All proper normal extensions of S5–square have the polynomial size model property

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Abstract

It is shown that all proper normal extensions of the bi-modal system $\mathbf{S5}^2$ have the poly-size model property. In fact, every normal proper extension L of $\mathbf{S5}^2$ is complete with respect to a class of finite frames $\mathbf{F}_L$. To each such class corresponds a natural number $b(L)$ – the bound of L. For every L, there exists a polynomial $P(\cdot)$ of degree $b(L)+1$ such that every L-satisfiable formula φ is satisfiable on an L-frame whose universe is bounded by $P(|\varphi|)$, for $|\varphi|$ the number of subformulas of φ. It is shown that this bound is optimal.

1 Introduction

We show that every normal proper extension L of $\mathbf{S5}^2$ is complete with respect to a class of finite frames $\mathbf{F}_L = \mathbf{F}_1 \sqcup \mathbf{F}_2 \sqcup \mathbf{F}_3$. To this class we correspond a natural number $b(L)$ – the bound of L. We will show that for every L, there exists a polynomial $P(\cdot)$ of degree $b(L)+1$ such that every L-satisfiable formula φ is satisfiable on an L-frame whose universe is bounded by $P(|\varphi|)$, where $|\varphi|$ is the number of subformulas of φ. At the end of the paper we show that this bound is optimal.

The modal logic $\mathbf{S5}^2$ is widely studied, under a variety of names. Its algebraic counterpart is the variety $\mathsf{Df}_2 = \mathsf{RDf}_2$ of (representable) diagonal free cylindric algebras of dimension two [4]. Segerberg [9] discusses an expansion under the name of two–dimensional modal logic. In Gabbay and Shehtman [2], $\mathbf{S5}^2$ is studied as a special case of taking products of modal logics.

One of the sources of interest for $\mathbf{S5}^2$ is that it corresponds to a "clean" equality free fragment of first order logic with two variables. Our work is best motivated from this angle.

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Let $\mathbf{S5}^n$ denote the modal logic of n commuting $\mathbf{S5}$ diamonds (meaning that for all i, j, $\Diamond_i \Diamond_j \phi \leftrightarrow \Diamond_j \Diamond_i \phi$ is valid). It is known that every satisfiable $\mathbf{S5}^1$ formula ϕ is satisfiable in a model of size $|\phi|+1$ (Ladner, [5]), that every satisfiable $\mathbf{S5}^2$ formula ϕ can be satisfied on a model of size $2^{O(|\phi|)}$ (Grädel, Kolaitis and Vardi [3]), and that both bounds are optimal. As a corollary, both logics are decidable. For $n \geq 3$, $\mathbf{S5}^n$ does not have the finite model property and is in fact undecidable, a result of Maddux [6]. To these results we add that in extensions L of $\mathbf{S5}^2$, every L satisfiable formula ϕ is satisfiable in an L model whose size is bounded by a polynomial function in $|\phi|$.

Recall that the language of $\mathbf{S5}^2$ is the propositional language equipped with two existential modalities $\Diamond_1$ and $\Diamond_2$ and two universal modalities $\Box_1$ and $\Box_2$.

Let $\mathbf{S5}^2$ be the set of formulas generated by the following axioms $i = 1, 2$:

1) All tautologies of the classical propositional calculus;
2) $\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$;
3) $\Box_i p \to p$;
4) $\Box_i p \to \Box_i \Box_i p$;
5) $\Diamond_i \Box_i p \to p$;
6) $\Box_1 \Box_2 p \leftrightarrow \Box_2 \Box_1 p$;
7) $\Box_i \varphi \leftrightarrow \neg \Diamond_i \neg \varphi$

and closed under the following rules of inference:

Modus Ponens (MP): from φ and $\varphi \to \psi$ infer ψ;
Necessitation $(\mathrm{N})_i$: from φ infer $\Box_i \varphi$.

(Since $\Box_i \varphi \leftrightarrow \neg \Diamond_i \neg \varphi$, for $i = 1, 2$, are axioms of $\mathbf{S5}^2$, we will subsequently assume that $\Box_i$ do not appear explicitly in the formulas.)

There are two kinds of models for this logic. Call a triple $\mathcal{F} = (W, E_1, E_2)$ an $\mathbf{S5}^2$-frame if W is a non-empty set, and E_1 and E_2 are equivalence relations on W such that $E_1 E_2(w) = E_2 E_1(w)$ and $E_1 E_2(w) = W$ for any $w \in W$ (in [4] by Df_2 is denoted the class of algebras generated by $\mathbf{S5}^2$-frames). Here $E_i(w) = \{v \in W : v E_i w\}$ and $E_i(V) = \bigcup_{w \in V} E_i(w)$ for $w \in W, V \subseteq W$ and $i = 1, 2$.

A valuation of a formula φ in an $\mathbf{S5^2}$-frame is, as usual, defined by induction. Interesting clauses are those of modal operators:

$$w \models \Box_i \psi \Leftrightarrow \forall v \in W (w E_i v \Rightarrow v \models \psi);$$
$$w \models \Diamond_i \psi \Leftrightarrow \exists v \in W (w E_i v \;\&\; v \models \psi).$$

An especially interesting subclass of this class is the class of frames which satisfy an additional condition: $E_1(w) \cap E_2(w) = \{w\}$ for any $w \in W$. These frames can be represented as products of two $\mathbf{S5}$-frames.

