Characterizing Interpolation Pairs in Infinitary Graded Logics

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1 Introduction

Interpolation is a desirable property linking syntax and semantics that a well behaved logic is supposed to have. However, there are useful logics where interpolation fails and for which we would be interested in a semantical characterization of pairs of formulae having an interpolant in the logic. More precisely, for these logics we look for a semantical property of pairs of formulae which is stronger than entailment but equivalent to the existence of an interpolant. In [4], van Benthem and Barwise found this characterization for Infinitary Logic: here, to characterize interpolation pairs (ϕ, ψ) (that is: pairs of formulae having an interpolant) it suffices to strengthened the condition of entailment $\phi \models \psi$ to the condition of entailment along potential isomorphism in the common language. Since two structures $\mathcal{M}, \mathcal{N}$ are potentially isomorphic w.r.t. a given language $\mathcal{L}$ iff they satisfy the same infinitary formulae of $\mathcal{L}$ (notation: $\mathcal{M} \equiv_{\mathcal{L}_\infty} \mathcal{N}$), we have that ϕ entails ψ along potential isomorphism iff for all models $\mathcal{M}, \mathcal{N}$ if $\mathcal{M} \models \phi$ and $\mathcal{M} \equiv_{(\mathcal{L}(\phi) \cap \mathcal{L}(\psi))_\infty} \mathcal{N}$, then $\mathcal{N} \models \psi$ (notation $\phi \models_{\equiv_\infty} \psi$). The Interpolation Theorem for Infinitary Logic can be stated as follows:

the pair (ϕ, ψ) has an interpolant iff $\phi \models_{\equiv_\infty} \psi$.

In general, we say that a logic L has elementary interpolation if interpolation pairs can be characterized as the pairs (ϕ, ψ) for which $\phi \models_{\equiv_L} \psi$ (here the relation $\models_{\equiv_L}$ is defined as above with the logic L instead that $\mathcal{L}_\infty$). In the realm of finitary logics this is a weak condition, because it can be proved that any compact logic enjoys elementary interpolation. On the other hand, this is not always true when the logic is not compact, and Monadic Second Order Logic and First Order Logic over finite models are example of logics for which even elementary interpolation fails (see [7]).

The proof of elementary interpolation for Infinitary Logic (and for its modal fragment) is given in [4] by using the Boundedness Theorem, and the proof has been generalized in [3] to deal with interpolation in *nice* fragment of Infinitary Logic. A different technique, generalizing the notion of consistency property for Countable Infinitary Logic, is proposed in [6]. This is a method applying to other logics than Infinitary Logic: for example, it can be used to prove elementary interpolation for Infinitary Modal Logic or the finite variable fragment of First Order Logic. The general idea underlying this technique is that for certain logics the relation of *elementary equivalence* between models (i.e. to satisfying the same formulae of the logic) is equivalent to the existence of a relation between the two models satisfying certain *combinatorial* properties (i.e. potential isomorphism for Infinitary logic, or bisimulation for Infinitary Modal Logic). If ϕ, ψ are formulae of the logic which do not have an interpolant, then one can use the generalized consistency properties to build *two related models* (which are then elementary equivalent) one satisfying ϕ and the other $\neg\psi$. This implies that $\phi \not\models_{\equiv_L} \psi$ and the non-trivial implication of elementary interpolation is proved. This method has been extensively applied to prove interpolation and preservation theorems for various logics in [6], and in [20] to obtain these results for Countable Infinitary Modal Logic.

In this paper we apply the generalized consistency method to prove elementary interpolation for the family of countable infinitary graded modal logics. The problem of interpolation for the first order version of these logics has already been considered by Andréka in [1]. She showed that the behavior of the logics in the class is not uniform: logics with a finite number of graded operators do not enjoy interpolation while the whole logic of Finite Graded Modalities does. On the other hand, being compact, all these logics have elementary interpolation, but the same question for the infinitary versions cannot be solved so easily. In this paper we prove elementary interpolation for the family of countable infinitary graded modal logics together with a preservation theorem w.r.t. their classical environments. After finishing a preliminary version of this paper we discovered that these results can also be obtained by applying the general technique described in [3]. In this paper we also improve the result of elementary interpolation for the case of the infinitary countable graded logic containing all finite graded operators.

Let us briefly recall the proof of interpolation in Countable Infinitary First Order Logic, to see how it can be modified to prove elementary interpolation for graded modal logics.

If a pair (ϕ, ψ) of countable infinitary first order formulae does not have an interpolant, a model for $\phi \wedge \neg\psi$, showing that $\not\models \phi \rightarrow \psi$, can be constructed as follows. Starting from the set $s_0 = \{\phi, \neg\psi\}$, we generate an increasing sequences $s_0 \subseteq s_1, \ldots, \subseteq s_n, \ldots$, of sets of formulae in such a way that each s_n can be divided into two subsets, s_n^0, s_n^1 with the following properties: all formulae in s_n^0 and s_n^1 are in the language of ϕ, ψ, respectively, augmented with new constants, and the pair $(\wedge s_n^0, \neg \wedge s_n^1)$ does not have an interpolant in the common language. If $s_n = s_n^0 \cup s_n^1$ is given, the set s_{n+1} is obtained by *reducing* a formula in one of s_n^0, s_n^1, towards atomic sentences, being careful to preserve the non existence of an interpolant between two subsets of s_{n+1}. For example, if $\exists x\theta$ belongs to the set s_n^0, we can add a constant c to the language, taken from a countable set of C of new constants and consider the sets $s_{n+1}^0 = s_n^0 \cup \{\theta(c)\}$, $s_{n+1}^1 = s_n^1$, and $s_{n+1} = s_{n+1}^0 \cup s_{n+1}^1$. If $\wedge s_n^0$ and $\neg \wedge s_n^1$ do not have an interpolant, then $\wedge s_{n+1}^0$ and $\neg \wedge s_{n+1}^1$ do not have one as well. One can then prove that in ω-steps a set $s^* = \bigcup s_n$ is reached, whose sentences are either atomic or have been reduced in previous steps. The atomic sentences of s^* are then used to build a model from the set of constants C which verifies all sentences in s^*. In particular, this model verifies $\phi \wedge \neg\psi$.

If a logic does not have interpolation then there is a pair of formulae (ϕ, ψ) in the logic with $\phi \models \psi$ such that ϕ and ψ do not have an interpolant: if we start with the previous construction from this pair, we certainly cannot hope to build a model for $\phi \wedge \neg\psi$, but if we reduce our expectations then in certain cases we can adapt the method of consistency property to prove elementary interpolation. In case of the graded modal logics considered in this paper, we first give a notion of *counting bisimilar* models (the combinatorial property announced above) characterizing elementary equivalence between models. Then, starting from a pair ϕ, ψ without an interpolant in the logic, we shall build two models $\mathcal{M}, \mathcal{N}$ which are counting bisimilar w.r.t. the language $\mathcal{L}(\phi) \cap \mathcal{L}(\psi))$, one verifying ϕ and the other verifying $\neg\psi$. This suffices to prove elementary interpolation for the logic in consideration. The construction of the two models is similar to the construction of a single model for $\phi \wedge \neg\psi$ in the case of Countable Infinitary First Order Logic, but we now use two disjoint sets of constants C, D: one will be used to build $\mathcal{M}$ and the other to build $\mathcal{N}$; moreover, during this construction we also fix *links* between C, D in such a way that at the end we can use these links to recognize a counting bisimulation between $\mathcal{M}$ and $\mathcal{N}$.

This is just the general idea of a proof of interpolation in the family of graded logics using consistency property modulo bisimulation. As in [6], this technique can also be used to prove a preservation theorem for Countable Infinitary First Order Logic versus its Graded Fragment. For the sake of simplicity, we divide the proof in two parts. In Section 3 we deal with a general notion of consistency property modulo bisimulation and in Section 4 we apply this notion to give a proof of preservation and elementary interpolation for countable infinitary graded logics. Finally, in Section 5 we consider the Infinitary Logic of Finite Graded Modalities $\mathcal{L}^{\diamond<\aleph_0}_{\omega_1}$: in this case we improve the result of the previous section and show that this logic enjoys Craig interpolation.

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2 Notation and Preliminaries

In this part we introduce the basic material (definitions, notations, and preliminary results) that are needed to read this paper.

All languages considered contain a countable set of unary predicates PROP, a countable set of binary predicates (indexed by a set A of atomic action), one constant, and no function symbols. If $\Sigma \subseteq$ PROP and $\Lambda \subseteq A$, we denote by $\mathcal{L} = (\Sigma, \Lambda)$ the *language* built up from Σ and Λ as follows: there is a single constant r (for 'root'), each $p \in \Sigma$ is a unary predicate symbol of $\mathcal{L}$ and for each $a \in \Lambda$ there is a binary relation symbol R_a in $\mathcal{L}$.

If $\mathcal{L} = (\Sigma, \Lambda)$ is a language, then a *structure* $\mathcal{M}$ on $\mathcal{L}$ consists of a domain $D^{\mathcal{M}}$, a root $r^{\mathcal{M}} \in D^{\mathcal{M}}$, a set of unary relations $p^{\mathcal{M}}$ on $D^{\mathcal{M}}$, for $p \in \Sigma$, and a set of binary predicates $R_a^{\mathcal{M}}$ on $D^{\mathcal{M}}$, for $a \in \Lambda$. We write $\mathcal{M}$ instead of $D^{\mathcal{M}}$ and wR_aw' instead of $wR_a^{\mathcal{M}}w'$. We also use this convention when two models $\mathcal{M}, \mathcal{N}$ are present and we avoid confusion on the binary relation R_a by stipulating that the domains of $\mathcal{M}, \mathcal{N}$ are disjoint: hence uR_au' means $uR_a^{\mathcal{M}}u'$ if $u \in \mathcal{M}$ and $uR_a^{\mathcal{N}}u'$ if $u \in \mathcal{N}$. If $w \in \mathcal{M}$ and $a \in \Lambda$, we write $R_a(w)$ for the set $\{v \in \mathcal{M} : wR_av\}$. If $\mathcal{M}$ is an $\mathcal{L}$-structure, we denote by $(\mathcal{M}, s)$ the structure $\mathcal{M}$ with s as a new root.

