Skills and mathematical knowledge

Benedikt Löwe[1,2,3], Thomas Müller[4,5]*

[1] Institute for Logic, Language and Computation, Universiteit van Amsterdam, Postbus 94242, 1090 GE Amsterdam, The Netherlands

[2] Department Mathematik, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany

[3] Mathematisches Institut, Rheinische Friedrich-Wilhelms-Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

[4] Departement Wijsbegeerte, Universiteit Utrecht, Heidelberglaan 6, 3584 CS Utrecht, The Netherlands

[5] Institut für Philosophie, Universität Bonn, Lennéstraße 39, 53113 Bonn, Germany

E-mail: `bloewe@science.uva.nl, Thomas.Mueller@phil.uu.nl`

1 Introduction

In (Löwe and Müller, 2008), we discuss the traditional modal view of (propositional) mathematical knowledge that reduces knowledge claims of the form "S knows that p" to the ability of S to produce a formal derivation of p. We argue that such a modal definition of knowledge cannot be given in a context-insensitive way and that the (now contextually determined) modality will have to be interpreted with respect to *skills* of the subject S. When looking at actual knowledge attributions in mathematics, it becomes clear that the access to proof that is purportedly behind mathematical knowledge, has to be of a dispositional nature: nobody has current conscious or physical access to proofs of all, or even of a small fraction of, the items of mathematical knowledge that can be truthfully attributed to her. This modal or dispositional element is present in many other accounts of knowledge, e.g., in Aristotle's conception of knowledge as a ἕξις (*Cat.* 8).

The crucial question is how to make this modalization precise. Our analysis (Löwe and Müller, 2008, p. 104) rests the modalization on the notion of "mathematical skill":

> S *knows* that p iff S's current mathematical skills are sufficient to produce the form of proof or justification for p required by the actual context. (‡)

Skill is both a modal notion (what somebody is able to do even while not doing it) and has an empirical side (skills can be tested). Skill levels can be characterised independently of any conceptual models for mathematical

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knowledge. Mathematical practice affirms that the concept of mathematical skill is well entrenched as it is customary to comment on students' or researchers' skills, and it is often possible to rank people with respect to their skills. Skills are tested in exams and job talks, and it may well be that the aim of mathematics education is best characterised not as instilling mathematical knowledge, but as teaching mathematical skills. In the mentioned paper we do not discuss this in detail, but instead refer to the Dreyfus-Dreyfus model of skill acquisition as a semi-formal theory of skill levels and relegate a more detailed discussion to future work. In this paper, we provide the necessary background and continue the discussion from (Löwe and Müller, 2008).

In § 2, we shall give a general discussion of the role of skills in epistemology, specializing to the Dreyfus-Dreyfus model of skills in § 3. The original applications of the Dreyfus-Dreyfus model were (relatively) homogeneous skills such as car driving (Dreyfus and Dreyfus, 1986, p. 24) and playing chess (p. 25). Mathematics is much more multi-faceted; in fact, we propose to see mathematics as involving a *professional skill*. There is a well-known treatment of a professional (set of) skills using the Dreyfus-Dreyfus model, viz. Benner's (1984) treatment of nursing skills which we discuss in detail in § 4. After having seen the example of nursing, we return to mathematical skills in § 5, asking a number of questions with very few concrete answers. In our concluding § 6, we summarize the discussion of this paper.

2 Skills

It is sometimes claimed that mathematical knowledge is mostly propositional knowledge: knowledge *that*, e.g., a specific mathematical proposition p is true or false; and it is this type of knowledge that we have investigated (Löwe and Müller, 2008). Rav (1999) has argued that mathematics is not really about knowing the truth values of theorems ("knowing that"), but about knowing the techniques and ideas behind their proofs ("knowing how"). Rather than viewing this as a strict dichotomy, we are interested mostly in the role of skills—knowing how—for propositional mathematical knowledge.