Recall that for given two **S5**-frames $\mathcal{F} = (W, E)$ and $\mathcal{F}' = (W', E')$, the product $\mathcal{F} \times \mathcal{F}'$ is defined as the triple $(W \times W', E_1, E_2)$, where

$$(w, w')E_1(v, v') \text{ iff } w = v \text{ and } w'E'v';$$

$$(w, w')E_2(v, v') \text{ iff } wEv \text{ and } w' = v'.$$

$\mathbf{S5}^2$ is complete with respect to both classes (cf., [4] or [2]), that is for every formula φ of the language of $\mathbf{S5}^2$ we have

$$\mathbf{S5}^2 \vdash \varphi \text{ iff } \varphi \text{ is valid in every } \mathbf{S5}^2 \text{ frame iff } \varphi \text{ is valid in every product-frame.}$$

(Note that in [4] by RDf_2 is denoted the class of algebras generated by product-frames.)

One of the main interests towards $\mathbf{S5}^2$ is that $\mathbf{S5}^2$ axiomatizes the two-variable substitution-free fragment of the First Order Classical Logic, FOL for short. Indeed, consider the following translation of the formulas of the language of $\mathbf{S5}^2$ to the formulas of the language of FOL:

$p^t = P(x_1, x_2)$,
$(\cdot)^t$ is a homomorphism for the Booleans,
$(\Diamond_1\varphi)^t = \exists x_1 \varphi^t$,
$(\Diamond_2\varphi)^t = \exists x_2 \varphi^t$.

Then one can show (see e.g. [4]) that this translation is faithful for all formulas φ: $\mathbf{S5}^2 \vdash \varphi$ iff $FOL \vdash \varphi^t$.

It is relevant here to recall that **S5** axiomatizes the one-variable fragment of FOL (see [10]). The lattice of (normal) extensions of **S5** is rather easy to describe: it is a $(\omega + 1)$-chain, every (normal) extension of **S5** is semantically characterized by the **S5**-frame shown in Figure 1 below, all of them are finitely axiomatizable, enjoy the finite model property and are decidable (see Scroggs [8]). Actually, every (normal) extension of **S5** has the linear-size model property, and every proper (normal) extension of **S5** is tabular.

Similarly to $\mathbf{S5}^2$, one can show that the logic of n-times products of **S5**-frames is a subfragment of the n-variable FOL and the translation is defined in the same way as for $\mathbf{S5}^2$ above.

However, unlike **S5**, in the case of $\mathbf{S5}^n$, for $n \geq 3$, the lattice of extensions of $\mathbf{S5}^n$ is much more complicated. It is shown by Maddux [6] that already $\mathbf{S5}^3$ is undecidable. Moreover, every logic in between $\mathbf{S5}^3$ and the logic of 3-times products of **S5**-frames is undecidable and there exist infinitely many logics in that interval which do not have the finite model property.

For every formula φ, let $Sub(\varphi)$ denote the set of all subformulas of φ, and let $|\varphi|$ denote the cardinality of the set $Sub(\varphi)$. A logic L is said to have the *poly-size model property* if there exist a polynomial $P(\cdot)$ such that every L-consistent formula φ is satisfiable in an L–frame containing at most $P(|\varphi|)$ points. If $P(\cdot)$ is a linear function, then L is said to have the *linear-size model property*.

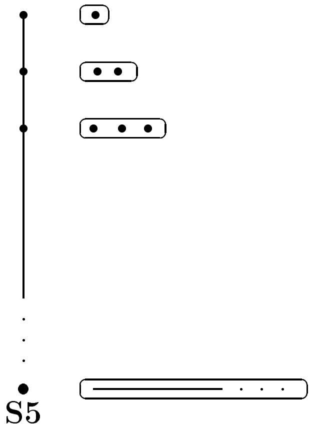

Figure 1: The proper extensions of **S5**.

We will show that all normal proper extensions of $\mathbf{S5}^2$ have the finite, indeed the poly-size model property.

Finally, let us mention that our notation is a bit different from the standard one in [4]: instead of c_0 and c_1 we use $\Diamond_1$ and $\Diamond_2$ respectively, and instead of T_1 and T_2 – E_1 and E_2.

2 NExtS5^2

Given two $\mathbf{S5}^2$-frames $\mathcal{F}_1 = (W, E_1, E_2)$ and $\mathcal{F}_2 = (W', E'_1, E'_2)$, a function $f : W \to W'$ is said to be a *p-morphism*, if, for any $w \in W, w' \in W'$ and $i = 1, 2$, $f(w)E'_i w'$ iff $\exists v \in W(wE_i v \ \&\ f(v) = w')$. $\mathcal{F}_2$ is said to be a *p-morphic image* of $\mathcal{F}_1$, if there is a p-morphism from W onto W'. It is well-known that p-morphic images preserve validity of formulas.

For a given $\mathbf{S5}^2$-frame $\mathcal{F} = (W, E_1, E_2)$ and $w \in W$, we call the sets of the form $E_i(w)$, $i = 1, 2$, *E_i-clusters*, the sets of the form $E_1(w) \cap E_2(w)$ – *E_0-clusters*, and denote them by $E_0(w)$. (It should be clear that $E_0 = E_1 \cap E_2$ is also an equivalence relation.)

Later we need the following result about p–morphisms.