We can express various properties of an $\mathcal{L}$-structure by means of *Countable Infinitary Logic*, which is the extension of First Order Logic on $\mathcal{L}$ that allows infinitary conjunctions on countable sets of formulae. We denote it by $\mathcal{L}_{\omega_1}$. The set of formulae of *Basic Modal Logic* based on $\mathcal{L} = (\Sigma, \Lambda)$ is the smallest set containing any $p \in \Sigma$ and closed under the operators $\phi \wedge \psi, \neg\phi$, and $\langle a \rangle \phi$, for any $a \in \Lambda$. We denote the Basic Modal Logic of $\mathcal{L}$ by $\mathcal{L}^{\diamond}$, and we interpret its formulae on a structure $\mathcal{M}$ in the standard way (e.g. $\mathcal{M} \models p$ iff $r^{\mathcal{M}} \in p^{\mathcal{M}}$, $\mathcal{M} \models \langle a \rangle \phi$ iff there exists w with $r^{\mathcal{M}}R_aw$ and $(\mathcal{M}, w) \models \phi$.) We also use the defined operator $\phi \vee \psi, \phi \to \psi, [a]\phi := \neg\langle a \rangle \neg\phi$, and the extension $\mathcal{L}^{\diamond}_{\omega_1}$ of $\mathcal{L}^{\diamond}$ which allow infinitary conjunctions on countable sets of formulae. Via the above semantics, one can easily define a translation ST (known as the standard translation) from the logic $\mathcal{L}^{\diamond}_{\omega_1}$ into $\mathcal{L}_{\omega_1}$.

The idea of graded operators is introduced in the seventies (see [11], [14], [10]) and studied in the eighties in [9], where a sound and complete system for the logic $\mathcal{L}^{\diamond<\aleph_0}$ defined below is proposed; it is a simple and pervasive notion, that can be naturally connected with many other formalisms coming from the areas of generalized quantifiers theory and knowledge representation. A survey of such connections is presented in [13]. Many technicals results and classical tools from modal model theory can be applied to graded modalities. A detailed study of expressiveness issues using such instruments can be found in [12].

The basic idea of the graded modalities is to extend Basic Modal Logic with *graded operators* that give the possibility of counting successors: we define, for any natural number n and action $a \in \Lambda$, the operator $\langle a \rangle_n \phi$, whose interpretation in a structure $\mathcal{M}$ is given by $\mathcal{M} \models \langle a \rangle_n \phi$ iff the cardinality of the set $\{w \in R_a(r^{\mathcal{M}}) : (\mathcal{M}, w) \models \phi\}$ is greater than n. The *Logic of Finite Graded Modalities* $\mathcal{L}^{\diamond<\aleph_0}$is defined as Basic Modal Logic extended with the operators $\langle a \rangle_n \phi$ for $n < \omega$ and $a \in \Lambda$. The logic where we only allow the use of a graded operator $\langle a \rangle_m$ if $m < n$, for a fixed natural number n, is denoted by $\mathcal{L}^{\diamond<n}$.

We denote the extension of $\mathcal{L}^{\diamond<\aleph_0}$ and $\mathcal{L}^{\diamond<n}$ where we allow conjunctions on countable sets of formulae by $\mathcal{L}^{\diamond<\aleph_0}_{\omega_1}$, $\mathcal{L}^{\diamond<n}_{\omega_1}$. For k finite or $\aleph_0$, the logics $\mathcal{L}^{\diamond<k}_{\omega_1}$ can be seen as fragments of $\mathcal{L}_{\omega_1}$ (via an obvious extension of the standard translation ST above). More generally, if h, h' are cardinals, we may define the logic $\mathcal{L}^{\diamond<h}_{h'}$, where we allow the use of the graded operators $\langle a \rangle_k$ for $k < h$ and conjunctions on sets of formulae of cardinality strictly smaller than h'. Finally, we will consider the full infinitary logic of graded modalities $\mathcal{L}^{\mathsf{grad}}_{\infty}$, where we allow arbitrary conjunctions on sets of formulae, and graded operators $\langle a\rangle_h\phi$, for any cardinal h (to be interpreted in the natural way).

To prove our interpolation theorem we also make use of the (apparently) stronger operator $\{a\}_m$ (for a natural number m and $a \in \Lambda$), which applies to a sequence $\phi_0, \ldots, \phi_{m-1}$ of formulae: its semantics in a given point s of a model $\mathcal{M}$ is given by

$$s \models \{a\}_m(\phi_0, \ldots, \phi_{m-1})$$
$$\Updownarrow$$
there are m different successors $y_0, \ldots, y_{m-1}$ of s with $y_i \models \phi_i$, for all $i < m$.

This seems a more difficult task than asking for m different successors all satisfying a single formula ϕ (expressed by $\langle a\rangle_m\phi$); in the proof of preservation and elementary interpolation for graded modal logics (Theorem 4.3) we shall use the result (see [1] and [17], Th 1.1.6.) that the operator $\{a\}_m$ can be expressed using the operators $\langle a\rangle_h$, for $h \leq m$. The proof of this fact is an interesting example of an application to Modal Logic of a combinatorial result known as *the Marriage Theorem*, which we state in the following. Let $\{A_0, \ldots, A_{m-1}\}$ be a finite family of sets. A set of *distinct representatives* for this family is a set $\{a_0, \ldots, a_{m-1}\}$ such that $a_i \in A_i$ for all $i < m$ and $a_i \neq a_j$ for $i \neq j$, $i < m, j < m$. The Marriage Theorem gives a necessary and sufficient condition that a family of set must satisfy in order to have a set of distinct representatives.

Theorem 2.1 (Marriage Theorem)
A finite family $\{A_0, \ldots, A_{m-1}\}$ of sets has a set of distinct representatives iff for all $k \leq m$, $0 \leq i_1 < \ldots < i_k < m$, the set $A_{i_1} \cup \ldots \cup A_{i_k}$ has cardinality greater than or equal to k.

Proposition 2.2 *If $m < n$ and $\phi_0, \ldots, \phi_{m-1}$ are in $\mathcal{L}^{\diamond<n}$, then the formula $\{a\}_m(\phi_0, \ldots, \phi_{m-1})$ is equivalent to a formula in $\mathcal{L}^{\diamond<n}$.*

Proof.
Using the Marriage Theorem, we prove that the formula $\{a\}_m(\phi_0, \ldots, \phi_{m-1})$ is equivalent to the formula

$$\psi = \bigwedge_{i<m} \langle a\rangle\phi_i \wedge \bigwedge_{i_1<i_2<m} \langle a\rangle_2(\phi_{i_1} \vee \phi_{i_2}) \wedge \ldots \wedge$$
$$\bigwedge_{i_1<\ldots<i_{m-1}<m} \langle a\rangle_{m-1}(\phi_{i_1} \vee \ldots \vee \phi_{i_{m-1}}) \wedge \langle a\rangle_m(\phi_0 \vee \ldots \vee \phi_{m-1}).$$

It is clear that ϕ implies ψ. Vice versa, if $\mathcal{M}, w \models \psi$, then apply the Marriage Theorem to the sets $A_i = \{w' : wR_aw' \wedge w' \models \phi_i\}$. $\square$

Finally, we introduce a notion of bisimulation which is adequate for graded logics.

Definition 2.3 *Let k be finite or $\aleph_0$. A $(< k)-$*counting bisimulation *between two models $\mathcal{M}, \mathcal{N}$ of a language $\mathcal{L}$ is a relation $Z \subseteq \mathcal{M} \times \mathcal{N}$ such that:*

1. *$r^{\mathcal{M}} Z r^{\mathcal{N}}$;*
2. *if wZv, then w, v validate the same propositional constants of $\mathcal{L}$;*
3. *if wZv, then for any action $a \in \Lambda$, if X is a subset of $R_a(w)$ of cardinality strictly smaller than k, then*
$$|\{y \in R(v) : \exists x \in X \;\; xZy\}| \geq |X|.$$
4. *and vice versa, if wZv, then for any action $a \in \Lambda$, if X is a subset of $R_a(v)$ of cardinality strictly smaller than k, then*
$$|\{y \in R(w) : \exists x \in X \;\; yZx\}| \geq |X|.$$

□

The corresponding notion of bisimulation for the logic $\mathcal{L}_\infty^{grad}$, which we simply call *counting bisimulation*, is obtained from the definition above by deleting cardinality restriction :

Definition 2.4 *A counting bisimulation between two models $\mathcal{M}, \mathcal{N}$ of a language $\mathcal{L}$ is a relation $Z \subseteq \mathcal{M} \times \mathcal{N}$ such that:*

1. *$r^{\mathcal{M}} Z r^{\mathcal{N}}$;*
2. *if wZv, then w, v validate the same propositional constants of $\mathcal{L}$;*
3. *if wZv, then for any action $a \in \Lambda$, if X is a subset of $R_a(w)$, then*
$$|\{y \in R(v) : \exists x \in X \;\; xZy\}| \geq |X|.$$
4. *and vice versa, if wZv, then for any action $a \in \Lambda$, if X is a subset of $R_a(v)$, then*
$$|\{y \in R(w) : \exists x \in X \;\; yZx\}| \geq |X|.$$

□

If Z is a $((< k)$-)counting bisimulation between $\mathcal{M}$ and $\mathcal{N}$, we say that $\mathcal{M}$ and $\mathcal{N}$ are $((< k)$-)*counting bisimilar.* If $\mathcal{M}$ is a structure for a language $\mathcal{L}$ and $\mathcal{N}$ is a structure for a language $\mathcal{L}'$ we also consider $((< k)$-)counting bisimulation w.r.t. a language $\mathcal{L}_1$ contained in $\mathcal{L} \cap \mathcal{L}'$: the definition is as above except that in 2) the nodes w, v validate the same propositional constants of $\mathcal{L}_1$ and 3),4) are restricted to atomic actions of this language.

Lemma 2.5 *Let k be finite or $\aleph_0$. If $\mathcal{M}$ is an $\mathcal{L}$ structure, $\mathcal{N}$ is an $\mathcal{L}'$ structure, $\mathcal{L}_1 \subseteq \mathcal{L} \cap \mathcal{L}'$, and Z is a $(< k)-$counting bisimulation (or a counting bisimulation) between $(\mathcal{M}, w)$ and $(\mathcal{N}, v)$ w.r.t. $\mathcal{L}_1$, then w and v satisfy the same $(\mathcal{L}_1)_\infty^{\diamond<k}$ formulae (the same $(\mathcal{L}_1)_\infty^{grad}$ formulae, respectively).*

A kind of converse of the Lemma 2.3, 2.4 is also true. If $\mathcal{M}$ is an $\mathcal{L}$-structure and k is a cardinal such that $|R(w)| < k$ for all $w \in \mathcal{M}$, then for any other $\mathcal{L}$-structure $\mathcal{N}$ the relation $Z = \{(w,v) : w, v$ satisfy the same formulae of $\mathcal{L}_k^{<k^+}\}$ is a counting bisimulation (in particular, a $(<k)$-counting bisimulation) between $\mathcal{M}$ and $\mathcal{N}$ (for a proof of the above results the reader can consult [7], in particular the claim in the proof of Theorem 4.2.3).

