligned} B &\coloneqq L_6 + L_8 + L_2 + L_{10} + L_2 + L_{16} \\ &= L_6 + (L_1 + L_8) + (L_1 + L_2) + (L_1 + L_{10}) + (L_2 + L_{16}) \end{aligned}$$

  is an 8-compression of A. This is witnessed by the partition $\mathcal{T}_1 = \langle L_6 \rangle$, $\mathcal{T}_2 = \langle L^u, L_u \rangle$, $\mathcal{T}_3 = \langle L_2 \rangle$, $\mathcal{T}_4 = \langle L_{10} \rangle$, $\mathcal{T}_5 = \langle L_u^u, L_4 \rangle$ and the algebras $C_1 = L_6$, $C_2 = L_8$, $C_3 = L_2$, $C_4 = L_{10}$, $C_5 = L_{16}$ and $D_2 = D_3 = L_1$, $D_4 = L_2$.

- The algebra C:

$$\begin{aligned} C &\coloneqq L_6 + L_2 + L_8 + L_2 + L_{10} + L_2 + L_{16} \\ &= L_6 + (L_2 + L_8) + (L_1 + L_2) + (L_1 + L_{10}) + (L_2 + L_{16}) \end{aligned}$$

  is not an 8-compression of A – we are not allowed to have an L_2 summand between L_6 and L_8.

[1] In this sum, we allow contractible L_1 summands. See Remark 3.3.12.

Let $D \coloneqq L_4 + L^u + L^u_u + L^u_u + L_2$.

- The algebra E:

$$\begin{aligned} E &\coloneqq L_4 + L_{50} \\ &= L_4 + (L_1 + L_{50}) \end{aligned}$$

  is an 18-compression of D. This is witnessed by the partition $\mathcal{T}_1 = \langle L_4 \rangle$, $\mathcal{T}_2 = \langle L^u, L^u_u, L^u_u, L_2 \rangle$.

- The algebra $F \coloneqq L_{20}$ is not a 10-compression of D, because L^u cannot appear in the middle of a partition block.

The following theorem gives us a full characterisation of subvarieties of bi-KG with the FMP.

Theorem 4.1.9. *Let $V = \mathbb{V}(\mathcal{B})$ where $\mathcal{B} \subseteq FGS(\text{bi-KG})$ is a class closed under subalgebras. Then V has the FMP if and only if for all $A \in \mathcal{B}$ and for all $m \in \mathbb{N}$, there exists an m-compression of A in $\mathcal{B}$.*

Proof. We outline only the most important steps and observations for the sake of comprehensibility.

($\Longrightarrow$) Let $A \in FGS(V)$ with a prime decomposition $A = P_1 + \cdots + P_n$ and let $m \in \mathbb{N}$. Let X be a normal local subgraph of A such that for every infinite prime summand P_j of A, we have $|X \cap P_j| \geq max\{m, 6\}$. By the assumption that V has the FMP, Proposition 4.1.4 and Theorem 3.4.7, we know $A \overset{loc}{\hookrightarrow} FinS(V)$. Hence there exists an embedding f of X into some $B \in FinS(V)$. Again by Theorem 3.4.7, $B \in FinS(V) = \mathbb{V}(\mathcal{B})$ implies that $B \overset{loc}{\hookrightarrow} \mathcal{B}$. By finiteness of B, this means $B \in \mathbb{S}(\mathcal{B})$. From the closure of $\mathcal{B}$ under subalgebras, we deduce $B \in \mathcal{B}$.

Let C be the subalgebra of B generated by $f[X]$. Note that, similarly to the proof of Proposition 3.4.3, if $a, b \in B$ are in different prime summands, then they generate no new points, and if $a, b \in B$ lie in the same prime summand, then all new points that they generate are in this prime summand.

We know that $C \in \mathcal{B}$, so showing that C is an m-compression of A would suffice for this direction of the proof. We do this by presenting the $\mathcal{T}_l$, C_l and D_l witnesses.

We define $\mathcal{T}_1, \ldots, \mathcal{T}_k$ inductively on $P_1, \ldots, P_n$. Suppose we have partitioned $P_1, \ldots, P_{j-1}$ into $\mathcal{T}_1, \ldots, T_{l-1}$. We define $\mathcal{T}_l$ as follows.

