Toward a Dynamic Logic of Questions

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1 Introduction and motivation

Questions are different from statements, but they are just as important in driving reasoning, communication, and general processes of investigation. The first logical studies merging questions and propositions seem to have come from the Polish tradition: cf. [24]. A forceful modern defender of this dual perspective is Hintikka, who has long pointed out how any form of inquiry depends on an interplay of inference and answers to questions. Cf. [17] and [16] on the resulting 'interrogative logic', and the epistemological views behind it. These logics are mainly about *general inquiry* and learning about the world. But there is also a related stream of work on the *questions* in natural language, as important speech acts with a systematic linguistic vocabulary. Key names are Groenendijk & Stokhof: cf. [14, 12], and the recent 'inquisitive semantics' of [13] ties this in with a broader information-oriented 'dynamic semantics'. Logic of inquiry and logic of questions are related, but there are also differences in thrust: a logic of 'issue management' that fits our intuitions is not necessarily the same as a logic of speech acts that must make do with what natural language provides.

In this paper, we do not choose between these streams, but we propose a different technical approach. Our starting point is a simple observation. Questions are evidently important informational actions in human agency. Now the latter area is the birth place of *dynamic-epistemic logic* of explicit events that make information flow. But surprisingly, existing dynamic-epistemic systems do not give an explicit account of what questions do! In fact, central examples in the area have questions directing the information flow (say, by the Father in the puzzle of the Muddy Children) – but the usual representations in systems like PAL or DEL leave them out, and merely treat the answers, as events of public announcement. Can we make questions themselves first-class citizens in dynamic-epistemic logic, and get closer to the dynamics of inquiry? We will show that we can, following exactly the methodology that has already worked in other areas, and pursuing the same general issues: what are natural acts of inquiry, and how can dynamic logics bring out their structure via suitable recursion axioms? Moreover, by doing so, we at once get an account of non-factual questions, multi-agent aspects, temporal sequences, and other themes that have already been studied in a DEL setting.

Our analysis starts with Section 2 on dynamic logic of public questions and public inquiry, a natural companion to the public announcement logic PAL. On a suitable static base logic of information and issues, we identify key dynamic actions of 'issue management', discuss some of their properties, and present a complete dynamic logic. This is the bulk of the present paper, but we follow up with two themes showing how this system leads to natural extensions inside DEL: temporal protocols that regulate what questions can be asked (and answers can be given) in Section 3, and multi-agent perspectives on public and private questions in Section 4. In Section 5, we discuss where we stand with this 'proof of concept', and which further directions look promising. This view of extensions is especially pressing since we do not view the semantic model in this paper as an end in itself. We feel that realistic management of questions and issues involve 'intensional' structure beyond the usual semantic models, and we give an opening toward the more finely-grained 'agenda dynamics' that we intend to pursue later. Section 6 is a very brief comparison with existing 'logic of interrogation' and 'inquisitive logic', to see where our approach diverges.

2 A toy-system of asking and announcing

The methodology of dynamic-epistemic logic starts with a static base logic describing states of some informational phenomenon, and identifies relevant informational state-changing events. Then, dynamic modalities are added to the base language, and their complete logic is determined on top of the given logic of the static models. To work in the same style, we need a convenient static semantics to 'dynamify'. We take such a model from existing semantics of public questions, considering only one agent first, for simplicity. We work in the style of public announcement logic PAL, though our logic of questions will also have its differences.

2.1 Epistemic issue models

A simple framework for representing questions uses an equivalence relation over some relevant domain of alternatives, that we will call the 'issue relation'. This idea is found in many places, from linguistics (cf. [14]) to learning theory (cf. [18]): the current 'issue' is a partition of the set of options, with partition cells standing for the areas where we would like to be. This partition may be induced by a conversation whose current focus are the issues that have been put on the table, or a game where finding out about certain issues has become important to further play, a learning scenario for the language fed to us by our environment, or even a whole research program with an agenda determining what is currently under investigation. The 'alternatives' or worlds may range here from simple finite settings like deals in a card game to complex infinite histories representing a total life experience. Formally, all this reduces to the following structure:

Definition 1 (Epistemic Issue Model)**.** *An* epistemic issue model *is a structure* $M = \langle W, \sim, \approx, V \rangle$ *with:*

- *W is a set of possible worlds or states (epistemic alternatives),*
- *$\sim$ is an equivalence relation on W (epistemic indistinguishability),*
- *$\approx$ is an equivalence relation on W (the abstract issue relation),*
- *$V : \mathtt{P} \to \wp(W)$ is a valuation for atomic propositions $p \in \mathtt{P}$.*

We could introduce models with more general structure, but equivalence relations will suffice for the points that we are trying to make in this paper.

2.2 Static language of information and issues

To work with these structures, we need matching modalities in our language. Here we make a minimal choice of modal and epistemic logic for state spaces plus two modalities describing the issue structure. First, $K\varphi$ talks about knowledge or semantic information of an agent, its informal reading is "φ is known", and its explanation is as usual: "φ holds in all epistemically indistinguishable worlds". To describe our models a bit more further, we add a universal modality $U\varphi$ saying that "φ is true in all worlds". Next, we use $Q\varphi$ to say that, locally in our current world, the current issue validates φ: "φ holds in all issue-equivalent worlds". While convenient, this local notion does not express the global assertion that the current issue *is* φ, which will be defined later. Finally, we find a need for a notion that mixes the epistemic and issue relations, talking (roughly) about what would be the case if the issue were resolved given what we already know. Technically, we add an intersection modality $R\varphi$ saying that "φ holds in all epistemically indistinguishable and issue equivalent worlds". While such modalities are frequent in many settings, they complicate axiomatization. We will assume the standard device of adding *nominals* naming single worlds (cf. [11, 19] for recent instances of this technique in the *DEL* setting).[1]

[1]As one illustration, working with nominals requires a modified valuation function in Definition 1, to a $V : \mathtt{P} \uplus \mathtt{N} \to \wp(W)$ mapping every proposition $p \in \mathtt{P}$ to a set of states

Definition 2 (Static Language)**.** *The language* $\mathcal{L}_{\mathbf{EL_Q}}(\mathtt{P}, \mathtt{N})$ *has disjoint countable sets* $\mathtt{P}$ *and* $\mathtt{N}$ *of propositions and nominals, respectively, with* $p \in \mathtt{P}$, $i \in \mathtt{N}$. *Its formulas are defined by the following inductive syntax rule:*

$$\bot \mid p \mid i \mid \neg\varphi \mid (\varphi \wedge \psi) \mid K\varphi \mid Q\varphi \mid R\varphi \mid U\varphi$$

Modal formulas of this static language are interpreted in the following way:

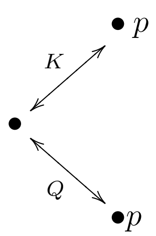
Definition 3 (Interpretation)**.** *Formulas are interpreted in models* M *at worlds* w *with the usual Boolean clauses, and the following modal ones:*

$$\begin{array}{lll} M \models_w K\varphi & \text{iff} & \text{for all } v \in W : w \sim v \text{ implies } M \models_v \varphi, \\ M \models_w Q\varphi & \text{iff} & \text{for all } v \in W : w \approx v \text{ implies } M \models_v \varphi, \\ M \models_w R\varphi & \text{iff} & \text{for all } v \in W : w \,(\sim \cap \approx)\, v \text{ implies } M \models_v \varphi, \\ M \models_w U\varphi & \text{iff} & \text{for all } w \in W : M \models_w \varphi, \end{array}$$

This semantics validates a number of obvious principles reflecting connections between our modalities. In particular, the following are valid:

$$U\varphi \to K\varphi, U\varphi \to Q\varphi, U\varphi \to R\varphi, \text{ and also } K\varphi \to R\varphi, Q\varphi \to R\varphi$$

Corresponding facts hold for existential modalities $\widehat{U}$, etc., defined as usual.