3 Consistency Properties for Graded Modalities

Let k be a natural number or $\aleph_0$. We consider two languages $\mathcal{L}, \mathcal{L}'$ and their extensions $\mathcal{L}_C$ and $\mathcal{L}'_D$ relative to two countable sets of constants C, D with $\{r\} = C \cap D$. Our *motivating* example for the notion of a consistency property consists here in the set S of triples (Z, Σ, Δ), where $\Sigma \subseteq (\mathcal{L}_C)_{\omega_1}$, $\Delta \subseteq (\mathcal{L}_D)_{\omega_1}$, and $Z \subseteq C \times D$, such that:

- there exists a model $\mathcal{M}$ for Σ with domain $\{c^{\mathcal{M}} : c \in \mathcal{M}\}$,
- there exists a model $\mathcal{N}$ for Δ with domain $\{d^{\mathcal{N}} : d \in D\}$,
- Z is a graded bisimulation w.r.t. $\mathcal{L} \cap \mathcal{L}'$ between $\mathcal{M}$ and $\mathcal{N}$.

We can extrapolate some natural properties of the set of triple S from this example: e.g., if $p(c) \in \Sigma$, for $p \in \mathcal{L} \cap \mathcal{L}'$ and $(c,d) \in Z$, then the triple $(Z, \Sigma, \Delta \cup \{p(d)\})$ belongs to S. This leads to the the notion of consistency properties modulo bisimulation (Definition 3.1). We then prove a Model Existence Lemma which says that any element (Z, Σ, Δ), belonging to such a consistency property is *satisfiable* in the following sense: there are models $\mathcal{M}, \mathcal{N}$ whose domains are given by (equivalence classes in) C, D, respectively, such that

1. $\mathcal{M} \models \Sigma$;
2. $\mathcal{N} \models \Delta$;
3. Z describes a graded bisimulation w.r.t. $\mathcal{L} \cap \mathcal{L}'$ between $\mathcal{M}$ and $\mathcal{N}$.

We are now ready for details. Let C, D be countable sets of constants symbols which are disjoint, except for the root constant $r \in C \cap D$. Without loss of generality we suppose that the formulae of $(\mathcal{L}_C)_{\omega_1}$ and $(\mathcal{L}'_D)_{\omega_1}$ are constructed from atoms and negated atoms using $\forall, \exists$, countable conjunctions and disjunctions. If ϕ is a formula, we denote by $\neg\phi$ the formula obtained by 'moving the negation inside' (e.g. $\neg(p(x) \wedge q(x)) \equiv \neg p(x) \vee \neg q(x)$).

Definition 3.1 *Let k be a natural number or $\aleph_0$. A* consistency property modulo $(<k)-$counting bisimulation *is a set S of triples (A, Σ, Δ) where*

- *$(r, r) \in A \subseteq C \times D$,*

- Σ *is a countable set of* $(\mathcal{L}_C)_{\omega_1}$*-sentences and* Δ *is a countable set of* $(\mathcal{L}'_D)_{\omega_1}$*-sentences, both satisfying the following* closure properties *(divided into six groups of left and right consistency, equality, and bisimulation properties).*

Left Consistency Properties:

(c_1) *If* ϕ *is an atomic sentence of* $(\mathcal{L}_C)_{\omega_1}$*, then either* $(A, \Sigma \cup \{\phi\}, \Delta) \in S$ *or* $(A, \Sigma \cup \{\neg\phi\}, \Delta) \in S$*;*

(c_2) *If* ϕ *is an atomic sentence of* $(\mathcal{L}_C)_{\omega_1}$*, either* $\phi \notin \Sigma$ *or* $\neg\phi \notin \Sigma$*;*

(c_3) *if* $\bigwedge \Phi \in \Sigma$ *then for all* $\phi \in \Phi$ $(A, \Sigma \cup \{\phi\}, \Delta) \in S$*;*

(c_4) *if* $\forall x \phi(x) \in \Sigma$*, then for all* $c \in C$ $(A, \Sigma \cup \{\phi(c/x)\}, \Delta) \in S$*;*

(c_5) *if* $\bigvee \Phi \in \Sigma$*, then there is a* $\phi \in \Phi$ *such that* $(A, \Sigma \cup \{\phi\}, \Delta) \in S$*;*

(c_6) *if* $\exists x \phi \in \Sigma$*, then there is a* $c \in C$ *such that* $(A, \Sigma \cup \{\phi(c/x)\}, \Delta) \in S$*.*

Right Consistency Properties:

The same as above, with Δ*,* $\mathcal{L}'$*, and* D *instead of* Σ*,* $\mathcal{L}$*, and* C*.*

Left Bisimulation Properties:

(b_1) *if* $(c, d) \in A$*,* $p \in \mathcal{L} \cap \mathcal{L}'$*, and* $p(c) \in \Sigma$ *then* $(A, \Sigma, \Delta \cup \{p(d)\}) \in S$*;*

(b_2) *for all* $h < k$*, if* $(c, d) \in A$ *and* $c_0, \ldots, c_{h-1}$ *are constants in* C *such that*

$$\{R_a(c, c_i) : i < h\} \cup \{c_i \neq c_j : i < j < h\} \subseteq \Sigma,$$

then there are constants $d_0, \ldots, d_{h-1}$ *in* D *such that*

$$(A \cup \{(c_i, d_i) : i < h\}, \Sigma, \Delta \cup \{R_a(d, d_i) : i < h\} \cup \{d_i \neq d_j : i < j < h\}) \in S.$$

Right Bisimulation Properties

The same as above exchanging the role of Σ*,* C *with* Δ*,* D*.*

Left Equality Properties:

($=_1$) *If* $c = c' \in \Sigma$ *then* $(A, \Sigma \cup \{c' = c\}, \Delta) \in S$*;*

($=_2$) *if* $c = c' \in \Sigma$ *and* $\phi(c') \in \Sigma$ *then* $(A, \Sigma \cup \{\phi(c)\}, \Delta) \in S$*;*

($=_3$) *for all* $c \in C$*,* $(A, \Sigma \cup \{c = c\}, \Delta) \in S$*;*

($=_4$) *If* $c, c' \in C$*,* $d \in D$*,* $(c, d) \in A$*, and* $(c = c') \in \Sigma$*, then the triple* $(A \cup \{(c', d)\}, \Sigma, \Delta)$ *belongs to* S*.*

Right Equality Properties:

The same as above exchanging the role of Σ *and* Δ. $\square$

Notice how these properties are extrapolated from our first example, in which S is the set of triples (Z, Σ, Δ) such that there exist two $(< k)$-counting bisimilar models $\mathcal{M}$, $\mathcal{N}$ with $\mathcal{M} \models \Sigma$, $\mathcal{N} \models \Delta$, and $D^{\mathcal{M}} = \{c^{\mathcal{M}} : c \in C\}, D^{\mathcal{N}} = \{d^{N} : d \in D\}$. On the other hand, we can prove that this example of a consistency property is *paradigmatic*, in the sense given by the following lemma. We give the proof of this lemma in full details and this will keep us busy until the end of the section.

Lemma 3.2 *If S is a consistency property modulo $(< k)$-counting bisimulation and $(A, \Sigma, \Delta) \in S$ then there are:*

- *a countable model $\mathcal{M}$ of $\mathcal{L}_C$ such that $\mathcal{M} \models \Sigma$,*
- *a countable model $\mathcal{N}$ of $\mathcal{L}'_D$ such that $\mathcal{N} \models \Delta$,*
- *a $(< k)$-counting bisimulation Z w.r.t. $\mathcal{L} \cap \mathcal{L}'$ between $\mathcal{M}$ and $\mathcal{N}$ such that for all $(c, d) \in A$, the pair $(c^{\mathcal{M}}, d^{\mathcal{N}}) \in Z$.*

Proof.
For the sake of readability we divide the proof in three parts. The first part regards a preliminary *make up* of our consistency property S and the construction of an enumeration of formulas and of certain triples. This enumeration will be used in the second part of the proof as a guide in the construction of the two models $\mathcal{M}, \mathcal{N}$. In the same part we also verify that $\mathcal{M} \models \Sigma, \mathcal{N} \models \Delta$, while the third part is devoted to the construction of the $(< k)$-counting bisimulation between $\mathcal{M}$ and $\mathcal{N}$.

(1): Preliminary Steps

We say that a consistency property S is *closed* if whenever $(A, \Sigma, \Delta) \in S$ and $A' \subseteq A$, $\Sigma' \subseteq \Sigma$ and $\Delta' \subseteq \Delta$ then $(A', \Sigma', \Delta') \in S$. Notice that if S is a consistency property modulo $(< k)$-counting bisimulation then

$$S' = \{(A', \Sigma', \Delta') : \exists (A, \Sigma, \Delta) \in S \text{ with } \ (r, r) \in A' \subseteq A, \Sigma' \subseteq \Sigma, \Delta' \subseteq \Delta\}$$

is also a consistency property modulo $(< k)$-counting bisimulation. Just to give an example, we check that property (c_1) holds for $(A', \Sigma', \Delta') \in S'$. By definition of S', there exists $(A, \Sigma, \Delta) \in S$ such that $A' \subseteq A$, $\Sigma' \subseteq \Sigma, \Delta' \subseteq \Delta$. If ϕ is an atomic sentence of $(\mathcal{L}_C)_{\omega_1}$, then either $(A, \Sigma \cup \{\phi\}, \Delta) \in S$ or $(A, \Sigma \cup \{\neg\phi\}, \Delta) \in S$, since (A, Σ, Δ) satisfies c_1). In the first case, we have $(A', \Sigma' \cup \{\phi\}, \Delta') \in S'$, while in the second $(A', \Sigma' \cup \{\neg\phi\}, \Delta') \in S'$.

Hence, we may suppose without loss of generality that S is closed.