1. If $P_j = L_p$ for $p > 4$, we define $T_l \coloneqq \langle P_j \rangle$ and $C_l \coloneqq P_j$. Note that, by Proposition 3.5.11, $f[P_j]$ is an L_p summand in B, so it is also a prime summand in C.

2. Otherwise, suppose P_j is L_2 or L_4. Let Q be the first infinite prime summand in $P_j, \dots, P_n$. The condition $m \geq 6$, together with Proposition 3.5.9, ensures that $f[Q]$ lies inside a single prime summand R of B. Moreover, $f[Q]$ generates the whole of R, so R is a prime summand of C. If $f[P_j] \nsubseteq R$, then define $\mathcal{T}_l \coloneqq \langle P_j \rangle$ and $C_l = P_j$.

3. Suppose P_j is L_1, L_2, L_4 or infinite. Let Q and R be defined as above and suppose that this time $f[P_j] \subseteq R$. Let $P_j, \dots, P_{j+p}$ be all prime summands of A that lie inside R. Define $\mathcal{T}_l \coloneqq \langle P_j, \dots, P_{j+p} \rangle$ and $C_l \coloneqq R$.

Now notice the following properties hold for all $l \in \{1, \dots, k\}$.

- If T_l contains a single finite summand, then this finite element is equal to C_l. This follows from items (1) and (2).

- If T_l contains an infinite summand, then T_l can only contain L^u as its first member and only contain L_u as its last member. This follows from item (3) and the fact that the part of X contained in an L^u summand (which is of size at least $m \geq 6$) can only embed into an initial segment of R and the part of X contained in an L_u summand can only embed into a final segment of R (by the proof of Proposition 3.5.12 and Proposition 3.5.13). Moreover, T_l does not contain L_p members for $p > 4$, because of Proposition 3.5.11.

- C_l is a prime summand of C. In items (1) this has already been observed and in item (3) it follows from the way we obtained R. In item (2), we know that C_l is contained neither in the summands that follow, nor in the preceding summands of C. And for $P_j = L_2$ or $P_j = L_4$, we know that $f[P_j]$ does not generate new elements.

- If $l < k$, then C_l and C_{l+1} are either consecutive prime summands of C (if they share an element) or there is exactly one L_2 summand between them (if they are disjoint). Moreover, the latter can only happen if the last element of $\mathcal{T}_l$ is unbounded from above or the first element of $\mathcal{T}_{l+1}$ is unbounded from below.

For $l < k$, if C_l and C_{l+1} are consecutive in the prime decomposition of C, define $D_{l+1} \coloneqq L_1$. Otherwise, define $D_{l+1} \coloneqq L_2$. From the observations that we made, it follows that $C = C_1 + \Sigma_{j \in \{2,\ldots,k\}}(D_j + C_j)$ and thus C is an m-compression of A.

($\Longleftarrow$) We prove this direction by showing that for every $A \in FGS(V)$:

$$A \overset{loc}{\hookrightarrow} \{B \in \mathcal{B} \mid m \in \mathbb{N} \text{ and } B \text{ is an } m\text{-compression of } A\}.$$

Let X be a normal local subgraph of A. Take $m \coloneqq 2|X|$ and choose an m-compression $C \in \mathcal{B}$ of A. Let $P_1, \ldots, P_n$ be the prime decomposition of A. We know C is induced by some partition $\mathcal{T}_1, \ldots, \mathcal{T}_k$ of $P_1, \ldots, P_n$ and the corresponding $\{C_1, \ldots, C_k\}$ and $\{D_2, \ldots, D_k\}$.

Suppose $l \in \{1, \ldots, k\}$, $T_l = \{P_{i_{l-1}+1}, \ldots, P_{i_l}\}$ and define $U_l \coloneqq P_{i_{l-1}+1} \cup \cdots \cup P_{i_l}$. We claim that $X \cap U_l$ embeds into C_l. Indeed, if $\mathcal{T}_l$ contains a single finite summand, then U_l is isomorphic to C_l. Otherwise, $\mathcal{T}_l$ contains only L_1, L_2, L_4 and infinite prime summands and $C_l = L_p$ for some $p \geq m$. In the proof of Propositions 3.5.7 and 3.5.14, we saw that for a sufficiently large $p \in \mathbb{N}$, L_p embeds finitely many L_2, L_4 and intervals of L_u^u. Furthermore, in Propositions 3.5.12 and 3.5.13 we saw that such an L_p also embeds initial segments of L^u and final segments of L_u. This is possible here, because in the definition of an m-compression, we imposed the restriction that L^u summands can only appear at the beginning of a partition block and L_u summands can only appear at the end of a partition block. Now the choice $m \geq 2|X|$ guarantees that C_l is indeed sufficiently large to embed U_l.