Lemma 1 *Let $\mathcal{F}_1 = (W, E_1, E_2)$ and $\mathcal{F}_2 = (W', E'_1, E'_2)$ be finite $\mathbf{S5}^2$-frames, and $f : W \to W'$ a surjection. Define an equivalence relation E on W by wEv iff $f(w) = f(v)$. Then the following holds:*

1) If $E \subseteq E_0$ (that is E identifies only the points from E_0-clusters), then f is a p-morphism.

2) Let $\{C_i\}_{i=1}^k$ and $\{C^j\}_{j=1}^m$ be the sets of all E_1 and E_2-clusters of W respectively.

a) Let there exist E_1-clusters C_1, C_2 such that the corresponding E_0-clusters $|C_1 \cap C^j|$ and $|C_2 \cap C^j|$ contain the same number of points for every $j = 1, \ldots, m$. Suppose $C_1 \cap C^j = \{w_{j_1}, \ldots, w_{j_n}\}$ and $C_2 \cap C^j = \{v_{j_1}, \ldots, v_{j_n}\}$. If E is the smallest equivalence relation such that $w_{j_s} E v_{j_s}$ holds for all $s = 1, \ldots, n$ and $j = 1, \ldots, m$, then f is a p-morphism.

b) Let there exist E_2-clusters C^1, C^2 such that the corresponding E_0-clusters $|C_i \cap C^1|$ and $|C_i \cap C^2|$ contain the same number of points for every $i = 1, \ldots, k$. Suppose $C_i \cap C^1 = \{w_{i_1}, \ldots, w_{i_n}\}$ and $C_i \cap C^2 = \{v_{i_1}, \ldots, v_{i_n}\}$. If E is the smallest equivalence relation such that $w_{i_s} E v_{i_s}$ holds for all $s = 1, \ldots, n$ and $i = 1, \ldots, k$, then f is a p-morphism.

Proof. Obviously f is a p-morphism iff $EE_i(w) = E_iE(w)$, for any $w \in W$ and $i = 1, 2$.

1) Now we have $EE_i(w) \subseteq E_0E_i(w) \subseteq E_iE_i(w) \subseteq E_i(w) \subseteq E_iE(w)$. Hence $EE_i(w) \subseteq E_iE(w)$ $i = 1, 2$. The converse can be proved analogously. Thus f is a p-morphism.

2) a) From the definition of E it follows that for any $w \in W - (C_1 \cup C_2)$, $E(w) = \{w\}$. Hence, for any $w \in W - (C_1 \cup C_2)$, $E_1E(w) = E_1(w)$. On the other hand, $E_1(w) \subseteq W - (C_1 \cup C_2)$. Therefore $EE_1(w) = E_1(w)$, and thus, for any $w \in W - (C_1 \cup C_2)$, $E_1E(w) = E_1(w) = EE_1(w)$.

Further, it is obvious that $E_2E(w) = E_2(w)$, for any $w \in W - (C_1 \cup C_2)$, and since E does not identify the points from different E_2-clusters, then $EE_2(w)$ is also equal to $E_2(w)$. Hence $E_2E(w) = EE_2(w)$.

Furthermore, if $w \in C_1 \cup C_2$, then without loss of generality we can assume that $w \in C_1$. But then $E(w) = \{w, v\}$ where $v \in C_2$. Obviously $E_1E(w) = E_1(\{w, v\}) = C_1 \cup C_2$. On the other hand, $E_1(w) = C_1$ and from the definition of E it follows that $E(C_1) = C_1 \cup C_2$. So, $EE_1(w) = E(C_1) = C_1 \cup C_2$, and $E_1E(w) = EE_1(w)$.

Finally, it is obvious that $E_2E(w) = E_2(\{w, v\}) = E_2(w)$, for $w \in C_1 \cup C_2$, and the same arguments as above ensure that $E_2E(w) = E_2(w) = EE_2(w)$.

Thus, f is a p-morphism.

b) is completely analogous to a). $\square$

Denote by NExt$\mathbf{S5^2}$ the lattice of all normal extensions of $\mathbf{S5^2}$. For every $L \in$NExt$\mathbf{S5^2}$, we say that an $\mathbf{S5^2}$-frame $\mathcal{F}$ is an *L-frame* if $\mathcal{F}$ validates all the theorems of L. Let $\mathbf{F}_L$ denote the class of all finite L-frames. We have the following

Theorem 2 [1] *Every $L \in$ NExt$\mathbf{S5^2}$ is complete with respect to $\mathbf{F}_L$, and hence, every $L \in$ NExt$\mathbf{S5^2}$ enjoys the finite model property.*

A given finite frame $\mathcal{F}$ is said to be of the *E_i-depth n* $(i = 1, 2)$, written as $d_i(\mathcal{F}) = n$, if it contains precisely n E_i-clusters.

A class $\mathbf{F}$ of finite $\mathbf{S5}^2$-frames is said to be of the *E_i-depth n* $(i = 1, 2$ and $n \in \omega)$, written as $d_i(\mathbf{F}) = n$, if there is a member of $\mathbf{F}$ of the E_i-depth n, and the E_i-depth of every other member of $\mathbf{F}$ is less than or equal to n. $\mathbf{F}$ is said to be of the *E_i-depth ω*, written as $d_i(F) = \omega$, if the E_i-depth of members of $\mathbf{F}$ is not restricted to any natural n.

Denote by $\mathbf{n}$ the $\mathbf{S5}$-frame (W_n, E), where W_n is an n-element set, and $E = W_n \times W_n$. Also denote by $\mathbf{n} \times \mathbf{n}$ the product of $\mathbf{n}$ on itself. Obviously $\mathbf{n} \times \mathbf{n}$ is an $\mathbf{S5}^2$-frame. As mentioned in the introduction, $\mathbf{S5}^2$ is complete with respect to the class of all finite product frames.