Skills aren't new to the philosophical scene. Ryle (1949, Chap. 2) has famously argued for the separation of knowing *how* (which he uses synonymously with "skill") from knowing *that*. Ryle's overall aim is to fight the "intellectualist doctrine which tries to define intelligence in terms of the apprehension of truths, instead of the apprehension of truths in terms of intelligence" (Ryle, 1949, p. 27). He claims that intellectualism is implicit in much of philosophy, but that "'[i]ntelligent' cannot be defined in terms of 'intellectual' or 'knowing *how*' in terms of 'knowing *that*'" (p. 32). If a reduction of one to the other has any chance of success, it should be the other way round—but here also a danger lurks. Skill is a modal notion and thus close to dispositions, but human know-how is different from physical dispositions like the solubility of sugar, which can (arguably) be tested by uniform behavior in specific conditions.[1] This creates a problem for a direct reduction of knowing that to knowing how:

> Epistemologists, among others, often fall into the trap of expecting dispositions to have uniform exercises. For instance, when they recognise that the verbs 'know' and 'believe' are ordinarily used dispositionally, they assume that there must therefore exist one-pattern intellectual processes in which these cognitive dispositions are actualised. Flouting the testimony of experience, they postulate that, for example, a man who believes that the earth is round must from time to time be going through some unique proceeding of cognising, 'judging', or internally re-asserting, with a feeling of confidence, 'The earth is round'. (Ryle, 1949, p. 44)

The role of triggering conditions or, more generally, the role of actual performance for skill assessment is certainly more diverse.

Dreyfus and Dreyfus affirm Ryle's point that "know-how is not accessible to you in the form of facts and rules" (Dreyfus and Dreyfus, 1986, p. 16). From this observation they draw the important conclusion that the genesis of skills contains the key to a better understanding of know-how.[2] Consequently, they choose to focus their investigation on a phenomenologically detailed study of skill acquisition.

3 The Dreyfus-Dreyfus model of skills

The philosopher Hubert Dreyfus and the mathematician Stuart Dreyfus propose their five-step skill acquisition model in their book *Mind over Machine* (Dreyfus and Dreyfus, 1986), which itself forms an important contribution to the discussion about symbolic Artificial Intelligence in the 1980s. Their model is grounded in phenomenological observations about the acquisition of various human skills, on the one hand, and in philosophical theories about human practices going back to Heidegger, Merleau-Ponty and the late Wittgenstein (Dreyfus and Dreyfus, 1986, p. 11).

Dreyfus and Dreyfus discern five skill steps in the development of skills in humans—stressing, of course, that not everyone acquiring a skill will necessarily reach the highest, expert level (Dreyfus and Dreyfus, 1986, Chap. 1; summary p. 50):

[1] As the discussion about so-called *ceteris paribus* laws and "finkish dispositions" in philosophy of science shows, this assumption about testing may have to be qualified.

[2] Ryle also makes an observation that points in this direction: "Learning *how* or improving in ability is not like learning *that* or acquiring information" (Ryle, 1949, p. 59).

1. *Novice*. Application of context-free rules through information processing.

2. *Advanced Beginner*. Application of rules, also based on perceived similarity with prior examples.

3. *Competence*. Application of a hierarchical procedure of decision-making ("problem solving"; p. 26).

4. *Proficiency*. Deep involvement, experiencing situations from a perspective ("holistic similarity recognition"; p. 28); decisions grounded analytically.

5. *Expertise*. No need for rules. "*[E]xperts don't solve problems and don't make decisions; they do what normally works*" (p. 30f.).

It is part of Dreyfus and Dreyfus's argument against symbolic AI that explicitly rule-based schemes, even if rules include heuristics polled from human experts, will never allow computer programs to advance to proficiency or expertise. According to them, higher skill levels are only reached through repeated *in situ* experience. The Dreyfus-Dreyfus skill model is a situational model offering "no context-free criteria to identify persons as possessing talents or traits indicative of expertise" (Benner, 1984, p. 15).

Note that Hubert Dreyfus (2001) extends the Dreyfus-Dreyfus skill model by two further levels called *Mastery* and *Practical Wisdom*. We shall focus mostly on the original five-level model and only mention some issues of Mastery in our concluding § 6.