Finally, notice that $D_l = L_2$ only if U_{l-1} is unbounded above or U_l is unbounded below, i.e., $U_{l-1} \cap U_l = \emptyset$. This ensures that the individual embeddings can be pasted together into a single embedding of X into C. ∎

As a consequence of this theorem, we obtain a significantly less general proposition, which is nevertheless useful, because it bypasses the heavy technicalities of m-compressions.

Corollary 4.1.10. Let $V = \mathbb{V}(\{B_1, \ldots, B_n\})$ for some algebras $B_1, \ldots, B_n \in FGS(\textsf{bi-KG})$. Then V has the FMP if and only if $B_1, \ldots, B_n$ are finite.

Proof. ($\Longrightarrow$) We prove this direction by contraposition. Assume that there exists an infinite algebra B among $B_1, \ldots, B_n$. Since $B_1, \ldots, B_n$ are finitely many and each of them contains finitely many prime summands, there exists $4 < m \in \mathbb{N}$ such that every finite prime summand in $B_1, \ldots, B_n$ is of size less than m. Take $\mathcal{C} \coloneqq \mathbb{S}(\{B_1, \ldots, B_n\})$. By Proposition 3.5.11, if L_p occurs as a summand in a member of $\mathcal{C}$, then it occurs in a member of $\{B_1, \ldots, B_n\}$. Therefore every finite prime summand in $\mathcal{C}$ is also of size less than m.

Since B is infinite, we know that for $k \geq m$, its finite k-compressions contain at least one L_p with $p \geq k \geq m$. This means that $\mathcal{C}$ does not contain any k-compressions of B for $k \geq m$. Therefore, by Theorem 4.1.9, V does not have the FMP.

($\Longleftarrow$) Follows directly from $V = \mathbb{V}(\{B_1, \ldots, B_n\})$ for finite $B_1, \ldots, B_n$. ■

We finish this section with the FMP in two particular varieties. We show that bi-KG has the FMP, but the variety $\mathbb{V}(L)$ generated by the bi-Heyting Rieger-Nishimura lattice lacks the FMP. Not only are these facts noteworthy on their own, but they will also be useful for our discussion of degrees of FMP in the next section.

Corollary 4.1.11. The variety bi-KG has the FMP.

Proof. We know bi-KG $= \mathbb{V}(FSG(\text{bi-KG}))$. But $FGS(\text{bi-KG})$ is closed under subalgebras and contains all m-compression algebras. By Theorem 4.1.9, this means bi-KG has the FMP. ■

Corollary 4.1.12. The variety $\mathbb{V}(\{L^u + L_1\})$ generated by the bi-Heyting Rieger-Nishimura lattice lacks the FMP.

Proof. Follows directly from Corollary 4.1.10. ■

This is a striking difference with the intuitionistic case. There the variety generated by the Rieger-Nishimura lattice has the FMP and, moreover, each of its subvarieties has the FMP (see, e.g., [4]).

4.2 Degrees of FMP relative to bi-KG

We move to the task of characterising the possible degrees of FMP relative to bi-KG. Here we give the formal definition of this notion.

Definition 4.2.1 (Degree of FMP of a variety)**.** Let U and V be varieties such that $U \subseteq V$. We define:

$$deg_V(U) = |\{W \subseteq V : W \text{ is a variety with } Fin(W) = Fin(U)\}|$$

and call it the *degree of FMP* of U relative to V.

Degrees of FMP can also be assigned to superintuitionistic and bi-superintuitionistic logics, through the finite frames of a logic $FinFr$ (recall Definition 2.5.4).

Definition 4.2.2 (Degree of FMP of a logic)**.** Let L' be a (bi-)superintuitionistic logic and let the logic L'' extend L'. We define:

$$deg_{L'}(L'') \coloneqq |\{M \supseteq L' | M \text{ is a logic such that } FinFr(M) = FinFr(L'')\}|$$

and call it the *degree of FMP* of L'' relative to L'.

