Information Transfer across Chu Spaces

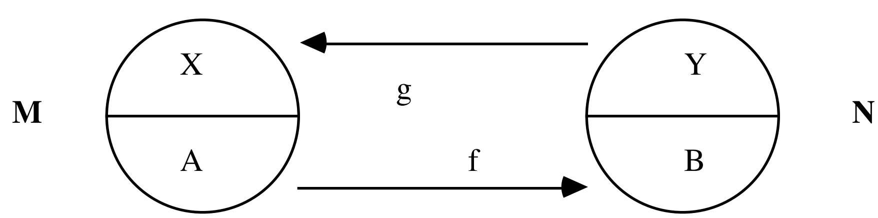
Chu spaces are a new model for information structure (cf. J. Barwise & J. Seligman, "Information Flow", Cambridge UP, 1997) and for mathematical structure in general (cf. Vaughan Pratt's ongoing work at the homepage http://boole.stanford.edu/live). Their properties are usually developed as a form of category theory. In this note, we show how they may also be viewed as models for a two-sorted first-order language, and we determine the exact flow of information across the natural Chu transforms. Our analysis is akin to that of process graphs via bisimulation and modal formulas.

1 Chu Spaces

A two-valued *Chu Space* is any structure (A, X, R) with two domains A, X and a binary relation R inside AxX. Examples: A = objects, X = sets, R = ∈, or A = models, X = formulas, R = |= , or A = 'tokens', X = 'types', R = 'classification by'. Such spaces are naturally viewed as models for a *two-sorted first-order language* with variables a over (as we shall say) 'objects' and variables x over 'types'. Of course, one can also use other languages extending first-order logic here, such as infinitary or second-order ones. General Chu Spaces have a k-valued relation R (which makes them 'fuzzy' rather than crisp 2-valued classification structures), but in practice two-sorted examples predominate.

2 Chu Transforms

A *Chu transform* between two Chu spaces **M** = (A, X, ∈), **N** = (B, Y, ∈) (we shall use the same notation '∈' for convenience across Chu spaces) is a pair of functions f: A→B, g: Y→X (note the inversion in direction!) satisfying the following condition:



for all a∈A, y∈Y: $f(a) \in y \;\Leftrightarrow\; a \in g(y)$

There are motivations galore for this 'contravariant ' equivalence, for instance, in the logical theory of relative interpretation (as emphasized by Barwise & Seligman).

3 Preservation and Flow Formulas

What information is preserved in switching between Chu spaces connected by such a transform? We can view this as a standard question in model theory, asking for a *preservation theorem*. The following syntactic notion is obvious from some reflection on what we have, and do not have, in the above diagram:

> a *flow formula* is any first-order formula produced
> by the schema $a{\in}x \mid \neg\, a{\in}x \mid \& \mid \vee \mid \exists a \mid \forall x$

Flow formulas $\phi\,(a_1, .., a_k, x_1, .., x_m)$ can define many useful notions on Chu spaces: in general, relations between k objects and m types. Here are some examples:

$\forall x\,(\neg\, a_1{\in}x \vee a_2{\in}x)$	$\leq$	'object inclusion'
$\forall x\,(\neg\, a_1{\in}x \vee \neg\, a_2{\in}x)$	▭	'object incompatibility'
$\exists a\,(a{\in}x_1\ \&\ a{\in}x_2)$	***o***	'type overlap'

Let us call a first-order formula ϕ *Chu-preserved* if we have (with bold-face symbols indicating 'fitting' finite tuples of objects and types):

> $\mathbf{M},\ \mathbf{a},\ \mathbf{g(y)} \models \phi$ *only if* $\mathbf{N},\ \mathbf{f(a)},\ \mathbf{y} \models \phi$
> whenever (f, g) is a Chu transform between **M** and **N**.

Of course, this notion also makes sense for non-first-order formulas ϕ.

<u>Proposition</u> All flow formulas are Chu preserved.