Let Y be the least set of sentences of $(\mathcal{L}_C)_{\omega_1}$ and $(\mathcal{L}'_D)_{\omega_1}$ such that:

- $\Sigma \subseteq Y$, $\Delta \subseteq Y$;
- Y is closed under C-substitution and D-substitution instances of subformulae;
- if $c, c' \in C$ and $d, d' \in D$ then $c = c'$, $c \neq c'$, $d = d'$, and $d \neq d'$ belong to Y;
- For all $c, c' \in C, d, d' \in D$ if $\phi(c) \in Y$ then $\phi(c') \in Y$, if $\psi(d) \in Y$ then $\psi(d') \in Y$.

Notice that Y is countable, since Σ, Δ, C, and D are all countable.

Let T be a set of tuples containing:

L_1) all tuples (c, c', d), with $c, c' \in C$, $d \in D$;

R_1) all tuples (d, d', c), with $d, d' \in D$, $c \in C$;

L_2) all tuples $(a, c, d, c_0, \ldots, c_{h-1})$, where $h < k$, a is an action in $\mathcal{L} \cap \mathcal{L}'$, $c, c_0, \ldots c_{h-1}$ in C, and d in D;

R_2) all tuples $(a, c, d, d_0, \ldots, d_{h-1})$, where $h < k$, a is an action in $\mathcal{L} \cap \mathcal{L}'$, c in C, and $d, d_0, \ldots, d_{h-1}$ in D;

L_3) all tuples (c, d, p), where $c \in C$, $d \in D$, and p is unary predicate symbol of $\mathcal{L} \cap \mathcal{L}'$;

R_3) all tuples (d, c, p), where $c \in C$, $d \in D$, and p is unary predicate symbol of $\mathcal{L} \cap \mathcal{L}'$.

These tuples correspond to the various requirements that a $(< k)-$counting bisimulation Z between models $\mathcal{M}, \mathcal{N}$ must satisfy, on the left (L) and on the right (R). Notice that T is countable, since $\mathcal{L}, \mathcal{L}', C$, and D are all countable. We will construct our models and our bisimulation step by step, following an enumeration that contains these tuples. Whenever a tuple of type L_1 is met, we continue the construction in such a way that if $(c^{\mathcal{M}}, d^{\mathcal{N}}) \in Z$ and $c^{\mathcal{M}} = c'^{\mathcal{M}}$ then $(c'^{\mathcal{M}}, d^{\mathcal{N}}) \in Z$ (and similarly for tuples of type R_1). The requirements of type L_2, R_2 corresponds to the zig-zag property of a $(< k)-$counting bisimulation and the L_3, R_3 conditions corresponds to preservation of propositions: the tuple (c, d, p) is used to remind us that if $(c^{\mathcal{M}}, d^{\mathcal{N}}) \in Z$ and $\mathcal{M} \models p(c)$ then $\mathcal{N} \models p(d)$ must hold, while the tuple (d, c, p) means that if $(c^{\mathcal{M}}, d^{\mathcal{N}}) \in Z$ and $\mathcal{M} \models p(d)$ then $\mathcal{N} \models p(c)$ must be true.

Since $Y \cup T$ is countable, we can list it as $G_0, G_1, \ldots, G_n, \ldots$.

We shall now construct a sequence $t_0 = (A_0, \Sigma_0, \Delta_0), t_1 = (A_1, \Sigma_1, \Delta_1), \ldots,$ of triples in S as follows. The triple $(A_0, \Sigma_0, \Delta_0)$ is the given (A, Σ, Δ). If $(A_n, \Sigma_n, \Delta_n)$ is already defined, then $(A_{n+1}, \Sigma_{n+1}, \Delta_{n+1})$ is constructed using the definition of a consistency property, according to the element G_n. More precisely, if G_n is a formula in $Y \subseteq (\mathcal{L}_C)_{\omega_1}$ then:

- if $(A_n, \Sigma_n \cup \{G_n\}, \Delta) \notin S$, then $t_{n+1} = t_n$;
- if G_n is an atomic formula, or $G_n = \bigwedge \Phi$, or $G_n = \forall x\phi$ and $(A_n, \Sigma_n \cup \{G_n\}, \Delta) \in S$, then $t_{n+1} = (A_n, \Sigma_n \cup \{G_n\}, \Delta)$;
- if $G_n = \bigvee \Phi$ and $(A_n, \Sigma_n \cup \{G_n\}, \Delta) \in S$ then by (c_5) there is a $\phi \in \Phi$ such that $(A_n, \Sigma_n \cup \{G_n\} \cup \{\phi\}, \Delta)$ belongs to S. We take this triple as t_{n+1};
- if $G_n = \exists x\phi$ and $(A_n, \Sigma_n \cup \{G_n\}, \Delta) \in S$ then by (c_6) there is a $c \in C$ such that $(A_n, \Sigma_n \cup \{G_n\} \cup \{\phi(c)\}, \Delta)$ belongs to S. We take this triple as t_{n+1};

The same construction is performed if G_n is a formula in $(\mathcal{L}'_D)_{\omega_1}$, using the right consistency properties of S.

On the other hand, if G_n is a tuple in T then:

- if $G_n = (c, c', d)$ with $c, c' \in C$, $d \in D$, and the triple

  $$(A_n \cup \{(c, d)\}, \Sigma_n \cup \{c = c'\}, \Delta_n)$$

  belongs to S, then by $(=_4)$ the triple

  $$(A_n \cup \{(c, d), (c', d)\}, \Sigma_n \cup \{c = c'\}, \Delta_n)$$

  belongs to S as well and we choose t_{n+1} equal to this triple.

  If $(A_n \cup \{(c, d)\}, \Sigma_n \cup \{c = c'\}, \Delta_n) \notin S$, choose $t_{n+1} = t_n$.
- If $G_n = (c, d, p)$ with $c \in C, d \in D$, and p is a unary predicate of $\mathcal{L} \cap \mathcal{L}'$, then if $(A_n \cup \{(c, d)\}, \Sigma_n \cup \{p(c)\}, \Delta_n)$ belongs to S, by the left bisimulation property (b_1) we can take

  $$t_{n+1} = (A_n \cup \{(c, d)\}, \Sigma_n \cup \{p(c)\}, \Delta_n \cup \{p(d)\}).$$

  If $(A_n \cup \{(c, d)\}, \Sigma_n \cup \{p(c)\}, \Delta_n)$ does not belongs to S, take $t_{n+1} = t_n$.
- if $G_n = (a, c, d, c_0, \ldots, c_{h-1})$ where $h < k$, a is an action in $\mathcal{L} \cap \mathcal{L}'$, $c, c_0, \ldots, c_{h-1}$ belong to C, $d \in D$, and the triple $(A^*_n, \Sigma^*_n, \Delta_n)$ belongs to S, where $A^*_n = A_n \cup \{(c, d)\}, \Sigma^*_n = \Sigma_n \cup \{R_a(c, c_i) : i < h\} \cup \{c_i \neq c_j : i < j < h\}$, then by the left bisimulation property (b_2) there are $d_0, \ldots, d_{h-1}$ such that the triple

  $$(A^*_n, \Sigma^*_n, \Delta_n \cup \{R_a(d, d_i) : i < h\} \cup \{d_i \neq d_j : i < j < h\})$$

  belongs to S. We take this triple as t_{n+1}. If $(A^*_n, \Sigma^*_n, \Delta_n)$ does not belong to S, take $t_{n+1} = t_n$.

The same construction is performed on the right, when G_n is a tuple of type (d, d', c) or (d, c, p) or $(c, d, d_0, \dots, d_{h-1})$.

Finally, we define

$$A_\omega = \bigcup_{n<\omega} A_n, \qquad \Sigma_\omega = \bigcup_{n<\omega} \Sigma_n, \qquad \Delta_\omega = \bigcup_{n<\omega} \Delta_n.$$

Remark 3.3 *a) If G_n is a sentence in Σ_ω, then $(A_n, \Sigma_n \cup \{G_n\}, \Delta_n)$ belongs to S and $G_n \in \Sigma_{n+1}$.*

b) If G_n is a sentence and $r \geq n$ is such that $(A_r, \Sigma_r \cup \{G_n\}, \Delta_r)$ belongs to S, then $G_n \in \Sigma_{n+1}$.

c) If G_n is a sequence (c, d, p) with $(c, d) \in A_\omega$, $p \in \mathcal{L} \cap \mathcal{L}'$, and $p(c) \in \Sigma_\omega$ then $p(d) \in \Delta_{n+1}$;

d) If G_n is a sequence $(a, c, d, c_0 \dots, c_{h-1})$ where $h < k$, a is an action in $\mathcal{L} \cap \mathcal{L}'$, $(c, d) \in A_\omega$ and $\{R_a(c, c_i) : i < h\} \cup \{c_i \neq c_j : i < j < h\} \subseteq \Sigma_\omega$, then there are $d_0, \dots, d_{h-1}$ in D such that $\{R_a(d, d_i) : i < h\} \cup \{d_i \neq d_j : i < j < h\} \subseteq \Delta_{n+1}$ and $\{(c_i, d_i) : i < h\} \subseteq A_{n+1}$.

Proof.

The same properties hold 'on the right'.

We just prove a) and d).

a) If $G_n \in \Sigma_\omega$, let $r \geq n$ be such that $G_n \in \Sigma_r$. Then $\Sigma_n \cup \{G_n\} \subseteq \Sigma_r$ and since S is closed the triple $(A_n, \Sigma_n \cup \{G_n\}, \Delta_n)$ belongs to S. By construction, $G_n \in \Sigma_{n+1}$.

d) Let $r \geq n$ be such that $(c, d) \in A_r$ and $\{R_a(c, c_i) : i < h\} \cup \{c_i \neq c_j : i < j < h\} \subseteq \Sigma_r$. Then $A_n \cup \{(c, d)\} \subseteq A_r$, $\Sigma_n \cup \{R_a(c, c_i) : i < h\} \cup \{c_i \neq c_j : i < j < h\} \subseteq \Sigma_r$, and, since S is closed, the triple

$$(A_n \cup \{(c, d)\}, \Sigma_n \cup \{R_a(c, c_i) : i < h\} \cup \{c_i \neq c_j : i < j < h\}, \Delta_n)$$

belongs to S. By construction of t_{n+1} there are $d_0, \dots, d_{h-1}$ in D such that $\{R_a(d, d_i) : i < h\} \cup \{d_i \neq d_j : i < j < h\} \subseteq \Delta_{n+1}$ and $\{(c_i, d_i) : i < h\} \subseteq A_{n+1}$.