Once again, the algebraic and logical notions coincide, as witnessed by the following proposition.

Proposition 4.2.3. Let U and V be a variety of Heyting or by-Heyting algebras with $U \subseteq V$. Then:

$$deg_V(U) = deg_{\mathcal{L}(V)}(\mathcal{L}(U)).$$

Consequently, we will formulate and prove results on degrees of FMP for varieties, while only stating the corresponding results for logics as corollaries.

For each cardinal $\kappa \in \{1, \ldots, 2^{\aleph_0}\}$, we are interested whether there exists a variety $V \subseteq \mathsf{bi\text{-}KG}$ such that $deg_{\mathsf{bi\text{-}KG}}(V) = \kappa$. We elaborate on the choice of the interval $\{1, \ldots, 2^{\aleph_0}\}$. It follows from the definition that degrees of FMP are always at least 1. The other bound, $2^{\aleph_0}$, is justified by the following proposition.

Proposition 4.2.4. If V is a variety in a countable language, degrees of FMP relative to V are at the most the continuum.

Proof. It suffices to prove that the total number of varieties in the language of V does not exceed $2^{\aleph_0}$. By Theorem 2.2.31, the number of varieties in a fixed language is bounded by the number of sets of equations in $\aleph_0$-many variables in the language. But since the language is assumed to be countable, there are countably many such equations and therefore continuum many sets of equations. ■

Corollary 4.2.5. Relative to HA, KG, bi-HA and bi-KG, degrees of FMP are at most the continuum.

Proof. The listed varieties are defined in a finite language. ■

Recall that relative to the intuitionistic KG, every cardinal κ with $\kappa \leq \aleph_0$ is a degree of FMP of some subvariety of KG. In complete contrast to this, it turns out that the only possible degrees of FMP relative to bi-KG are 1 and $2^{\aleph_0}$. Our strategy for showing this is to display a variety with degree 1 and one with degree greater than 1 and prove that any variety with degree greater than 1 has degree the continuum.

We begin by giving the witness of a subvariety of bi-KG with degree 1 and a subvariety with a degree greater than 1. In fact, the variety with degree 1 is bi-KG itself and the variety with a greater degree is $\mathbb{V}(L^u + L_1)$, the variety generated by the bi-Heyting Rieger-Nishimura lattice L.

Proposition 4.2.6. The degree of FMP of bi-KG relative to bi-KG is 1.

Proof. Recall from Corollary 4.1.11 that bi-KG has the FMP. Suppose $V \subseteq$ bi-KG satisfies $Fin(V) = Fin(\text{bi-KG})$. We have:

$$\text{bi-KG} = \mathbb{V}(Fin(\text{bi-KG})) = \mathbb{V}(Fin(V)) \subseteq V \subseteq \text{bi-KG}.$$

As a result, $V =$ bi-KG and we conclude $deg_{\text{bi-KG}}(\text{bi-KG}) = 1$. ∎

Proposition 4.2.7. The degree of FMP of $\mathbb{V}(L^u + L_1)$ relative to bi-KG is greater than 1.

Proof. Denote $V := \mathbb{V}(L^u + L_1)$. Let $U := \mathbb{V}(Fin(V))$. We know that $Fin(V) \subseteq V$, so $U \subseteq V$. Thus it turns out that U is a subvariety of V with the same finite members. But since U is generated by finite algebras and by Corollary 4.1.12, V does not have the FMP, we conclude $V \neq U$ and so $deg_{\text{bi-KG}}(V) > 1$. ∎

The main challenge of this section, which we address next, is proving that there do not exist any degrees of FMP relative to bi-KG strictly between 1 and $2^{\aleph_0}$. The way we do this is by explicitly constructing continuum many varieties with the same finite members as a given variety V with $deg_{\text{bi-KG}}(V) > 1$.

The following notion of a prime skeleton is a technical preparation for proving the main theorem. It is a formal way to describe taking the sequence of all L^u and L_u summands in the prime decomposition of an algebra $A \in FGS(\text{bi-KG})$.