Now, for every proper normal extension L of $\mathbf{S5}^2$, we are in a position to describe the class $\mathbf{F}_L$.

Suppose L is a proper normal extension of $\mathbf{S5}^2$. Then there exists a product frame $\mathbf{n}\times\mathbf{n}$ such that $\mathbf{n}\times\mathbf{n}\notin\mathbf{F}_L$. Let n be the minimal number such that $\mathbf{n}\times\mathbf{n}\notin\mathbf{F}_L$. Consider three subclasses of $\mathbf{F}_L$: $\mathbf{F}_1=\{\mathcal{F}\in\mathbf{F}_L : d_1(\mathcal{F})>n\}$, $\mathbf{F}_2=\{\mathcal{F}\in\mathbf{F}_L : d_2(\mathcal{F})>n\}$ and $\mathbf{F}_3=\{\mathcal{F}\in\mathbf{F}_L : d_1(\mathcal{F}),d_2(\mathcal{F})\leq n\}$. It is obvious that if $\mathcal{F}\in\mathbf{F}_1$, then $d_2(\mathcal{F})\leq n$, that if $\mathcal{F}\in\mathbf{F}_2$, then $d_1(\mathcal{F})\leq n$, and that $\mathbf{F}_L=\mathbf{F}_1\sqcup\mathbf{F}_2\sqcup\mathbf{F}_3$. Moreover, $d_1(\mathbf{F}_1)\leq\omega$, $d_2(\mathbf{F}_1)\leq n$, $d_1(\mathbf{F}_2)\leq n$, $d_2(\mathbf{F}_2)\leq\omega$, and both $d_1(\mathbf{F}_3),d_2(\mathbf{F}_3)\leq n$. Hence we obtain the following useful

Theorem 3 *For every proper normal extension L of $\mathbf{S5}^2$ there exists a natural number n such that $\mathbf{F}_L=\mathbf{F}_1\sqcup\mathbf{F}_2\sqcup\mathbf{F}_3$, where $d_1(\mathbf{F}_1),d_2(\mathbf{F}_2)\leq\omega$, and $d_2(\mathbf{F}_1),d_1(\mathbf{F}_2),d_1(\mathbf{F}_3),d_2(\mathbf{F}_3)\leq n$. (Note that any two of the classes $\mathbf{F}_1,\mathbf{F}_2$ and $\mathbf{F}_3$ may be empty.)*

3 Poly-size model property

Now for every proper normal extension L of $\mathbf{S5}^2$ let us introduce the following parameters: $p_1(L)=d_1(\mathbf{F}_1)$, $p_1'(L)=d_2(\mathbf{F}_1)$, $p_2(L)=d_1(\mathbf{F}_2)$, $p_2'(L)=d_2(\mathbf{F}_2)$, $p_3(L)=d_1(\mathbf{F}_3)$ and $p_3'(L)=d_2(\mathbf{F}_3)$. Call a parameter finite, if it is not ω. Note that the only parameters which may be infinite are $p_1(L)$ and $p_2'(L)$. Denote by $b(L)$ the maximum between all the finite parameters of L, and call it the *bound* of L. Note that if $p_1(L)$ and $p_2'(L)$ are ω, then $b(L)=n$, where n is the minimal natural number such that $\mathbf{n}\times\mathbf{n}\notin\mathbf{F}_L$.

We are in a position now to prove the main result of the paper, that every proper normal extension of $\mathbf{S5}^2$ has the poly-size model property. Let $|\varphi|$ denote the *modal size* of φ, that is the number of subformulas of φ of the form $\Diamond_1\psi$ and $\Diamond_2\chi$. Recall that a polynomial $P(n)$ is said to be of *degree* k, if n^k occurs in $P(n)$ and n^m does not occur in $P(n)$ for any $m>k$.

Theorem 4 *Let L be a proper normal extension of $\mathbf{S5}^2$ with the bound $b(L)$. Then every L-satisfiable formula φ, is satisfiable in an L-frame of the size $P(|\varphi|)$, where P is a polynomial of degree $b(L)+1$ in $|\varphi|$. Moreover, if all the parameters of L are finite, then P is linear in $|\varphi|$.*

Proof. Let L be as in the theorem with bound $b(L)$. Let φ be L–satisfiable. Then there exist an L-frame $\mathcal{F}=(W,E_1,E_2)$, a valuation $\models_{\mathcal{F}}$ on $\mathcal{F}$ and $w\in W$ such that $w\models_{\mathcal{F}}\varphi$. Now we will find a small satisfiable model for φ. As was already shown in Theorem 3, $\mathbf{F}_L=\mathbf{F}_1\sqcup\mathbf{F}_2\sqcup\mathbf{F}_3$. Hence, either $\mathcal{F}\in\mathbf{F}_1$, $\mathcal{F}\in\mathbf{F}_2$, or $\mathcal{F}\in\mathbf{F}_3$.

Case 1: First suppose $\mathcal{F}\in\mathbf{F}_3$. Let $\{C_i\}_{i=1}^k$ and $\{C^j\}_{j=1}^m$ be the classes of all E_1 and E_2-clusters of W respectively. Obviously $k,m\leq b(L)$. Let C_i^j denote the E_0-cluster $C_i\cap C^j$.