$\square$

(2): Construction of the Models $\mathcal{M}$, $\mathcal{N}$

To define the model $\mathcal{M}$ we use the set Σ_ω defined in the previous part. Let E be the following relation on C:

$$cEc' \qquad \text{iff} \;\; c = c' \in \Sigma_\omega.$$

From the equality properties of S it follows that E is an equivalence relation. For example, suppose cEc' and $c'Ec''$. Let n be such that G_n is $c = c''$, and let $r \geq n$ be such that $c = c'$ and $c' = c''$ belong to Σ_r. By applying $(=)_2$ with $\phi(c')$ equal to $c' = c''$ we obtain that the triple

$$(A_r, \Sigma_r \cup \{c = c''), \Delta_r)$$

belongs to S. By Remark 3.3.b we obtain $c = c'' \in \Sigma_{n+1}$. Hence, cEc''. Reflexivity and symmetry of E are proved in the same way using properties $(=)_1, (=)_3$.

For $c \in C$, let $[c]$ be its equivalence class w.r.t. E. The domain of $\mathcal{M}$ is the set $\{[c] : c \in C\}$ and each constant $c \in C$ is interpreted in $\mathcal{M}$ as $[c]$. By the equality property $(=)_2$, if $R_a(c_1, c_2) \in \Sigma_\omega$ and c_1Ec_1', c_2Ec_2', then $R_a(c_1', c_2') \in \Sigma_\omega$ and the definition

$$R_a^{\mathcal{M}}([c_1],[c_2]) \qquad \text{iff} \qquad R_a(c_1, c_2) \in \Sigma_\omega$$

does not depend on the representatives of the classes $[c_1], [c_2]$.

The same is true if we define

$$p^{\mathcal{M}}([c]) \qquad \text{iff} \qquad p(c) \in \Sigma_\omega,$$

for all unary predicate $p \in \mathcal{L}$. Finally, let $[r]$ be the root $r^{\mathcal{M}}$ of $\mathcal{M}$.

We now show that all formulae of Σ_ω are true in $\mathcal{M}$. Since $\Sigma \subseteq \Sigma_\omega$, we obtain $\mathcal{M} \models \Sigma$. If ϕ is an atomic formula, then by definition we have

$$\phi \in \Sigma_\omega \qquad \text{iff} \qquad \mathcal{M} \models \phi.$$

Using the consistency properties $(c_1), (c_2)$ we can easily extend this fact to literals. For example, if ϕ is $c_i \neq c_j$ and $\phi \in \Sigma_\omega$, then $\mathcal{M} \models \phi$, because otherwise $c_i = c_j$ would also belong to Σ_ω, contradicting (c_2). Vice versa, if $\mathcal{M} \models \phi$, G_n is $c_i = c_j$, and G_m is $c_i \neq c_j$, consider $r \geq m, n$. The triple $(A_r, \Sigma_r \cup \{G_n\}, \Delta_r)$ does not belong to S otherwise by Remark 3.3.b we would have $G_n \in \Sigma_{n+1}$ and $\mathcal{M} \models G_n$, which is not. By (c_1) the triple $(A_r, \Sigma_r \cup \{G_m\}, \Delta_r)$ belongs to S and by Remark 3.3.b it follows that $G_m \in \Sigma_{m+1} \subseteq \Sigma_\omega$.

In case of more complex sentences, we can prove by induction on the complexity of a sentence ϕ *belonging to* Y that

$$\phi \in \Sigma_\omega \qquad \text{implies} \qquad \mathcal{M} \models \phi.$$

- If ϕ is a literal, we have already proved a stronger result.
- If $\phi = \forall x \psi \in \Sigma_\omega$ and $c \in C$, let $G_n = \psi(c)$ ($\psi(c)$ is in Y because Y is closed under C-substitution instances of subformulae). If $r \geq n$ is such that $\forall x \psi \in \Sigma_r$, by (c_4) we have that the triple $(A_r, \Sigma_r \cup \{\psi(c)\}, \Delta_r)$ belongs to S. From Remark 3.3.b it follows that $\psi(c) \in \Sigma_{n+1} \subseteq \Sigma_\omega$ and by induction $\mathcal{M} \models \psi(c)$. Since this holds for all $c \in C$ and any element of $\mathcal{M}$ is denoted by an element in C, it follows that $\mathcal{M} \models \forall x \psi$.

- If $\phi = G_n = \exists x\psi$ belongs to Σ_ω, then by Remark 3.3.a we have that the triple $(A_n, \Sigma_n \cup \{\exists x\psi\}, \Delta_n)$ belongs to S. By construction of t_{n+1}, there is a $c \in C$ such that $\psi(c) \in \Sigma_{n+1} \subseteq \Sigma_\omega$ and by induction $\mathcal{M} \models \psi(c)$. Hence, $\mathcal{M} \models \exists x\psi$.

- The remaining cases are similar and left to the reader.

In this way we proved that $\mathcal{M}$ is a model for all sentences in Σ_ω. In a similar way we can prove that there is a model $\mathcal{N}$, whose definition is based on the equivalence relation

$$d \sim d' \qquad \text{iff} \qquad d = d' \in \Delta_\omega,$$

which is a model for all sentences in Δ_ω (in particular, it is a model for all sentences in Δ).

(3): Construction of the $(< k)-$Counting Bisimulation Z between $\mathcal{M}$ and $\mathcal{N}$.

Here we use the sets A_ω defined in the first part of the proof. Let

$$[c]Z[d] \qquad \text{iff } \ (c,d) \in A_\omega.$$

We prove that the definition of Z does not depend on the representatives of $[c],[d]$. By the equality property $(=_4)$ we have that if $(c,d) \in A_\omega$ and $c = c' \in \Sigma_\omega$, then $(c',d) \in A_\omega$: if $G_n = (c,c',d)$ and $r \geq n$ is such that $(c,d) \in A_r$, $c = c' \in \Sigma_r$, then, since S is closed, the triple $(A_n \cup \{(c,d)\}, \Sigma_n \cup \{c = c'\}, \Delta_n)$ belongs to S. By definition of t_{n+1} we have $(c',d) \in A_{n+1} \subseteq A_\omega$. In the same way we prove that if $(c,d) \in A_\omega$ and $d = d' \in \Delta_\omega$, then $(c,d') \in A_\omega$.

Finally, we show that Z is a graded bisimulation w.r.t. $\mathcal{L} \cap \mathcal{L}'$ between $\mathcal{M}$ and $\mathcal{N}$.

- From $(r,r) \in A_0 \subseteq A_\omega$ we obtain $r^{\mathcal{M}} Z r^{\mathcal{N}}$.

- If $[c]Z[d]$ and $\mathcal{M} \models p(c)$ for $p \in \mathcal{L} \cap \mathcal{L}'$, then $(c,d) \in A_\omega$, $p(c) \in \Sigma_\omega$, and by Remark 3.3.c it follows that $p(d) \in \Delta_{n+1}$. Hence we also have $\mathcal{N} \models p(d)$.

- If $[c]Z[d]$, $a \in \mathcal{L} \cap \mathcal{L}'$, and X is subset of $R_a([c])$ with $|X| = h < k$, then $X = \{[c_0], \ldots, [c_{h-1}]\}$ with $\mathcal{M} \models R_a(c, c_i)$, for all $i < h$, and $\mathcal{M} \models c_l \neq c_j$, for $l < j < h$. Then $R_a(c,c_i) \in \Sigma_\omega$ and $(c_l \neq c_j) \in \Sigma_\omega$ (remember that for literals we have $\phi \in \Sigma_\omega \Leftrightarrow \mathcal{M} \models \phi$). By Remark 3.3.d there are $d_0, \ldots, d_{h-1}$ in D such that $\{R_a(d,d_i) : i < h\} \cup \{d_i \neq d_j : i < j < h\} \subseteq \Delta_\omega$ and $\{(c_i,d_i) : i < h\} \subseteq A_\omega$. It follows that $\mathcal{N} \models R_a(d,d_i)$, $\mathcal{N} \models d_l \neq d_j$, and $[c_i]Z[d_i]$ for $i < h$, $l < j < h$. This implies that

$$|\{y \in R_a([d]) : \exists x \in X \ xZy\} \geq h,$$

as required to a $(< k)-$counting bisimulation.

The last clause of a $(< k)-$counting bisimulation is verified in the same way.

Hence, if $(A, \Sigma, \Delta) \in S$ we showed how to construct two models $\mathcal{M}, \mathcal{N}$ with $\mathcal{M} \models \Sigma$, $\mathcal{N} \models \Delta$, and how to *expand* A to a $(< k)$-counting bisimulation Z between $\mathcal{M}$ and $\mathcal{N}$. This ends the proof of the Lemma. $\square$

4 Preservation and Elementary Interpolation

In this section we characterize the Graded (Infinitary) Logic $\mathcal{L}_{\omega_1}^{\diamond<k}$, for $k \leq \aleph_0$, as the fragment of Infinitary Logic $\mathcal{L}_{\omega_1}$ which is invariant for $(< k)-$counting bisimulation and prove elementary interpolation for it. All proofs can be specialized to the finitary fragment $\mathcal{L}^{\diamond<k}$.

First of all, let us recall how consistency properties are used in $\mathcal{L}_{\omega_1}$ to prove interpolation. Suppose $\phi \models \psi$ and suppose by contradiction that the pair (ϕ, ψ) does not have an interpolant. Consider the set S of all finite sets of sentences s which can be decomposed as $s = s_1 \cup s_2$, with:

1. $\mathcal{L}(s_1) \subseteq \mathcal{L}(\phi)$, $\mathcal{L}(s_2) \subseteq \mathcal{L}(\psi)$;

2. $\wedge s_1$, $\neg \wedge s_2$ do not have an interpolant.

It can be proved that the set S defined above is a consistency property; since by hypothesis the pair $(\{\phi\}, \{\neg\psi\})$ belongs to S, the Model Existence Lemma tells us that there exists a model for $\phi \wedge \neg\psi$, a contradiction.

We generalize this idea in the proof of Theorem 4.3. In this context the interpolant is a graded formula, but for technical reasons that will become clear during the proof, we need an extension inside $\mathcal{L}_{\omega_1}$ of the graded fragment, the so-called extended graded formulae over a set of variables X (these formulae were first introduced for the modal case by van Benthem). Intuitively, extended graded formulae consist on boolean combination of ordinary graded modal formulae *evaluated* at different nodes.