Definition 4.2.8 (Prime skeleton)**.** Let $A \in FGS(\text{bi-KG})$. Let $P_1, \ldots, P_n \in \{L^u, L_u\}$ and $B_0, \ldots, B_n \in FinSum(\mathcal{P})$ be such that for every $i \in \{1, \ldots, n\}$, B_i does not contain any L^u and L_u prime summands. Suppose:

$$A = B_0 + \Sigma_{i \in \{1,\ldots,n\}}(P_i + B_i),$$

where we do allow sum contraction (see Remark 3.3.12). In this case, we call $P_1, \ldots, P_n$ the prime skeleton of A.

The following lemma gives us, under certain circumstances, a substantial restriction on the form of embeddings by requiring that prime skeletons are mapped to prime skeletons. This will be useful in proving the non-existence of embeddings.

Lemma 4.2.9. *Let $\{A\} \cup \mathcal{B} \subseteq FGS(\textsf{bi-KG})$ be a set of algebras with the same prime skeleton $P_1, \ldots, P_n$. Moreover, assume there exists $m \in \mathbb{N}$ such that $\mathcal{B}$ does not contain L_k summands for $k \geq m$. Then there exists a normal local subgraph X of A such that if X embeds into some $B \in \mathcal{B}$ via the function f, then $f[X \cap P_i] \subseteq P_i$ for every $i \in \{1, \ldots, n\}$.*

Proof. Without loss of generality, assume $m > 2$. Take X to be a normal local subgraph of A that contains at least m elements from every P_i for $i \in \{1, \ldots, n\}$ and assume X embeds into some $B \in \mathcal{B}$ via a function f. By Proposition 3.5.9, $f[X \cap P_i]$ is contained a single prime summand of B, for each $i \in \{1, \ldots, n\}$. Because of the choice of m, these single prime summands must be infinite. Now from the proof of Proposition 3.5.12, it follows that if P_i is a L^u summand, then $X \cap P_i$ embeds into an initial segment of an L^u summand and if P_i is an L_u summand, then $X \cap P_i$ embeds into a final segment of a L_u summand. This shows that for every $i \in \{1, \ldots, n\}$ there exists $j \in \{1, \ldots, n\}$ such that $f[X \cap P_i] \subseteq P_j$. Moreover, if $i_1 \neq i_2$ are two indices in $\{1, \ldots, n\}$, then $X \cap P_{i_1}$ and $X \cap P_{i_2}$ embed into different prime summands. By monotonicity of f, it follows that $f[X \cap P_i] \subseteq P_i$ for every $i \in \{1, \ldots, n\}$. ■

Intuitively, we will construct continuum many different varieties with the same finite members by taking different subsets of ω. More specifically, we will have a collection of countably many algebras and will generate varieties from subcollections of these algebras. In order to guarantee that distinct subcollections of algebras indeed generate distinct varieties, we want the algebras to be incomparable in some sense. In this context, comparability means local embeddability. So we are looking for countably many algebras that do not mutually locally embed into each other. A good candidate is the sequence $L_4, L_4 + L_4, L_4 + L_4 + L_4, \ldots$. Let us now see how to formalise these ideas.

Theorem 4.2.10. *Let $V \subseteq \textsf{bi-KG}$ be a variety with $deg_{\textsf{bi-KG}}(V) > 1$. Then $deg_{\textsf{bi-KG}}(V) = 2^{\aleph_0}$.*

Proof. Let $U := \mathbb{V}(FinS(V))$. We have $U \subseteq V$, hence $FinS(U) \subseteq FinS(V)$. In addition, by definition of U, we know $FinS(V) \subseteq FinS(U)$, therefore $FinS(V) = FinS(U)$. Note that every variety with the same finite simple members as V contains U. Now $deg_{\mathsf{bi\text{-}KG}}(V) > 1$ implies the existence of a variety $W \subseteq \mathsf{bi\text{-}KG}$ with $W \neq U$ and $FinS(W) = FinS(U)$. By Theorem 3.4.7, there exists $A \in FGS(W)$ with $A \overset{loc}{\not\hookrightarrow} FinS(V)$.

From $A \overset{loc}{\not\hookrightarrow} FinS(V)$, we know that A is infinite. Since A is the sum of finitely many prime summands, it contains at least one infinite prime summand in its prime decomposition. Let P be the first infinite summand in the decomposition of A into prime summands. We consider as two separate cases $P = L_u$ and $P \neq L_u$.