We have chosen to focus on the Dreyfus-Dreyfus skill model because it is general, explicit, and empirically grounded. However, despite the fact that Stuart Dreyfus himself is a mathematician, that model has not been applied to the case of mathematics itself.[3] Early applications of the model were skills or skill sets that are clearly delineated such as playing chess and driving a car (or, slightly more complex, the education of airplane pilots: Dreyfus and Dreyfus, 1977). Car-driving and chess are skills that are needed in specific situations; there is a fairly clear distinction to be made between the skills involved in such settings and more general enabling conditions or auxiliary skills. For instance, your car-driving skill can be assessed independently from your competence in using the CD player in the car, even though handling of the CD player is something that is typically done by drivers. Or, in the case of chess, a world class chess player needs to travel to tournaments and many everyday skills (such as booking flights and hotels) are required for this, but we feel confident in completely separating this from the chess-playing skill of an individual: if the world champion of

[3] Hubert Dreyfus, personal communication, June 2003.

chess did not know how to book a flight or a hotel, this would not affect the level of his or her skill.

In the case of mathematics, this separation is more difficult: while we think that booking flights or hotels are not parts of the skill set that defines a mathematician (after all, we know research mathematicians who do not go to conferences), this is much less clear for being able to write up proofs intelligibly or to explain proofs to graduate students, etc. (for more discussion, see § 5.1 below).

Later applications of the Dreyfus-Dreyfus skill model have been dealing with skills more complex than car-driving and chess. Examples of this are the famous studies on nursing by Patricia Benner (cf. § 4) and Flevbjerg's studies on social workers (Flyvbjerg, 2001). In order to provide a background for our discussions about mathematical skills, we shall discuss the prototypical case of nursing in more detail in the following section.

4 Nursing

Patricia Benner's analysis of the professional skills of nursing in terms of the five-level Dreyfus-Dreyfus model of skills (Benner, 1984) has been called one "among the most sustained, thoughtful, deliberate, challenging, empowering, influential, empirical [. . .], and research-based bodies of nursing scholarship" (Darbyshire, 1994, p. 760). Her approach has had a substantial influence on the practice of teaching nursing, as witnessed by a number of papers published in the 2001 commemorative edition of the 1984 book (cf. Gordon, 2001; Huntsman et al., 2001; Ullery, 2001; Fenton, 2001; Dolan, 2001). An overview of the impact of Benner's study can be found in (Brykczynski, 2006).

Benner (1984) gave a detailed description of the five stages in the learning of nursing skills based on paired interviews, individual interviews, and participant observation. She covers the stages of *Novice* (Benner, 2001, pp. 20–22), *Advanced Beginner* (pp. 22–25), *Competence* (pp. 25–27), *Proficiency* (pp. 27–31), and *Expert* (pp. 31–36), including implications for teaching of nursing students at the particular levels.[4]

Note that since the Dreyfus-Dreyfus model is a situational model, you cannot expect criteria to determine the skill level of a nurse in an objective, context-independent way. Rather, the study exhibits examples of behaviour and insight indicative of particular skill levels.[5]

[4]For instance, for *advanced beginners* she notes that "their nursing care needs to be backed up by nurses" (Benner, 2001, p. 25) and for *proficient* nurses, she concludes that they "are best taught by the use of case studies" and that "the proficient performer is best taught inductively" (p. 30).

[5]Cf. (Benner, 2001, p. 15): "No attempt was made to classify the nurses themselves according to proficiency levels".

As we have argued above, nursing is a good example for what we aim to do in § 5 as it is a much richer activity than the one-dimensional examples of chess playing or car driving. Benner identifies a number of skill sets that are all part of the skills of a nurse, including providing comfort (Benner, 2001, § 4) and interpretation for patients (§ 5), diagnostics (§ 6), situation management (§ 7), medication (§ 8), quality control (§ 9), and organization (§ 10). The notion of a skilled nurse is related, ultimately, to a nurse's job description, which has developed historically. We are not concerned here with a natural kind of human beings, nurses, of which there are more and less skilled ones. Rather, we are assessing human beings who have chosen a specific profession, as more or less skilled as required by the (historically and sociologically contingent and changing) requirements of that profession. Nursing skills are *professional skills*.

Not every highly skilled nurse will be equally good at all of these skill sets. We can imagine highly accomplished nurses with decades of experience who are not very good at particular parts of the job description. This is why the Dreyfus-Dreyfus model should not be seen as assigning skill levels to individuals but to performance patterns in a given situation.[6] We shall see this phenomenon in more detail in the case of mathematical skills in the next section.