<u>Proof</u> This is a straightforward induction on the above definition, starting from the above characteristic Chu equivalence $f(a){\in}y \Leftrightarrow a{\in}g(y)$ for literals.

<u>Application</u> Chu transforms are *monotone* with respect to object inclusion, as the latter relation was defined by a flow formula.

<u>Comment</u> We have described preservation in the 'f-direction' only. But in the opposite 'g-direction', we have the following syntactic description of preserved formulas (pushing negations inward for the equivalent implication "$\mathbf{N} \models \neg\,\phi \Rightarrow \mathbf{M} \models \neg\,\phi$"):

$a \in x \mid \neg\, a \in x \mid \& \mid \vee \mid \forall a \mid \exists x$

This outcome is precisely what one would predict by the obvious *duality* of Chu Spaces, where interchanging of the roles of A and X makes no difference.

4 Application: Rigid Chu Spaces

The preceding analysis systematizes several separate observations about Chu transforms. Take the following 'rigid Chu space' constructed by Pratt. Objects A = {1, 2, 3, 4}, types X = {x, y, z, u}, classification $\in$ as in the following diagram:

x: {1}	y: {2}
z: {1, 3}	u: {1, 2, 4}

Plotkin and Pratt have shown that the only Chu transform sending (A, X, $\in$) to itself must be the *identity*. We can explain this by observing that each object $a \in A$ is *uniquely definable* by a flow formula, and therefore, it must 'land on itself' by our Proposition. Let the relations $\leq$, $=$ be as in Section 3 above. Here are the definitions:

(a) $\exists a_1 a_2\, (a_1 \leq \mathbb{1} \ \&\ a_2 \leq \mathbb{1} \ \&\ a_1 = a_2)$ is unique for object $\mathbb{1}$

(b) $\forall x\, (\mathbb{1} \in x \vee \mathbb{2} \in x)$ is unique for object $\mathbb{2}$
(one uses the flow definition for object $\mathbb{1}$ here:
$\exists a\, (DEF_{\mathbb{1}}(a) \ \&\ \forall x\, (a \in x \vee \mathbb{2} \in x))$.)

(c) $\forall x\, ((\mathbb{1} \in x \ \&\ \mathbb{2} \in x) \rightarrow \neg\, \mathbb{3} \in x)$ is unique for object $\mathbb{3}$

(d) $\mathbb{4} \leq \mathbb{1} \ \&\ \mathbb{4} \leq \mathbb{1}$ is unique for object $\mathbb{4}$

5 Flow Preservation Implies Chu Tranform on Finite Models

Now let us convert the result, showing that the above is 'best possible'. Up to logical equivalence, *only* flow formulas are preserved under Chu transforms. We will formulate this as a preservation theorem in Section 6. But before proving this, we give a warm-up result inspired by an analogy with bisimulation and modal logic. (Some further aspects to this modal analogy that can be usefully exploited have been pointed out by Martin Otto.) The proof that follows here contains some key ideas for the later one.

<u>Proposition</u> For *finite* Chu spaces **M**, **N**, the following assertions are equivalent:

(i) there exists a Chu transform from **M** to **N**

(ii) every flow sentence true in **M** is also true in **N**

<u>Proof</u> (i) $\Rightarrow$ (ii) is a special case of our earlier Proposition. (ii) $\Rightarrow$ (i) works as follows. Enumerate A as $\{a_1, ..., a_k\}$ and Y as $\{y_1, ..., y_m\}$. We do one case of a stepwise construction for the desired function f . (The remaining case, as well as the construction of the contravariant companion function g , are similar.) The idea is that, progressively, f should assign some object $f(a) \in B$ to $a \in A$ which satisfies the same flow properties (where the latter may involve parameters for objects which have already been matched).