Firstly, suppose $P = L_u$. We write a representation $A = A' + P + A''$, where A'' could be a contractible L_1. We construct from A countably many algebras $B_0, B_1, \ldots$ which will be used to generate continuum many varieties with the same finite simple members as V. Define for every $n \in \mathbb{N}$:

$$B_n \coloneqq A' + L_u + \Sigma_{i \in \{1,\ldots,n\}} L_4 + L^u + P + A''.$$

A visual representation of this construction can be seen on Figure 4.1. Using these algebras, define for every $I \subseteq \mathbb{N}$:

$$V_I \coloneqq \mathbb{V}(\{B_i \mid i \in I\} \cup FinS(V)).$$

Our purpose is to show that $|\{V_I \mid I \subseteq \mathbb{N}\}| = 2^{\aleph_0}$ and that all of these varieties have the same finite simple members as V. For the first claim, let $I, J \subseteq \mathbb{N}$ with $I \neq J$. We prove that $V_I \neq V_J$. Since I and J are different, without loss of generality, we have $i \in I \setminus J$. By Theorem 3.4.7, it suffices to show that $B_i \overset{loc}{\not\hookrightarrow} \{B_j \mid j \in J\} \cup FinS(V)$. It is straightforward to see that $A \leq B_i$. So if we assume $B_i \overset{loc}{\hookrightarrow} FinS(V)$, this would imply $A \overset{loc}{\hookrightarrow} FinS(V)$, which is in contradiction with the choice of A.

As a result, it only remains to see that $B_i \overset{loc}{\not\hookrightarrow} \{B_j \mid j \in J\}$. Let X be the normal local subgraph given by Lemma 4.2.9 by taking $A \coloneqq B_i$ and $\mathcal{B} \coloneqq \{B_j \mid j \in J\}$. Let Y be the local subgraph of B_i obtained by extending X with the i copies of L_4 between $A' + L_u$ and $L^u + P + A''$.

Assume towards a contradiction that Y embeds into some B_j for some $j \in J$ via a function f. Denote:

$$\begin{aligned} a &\coloneqq max\{A' + L_u\} \text{ in } B_i \\ a' &\coloneqq max\{A' + L_u\} \text{ in } B_j \\ b &\coloneqq min\{L^u + P + A''\} \text{ in } B_i \\ b' &\coloneqq min\{L^u + P + A''\} \text{ in } B_j. \end{aligned}$$

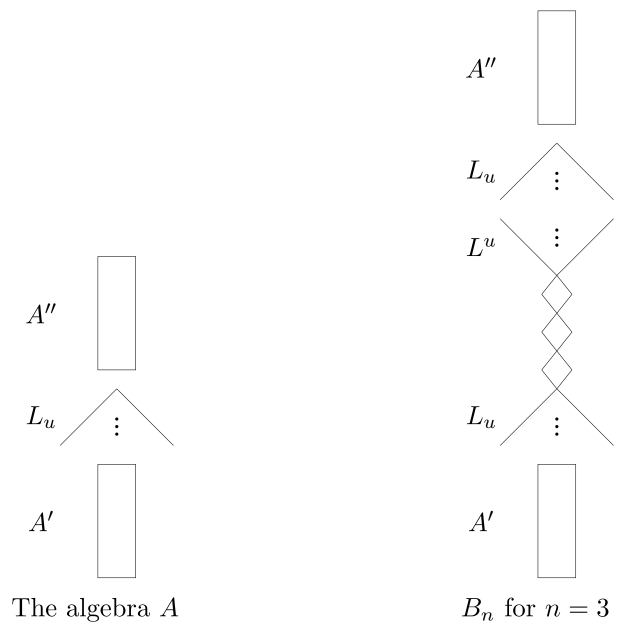

Figure 4.1: Constructing algebras from A

Since $a, b \in X \subseteq Y$, we get $f(a) = a'$ and $f(b) = b'$, due to the choice of X. Therefore f, restricted to the i copies of L_4, is an embedding:

$$g : \Sigma_{k \in \{1,\ldots,i\}} L_4 \to \Sigma_{k \in \{1,\ldots,j\}} L_4.$$

We show that such an embedding cannot exist. Note that if $i < j$, we have:

$$|\Sigma_{k \in \{1,\ldots,i\}} L_4| < |\Sigma_{k \in \{1,\ldots,j\}} L_4|,$$

so the embedding is impossible due to the requirement of injectivity. We know $i \neq j$, so the only option left is $i < j$. For convenience, denote:

$$C \coloneqq \Sigma_{k \in \{1,\ldots,i\}} L_4$$
$$D \coloneqq \Sigma_{k \in \{1,\ldots,j\}} L_4.$$

