Simulating polyadic modal logics by monadic ones

George Goguadze[*] Carla Piazza[†] Yde Venema[‡]

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Abstract

We define an interpretation of modal languages with polyadic operators in modal languages that use monadic operators (diamonds) only. We also define a simulation operator which associates a logic Λ^{sim} in the diamond language with each logic Λ in the language with polyadic modal connectives. We prove that this simulation operator transfers several useful properties of modal logics, such as finite/recursive axiomatizability, frame completeness and the finite model property, canonicity and first-order definability.

Keywords modal logic, polyadic operators, interpretation, simulation, transfer results.

[*]Faculty of Computer Science, University of Saarland, Im Stadwald/36.1, Saarbrücken, Germany. E-mail: `george@ags.uni-sb.de`.

[†]Dipartimento di Matematica e Infomatica, Università degli Studi di Udine, Via delle Scienze 206, I-33100 Udine, Italy. E-mail: `piazza@dimi.uniud.it`.

[‡]Institute of Logic, Language and Computation, University of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam. E-mail: `yde@science.uva.nl`.

1 Introduction

Modern modal logic originated as a branch of philosophical logic in which the concepts of 'necessity' and 'possibility' were investigated by means of a single pair of dual, monadic operators ($\Box$ and $\Diamond$) that are added to a propositional or first order language. In subsequent years, modal logic has received attention as an attractive approach towards formalizing such diverse notions as time, knowledge, or action. Nowadays, modal logics are applied in various disciplines, ranging from economics to linguistics and computer science. Consequently, there is by now a large variety of modal languages, with an even greater wealth of interpretations. In particular, many applications require a framework consisting of a language with a family of modal connectives, and a semantics in which the associated accessibility relations are somehow related.

Because of this enormous diversity in appearance and applications, there is a need for unifying results in the mathematical theory of modal logics. Fortunately, this need has been recognized and addressed by modal logicians. Roughly speaking, one can distinguish two approaches in the literature on such unifying results in modal logic. One way is to *abstract* from the number and arity of the modal operators and develop the *general* theory of modal logic. This perspective was already taken in the classic paper Jónsson & Tarski [8] and followed by for instance Goldblatt [5]; it also prevails in modern works such as the monograph Kracht [9] and the recent textbook Blackburn et alii [3]. A second direction is to *compare* various classes of modal logics, cf. Kracht & Wolter [10] for a survey. One can distinguish two research lines in this direction; *transfer theory* comprises investigations of the effect of extending modal languages with certain operators that are somehow related to the old ones. In the second line of research, to which the present paper forms a contribution, one studies *simulations* of one modal logic by another.

Let us briefly explain what a simulation is (a more precise definition is given in section 2). Suppose that we are dealing with two modal languages, $\mathcal{L}_1$ and $\mathcal{L}_2$. A translation mapping $\mathcal{L}_1$-formulas to $\mathcal{L}_2$-formulas is called an *interpretation* if it meets certain uniformity conditions. Now we say that an $\mathcal{L}_2$-logic Λ_2 *simulates* an $\mathcal{L}_1$-logic Λ_1 with respect to the interpretation $(\cdot)^t$ if for all formulas φ in $\mathcal{L}_1$ we have that

$$\varphi \in \Lambda_1 \iff \varphi^t \in \Lambda_2.$$

A *simulation* is a function mapping $\mathcal{L}_1$-logics to simulating $\mathcal{L}_2$-logics; simulations can and have been used to derive results about (families of) $\mathcal{L}_1$-logics from results about $\mathcal{L}_2$-logics, and vice versa. Obviously, the more properties a simulation preserves or reflects, the more useful it is.

Gödel's translation of intuitionistic logic in Grzegorczyk's logic, cf. [4], provides a well-known early example of a simulation. Important results in modal logic were obtained by Thomason in the early seventies, cf. [16, 17]. Thomason showed how polymodal logics (that is, normal modal logics in a language with a number of *diamonds* or unary modalities) can be simulated by monomodal ones, and applies this result to prove certain (negative) results concerning monomodal logics.

Thomason's approach was taken up and developed further in Kracht & Wolter [10, 11]. In the second paper, which was in fact written earlier, the authors show that normal monomodal logics can simulate normal polymodal logics, (equational theories of lattices) and non-normal monomodal logics. Concerning the first simulation operation, which is based on Thomason's ideas, Kracht and Wolter prove that several properties of modal logics are transferred; that is, they show for various properties P that the normal polymodal logic has property P if and only its simulating logic has P. Examples include various kinds of axiomatizability, canonicity, first-order definability and interpolation. Although the results in [11] are significant, they do entirely justify the claim made in the paper's title that normal monomodal logics can simulate *all* others, since polyadic modal operators, which we will discuss in a moment, are not considered.

In their second paper [10], Kracht and Wolter fill this gap by making a proposal for a simulation of polyadic logics by monadic ones. However, this article is a survey paper rather than a research report, and the authors confine themselves to a very rudimentary investigation of properties that are preserved or reflected by this simulation. To be precise, they only prove reflection of canonicity and first-order definability. As far as we know, the only other interpretation of dyadic modalities by diamonds was developed in the context of categorial grammar, cf. KURTONINA [12]. The main result in this paper states that a certain simulation preserves Sahlqvist axiomatizability, but no other properties of logics are considered. In other words, the question whether there is an adequate simulation of polyadic modal logics by monadic ones is still open. It is our intention to provide a positive answer to this question.

What *are* polyadic modal operators? Syntactically, an n-ary modal operator is just an n-ary connective; what makes it modal is its intended interpretation, which uses accessibility relations of arity $n+1$. Generalizing the definition of ordinary modal logic, the truth condition for an n-ary operator ∇ reads as follows:

$$\mathcal{M}, w \Vdash \nabla(\varphi_1, \ldots, \varphi_n) \quad \text{iff} \quad \text{there are } v_1, \ldots, v_n \text{ such that } R_\nabla w v_1 \ldots v_n \\ \text{and } \mathcal{M}, v_j \Vdash \varphi_i \text{ for each } j.$$

Here R_∇ denotes the $n+1$-are relation associated with ∇. Alternatively, one could give a syntactic specification by defining the notion of a normal, polyadic modal *logic*, cf. section 2, or start from an algebraic perspective by considering n-ary *operators* on Boolean algebras, cf. JÓNSSON & TARSKI [8]. Examples of such (binary) operators can be found in categorial grammar (see LAMBEK [13], ROORDA [15]), in temporal logic of intervals (see VENEMA [18], HANSEN & ZHOU [6]) or in multidimensional modal logics such as arrow logic (see MARX ET ALII [14]). A prime example in the theory of Boolean algebras with operators is the composition operator in relation algebras, see HODKINSON AND HIRSCH [7].

If we are after a simulation of polyadic modal logics by monomodal ones, by the results of Kracht and Wolter it suffices to find out how to simulate a polyadic modal logic by a polymodal one. And since simulation results are usually based on semantic intuitions, the starting point would be to encode an $n+1$-ary relation ($n \geq 2$) using a number of binary relations. There are of course various ways to do so; our approach, which is independent and different from the one in [10], and also differs from the approach followed in KURTONINA [12], is described in detail in section 3. We confine ourselves to a polyadic language $\mathcal{L}_\nabla$ with one single dyadic operator ∇; our basic idea is to decompose the ternary accessibility relation T into four binary relations that we will call S, R_0, R_1 and R_2; accordingly, the polymodal language

$\mathcal{L}_\diamond$ in which we interpret $\mathcal{L}_\nabla$ will have four diamonds $\diamond_s$, $\diamond_0$, $\diamond_1$ and $\diamond_2$. The semantic idea of simulating $\mathcal{L}_\nabla$-structures by $\mathcal{L}_\diamond$-ones has a syntactic counterpart in an interpretation function $(\cdot)^\diamond$ mapping $\mathcal{L}_\nabla$-formulas to $\mathcal{L}_\diamond$-formulas. Finally, using the completeness of normal modal logics with respect to descriptive general frames, we are able to define a function $(\cdot)^{sim}$ mapping normal $\mathcal{L}_\nabla$-logics to $\mathcal{L}_\diamond$-logics.

The main results of our paper are stated in the Theorems 5.2 and 5.3; the first theorem states that the function $(\cdot)^{sim}$ is indeed a simulation, while the second result lists a number of important properties of modal logics that are transferred by this simulation — a concise survey is provided in Table 1.

Finally, the only reason why we confine ourselves to a polyadic modal language with a single modality of arity 2 is to keep our notation as simple as possible. It is entirely obvious how to extend our results to modalities of arity bigger than two. Our approach does not cover the case of a nullary modality or *modal constant*; however, the unary relation U_c that is the semantic counterpart of such a constant c, can easily be encoded as the binary relation $Id_{U_c} = \{(s,s) \mid s \in U_c\}$. Following this lead one easily shows how to simulate modal logics in such a language by a normal modal logic in a diamond language. Since it is also obvious that one can generalize our result to languages with a (possibly infinite) number of modalities of various rank, this shows that normal polymodal logics can adequately simulate normal modal logics in arbitrary modal languages. Combining this with the results of Kracht and Wolter, it follows that (provided we confine ourselves to finite languages),

> normal monomodal logics can indeed simulate all others.

It also follows that using our simulation, we can extend any interpretation of monomodal logic to polyadic modal logics. As an example we mention the 'box-as-power set' interpretation which maps the notion of derivability from a monomodal logic into the first-order notion of derivability from the set theory Ω; we refer to van Benthem et alii [1, 2] for more details.

Overview The next section provides some background information on the theory of modal logics with operators of arbitrary rank, and on the notion of one logic simulating another; we also mention some specific definitions and notational conventions. In section 3 and 4 we explain the basic semantic ideas underlying our approach, defining simulation and unsimulation maps that turn $\mathcal{L}_\nabla$-structures into $\mathcal{L}_\nabla$-structures, and vice versa. Section 5 forms the pivot of the paper, containing the definition of the simulation operation $(\cdot)^{sim}$ which maps $\mathcal{L}_\nabla$-logics to $\mathcal{L}_\diamond$-logics, and the two main theorems of the paper; this section also has the proofs of the easy parts of these theorems, while the remaining sections are devoted to proofs of the harder parts.

2 Preliminaries

Since not every reader may be familiar with polyadic modal operators, in this preliminary section we briefly recall some basic notions concerning modal languages and logics with poyadic operators; for more detailed information, the reader is referred to [3]. We also fix some notational conventions concerning the particular languages that we will be dealing with in this paper.

Modal Languages A *modal similarity type* τ is a pair $\tau = \langle O, \rho \rangle$, where O is a non-empty set, whose elements are called modal operators, and $\rho : O \to \mathbb{N}$ is an arity function. With a modal similarity type τ and a set of propositional variables P we associate a *modal language* $\mathcal{L}_\tau(P)$ whose symbols range over $P \cup O \cup \{\top, \wedge, \neg\} \cup \{(,)\}$. Often, we will not be specific concerning the set of propositional variables and write $\mathcal{L}_\tau$ instead of $\mathcal{L}_\tau(P)$. We will identify the modal language with its set of *formulas* which we define inductively as the smallest superset of $P \cup \{\top\}$ which contains $\neg\varphi$ and $\varphi \wedge \psi$ if φ and ψ are formulas, and contains $\nabla(\varphi_1, \ldots, \varphi_n)$ if ∇ is an n-adic modal operator and $\varphi_1, \ldots, \varphi_n$ are formulas.

Besides the standard abbreviations for the Boolean connectives, we use the symbol Δ for the dual operator to ∇; that is, $\Delta(\varphi_1, \ldots, \varphi_n)$ abbreviates $\neg\nabla(\neg\varphi_1, \ldots, \neg\varphi_n)$. We will use *diamonds*, i.e., symbols of the form $\Diamond_j$, for monadic modal operators; their duals are *boxes*: $\Box_j\varphi$ abbreviates $\neg\Diamond_j\neg\varphi$.

Modal Semantics Given a modal language $\mathcal{L}_\tau$, a *Kripke frame over* $\mathcal{L}_\tau$, or briefly: an $\mathcal{L}_\tau$*-frame*, is a tuple $\mathcal{F} = (W, R_\nabla)_{\nabla \in \tau}$, where W is a nonempty set, and every R_∇ is a relation on W of arity $\rho(\nabla) + 1$. A *model* is a pair $\mathcal{M} = (\mathcal{F}, V)$, where $\mathcal{F}$ is a frame and V is a *valuation*; that is, a function assigning to each propositional variable $p \in P$ a subset $V(p)$ of W. The notion of truth of a formula φ in $\mathcal{M}$ at a world w is defined in a standard way, the clause for an n-adic modal operator ∇ being

$$\mathcal{M}, w \Vdash \nabla(\varphi_1, \ldots, \varphi_n) \quad \text{iff} \quad \begin{array}{l}\text{for some } v_1, \ldots, v_n \text{ such that } R_\nabla w v_1 \ldots v_n \\ \text{we have } \mathcal{M}, v_i \Vdash \varphi_i.\end{array}$$

The definitions of other semantic notions are also as usual. We would like to stress however that in this paper we only consider the *local* consequence relation. For instance, if C is a class of $\mathcal{L}_\tau$-frames, Σ is a set of $\mathcal{L}_\tau$-formulas, and φ is a $\mathcal{L}_\tau$-formula, then we say that φ is a consequence of Σ over C if for every model $\mathcal{M}$ that is based on a frame in C, and every point w in $\mathcal{M}$, we have that $\mathcal{M}, w \Vdash \Sigma$ only if $\mathcal{M}, w \Vdash \varphi$.