Suppose that no $b \in B$ satisfies all flow properties that hold for a_1 in **M**. That is, for some flow formula γ_b , we have that

$$\mathbf{M}, a_1 \models \gamma_b \quad \text{but } \textit{not } \mathbf{N}, b \models \gamma_b$$

Altogether, **M** then satisfies the flow formula (here we use $\exists$ | & – closure)

$$\exists a \ \&_{b \in B} \ \gamma_b$$

But by condition (ii), the latter formula should hold in **N**. But, each $b \in B$ is disqualified as a witness for this, since it lacks 'its' conjunct γ_b .Therefore, by reductio ad absurdum, a 'good choice' b_1 must exist after all, and we can set

$$f(a_1) = b_1$$

This argument can be repeated to produce the successive values for f on all of A. Moreover, it can also be used in the opposite direction to find values for the function g, again maintaining the 'invariant' that flow formulas be preserved going from **M** to **N**. E.g., when searching for a matching type x for y_1, one assumes that each x_i fails for this purpose with a 'defect' δ_y , and then uses a flow formula of the form

$$\forall x \ \bigvee_{y \in Y} \ \delta_y$$

(with a dual use of $\forall$ | v – closure) to obtain a contradiction, going from **M** to **N**.

6 A First-Order Preservation Theorem

Instead of a standard preservation theorem, we formulate a slight strengthening in terms of 'generalized interpolation' (cf. Barwise & van Benthem 1998). Let us say that

ϕ *implies* ψ *along Chu transforms* if always

$$\mathbf{M}, \mathbf{a}, \mathbf{g}(\mathbf{y}) \models \phi \quad \textit{only if} \quad \mathbf{N}, \mathbf{f}(\mathbf{a}), \mathbf{y} \models \psi$$

Then we have the following result:

<u>Theorem</u> For all first-order formulas ϕ,ψ, the following statements are equivalent:

(i) ϕ implies ψ along Chu transforms

(ii) there exists a flow formula α such that $\phi \models \alpha \models \psi$

<u>Proof</u> (ii) $\Rightarrow$ (i) is again essentially the earlier Proposition. As for (i) $\Rightarrow$ (ii), assume that ϕ implies ψ along Chu transforms. First, define

$$FF(\phi) =_{def} \{ \alpha \text{ a flow formula} \mid \phi \models \alpha \}$$

It suffices to prove that

<u>Claim</u> $FF(\phi) \models \psi$

The required flow interpolant then exists by Compactness. So consider any countable model $\mathbf{N} = (B, Y, \in)$ for $FF(\phi)$. (This case suffices by the Löwenheim-Skolem theorem.) Let $Th_{\neg FF}(\mathbf{N})$ be the set of all $\mathbf{N}$-true *negations of* flow sentences. By a routine argument, using the closure of flow formulas under disjunctions, we have that

$Th_{\neg FF}(\mathbf{N}) \cup \{\phi\}$ is *finitely satisfiable*

Therefore, there is a (countable) model $\mathbf{M} = (A, X, \in)$ for $Th_{\neg FF}(\mathbf{N}) \cup \{\phi\}$, so that the following implication holds for all flow formulas γ :

$$\mathbf{M} \models \gamma \Rightarrow \mathbf{N} \models \gamma$$

Without loss of generality, we can even assume that $(\mathbf{M}, \mathbf{N})$ is a *recursively saturated model pair* with the same transfer property. But then we can mimick the earlier argument for finite models, this time, using the recursive saturation. Enumerate A as $\{a_1, a_2, ...\}$ and Y as $\{y_1, y_2, ...\}$. In the general case, suppose that some finite part of the pair (f, g) has already been constructed. Moreover, assume that all flow formulas whose free variables are set to objects a in the domain of f and types $g(y)$ in the range of g in $\mathbf{M}$, and to the corresponding items $f(a)$, y in $\mathbf{N}$, satisfy the following implication:

$$\mathbf{M}, \mathbf{a}, \mathbf{g(y)} \models \gamma \quad \textit{only if} \quad \mathbf{N}, \mathbf{f(a)}, \mathbf{y} \models \gamma \qquad \#$$