Next, the *intersection modality* $R\varphi$ cannot be defined in terms of others. In particular, $\widehat{R}\varphi$ is not equivalent with $\widehat{K}\varphi \wedge \widehat{Q}\varphi$, witness the counterexample in the figure in the left. However, the use of so-called 'nominals' i from hybrid logic helps us to completeness, by the valid converse

$$\widehat{K}(i \wedge \varphi) \wedge \widehat{Q}(i \wedge \varphi) \to \widehat{R}\varphi$$

Our modal language can define various basic global statements describing the current structure of inquiry. For instance, here is how it says which propositions φ are 'settled' by the current issue:

Definition 4 (Settlement)**.** *The current issue* settles *fact* φ *iff* $U(Q\varphi \vee Q\neg\varphi)$.

2.3 Static base logic of information and issues

As for reasoning with our language, we write $\models \varphi$ if the static formula φ is true in every model at every world. The static epistemic logic $\mathbf{EL_Q}$ of questions in our models is the set of all validities:

$$\mathbf{EL_Q} = \{\varphi \in \mathcal{L}_{\mathbf{EL_Q}} : \ \models \varphi\}$$

All instances of propositional tautologies	
$\mathbf{K}, \mathbf{T}, \mathbf{B}, \mathbf{4}, \mathbf{5}$ axioms for U, K, Q and R	Epistemic S5
$\widehat{U}(i \wedge \varphi) \to U(i \to \varphi)$	Nominals
$U\varphi \to K\varphi$, $U\varphi \to Q\varphi$, $U\varphi \to R\varphi$	Inclusion
$\widehat{R}i \leftrightarrow \widehat{K}i \wedge \widehat{Q}i, K\varphi \to R\varphi, Q\varphi \to R\varphi$	Intersection
From φ infer $U\varphi$	Necessitation
From φ and $\varphi \to \psi$ infer ψ	Modus Ponens

Table 1: The proof system $\mathbf{EL_Q}$.

We write $\vdash_s \varphi$ if φ is provable in the system of Table 1 above. These laws of reasoning derive many intuitive principles. For instance, here is how agents have introspection about the current public issue:

$$U(Qp \vee Q\neg p) \vdash UU(Qp \vee Q\neg p) \vdash KU(Qp \vee Q\neg p)$$

Theorem 1 (Completeness of $\mathbf{EL_Q}$)**.** *For every formula* $\varphi \in \mathcal{L}_{\mathbf{EL_Q}}(\mathtt{P}, \mathtt{N})$*:*

$$\models_p \varphi \quad \textit{if and only if} \quad \vdash_s \varphi$$

Proof. By standard techniques for multi-modal hybrid logic. □

2.4 Dynamic actions of issue management

Now we look into basic actions that change the issue relation in a given model. We do this first by some pictures. In the figures, epistemic indistinguishability is represented by lines linking possible worlds, and the issue relation is represented by partition cells. For simplicity, we start with the initial issue as the universal relatin, represented as a frame border.

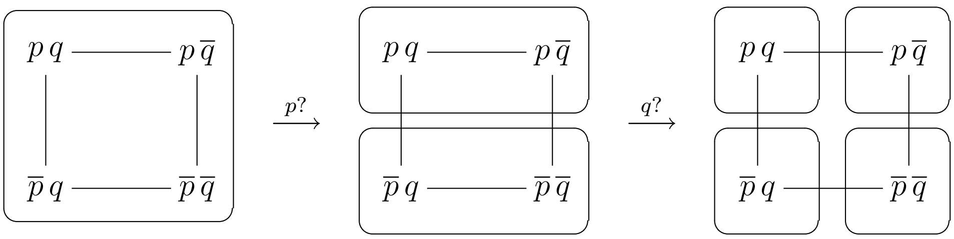


Figure 1: Effects of Asking Yes/No Questions.

$V(p) \subseteq W$, but every nominal $i \in \mathtt{N}$ to a singleton set $V(i) \in W$.

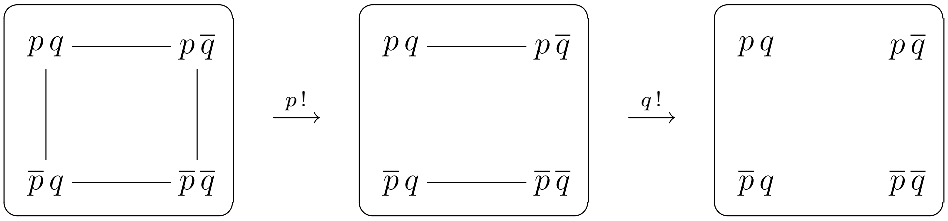


Figure 2: Almost Symmetrical Effects of 'Soft' Announcing.

In Figure 1, the first transition illustrates the effect of asking a question: the issue relation is split into p and $\neg p$ cells. The second transition illustrates the effect of asking a second question: the issue partition is further refined.

In Figure 2, the first transition gives the effect of making an announcement: the indistinguishability links between p and $\neg p$ worlds are removed. The second transition illustrates the effect of making a second announcement, the epistemic partition is further refined. Here we use a special sort of event that is congenial to this setting, viz. the *link-cutting announcements* of van Benthem & Liu [6] that do not throw away worlds.

There is a certain symmetry between asking a question and making a soft announcement. One refines the issue, the other the information partition:

Definition 5 (Questions & Announcements)**.** *The execution of a $\varphi?$ action in a given model M results in a changed model $M_{\varphi?} = \langle W_{\varphi?}, \sim_{\varphi?}, \approx_{\varphi?}, V_{\varphi?}\rangle$, with $\stackrel{\varphi}{\equiv}_M = \{(w,v) \mid \|\varphi\|^M_w = \|\varphi\|^M_v\}$. Likewise, a $\varphi!$ action results in $M_{\varphi!} = \langle W_{\varphi!}, \sim_{\varphi!}, \approx_{\varphi!}, V_{\varphi!}\rangle$, and we then have:*

$$\begin{array}{rcl} W_{\varphi?} & = & W \\ \sim_{\varphi?} & = & \sim \\ \approx_{\varphi?} & = & \approx \cap \stackrel{\varphi}{\equiv}_M \\ V_{\varphi?} & = & V \end{array} \qquad\qquad \begin{array}{rcl} W_{\varphi!} & = & W \\ \sim_{\varphi!} & = & \sim \cap \stackrel{\varphi}{\equiv}_M \\ \approx_{\varphi!} & = & \approx \\ V_{\varphi!} & = & V \end{array}$$

The symmetry in this mechanism is lost if we let $p!$ be an executable action only if it is *truthful*, while the corresponding question $p?$ is executable in every world in a model, even those not satisfying p.

