Semantics of Informational Independence

Gabriel Sandu

October 21, 1999

First-order languages of imperfect information (IF first-order languages) have been introduced by Hintikka and Sandu [2]. These languages contained independent quantifiers and connectives of the form

$$(\exists x/W),\ (\forall x/W),\ (\vee/W),\ (\wedge/W)$$

where W is a set of variables ($x \notin W$), the values of which the quantifiers and the connectives on the left side of the slash are supposed to be independent of. The notion of a variable being free or bound is the same as in ordinary first-order logic. In the formula $(\exists x/W)\varphi$ the variables in W are free. In Hintikka and Sandu [3] only sentences receive a semantic interpretation, and this is done in terms of games $G(\varphi, A)$ of imperfect information played with an IF first-order *sentence* φ, and a model A. The game $G(\varphi, A)$ is played by two players, *Eloise* ($\exists$) and *Abelard* ($\forall$). The standard example which goes back to Henkin (1961) is with φ of the form $\forall x \exists y \forall z (\exists w/\{x\}) Rxyzw$. In this case the game can be pictured like this:

$$\begin{array}{ccc} \exists : & b & d/a \\ \hline \forall : & a & c \end{array}$$

Here a, b, c, d are individuals from the domain $\mid A \mid$ of the model, and 'd/a' means that when choosing the individual d *Eloise* did not "know" which individual a *Abelard* chose earlier. The notion of "not knowing" is made explicit in the strategies of the players: they are defined only on earlier "known" choices of the opponent. In other words, a strategy for *Eloise* in the game consists of two functions

$$f, g : \mid A \mid \to \mid A \mid$$

The truth of a sentence φ in a model A, $A \models \varphi$, is defined as the existence of a winning strategy for Eloise in the game $G(\varphi, A)$. In our particular example this means

$$A \models \varphi \Leftrightarrow A \models \exists f \exists g \forall x \forall z Rxf(x)zg(z)$$

The Hintikka-Sandu interpretation of IF first-order sentences in terms of games of imperfect information is clearly not compositional. Subformulas of a sentence, like for instance, $\exists w/\{x\}Rxyzw$, did not receive a semantical interpretation.

There exist at this moment three semantics for IF first-order sentences. Each of them is compositional and agrees with the Hintikka-Sandu interpretation on sentences. Each of them interprets an IF first-order $formula$ with respect to a set of assignments, and not with respect to a an assignment, as in ordinary first-order logic. Thus the key concept for all of them is

$$A \models_X \varphi$$

where X is a set of assignments. We regard an assignment f as a function $f : V \rightarrow \mid A \mid$, where V is a set of natural numbers.

The first to give such a semantics was Hodges [4]. The second-one is given by Caicedo and Krynicki [1], and the third one by Väänänen [6]. It is straightforward to show that the three interpretations are equivalent.

The basic idea in each of them should be clear from the following few clauses:

Definition 1 *(a)* $A \models_X \varphi \Leftrightarrow \forall x \in X(A \models \varphi(f(i_1), ..., f(i_m)))$, *with* $\varphi(v_{i_1}, ..., v_{i_m})$ *an atomic or the negation of an atomic formula.*

(b) $A \models_X \varphi \wedge \psi \Leftrightarrow A \models_X \varphi$ *and* $A \models_X \psi$

(c) $A \models_X \varphi \vee \psi \Leftrightarrow A \models_{X_0} \varphi$ *and* $A \models_{X_1} \psi$ *for some* X_0, X_1 *such that* $X = X_0 \cup X_1$.

(d) $A \models_X \exists v_n/\{v_{i_1}, ..., v_{i_k}\}\psi \Leftrightarrow (\exists F : X \rightarrow A)(A \models_{X[F,n]} \psi$ *and* F *is* $\{i_1, ..., i_k\}-$*uniform.*

Here $X[F, n]$ is the set which is obtained from X by adding to each of its sequences f $F(f)$ as its n'th element. For $F : X \rightarrow \mid A \mid, X \in Pow(A^V)$, and $T \subseteq V$,we say that F is $T-$uniform, if for all $f, g \in X$ we have

$$(\forall n \in V \backslash T)(f(n) = g(n) \Rightarrow F(f) = F(g))$$

(The presentation here is from Väänänen [6], but the same idea is present in Hodges [5] and Caicedo and Krynicki [1].)

The present paper is devoted to extend the interpretation above to modal propositional IF-languages. In these languages we shall have modal operators of the form

$$(\Diamond_n/\{W\}), (\Box_n/\{W\})$$

where W is a set of natural numbers. Here are some examples of IF modal sentences:

$$\Box_1\Box_2(\Diamond_3/\{1,2\})p,\ \Box_1(p \to \Box_2(\Diamond_3/\{1\})q)$$

The game interpretion for such *sentences* is completely analoguous to that for IF first-order sentences. The game is played on a model $M = (T, (R_i)_{i<n}, V)$, with T a set of possible worlds, R_i a binary relation on T, and V a valuation function, and starting in a possible world $w \in T$. The choices of *Abelard* and *Eloise* consists of possible worlds standing in the relevat accessibility relations. The interesting new element is to define a compositional semantics for these languages. This will be done *via* the concept

$$M \models_X \varphi$$

where X is a set of "histories" in M.

Bibliography

[1] X. Caicedo and M. Krynicki, forthcoming, 'Quantifiers for reasoning with imperfect information and Σ^1_1-logic'.

[2] J. Hintikka and G. Sandu, 1989, 'Informational Independence as a semantical phenomenon', in J. Fenstad et al. Logic, Methodology, and Philosophy of Science VIII, 571-589

[3] J. Hintikka and G. Sandu, 'Game-theoretical semantics', in J. van Benthem and A. ter Meulen (eds.) Handbook of logic and language, Elsevier Science B.V. 361-410.

[4] W. Hodges, 1997a, 'Compositional semantics for a language of imperfect information', Journal of the IPGL 5 (4), pp. 539-563.

[5] W. Hodges, 1997b, 'Some strange quantifiers', in J. Mycielski et al. (eds) Structures in logic and computer science, number 1261 in Lecture Notes in computer science, Springer, Berlin, pp. 51-65.

[6] J. Väänänen, forthcoming, 'On the semantics of informational independence'.