Then we can extend this situation both ways. Here is the case for objects in **M** (that for types in **N** is similar). Let a^* be the first object in A without an f-value. Consider the recursive(!) set of all formulas of the following shape, where γ runs over flow formulas as in the preceding implication – except that there is one free object variable $\underline{a}$ on the right:

$$\gamma^M (\mathbf{a}, \mathbf{g(y)}, a^*) \to \gamma^N (\mathbf{f(a)}, \mathbf{y}, \underline{a})$$

This set is finitely satisfiable in the model pair (**M**, **N**), because for any finite number of flow properties γ_i of a^* in **M**, we can form the flow statement (by $\exists$ | & – closure)

$$\exists a \ \&_i \ \gamma_i (\mathbf{a}, \mathbf{g(y)}, a)$$

which is also true in **N** (by the earlier transfer implication #). So, we can find a value for $\underline{a}$ in B satisfying all these finitely many implications. But then, because of recursive saturation (our set is recursive, with only finitely many parameters from the domain), there is even some $b \in B$ satisfying this whole set of flow implications *simultaneously*, and we can choose this object to be the desired f-value for the object a^*. The argument in the opposite direction, producing a suitable g-value in **M** for the first virgin type $y \in Y$, is analogous, but now using the $\forall$ | $\vee$ – closure of flow formulas.

Then, finally, we have $\mathbf{M} \models \phi$, **N** is a Chu transform of **M**, and so $\mathbf{M} \models \psi$:

Remark Sol Feferman (Stanford Logic Seminar, June 1998) has given an alternative proof for this preservation result using his interpolation theorem for many-sorted first-order logic. At its present state, this argument only covers Chu transforms with injective object maps. But it can presumably be modified to deal with the full case. Martin Otto has subsequently generalized this analysis using a translation with subset-bounded quantifiers ("An Interpolation Theorem", department of computer science, RWTh. Aachen), to deal with full Chu transfer, as well as other more refined aplications of interpolation.

7 Applications, Variations and Extensions

The preceding result tells us precisely *how much* (or perhaps, *what little*) information is passed between Chu spaces that are related by their 'natural equivalence', at least for their *first-order* language. Here are two standard mathematical examples of this transfer, using flow syntax to analyse intuitive notions of topological or category-theoretic property.

(1) *Continuous maps between topological spaces*. In this case, structures (A, X) consist of the points A of the space, plus the open sets X , and their relation is membership. Any continuous point-map f from a space (A, X) to (B, Y) induces a Chu transform, as its inverse is a map from Y to X (inverse images of open sets are open) standing in the right relationship: $f(a) \in y$ iff $a \in f^{-1}(y)$. It is often said that 'continuous maps preserve important topological structure'. Our analysis gives this a syntactic sense. E.g., continuous maps preserve boundary points, and this may be seen by the flow formula

$$\forall y\, (a{\in}y \rightarrow \exists b(b{\in}y \;\&\; b{\in}x)) \qquad (\text{" } a \text{ is a boundary point of } x \text{ "})$$

By contrast, the topological property of being T_2 is not flow-definable, and indeed not preserved under continuous maps. Other properties require flow syntax with infinitary conjunctions or disjunctions, discussed later. E.g., surjective continuous maps preserve compactness of topological spaces, which can be formulated in infinitary flow form.

(2) *Adjunctions in category theory*. Again, these are connections between categories, that are said to transfer 'natural categorial properties'. Consider the special case of *pre-orders* **M**, **N** with a left-adjoint functor pair between them: $Fa \leq b$ iff $a \leq Gb$. The relevant Chu spaces have just one set, doubling for A and X – while the relation is the pre-order $\leq$. We will analyze the syntax of this in Section 12 below, but note for now that the typical examples of category-theoretic transfer in this setting are finite (co-)limit constructions. And indeed, a basic construction like " a is the *supremum* of b, c ", which is typically preserved under left-adjunctions, has the following flow definition:

$$b{\leq}a \;\&\; c{\leq}a \;\&\; \forall x\, ((b{\leq}x \;\&\; c{\leq}x) \rightarrow a{\leq}x)$$

Next, we consider some variations and extensions.