Given an $n+1$-ary relation R on a set W, we define the operation $m_R : (\mathcal{P}(W))^n \to \mathcal{P}(W)$ by

$$m_R(a_1, \ldots, a_n) = \{w \in W \mid Rwv_1 \ldots v_n \text{ for some } v_1 \in a_1, \ldots, v_n \in a_n\}.$$

A *general* $\mathcal{L}_\tau$*-frame* is a pair $\mathcal{G} = (\mathcal{F}, A)$, where $\mathcal{F}$ is an $\mathcal{L}_\tau$-frame and A is a collection of subsets of W which is closed under the Boolean operations and under the operations m_{R_∇} for each modal operator ∇. The definition of notions such as *admissible sets* and *validity on a general frame* is as usual in modal logic.

A general $\mathcal{L}_\tau$-frame $\mathcal{G} = (\mathcal{F}, A)$ is called *differentiated* if for all distinct points there is an admissible set a containing precisely one of the two points; *tight* if for every modality ∇ and for all points $w_1, \ldots, w_n \in W$ for which $R_\nabla w w_1 \ldots w_n$ does not hold, there are admissible sets $a_1, \ldots, a_n$ satsifying $w_i \in a_i$ for all i but not $w \in m_{R_\nabla}(a_1, \ldots, a_n)$; *compact* if every subset A_0 of A with the finite intersection property has a non-empty intersection. A general frame is called *refined* if it is differentiated and tight, and *descriptive* if it is refined and compact. D_Σ denotes the class of descriptive general frames in which Σ is valid.

Normal Modal Logics The *minimal normal modal logic* $\mathbf{K}_\tau$ of similarity type τ is the minimal set of formulas of the language $\mathcal{L}_\tau$ which is closed under the rules of Modus Ponens, Uniform Substitution, and Necessitation:

$$\text{from } \varphi, \text{ infer } \Delta(\bot, \ldots, \varphi, \ldots, \bot),$$

while containing all propositional tautologies and the axioms
(K_∇) $\quad \Delta(p_1 \to q_1, \ldots, p_n \to q_n) \to (\Delta(p_1, \ldots, p_n) \to \Delta(q_1 \ldots q_n))$ and
(Dual) $\quad \nabla(p_1, \ldots, p_n) \leftrightarrow \neg\Delta(\neg p_1, \ldots, \neg p_n)$
(The latter axiom is needed since we have taken the existential ∇ as our primitive operator.)

A *normal modal $\mathcal{L}_\tau$-logic* Λ is a set of formulae which contains $\mathbf{K}_\tau$ and is closed under Modus Ponens, Uniform Substitution and Necessitation. If Γ is a set of formulae, we define $\mathbf{K}_\tau.\Gamma$ as the minimal normal modal logic containing Γ.

Given a normal logic Λ we say that a formula φ is a *theorem* of Λ, and we write $\vdash_\Lambda \varphi$, if φ belongs to Λ. In accordance with our use of the local paradigm in the semantics, we say that a formula φ is *derivable* from a set of formulas Σ in a normal modal logic Λ, notation: $\Sigma \vdash_\Lambda \varphi$, if there exists a finite subset $\{\sigma_1, \ldots, \sigma_n\}$ of Σ such that $\vdash_\Lambda (\sigma_1 \wedge \ldots \wedge \sigma_n) \to \varphi$.

For properties of normal modal logics such as elementarity, completeness or canonicity, the usual definitions apply.

Simulations The following definitions are taken from Kracht & Wolter's [11], but restricted to our context. Given two modal languages $\mathcal{L}_\tau$ and $\mathcal{L}_{\tau'}$, an *interpretation* of $\mathcal{L}_\tau$ in $\mathcal{L}_{\tau'}$ is a map $(\cdot)^F : \mathcal{L}_\tau \to \mathcal{L}_{\tau'}$ which satisfies the following uniformity conditions (for all proposition letters, $\mathcal{L}_\tau$-formulas and τ-operators involved):

$$\begin{array}{lcl} q^F & = & p^F[q/p] \\ (\varphi_1 \wedge \varphi_2)^F & = & (p_1 \wedge p_2)^F[\varphi_i^F/p_i] \\ (\neg\varphi)^F & = & (\neg p)^F[\varphi^F/p] \\ (\nabla(\varphi_1, \ldots, \varphi_n))^F & = & (\nabla(p_1, \ldots, p_n))^F[\varphi_i^F/p_i]. \end{array}$$

Given two normal modal logics Λ and Λ' in $\mathcal{L}_\tau$ and $\mathcal{L}_{\tau'}$, respectively, we say Λ' *simulates* Λ *with respect to* $(\cdot)^F$, if for all $\Sigma \subseteq \mathcal{L}_\tau$ and $\varphi \in \mathcal{L}_\tau$:

$$\Sigma \vdash_\Lambda \varphi \quad \text{iff} \quad \Sigma^F \vdash_{\Lambda'} \varphi^F.$$

A *simulation* is a function mapping normal modal logics in one language to simulating logics in another. For a property P of normal modal logics, we say that a simulation $(\cdot)^s$ *preserves* P if it holds that if Λ has P, then Λ^s has P. Similarly we say that $(\cdot)^s$ *reflects* P if Λ has P whenever Λ^s does. Finally, $(\cdot)^s$ *transfers* P if it both preserves and reflects P.

Specific definitions and notational conventions In this paper we propose a scheme of interpretations to map modal languages with polyadic operators to modal languages with diamonds only, and for each interpretation we define an associated simulation. For notational convenience, we will be dealing exclusively with two modal languages, $\mathcal{L}_\nabla$ and $\mathcal{L}_\Diamond$, respectively. $\mathcal{L}_\nabla$ has a single binary operator ∇, while $\mathcal{L}_\Diamond$ has four diamonds: $\Diamond_s$, $\Diamond_0$, $\Diamond_1$ and $\Diamond_2$;

throughout this paper we reserve the symbol i to refer to an element of the set $\{0, 1, 2\}$. In the semantics of $\mathcal{L}_\nabla$, we will denote the ternary accessibility relation for ∇ by T, and in the semantics of $\mathcal{L}_\Diamond$, we will denote the relations for $\Diamond_s$, $\Diamond_0$, $\Diamond_1$ and $\Diamond_2$ with S, R_0, R_1 and R_2, respectively. Given an $\mathcal{L}_\Diamond$-frame $\mathcal{F} = (W, S, R_0, R_1, R_2)$ we will denote the associated operations m_S, m_{R_0}, m_{R_1}, m_{R_2} as m_s, m_0, m_1 and m_2, respectively.

Throughout this paper we use the following notational convention: whenever we define a map $(\cdot)^q : A \to B$, then for any subset X of A we let X^q denote the set $\{x^q \mid x \in X\}$. The same convention applies, mutatis mutandis, to lifting operations on structures to operations on classes of structures.

3 Simulating $\mathcal{L}_\nabla$-structures

3.1 The basic idea

The basic idea behind our simulation is the following. In the Kripke semantics of $\mathcal{L}_\nabla$, we interpret the modal operator ∇ using a ternary accessibility relation T. If we want to simulate ∇ using diamonds, we have to encode the ternary relation using binary ones. There are a number of ways to do this; of these options we have chosen the following.

Definition 3.1 Let $\mathcal{F}$ be the $\mathcal{L}_\nabla$-frame (W, T). Its *simulation frame* $\mathcal{F}^\bullet$ is given as the $\mathcal{L}_\Diamond$-frame $\mathcal{F}^\bullet = (W^\bullet, S, R_0, R_1, R_2)$ where $W^\bullet, S, R_0, R_1$ and R_2 are defined as follows:

$$\begin{array}{rcl} W^\bullet & = & W \cup T, \\ wSt & \text{iff} & t \in T \text{ and } t_0 = w, \\ tR_iw & \text{iff} & t \in T \text{ and } t_i = w. \end{array}$$

Here and in the sequel we use the following notation: whenever t is a triple of elements of W, we let t_i denote the i-th coordinate of t; that is, we can write $t = (t_0, t_1, t_2)$.

When we are discussing such a simulation frame $\mathcal{F}^\bullet$, the elements of W will be called *old* or *base* points, the elements of T, *new* or *middle* points. $\lhd$

In words, to obtain $\mathcal{F}^\bullet$ from $\mathcal{F}$, we add each triple (u, v, w) of points from W to the universe, provided that the relation T holds of u, v and w. The relation S holds of two points w and t if w is an old point, t is a triple in T and $w = t_0$; and for each i, the relation R_i holds of two points t and w provided that w is an old point, t is a triple in T and w is the i-th coordinate of t. Thus each T-triple (u, v, w) is added as a *witness* to the fact that (u, v, w) indeed belongs to T, with the new relations holding as indicated in Figure 1. [htb]

This picture also explains our use of the terms 'base' and 'middle' points. Throughout this paper we will make use of the fact that in any simulation frame we can distinguish the base points from the middle points through $\mathcal{L}_\Diamond$-formulas. Let m denote the formula $\Diamond_0\top$, and b the formula $\neg\mathsf{m}$; it is immediate from the definitions that for any model $(\mathcal{F}^\bullet, V)$ and any point w in $\mathcal{F}^\bullet$ we have

$$\begin{array}{rcl} \mathcal{F}^\bullet, V, w \Vdash \mathsf{m} & \text{iff} & w \text{ is a middle point}, \\ \mathcal{F}^\bullet, V, w \Vdash \mathsf{b} & \text{iff} & w \text{ is a base point}. \end{array}$$

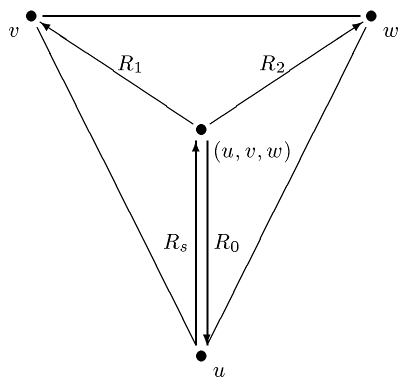

Figure 1: Adding a middle point

The basic translation function mapping $\mathcal{L}_\nabla$-formulas to $\mathcal{L}_\diamond$-formulas is closely related to the semantic intuition of simulating a frame.

Definition 3.2 The translation function $(\cdot)^t : \mathcal{L}_\nabla \to \mathcal{L}_\diamond$ is defined by the following formula induction:

$$\begin{array}{rcl} p^t &=& p \\ (\neg\varphi)^t &=& \neg\varphi^t \\ (\varphi_1 \wedge \varphi_2)^t &=& \varphi_1^t \wedge \varphi_2^t \\ (\varphi_1 \nabla \varphi_2)^t &=& \diamond_s(\diamond_1\varphi_1^t \wedge \diamond_2\varphi_2^t). \end{array}$$

The map $(\cdot)^\diamond : \mathcal{L}_\nabla \to \mathcal{L}_\diamond$ is defined as follows:

$$\varphi^\diamond = \mathsf{b} \to \varphi^t.$$

$\lhd$

Remark 3.3 The map $(\cdot)^\diamond$ is not an interpretation in the strict sense of the word — the reader can easily check that, for instance, $(\neg q \nabla \neg q)^\diamond \neq (p_1 \nabla p_2)^\diamond[(\neg q)^\diamond / p_i]$. Nevertheless, we will pretend that it is an interpretation and also *call* it an interpretation.

Our justification for this improper use of terminology is the following. Consider the translation $(\cdot)^+ : \mathcal{L}_\nabla \to \mathcal{L}_\diamond$ defined as follows:

$$\begin{array}{rcl} p^+ &=& \mathsf{b} \to p \\ (\neg\varphi)^+ &=& \mathsf{b} \to \neg\varphi^+ \\ (\varphi_1 \wedge \varphi_2)^+ &=& \mathsf{b} \to (\varphi_1^+ \wedge \varphi_2^+) \\ (\varphi_1 \nabla \varphi_2)^+ &=& \mathsf{b} \to \diamond_s(\diamond_1\varphi_1^+ \wedge \diamond_2\varphi_2^+). \end{array}$$

Clearly, $(\cdot)^+$ is an interpretation in the strict sense of the word. The point is that we could have been working with $(\cdot)^+$ instead of with $(\cdot)^\diamond$, and obtain the very same results. In fact, the map $(\cdot)^\diamond$ can be seen as a simplified version of $(\cdot)^+$; the difference, which lies in the fact that in $(\cdot)^\diamond$ we do not restrict to base points in every clause of the inductive definition, makes that $(\cdot)^\diamond$ is much easier to work with in practice.

The following proposition shows that the translation map $(\cdot)^t$ is truth preserving.

Proposition 3.4 *Let $\mathcal{F}$ be an $\mathcal{L}_\nabla$-frame, and let V and V' be valuations on $\mathcal{F}$ and $\mathcal{F}^\bullet$, respectively, such that V and V' agree on the base points of $\mathcal{F}^\bullet$. Then for all states w of $\mathcal{F}$, and for all $\mathcal{L}_\nabla$-formulas φ we have*

$$\mathcal{F}, V, w \Vdash \varphi \text{ iff } \mathcal{F}^\bullet, V', w \Vdash \varphi^t.$$

Proof. This proposition can be proved by a straightforward induction on the complexity of φ. For the atomic case, we have that $\mathcal{F}, V, w \Vdash p$ iff $w \in V(p)$ iff $w \in V'(p)$ iff $\mathcal{F}^\bullet, V', w \Vdash \varphi^\diamond$.