For every E_1-cluster C_i $(1 \leq i \leq k)$ and every $\Diamond_1\psi \in Sub(\varphi)$, we choose a point $x \in C_i$ such that $x \models_{\mathcal{F}} \psi$ (if such a point exists at all). We do the same for E_2-clusters. Moreover, if there are E_0-clusters of W which do not contain any selected points, we choose one point from each of them.

Denote by W_0 the set of all selected points and consider $V = W_0 \cup \{w\}$. Then $|V| \leq k|\varphi| + m|\varphi| + km + 1 \leq 2b(L)|\varphi| + b(L)^2 + 1$. Indeed, there exist k-many E_1-clusters and m-many E_2-clusters of W. From every E_i-cluster, $i = 1, 2$, we select at most $|\varphi|$ points. So, we select $(k|\varphi| + m|\varphi|)$-many points, and then from every E_0-cluster which does not contain any selected point, we choose an additional point. Obviously there are km-many E_0-clusters in W, hence $|W_0| \leq k|\varphi| + m|\varphi| + km$. (Note that if $\mathcal{F}$ is a product-frame, then $W = V$.) Consider the model $(\mathcal{G}, \models_{\mathcal{G}})$, where $v \models_{\mathcal{G}} p$ iff $v \models_{\mathcal{F}} p$, for every $v \in V$ and every propositional letter p occurring in φ.

Claim 1: For every $v \in V$ and $\psi \in Sub(\varphi)$, $v \models_{\mathcal{G}} \psi$ iff $v \models \psi$.

Proof By induction on the size of $\psi \in Sub(\varphi)$. The boolean clauses are trivial. Let $\psi = \Diamond_i\chi$, $i = 1, 2$. Then $v \models_{\mathcal{G}} \Diamond_i\chi$ implies that there exists $v' \in \mathcal{G}$ such that vE_iv' and $v' \models_{\mathcal{G}} \chi$. But then by the induction hypothesis $v' \models_{\mathcal{F}} \chi$, and hence $v \models_{\mathcal{F}} \Diamond_i\chi$. Conversely, from $v \models_{\mathcal{F}} \Diamond_i\chi$ it follows that χ is satisfied in $E_i(v)$. From the construction of $\mathcal{G}$ it follows that there exists $v' \in \mathcal{G}$ such that vE_iv' and $v' \models_{\mathcal{F}} \chi$. But then by the induction hypothesis $v' \models_{\mathcal{G}} \chi$, and hence $v \models_{\mathcal{G}} \Diamond_i\chi$. □

Hence $w \models_{\mathcal{G}} \varphi$, and φ is satisfiable in $\mathcal{G}$.

It is left to be shown that $\mathcal{G}$ is an L-frame. For this we show that there exists a p-morphism $f : W \to V$. Recall that for every E_0 cluster C_i^j in W there exists at least one element $v \in C_i^j$ in the chosen subset V. Let $witness(C_i^j)$ be a function which chooses an element in $C_i^j \cap V$. Now define $f : W \longrightarrow V$ as

$$f(w) = \begin{cases} witness(C_i^j) & \text{for } w \in C_i^j \text{ but } w \notin V \\ w & \text{otherwise} \end{cases}$$

Let $E \subseteq W \times W$ be such that xEy iff $f(x) = f(y)$. Then $E(w) \subseteq E_0(w)$ thus by Lemma 1.1, f is a p-morphism.

Hence φ is satisfiable in an L–frame containing at most $P(|\varphi|)$-many points, for $P(|\varphi|) = 2b(L)|\varphi| + b(L)^2 + 1$, a linear function.

It is worth to be noted that if $p_1(L)$ and $p_2'(L)$ are also finite, then the other cases, that is when $\mathcal{F} \in \mathbf{F}_2$ or $\mathcal{F} \in \mathbf{F}_1$, are proved completely analogously to Case 1. Hence, if all the parameters of L are finite, then φ is satisfiable in a model containing at most $P(|\varphi|)$-many points, where P is linear in $|\varphi|$.

Case 2: Now let $p_2'(L) = \omega$, $\mathcal{F} \in \mathbf{F}_2$, and $\{C_i\}_{i=1}^k$ and $\{C^j\}_{j=1}^t$ be the classes of all E_1 and E_2-clusters of W respectively. Here $k \leq b(L)$, while t may be any natural number.

For every E_1-cluster C_i $(1 \leq i \leq k)$ and every $\Diamond_1\psi \in Sub(\varphi)$, we choose a point $x \in C_i$ such that $x \models_{\mathcal{F}} \psi$ (if such a point exists at all). Denote the obtained set together with w by W_0. From every E_1-cluster we select not more than $|\varphi|$ points; there are only k many

E_1-clusters in $\mathcal{F}$, hence $|W_0| \leq k|\varphi|+1$. $E_2(W_0)$ contains no more than $k|\varphi|+1$ E_2-clusters. For every such E_2-cluster C^j and every $\Diamond_2\psi \in Sub(\varphi)$, we choose a point $y \in C^j$ such that $y \models_{\mathcal{F}} \psi$ (again, if such a point exists at all). Thus the number of selected points from $E_2(W_0)$ is bounded by $(k|\varphi|+1)+(k|\varphi|+1)|\varphi|$. Finally we choose a single point from each E_0-cluster C_i^j which does not contain a selected point yet. Denote by W' the set of all selected points from W (in all three steps). Our aim now is to get a small frame out of W'.