One attractive thing about our setting is that it can accomodate further natural dynamic operations on information and issues.

In particular, Figure 3 contains two more issue management actions. In the first example two Yes/No questions $p?$ and $q?$ are asked and a *resolving* action follows on the epistemic relation. In the second, two announcements $p!$ and $q!$ are made, and a *refinement* action follows on the issue relation,

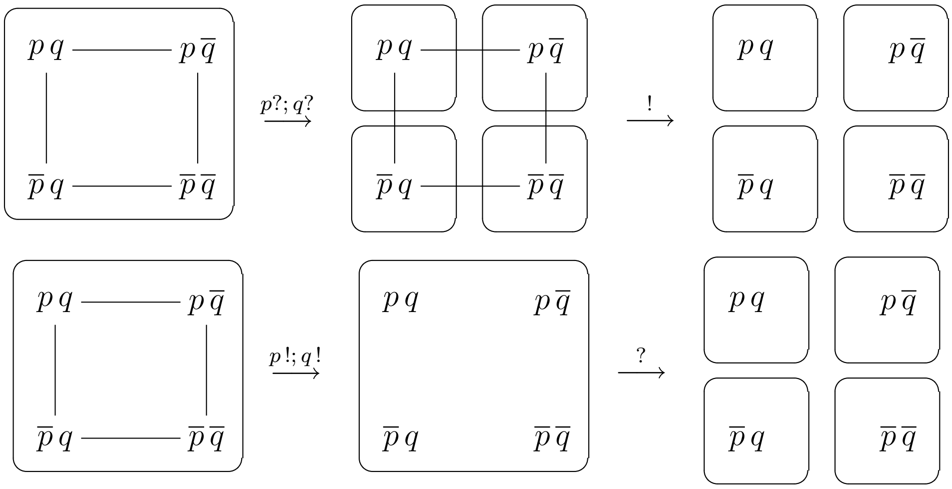


Figure 3: Resolving and Refining Actions.

adjusting it to what agents already know. These operations are natural generalizations of asking and announcing, that need not have natural language correspondents. They can be formally defined as:

Definition 6 (Resolution and Refinement)**.** *Execution of a 'resolve' action* ?, *and of a 'refine' action* ! *in model* M *results in changed models* $M_? = \langle W_?, \sim_?, \approx_?, V_? \rangle$, *respectively,* $M_! = \langle W_!, \sim_!, \approx_!, V_! \rangle$ *with:*

$$\begin{array}{rcl} W_? & = & W \\ \sim_? & = & \sim \\ \approx_? & = & \approx \cap \sim \\ V_? & = & V \end{array} \qquad \begin{array}{rcl} W_! & = & W \\ \sim_! & = & \sim \cap \approx \\ \approx_! & = & \approx \\ V_! & = & V \end{array}$$

Again, these two actions are symmetric - suggesting that we could view, say, the 'issue manager' as an epistemic information agent.

In light of he previous observations we can also note that the 'refine' action behaves analogously to the intersection of indistinguishability relations in the usual treatment of "Distributed group knowledge" in the literature. This notion of group knowledge represents what the agents would know by pooling their information. This observation will become even more relevant in a multi-agent setting and in combination with other group notions for issue managment, defining an issue that is common to a group of agents.

2.5 Dynamic language of issue management

In order to talk about the above changes, dynamic modalities are added to the earlier modal language of static epistemic situations:

Definition 7 (Dynamic Language)**.** *Language* $\mathcal{L}_{\mathbf{DEL_Q}}(\mathtt{P},\mathtt{N})$ *is defined by adding the following clauses to Definition 2:* $\cdots \mid [\varphi!]\psi \mid [\varphi?]\psi \mid [?]\varphi \mid [!\,]\varphi$

These are interpreted by adding the following clauses to Definition 3:

Definition 8 (Interpretation)**.** *Formulas are interpreted in* M *at* w *by the following clauses, where models* $M_{\varphi?}$, $M_{\varphi!}$, $M_?$ *and* $M_!$ *are as defined above:*

$$
\begin{array}{lll}
M \models_w [\varphi!]\psi & \textit{iff} & M \models_w \varphi \textit{ implies } M_{\varphi!} \models_w \psi, \\
M \models_w [\varphi?]\psi & \textit{iff} & M_{\varphi?} \models_w \psi, \\
M \models_w [?]\varphi & \textit{iff} & M_? \models_w \varphi \\
M \models_w [!\,]\varphi & \textit{iff} & M_! \models_w \varphi
\end{array}
$$

In this dynamic language, useful notions about questions and their relation with answers or announcements can be expressed that have been previously proposed in the literature. Without further explanation here, we only mention two of them:

Definition 9 (Question Entailment)**.** *For formulas* $\varphi_0, \ldots, \varphi_n, \psi \in \mathcal{L}^{\text{prop}}_{\mathbf{DEL_Q}}$ $?\varphi_0, \ldots, ?\varphi_n$ entail $?\psi$ *iff* $\models_p [\varphi_0?] \cdots [\varphi_n?] U((\psi \to Q\psi) \land (\neg\psi \to Q\neg\psi))$

Definition 10 (Answer Compliance)**.** *For formulas* $\varphi_0, \ldots, \varphi_n, \psi \in \mathcal{L}^{\text{prop}}_{\mathbf{DEL_Q}}$ $?\varphi_0; \cdots ; ?\varphi_n$ license $!\psi$ *iff* $\models_p [\varphi_0?] \cdots [\varphi_n?] \neg((\neg\psi \land \widehat{Q}\psi) \lor (\psi \land \widehat{Q}\neg\psi))$

An interesting property, also observed in a preference change context in [6] and [5], is that, unlike for standard announcements, implemented by world elimination, there is no 'action contraction' principle equating two or more successive questions to a single one with identical effect:

Fact 1 (Proper Iteration)**.** *There is no question composition principle.*

Proof. If there were one single assertion having just the same effect as a sequence $\varphi?; \psi?$, then, starting with the issue configured as the universal relation on the domain of a model, such a sequence will always induce a two, not four, element partition; this refutation is also depicted in Figure 3. □

Another interesting property consists in the fact that validities encode reasoning in advance about later epistemic effects of asking questions and answering them. Here is an example, without proof:

Fact 2 (Questioning Thrust)**.** *The formula* $K[\varphi?][\,!\,]U(K\varphi \lor K\neg\varphi)$ *is valid.*

This says that agents know that the effect of a question followed by resolution is knowledge. Such results can be generalized to more complex types of questions. These and other examples show how our logic encodes a formal base theory of question answering, some parts of which have been previously investigated in a more ad-hoc manner in the literature.