5 Mathematical skills

In the mathematics education literature, Ryle's distinction of *knowing that* and *knowing how* has been embraced as a fundamental dichotomy for mathematical epistemology, and has given rise to a number of related (and yet subtly different) dichotomies for mathematical skills. We find examples in Sfard's *structural* vs. *operational* duality,[7] in Anderson's *declarative* vs. *procedural* distinction,[8] and in the notions of *functional* and *predicative thinking* in the work done at the Osnabrück *Institut für kognitive Mathe-*

[6] Compare the recent proposal of applying the Dreyfus-Dreyfus model for the profession of infection preventionists by Marx (2009). Here, the skill levels are objective and context-invariant properties of the individuals; e.g., "The Infection Preventionist-Competent would have more experience (> 2 years) and be certified in Infection Control (CIC) OR have a Masters or Doctorate in a healthcare field, > 6 months experience and be certified in Infection Control (CIC)" (p. E157). This is clearly not in line with the set-up of the Dreyfus-Dreyfus skill model and no such attempt should be made for the case of mathematics.

[7] Cf. (Sfard, 1991, p.4): "whereas the structural conception is static, [...], instantaneous, and integrative, the operational is dynamic, sequential, and detailed." Sfard stresses that there is no dichotomy between the structural and the operational approach, but rather a duality of "inseparable, though dramatically different, facets of the same thing" (Sfard, 1991, p. 9).

[8] Cf. (Anderson, 1993, p. 18): "Intuitively, declarative knowledge is factual knowledge that people can report or describe, whereas procedural knowledge is knowledge people can only manifest in their performance."

matik by Cohors-Fresenburg, Schwank, and their collaborators (Schwank, 2003). Combining the two sides of the duality, we also find the notion of *procept* (a combination of 'process' and 'concept') in (Gray and Tall, 1994).

A lot of interesting educational and empirical research has come out of these dichotomies and dualities, e.g., proposals for supporting certain talents of students based on the *functional* vs. *predicative* distinction (Cohors-Fresenborg and Schwank, 1992) or studies using eye-tracking as an indicator for such talent focus (Cohors-Fresenborg et al., 2003). However, we do not think that the classification of mathematical skill into a very small number of basic mathematical aptitudes is appropriate for the analysis that we are aiming for here.

We shall approach the issue of mathematical skills and their role in mathematical knowledge by studying three interrelated questions:

1. What kind of skills are mathematical skills? What is their principle of unity? Are they linked to the mathematical profession, or are they rather a type of natural skills?

2. How are mathematical skills individuated? Is it useful to distinguish mathematical skills very finely, in line with the division of the subject itself, or is there an overarching principle of unity?

3. How are mathematical skills measured and assessed? Which indicators are employed in practice; what makes mathematical skills empirically accessible?

We shall pursue these issues, in the above order, in sections 5.1 through 5.3.

5.1 What kind of skills are mathematical skills?

When we talk about skills, we group them according to different principles, the spectrum ranging from purely task-related clustering (e.g., when assessing someone's skill at repairing bicycles or driving a car) to clusters that can lay a claim to resonating with some natural subdivision of the activities of a human being (when, e.g., assessing someone's musical skills, or her skills at bringing people together). The latter skills seem to be more strongly associated with the notion of talent than the former ones. Further subdivisions in all these areas are possible and sometimes useful, according to demands set by the context (e.g., even a skilled bicycle mechanic may be poor at some specific task like adjusting the headlights).

Where do mathematical skills lie in the above spectrum? Both extremes of the spectrum have a certain appeal. In a somewhat romantic fashion that pervades the public image of research mathematics, propounded by popular culture,[9] one may picture mathematical skills as a combination of different natural talents ranging from an analytical mindset to powerful visual intuition. At the other extreme, one may view mathematical skills simply as the skills a research mathematician needs for his or her job, which would mean, in most cases, that filling in one's travel expense declarations and LaTeX typesetting are as much part of the deal as are finding and checking proofs.