For the inductive step, the case of the Boolean connectives is trivial. Assume then that φ is of the form $\varphi_1 \nabla \varphi_2$. First we suppose that $\mathcal{F}, V, w \Vdash \varphi$. This implies the existence of points u_1, u_2 in W such that Twu_1u_2, and $\mathcal{F}, V, u_i \Vdash \varphi_i$. From the inductive hypothesis we may infer that $\mathcal{F}^\bullet, V', u_i \Vdash \varphi_i$. It follows from Twu_1u_2 that the triple $t = (w, u_1, u_2)$ is a middle point in $\mathcal{F}^\bullet$; it is obvious that wSt, tR_iu_i. But then we easily find that $\mathcal{F}^\bullet, V', w \Vdash \diamond_s(\diamond_1\varphi_1^t \wedge \diamond_2\varphi_2^t)$, which is what we were after.

For the other direction, suppose that $\mathcal{F}^\bullet, V', w \Vdash \varphi^t$. It is easy to see that this implies the existence of points t, u_1 and u_2 in $\mathcal{F}^\bullet$ such that wSt, tR_iu_i and $\mathcal{F}^\bullet, V', u_i \Vdash \varphi_i$. Using the definition of $\mathcal{F}^\bullet$ we find that t is a middle point, that u_1 and u_2 are base points, that in fact, t is the triple (w, u_1, u_2), and that since t belongs to T we have Twu_1u_2. The inductive hypothesis gives that $\mathcal{F}, V, u_i \Vdash \varphi_i$. Hence we may conclude that indeed $\mathcal{F}, V, w \Vdash \varphi_1 \nabla \varphi_2$. QED

As an immediate corollary we obtain that the map $(\cdot)^\diamond$ transfers validity between a frame and its simulation.

Proposition 3.5 *Let $\mathcal{F}$ be an $\mathcal{L}_\nabla$-frame, and φ an $\mathcal{L}_\nabla$-formula. Then*

$$\mathcal{F} \Vdash \varphi \text{ iff } \mathcal{F}^\bullet \Vdash \varphi^\diamond.$$

Proof. We only prove the direction from left to right. Suppose that $\mathcal{F} \Vdash \varphi$ but, for contradiction, that $\mathcal{F}^\bullet \nVdash \varphi^\diamond$. That is, for some valuation V' on $\mathcal{F}^\bullet$ and some point w in $\mathcal{F}^\bullet$ we have that $\mathcal{F}^\bullet, V', w \nVdash \mathsf{b} \to \varphi^t$. It follows that $\mathcal{F}^\bullet, V', w \Vdash \mathsf{b}$ whence w must be a base point. Now define the valuation V on $\mathcal{F}$ as follows:

$$V(p) = V'(p) \cap W,$$

where W is the collection of old points (that is, the universe of $\mathcal{F}$). Obviously, we may now use Proposition 3.4 to infer from $\mathcal{F}^\bullet, V', w \nVdash \varphi^t$ that $\mathcal{F}, V, w \nVdash \varphi$. This shows that $\mathcal{F} \nVdash \varphi$. QED

3.2 Simulating general frames

It is well-known that Kripke frames do not form a mathematically adequate semantics for modal logics; hence, if we want to simulate arbitrary modal ∇-logics, including the ones that are not complete, it would be good to extend our simulation to the kind of structures that do provide a good semantics for modal logics, such as modal algebras or general frames. In this paper we concentrate on general frames.

Definition 3.6 Let $\mathcal{G} = (\mathcal{F}, A)$ be a general $\mathcal{L}_\nabla$-frame. Its *simulation* $\mathcal{G}^\bullet$ is defined as the structure $(\mathcal{F}^\bullet, A^\bullet)$, where $A^\bullet$ is the collection of subsets of $\mathcal{F}^\bullet$ given by

$$A^\bullet = \{a \cup (T \cap \bigcup_{j \in J} a_j \times b_j \times c_j) \mid a, a_j, b_j, c_j \in A,\ J \text{ finite } \}.$$

$\lhd$

Proposition 3.7 *Let $\mathcal{G} = (\mathcal{F}, A)$ be a general $\mathcal{L}_\nabla$-frame. Then $\mathcal{G}^\bullet$ is a general $\mathcal{L}_\Diamond$-frame; in fact, it is the general $\mathcal{L}_\Diamond$-frame based on $\mathcal{F}^\bullet$ which is generated by A.*

Proof. In order to prove that $\mathcal{G}^\bullet$ is a general frame, it suffices to show that the collection $A^\bullet$ contains $\varnothing$ and $W \cup T$, and that it is closed under the boolean and modal operations. We only prove the part concerning closure under the modal operations.

Let d be an arbitrary element of $A^\bullet$, say, $d = a \cup (T \cap \bigcup_{j \in J} a_j \times b_j \times c_j)$. We have to show that $m_s d$ and $m_i d$ for $i = 0, 1, 2$ belong to $A^\bullet$. For m_i this follows from

$$m_i(T \cap a_j \times b_j \times c_j) = \varnothing,$$

which gives that $m_i d = m_i a = \{t \in T \mid t_i \in a\}$. Thus we have that $m_0 d = T \cap (a \times W \times W)$, $m_1 d = T \cap (W \times a \times W)$ and $m_2 d = T \cap (W \times W \times a)$. In each case we obtain a subset of $W^\bullet$ which clearly belongs to $A^\bullet$. For closure under m_s, we need the fact that

$$\begin{aligned}
m_s(T \cap a_0 \times a_1 \times a_2) &= \{w \in W \mid \text{ there is a } t \text{ in } T \cap a_0 \times a_1 \times a_2 \text{ with } wSt\} \\
&= \{w \in W \mid \text{there are } t_i \text{ in } W \text{ with } w = t_0, t_i \in a_i \text{ and } Tt_0t_1t_2\}, \\
&= \{t_0 \in a_0 \mid \text{there are } t_1 \in a_1,\ t_2 \in a_2 \text{ with } Tt_0t_1t_2\}, \\
&= a_0 \cap m_T(a_1, a_2).
\end{aligned}$$

For, this shows that

$$\begin{aligned}
m_s d &= m_s a \cup \textstyle\bigcup_j m_s(T \cap a_j \times b_j \times c_j) \\
&= \varnothing \cup \textstyle\bigcup_j (a_j \cap m_T(b_j, c_j)),
\end{aligned}$$

from which it is immediate that $m_s d$ belongs to $A \subseteq A^\bullet$.

Finally, we leave it for the reader to verify that $A^\bullet$ is the *smallest* set containing A which is closed under the boolean and modal operations. QED

The following proposition will play an important role when we prove our basic simulation result.

Proposition 3.8 *Let $\mathcal{G}$ be a general $\mathcal{L}_\nabla$-frame, and φ an $\mathcal{L}_\nabla$-formula. Then*

$$\mathcal{G} \Vdash \varphi \textit{ iff } \mathcal{G}^\bullet \Vdash \varphi^\diamond.$$

Proof. First we prove the direction from left to right. Assume that $\mathcal{G}$ is a general $\mathcal{L}_\nabla$-frame such that $\mathcal{G} \Vdash \varphi$. Now let V' be an arbitrary admissible valuation on $\mathcal{G}^\bullet$; define the valuation V on $\mathcal{G}$ as in the proof of Proposition 3.5; that is, put $V(p) = V'(p) \cap W$ for each proposition letter p. It is easy to see that V is admissible on $\mathcal{G}$: if $V'(p)$ is of the form $a \cup (T \cap \bigcup_{j \in J} a_j \times b_j \times c_j)$ with J finite and a and each a_j, b_j and c_j in A, then we find that $V(p) = a$ belongs to A. From the admissibility of V we may infer that $\mathcal{G}, V \Vdash \varphi$; hence, using Proposition 3.4 we find that $\mathcal{G}^\bullet, V' \Vdash \mathsf{b} \to \varphi^t$; since V' was arbitrary this gives that $\mathcal{G}^\bullet \Vdash \varphi^\diamond$.

The other direction is even simpler: assume that $\mathcal{G}^\bullet \Vdash \varphi^\diamond$, and consider an arbitrary admissible valuation V on $\mathcal{G}$. It is immediate by the definition of $\mathcal{G}^\bullet$ that V is admissible on $\mathcal{G}^\bullet$ as well. From this it is straightforward to prove the proposition. QED

3.3 Transfer of properties

In the sequel we will be working with *descriptive* general frames mainly. Hence, it is good to know that the simulation operation behaves well with respect to this property of general frames.

Proposition 3.9 *Let $\mathcal{G}$ be a general $\mathcal{L}_\nabla$-frame. Then $\mathcal{G}$ is descriptive if and only if $\mathcal{G}^\bullet$ is descriptive.*

Proof. The right to left direction of this proposition follows from more general results that we will prove later, cf. the Propositions 4.16 and 4.18 below.

In order to show that the simulation operation preserves descriptiveness, fix a general frame $\mathcal{G} = (W, T, A)$. We first prove two claims that concern the differentiation and tightness of $\mathcal{G}^\bullet$.

(1) If $\mathcal{G}$ is differentiated, then so is $\mathcal{G}^\bullet$.

Assume that $\mathcal{G}$ is differentiated and let u and v be two distinct points in $W^\bullet$. We have to find an admissible set $a \in A^\bullet$ such that one of the points belongs to a and the other one does not. This is easy if one of the points, say u, belongs to W, while the other belongs to T, since both $T = \varnothing \cup (T \cap (W \times W \times W))$ and $W = W \cup (T \cap (\varnothing \times \varnothing \times \varnothing))$ belong to $A^\bullet$. If both u and v belong to W then by assumption there is an $a \in A$ such that $u \in a$ and $v \notin a$; this suffices because $A \subseteq A^\bullet$. This leaves the case that both u and v belong to T. It follows from $u \neq v$ that $u_i \neq v_i$ for some $i \in \{0, 1, 2\}$. Since $\mathcal{G}$ is differentiated there is an $a \in A$ such that $u_i \in a$ and $v_i \notin a$. Clearly then $u \in m_i a \in A^\bullet$ while $v \notin m_i a$. This proves (1).

(2) If $\mathcal{G}$ is differentiated, then $\mathcal{G}^\bullet$ is tight.

Assume that $\mathcal{G}$ is differentiated. We first prove tightness with respect to the R_i relations. Let u and v be points in $W^\bullet$ such that for all $a \in A^\bullet$ we have that $v \in a$ implies $u \in m_i a$. Then it follows from $v \in W^\bullet$ that $u \in T = m_i W^\bullet$, and from $m_i T = \varnothing$ that $v \notin T$. Hence, v is a base point and u is a middle point of $W^\bullet$. Suppose, for contradiction, that R_iuv does not hold; by definition this means that $v \neq u_i$, so differentiation of $\mathcal{G}$ gives a set $a \in A$ such that $v \in a$, $u_i \notin a$. But then we have $v \in a$ and $u \notin m_i a$ which gives the desired contradiction.

Turning to the tightness of $\mathcal{G}$ with respect to the relation S, we consider two points u and v such that for all $a' \in A^\bullet$ we have that $v \in a$ implies $u \in m_s a$. It is fairly easy to see that u must be a base point, and v, a middle point. We claim that $u = v_0$, from which it is immediate that Suv. In order to prove that $u = v_0$, by the assumed differentation of $\mathcal{G}$ it suffices to show that for every $a \in A$ we have that $v_0 \in a$ implies $u \in a$. So let a be admissible in $\mathcal{G}$ and assume that v_0 belongs to a; then the triple v belongs to $a \times W \times W$, so by assumption we have that $u \in m_s(a \times W \times W)$. From $m_s(a \times W \times W) = a \cap m_T(W, W)$ we obtain the desired membership of u in a. This proves (2).

Now assume that $\mathcal{G}$ is descriptive. It follows from (1) and (2) that $\mathcal{G}^\bullet$ is refined, so we only have to show that $\mathcal{G}^\bullet$ is compact. Let $C \subseteq A^\bullet$ be a collection of admissible sets with the finite intersection property. We have to show that $\bigcap C \neq \varnothing$.

Define $C_b = \{c \cap W \mid c \in C\}$ and $C_m = \{c \cap T \mid c \in C\}$; obviously, at least one of these two sets has the finite intersection property. If this is the set C_b, then we may infer that $\bigcap C_b \neq \varnothing$ from $C_b \subseteq A$ and the compactness of $\mathcal{G}$. So, assume that C_m has the finite intersection property. Let τ be the topology on W that we obtain by taking A as a basis. It is well known that τ is a Stone space; hence, the product topology (W^3, τ^3) is a compact Hausdorff space. It follows from the tightness of $\mathcal{G}$ that T is a closed subset of W^3 with respect to this topology (in order to see this, show that its complement $W^3 - T$ is open by proving that $W^3 - T = \bigcup\{-m_T(a,b) \times a \times b \mid a, b \in A\}$). But then the restriction of τ^3 to T is compact; from this it is immediate that $\bigcap C_m \neq \varnothing$. This proves the compactness, and hence, the descriptiveness, of $\mathcal{G}$. QED

It may have attracted the attention of the reader that in our proof of Proposition 3.9 we did not show that the constituting properties of descriptiveness (differentiation, tightness and compactness) are preserved one by one. For instance, when proving that $\mathcal{G}^\bullet$ is compact, we needed $\mathcal{G}$ to be differentiated and tight as well. In fact, we have a number of counterexamples showing that not all properties of general frames are transferred by the operation $(\cdot)^\bullet$.