Finally, this system brings to light phenomena reminiscent of DEL. For instance, asking the same question repeatedly can have different effects on a model, as illustrated in Figure 4, where:

$$\xi := (\widehat{Q}i \to (j \vee k)) \wedge ((\widehat{Q}j \wedge p) \to \widehat{Q}i);$$

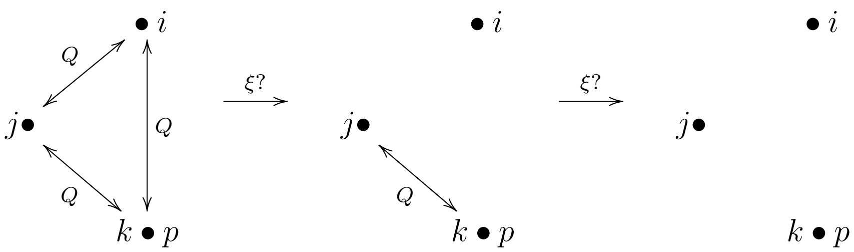


Figure 4: Asking the Same Question Once or Twice.

2.6 Complete dynamic logic of informational issues

The examples discussed so far show that predicting epistemic effects of asking questions is not always easy to keep straight, but they also suggest there is an interesting algebra of operations on models that deserves separate study. For both purposes, we propose a complete axiomatization for the dynamic epistemic logic of questions.

Satisfaction and validity are defined as before. The dynamic epistemic logic of questioning based on a partition modeling (henceforth, $\mathbf{DEL_Q}$) is defined as the set of all validities:

$$\mathbf{DEL_Q} = \{\varphi \in \mathcal{L}_{\mathbf{DEL_Q}}(\mathtt{P}, \mathtt{N}) : \ \models \varphi\}$$

We introduce a new proof system by adding the reduction axioms in Table 2 to the proof system for the static fragment from Table 1. We write $\vdash \varphi$ iff φ is provable in the system from Tables 1 and 2.

To save some space, in this short paper, Table 2 only lists axioms for two of the four dynamic modalities. *Soft announcements* $\langle \varphi ! \rangle$ satisfy the usual

PAL-style axioms given in [6], plus principles for its interaction with the two new base modalities involving questions, such as

$$\langle\varphi!\rangle\widehat{Q}\psi \leftrightarrow (\varphi \wedge \widehat{Q}\langle\varphi!\rangle\psi)$$

Also, the axioms for the *refinement action* [?] or are those listed below for the 'resolving' action with the modalities K and Q interchanged.

$[\varphi?]a \leftrightarrow a$	*Asking & Atoms*
$[\varphi?]\neg\psi \leftrightarrow \neg[\varphi?]\psi$	*Asking & Negation*
$[\varphi?](\psi \wedge \chi) \leftrightarrow [\varphi?]\psi \wedge [\varphi?]\chi$	*Asking & Conjunction*
$[\varphi?]K\psi \leftrightarrow K[\varphi?]\psi$	*Asking & Knowledge*
$[\varphi?]Q\psi \leftrightarrow (\varphi \wedge Q(\varphi \to [\varphi?]\psi)) \vee (\neg\varphi \wedge Q(\neg\varphi \to [\varphi?]\psi))$	*Asking & Partition*
$[\varphi?]R\psi \leftrightarrow (\varphi \wedge R(\varphi \to [\varphi?]\psi)) \vee (\neg\varphi \wedge R(\neg\varphi \to [\varphi?]\psi))$	*Asking & Intersection*
$[\varphi?]U\psi \leftrightarrow U[\varphi?]\psi$	*Asking & Universal*
$[\,!\,]a \leftrightarrow a$	*Resolving & Atoms*
$[\,!\,]\neg\varphi \leftrightarrow \neg[\,!\,]\varphi$	*Resolving & Negation*
$[\,!\,](\psi \wedge \chi) \leftrightarrow [\,!\,]\psi \wedge [\,!\,]\chi$	*Resolving & Conjunction*
$[\,!\,]K\varphi \leftrightarrow R[\,!\,]\varphi$	*Resolving & Knowledge*
$[\,!\,]Q\varphi \leftrightarrow Q[\,!\,]\varphi$	*Resolving & Partition*
$[\,!\,]R\varphi \leftrightarrow R[\,!\,]\varphi$	*Resolving & Intersection*
$[\,!\,]U\varphi \leftrightarrow U[\,!\,]\varphi$	*Resolving & Universal*

Table 2: Reduction axioms for $\mathbf{DEL_Q}$.

Theorem 2 (Soundness). *The reduction axioms in Table 2 are sound.*

Proof. By standard modal arguments. We discuss two cases that go beyond mere commutation of operators. The first (*Asking & Partition*) explains how questions refine a partition:

$$[\varphi?]Q\psi \leftrightarrow (\varphi \wedge Q(\varphi \to [\varphi?]\psi)) \vee (\neg\varphi \wedge Q(\neg\varphi \to [\varphi?]\psi))$$

(from left to right) Assume $M \models_w [\varphi?]Q\psi$ then we also have $M_{\varphi?} \models_w Q\psi$. In case $M \models_w \varphi$, suppose $M \models_w (\varphi \wedge Q(\varphi \to [\varphi?]\psi)) \vee (\neg\varphi \wedge Q(\neg\varphi \to [\varphi?]\psi))$ does not hold, then we can proceed by cases. If $M \models_w \neg Q(\varphi \to [\varphi?]\psi)$ and $M \models_w \varphi$, then we have $\exists v \in W : w \approx v$ and $M \models_v \varphi \wedge \neg[\varphi?]\psi$, therefore, $w \stackrel{\varphi}{\equiv} v$, and from this we have $w \approx_{\varphi?} v$. But we also have $M_{\varphi?} \models_v \neg\psi$, hence $M\varphi? \models_w \neg Q\psi$, which contradicts our initial assumption. For the remaining interesting case $M \models_w \neg Q(\varphi \to [\varphi?]\psi)$ and $M \models_w \neg Q(\neg\varphi \to [\varphi?]\psi)$ the argument is similar. In case $M \models_w \neg\varphi$ we can reason analogously.