Let $V = E_2(W_0) \cap W'$. Since there exist at most $k(k|\varphi|+1)$-many E_0-clusters in $E_2(W_0)$, $|V| \leq (k|\varphi|+1)+(k|\varphi|+1)|\varphi|+k(k|\varphi|+1)$ (note that this bound directly follows from the bound in Case 1, where m is substituted by $k|\varphi|+1$).

Claim 2: Suppose $U \subseteq W'$ satisfies the following two conditions: $V \subseteq U$ and for every E_2-cluster C^j of W, $U \cap C^j \neq \emptyset$ implies $C^j \cap W' \subseteq U$. Then for every $v \in U$ and $\psi \in Sub(\varphi)$ we have $v \models_{\mathcal{U}} \psi$ iff $v \models_{\mathcal{F}} \psi$, where $\mathcal{U}$ is the model whose underlying frame is obtained by restricting the equivalences E_1 and E_2 to the set U, and $v \models_{\mathcal{U}} p$ iff $v \models_{\mathcal{F}} p$ for every $v \in U$.

Proof can be carried over similarly to Claim 1. We only consider the non-trivial direction for the cases $\psi = \Diamond_i\chi$, $i = 1, 2$. Indeed, if $v \models_{\mathcal{F}} \Diamond_1\chi$, then χ is satisfied in $E_1(v)$. But then from the construction of V it follows that there exists $v' \in V$ such that vE_1v' and $v' \models_{\mathcal{F}} \chi$. Since $V \subseteq U$, $v' \in U$ and by the induction hypothesis $v' \models_{\mathcal{U}} \chi$. Hence $v \models_{\mathcal{U}} \Diamond_1\chi$.

Now suppose $v \models_{\mathcal{F}} \Diamond_2\chi$. Then χ is satisfied in $E_2(v)$ and from the construction of W' it follows that there exists $v'' \in W'$ such that vE_2v'' and $v'' \models_{\mathcal{F}} \chi$. Now since $E_2(v) \cap U \neq \emptyset$, $E_2(v) \cap W' \subseteq U$. Therefore $v'' \in U$ and by the induction hypothesis $v'' \models_{\mathcal{U}} \chi$. Hence $v \models_{\mathcal{U}} \Diamond_2\chi$. □

The frame $(V, E_1|_V, E_2|_V)$ is an $\mathbf{S5}^2$-frame though it need not to be an L-frame. We solve this by adding some extra points to V.

We say that two E_2-clusters C^p and C^q from $W - E_2(W_0)$ are *φ-equivalent*, if $|W' \cap C_i^p| = |W' \cap C_i^q|$ $(1 \leq i \leq k)$. The number of non-φ-equivalent E_2-clusters is bounded by $|\varphi|^k$. Indeed, to every E_2-cluster C^p from $W - E_2(W_0)$ corresponds the sequence of natural numbers $(n_1, \ldots, n_k)$, where $n_1 = |W' \cap C_1^p|, \ldots, n_k = |W' \cap C_k^p|$. Obviously, $n_j \leq |\varphi|$ for $1 \leq j \leq k$, and to φ-equivalent E_2-clusters correspond the same sequences. Now since there exist only $|\varphi|^k$-many different sequences $(n_1, \ldots, n_k)$, then there exist only $|\varphi|^k$-many non-φ-equivalent E_2-clusters.

We choose from each φ-equivalence class a representative C^p, let $C'^p = C^p \cap W'$ and consider the new set $\overline{V} = V \cup \bigcup_{p \leq |\varphi|^k} C'^p$. Let $\mathcal{G}$ be $(\overline{V}, E_1|_{\overline{V}}, E_2|_{\overline{V}})$, and consider the model $(\mathcal{G}, \models_{\mathcal{G}})$, where $v \models_{\mathcal{G}} p$ iff $v \models_{\mathcal{F}} p$, for every $v \in \overline{V}$ and every propositional letter p occurring in φ. Obviously $|\overline{V}| \leq (k|\varphi|+1)+(k|\varphi|+1)|\varphi|+k(k|\varphi|+1)+|\varphi|^k(|\varphi|+k) \leq (b(L)|\varphi|+1)+(b(L)|\varphi|+1)|\varphi|+b(L)(b(L)|\varphi|+1)+|\varphi|^{b(L)}(|\varphi|+b(L))$.

From Claim 2 it follows directly that for any $v \in \overline{V}$, and any subformula ψ of φ, $v \models_{\mathcal{G}} \psi$ iff $v \models_{\mathcal{F}} \psi$. Hence $w \models_{\mathcal{G}} \varphi$, and φ is satisfiable in $\mathcal{G}$.

Finally, almost the same construction as before will provide us with a p-morphism from $\mathcal{F}$ to $\mathcal{G}$. Define f as a composition of two maps $g : W \to W'$ and $h : W' \to \overline{V}$. The map g

is defined as the map f in case 1, whence is a p–morphism.

Further, for every E_2-cluster $C^j \subseteq W$ let $C'^j = C^j \cap W'$. Suppose $C'^j \subseteq W' - V$, where $1 \leq j \leq t' < t$, and let C^p be a E_2-cluster which is $|\varphi|$-equivalent to C^j and $C'^p \subseteq \overline{V}$ (from the construction of $\overline{V}$ it directly follows that such C^p exists). Then the E_0-clusters $|C_i \cap C'^j|$ and $|C_i \cap C'^p|$ contain the same number of points for every $i = 1, \ldots, k$. Suppose $C_i \cap C'^j = \{w_{i_1}, \ldots, w_{i_n}\}$ and $C_i \cap C'^p = \{v_{i_1}, \ldots, v_{i_n}\}$. Let E' be the smallest equivalence relation such that $w_{i_s} E v_{i_s}$ holds for all $s = 1, \ldots, n$, $i = 1, \ldots, k$ and $j = 1, \ldots, t'$.