Remark 3.10 It can be the case that $\mathcal{G}$ is tight, while $\mathcal{G}^\bullet$ is not. For instance, suppose that $\mathcal{G}$ is given as the structure (W, T, A) with

$$\begin{aligned} W &= \{x, y, y', z\}, \\ T &= \{(x, y, z), (x, y', z)\}, \\ A &= \{a \subseteq W \mid y \in a \text{ iff } y' \in a\}. \end{aligned}$$

We leave it for the reader to verify that $\mathcal{G}$ is tight. In order to describe the general simulation frame $\mathcal{G}^\bullet$, let w denote the triple (x, y, z) and w' the triple (x, y', z). Then we have that $\mathcal{G}^\bullet$ is of the form $(W^\bullet, S, R_0, R_1, R_2, B)$ with $W^\bullet = \{x, y, y', z, w, w'\}$, the binary relations as always, and B is given as the set of subsets of $W^\bullet$ satisfying both $y \in b$ iff $y' \in b$ and $w \in b$ iff $w' \in b$. It is then easy to see that for all $b \in B$ we have that $y \in b$ implies $w' \in m_1 b$, while it certainly does not hold that $w' R_1 y$. This shows that $\mathcal{G}^\bullet$ is not tight.

Conversely, since we only needed differentiation of $\mathcal{G}$ in order to prove tightness of $\mathcal{G}^\bullet$, there are many non-tight general frames with a tight simulation. For instance, consider the structure $\mathcal{H} = (W', T', A')$ with

$$\begin{array}{rcl} W' & = & \mathbb{N} \cup \{\omega\}, \\ T' & = & \{(n,n,n) | n \in \mathbb{N}\}, \\ A' & = & \{a \subseteq W' | a \text{ is finite and } \omega \notin a, \text{ or } a \text{ is co-finite and } \omega \in a\}. \end{array}$$

$\mathcal{H}$ is not tight, since we have $\omega \in m_{T'}(a,b)$ whenever $\omega \in a$ and $\omega \in b$, while $T'\omega\omega\omega$ does not hold. But $\mathcal{H}$ is clearly differentiated, whence $\mathcal{H}^\bullet$ is tight by (2).

$\mathcal{H}$ is also a counterexample to the conjecture that the operation $(\cdot)^\bullet$ preserves compactness. For, it is not difficult to see that $\mathcal{H}$ is compact; now consider the collection $B \subseteq A^\bullet$ of admissible sets of the form $T \cap (a_0 \times a_1 \times a_3)$ such that each a_i is co-finite. It is not difficult to show that B has the finite intersection property; in order to show that the intersection of finitely many elements of B is not empty one just has to look for a large enough natural number. However, the set $\bigcap B$ is empty; for, (ω,ω,ω) is the only triple (t_0, t_1, t_2) such that each t_i belongs to all co-finite sets. The problem is that (ω,ω,ω) does not belong to T. This problem would not have occurred if $\mathcal{H}$ had been tight, as the last part of the proof of Proposition 3.9 shows.

4 Unsimulating $\mathcal{L}_\diamond$-structures

For various reasons it is not sufficient to have a structural operation that moves in one direction only, taking $\mathcal{L}_\nabla$-structures into $\mathcal{L}_\diamond$-structures. We also need an 'unsimulation' operation that maps $\mathcal{L}_\diamond$-structures to $\mathcal{L}_\nabla$-ones.

4.1 Axiomatizing simulation frames

One of the main applications of this unsimulation map is to help find a characterization of the simulation frames, i.e., the $\mathcal{L}_\diamond$-frames that are the simulation of some $\mathcal{L}_\nabla$-frame.

Definition 4.1 Consider the following set of first order axioms:

$$\begin{array}{ll} (S0) & \forall xy\,(Sxy \leftrightarrow R_0yx) \\ (S1)_i & \forall xyy'(R_ixy \wedge R_ixy' \rightarrow y = y') \\ (S2)_{ij} & \forall x\,(\exists y R_ixy \rightarrow \exists y' R_jxy') \\ (S3)_{ij} & \neg\exists xyz\,(R_ixy \wedge R_jyz) \\ (S4) & \forall x_0x_1x_2yy'\,((\bigwedge_i R_iyx_i \wedge \bigwedge_i R_iy'x_i) \rightarrow y = y'). \end{array}$$

The formula $S2$ is defined as the conjunction $\bigwedge_{ij} S2_{ij}$, and similar definitions apply to $S1$ and $S3$. Let *Sim* denote the set $\{S0,\ldots,S3\}$, and Sim^+ the set $\{S0,\ldots,S4\}$. An $\mathcal{L}_\diamond$-frame satisfying *Sim* is called a *Sim*-frame. $\lhd$

In words, these axioms express the following. S0 states that the relations S and R_0 are each other's converse; S1, that each relation R_i is a partial function; S2, that if a state belongs to the domain of one of these three partial functions, then it belongs to the domain of each of them. According to S3, if a point y belongs to the range of some of these partial functions, then it cannot belong to the domain of any of them. Finally, given the other axioms, $S4$ expresses the fact that points that belong to the domain of each R_i are completely determined by the triple consisting of their R_0, R_1 and R_2-successors.

It is not very difficult to prove that Sim^+ completely characterizes (up to isomorphism) the class of simulation frames, cf. Proposition 4.4 below. But if the Sim^+-frames form the class of simulation frames (modulo isomorphism) and, obviously, not all Sim-frames are Sim-frames, why are we interested in the Sim-frames? The reason is that although we have a *first order* characterization of the class of simulation frames, we cannot characterize it in terms of *modal* formulas — this follows from Proposition 4.10 below, together with the fact that the classes of Sim-frame and Sim^+-frames are distinct. The class of Sim-frames happens to be a good approximation for the class of simulation frames, and it *does* allow a modal characterization. In fact, Proposition 4.10 states that the Sim-frames form the class of bounded morphic images of simulation frames. This explains the important role of this class in our work.

As we mentioned already, in order to prove the first order characterization result concerning the simulation frames, we need the definition of the *unsimulation* of a simulation frame. However, we need to define this notion for arbitrary Sim-frames.

Definition 4.2 Let $\mathcal{F} = (W, S, R_0, R_1, R_2)$ be a Sim-frame. A point w of $\mathcal{F}$ is called a *middle point* if it has an R_0-successor (or, equivalently, an R_1 or an R_2-successor) and a *base point* if it has no R_0-successor. The collection of base points is denoted as $W_\bullet$, the collection of middle points by W^m.

It follows from the axioms $S1$ and $S4$ that for each i we may assume the existence of a map $r_i : W^m \to W$ mapping middle points to their unique R_i-successors.

Given a Sim-frame, we define its *unsimulation* as the $\mathcal{L}_\nabla$-frame $\mathcal{F}_\bullet = (W_\bullet, T)$, where $Tuvw$ holds of three base points if there is a point m such that R_0mu, R_1mv and R_2mw. $\lhd$

In the following proposition we gather some useful information about Sim-frames. In the sequel, these facts will be used without warning.

Proposition 4.3 *Let $\mathcal{F} = (W, S, R_0, R_1, R_2)$ be a Sim-frame.*

1. *Each R_i-successor (of any point) is a base point; that is, the range of each map r_i is included in $W_\bullet$.*
2. *Each S-successor (of any point) is a middle point.*
3. *No point can have both S-successors and R_i successors for some i.*

Proof. Left to the reader. QED

We leave the proof of Proposition 4.3 to the reader and turn to the first order characterization of the simulation frames.

Proposition 4.4 *Let $\mathcal{F}$ be an $\mathcal{L}_\diamond$-frame. Then $\mathcal{F} \models Sim^+$ if and only if $\mathcal{F}$ is isomorphic to a simulation frame.*

Proof. It is easy to check that each of the Sim^+-axioms holds of every simulation frame, so we do not go into detail here. For the other direction of the proof, consider an $\mathcal{L}_\diamond$-frame $\mathcal{F} = (W, S, R_0, R_1, R_2)$ such that $\mathcal{F} \Vdash Sim^+$. We claim that

$$\mathcal{F} \cong (\mathcal{F}_\bullet)^\bullet.$$

Clearly, for any middle point m of $\mathcal{F}$ the triple $(r_0 m, r_1 m, r_2 m)$ belongs to the relation T of $\mathcal{F}_\bullet$. This means that the following map $f : W \to (W_\bullet)^\bullet$ is well-defined:

$$f(w) = \begin{cases} w & \text{if } w \text{ is a base point,} \\ (r_0 m, r_1 m, r_2 m) & \text{if } w \text{ is a middle point.} \end{cases}$$

It is fairly easy to see that f is a bijection: surjectivity is immediate by the definition of T, while injectivity follows from axiom $S4$.

The map is a homomorphism. To see this, first suppose that $R_i mw$. Then m is a middle point and w is a base point. Since w is the unique successor of m, we have that $w = r_i m$. This shows that in $(\mathcal{F}_\bullet)^\bullet$ we have $R_i f(m) f(w)$, as required. But if f is a homomorphism with respect to each R_i then also with respect to S, since in both $\mathcal{F}$ and $(\mathcal{F}_\bullet)^\bullet$, R_0 and S are each other's converse.

Finally, we show that f^{-1} is a homomorphism as well. Suppose that $R_i f(m) f(w)$ in $(\mathcal{F}_\bullet)^\bullet$. By definition of $(\cdot)^\bullet$, this means that $f(m)$ is a triple $t = (t_0, t_1, t_2)$ of points in $W_\bullet$ such that $t \in T$ and $f(w) = t_i$. By definition of T in $\mathcal{F}_\bullet$, there must be a middle point m' such that $R_0 m' t_0$, $R_1 m' t_1$ and $R_2 m' t_2$; this shows that $f(m') = t = f(m)$, so by injectivity of f we find that m and m' are identical. Also, from $f(w) = t_i$ we easily infer that w is a base point and that in fact, $w = t_i$. We already saw that $R_i m' t_i$; now we know that this really boils down to $R_i mw$. Hence, f^{-1} is a homomorphism with respect to the R_i relations; but just like for f we then easily infer f^{-1} to be a homomorphism with respect to S as well.

This shows that f is an isomorphism between $\mathcal{F}$ and $(\mathcal{F}_\bullet)^\bullet$ and thus finishes the proof. QED

Remark 4.5 It is possible to extend the simulation and unsimulation operations to adjoint functors between the categories of $\mathcal{L}_\nabla$-frames and *Sim*-frames (both with bounded morphisms as arrows). Using these functors one can show that the category of $\mathcal{L}_\nabla$-frames is equivalent to the category of (isomorphic copies of) simulation frames. Since we will not need these results further on, we do not provide the details here.

We *do* need the fact that the unsimulation operation behaves well with respect to the translation maps we defined in the previous section.

Proposition 4.6 *Let $\mathcal{F}$ be a Sim-frame, and let V and V' be valuations on $\mathcal{F}$ and $\mathcal{F}_\bullet$, respectively, such that V and V' agree on $W_\bullet$. Then for all $\mathcal{L}_\nabla$-formulas φ and all base points w we have*

$$\mathcal{F}, V, w \Vdash \varphi^t \textit{ iff } \mathcal{F}_\bullet, V', w \Vdash \varphi.$$

Proof. This proposition is proved by a straightforward induction on the complexity of $\mathcal{L}_\nabla$-formulas. We only treat, of the inductive step, the case that φ is of the form $\psi_1 \nabla \psi_2$. The claim follows by the following chain of equivalences:

$$\begin{array}{ll}
\multicolumn{2}{l}{\mathcal{F}_\bullet, V', w \Vdash \psi_1 \nabla \psi_2} \\
\text{iff} & \text{there are } w_1,\, w_2 \text{ with } Tww_1w_2 \text{ and } \mathcal{F}_\bullet, V', w_i \Vdash \psi_i, \\
\text{iff (def. } T) & \text{there are } m,\, w_1,\, w_2 \text{ with } R_0mw,\, R_imw_i \text{ and } \mathcal{F}_\bullet, V', w_i \Vdash \psi_i, \\
\text{iff (Ind. Hyp.)} & \text{there are } m,\, w_1,\, w_2 \text{ with } R_0mw,\, R_imw_i \text{ and } \mathcal{F}, V, w_i \Vdash \psi_i^t, \\
\text{iff} & \mathcal{F}, V, w_i \Vdash \Diamond_s(\Diamond_1\psi_1^t \wedge \Diamond_2\psi_2^t).
\end{array}$$

QED

As an immediate corollary we obtain the following proposition.

Proposition 4.7 *Let $\mathcal{F}$ be a Sim-frame. Then for all $\mathcal{L}_\nabla$-formulas φ we have*

$$\mathcal{F} \Vdash \varphi^\Diamond \textit{ iff } \mathcal{F}_\bullet \Vdash \varphi.$$

To finish off this subsection, we discuss the precise relation between the *Sim*-frames and the simulation frames. We will show in Proposition 4.10 below that the *Sim*-frames form the class of bounded morphic images of the simulation frames. In our proof of this result we will use the construction of *unravelling* a frame and a model. Since we will use this construction later as well, we isolate the following definition and proposition.