Our second illustration (*Resolving & Knowledge*) shows how resolution changes knowledge making crucial use of our intersection modality:

$$[\,!\,]K\varphi \leftrightarrow R[\,!\,]\varphi$$

Let $M \models_w [\,!\,]K\varphi$. Then we have equivalently, $M_Q \models_w K\varphi$ and from this we get $\forall v \in W_Q : w \sim_Q v$ implies $M_Q \models_v \varphi$. As $\sim_Q = \sim \cap \approx$, we can obtain equivalently $\forall v \in W : w\,(\sim \cap \approx)\,v$ implies $M_Q \models_v \varphi$, and from this, by the semantics of our dynamic modality, we get $M \models_w R[\,!\,]\varphi$ as desired. □

Theorem 3 (Completeness of $\mathbf{DEL_Q}$). *For every formula $\varphi \in \mathcal{L}_{\mathbf{DEL_Q}}(\mathtt{P}, \mathtt{N})$:*

$$\models_p \varphi \quad \textit{if and only if} \quad \vdash \varphi.$$

Proof. Proceeds by a standard DEL-style translation argument. Working inside out, the reduction axioms translate dynamic formulas into corresponding static ones, in the end completeness for the static fragment is invoked. □

Remark (Hidden validities). Finally, although the system DEL_Q is complete, like PAL, it still leaves something to be desired. We have said already that our new model operations of issue management have a nice algebraic structure. For instance, it is easy to see that resolving is idempotent: $!; ! = !$, while it commutes with refinement: $!; ? = ?; !$. But our logic does not state such facts explicitly, since, by working from innermost occurrences of dynamic modalities, our completeness argument needed no recursion axioms with stacked modalities like [!][!]. Nevertheless, this is obviously crucial information for a logic of issue management, and hence, the structure of such schematic validities concerning operator stacking remains to be investigated.

3 Temporal protocols with questions

Single announcements usually only make sense in a longer-term temporal perspective of some ongoing informational process: a conversation, an experimental protocol, a learning mechanism, and so on. To make this procedural information explicit, van Benthem, Gerbrandy, Hoshi & Pacuit 2009 [4] have introduced *protocols* into dynamic-epistemic logic. This results in modified versions of the public announcement logic PAL, which now encode procedural as well as factual and epistemic information.

But the very same move applies to questions: not everything can be asked, either because of social convention, or the limitations on our measuring apparatus, or financial resources. Thus, it makes sense to adapt our dynamic logic to a protocol setting, and we will now show how this can be done, resulting in a more realistic logical theory of inquiry.

Definition 11 ($\mathrm{DEL_Q}$ Protocol)**.** *Let Σ be an arbitrary set of epistemic events (questioning actions). Let Σ^* be the set of finite strings over Σ (finite histories of questioning events). A* questioning protocol *is a set $\mathcal{H} \subseteq \Sigma^*$ (containing all non-empty finite histories and all their prefixes, or rooted sub-histories) such that $\mathsf{FinPre}_{-\lambda}(\mathcal{H}) = \{h \mid h \neq \lambda, \exists h' \in \mathcal{H} : h \preceq h'\} \subseteq \mathcal{H}$.*

During the construction in the following definition the only sequences considered are those of the form $w\sigma$, where w is a world in the initial model M, and σ a sequence in the protocol Q, σ_n denotes the sequence σ up to its n-th position and $\sigma_{(n)}$ denotes the n-th element in the sequence.

Definition 12 (Q-Generated Model)**.** *Let $M = \langle W, \sim, \approx, V\rangle$ be an arbitrary model and let Q be an arbitrary $\mathrm{DEL_Q}$ protocol over model M (a prefix-closed set of finite sequences of questioning events). The Q-*Generated Model *at level n, $M^n_Q = \langle W^n_Q, \sim^n_Q, \approx^n_Q, V^n_Q\rangle$ is defined by induction on n as follows:*

1 $W^0_Q = W, \quad \sim^0_Q = \sim, \quad \approx^0_Q = \approx, \quad V^0_Q = V,$

2 $w\sigma \in W^{n+1}_Q$ *if and only if* $w \in \mathrm{dom}(M)$, $\sigma \in Q$, $\mathtt{len}(\sigma) = n+1$, *and* $w\sigma_n \in W^n_Q$,

3 If $\sigma_{(n+1)} = \langle ! \rangle$ *then: (a)* $(w\sigma, v\sigma') \in \sim^{n+1}_Q$ *iff* $(w\sigma_n, v\sigma'_n) \in \sim^n_Q$, $(w\sigma_n, v\sigma'_n) \in \approx^n_Q$, *and* $\sigma_{(n+1)} = \sigma'_{(n+1)}$; *and (b)* $(w\sigma, v\sigma') \in \approx^{n+1}_Q$ *iff* $(w\sigma_n, v\sigma'_n) \in \approx^n_Q$, *and* $\sigma_{(n+1)} = \sigma'_{(n+1)}$,

4 If $\sigma_{(n+1)} = \langle \varphi ? \rangle$ *then: (a)* $(w\sigma, v\sigma') \in \approx^{n+1}_Q$ *iff* $\sigma_{(n+1)} = \sigma'_{(n+1)}$, $(w\sigma_n, v\sigma'_n) \in \approx^n_Q$, *and* $(\sigma_{(n+1)}, \sigma'_{(n+1)}) \in \equiv_{M^n_Q}$; *and (b)* $(w\sigma, v\sigma') \in \sim^{n+1}_Q$ *iff* $(w\sigma_n, v\sigma'_n) \in \sim^n_Q$, *and* $\sigma_{(n+1)} = \sigma'_{(n+1)}$.

The class of structures $\mathsf{Forest}(\mathrm{TDEL_Q})$ *consists of all models* $\mathsf{Forest}(M, Q)$ *for some arbitrary model M and some arbitrary* $\mathrm{TDEL_Q}$ *protocol Q.*

Next we give a truth definition for a suitable dynamic language, where we will assume that all dynamic actions involve formulas from the static base language only.

Definition 13 (Interpretation)**.** *Truth of formulas is given at state h in model* $\mathsf{Forest}(M, \mathcal{Q}) := \mathsf{Fr}(M, \mathcal{Q}) = \langle \mathcal{H}, \sim, \approx, V\rangle$, *by the following definition:*

- $\mathsf{Fr}(M, \mathcal{Q}) \models_h K\varphi$ *iff* $\forall h \in \mathcal{H} : h \sim h'$ *implies* $\mathsf{Fr}(M, \mathcal{Q}) \models_{h'} \varphi$

- $\mathsf{Fr}(M,\mathcal{Q}) \models_h Q\varphi$ *iff* $\forall h \in \mathcal{H} : h \approx h'$ *implies* $\mathsf{Fr}(M,\mathcal{Q}) \models_{h'} \varphi$

- $\mathsf{Fr}(M,\mathcal{Q}) \models_h R\varphi$ *iff* $\forall h \in \mathcal{H} : h\,(\sim \cap \approx)\,h'$ *implies* $\mathsf{Fr}(M,\mathcal{Q}) \models_{h'} \varphi$

- $\mathsf{Fr}(M,\mathcal{Q}) \models_h \langle q\rangle\varphi$ *iff* $hq \in \mathcal{H}$ *and* $\mathsf{Fr}(M,\mathcal{Q}) \models_{hq} \varphi$

Definition 14 ($\mathrm{TDEL_Q}$)**.** *The logic* $\mathrm{TDEL_Q}$ *is the set of formulas derivable in the system of Table 3.*

One feature that distinguishes $TDEL_Q$ from our earlier system is this. Even "Yes/No" questions φ? with tautological preconditions $\varphi \vee \neg\varphi$ need not always be available for inquiry. Thus, as in PAL with protocols, the earlier recursion axioms of DEL_Q have to be modified as in Table 3.