Definition 4.1 *A formula $\theta(X) \in \mathcal{L}_{\omega_1}$ is an* extended k-graded formula *over the set of variables X if there are graded infinitary formulae $\phi_{j,l} \in \mathcal{L}_{\omega_1}^{\diamond<k}$ and variables $x_{j,l} \in X$ for $j \in J, l \in L$ such that*

$$\theta = \bigvee_{j \in J} \bigwedge_{l \in L} ST(\phi_{j,l})(x_{j,l}/x).$$

$\square$

Notice that if $|X| = 1$ the extended graded formulae over X are equivalent to standard translation formulae in $\mathcal{L}_{\omega_1}^{\diamond<k}$.

In order to state the interpolation and preservation theorems, we need the following definition.

Definition 4.2 *Let ϕ, ψ be sentences in $\mathcal{L}_{\omega_1}$ and let k be finite or $\aleph_0$. We say that ϕ* entails ψ along $(< k)-$counting bisimulation *if whenever $\mathcal{M}$ is an $\mathcal{L}(\phi)$ model such that $\mathcal{M} \models \phi$ and $\mathcal{N}$ is an $\mathcal{L}(\psi)$ models which is $(< k)-$counting bisimilar to $\mathcal{M}$ with respect to $\mathcal{L}(\phi) \cap \mathcal{L}(\psi)$, then $\mathcal{N} \models \psi$.* $\square$

Suppose that the two sentences ϕ, ψ have a graded interpolant, that is, that there exists a formula $\theta \in (\mathcal{L}(\phi) \cap \mathcal{L}(\psi))_{\omega_1}^{\diamond<k}$ such that

- $\phi \models \theta$
  (i.e. if $\mathcal{M} \models \phi$ then $\mathcal{M} \models \mathsf{ST}(\theta)(r)$, where ST is the standard translation),
- $\theta \models \psi$
  (i.e. if $\mathcal{N} \models \mathsf{ST}(\theta)(r)$ then $\mathcal{N} \models \psi$).

If ϕ, ψ have a graded interpolant then it is easy to show that ϕ entails ψ along $(< k)-$counting bisimulation: if $\mathcal{M} \models \phi$ and $\mathcal{N}$ is $(< k)-$counting bisimilar to $\mathcal{M}$ w.r.t. $\mathcal{L}(\phi) \cap \mathcal{L}(\psi)$, then $\mathcal{M} \models \theta$, $\mathcal{N} \models \theta$ (because k-graded formulae are preserved under $(< k)$-counting bisimulation) and $\mathcal{N} \models \psi$.

The converse is given by the following theorem.

Theorem 4.3 *Let ϕ, ψ be sentences in $\mathcal{L}_{\omega_1}$. If ϕ entails ψ along $(< k)$-counting bisimulation, then there is a graded infinitary formula $\theta \in (\mathcal{L}(\phi) \cap \mathcal{L}(\psi))_{\omega_1}^{\diamond<k}$ such that $\phi \models \theta$ and $\theta \models \psi$.*

Proof.
By contraposition. If ϕ and ψ are infinitary sentences having no graded interpolant, then using the notion of consistency property modulo $(< k)-$counting bisimulation we prove that there are two models $\mathcal{M}, \mathcal{N}$ which are $(< k)-$counting bisimilar w.r.t. $\mathcal{L}(\phi) \cap \mathcal{L}(\psi)$, such that $\mathcal{M} \models \phi$, $\mathcal{N} \models \neg\psi$.

Let C, D be countable sets of constant symbols with $\{r\} = C \cap D$. Consider the set S consisting of all triples (A, Σ, Δ) such that

- $(r, r) \in A$ and A is a finite subset of $C \times D$;
- Σ is a finite set of C-substitution instances of subformulae of ϕ, or of type $c = c'$, $c \neq c'$, for $c, c' \in C$;
- Δ is a finite set of D-substitution instances of subformulae of $\neg\psi$, or of type $d = d'$, $d \neq d'$, for $d, d' \in D$.
- there is no extended graded formula $\theta(X)$ over variables X in the language $\mathcal{L}(\phi) \cap \mathcal{L}(\psi)$ and no substitution functions $\sigma_1 : X \to C$, $\sigma_2 : X \to D$ such that:

  for each $x \in X$ the pair $(\sigma_1 x, \sigma_2 x)$ is in A;

  $\Sigma \models \theta(\sigma_1)$ and $\Delta \models \neg\theta(\sigma_2)$.

Notice that only finitely many constants appear in Σ, Δ, and A (this is because if χ is a sentence of $\mathcal{L}_{\omega_1}$ then its subformulae contain only a finite number of free variables).

Proposition 4.4 *S is a consistency property modulo ($< k$)-counting bisimulation.*

Proof.
We prove some of the left properties of Definition 3.1, leaving the others and the right properties to the reader.

(c_1) If ϕ is an atomic sentence in $(\mathcal{L}_C)_{\omega_1}$, suppose by contradiction that $(A, \Sigma \cup \{\phi\}, \Delta) \notin S$, and $(A, \Sigma \cup \{\neg\phi\}, \Delta) \notin S$. Then there are extended graded formulae $\alpha(X), \beta(Y)$ over set of variables X, Y (which we may suppose disjoint) and substitutions $\delta_1, \delta_2, \tau_1, \tau_2$ such that

$$\Sigma \cup \{\phi\} \models \alpha(\delta_1), \qquad \Delta \models \neg\alpha(\delta_2),$$

$$\Sigma \cup \{\neg\phi\} \models \beta(\tau_1), \qquad \Delta \models \neg\beta(\tau_2).$$

If $\theta(X, Y) = \alpha(X) \vee \beta(Y)$ and $\sigma_1 = \delta_1 \cup \tau_1$, $\sigma_2 = \delta_2 \cup \tau_2$, then

$$\Sigma \models \theta(\sigma_1), \qquad \Delta \models \neg\theta(\sigma_2),$$

in contradiction with $(A, \Sigma, \Delta) \in S$.

(c_5) If $\bigvee \Phi \in \Sigma$ and for no $\phi \in \Phi$ the triple $(A, \Sigma \cup \{\phi\}, \Delta)$ belongs to S, then for each $\phi \in \Phi$ there is an extended graded formula θ_ϕ over a set of variables X_ϕ and two substitutions $\sigma_1^\phi, \sigma_2^\phi$ such that

$$\Sigma \cup \{\phi\} \models \theta_\phi(\sigma_1^\phi),$$

$$\Delta \models \neg\theta_\phi(\sigma_2^\phi),$$

and $(\sigma_1^\phi(x), \sigma_2^\phi(x)) \in A$, for all $x \in X_\phi$. We may suppose without loss of generality that the sets of variables are pairwise disjoint.

Consider the formula $\theta = \bigvee_{\phi \in \Phi} \theta_\phi$ over the set of variables $X = \bigcup_{\phi \in \Phi} X_\phi$ and the two substitutions $\sigma_1 = \bigcup_{\phi \in \Phi} \sigma_1^\phi$, $\sigma_2 = \bigcup_{\phi \in \Phi} \sigma_2^\phi$. Then

$$\Sigma \models \theta(\sigma_1),$$

$$\Delta \models \neg\theta(\sigma_2),$$

and $(\sigma_1(x), \sigma_2(x)) \in A$, for all $x \in X$, proving that $(A, \Sigma, \Delta) \notin S$.

(c_6) If $\exists x\phi \in \Sigma$, consider a $c \in C$ which is new for A, Σ (this is possible because A and Σ only contain a finite number of constants). We prove that $(A, \Sigma \cup \{\phi(c/x)\}, \Delta) \in S$. If $(A, \Sigma \cup \{\phi(c/x)\}, \Delta) \notin S$ then there are an extended graded formula $\theta(X)$ and substitutions σ_1, σ_2 such that $\Sigma \cup \{\phi(c/x)\} \models \theta(\sigma_1)$, $\Delta \models \neg\theta(\sigma_2)$, and $(\sigma_1 x, \sigma_2 x) \in A$, for all $x \in X$. Since c is new for A, Σ we also have $\Sigma \models \theta(\sigma_1)$, $\Delta \models \neg\theta(\sigma_2)$, a contradiction.

We prove the left bisimulation properties.

(b_1) If $(c,d) \in A$, $p \in \mathcal{L}(\phi) \cap \mathcal{L}(\psi)$, and $p(c) \in \Sigma$, suppose by contradiction that there are an extended graded formula $\theta(X)$ and substitutions σ_1, σ_2 such that $\Sigma \models \theta(\sigma_1)$, $\Delta \cup \{p(d)\} \models \neg\theta(\sigma_2)$, and $(\sigma_1 x, \sigma_2 x) \in A$, for all $x \in X$.

Then $\Sigma \models p(c) \wedge \theta(\sigma_1)$, $\Delta \models \neg(p(d) \wedge \theta(\sigma_2))$, and $(A, \Sigma, \Delta) \notin S$, witnessed by the extended formula $\theta'(y, X) = p(y) \wedge \theta(X)$ (which can be rewritten in the form required to be an extended graded formula by moving $p(y)$ inside over disjunctions) and by the substitutions $\sigma_1 \cup \{(y,c)\}$, $\sigma_2 \cup \{(y,d)\}$.

(b_2) for all $h < k$, if $(c,d) \in A$, and a is an action in $\mathcal{L}(\phi) \cap \mathcal{L}(\psi)$ with

$$\{R_a(c, c_i) : i < h\} \cup \{c_i \neq c_j : i < j < h\} \subseteq \Sigma,$$

then consider some constants $d_0, \ldots, d_{h-1}$ new for A, Δ.