Definition 4.8 Let $\mathcal{F} = (W, S, R_0, R_1, R_2)$ be an $\mathcal{L}_\Diamond$-frame, and let r be some point in $\mathcal{F}$. We define the *unravelling of $\mathcal{F}$ from r*, notation: $\vec{\mathcal{F}}_r$, to be the frame $(W', S', R_0'R_1', R_2')$ given as follows. W' is the set of all sequences $w_0 \dots w_n$ $(n \geq 0)$ of elements in W such that $w_0 = r$ and for all $i < n$: either w_iSw_{i+1} or $w_iR_1w_{i+1}$ or $w_iR_2w_{i+1}$. Let $\mathit{last}(\bar{w})$ denote the last item of the sequence $\bar{w}$, and let $\bar{w}s$ denote the result of adding s as a new last item to the sequence $\bar{w}$. The relations of $\mathcal{F}'$ are defined as follows:

$$\begin{array}{rll}
\bar{w}S'\bar{v} & \text{if} & \bar{v} = \bar{w}s \text{ and } \mathit{last}(\bar{w})Ss \text{ for some point } s, \\
\bar{w}R_0'\bar{v} & \text{if} & \bar{v}S'\bar{w}, \\
\bar{w}R_1'\bar{v} & \text{if} & \bar{v} = \bar{w}s \text{ and } \mathit{last}(\bar{w})R_1s \text{ for some point } s, \\
\bar{w}R_2'\bar{v} & \text{if} & \bar{v} = \bar{w}s \text{ and } \mathit{last}(\bar{w})R_2s \text{ for some point } s.
\end{array}$$

For an $\mathcal{L}_\Diamond$-model $\mathcal{M} = (\mathcal{F}, V)$, its *unravelling from r* is defined as the model $\vec{\mathcal{M}}_r = (\vec{\mathcal{F}}_r, \vec{V})$ where $\vec{V}$ is the valuation given by $\vec{V}(p) = \{\bar{w} \in \vec{W}_r \mid \mathit{last}(\bar{w}) \in V(p)\}$. $\lhd$

Proposition 4.9 *Let $\mathcal{F} = (W, S, R_0, R_1, R_2)$ be an $\mathcal{L}_\Diamond$-frame, let $\mathcal{M} = (\mathcal{F}, V)$ be a model and let r be some point in $\mathcal{F}$. Then*

1. *the map $\mathit{last} : \vec{W}_r \to W$ constitutes a bounded morphism from $\vec{\mathcal{F}}_r$ to $\mathcal{F}$.*
2. *the map $\mathit{last} : \vec{W}_r \to W$ constitutes a bounded morphism from $\vec{\mathcal{M}}_r$ to $\mathcal{M}$ mapping (the sequence) r to r.*

3. *if $\mathcal{F}$ is a Sim-frame, and r is a base point, then $\vec{\mathcal{F}}_r$ is isomorphic to a simulation frame.*

Proof. The first two claims of the proposition are standard.

For the third part, it follows immediately from the definitions that unravellings of *Sim*-frames validate the axiom $S4$. It is also fairly straightforward to check that if $\mathcal{F}$ is a *Sim*-frame, then so is $\vec{\mathcal{F}}_r$ — we leave the details to the reader. QED

This relation between the simulation frames and their modal approximations, the *Sim*-frames, can be concisely formulated as follows.

Proposition 4.10 *The Sim-frames form the class of bounded morphic images of the simulation frames.*

Proof of Proposition 4.10. First suppose that $\mathcal{F} = (W, S, R_0, R_1, R_2)$ is a *Sim*-frame. It is easy to prove from Proposition 4.9 that $\mathcal{F}$ is a bounded morphic image of the disjoint union of the collection $\{\vec{\mathcal{F}}_r \mid r \in W_\bullet\}$ of its unravellings from base points. Since each of these unravellings is isomorphic to a simulation frame, say $\vec{\mathcal{F}}_r \cong \mathcal{E}_r^\bullet$, and an easy proof reveals that $\biguplus_{r\in W_\bullet} \mathcal{E}_r^\bullet \cong \left(\biguplus_{r\in W_\bullet} \mathcal{E}_r\right)^\bullet$, this shows that $\mathcal{F}$ is a bounded morphic image of the simulation frame $\left(\biguplus_{r\in W_\bullet} \mathcal{E}_r\right)^\bullet$.

For the other direction of the proposition, it suffices to prove that the class of **Sim**-frames is closed under taking bounded morphic images. This can be proved directly, but it also follows from the fact that this class allows a modal characterization (cf. Proposition 4.12) and the fact that modally definable classes are closed under taking bounded morphic images. QED

4.2 A modal axiomatization

It is fairly easy to give a modal axiomatization of the class of *Sim*-frames.

Definition 4.11 Consider the following set of modal axioms:

$(A0)$ $\quad (p \to \Box_s \Diamond_3 p) \wedge (p \to \Box_3 \Diamond_s p)$

$(A1)_i$ $\quad \Diamond_i p \to \Box_i p$

$(A2)_{ij}$ $\quad \Diamond_i \top \to \Diamond_j \top$

$(A3)_{ij}$ $\quad \Box_i \Box_j \bot$

Let Ax be the set $\{A0, A1, A2, A3\}$ (with similar definitions of $A1$, $A2$ and $A3$ as in Definition 4.1). We define **Sim** to be the basic modal logic $\mathbf{K}_\Diamond$ extended with the axiom set Ax. ◁

Proposition 4.12 *Each of the axioms Ak_{ij} is a Sahlqvist formula; its first order correspondent is the formula Sk_{ij}. As a corollary, the set Ax modally characterizes the class of Sim-frames, and the logic* **Sim** *is canonical and strongly sound and complete with respect to the class of* **Sim***-frames.*

Proof. Immediate by the syntactic shape of the modal axioms, and the properties of Sahlqvist formulas. QED

From this result and the earlier established close connection between *Sim*-frames and simulation frames, the following proposition is almost immediate.

Proposition 4.13 **Sim** *is strongly sound and complete with repect to the class of simulation frames.*

Proof. Soundness is a consequence of Proposition 4.12. For completeness, suppose that Σ is a set of $\mathcal{L}_\Diamond$-formulas and φ is an $\mathcal{L}_\Diamond$-formula such that $\Sigma \not\vdash_{\mathbf{Sim}} \varphi$. It follows from Proposition 4.12 that there is a model $\mathcal{M} = (\mathcal{F}, V)$ based on a **Sim**-frame $\mathcal{F}$, and a point r in $\mathcal{M}$ such that $\mathcal{M}, r \Vdash \Sigma$ while $\mathcal{M}, r \not\Vdash \varphi$. It easily follows from the results in Proposition 4.9 and the invariance of modal truth under bounded morphisms (on the level of models) that $\vec{\mathcal{M}}_r, r \Vdash \Sigma$ while $\vec{\mathcal{M}}_r, r \not\Vdash \varphi$. Since $\vec{\mathcal{F}}_r$ is isomorphic to a simulation frame, this shows that φ is not a consequence of Σ on the class of simulation frames. QED

4.3 Unsimulating general frames

Now we extend the unsimulation to the level of general frames.

Definition 4.14 A *general Sim-frame* is a general frame based on a *Sim*-frame. Given such a general *Sim*-frame $\mathcal{G} = (\mathcal{F}, A)$, its *unsimulation* $\mathcal{G}_\bullet$ is defined as the structure $(\mathcal{F}_\bullet, A_\bullet)$, where $A_\bullet$ is the set $\{a \cap W_\bullet \mid a \in A\}$. $\lhd$

Proposition 4.15 *Let $\mathcal{G}$ be a general Sim-frame; then $\mathcal{G}_\bullet$ is a general $\mathcal{L}_\nabla$-frame.*

Proof. Let $\mathcal{G} = (\mathcal{F}, A)$ be a general *Sim*-frame. We have to show that $A_\bullet$ contains the sets $W_\bullet$ and $\varnothing$ and that it is closed under the Boolean operations and under the operation m_T. We only treat the case of m_T.

Suppose that a_1 and a_2 belong to $A_\bullet$; it suffices to prove that $m_T(a_1, a_2) \in T$. It is straightforward to show that in a *Sim*-frame we have that

$$m_T(a_1, a_2) = m_s(m_1 a_1 \cap m_2 a_2).$$

From this it follows that $m_T(a_1, a_2)$ belongs to A; but since in a *Sim*-frame it also holds that $m_s a \subseteq W_\bullet$ for any $a \subseteq W$, we find that $m_T(a_1, a_2) = m_T(a_1, a_2) \cap W_\bullet$. This means that $m_T(a'_1, a'_2) \in A_\bullet$. QED

That the name *unsimulation* is well chosen follows from the following proposition.

Proposition 4.16 *Let $\mathcal{G}$ be a general $\mathcal{L}_\nabla$-frame; then $\mathcal{G}$ is isomorphic to $(\mathcal{G}^\bullet)_\bullet$.*

Proof. Let $\mathcal{G} = (\mathcal{F}, A)$ be a general $\mathcal{L}_\nabla$-frame. Since $\mathcal{F}^\bullet$ is a *Sim*-frame, the unsimulation operation is defined on both $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$. It is straightforward to derive from the definitions that $\mathcal{F}$ is isomorphic to $(\mathcal{F}^\bullet)_\bullet$, and that $A = (A^\bullet)_\bullet$. From this the proposition follows immediately. QED

Proposition 4.17 *Let $\mathcal{G}$ be a general Sim-frame. Then for all $\mathcal{L}_\nabla$-formulas φ we have*

$$\mathcal{G} \Vdash \varphi^\Diamond \textit{ iff } \mathcal{G}_\bullet \Vdash \varphi.$$

Proof. Let $\mathcal{F}$ be the underlying frame of $\mathcal{G}$.

First suppose that $\mathcal{G} \not\Vdash \varphi^{\diamond}$. That is, for some admissible valuation V and some point w in $\mathcal{G}$ we have $\mathcal{F}, V, w \not\Vdash \mathsf{b} \to \varphi^t$. It follows immediately that w must be a base point. Define the valuation V' on $\mathcal{F}$ by putting $V'(p) = V(p)' \cap W_\bullet$. Clearly, V' is admissible on $\mathcal{G}_\bullet$, and since V and V' meet the conditions of Proposition 4.6, we find that $\mathcal{F}_\bullet, V', w \not\Vdash \varphi$. This shows that $\mathcal{G}_\bullet \not\Vdash \varphi$.

For the other direction, suppose that $\mathcal{G}_\bullet \not\Vdash \varphi$. Then there is an admissible valuation V' on $\mathcal{G}_\bullet$ and a point w in $\mathcal{F}_\bullet$ such that $\mathcal{F}_\bullet, V', w \not\Vdash \varphi$. Since the valuation V is admissible on $\mathcal{G}$ as well, and since w, being a point of $W_\bullet$, is a base point of $\mathcal{F}$, we may apply Proposition 4.6 which immediately gives $\mathcal{F}, V, w \not\Vdash \neg\varphi^t$. Using the fact that $\mathcal{F}, V, w \Vdash \mathsf{b}$ we find that $\mathcal{G} \not\Vdash \varphi^{\diamond}$.
QED

4.4 Transfer of properties of general frames

Proposition 4.18 *Let $\mathcal{G}$ be a general Sim-frame. If $\mathcal{G}$ is descriptive, then so is $\mathcal{G}_\bullet$.*

Proof. It is straightforward to show that the unsimulation operations preserves differentiation and compactness, so we concentrate on tightness. Let $\mathcal{G} = (W, S, R_0, R_1, R_2, A)$ be a descriptive general *Sim*-frame. In order to prove that $\mathcal{G}_\bullet$ is tight, let u_0, u_1 and u_2 be points in $W_\bullet$ such that for all a_1 and a_2 in $A_\bullet$ we have that $u_1 \in a_1$ and $u_2 \in a_2$ imply that $u_0 \in m_T(a_1, a_2)$. We will prove that $Tu_0u_1u_2$.

Consider the set

$$D \;=\; \{m_0a_0 \mid u_0 \in a_0 \in A_\bullet\} \;\cup\; \{m_1a_1 \mid u_1 \in a_1 \in A_\bullet\} \;\cup\; \{m_2a_2 \mid u_2 \in a_2 \in A_\bullet\}.$$

Our first claim is that D has the finite intersection property. Since for each i the set $\{m_ia \mid u_i \in a\}$ is closed under taking intersections, it suffices to show that for each triple a_0, a_1, a_2 in $A_\bullet$ such that $u_0 \in a_0$, $u_1 \in a_1$ and $u_2 \in a_2$ we have that $m_0a_0 \cap\; m_1a_1 \cap m_2a_2 \neq \varnothing$. Take such a triple; it follows from $u_1 \in a_1$ and $u_2 \in a_2$ that $u_0 \in m_T(a_1, a_2) = m_s(m_1a_1 \cap m_2a_2)$. Hence, there is a point t in W such that R_0tu_0 and $t \in m_1a_1 \cap m_2a_2$. From Rtu_0 if follows that Su_0t, since $\mathcal{G}$ is based on a *Sim*-frame; hence, from $u_0 \in a_0$ we may infer that $t \in m_0a_0$. This shows that $t \in m_0a_0 \cap\; m_1a_1 \cap m_2a_2$, so indeed, the latter set is not empty.

But then by compactness, we have that $\bigcap D \neq \varnothing$. Let t be an arbitrary point in the intersection; obviously, t is a middle point of $\mathcal{G}$. Fix an $i \in \{0, 1, 2\}$; it follows from $t \in \bigcap\{m_ia \mid u_i \in A^\bullet\}$ and the fact that $m_ia = m_i(a \cap W^\bullet)$ for each $a \in A$, that $t \in m_ia$ for each $a \in A$ such that $u_i \in a$. Then from tightness of $\mathcal{G}$ we may infer that R_itu. Since this applies to each i, we find that t is a middle point of u_0, u_1 and u_2; by definition of T we thus get that $Tu_0u_1u_2$. QED

In the above proof, we used compactness of $\mathcal{G}$ in order to prove that $\mathcal{G}_\bullet$ is tight; this is necessary, as is witnessed by the general $\mathcal{L}_\nabla$-frame $\mathcal{H}$ of Remark 3.10. For, the general $\mathcal{L}_\diamond$-frame $\mathcal{H}^\bullet$ is tight (but not compact), while its unsimulation $\mathcal{H} = (\mathcal{H}^\bullet)_\bullet$ is not tight.

Finally, there are not many properties of general frames that are *reflected* by the unsimulation operation. For instance, it is fairly easy to see that differentiation and compactness are not reflected, by considering general frames that are based on a *Sim*-frame with three base points and infinitely many middle points. Tightness is not reflected either, since it is not even preserved by the simulation operation (cf. the general frame $\mathcal{G}$ of Remark 3.10).