All instantiations of propositional tautologies	
$S5$ and *hybrid* axioms for $\langle\varphi?\rangle$, $\langle\,!\,\rangle$ and K, R, Q	
intersection axioms for R relative to K and Q	
$\langle\varphi?\rangle p \leftrightarrow \langle\varphi?\rangle\top \wedge p$	*A&A*
$\langle\varphi?\rangle\neg\psi \leftrightarrow \langle\varphi?\rangle\top \wedge \neg\langle\varphi?\rangle\psi$	*A&N*
$\langle\varphi?\rangle K\psi \leftrightarrow \langle\varphi?\rangle\top \wedge K\langle\varphi?\rangle\psi$	*A&K*
$\langle\varphi?\rangle Q\psi \leftrightarrow \langle\varphi?\rangle\top \wedge ((\varphi \wedge Q(\varphi \to \langle\varphi?\rangle\psi)) \vee (\neg\varphi \wedge Q(\neg\varphi \to \langle\varphi?\rangle\psi)))$	*A&P*
$\langle\varphi?\rangle R\psi \leftrightarrow \langle\varphi?\rangle\top \wedge ((\varphi \wedge R(\varphi \to \langle\varphi?\rangle\psi)) \vee (\neg\varphi \wedge R(\neg\varphi \to \langle\varphi?\rangle\psi)))$	*A&R*
$\langle !\rangle p \leftrightarrow \langle !\rangle\top \wedge p$	*R&A*
$\langle !\rangle\neg\varphi \leftrightarrow \langle !\rangle\top \wedge \neg\langle !\rangle\varphi$	*R&N*
$\langle !\rangle K\varphi \leftrightarrow \langle !\rangle\top \wedge R\langle !\rangle\varphi$	*R&K*
$\langle !\rangle Q\varphi \leftrightarrow \langle !\rangle\top \wedge Q\langle !\rangle\varphi$	*R&P*
$\langle !\rangle R\varphi \leftrightarrow \langle !\rangle\top \wedge R\langle !\rangle\varphi$	*R&R*
Necessitation rules for K, R, Q, $\langle\cdot?\rangle$ and $\langle !\rangle$	
Modus Ponens and suitable *Sorted Substitution* rules	

Table 3: The axiomatization $\mathrm{TDEL_Q}$

These new axioms describe the procedural restrictions that drive conversations or processes of inquiry and discovery. In $TDEL_Q$, $\langle\varphi?\rangle\top$ means that the question φ? can be asked. In general, $\langle q\rangle\top$ will mean that the issue management action q is available for execution.[2]

Theorem 4 (Completeness of $\mathrm{TDEL_Q}$)**.** *For every formula* $\varphi \in \mathrm{TDEL_Q}$*:*

$$\models \varphi \quad \textit{if and only if} \quad \vdash \varphi$$

[2]We have considered here only uniform protocols restricted to asking φ and resolution questioning actions. Of course, it is possible to add the remaining questioning actions of announcing φ and refinement to this setting in a standard way.

Proof. This is analogous to the proof in [4]. The key points to observe are the right procedural changes in the recursion axiom for (a) atomic statements, and (b) the modalities affected by the action. □

4 Multi-agent scenarios

Questions typically involve more than one person. Thus, interaction between agents is a crucial motivation for our dynamic logic of issues. Indeed, our system is easily extended in this way, providing aspects that are lacking in the usual single-agent approaches.

4.1 Multi-agent DQL with public issues

It is easy to generalize earlier definitions of models and languages to a multi-agent version, providing accessibility relations and modalities with subscripts. Complete logics will also be as before, since as in epistemic logic, we do not expect the logic itself to enforce significant interaction principles tying agents together. But of course, we can formulate specific agent interactions in the extended framework.

One obvious way where the multi-agent setting is essential concerns *pre-conditions* of questions. A question event is usually linked to a questioner and an answerer. For instance,

one precondition of $e_1 =$ "b asks φ" is $\neg K_b\varphi \wedge \neg K_b\neg\varphi$.

The questioner must not know the answer to the question she asks. But questions are also asked to be answered, in general, by another agent:

a complex epistemic event $e_3 =$ "b asks φ to a" also has the precondition that the questioner must consider it possible that the answerer knows the answer: $\widehat{K}_b(K_a\varphi \vee K_a\neg\varphi)$.

These observations suggest that the following definition might be useful:

$$M \models_w \langle\varphi?\rangle_a^b\psi \textit{ iff } M \models_w (\neg K_a\varphi \wedge \neg K_a\neg\varphi) \wedge \widehat{K}_a(K_b\varphi \vee K_b\neg\varphi) \text{ and } M_{\varphi?} \models_w \psi$$

Our logic of information and issue update can obviously describe such more realistic questioning actions – as well as many others suitable for various concrete scenarions of interaction.[3]

[3] Indeed, the logic should be flexible here. Different types of question can have different preconditions: e.g., rhetorical questions have none of the above.

There are also interesting further issues here. In the above event, can we *separate* the informative preconditions from the questions per se? That is, we first announce the precondition, and then perform the issue change? For factual assertions, this works well, since the sequential order has the same effect as doing the two simultaneously. In general, however, there can be a problem, since announcing the precondition may change the correct answer to the question. Thus, we may have to analyze complex question events like the above as one unit, writing their recursion axioms separately.[4]

4.2 Multi-agent DQL with private issues

Extending our earlier logic in this manner will deal with questions considered as public events. But in many scenarios, there may be private aspects, reflecting partial observation or other limitations.

One obvious scenario is a public question followed by a private answer, like what happens in many classrooms. This is easily dealt with by attaching our logic of public questions to the logic DEL of private announcements. But there can also be private questions, with either private or public answers. For instance, agent a can ask b if φ, while c does not hear it. Or, c may just have been unable to hear if the question asked was P? or Q?. Such scenarios call for events that modify the issue relation in ways that are different for different agents. In the extended version of this paper (van Benthem & Minică 2009 [7]), we give a generalization of the *product update* mechanism of DEL to deal with issue management in the presence of privacy.[5]

Other multi-agent issues that we plan to study concern the formation of *groups* of agents. In particular, when many agents have different views of what the issues are, they may *merge* their separate issue relations to one 'common refinement' of their individual equivalence relations. This relation is a first natural candidate for the 'collective issue' for the whole group, and thus, we can now also form group versions of our modalities K, Q, R, linking common issues to common knowledge.

[4]Incidentally, a multi-agent setting may also change our views of the effects of answers. For instance, as observed in van Benthem (One is a Lonely Number) [21], an *answer* like "I do not know" can be highly informative!

