Informational independence

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Introduction In game theoretical semantics the truth of a formula is determined by a game between two players, $\exists$loise who tries to verify the formula, and $\forall$belard to refute it. He chooses on $\wedge$ and $\forall$, she on $\vee$ and $\exists$. An extension of such games, introduced by J. Hintikka, is IF logic: independence friendly logic. It contains the quantifier $\exists(y/x)$ which means that $\exists$loise has to choose y independent of x, and $\psi(\vee/x)\theta$ which means that she has to choose a subformula has independent of x. A formula is true if $\exists$loise has a winning strategy. Hodges has given a compositional interpretation for the logic: trump semantics. His interpretation gives results that are not in accordance with intuitions concerning independence of information. Therefore an alternative interpretation will be proposed.

Examples Two examples (over $\mathbf{N}$) which illustrate the aims of IF logic are:

1. $\forall x \exists(y/x)[x = y]$

When $\forall$belard has chosen, $\exists$loise does not know the value of x, and therefore there it may happen that she selects another value. Hence $\exists$loise has no winning strategy, so the formula is 'not true'. The variant $\forall x \exists(y/x)[y \leq x]$ is true: choose $y := 0$ ('choose y to become 0').

2. $\forall x_1 \exists y_1 \forall x_2 \exists(y_2/x_1)[x_1 < y_1 \wedge x_2 < y_2]$

The choice of y_2 is independent of x_1, but may depend on x_2. The formula is true (on $\mathbf{N}$). A more familiar representation of this example is by means of a branching quantifier:

$$\begin{pmatrix} \forall x_1 & \exists y_1 \\ \forall x_2 & \exists y_2 \end{pmatrix} [x_1 < y_1 \wedge x_2 < y_2]$$

Problems Some examples which illustrate the problems of trump semantics are:

1. $\forall x[x \neq 2 \vee (\exists y/x)[x = y]]$

The strategy which $\exists$loise follows in trump semantics is to choose $y := 2$. Her strategy for the disjunction is: if x happens to be different from 2, she chooses the left side of the disjunction (which then is true), and otherwise she chooses the right hand side (which then becomes true by her choice for y). So $\exists$loise wins because she deduces from the context, when $\exists x/y$ is played, that $x = 2$. Since she is supposed *not* to use the value of x, this is cheating.

2. $\forall x[(\exists y/x)[x \neq y] \vee (\exists y/x)[x \neq y]]$

If you cannot find a y independent of x, you also cannot find it if you have to do that twice. But in trump semantics that is not the case. If $x = 3$, then $\exists$loise chooses left, and there she always chooses 4. Otherwise $\exists$loise chooses right, and there 3. In both cases the chosen disjunct is true. So ϕ is true.
This example shows a strange property of trump semantics: $\phi \vee \phi$ is not in all contexts equivalent with ϕ.

Discussion The examples given above illustrate that, although in trump semantics the strategy for $(\exists y/x)$ and $\vee/x$ indeed is not based directly on the value of x, any information about the value of x which can be deduced from other sources may be used: the value of other variables, the form of other parts the formula, or strategies used in other parts of the formula. For this reason trump semantics is not a formalization of 'informational independence', but of 'incomplete information'. It is not impossible that Hodges would agree with this opinion, because the title of his paper is 'Compositional semantics for a language of imperfect information', and *not* '... for a language of informational independence'.

The idea The basic idea of our approach is that subformulas are considered as subgames. So the notion 'game' is generalized to arbitrary formulas. A game is a pair consisting of a formula (maybe with free variables), and a dependency set, which for each x indicates which y's depend on it. Such a game can be played on its own. If the formula has free variables either a initial position (initial values for the free variables) is set up by the players, or it is given by previous moves in a larger game. Then the game is played.

$\exists$loise may have a strategy which describes how she will react on moves of $\forall$belard. A strategy for $\exists y$ is a function that yields a value for y which has as argument the start position of the subgame, and one for $\exists y/x$ is such a function without the value of x as argument. Analogously for $\vee$ and $\vee/x$. After application of such a strategy, a new subgame is entered.