5 Simulation results

In this pivotal section we define the simulation operation $(\cdot)^{sim}$ which maps $\mathcal{L}_\nabla$-logics to $\mathcal{L}_\Diamond$-logics, we prove that Λ^{sim} indeed simulates Λ and we state that several interesting properties of logics transfer under $(\cdot)^{sim}$. In this section we will also provide some of the easier transfer proofs; the more elaborate ones will be given in later sections.

In order to motivate our definition of the simulation operation, consider an $\mathcal{L}_\nabla$-logic Λ. Since Λ is determined by the class D_Λ of descriptive general $\mathcal{L}_\nabla$-frames for Λ, it seems natural to define Λ^{sim} as the modal theory of a class K of descriptive general $\mathcal{L}_\Diamond$-frames that is related to D_Λ. A first choice would be to take the class $(\mathsf{D}_\Lambda)^\bullet$ of simulations of frames in D_Λ. Unfortunately, although the theory of this class would constitute a simulation of Λ, it turned out to be difficult to prove nice transfer results using this approach. Therefore, we have chosen a slightly more involved definition.

Definition 5.1 Let C be a class of $\mathcal{L}_\nabla$-frames ; we define $\mathsf{C}^\star$ to be class of *Sim*-frames $\mathcal{F}$ such that $\mathcal{F}_\bullet$ belongs to C. For a class C of descriptive general $\mathcal{L}_\nabla$-frames, we define $\mathsf{C}^\star$ to be the class of descriptive general *Sim*-frames $\mathcal{G}$ such that $\mathcal{G}_\bullet$ belongs to C.

Now let Λ be an $\mathcal{L}_\nabla$-logic. Recall that D_Λ denotes the class of descriptive general $\mathcal{L}_\nabla$-frames for Λ. We define Λ^{sim}, the *simulation* of Λ, to be the set of formulas valid on $(\mathsf{D}_\Lambda)^\star$.
$\lhd$

The following theorem reveals that this is an appropriate definition.

Theorem 5.2 (Simulation Theorem) *Let Λ be an $\mathcal{L}_\nabla$-logic; then Λ^{sim} is indeed a simulation of Λ. That is, for any set Σ of $\mathcal{L}_\nabla$-formulas, and any $\mathcal{L}_\nabla$-formula φ we have*

$$\Sigma \vdash_\Lambda \varphi \textit{ iff } \Sigma^\Diamond \vdash_{\Lambda^{sim}} \varphi^\Diamond.$$

In particular, we have that

(3) $$\varphi \in \Lambda \textit{ iff } \varphi^\Diamond \in \Lambda^{sim}.$$

Proof. We first prove (3). For the left to right implication of (3), assume that $\varphi^\Diamond$ does not belong to Λ^{sim}. Then by definition of Λ^{sim}, there is a *Sim*-based descriptive general $\mathcal{L}_\Diamond$-frame such that $\mathcal{G}_\bullet \Vdash \Lambda$ and $\mathcal{G} \not\Vdash \varphi^\Diamond$. It follows from Proposition 4.17 that $\mathcal{G}_\bullet \not\Vdash \varphi$, whence by the soundness of Λ with respect to the class of general frames for Λ we obtain that φ does not belong to Λ.

For the other direction, assume that φ does not belong to Λ. By the completeness for descriptive general frames it follows that there is a descriptive general $\mathcal{L}_\nabla$-frame $\mathcal{G}$ for Λ such that $\mathcal{G} \not\Vdash \varphi$. It follows from Proposition 3.8 that $\mathcal{G}^\bullet \not\Vdash \varphi^\Diamond$, while we may infer from Proposition 3.9 that $\mathcal{G}^\bullet$ is descriptive. Since $(\mathcal{G}^\bullet)_\bullet \cong \mathcal{G}$ by Proposition 4.16, $\mathcal{G}^\bullet$ belongs to the class $(\mathsf{D}_\Lambda)^\star$, so we find that $\varphi^\Diamond$ does not belong to Λ^{sim}. This proves (3).

In order to derive the theorem from this, only propositional logic is required. For instance, suppose that $\Sigma \vdash_\Lambda \varphi$. By definition there are formulas $\sigma_1, \ldots, \sigma_n$ in Σ such that $\vdash_\Lambda (\sigma_1 \wedge \cdots \wedge \sigma_n) \to \varphi$. Thus by (3) we find that $\vdash_{\Lambda^{sim}} \mathsf{b} \to (\sigma_1^t \wedge \cdots \wedge \sigma_n^t) \to \varphi^t$. Some propositional manipulation reveals that this implies $\vdash_{\Lambda^{sim}} ((\mathsf{b} \to \sigma_1^t) \wedge \cdots \wedge (\mathsf{b} \to \sigma_n^t)) \to (\mathsf{b} \to \varphi^t)$. In other words, we have $\vdash_{\Lambda^{sim}} (\sigma_1^\diamond \wedge \cdots \wedge \sigma_1^\diamond) \to \varphi^\diamond$, which by definition means that $\Sigma^\diamond \vdash_{\Lambda^{sim}} \varphi^\diamond$. We leave the other direction as an exercise to the reader. QED

Obviously, a simulation operation becomes more interesting the more properties it preserves. That our operation $(\cdot)^{sim}$ behaves rather well follows from the following proposition.

Theorem 5.3 (Properties of the simulation) *The map $(\cdot)^{sim}$ maps the lattice of normal $\mathcal{L}_\nabla$-logics into the lattice of normal modal $\mathcal{L}_\diamond$-logics extending* **Sim**. *This map preserves, reflects and transfers properties as indicated in Table 1.*

Property	preservation	reflection	transfer
finite axiomatizability	yes (5.4)	yes (8.10)	yes
recursive axiomatizability	yes (5.4)	yes (8.10)	yes
Sahlqvist axiomatizability	yes (5.4)	??	??
completeness	yes (7.2)	yes (5.8)	yes
canonicity	yes (5.7)	yes (5.9)	yes
finite model property	yes (7.2)	yes (5.8)	yes
first-order definability	yes (5.6)	yes (8.5)	yes

Table 1: Properties of the simulation

Proof. Here we only prove the first part of the theorem — the statements in Table 1 concerning the preservation and reflection of properties are proved in the remainder of the paper, as indicated by the entries in the table. (Each entry in the transfer column of the table can easily be calculated as the conjunction of the preservation and reflection entries in the same row).

By definition, $\mathbf{K}_\nabla^{sim}$ is the logic of the class of all descriptive general *Sim*-frames. Hence, by Proposition 4.12, $\mathbf{K}_\nabla^{sim} = \mathbf{Sim}$. The inconsistent $\mathcal{L}_\nabla$-logic is mapped by $(\cdot)^{sim}$ to the inconsistent $\mathcal{L}_\diamond$-logic. It is immediate from Theorem 5.2 that the simulation map is injective.

It remains to be shown that $(\cdot)^{sim}$ is a lattice homomorphism. It easily follows from Proposition 5.4 that $(\Lambda_1 \sqcup \Lambda_2)^{sim} = \Lambda_1^{sim} \sqcup \Lambda_2^{sim}$. In order to show that $(\Lambda_1 \sqcap \Lambda_2)^{sim} = \Lambda_1^{sim} \sqcap \Lambda_2^{sim}$, we will use the well-known fact that

$$\Lambda_1 \sqcap \Lambda_2 = Th_\nabla(\mathsf{D}_{\Lambda_1} \cup \mathsf{D}_{\Lambda_2}).$$

Likewise, we have that $\Lambda_1^{sim} \sqcap \Lambda_2^{sim} = Th_\diamond(\mathsf{D}_{\Lambda_1^{sim}} \cup \mathsf{D}_{\Lambda_2^{sim}})$. But this gives that

$$\begin{aligned} \Lambda_1^{sim} \sqcap \Lambda_2^{sim} &= Th_\diamond(\mathsf{D}^\star_{\Lambda_1} \cup \mathsf{D}^\star_{\Lambda_2}) \\ &= Th_\diamond((\mathsf{D}_{\Lambda_1} \cup \mathsf{D}_{\Lambda_2})^\star) \end{aligned}$$

The statement then follows from the observation that for any class C of descriptive general $\mathcal{L}_\nabla$-frames it holds that $(Th_\nabla(\mathsf{C}))^{sim} = Th_\diamond(\mathsf{C}^\star)$, cf. Proposition 7.9. QED

Some properties are conspicuous by their absence in Table 1. To start with, we have not investigated the question whether the simulation reflects Sahlqvist axiomatizability; we conjecture that the answer to this question is positive, but the only proof that we can think of would involve a very tedious investigation of the properties of the residual map discussed in section 8. The question seems to be not very relevant since we can show that the most useful properties of Sahlqvist logics (canonicity, completeness and first-order definability) are reflected by the simulation operation anyway.

Second, as we already mentioned in the preliminaries, we confine ourselves in this paper to the local paradigm concerning the semantic consequence relation and our notion of deduction. We have not investigated the transfer of properties such as global completeness.

Finally, Table 1 does not mention any kind of interpolation property. Note that in our local paradigm, the relevant notion is that of *Craig's interpolation property* or CIP. A logic Λ has this property if for every pair of formulas φ and ψ such that $\vdash_\Lambda \varphi \to \psi$ there is an *interpolant*; that is, a formula θ such that $\vdash_\Lambda \varphi \to \theta$ and $\vdash_\Lambda \theta \to \psi$, while θ may only use proposition letters that occur in both φ and ψ. Using the algebraic counterpart of this property, viz. the *superamalgamation* property, it is fairly easy to show that our simulation map reflects CIP. We also have strong reasons to believe that $(\cdot)^{sim}$ preserves this property, but we hope to report on this at some other occasion.

In the remainder of this section we prove some of the easier results listed in Table 1.

5.1 Easy preservation results

We first see how we can find axiomatizations for Λ^{sim} given axiomatizations for Λ. The following result states that if $\Gamma \subseteq \mathcal{L}_\diamond$ axiomatizes a $\mathcal{L}_\nabla$-logic Λ, then Λ^{sim} is axiomatized over **Sim** by $\Gamma^\diamondsuit$.

Proposition 5.4 *Let Γ be a set of $\mathcal{L}_\nabla$-formulas. Then*

$$(\mathbf{K}_\nabla.\Gamma)^{sim} = \mathbf{Sim}.\Gamma^\diamondsuit. \tag{4}$$

Hence, if a $\mathcal{L}_\nabla$-logic Λ has a finite, recursive or Sahlqvist axiomatization, respectively, then so has Λ^{sim}.

Proof. We first prove (4); let Λ denote the logic $\mathbf{K}_\nabla.\Gamma$. In order to show that $\Lambda^{sim} = \mathbf{Sim}.\Gamma^\diamondsuit$, by the general completeness result of modal logics with respect to descriptive general frames, it suffices to prove that for every descriptive general $\mathcal{L}_\diamond$-frame $\mathcal{G}$ we have the following equivalence:

$$\mathcal{G} \Vdash \Lambda^{sim} \text{ iff } \mathcal{G} \Vdash \mathbf{Sim}.\Gamma^\diamondsuit. \tag{5}$$

In order to prove the left to right direction of (5), let $\mathcal{G}$ be a descriptive general $\mathcal{L}_\diamond$-frame such that $\mathcal{G} \Vdash \Lambda^{sim}$. It follows immediately from the definition of Λ^{sim} that $\mathcal{G} \Vdash \mathbf{Sim}$, so by Proposition 4.12, $\mathcal{G}$ is based on a *Sim*-frame. Hence, the unsimulation of $\mathcal{G}$ is defined. From $\mathcal{G} \Vdash \Lambda^{sim}$ and (3) it follows that $\mathcal{G} \Vdash \Lambda^\diamondsuit$; in particular this gives that $\mathcal{G} \Vdash \Gamma^\diamondsuit$ since $\Gamma^\diamondsuit \subseteq \Lambda^\diamondsuit$.

For the other direction, assume that $\mathcal{G} \Vdash \mathbf{Sim}.\Gamma^{\diamond}$. It follows from $\mathcal{G} \Vdash \mathbf{Sim}$ and Proposition 4.12 that $\mathcal{G}$ is based on a *Sim*-frame, whence $\mathcal{G}_{\bullet}$ is defined. From $\mathcal{G} \Vdash \Gamma^{\diamond}$ and Proposition 4.17 we obtain that $\mathcal{G}_{\bullet} \Vdash \Gamma$. Since Γ axiomatizes Λ this gives that $\mathcal{G}_{\bullet} \Vdash \Lambda$, whence $\mathcal{G}$ belongs to the class $(\mathsf{D}_{\Lambda})^{\star}$. Thus by definition of Λ^{sim} we have that $\mathcal{G} \Vdash \Lambda^{sim}$.

The second part of the proposition is immediate from (4). QED

The second 'easy' preservation result concerns first-order definability: we will prove that if Λ defines an elementary class of frames, then so does Λ^{sim}. Let $\mathcal{L}^{fo}_{\nabla}$ and $\mathcal{L}^{fo}_{\diamond}$ be the first order frame languages (with equality) corresponding to $\mathcal{L}_{\nabla}$ and $\mathcal{L}_{\diamond}$, respectively.

Definition 5.5 We define a translation mapping $\mathcal{L}^{fo}_{\nabla}$-formulas to $\mathcal{L}^{fo}_{\diamond}$-formulas by the following induction:

$$\begin{array}{rcl}
(x=y)^e & = & (x=y) \\
(T(x,y,z))^e & = & \exists w (R_0 wx \wedge R_1 wy \wedge R_2 wz) \\
(\alpha \wedge \beta)^e & = & \alpha^e \wedge \beta^e \\
(\neg \alpha)^e & = & \neg \alpha^e \\
(\exists x\, \alpha)^e & = & \exists x (B(x) \wedge \alpha^e)
\end{array}$$

Here we use $B(x)$ as an abbreviation for the $\mathcal{L}^{fo}_{\diamond}$-formula $\neg \exists y\, R_0 xy$; clearly, $B(x)$ is an indication that x is a base point.