[5]Of course, this requires refined epistemic issue models where the structure of the issue for different agents is no longer common knowledge.

5 Further directions

In this paper, we have shown how dynamic logics of (mostly) public questions can be designed, and used to analyze various aspects of private and public inquiry. These systems fit entirely within the methodology of dynamic-epistemic logic, and they seem to form a natural complement to what already exists in this area, making the questions explicit that drive public announcements and other informational events. In line with this first finding, many lines of investigation open up. We mention a few that we are currently exploring, even though they go beyond the compass of this paper.

Further agent attitudes: beliefs and preferences. We have studied the interaction of questions with knowledge. But of course, agents' *beliefs* are just as important, and we can also merge the preceding analysis with dynamic logics of belief change. In fact, in addition to conveying hard information, asking a question can also be a subtle way of influencing beliefs of agents. For instance, we said earlier that not all questions impart knowledge that the speakers does not know the answer. But we might say that, barring further information, they induce a defeasible belief of the audience that this is the case. Thus, our question dynamics might be added to the DEL-style belief logics of van Benthem 2007 [22] and Baltag & Smets 2007 [2].

Beyond beliefs, questions can also affect other agent attitudes. For instance, a question can give us information about other agents' goals and *preferences*, and indeed, "Why" questions explicitly concern such reasons for behavior. Just as information dynamics does not stop at purely informational attitudes, but also extends to the way in which agents evaluate situations and actions, the same extension makes sense for questions. This would come out concretely by adding question dynamics to the preference logics of Girard 2008 [11] and Liu 2008 [19].[6]

An interesting further development is an application of our analysis of questions in a dynamic epistemic setting to epistemic games like those developed for public announcements by Ågotnes and van Ditmarsch in [1]. In Public Announcement Games players have to find the optimal announcement to make in order to reach their epistemic goals given their knowledge. In [1] interesting and innovative solution concepts are proposed for such games and a connection with the analysis of questions in [10] is already suggested. Considering games in which the available moves for the players include both

[6]Indeed, there are formal analogies between our question update operation and the 'ceteris paribus' preferences of van Benthem, Girard & Roy 2008 [5].

announcements and questions is a way in which the value of a question can receive a precise game-theoretical definition. In strategic interactions the optimal question to ask doesn't have to be the most informative one, and different reasons can determine a player to prefer one question over another in various epistemic scenarios. We are investigating question versions of such games in collaboration with Ågotnes and van Ditmarsch.

Update, inference, and syntactic awareness dynamics. While DEL has been largely about observation-based semantic information, some recent proposals have extended it to include more finely grained information produced by inference or introspection. The same sort of move makes sense in our current setting. For instance, yet another effect of asking a question is of making agents aware that something is an issue. This does not just make sense in the above epistemic logic-based environment of semantic information. Raising an issue may also just make an agent aware that some proposition is important. In that case, we can think of a finer dynamics of questions, where they increase some current set of 'relevant propositions' whose truth value needs to be determined. This would work well in the syntactic approach to inferential and other fine-grained information in van Benthem & Quesada 2009, [8] with questions providing one reason for their acts of 'awareness promotion'. The latter take would also fit well with Hintikka's emphasis on the combination of questions and deductions as driving inquiry. In a dynamic perspective, merging semantic observational information and inferential syntactic information will become even more natural when questions come into play.

Multi-agent behavior over time. We have already indicated that, just as with assertions, questions make most sense in the context of some longer temporal process. A single question is hard to 'place' outside of the setting of some scenario. For instance, questions as much as arguments drive argumentation, and serve as ways of either underpinning assertions, or calling them into doubt. To deal with this formally, we need temporal logics that can talk about sequences of questions and their effects on the current history. Our study of *protocols* was one step in this direction, but obviously, we also need to make our dynamic logics of questions work in analyses of extended conversation, or especially, *games*. Another long-term perspective where this makes eminent sense are *learning* scenarios, where asking successive local questions would seem a very natural addition to the usual input streams of answers (cf. Kelly 1996 [18]) to one unchanging grand question which global hypothesis about the actual history is the correct one.

Structured issues and agenda dynamics. To us, the most striking limitation of our current approach is the lack of structure in our epistemic issue models. Surely, both in conversation and in general investigation, the *agenda* of relevant issues is much more delicate than just some equivalence relation. For instance, there are more important basic issues to be solved, and less important secondary issues. Again, this reflects a more general point also on the purely propositional informational side, where logics of ordered propositions have been used to model belief revision and preference ordering.

While it may be possible to define some such order in the semantics, we feel that such an approach would be misguided. The primary fact seems to be rather that we are usually maintaining a 'structured agenda' – and it is this agenda that gets modified by successive events of either resolving old questions, or raising new ones. If we are to have any realistic logical account of, say, the development of research programs, we need to understand this more finely-grained dynamics.

Moreover, there are already models that allow for this sort of dynamics. Girard 2008 [11], Liu 2008 [19] consider, essentially, 'priority graphs' of ordered relevant propositions (first proposed and studied in Andreka, Ryan & Schobbens 2001 [15]) that can be used for this purpose. Priority graphs can encode a structured family of issues, and they allow for a larger repertoire of inserting or deleting questions. The cited authors have suggested that they would be suited for studying the structured agendas of research programs in the philosophy of science, and indeed, it is an appealing thought that 'theories' in science consist not just of propositions encoding the answers to past questions, but also a representation of those guiding questions themselves.[7]

For a first system of this more structured sort in our DEL-style setting, we refer to the extended version of this paper (van Benthem & Minică 2009 [7]).

6 Comparisons with other approaches

We have mentioned several other approaches to the logic of questions. There is the tradition of erotetic logic in the sense of Wiśniewski [24], which we have not yet studied. Likewise, it would be good to go back to the slightly later classic Belnap & Steele 1976 [20] in the light of current dynamic logics.

More directly connected to our approach, we have mentioned the still active program of Hintikka for *interrogative logic* [17]. Questions are treated here as requests for new information, which function intertwined with deductive indicative moves in 'interrogative tableaux'. There is an extensive

[7] Being good at research seems to imply being able to ask good questions just as much as giving clever answers.

theory of answerhood, as well as an analysis of various types of question in a predicate-logical setting, beyond what we have done here. The framework has a number of nice theoretical results, including meta-theorems about the scope of questioning in finding the truth about some given situation. Clearly, several of these results would also be highly relevant to what we are doing here, and a merge of the two approaches might be of interest, bringing out Hintikka's concerns even more explicitly in a dynamic epistemic setting.

But the closest comparison to our approach is the inquisitive semantics ([13, 9]). Inquisitive semantics gives propositions an 'interrogative meaning' defined in a universe of information states over propositional valuations, with sets of valuations expressing issues. First, a compositional semantics is given, evaluating complex propositions in sets of worlds, viewed as information states. For instance, a conditional is true if every subset (stronger information state) supporting the antecedent also supports the consequent. Eventually, propositions are associated with the family of maximal information states supporting them. In particular, disjunctions then introduce possibly different maximal states, leading to a generalization of the partition picture in the present paper. Based on this semantics, a propositional logic arises that describes valid consequence and other relations between questions, and questions with answers.