∃loise may have a strategy for a subgame which guarantees her a win. Such a winning strategy can, of course, not depend on variables that do not occur in the subgame. Also the larger context in which the subformula occurs, is irrelevant: the strategy must be winning for the subgame as such. A winning strategy for $\exists y/x$ must be one that works for all values of x, provided the values of y's that depend on x are adapted.

A winning strategy for an entire game is formed by a combination of winning strategies for its subgames. One might describe the situation as follows: associated with a (sub)game there is a shelf of plans, and when a subgame is entered in a certain initial position, a plan is taken from the shelf which fits on that position, and it is followed until a new subgame is entered. This concept of a winning strategy means that not all conceivable strategies are available; only those which are built from strategies that would also work if the game was played in isolation or as subgame of a larger game. So in comparison with Hodges' approach, less strategies are possible.

Maybe it is useful to emphasize the distinction with other interpretations of 'independence':

1. It is not assumed that there are players who forget a value for a variable and may remember it later.
2. It is not assumed that there are teams of players in which for each new variable a new player is introduced who gets only partial information
3. It is not based on equivalence classes of information sets within the choice tree of a given game.

One might say that in the present approach equivalence classes are introduced among the information sets in *all* games in which the subgame arises.

Examples Some examples illustrate the kind of analysis arising with the subgame:

1. $\exists y/x\,[x \neq y]$

No variables depend on x, hence the dependency set is empty. Let us assume that this game is played in start position $x = n$. First ∃loise has to select a value for y. Would there be a winning strategy for her? A strategy for $\exists y/x$ is a function that does not depend on x, there are no other variables in the domain, so the strategy is a constant function. A candidate is $y := n + 1$, which is winning in the given start position. But would it also be winning for other values of x? The answer evidently is 'No': for $x = n + 1$ the strategy $y := n + 1$ looses. Since there is no choice which works for all values of x, there is no winning strategy for ∃loise in the given start position. Neither there is a winning strategy in other start positions.

2. $\forall x[\,\exists y/x\,[x \neq y]]$

Also here the dependency set is empty, and no start values have to be given. The game starts with ∀belard choosing a value for x, say $x := n$. Then the subgame $\exists y/x\,[x \neq y]$ is played, from start position $x = n$. As we have seen before, there is no winning strategy for this subgame. Hence there is no winning strategy for the entire game. The formula is not 'true'.

3. $\forall x[\,\exists y/x\,[x \neq y] \vee \exists y/x\,[x \neq y]]$

First ∀belard chooses, say $x = n$. Next ∃loise has to choose L or R. If she chooses L then she has to play a subgame for which she has no winning strategy. The same of she chooses R. Hence she has no winning strategy for the entire game. So the formula is not true.

It may be instructive to consider the difference with Hodges' approach. There in the left game the strategy $y := 4$ was followed. Since ∃loise plays that subgame only when $x = 3$, she wins that subgame. But $y := 4$ is not a winning strategy for the subgame because it does not work for all values of x. Therefore it can, in our approach, not be part of a winning strategy in the entire game. So the difference with Hodges' approach is that he does not require that a winning strategy is built from winning strategies for subgames.

4. $\forall x_1 \exists y_1 \forall x_2\ \exists y_2/x_1\,[x_1 < y_1 \wedge x_2 < y_2]$

First we consider a subgame: $\exists y_2/x_1\,[x_1 < y_1 \wedge x_2 < y_2]$. The dependency information we have here, is that y_1 depends on x_1. Assume this subgame to be played from a position where $x_1 = n, y_1 = n + 1$, and $x_2 = m$. There is a strategy that gives ∃loise a win: let $y_2 := x_2 + 2$. This strategy is allowed because y_2 does not depend on x_1 as is required by $\exists y_2/x_1$. Furthermore, if x_1 has another value (say p), then the same strategy brings ∃loise a winning position, provided the value of y_1 is changed to $p + 1$. This gives ∃loise a winning strategy that can be used for the entire game. After ∀belard's choice for x_1, she chooses $y_1 := x_1 + 1$, and after his choice for x_2, she follows the just describes winning strategy for the subgame.

Conclusion Informational independence can be formalized by a quantification over a class of games.