Suppose there is an extended graded formula $\theta(X)$ and substitutions σ_1, σ_2 such that

$$\Sigma \models \theta(\sigma_1), \qquad \Delta \cup \{R_a(d, d_i) : i < h\} \cup \{d_i \neq d_j : i < j < h\} \models \neg\theta(\sigma_2),$$

and $(\sigma_1 x, \sigma_2 x) \in A \cup \{(c_i, d_i) : i < h\}$, for all $x \in X$. For each $i < h$, let $Y_i = \{x \in X : \sigma_1 x = c_i, \sigma_2 x = d_i\}$ and consider the set of variables $X' = X \setminus \bigcup_{i<h} Y_i$. If $\theta'(X' \cup \{x\})$ is defined as the formula

$$\exists y_0 \ldots y_{h-1} \ (\bigwedge_{i<h} x R_a y_i \wedge \bigwedge_{i<j} y_i \neq y_j \wedge \theta(y_0/Y_0, \ldots, y_{h-1}/Y_{h-1})),$$

(where each y_i substitutes all variables in Y_i) and σ'_1, σ'_2 are the restrictions of σ_1, σ_2 to X', then

$$\Sigma \models \theta'(\sigma'_1 \cup \{(x,c)\}),$$
$$\Delta \models \neg\theta'(\sigma'_2 \cup \{(x,d)\}).$$

However, θ' is not in the form of an extended graded formula. The problem is that in θ' we require the existence of different successors $y_0 \ldots y_{h-1}$ satisfying the formula θ, where all y_i may occur simultaneously. Since θ *is* an extended graded formula, we can divide out θ so that each y_i has to satisfy a formula without the occurrence of other variables and then we can use Proposition 2.2 to show that the obtained formula is equivalent to an extended graded formula over X'.

Formally, if $\theta = \bigvee_{l \in L} \bigwedge_{j \in J} ST\phi_{l,j}(x_{l,j})$, then θ' is equivalent to the formula

$$\bigvee_{l \in L} \exists y_0 \ldots y_{h-1} \ (\bigwedge_{i<h} x R_a y_i \wedge \bigwedge_{i<j} y_i \neq y_j \wedge \bigwedge_{j \in J} ST\phi_{l,j}(x_{l,j})[\ldots, y_i/Y_i, \ldots]).$$

A variable $x_{l,j}$ is substituted by a y-variable only if $x_{l,j} \in \bigcup_{i<h} Y_i$, proving the equivalence with the formula

$$\bigvee_{l \in L} \exists y_0 \dots y_{h-1} \; (\bigwedge_{i<h} x R_a y_i \wedge \bigwedge_{i<j} y_i \neq y_j \wedge$$

$$\bigwedge_{x_{l,j} \notin \bigcup_{i<h} Y_i} ST\phi_{l,i}(x_{l,j}) \wedge \bigwedge_{i<h} \bigwedge_{x_{l,j} \in Y_i} ST\phi_{l,j}(y_i)),$$

which is in turn equivalent to

$$\bigvee_{l \in L} \bigwedge_{x_{l,j} \notin \bigcup_{i<h} Y_i} ST\phi_{l,i}(x_{l,j}) \wedge \gamma_l(x),$$

where

$$\gamma_l(x) = \exists y_0 \dots y_{h-1} \; (\bigwedge_{i<h} x R_a y_i \wedge \bigwedge_{i<j} y_i \neq y_j \wedge \bigwedge_{i<h} \bigwedge_{x_{l,j} \in Y_i} ST\phi_{l,j}(y_i)).$$

If $\{a\}_h$ is the operator defined in the preliminaries, then $\gamma_l(x)$ is equivalent to the formula

$$\{a\}_h(\bigwedge_{x_{i_j} \in Y_0} \phi_{i_j} \wedge \dots \wedge \bigwedge_{x_{i_j} \in Y_{n-1}} \phi_{i_j}).$$

Since the formulae ϕ_{i_j} belongs to $\mathcal{L}_{\omega_1}^{\diamond<k}$, by Proposition 2.2 it follows that $\gamma_l(x)$ is equivalent to the (standard translation of) a formula ϕ_l in $\mathcal{L}_{\omega_1}^{\diamond<k}$. Hence, θ' is equivalent to the k-extended graded formula

$$\bigvee_{l \in L} \bigwedge_{x_{l,j} \notin \bigcup_{i<h} Y_i} ST\phi_{l,i}(x_{l,j}) \wedge ST\phi_l(x).$$

Then, $(A, \Sigma, \Delta) \notin S$.

Next, we prove the equality property $(=)_4$, leaving the other properties to the patient reader.

$(=_4)$ If $(c, d) \in A$ and $(c = c') \in \Sigma$, suppose that there exists an extended graded formula $\theta(X)$ substitutions σ_1, σ_2 such that

$$\Sigma \models \theta(\sigma_1), \qquad \Delta \models \neg\theta(\sigma_2),$$

and $(\sigma_1 x, \sigma_2 x) \in A \cup \{(c', d)\}$, for all $x \in X$. Consider the set $Y = \{x \in X : \sigma_1 x = c', \sigma_2 x = d\}$, and the substitution

$$\tau_1 x \quad = \quad \begin{cases} \sigma_1 x & \text{if } \; x \in X \setminus Y \\ c & \text{otherwise;} \end{cases}$$

Then $\Sigma \models \theta(\tau_1)$, $\Delta \models \neg\theta(\sigma_2)$, and $(\tau_1 x, \sigma_2 x) \in A$, for all $x \in X$, contradicting $(A, \Sigma, \Delta) \in S$.

Having proved that S is a consistency property modulo $(< k)-$counting bisimulation, we can conclude the proof of our theorem. If ϕ, ψ are sentences in $(\mathcal{L}_C)_{\omega_1}$ that have no graded interpolant, the triple $\{(r,r), \phi, \neg\psi\}$ belongs to S. Otherwise there are an extended graded formula $\theta(X)$ and two substitutions σ_1, σ_2 such that

$$\phi \models \theta(\sigma_1), \qquad \neg\psi \models \neg\theta(\sigma_2),$$

and $(\sigma_1 x, \sigma_2 x) = (r,r)$, for all $x \in X$. Hence $\sigma_1(x) = r, \sigma_2(x) = r$, for all $x \in X$, and we may suppose that $|X| = 1$. Therefore $\theta(\sigma_1) = \theta(\sigma_2)$ is equivalent to a graded formula θ which is an interpolant for ϕ, ψ, a contradiction. Hence $\{(r,r), \phi, \neg\psi\}$ belongs to S and by Lemma 3.2 it follows that there are two $(< k)$-counting bisimilar models $\mathcal{M}, \mathcal{N}$ with $\mathcal{M} \models \phi$, $\mathcal{N} \models \neg\psi$. This proves that ϕ does not entails ψ modulo $(< k)-$counting bisimulation. $\square$

Corollary 4.5 *Let k be a natural number or $\aleph_0$.*

(Invariance) A sentence $\phi \in \mathcal{L}_{\omega_1}$ is invariant for $(< k)-$counting bisimulation iff it is equivalent to a formula in $\mathcal{L}_{\omega_1}^{\diamond<k}$. If ϕ is a first-order sentence, then the equivalent formula can be found in $\mathcal{L}^{\diamond<k}$.

(Interpolation) If ϕ, ψ are sentences in $\mathcal{L}_{\omega_1}^{\diamond<k}$ and ϕ entails ψ along $(< k)$-counting bisimulation w.r.t. $\mathcal{L}(\phi) \cap \mathcal{L}(\psi)$, there is a graded infinitary formula $\theta \in (\mathcal{L}(\phi) \cap \mathcal{L}(\psi))_{\omega_1}^{\diamond<k}$ such that $\phi \models \theta$ and $\theta \models \psi$ (and the finitary version as above).

Proof.
For the first part, notice that a sentence $\phi \in \mathcal{L}_{\omega_1}$ is invariant for $(< k)-$counting bisimulation iff ϕ entails ϕ along $(< k)-$counting bisimulation.

The second part is just Theorem 4.3 applied to the sentences $\phi, \psi \in \mathcal{L}_{\omega_1}^{\diamond<k}$, which can be considered $\mathcal{L}_{\omega_1}$ sentences via the standard translation. $\square$The preservation part of this corollary extends a result [18] characterizing the first order sentences which are equivalent to graded modal formulae. Notice that this corollary gives elementary interpolation for the logic $\mathcal{L}_{\omega_1}^{\diamond<k}$: since two $(< k)$-counting bisimilar structures satisfy the same $\mathcal{L}_{\omega_1}^{\diamond<k}$-sentences, if ϕ entails ψ along $(\mathcal{L}(\phi) \cap \mathcal{L}(\psi))_{\omega_1}^{\diamond<k}$-equivalence, then ϕ entails ψ along $(< k)$-counting bisimulation w.r.t. $\mathcal{L}(\phi) \cap \mathcal{L}(\psi)$, and we can apply the corollary to get the interpolant.

Finally, we remark that it is not possible to improve Corollary 4.5 to full interpolation in the logics $\mathcal{L}_{\omega}^{\diamond<n}$ for a natural $n \geq 3$: just consider the two formulae

$$\phi(p) \equiv (\langle a \rangle_{n-1} p \wedge \langle a \rangle \neg p),$$
$$\psi(q) \equiv (\langle a \rangle_2 q \vee \langle a \rangle_{n-1} \neg q).$$

We have $\phi(p) \models \psi(q)$, but $\phi(q)$ does not entail $\psi(q)$ along $(< n)-$counting bisimulation. Hence ϕ, ψ cannot have an interpolant. The situation is different if we allow the use of the operators $\langle a \rangle_n$ for all natural n: indeed, an interpolant for the formulae ϕ, ψ above is the formula $\langle a \rangle_n \top$. In the next section we prove Craig interpolation for the Infinitary Logic $\mathcal{L}_{\omega_1}^{\diamond<\aleph_0}$.

5 Craig Interpolation for $\mathcal{L}_{\omega_1}^{\diamond<\aleph_0}$

In the previous section we proved a weak form of interpolation for the graded logics $\mathcal{L}_{\omega_1}^{\diamond<k}$, $\mathcal{L}^{\diamond<k}$, for $k \leq \aleph_0$, and we provided a counterexample to interpolation in the logics $\mathcal{L}_{\omega_1}^{\diamond<n}$ for a natural number n. The situation is different for the logics $\mathcal{L}^{\diamond<\aleph_0}$, $\mathcal{L}_{\omega_1}^{\diamond<\aleph_0}$: these logics enjoy the classical Craig interpolation property. This result is already known for $\mathcal{L}^{\diamond<\aleph_0}$ ([1]) and it will be proved here for the logic $\mathcal{L}_{\omega_1}^{\diamond<\aleph_0}$, by using Corollary 4.5 and the notion of unraveling (see Definition 5.1).