The map h corresponding to the equivalence relation E' is a p-morphism, by Lemma 1.2b. But then f is a p-morphism as well, as a composition of two p-morphisms.

Thus φ is satisfiable in an L-frame containing at most $P(|\varphi|)$-many points, for $P(\cdot)$ a polynomial of degree $b(L) + 1$.

Case 3 The case when $p_1(L) = \omega$ and $\mathcal{F} \in \mathbf{F}_1$ is proved completely analogously to Case 2.

The theorem is proved. □

Corollary 5 *Every proper normal extension of* $\mathbf{S5}^2$ *has the poly-size model property. Moreover, if all parameters of* L *are finite, then* L *has the linear-size model property.* □

4 Proper extensions of S5^2 without linear-size model property

In the previous section, we have shown that proper extensions of $\mathbf{S5}^2$ have the poly-size model property. In this section we show that our bound is indeed optimal by constructing extensions L_k of $\mathbf{S5}^2$ and formulas φ_k^n such that the size of the smallest L_k-frame satisfying φ_k^n is a polynomial of degree $b(L_k)+1$ in $|\varphi_k^n|$. (Of course, the logics L_k will have an infinite parameter, namely $p_2'(L_k)$ will be ω.)

Let a finite $\mathbf{S5}^2$-frame $\mathcal{F}$ be given and let $\{C_i\}_{i=1}^n$ and $\{C^j\}_{j=1}^m$ be the sets of the E_1 and the E_2-clusters of $\mathcal{F}$, respectively. We say that two distinct E_2-clusters C^p and C^q are *equivalent* if for any $i = 1, \ldots, n$ $|C_i^p| = |C_i^q|$, where as usual $C_i^p = C_i \cap C^p$ and $C_i^q = C_i \cap C^q$.

Fix any natural number $k \geq 2$. For any natural number n, let $\mathcal{G}_k^n$ be an $\mathbf{S5}^2$-frame of the E_1-depth k such that every E_2-cluster of $\mathcal{G}_k^n$ contains exactly $k + n$ points and every two distinct E_2-clusters of $\mathcal{G}_k^n$ are not equivalent to each other. Note that $\mathcal{G}_k^n$ is not unique, since there are several (though finitely many) frames with this property. Let $\mathcal{F}_k^n$ be the maximal one with this property, that is $|\mathcal{G}_k^n| \leq |\mathcal{F}_k^n|$, for any $\mathcal{G}_k^n$.

Denote by L_k the logic of frames $\mathcal{F}_k^1, \ldots, \mathcal{F}_k^n, \ldots$, that is $L_k = \bigcap_{n\in\omega} L(\mathcal{F}_k^n)$, where $L(\mathcal{F}_k^n)$ is the logic of the frame $\mathcal{F}_k^n$, $n \in \omega$. Obviously $p_2'(L_k) = \omega$ and $b(L_k) = k$.

The cases for $k = 2$ and $k = 3$ are depicted in Figure 2 below.

Now for $n > k$, let $\varphi_k^n = Q_k \wedge \psi^n$, where

$$\begin{aligned} Q_k &= \textstyle\bigwedge_{i=1}^k \Diamond_1 \Diamond_2 p_i \wedge \Box_1 \Box_2 [\bigwedge_{i=1}^k (\Diamond_1 p_i \leftrightarrow p_i) \wedge \bigwedge_{1 \leq i \neq j \leq k} \neg (p_i \wedge p_j)], \\ \psi^n &= \Box_1 [\textstyle\bigwedge_{i=1}^n \Diamond_2 q_i \wedge \Box_2 (\bigwedge_{1 \leq i \neq j \leq n} \neg (q_i \wedge q_j))]. \end{aligned}$$