In addition, we write Q_k for the k-th L_4 copy in C and R_k for the k-th L_4 copy in D. We prove by induction on $k \in \{1, \ldots, i\}$ that $g[Q_k] = R_k$.

- Base case $k = 1$. By Proposition 3.5.9, we know that Q_1 embeds into a single prime summand of D. Since $g(a) = a'$, this single prime summand has to be the first one, i.e, R_1.

- Induction step. Suppose $k < i$ and the $g[Q_k] = R_k$. In particular, the top element c of Q_k is mapped to the top element c' of R_k. Once again, we apply Proposition 3.5.9 and deduce that Q_{k+1} is mapped into a single prime summand in D. But c is the bottom element of the Q_{k+1} and c' is the bottom element of R_{k+1}, so this single prime summand has to be R_{k+1}.

Consequently, $g[Q_i] = R_i \neq R_j$, which is a contradiction with $g(b) = b'$. Therefore, we found a local subgraph Y of B_i that does not embed into $\{B_j \mid j \in J\}$, so $B_i \overset{loc}{\not\to} \{B_j \mid j \in J\}$ and $V_I \neq V_J$.

It remains to show $FinS(V_I) = FinS(V)$ for every $I \subseteq \mathbb{N}$. By definition of V_I, it follows directly that $FinS(V) \subseteq FinS(V_I)$. For the other inclusion $FinS(V_I) \subseteq FinS(V)$, suppose $E \in FinS(V_I)$. Take X to be the total local subgraph of E, then we know that X embeds into a member of $\{B_i \mid i \in I\} \cup FinS(V)$. If X embeds into a member of $FinS(V)$, we immediately get $E \in FinS(V)$. Otherwise, X embeds into a member of $\{B_i \mid i \in I\}$, so suppose $i \in I$ is such that X embeds into B_i via a function f. We prove that X embeds into A.

We name the following subsets of B_i:

$$B' \coloneqq B_i \cap (A' \cup P \cup A''),$$
$$B'' \coloneqq B_i \cap (L_u + \Sigma_{i\in\{1,\ldots,n\}} L_4 + L^u),$$

i.e., B' is the A-segment of B_i and B'' contains the rest of the summands. Since there is no point identification between the summands A' and L_u and between the summands L^u and P in B_i, we know that $B' \cap B'' = \emptyset$. Hence for every point in $x \in X$, we know that $f(x)$ belongs to exactly one of B' or B''.

We construct an embedding $g : X \to A$ in two steps.

- For the points $x \in X$ with $f(x) \in B'$, since B' is isomorphic to A, we can take $g(x)$ to be the point in A corresponding to $f(x)$.

- The remaining points of X are $Y \coloneqq \{x \in X \mid f(x) \in B''\}$. Notice that B'' does not contain any L_m summands for $m > 4$, so by Proposition 3.5.11, Y does not contain any L_m summands for $m > 4$ either. This means that Y is a finite sum of L_2 and L_4. Since $P = L_u$ is infinite, there exists a final segment Q of P with $f^{-1}[Q] = \emptyset$. From the proof of Proposition 3.5.7, we know that Q embeds all finite sums of L_2 and L_4. Therefore we can define g to embed Y into Q.

Now g witnesses $E \overset{loc}{\hookrightarrow} A$, thus $E \in FinS(W) = FinS(V)$.

This completes the proof that $\{V_I \mid I \subseteq \mathbb{N}\}$ is a set of continuum many varieties with the same finite simple members as V. So, by Proposition 4.1.2, they all have the same finite members. It follows that $deg_{\textsf{bi-KG}}(V) \geq 2^{\aleph_0}$. Together with Proposition 4.2.4, we conclude $deg_{\textsf{bi-KG}}(V) = 2^{\aleph_0}$.