For the sake of simplicity, we just deal with languages containing as binary relation symbols only one binary relation R_a, which we denote by R. Similarly, the graded operators $\langle a \rangle_h$ are denoted by $\diamond_h$. However, this restriction is not necessary and the proof could be carried out with minor adaptations also in the case of a countable number of such relations.

By Corollary 4.5 we know that a (necessary and) sufficient condition for a pair (ϕ, ψ) in $\mathcal{L}_{\omega_1}^{\diamond<\aleph_0}$ to have an interpolant is that ϕ entails ψ along $(<\aleph_0)$-counting bisimulation w.r.t. the language $\mathcal{L}_1 = \mathcal{L}(\phi) \cap \mathcal{L}(\psi)$. To prove interpolation for $\mathcal{L}_{\omega_1}^{\diamond<\aleph_0}$ it is then enough to prove that $\phi \models \psi$ implies that ϕ entails ψ along $(<\aleph_0)$-counting bisimulation w.r.t. the language $\mathcal{L}_1$. This can be proved by contraposition. If ϕ does not entails ψ along $(<\aleph_0)$-counting bisimulation w.r.t. $\mathcal{L}_1$, there are two models $\mathcal{M}, \mathcal{N}$ which are $(<\aleph_0)$-counting bisimilar w.r.t. $\mathcal{L}_1$, such that $\mathcal{M} \models \phi$ and $\mathcal{N} \models \neg\psi$. We prove that ϕ does not entails ψ, either: starting from $\mathcal{M}, \mathcal{N}$ we construct a third model $\mathcal{K}$ in the language $\mathcal{L}(\phi) \cup \mathcal{L}(\psi)$ which is $(<\aleph_0)$-counting bisimilar to $\mathcal{M}$ and $\mathcal{N}$ (w.r.t. the languages $\mathcal{L}(\phi)$, $\mathcal{L}(\psi)$, respectively). This implies that $\mathcal{K} \models \phi \wedge \neg\psi$. The idea of the proof is to construct $\mathcal{K}$ using the notion of unraveling:

Definition 5.1 *For any structure $\mathcal{M}$ in a language (Σ, Λ), the* unraveling $\mathcal{M}^U$ *of $\mathcal{M}$ is defined as follows.*

- *Its domain $\mathcal{M}^U$ consists of all finite sequences $v_0, v_1, \ldots, v_k$ where:*

  $v_0 = r^{\mathcal{M}}$;
  for all $i < k$ we have $v_i R^{\mathcal{M}} v_{i+1}$.

  We define the function end : $\mathcal{M}^U \to \mathcal{M}$ as follows: $end(r^{\mathcal{M}}) = r^{\mathcal{M}}$, $end(\sigma w) = w$.

- *The accessibility relations are defined by*

$$\sigma R^{\mathcal{M}^U} \tau \text{ iff } \tau = \sigma w.$$

- *The root is the sequence $r^{\mathcal{M}}$ having only one element.*
- *The unary predicates are defined by*

$$p^{\mathcal{M}^U}(\sigma) \Leftrightarrow p^{\mathcal{M}}(end(\sigma)).$$

□

One can easily prove that the unraveling $\mathcal{M}^U$ of a model $\mathcal{M}$ is counting bisimilar (in particular, $(< \aleph_0)-$counting bisimilar) to $\mathcal{M}$. Suppose the two models $\mathcal{M}$ and $\mathcal{N}$ in the discussion above are such that $\mathcal{M}^U, \mathcal{N}^U$ are isomorphic; then we can take $\mathcal{K} := \mathcal{M}$ and consider it as a $\mathcal{L}(\psi)$-model by *copying* the evaluation of the propositional constants in $\mathcal{L}(\psi)$ from $\mathcal{N}^U$, via the isomorphism between $\mathcal{M}^U$ and $\mathcal{N}^U$. Then $\mathcal{K}$ would be bisimilar to both $\mathcal{M}$ and $\mathcal{N}$. However, it is not true in general that two $(< \aleph_0)$-counting bisimilar models have an isomorphic unraveling (as a counterexample, consider a node with $\aleph_1$-successors and a node with $\aleph_0$-successors), but the result is true when restricted to countable models. This can be seen as a corollary of the more general result that the unraveling of a model can be characterized by the full infinitary logic $\mathcal{L}_\infty^{\mathsf{grad}}$ of graded modalities of the model (see [7, 8], where graded modalities and unravelings are used to model Scott's non-well founded theory of sets). More precisely:

Theorem 5.2 *Two models $\mathcal{M}, \mathcal{N}$ satisfy the same formulae of $\mathcal{L}_\infty^{\mathsf{grad}}$ iff their unravelings $\mathcal{M}^U, \mathcal{N}^U$ are isomorphic.*

Corollary 5.3 *If $\mathcal{M}, \mathcal{N}$ are $(< \aleph_0)$-counting bisimilar countable models then their unravelings $\mathcal{M}^U, \mathcal{N}^U$ are isomorphic.*

Proof.
Since $\mathcal{M}, \mathcal{N}$ are countable, we can prove that any $(< \aleph_0)$-counting bisimulation Z between the two models is a full counting bisimulation. If wZv and $X \subseteq R(w)$ then X is either finite or $|X| = \aleph_0$. Writing $X = \{w_1, \ldots, w_n, \ldots\}$, we can consider the finite sets $X_n = \{w_1, \ldots, w_n\}$: since Z is an $(< \aleph_0)$-counting bisimulation we have that

$$|\{y \in R(v) : \exists x \in X_n \; xZy\}| \geq |X_n|,$$

from which we deduce that

$$|\{y \in R(v) : \exists x \in X \; xZy\}| \geq |X|.$$

Having proved that there is a full counting bisimulation between $\mathcal{M}, \mathcal{N}$, we know from Lemma 2.5 that the two models satisfy the same formulae of $\mathcal{L}_\infty^{grad}$. From Theorem 5.2 it follows that the unravelings $\mathcal{M}^U$ and $\mathcal{N}^U$ are isomorphic.
□

We can now reconsider the informal discussion given before Definition 5.1 to give a proof of:

Lemma 5.4 *The logic $\mathcal{L}_{\omega_1}^{<\aleph_0}$ enjoys interpolation.*

Proof.
Suppose that ϕ and ψ do not have an interpolant. We prove that ϕ does not entails ψ. By Corollary 4.5 we know that ϕ does not entails ψ along $(<\aleph_0)$-counting bisimulation w.r.t. $\mathcal{L}_1$, that is, there are two models $\mathcal{M}, \mathcal{N}$ which are $(<\aleph_0)$-counting bisimilar w.r.t. $\mathcal{L}_1$, such that $\mathcal{M} \models \phi$ and $\mathcal{N} \models \neg\psi$. We can suppose without loss of generality that $\mathcal{M}$ and $\mathcal{N}$ are countable. To show this, consider the language $L = \{M, N, r_M, r_N, R, Z\} \cup \{p : p \in \mathcal{L}_1\}$, where M, N are unary predicates, R, Z are binary, r_M, r_N are constants, and $\{p : p \in \mathcal{L}_1\}$ is the set of unary predicates of $\mathcal{L}_1$. We can then write a $\mathcal{L}_{\omega_1}$- sentence Θ which expresses the existence of models $\mathcal{M}, \mathcal{N}$ as above. Since Θ is a consistent $\mathcal{L}_{\omega_1}$-sentence, by the Lowenheim-Skolem Theorem for $\mathcal{L}_{\omega_1}$ we obtain that Θ has a countable model. From this model we can easily extract two *countable* models $\mathcal{M}, \mathcal{N}$ which are $(<\aleph_0)$-counting bisimilar w.r.t. $\mathcal{L}_1$, such that $\mathcal{M} \models \phi$ and $\mathcal{N} \models \neg\psi$. By Corollary 5.3, we know that the unravelings $\mathcal{M}^U$ and $\mathcal{N}^U$ are isomorphic. Consider the $\mathcal{L}(\phi) \cup \mathcal{L}(\psi)$-model $\mathcal{K}$ which is equal to $\mathcal{M}^U$ as a model of the language $\mathcal{L}(\phi)$, while the interpretation of the propositional constants in $\mathcal{L}(\psi)$ is obtained by copying the interpretation of these constants in $\mathcal{N}^U$, via the isomorphism of this model with $\mathcal{M}$. In this way we obtain a model $\mathcal{K}$ for the language $\mathcal{L}(\phi) \cup \mathcal{L}(\psi)$ which is isomorphic to both the unravelings of $\mathcal{M}$ and $\mathcal{N}$ w.r.t. the language $\mathcal{L}(\phi) \cup \mathcal{L}(\psi)$. Then, again by Theorem 5.2, $\mathcal{K}$ satisfies the same formulae of $(\mathcal{L}(\phi) \cup \mathcal{L}(\psi))_{\infty}^{grad}$ as $\mathcal{M}$ and $\mathcal{N}$, hence $\mathcal{K} \models \phi$ and $\mathcal{K} \models \neg\psi$. This proves that ϕ does not entails ψ. □

6 Concluding remarks.

In this paper we investigate interpolation in the family of (infinitary) graded modal logics, proving elementary interpolation and Craig interpolation whenever possible. One question we left unsolved here is whether the logic $\mathcal{L}_{\infty}^{grad}$ enjoys interpolation or not (in a previous version of this work (see [7]) I erroneously claimed to have a proof for it, but Alexandru Baltag pointed me a mistake in the construction). We also have some general questions regarding the method of consistency property modulo bisimulation. We list them below.

- Can we fruitfully use this technique to prove results in other logics as well? In any logic which is invariant under bisimulation, the Amalgamation Lemma allows to prove interpolation by showing that if ϕ entails ψ *modulo bisimulation* then ϕ, ψ have an interpolant. In this paper we showed how this weak form of interpolation can be proved via consistency property. What about this technique in PDL?
- In this chapter we characterized pairs of formulae having an interpolant in logics which do not have Craig interpolation. What about doing the same with uniform interpolation? For example, we know that PDL does

not enjoy uniform interpolation. Can we characterize formulae of PDL having a uniform interpolant? Between them we certainly have all basic modal logic formulae, but these are not all. For example, $\langle a^* \rangle p$ is not a basic modal logic formula but it has $\top$ as uniform interpolant w.r.t. the language $\{a\}$ (more generally, all positive $\phi(p)$ have $\phi(\top)$ as a uniform interpolant).

- Another interesting application of the method of generalized consistency property is to use it to obtain a generalized Gentzen Proof Theory, with inference rules manipulating ternary sequents as described in [6].

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