(3) The result goes through for special classes of Chu spaces, provided that these have first-order definitions. This holds in particular for *bi-extensional* Chu spaces, satisfying

$$\forall a \forall b\, (a{=}b \leftrightarrow \forall x(a{\in}x \leftrightarrow b{\in}x))$$
$$\forall x \forall y\, (x{=}y \leftrightarrow \forall a(a{\in}x \leftrightarrow a{\in}y))$$

In practice, this means that for bi-extensional Chu spaces, one can use two further atoms in flow formulas, without affecting preservation:

$$\neg\, x{=}y, \;\; a{=}b$$

(2) But also, the above proof itself can easily be modified to yield further preservations. For instance, if we know that the f-map in a Chu transform is *surjective*, then we can add *universal object quantifiers* $\forall a$ in the construction of flow formulas, and likewise, *existential type quantifiers* $\exists x$ if g is surjective.

(3) Finally, a first-order perspective also suggests *other equivalences* for Chu spaces, such as *elementary equivalence*, *potential isomorphism*, or pebble game variants thereof. In particular, such notions need no longer be based on function pairs. Even Chu transfer itself might be formulated using many-valued relations of some sort. For a concrete illustration, consider again topology. Besides continuous maps, there are the stronger *homeomorphisms*, preserving more structure. But there are also weaker relations of topological *bisimulation* (M. Aiello & J. van Benthem, "Visual Structure as Modality", ILLC, Amsterdam, 1999) that match points and sets in a much looser fashion, and yet transfer significant statements. From the perspective of information flow, there is no need for one unique model equivalence: the more structure preservation one can get, the better!

8 Infinitary Versions: Information Sequents

The preservation proposition of Section 3 still holds for arbitrary *infinitary* conjunctions and disjunctions in flow formulas. This explains the observations found in Barwise & Seligman 1996 on transfer (and non-transfer) of 'local logics' along 'infomorphisms'. In their terminology, let U, V be sets of types in **M** . We define true sequents:

$$U \vdash_{\mathbf{M}} V \qquad \text{if} \qquad \forall a\colon a \in \bigcap U \rightarrow a \in \bigcup V$$

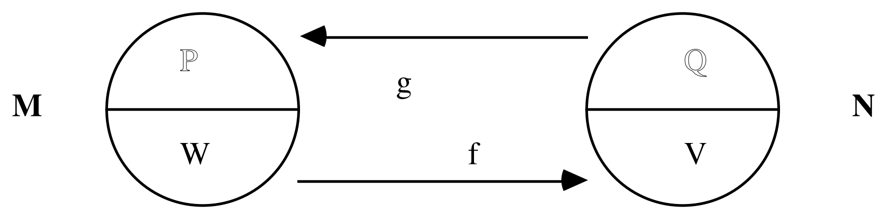
This infinitary definition is not an flow formula (as their maps f need not be surjective, universal object quantifiers are not allowed), and it is not preserved by Chu transforms. The implication will only hold in **N** on the image of f, the so-called 'normal tokens' in **N**. Thus, logically true sequents do not transfer in the f-direction. But they do transfer in the opposite g-direction, as the negation $\exists a\colon a \in \bigcap U \ \&\ \neg\, a \in \bigcup V$ *is* equivalent to an infinitary flow formula. (Barwise & Seligman do not consider further flow properties.)

This application increases the interest of an infinitary version of our preservation result. We conjecture that this is the case. But so far, we have only checked that the techniques of Barwise & van Benthem 1996 go through. These apply to model relations that can be cast in the form of pebble games. Applied to Chu transforms, this means the following. Instead of total maps, we now have a non-empty family of finite partial maps (f, g) between **M**, **N**, which satisfy the basic Chu equivalence for atoms, such that two back-and-forth properties hold, one extending each f-domain with an object from **M**, and one extending each g-domain with a type from **N**. Let us call these *potential Chu transforms*.

<u>Theorem</u> The above preservation theorem extends to formulas ϕ, ψ in the infinitary language $L_{\infty\omega}$, when we require preservation along potential Chu transforms.