In this spectrum we lean towards viewing mathematical skills as *professional skills* whose unity is defined by the job that a mathematician is doing as a researcher. In the process of mathematical research, a lot of skills are involved in a successful research episode: a mathematician tackles a research question, asks the right people who give her ideas helping on her way to the correct proofs, finally finds the proof, writes it up in a way that she can communicate it to the experts, gives a number of seminar talks on the proof, receives comments from peers in these talks, fixes a number of inaccuracies and uncertainties in the proof, types a journal paper, submits the paper, goes to international conferences reporting on the result, receives a referee report with revisions, revises the paper, and finally publishes it. In this overview of the mathematical research process, a number of skills are needed that are central, others that are less central, and yet others that are peripheral. The skills involved in finding the proof are certainly central, but being able to communicate the proof to the experts (i.e., knowing how the community expects proofs to be communicated) is equally important. Giving presentations is still relatively important, but probably more dispensable than the earlier mentioned skills. Being able to write a paper in a form that is acceptable for a referee is slightly further down the scale, and somewhere at the end of the scale we find skills such as involved in filling in travel expense forms for the trip to the international conference. While the extremes on this spectrum (the ones that clearly are part of the mathematical research skills and the ones that clearly aren't) are easily identified, we do not think that an objective stable core of skills can be identified that could usefully take the place of the profession as a unifying principle.

To illustrate this, let us give an example from actual practice. A few years ago, the first author had an autistic student who took several advanced mathematical classes at the graduate level. The student would not hand in homework exercises, so there was no ordinary means of assessing the student's understanding of the material. However, even in complicated

[9] In a recent episode of the TV series NUMB3RS (episode 6.01, "Hangman", aired 25 September 2009), mathematical skill of the protagonist math professor was manifested by being able to calculate the position of the attacker while under sniper fire. Similar topoi, such as being able to detect hidden patterns very quickly in large amounts of unstructured data, can be found in other mainstream movies such as "A beautiful mind" (2001). For a detailed discussion of the public representation of mathematicians, cf. (Osserman, 2005).

proofs, the autistic student was able to correct mistakes on the blackboard by shouting corrections, indicating an understanding of the proofs. At the end of the lecture, the student took a personalized exam in which he performed very well on questions that essentially required a binary answer or just an intuitive idea. He performed badly on questions that required the student to give a proper mathematical argument, so for most contexts of research mathematics, the autistic student described would get a low assessment of skill level. We see this as an example corroborating the fact that pure understanding for the mathematical structures under investigation is not enough to have a high level of mathematical skill if this is not paired with the appropriate (historically and culturally determined) skills of communicating why the insights are true.

5.2 How are mathematical skills individuated?

The granularity of mathematical skills also leaves open a spectrum of options for analysis. The mathematical community usually puts great emphasis on the unity of the subject, which is indeed one of the special traits of the historical development of mathematics (cf. François and Van Bendegem, 2010). Furthermore, interrelations between seemingly disconnected areas of mathematics are constantly discovered,[10] so there is a strong empirical basis for claiming the unity of the subject and thus, for expecting one unified notion of mathematical skills. On the other hand, mathematicians themselves of course differentiate when it comes to specific aspects of a colleague's skills, and such aspects may also be epistemically important. Classifications of types of mathematicians have been around for a long time and are not a result of the diversification of the mathematical field.[11]

[10] To give a famous example, consider Gerhard Frey's 1984 observation that a solution to the Fermat equation would yield a counterexample to the Taniyama-Shimura conjecture, thus linking number theory to the area of elliptic curves.

[11] Cf., e.g., Felix Klein's recommendations for choosing among candidates for a vacant position in Berlin 1892, in the course of which he gives his view of the required balance of skills in a math department: "Bei der Mannigfaltigkeit der Individualitäten kann man ja nicht schematisieren, aber im grossen und ganzen sollten folgende Typen vorhanden sein:

1. Der Philosoph, der von den Begriffen construirt,
2. Der Analytiker, der wesentlich mit der Formel operirt,
3. Der Geometer, der von der Anschauung ausgeht."

(Letter dated 6 January 1892; cited after (Siegmund-Schultze, 1996); translation: "Given the manifold of individualities one cannot press the discussion into a schema, but generally speaking, the following types should be present: 1. The philosopher who constructs conceptually, 2. The analyst who essentially operates with formulae, 3. The geometer who proceeds from intuition.")

Klein mentions Weierstraß and Cantor for type 1, Weber and Frobenius for type 2, and Schwarz and Lindemann for type 3. (We should like to thank to Dirk Schlimm for help related to this reference.)