We have covered the case $P = L_u$. For the other case, i.e., $P = L^u$ or $P = L^u_u$, redefine:

$$B_n = A' + P + L_u + \Sigma_{i \in \{1,\dots,n\}} L_4 + L^u + A''.$$

Now notice that the proof of the previous case $P = L_u$ relied only on the fact that L_u is infinite and unbounded below. And since in the present case P is infinite and unbounded above, we can run a symmetric proof. ∎

Remark 4.2.11. We constructed continuum many subvarieties of bi-KG with the same finite algebras. But since at most one of them can have the FMP, it follows that for every variety $V \subseteq$ bi-KG with $deg_{\textsf{bi-KG}}(V) > 1$, there exist continuum many subvarieties of bi-KG without the FMP.

As a consequence of the theorem, we obtain a full description of degrees of FMP relative to bi-KG.

Theorem 4.2.12. *Relative to bi-KG, all possible degrees of FMP are* 1 *and* $2^{\aleph_0}$.

Proof. By Proposition 4.2.6, there exists a subvariety of bi-KG of degree 1. By Proposition 4.2.7 and Theorem 4.2.10, there exists a subvariety of bi-KG of degree $2^{\aleph_0}$. By Theorem 4.2.10, there are no other degrees relative to bi-KG. ∎

Now we can use Proposition 4.2.3 to derive a corresponding statement about the logic **bi-KG**.

Corollary 4.2.13. Relative to the logic **bi-KG**, all possible degrees of FMP are 1 and $2^{\aleph_0}$.

In conclusion, we have obtained a *dichotomy theorem*, which is reminiscent of Blok's dichotomy [6, 7] and is in stark contrast with the antidichotomy theorem for KG [3].

Chapter 5

Conclusion

This thesis is centered around the variety bi-KG, understanding the lattice of its subvarieties and, ultimately, finding degrees of FMP of its subvarieties.

We began by defining bi-KG as the variety generated by the class $\mathcal{G}$ of finite sums of 1-generated Heyting algebras, endowed with a co-implication. We looked more closely into the universal class generated by $\mathcal{G}$ and found a representation of its members as sums of indecomposable algebras, which we called prime. This fact yielded a description of the subdirectly irreducible members of bi-KG and led to the discovery that bi-KG is semi-simple. This enabled us to compare subvarieties of bi-KG by examining their finitely generated simple members. In particular, we showed how to do this using local embeddability and developed technical tools for determining local embeddability for finitely generated simple bi-KG algebras.

Subsequently, we applied these findings to obtain a full characterisation of the FMP in subvarieties of bi-KG. As a result, bi-KG was found to satisfy the FMP, while the variety generated by the bi-Heyting Rieger-Nishimura lattice L does not. The latter was the first major difference from the intuitionistic case, where the variety generated by the Heyting Rieger-Nishimura lattice RN enjoys the FMP, together with all of its subvarieties.

Lastly, we found a complete description of the possible degrees of FMP relative to bi-KG. For every subvariety V of bi-KG with degree of FMP relative to bi-KG greater than 1, we were able to build continuum many distinct subvarieties of bi-KG with the same finite members as V. In this way, we proved a dichotomy theorem, stating that the set of degrees of FMP relative to bi-KG is $\{1, 2^{\aleph_0}\}$. This differs substantially from degrees of FMP in KG, where every cardinal κ with $\kappa \leq \aleph_0$ is the degree of FMP relative to KG of some variety.

We conclude with possible research directions extending our work.

- An important property of KG, described in [3], is that it can be axiomatised by Jankov-de Jongh formulas (see, e.g., [4, Section 3.3]). We defined the variety bi-KG by providing a class of generators, but we did not give any axiomatisation. We find it interesting whether a recursive axiomatisation exists for bi-KG and how it compares to the axiomatisation of KG.

- The problem of characterising degrees of FMP relative to bi-HA is still open. We believe our work in bi-KG highlights the differences between HA and bi-HA and hints at a possible dichotomy in bi-HA.

- On the side of modal logics, we noted in the introduction that temporal logic resembles bi-intuitionistic logic with its backwards-looking modalities. Therefore, we are interested whether our bi-intuitionistic insights can assist the study of degrees of FMP relative to temporal logic.

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