At some level of abstraction, the ideas in this system sound very close to ours: there is information dynamics, questions change current partitions, etcetera. But the eventual system turns out to be an intermediate propositional logic in between intuitionistic and classical logic. Comparing the two approaches is an enterprise we leave for another occasion, though Icard 2009 (seminar presentation, Stanford) has suggested that there might be both translations between the two systems, and natural merges. Here, we just make one observation that seems the key point of difference between the two approaches. Inquisitive semantics puts the dynamic information about questions in a new account of the meaning of interrogative sentences in a propositional language. This is not classical declarative meaning, and hence some deviant propositional logic emerges. By contrast, dynamic-epistemic logic wants to give an explicit account of questions and other actions of issue management, but it does so by means of dynamic modalities on top of a classical logical language. In particular, there is no meaning shift: but rather an expansion of the domain of study of classical logic. The distinction is similar to one in logic itself (van Benthem 1993, 'Reflections on Epistemic Logic' [3]). Intuitionistic logic studies knowledge and information *implicitly* by changing the meaning of the classical logical constants, and then picking a fight with classical logic in the set of 'validities'. By contrast, epistemic logic analyzes knowledge *explicitly* as an additional operator on top of classical propositional logic: there is no meaning shift, but agenda expansion. In our view, dynamic-epistemic logic of questions stands in exactly the same relationship to inquisitive semantics: it makes the dynamics explicit, and steers away from foundational issues of meaning and logic. Comparisons between the two approaches can be quite delicate (van Benthem 2008, 'The Information in Intuitionistic Logic' [23]), and the same may also be true here.

7 Conclusion

The dynamic calculi of questions in this paper show how dynamic-epistemic logic can incorporate a wide range of what we have dubbed 'issue management' beyond mere information handling. Our contribution is showing how this can be defined precisely, leading to complete dynamic logics that fit naturally with existing systems of DEL, broadly construed. Moreover, we have indicated how these systems can be used to explore properties of issue management beyond what is found in other logics of questions, including complex epistemic assertions, many agents, and explicit dynamics.[8]

Even so, we do feel that our systems are only a first step - still far removed from the complex structures of issues that give direction to rational agency. The insight itself that the latter are crucial comes from other traditions, as we have observed, but we hope to have shown that dynamic-epistemic logic has something of interest to contribute.

Acknowledgments We first started developing these ideas about half a year ago, inspired by the 'inquisitive semantics' of [13] which, to us, raised the issue how the phenomena covered there (and others) would be dealt with from a dynamic-epistemic perspective. In the meantime, we have profited from comments on various drafts of this paper from Viktoria Denisova, Solomon Feferman, Tomohiro Hoshi, Thomas Icard, Floris Roelofsen, Lena Kurzen, Cedric Degremont, Fernando Velazquez-Quesada, and George Smith.

[8]However, we have not arrived at any definite conclusion about the formal relationships between our dynamic logics and existing alternatives. Perhaps all of them are needed to get the full picture of issue management.

References

[1] Thomas Å gotnes and Hans van Ditmarsch. But what will everyone say? - public announcement games. (to appear), 2009.

[2] Alexandru Baltag and Sonja Smets. From conditional probability to the logic of doxastic actions. In *TARK '07: Proceedings of the 11th conference on Theoretical aspects of rationality and knowledge*, pages 52–61, New York, NY, USA, 2007. ACM.

[3] Johan van Benthem. Reflections on epistemic logic. *Logique et Analyse*, (34), 1993.

[4] Johan van Benthem, Jelle Gerbrandy, Tomohiro Hoshi, and Eric Pacuit. Merging frameworks of interaction. *Journal of Philosophical Logic*, 2009.

[5] Johan van Benthem, Patrick Girard, and Olivier Roy. Everything else being equal: A modal logic for ceteris paribus preferences. *Journal of Philosophical Logic*, (38:83-125), 2008.

[6] Johan van Benthem and Fenrong Liu. Dynamic logic of preference upgrade. *Journal of Applied Non-Classical Logic*, (17:157-182), 2007.

[7] Johan van Benthem and Ştefan Minică. Dynamic logic of questions. working paper ILLC, 2009.

[8] Johan van Benthem and Fernando Velazquez-Quesada. Inference, promotion and the dynamics of awareness. (working paper ILLC, to appear in KRA), 2009.

[9] Ivano Ciardelli and Floris Roelofsen. Generalized inquisitive logic. 2009.

[10] Hans van Ditmarsch. Knowledge games. *Ph.D. Thesis*, (ILLC), 2000.

[11] Patrick Girard. Modal logic for belief and preference change. *Ph.D. Thesis*, (Stanford University), 2008.

[12] Jeroen Groenendijk. The logic of interrogation: Classical version. In T Matthews and M Strolovitch, editors, *SALT IX: Semantics and Linguistic Theory*. 1999.

[13] Jeroen Groenendijk. Inquisitive semantics: Two possibilities for disjunction. *Seventh International Tibilisi Symposium on Language, Logic, and Computation*, 2008.

[14] Jeroen Groenendijk and Martin Stokhof. Questions. In Johan van Benthem and Alice ter Meulen, editors, *Handbook of the Logic and Language*. Elsevier, 1997.

[15] Andrka Hajnal, Ryan Mark, and Schobbens Pierre-Yves. Operators and laws for combining preference relations. *Journal of Logic and Computation*, 1(12), 2002.

[16] Jaakko Hintikka. *Socratic Epistemology: Explorations of Knowledge-seeking by Questioning*. Cambridge University Press, 2007.

[17] Jaakko Hintikka, Ilpo Halonen, and Arto Mutanen. Interrogative logic as a general theory of reasoning. In Dov Gabbay, Ralph Johnson, Hans Ohlbach, and John Woods, editors, *Handbook of the Logic of Argument and Inference*. Elsevier, 2002.

[18] Kevin Kelly. *The Logic of Reliable Inquiry*. Oxford University Press, 1996.

[19] Fenrong Liu. Changing for the better - preference dynamics and agent diversity. *Ph.D. Thesis*, (ILLC), 2008.

[20] Belnap Nuel and Steel Thomas. *The logic of questions and answers*. Yale University Press, 1976.

[21] Johan van Benthem. One is a lonely number. *Z. Chatzidakis, P. Koepke & W. Pohlers, eds., 2006, Logic Colloquium '02, ASL & A.K. Peters, Wellesley MA,*, (96 129), 2001.

[22] Johan van Benthem. Dynamic logic of belief revision. *Journal of Applied Non-Classical Logics*, 2(17), 2007.

[23] Johan van Benthem. The information in intuitionistic logic. *Synthese*, 2(167), 2008.

[24] Andrzej Wiśniewski. *The Posing of Questions*. Kluwer Academic Publishers, 1995.