For a set Σ of $\mathcal{L}^{fo}_{\nabla}$-formulas, we let Σ^e denote the set $\{\sigma^e \mid \sigma \in \Sigma\}$. $\lhd$

Proposition 5.6 *Let Λ be an $\mathcal{L}_{\nabla}$-logic. If the set Σ of $\mathcal{L}^{fo}_{\nabla}$-formulas axiomatizes the class of $\mathcal{L}_{\nabla}$-frames for Λ, then the set $Sim \cup \Sigma^e$ axiomatizes the class of $\mathcal{L}_{\diamond}$-frames for Λ^{sim}.*

Hence, if Λ is elementary (Δ-elementary), then so is Λ^{sim}.

Proof. The key part of the proof is to show that for any *Sim*-frame $\mathcal{F}$ and any assignment b mapping variables to base points of $\mathcal{F}$, we have for every $\mathcal{L}^{fo}_{\nabla}$-formula α:

$$\mathcal{F} \models \alpha^e[b] \text{ iff } \mathcal{F}_{\bullet} \models \alpha[b]. \tag{6}$$

The proof of (6) is by a straightforward induction on the complexity of α.

Given (6), the proposition follows from the following chain of equivalences, which hold for every $\mathcal{L}_{\diamond}$-frame $\mathcal{F}$:

$\mathcal{F}$ is a frame for Λ^{sim}
iff (Prop. 5.4) $\mathcal{F} \Vdash \mathbf{Sim}$ and $\mathcal{F} \Vdash \Lambda^{\diamond}$,
iff (Prop. 4.7) $\mathcal{F} \Vdash \mathbf{Sim}$ and $\mathcal{F}_{\bullet} \Vdash \Lambda$,
iff (assumption) $\mathcal{F} \Vdash \mathbf{Sim}$ and $\mathcal{F}_{\bullet} \models \Sigma$,
iff (Prop. 4.12) $\mathcal{F} \models Sim$ and $\mathcal{F}_{\bullet} \models \Sigma$,
iff (6) $\mathcal{F} \models Sim$ and $\mathcal{F} \models \Sigma^e$. QED

The final preservation result that we prove in this section concerns canonicity.

Proposition 5.7 *Let Λ be an $\mathcal{L}_{\nabla}$-logic. If Λ is canonical, then so is Λ^{sim}.*

Proof. Assume that Λ is a canonical $\mathcal{L}_\nabla$-logic, and take a descriptive general $\mathcal{L}_\diamond$-frame $\mathcal{G}$ such that $\mathcal{G} \Vdash \Lambda^{sim}$. We will show that $\mathcal{F} \Vdash \Lambda^{sim}$, where $\mathcal{F}$ is the underlying frame of $\mathcal{G}$.

To start with, it follows from the descriptiveness of $\mathcal{G}$, the inclusion $\mathbf{Sim} \subseteq \Lambda^{sim}$ and the canonicity of $\mathbf{Sim}$, that $\mathcal{F} \Vdash \mathbf{Sim}$, whence $\mathcal{F}$ is a *Sim*-frame. Hence, the unsimulation operation is defined for $\mathcal{G}$. It follows from Proposition 4.17 that $\mathcal{G}_\bullet \Vdash \Lambda$, and from Proposition 4.18 that $\mathcal{G}_\bullet$ is descriptive; hence, from the canonicity of Λ we may infer that Λ holds on the underlying frame $\mathcal{F}_\bullet$ of $\mathcal{G}_\bullet$. But then it follows from Proposition 3.5 that $\mathcal{F} \Vdash \Lambda^\diamond$, so by Proposition 5.4 we have that $\mathcal{F} \Vdash \Lambda^{sim}$. QED

5.2 Easy reflection results

In this subsection we state and prove two reflection result. First we treat completeness and the finite model property.

Proposition 5.8 *Let Λ be an $\mathcal{L}_\nabla$-logic, and C a class of $\mathcal{L}_\diamond$-frames. If Λ^{sim} is complete with respect to C, then Λ is complete with respect to $\mathsf{C}_\bullet$. Hence, if Λ^{sim} is complete, then so is Λ, and if Λ has the finite model property, then so does Λ.*

Proof. Let Λ and C be as in the formulation of the proposition, and assume that $\Lambda^{sim} = \mathrm{Th}_\diamond(\mathsf{C}_\bullet)$. It follows from the following chain of equivalences that $\Lambda = \mathrm{Th}_\nabla(\mathsf{C})$:

$\varphi \in \Lambda$
iff (Theorem 5.2) $\varphi^\diamond \in \Lambda^{sim}$,
iff (assumption) $\mathsf{C} \Vdash \varphi^\diamond$,
iff (Prop. 4.7) $\mathsf{C}_\bullet \Vdash \varphi$. QED

The last proposition of this section states that the simulation operation reflects canonicity.

Proposition 5.9 *Let Λ be an $\mathcal{L}_\nabla$-logic. If Λ^{sim} is canonical, then so is Λ.*

Proof. Assume that Λ is an $\mathcal{L}_\nabla$-logic such that Λ^{sim} is canonical. In order to prove that Λ itself is canonical, consider an arbitrary descriptive general $\mathcal{L}_\nabla$-frame $\mathcal{G}$ such that $\mathcal{G} \Vdash \Lambda$. We will show that Λ holds as well on the underlying frame $\mathcal{F}$ of $\mathcal{G}$.

Using the Propositions 3.8 and 5.4 we obtain that $\mathcal{G}^\bullet \Vdash \Lambda^{sim}$. Moreover, from Proposition 3.9 it follows that $\mathcal{G}^\bullet$ is descriptive. Since Λ^{sim} is canonical we find that $\mathcal{F}^\bullet \Vdash \Lambda^{sim}$, so Proposition 3.5 gives that $\mathcal{F} \Vdash \Lambda$. QED

6 Normal forms

In the following two sections we will be dealing with syntactic issues concerning $\mathcal{L}_\diamond$-formulas. Much of our considerations will be made simpler because we may assume that $\mathcal{L}_\diamond$-formulas are in a certain syntactic shape, viz., in *normal form*. In this section we will define these normal forms and prove that every $\mathcal{L}_\diamond$-formula can be effectively rewritten into an equivalent formula in normal form.

Recall that we use the symbol m to denote the $\mathcal{L}_\diamond$-formula $\diamond_0\top$, and that this formula is true precisely at the middle points of any $\mathcal{L}_\diamond$-model.

Definition 6.1 A formula of $\mathcal{L}_\Diamond$ is said to be a *propositional list* if it is a conjunction of propositional variables and negations of propositional variables. A formula of $\mathcal{L}_\Diamond$ is said to be in *base normal form* or a *bnf-formula* if it is a propositional formula, a Boolean combination of formulas in base normal form, or of the form

$$\Diamond_s(\pi \wedge \Diamond_1\theta_1 \wedge \Diamond_2\theta_2),$$

with π a propositional list and θ_1 and θ_2 bnf-formulas. The *depth* of a bnf-formula is defined using the above inductive definition: $depth(p) = 0$, $depth(\neg\theta) = depth(\theta)$, $depth(\theta_1 \wedge \theta_2) = \max(depth(\theta_1), depth(\theta_2))$ and

$$depth(\Diamond_s(\pi \wedge \Diamond_1\theta_1 \wedge \Diamond_2\theta_2)) \;=\; 1 + \max(depth(\theta_1), depth(\theta_2)).$$

An $\mathcal{L}_\Diamond$-formula is in *middle normal form* or an *mnf-formula* if it is of the form $\Diamond_i\theta$ with $i \in \{0,1,2\}$ and θ a bnf-formula. Finally, a $\mathcal{L}_\Diamond$-formula is in *normal form* if it has the shape

$$(\mathsf{b} \wedge \theta_0) \;\vee\; (\mathsf{m} \wedge \bigvee_{j\in J}(\pi_j \wedge \Diamond_0\theta_{0,j} \wedge \Diamond_1\theta_{1,j} \wedge \Diamond_2\theta_{2,j})), \tag{7}$$

where all the π formulas are propositional lists, all the θ formulas are in base normal form, and J is some finite index set. We call the formula $\mathsf{b} \wedge \theta_0$ the *base disjunct* or b-*disjunct* of (7), and the other disjunct, the m-*disjunct*. $\lhd$

Proposition 6.2 *1. There is an effective procedure that rewrites any $\mathcal{L}_\Diamond$-formula into a* **Sim***-equivalent formula which is a boolean combination of bnf-formulas and mnf-formulas.*

2. There is an effective procedure $(\cdot)^n$ that rewrites any $\mathcal{L}_\Diamond$-formula φ into a **Sim***-equivalent formula φ^n which is in normal form.*

Proof. Throughout the proof we will work modulo the logic **Sim**; that is, 'equivalent' will always mean 'equivalent in **Sim**'.

We first show how to prove the second part of the proposition from the first one. That is, we will prove how any Boolean combination of bnf- and mnf-formulas can be effectively rewritten into normal form. Let φ be such a Boolean combination; rewrite φ as a disjunction of conjunctions of proposition letters, bnf-formulas in the shape $\Diamond_s(\pi \wedge \Diamond_1\theta_1 \wedge \Diamond_2\theta_2)$, mnf-formulas, and negations of such formulas. Without loss of generality we may assume that every disjunct has either m or b as a conjunct (any disjunct ψ that would not have such a conjunct may be split into $(\psi \wedge \mathsf{m}) \vee (\psi \wedge \mathsf{b})$).

First consider the disjuncts containing b. Observe that any formula of the form $\mathsf{b}\wedge\Diamond_i\chi\wedge\gamma$ is **Sim**-inconsistent, and that any formula of the form $\mathsf{b}\wedge\Box_i\chi\wedge\gamma$ is equivalent to $\mathsf{b}\wedge\gamma$. (Readers having suspicions about these claims are advised to prove them using the Completeness of **Sim** with respect to the class of *Sim*-frames, see Proposition 4.13). Use these observations to replace each b-disjunct with a formula of the form $\mathsf{b} \wedge \theta_k$, with each θ_k being a conjunction of proposition letters, bnf-formulas in the form $\Diamond_s(\pi \wedge \Diamond_1\theta_1 \wedge \Diamond_2\theta_2)$, and negations of such formulas. Clearly then, each θ_k is a bnf-formula. Now group together all these disjuncts and take out the b conjunct, obtaining the equivalent formula $\mathsf{b} \wedge \bigvee_k \theta_k$ which will form the base disjunct of the normal form of φ.

Now we can turn to the disjuncts containing m as a conjunct. Consider an arbitrary such disjunct, say $\mathsf{m} \wedge \delta$. All we need to know about δ is that it is a conjunction of proposition letters, mnf-formulas, formulas of the form $\Diamond_s\theta$, and negations of such formulas. We will step by step simplify δ using the following **Sim**-equivalences (here, as always, i is one of the indices in $\{0, 1, 2\}$):

$$\begin{aligned} \mathsf{m} \wedge \Diamond_s\psi \wedge \chi &\sim \mathsf{m} \wedge \bot \\ \mathsf{m} \wedge \neg\Diamond_s\psi \wedge \chi &\sim \mathsf{m} \wedge \chi \\ \mathsf{m} \wedge \neg\Diamond_i\psi \wedge \chi &\sim \mathsf{m} \wedge \Diamond_i\neg\psi \wedge \chi. \end{aligned}$$

What we are left with in the end is a disjunct of the form $\mathsf{m} \wedge \delta'$ with δ' a conjunction of (negated) proposition letters and formulas in middle normal form. Then we may use the fact that the $\Diamond_i$-diamonds distribute over conjunctions to rewrite $\mathsf{m} \wedge \delta'$ into a formula of the form $\mathsf{m} \wedge \pi_j \wedge \Diamond_0\theta_{0,j} \wedge \Diamond_1\theta_{1,j} \wedge \Diamond_2\theta_{2,j}$ with π_j being a propositional list, and each θ in bnf-form. Finally, grouping all the m-disjuncts together in the same way as we did with the b-disjuncts, we find an m-disjunct of the form $\mathsf{m} \wedge \bigvee_j(\pi_j \wedge \Diamond_0\theta_{0,j} \wedge \Diamond_1\theta_{1,j} \wedge \Diamond_2\theta_{2,j})$.

Clearly, φ is equivalent to the disjunction of these two formulas; that is, φ is in normal form.

The first part of the proposition is proved by a straightforward formula induction. The base step and the boolean cases of the induction step are straightforward, so assume that φ is of the form $\Diamond\psi$, where $\Diamond$ is one of the diamonds of the language $\mathcal{L}_\Diamond$. By the inductive hypothesis we may assume that ψ is in normal form, say, ψ is the formula (7).

First suppose that $\Diamond$ is one of the diamonds $\Diamond_i$, with $i \in \{0, 1, 2\}$. Since $\Diamond_i$ distributes over disjunctions, it suffices to look at each of the two disjuncts of (7) separately. Treating the base disjunct $(\mathsf{b} \wedge \theta_0)$ first, it is easy to see that $\Diamond_i(\mathsf{b} \wedge \theta_0)$ is equivalent to the formula $\Diamond_i\theta_0$, a formula is in mnf-form. Now consider the m-disjunct of (7). It easily follows from the **Sim**-axiom $\Box_i\Box_j\bot$ that for any χ, the formula $\Diamond_i(\mathsf{m} \wedge \chi)$ is equivalent to $\bot$. From these observations it follows that φ is equivalent to the formula $\Diamond_i\theta_0 \vee \bot$, and hence, to $\mathsf{b} \wedge \Diamond_i\theta_0$ which meets the requirements.