9 Richer Chu Spaces: General Frames in Modal Logic

In modal logic, the natural Chu spaces are general frames $(W, \mathbb{P}, \in)$ with W a set of worlds, $\mathbb{P}$ a family of sets of worlds (the 'admissible propositions') and $\in$ membership. Here the natural equivalence is as in the following 'contravariant' picture (cf. Blackburn, de Rijke & Venema 1998, van Benthem 1985), where *f(w)* $\in$*) q iff w* $\in$ *g(q)* :



But here there is an additional requirement: f must be a *p-morphism* from **M** to **N**. I.e., it is a homomorphism for the accessibility relation $R^{\mathbf{M}}$, and it satisfies the zigzag clause

if $f(w) R^{\mathbf{N}} v$, then there exists some $u \in W$ such that $w R^{\mathbf{M}} u$ & $f(u) = v$

Moreover, the map g is just the set-theoretic inverse f^{-1} on $\mathbb{Q}$ (landing inside $\mathbb{P}$!), which is a *homomorphism* with respect to the natural 'modal algebra' structure on $\mathbb{Q}$, $\mathbb{P}$. (This 'parasitic' nature of g is also known from Chu spaces in general.) Modal logicians have proved preservation theorems in this setting. But of course, more is preserved now, as the 'quality' of f, g is higher than in the above. In particular, flow formulas will now also allow the usual modal constructions, or more precisely:

atoms Rab | bounded universal object quantifiers $\forall b\ (Rab \rightarrow$

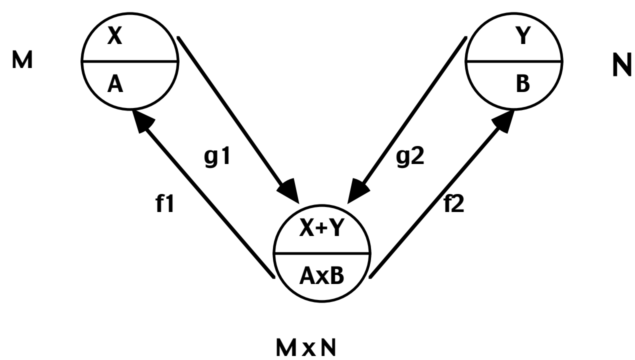
Combining this with the earlier syntax of flow formulas, we see that we get propositional literals p, ¬p, conjunction, disjunction, existential and universal modalities, plus arbitrary existential object quantifiers and universal propositional quantifiers. This includes all standard modal formulas, with a slight first-order and 'second-order' extra. This is surely an instance of a more general result, telling us how to 'load' our general Chu preservation with extra information from maps f, g that preserve special structure.

10 Constructions on Chu Spaces

The theory of Chu spaces gives a prominent place to (categorial) model *constructions*. One example is the *dual* operation taking (A, X, ∈) to (X, A, ∋). Another is the product **M**x**N** used extensively in Barwise & Seligman 1996:

new object	AxB (Cartesian product)
new types	X **+** Y (disjoint union)
new epsilon	(a, b) ∈ X^1 iff a∈X, (a, b) ∈ Y^2 iff b∈Y

Here the picture of natural connections is as follows:



Here, preservation results might characterize formulas φ such that, if both **M** |= φ and **N** |= φ, then **M**x**N** |= φ. (In particular, this holds for first-order Horn sentences.) But one does not want 'preservation' here so much as *combination* of information, or viewed in the other direction, *decomposition*. If we have a tight constructive definition of some operation on Chu spaces, then we can use it to reduce first-order evaluation.