At the very fine-grained level, there is the division of the subject of mathematics into subfields, e.g., according to the *Mathematics Subject Classification* of the American Mathematical Society (the 2010 version of this classification is a document of 46 pages just listing the names of the subareas). It makes good sense to ask, when confronted with a specific problem in one of these areas, who is an expert in that specific field, i.e., who is a skilled mathematician with respect to that subject. Does that mean that there are as many variants of mathematical skills as there are subfields of mathematics?

With respect to this spectrum, we support a unificationist stance: Mathematics is one subject, and for most purposes, it makes sense to view *general* mathematical skills as the pertinent level of granularity. For purposes of assessing knowledge claims, local dimensions of skill may however also play a role, depending on context.

Suppose that we have a given mathematical theorem p and a given context and would like to know whether a mathematician S satisfies our requirements for knowledge given in (‡) at the beginning of this paper. This will require us to describe a skill level in terms of "mathematical skills" that is sufficient for the task at hand. But if mathematical skills are so diverse, what part of the skill set will be relevant here? We claim that this will be given by the context and the nature of the theorem p in the same sense that a nurse's skill level is not an objectively defined property of a given human being, but situationally determined.

To give an example similar to the situations described in (Löwe and Müller, 2008) as part of the argument for the context-dependency of mathematical knowledge, let us consider a mathematician S and a theorem p that is not from his immediate research area, but a closely related area, and of which he has seen proof sketches, but never a full proof. We assume S to be of lower skill level than *expert* for the relevant area of mathematics.

Scenario 1. If we are looking at a context in which only the proof idea matters, then the cognitive access that S has to the proof sketches (by virtue of memory) is enough to satisfy the requirements of our definition (‡), and no skill for transforming the proof sketch into something else is needed. As a consequence, we would conclude that S knows that p in these contexts.

Scenario 2. Other contexts (for example, research contexts in which S needs to use parts of the details of the proof in order to generalize the proof to a different setting) need *expert* level skills in order to allow S to bring the proof sketches to which he has cognitive access to the level of detail needed for the context. As we assumed that S is not of *expert* level for the relevant area, in these contexts we would not say that S knows that p.

Scenario 3. Extending the example a bit, we can consider a context like in the last scenario (i.e., S does not know that p); now S asks an expert to give a more detailed proof sketch, gaining cognitive access to a new account of the proof for which a lower skill level than *expert* is enough in order to allow S to transform this detailed account into the level of detail needed for the context, thus creating knowledge.

Comparing the three scenarios in this example, we see that a lot depends on judgments about situations of the type "given cognitive access to X, you need skill level Y to produce a proof of type Z" or "S has skill level Y with respect to this particular situation". So, in order to make definition (‡) useable in practice, we need to be able to make assessments of this type.

5.3 How are mathematical skills measured and assessed?

Directly continuing our discussion of the examples in the last section, we consider the question: how do we assess a person's mathematical skills? It is of the essence of a modal or dispositional predicate that while performance in specific circumstances may be a valid indicator, skill also transcends recorded performance. Even a good bicycle rider may fall from her bike, and even a skilled musician can play out of tune (cf. also the long quote from Ryle in § 2 above).

We do not think that mathematical skills are special in this respect. They are dispositional, but performance is an indicator. Typical exam situations show the tension inherent in this: We believe that many mathematicians have experienced a situation in which they wanted to assess the *mathematical skills* of a student, but were forced by exam regulations (and, in some sense, also considerations of fairness) to give marks based solely on performance. When a skilled student underperforms, it is not rare to comment on the less than satisfactory grade by telling the student that one is convinced that she can do better than that. Those are not empty words—rather, such assessment highlights the fact that skill is dearer to most of us than "mere" performance. Of course, performance matters—we all know cases of people who have been promising for just a little too long. Assessment of skill and assessment of performance are not independent. But, as with many other dispositional traits, no strict statistical relationship appears to exist either.

The assessment of mathematical skills and of mathematical knowledge of course go hand in hand. When claiming, as we did above, that mathematical skills can be the key to an analysis of mathematical knowledge, we do not claim that we have access to mathematical skills completely independently of knowledge attributions. This seemingly circular structure is typical of modality—cf., e.g., David Lewis's remarks about the interrelation between formal models giving truth conditions for counterfactuals, on the one hand, and our intuitive assessment of counterfactuals on the other hand (Lewis, 1986, p. 43).