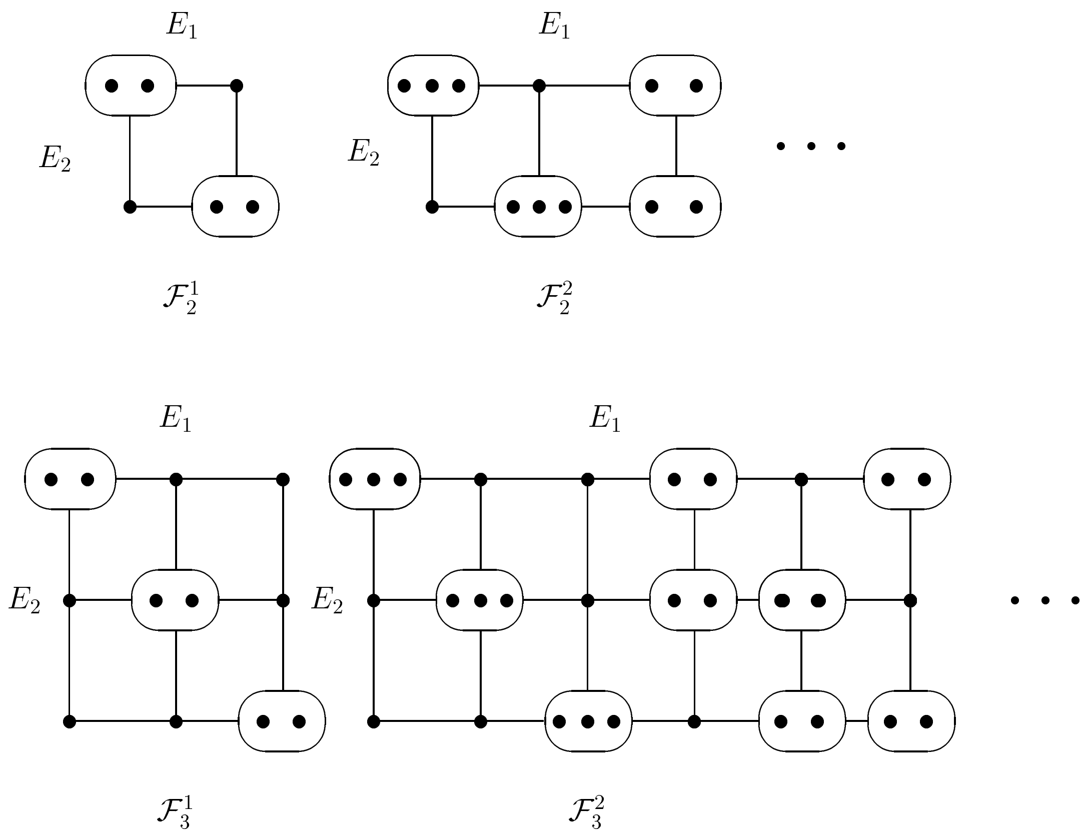

Figure 2: $\mathcal{F}_k^n$ frames for $k = 2$ and $k = 3$.

It is not hard to show that

$$Q_k \text{ is satisfiable in } \mathcal{F} \text{ iff } \mathcal{F} \text{ contains at least } k\text{-many } E_1\text{-clusters;} \tag{1}$$
$$\psi^n \text{ is satisfiable in } \mathcal{F} \text{ iff every } E_2\text{-cluster of } \mathcal{F} \text{ contains at least } n \text{ points.} \tag{2}$$

Thus the formula φ_k^n is satisfiable in the frame $\mathcal{F}_k^{n-k}$. The next claim states that in the logic L_k we cannot do better.

$$\mathcal{F}_k^{n-k} \text{ is the smallest } L_k\text{-frame satisfying } \varphi_k^n. \tag{3}$$

To prove (3), suppose φ_k^n is satisfiable in a finite L_k-frame $\mathcal{F}$. Then $\mathcal{F}$ is a p-morphic image of some $\mathcal{F}_k^i$, $i \in \omega$, that is, there is an onto p-morphism $f : \mathcal{F}_k^i \to \mathcal{F}$. As ψ^n is satisfied in $\mathcal{F}$, by (2), $i \geq n-k$.

Let $i = n-k$. The argument when $i > n-k$ is similar. Since Q_k is satisfiable in $\mathcal{F}$, (1) implies that $\mathcal{F}$ contains k-many E_1-clusters. Thus, f cannot identify points from different E_1-clusters of $\mathcal{F}_k^{n-k}$. Also note that since ψ^n is satisfiable in $\mathcal{F}$ and every E_2-cluster of $\mathcal{F}_k^{n-k}$ contains n points, f cannot identify points from the same E_2-cluster. Let us show that f cannot identify points from different E_2-clusters either. Indeed, suppose there exist $w \in C_i^p$ and $v \in C_i^q$ such that $f(w) = f(v)$. Since f is a p-morphism, for any $j = 1, \ldots, k$ and $w' \in C_j^p$ there exists $v' \in C_j^q$ such that $f(w') = f(v')$. Now since C^p is not equivalent to C^q, at least two points from some C_j^p will be identified by f. Hence the number of points of the E_2-cluster $f(C^p)$ of $\mathcal{F}$ is strictly less than n, which again contradicts the satisfability of ψ^n in $\mathcal{F}$. Therefore, f should be the identity map, and $\mathcal{F} = \mathcal{F}_k^{n-k}$.

Now we compute the size of $\mathcal{F}_k^{n-k}$. As in Theorem 4, to every E_2-cluster C^p of $\mathcal{F}_k^{n-k}$ we correspond the sequence of natural numbers $(m_1, \ldots, m_k)$, where $m_1 = |C_1^p|, \ldots, m_k = |C_k^p|$. From the definition of $\mathcal{F}_k^{n-k}$ it follows that $m_1 + \ldots + m_k = n$. But then the number of different sequences $(m_1, \ldots, m_k)$ will be

$$C_{n-1}^k = \frac{(n-1)!}{k!(n-(k+1))!} = \frac{(n-1)\ldots(n-k)}{k!} \geq \frac{(n-k)^k}{k!}.$$

Furthermore, every E_2-cluster of $\mathcal{F}_k^{n-k}$ contains precisely n points. So the size of $\mathcal{F}_k^{n-k}$ is at least $\frac{n(n-k)^k}{k!}$, hence

$$\text{The size of } \mathcal{F}_k^{n-k} \text{ is a polynomial of degree } k+1 \text{ in } n. \tag{4}$$

Putting (3) and (4) together yields

Theorem 6 *There exist infinitely many extensions L_k of $\mathbf{S5}^2$ and formulas φ_k^n such that the size of the smallest L_k-frame satisfying φ_k^n is a polynomial of degree $b(L_k)+1$ in $|\varphi_k^n|$.*

□

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