<u>Example</u> 'Every Type is Inhabited'

Here is a simple semantic calculation from the given definitions:

Mx**N** |= ∀x ∃a a∈x	iff
∀x∈X ∃a∈A ∃b∈B : (a, b)∈x & ∀y∈Y ∃a∈A ∃b∈B : (a, b)∈y	iff
∀x∈X ∃a∈A a∈x & ∀y∈Y ∃b∈B : b∈y	iff
M |= ∀x ∃a a∈x & **N** |= ∀x ∃a a∈x	

So in this particular (Horn-definable!) case, the property does reduce to its presence in the components. In general, however, we don't expect this. Nevertheless, the example suggests an *effective component reduction* for arbitrary first-order statements $\phi^{\mathbf{M}x\mathbf{N}}$:

(a) introduce a supply of marked variables with superscripts for A, B, X, Y
(b) replace object quantifiers $\exists a$ by $\exists a^A \exists b^B$, and replace corresponding atoms $a \in x$ in the formula by disjunctions $a^A \in x \vee b^B \in x$
(c) replace type quantifiers $\exists x$ by disjunctions $\exists x^X ... \vee \exists x^Y ...$
(d) replace (using the added markings) all 'heterogeneous' atoms $a^A \in x^Y$ or $b^B \in x^X$ by <u>false</u>

The result is a first-order formula which may be *separated* into an equivalent Boolean compound of separate fist-order assertions about **M** and **N**.

Sol Feferman has pointed out a more general background here. Chu dual and product satisfies the following preservation property (with '≡' for elementary equivalence):

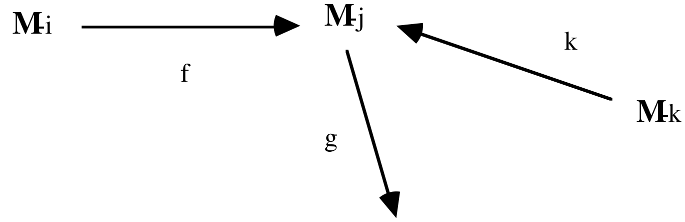
if $\mathbf{M} \equiv \mathbf{M'}$ and $\mathbf{N} \equiv \mathbf{N'}$, then $\mathbf{M}x\mathbf{N} \equiv \mathbf{M'}x\mathbf{N'}$

Most operations in abstract process algebra have this feature (Hollenberg 1998). On the other hand, product spaces in the usual mathematical sense, whose objects are functions, do not (cf. the references given in Section 4). One obvious conjecture is that Chu *tensor product* as defined by Pratt in his model for linear logic, lacks this preservation property.

Preservation of elementary equivalence is a consequence of the above effective reduction. But having an effective decomposition seems stronger. So we want to know about both.

11 Co-limits and Generalized Evaluation

Dual and product were just two examples. The natural general construction is an *inverse limit* of families of Chu spaces which may have Chu transfoms running between them:



'Objects' in the inverse limit $\mathbb{M}$ are tuples **a** having the right 'coherence': e.g.,

$f((\mathbf{a})_i) = (\mathbf{a})_j$, etcetera.

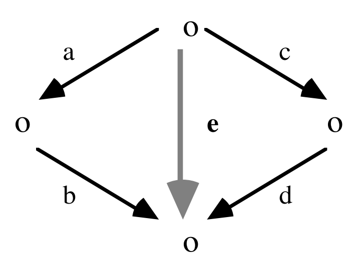
This setting makes it much harder to do a 'logical decomposition' as above. It rather suggests that we generalize our perspective once more. One could think of evaluation of formulas in $\mathbb{M}$ as a *generalized semantic*s, where we have a family of models available instead of just one. We then sometimes shift (a bit like in some recent semantics for modal predicate logic) from looking for an object in one model to some image in another. This theme of 'long-distance evaluation' is elaborated in Section 8 of J. van Benthem, "Dynamic Odds and Ends", ILLC, Amsterdam, in the study of 'information links', and also, as a strategy for 'decomplexifying' higher-order logic, in Section 10.3.