In § 5.1 we have argued that mathematical skills are best viewed as professional skills, i.e., as skills belonging to a specific, historically and sociologically contingent profession, the research mathematician. Thus, performance on the job is certainly another, but again defeasible, indicator of mathematical skills.

It seems that in contexts like the ones discussed at the end of § 5.2, a good test question whether a mathematician has the required skill level is the following: Suppose that I have a certain type X of proof at my cognitive disposal (e.g., a proof by testimony, a proof sketch, a handwritten proof with gaps, etc.) and a certain skill level Y is needed to transform this into the type Z of proof that I need in order to satisfy the definiens of (‡). In order to assess whether a mathematician S has skill level Y, I can ask myself the question: "Assuming that S has never heard about my problem before, if I give him the information at my cognitive disposal (of type X), will he be able to produce a proof of type Z?"

6 Conclusion

This paper is a specification of the general ideas starting in (Löwe and Müller, 2008), explaining how a link can be made between the Dreyfus-Dreyfus model of skills and our context-sensitive definition of mathematical knowledge. It raises a large number of questions and answers few of them. The Dreyfus-Dreyfus skill model is a situational model, not allowing for objective characterizations of individuals in terms of levels, but rather describing typical behaviour of individuals at certain levels of expertise in particular situations.

We have proposed that mathematical skill should be seen as a *professional skill*, largely delineated by the skills necessary for being a mathematical researcher, certainly a culturally and historical determined notion.[12] For each given knowledge assessment context, we need to determine which parts of the professional skill set are relevant. Certain parts of the skill set (the ones we called peripheral, such as booking flights to conferences) will almost certainly never of be relevance in mathematical knowledge claims; but we doubt that there is a clear definition of which part of the professional skill set forms the stable epistemological core.

[12] One of the consequences of this general perspective is that the set of skills that make a good mathematician can change. Some researchers in automated deduction claim that 25 years from now, proofs will not be checked by referees anymore, but mathematicians will write their proofs in codes checkable by automated theorem checkers. If they are right (witness the December 2008 issue of the *Notices of the American Mathematical Society* on formal proof for some indication of momentum gathering), it may become a central mathematical skill to be versatile in `HOL` programming, as much as it is now a central skill in mathematical physics to be able to use `Mathematica` or in statistics to use `R`.

We have also given examples of how the skill levels would be used in research situations, but the largest part of the empirical project remains: providing an empirical basis (similar to the empirical basis that Benner provided for the area of nursing) that allows us to identify various levels of expertise in research mathematicians. This is a long-term project and will require a lot of observation of research mathematicians in the style of Heintz (2000) and Greiffenhagen (2008). We hope that this paper can serve as a stepping stone for these investigations to come.

To close this paper, we would like to mention the issue of "Mastery". In this paper, we have based the discussion on the five-level Dreyfus-Dreyfus skill model, not on its extension that includes the level of *Mastery* (Dreyfus, 2001). Since mathematics is sometimes closer to an art than to a trade, and issues such as creativity can play a vital role, extending the five-level model by the additional level of *Mastery* seems particularly fitting for the case of mathematics. In mathematics, as in music or art, we run into situations that are difficult for the empirical researcher: there are very few people who understand the most complicated proofs in mathematics; there are very few people who can give us an insight into how the minds of the top-researchers of a field work; some of the best mathematicians claim theorems whose proofs are essentially uncheckable by anyone else.[13] If we want to get behind these epistemological conundra, a theory of *Mastery* could certainly help, in the same sense that it helps to understand singular phenomena in music and art.

However, we believe that currently, we first need to understand the levels of expertise of the non-exceptional research mathematicians (as part of the five-level skill model) before we move on to the more puzzling and exotic realms of exceptional talent and skill.

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[13]Cf. Thurston's discussion of how he struggled in communicating his proof of the geometrization theorem for Haken manifolds: "[S]ome concepts that I use freely and naturally in my personal thinking are foreign to most mathematicians I talk to. [...] At the beginning, this subject was foreign to almost everyone. It was hard to communicate—the infrastructure was in my head, not in the mathematical community" (Thurston, 1994, p. 174–175).

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