12 Appendix: Categorial Excursion

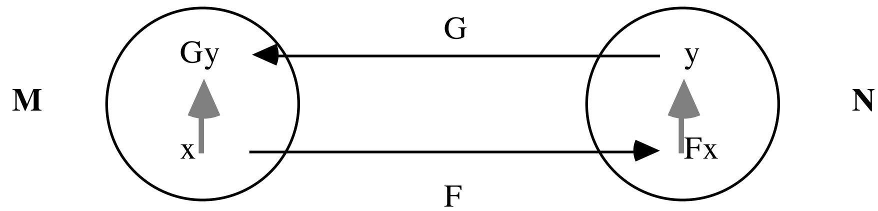
The Chu analysis of this paper may be viewed as a standard model-theoretic account of one form of *category-theoretic adjunction*. Can we carry this style of thinking further? First, we need to see a category as a logical model. Disregarding the issue of sets versus classes, we have again *two-sorted models*, with objects and morphisms (or 'arrows'). Here is one useful relational similarity type, suppressing – as a first stab – the usual uniqueness conditions of standard category theory (this is more like 'Arrow Logic'):

D ax	object x is the source of arrow a
R ax	object x is the target of arrow a
I a	a is an identity arrow
C a, bc	arrow a is a composition of b, c (in that order)

Categories impose first-order conditions definable in this language, such as associativity. And e.g., commutativity of a diagram with arrows a, b, c, d would be expressed by stating that some suitable arrow runs through the middle: $\exists e$: C e, ab & C e, cd:



Now, category theory has many notions of structure-preserving transformation: plain, faithful or full functors, functor pairs that form an isomorphism, adjunctions. These will in general preserve different categorial properties expressed in our language. Consider left or right *adjoint functor pairs*. The simple case of pre-orders is pictured as follows:



where $Fx \leq y$ iff $x \leq Gy$. As was pointed out in Section 7, there is a complete syntactic description of the first-order formulas that are preserved according to the schema:

$$\mathbf{M}, \mathbf{x}, \mathbf{G(y)} \models \phi \qquad \textit{only if} \qquad \mathbf{N}, \mathbf{F(x)}, \mathbf{y} \models \phi$$

We must make a distinction between two kinds of variables: a, ..., y, ... as noted before, even though they are running over the same domain. Then we may allow all forms

$$(\neg)a \leq y \mid \& \mid \vee \mid \exists a \mid \forall y$$

Moreover, we may freely use atoms $a \leq b$ as well, since these are definable in pre-orders via the flow formula $\forall y\, (b \leq y \rightarrow a \leq y)$, and the same applies to literals $\neg y \leq z$, since these are definable by the flow formula $\exists a\, (a \leq y \,\&\, \neg a \leq z)$. As is easily shown, the relation "supremum" is definable in this form, and so are all finite co-limit constructions.

But what about *arbitrary adjunctions*? Here one requires that the adjoint functors should also provide *natural bijections between the arrow sets* from x to Gy and from Fx to y. One can get a glimpse by analyzing specific preservation arguments in this setting, e.g. for co-limits. I do not know yet what to do in general, but here are some observations. Assume that F, G are also maps on arrows, not just on objects. Moreover, assume that the arrow isomorphism validates the following strong homomorphism properties:

(a) if $a: x \rightarrow y$, then $Fa: Fx \rightarrow Fy$,
if $b: Fx \rightarrow y$, then $Gb: x \rightarrow Gy$
plus all similar morphism/object fit principles

(b) if C Gd, ab , then C d, Fa Fb

These properties help us preserve all formulas constructed as follows. (Again a split in the variables is needed, now for both objects: x versus y , and morphisms: a versus b .)

atoms	Dax, Rax, Ray, Ca, a_1a_2, C b, a_1a_2
literals	¬Dbx, ¬Rby
operations	& ∣ v ∣ ∃x ∣ ∃a ∣ ∀y ∣ ∀b

where safe transitions now have the form

$$\mathbf{M}, \mathbf{a}, \mathbf{x}, \mathbf{G(b)}, \mathbf{G(y)} \models \phi \qquad \textit{only if} \qquad \mathbf{N}, \mathbf{F(a)}, \mathbf{F(x)}, \mathbf{b}, \mathbf{y} \models \phi$$

The best way of understanding how this preservation works is by drawing a picture for the case of categorial sums, which generalizes the earlier example of the supremum.