Now suppose that $\Diamond$ is the diamond $\Diamond_s$. Again, it suffices to look at each of the two disjuncts of (7) separately. Using the **Sim**-theorem $\Diamond_s\mathsf{b} \leftrightarrow \bot$ one easily sees that $\Diamond_s(\mathsf{b} \wedge \theta_0)$ is equivalent to $\bot$, whence we may turn to the m-disjunct of (7). We first rewrite this into the equivalent

$$\bigvee_j(\mathsf{m} \wedge \pi_j \wedge \Diamond_0\theta_{0,j} \wedge \Diamond_1\theta_{1,j} \wedge \Diamond_2\theta_{2,j});$$

now distribute $\Diamond_s$ over $\bigvee_j$, and treat each of the new disjuncts separately. Using the completeness theorem for **Sim** with respect to the class of simulation frames (Proposition 4.13), it is not difficult to prove that

$$\vdash_{\mathbf{Sim}} \Diamond_s(\mathsf{m} \wedge \pi_j \wedge \Diamond_0\theta_{0,j} \wedge \Diamond_1\theta_{1,j} \wedge \Diamond_2\theta_{2,j}) \ \leftrightarrow\ \theta_{0,j} \wedge \Diamond_s(\pi_j \wedge \Diamond_1\theta_{1,j} \wedge \Diamond_2\theta_{2,j}).$$

From this it follows that $\Diamond_s\psi$ is equivalent to the formula

$$\bigvee_j (\theta_{0,j} \wedge \Diamond_s(\pi_j \wedge \Diamond_1\theta_{1,j} \wedge \Diamond_2\theta_{2,j})),$$

which is clearly in bnf-form. Finally, we find that $\varphi = \Diamond_s\psi$ is equivalent to the formula $\mathsf{b} \wedge \bigvee_j(\theta_{0,j} \wedge \Diamond_s(\pi_j \wedge \Diamond_1\theta_{1,j} \wedge \Diamond_2\theta_{2,j}))$ which is in normal form. QED

7 Preservation of completeness

This section will be devoted to the fairly involved proofs concerning the preservation of completeness properties by our simulation map. In the first subsection we concentrate on frame completeness and the finite model property, in the second part we discuss completeness with respect to classes of general frames.

7.1 Frame completeness

In order to make the statement concerning the preservation of frame completeness a bit more precise we need the following definition.

Definition 7.1 An *Sim*-frame is *middle finite* if there are at most finitely many middle points in between any triple of base points. Given a class C of $\mathcal{L}_\nabla$-frames, we define $\mathsf{C}^\star_{mf}$ to be the class of middle finite *Sim*-frames $\mathcal{F}$ such that $\mathcal{F}_\bullet$ belongs to C. $\lhd$

Proposition 7.2 *Let Λ be a $\mathcal{L}_\nabla$-logic which is complete with respect to a class C of $\mathcal{L}_\nabla$-frames. Then Λ^{sim} is complete for $\mathsf{C}^\star_{mf}$. Hence, if Λ has the finite model property, then so does Λ^{sim}.*

The entire section is devoted to a proof of Proposition 7.2. Observe that the result concerning the finite model property follows from the main statement in the proposition and the fact that if C is a class of finite frames, then so is $\mathsf{C}^\star_{mf}$.

The aim of our proof is to show that any Λ^{sim}-consistent formula is satisfiable in $\mathsf{C}^\star_{mf}$. It follows from the proposition below that without loss of generality we may assume that the formula at stake is of the form $\mathsf{b} \wedge \theta$ with θ in *base normal form*.

Proposition 7.3 *Let γ be an $\mathcal{L}_\Diamond$-formula. Then there is a bnf-formula θ such that*

1. *γ is Λ^{sim}-consistent iff $\mathsf{b} \wedge \theta$ is Λ^{sim}-consistent.*
2. *γ is satisfiable in $\mathsf{C}^\star_{mf}$ iff $\mathsf{b} \wedge \theta$ is satisfiable in $\mathsf{C}^\star_{mf}$.*

Proof. Since the formula γ^n is equivalent to γ on the basis of **Sim**, we may work with γ^n instead of with γ. Suppose that γ^n is of the form (7). We claim that the formula

$$\theta := \theta_0 \vee \bigvee_{j\in J} (\theta_{0,j} \wedge \Diamond_s(\pi_j \wedge \Diamond_1\theta_{1,j} \wedge \Diamond_2\theta_{2,j})),$$

which clearly is in base normal form, satisfies the required properties. This follows from the fact that for any $\mathcal{L}_\Diamond$-model $\mathcal{M}$ based on a **Sim**-frame, we have that

(8) $\qquad \gamma^n$ is satisfiable in $\mathcal{M}$ iff θ is satisfiable at a base point of $\mathcal{M}$.

In order to prove (8), first suppose that γ^n is satisfiable in $\mathcal{M}$, say at the point w. If w is a base point, this means that $\mathcal{M}, w \Vdash \theta_0$, from which it is immediate that $\mathcal{M}, w \Vdash \theta$. If, on the other hand, w is a middle point, then we have that $\mathcal{M}, w \Vdash \pi_j \wedge \Diamond_0\theta_{0,j} \wedge \Diamond_1\theta_{1,j} \wedge \Diamond_2\theta_{2,j}$ for some j. In this case, recall that $r_0 w$ is the (unique) R_0-successor of w. It is easy to see that $\mathcal{M}, r_0 w \Vdash \theta_{0,j} \wedge \Diamond_s(\pi_j \wedge \Diamond_1\theta_{1,j} \wedge \Diamond_2\theta_{2,j})$, which makes that $\mathcal{M}, r_0 w \Vdash \theta$.

For the other direction, suppose that θ is satisfiable at the base point w in $\mathcal{M}$. If $\mathcal{M}, w \Vdash \theta_0$ it follows immediately that $\mathcal{M}, w \Vdash \gamma^n$, so suppose that for some j we have that $\mathcal{M}, u \Vdash \theta_{0,j} \wedge \Diamond_s(\pi_j \wedge \Diamond_1\theta_{1,j} \wedge \Diamond_2\theta_{2,j})$. By the truth definition there must be a middle point m such that wSm and $\mathcal{M}, m \Vdash \pi_j \wedge \Diamond_1\theta_{1,j} \wedge \Diamond_2\theta_{2,j}$. It then follows from wSm and $\mathcal{M}, w \Vdash \theta_{0,j}$ that $\mathcal{M}, m \Vdash \pi_j \wedge \Diamond_0\theta_{0,j} \wedge \Diamond_1\theta_{1,j} \wedge \Diamond_2\theta_{2,j}$ which immediately gives that $\mathcal{M}, m \Vdash \gamma^n$. This proves (8) and thus concludes the proof of the proposition. QED

For the remainder of the section we fix a formula θ in base normal form such that $\mathsf{b} \wedge \theta$ is Λ^{sim}-consistent; we will prove that there is a frame $\mathcal{F}$ in $\mathsf{C}^\star_{mf}$ in which $\mathsf{b} \wedge \theta$ is satisfiable. Let Θ be the collection of those subformulas of θ that are in base normal form; clearly, Θ is finite.

The idea of our proof is as follows. From the consistency of $\mathsf{b} \wedge \theta$ we may infer that θ holds at some base point Γ of the canonical Λ^{sim}-model. We will unravel this canonical model from Γ and gather some information concerning an initial part of this unraveled tree model; this information will be encoded in $\mathcal{L}_\nabla$-formulas, but in order to obtain sufficient information we have to expand the language with new proposition letters. The formula χ_r that we can thus associate with the root of the unraveled, expanded model, will be Λ-consistent, and thus by assumption satisfiable in some frame $\mathcal{F}$ that that belongs to C. Finally, we will use this frame $\mathcal{F}$ to construct a frame $\mathcal{F}'$ such that $\mathcal{F}'_\bullet = \mathcal{F}$ and in which θ is satisfiable.

First, we introduce some notational conventions. For a set of formulas Σ and a point s in a model $\mathcal{M}$, we define $\Sigma_s = \{\varphi \in \Sigma \mid \mathcal{M}, s \Vdash \varphi\}$ as the set of formulas in Σ that hold at s. (The model $\mathcal{M}$ will always be clear from context.) Given two sets of formulas A and B such that $A \subseteq B$, define

$$\Omega_{A|B} \;=\; \bigwedge_{\varphi \in A} \varphi \wedge \bigwedge_{\varphi \in B \setminus A} \neg\varphi.$$

In words, $\Omega_{A|B}$ says that of the formulas in B, precisely those in A are true. Note that it follows that for any set of formulas Σ and any point s in any model $\mathcal{M}$ we have that $\mathcal{M}, s \Vdash \Omega_{\Sigma_s|\Sigma}$.

Since $\mathsf{b} \wedge \theta$ is Λ^{sim}-consistent, the formula θ is satisfied at some base point Γ in the canonical model. Let $\mathcal{N} = (N, S, R_0, R_1, R_2, V)$ be the unravelling of the canonical model from this state Γ, cf. Definition 4.8. It easily follows from the properties of unravellings (cf. Proposition 4.9) that $\mathcal{N}$ itself is a model for Λ^{sim}; that $\mathcal{N}$ is based on a **Sim**-frame; and that its root r satisfies θ.

We need some terminology concerning points in $\mathcal{N}$. The *height* of a base point s in $\mathcal{N}$ is simply the distance from r to s (measured in number of $S \circ R_i$-steps). A base point is *near*

r if its height does not exceed the depth of θ (as defined in Definition 6.1); a middle point is *near* r if its immediate base successors are near r. Let N^- be the set of points in $\mathcal{N}$ that are near r. N^- is not necessarily finite since $\mathcal{N}$ will generally not be finitely branching. We say that a base point s is *on the edge of* N^-, or briefly: *on the edge*, if $height(s) = depth(\theta)$. Note that 'being on the edge of N^-' is not the same notion as 'having no successors in $\mathcal{N}$'.

We now extend the language, as follows. Let $P = \{p_1, \ldots, p_n\}$ be the finite set of propositional letters occurring in θ; we will use the letter p to refer to a generic element of P. Let $Q = \{q_{(I,\Theta_1,\Theta_2)} \mid I \subseteq P, \Theta_i \subseteq \Theta\}$ be a set of new propositional letters; observe that Q is finite. We will use the letter q to refer to a generic element of Q.

The idea underlying our use of the proposition letter $q_{(I,\Theta_1,\Theta_2)}$ is the following. In our proof we will need to transfer the information provided by $\mathcal{L}_\diamond$-formulas into the language $\mathcal{L}_\nabla$. It is obvious that in a bnf-formula of the form $\diamond_s(\pi \wedge \diamond_1\theta_1 \wedge \diamond_2\theta_2)$, the propositional information stored in π is relevant. Unfortunately, there is no direct way to transfer this information to $\mathcal{L}_\nabla$ since in $\mathcal{L}_\nabla$ there is no access to middle points. Even if θ_1 and θ_2 are the translations of $\mathcal{L}_\nabla$-formulas, say of ψ_1 and ψ_2, respectively, then the formula $\nabla(\psi_1, \psi_2)$ would be as close as we could get; however, in its translation $(\nabla(\psi_1, \psi_2)^t = \diamond_s(\diamond_1\theta_1 \wedge \diamond_2\theta_2)$ all propositional information about middle points is lost. It is here that the new propositional variables come in: the idea is simply to encode all relevant information (relevant in the sense of pertaining to subformulas of θ) into the q variables. To be a bit more precise, the variable $q_{(I,\Theta_1,\Theta_2)}$ should be read as the encoding of the formula $\diamond_s(\Omega_{I|P} \wedge \diamond_1\Omega_{\Theta_1|\Theta} \wedge \diamond_2\Omega_{\Theta_2|\Theta})$ into propositional logic.

The model $\mathcal{N}$ will be expanded to this new language, by defining the valuation $V' : P \cup Q \to \mathcal{P}(N)$ as follows:

$$
\begin{aligned}
V'(p) &= V(p) \\
V'(q_{(I,\Theta_1,\Theta_2)}) &= \{w \in \mathcal{N} \mid \mathcal{N}, w \Vdash \diamond_s(\Omega_{I|P} \wedge \diamond_1\Omega_{\Theta_1|\Theta} \wedge \diamond_2\Omega_{\Theta_2|\Theta})\}.
\end{aligned}
$$

The following lemma shows that even in the expanded language we are still dealing with a model for Λ^{sim} at the root of which the formula $\mathsf{b} \wedge \theta$ holds, while we have indeed encoded sufficient relevant information in the new propositional variables.

Proposition 7.4 *Let $\mathcal{N}'$ be the model (N, S, R_1, R_2, V').*

1. *$\mathcal{N}'$ is a model for Λ^{sim}.*
2. *Let u and v be two base points in $\mathcal{N}'$. Then $P_u = P_v$ and $Q_u = Q_v$ only if $\Theta_u = \Theta_v$.*
3. *The formula $\mathsf{b} \wedge \theta$ holds at the root r of $\mathcal{N}'$.*

Proof. For the first part of the proposition, consider a formula ψ in Λ^{sim}, and an arbitrary point v in $\mathcal{N}'$. For each $\mathcal{L}_\diamond$-formula χ, let χ' be the formula obtained from χ by uniformly substituting each variable $q_{(I,\Theta_1,\Theta_2)}$ by the formula $\diamond_s(\Omega_{I|P} \wedge \diamond_1\Omega_{\Theta_1|\Theta} \wedge \diamond_2\Omega_{\Theta_2|\Theta})$. Since ψ' then belongs to Λ^{sim} as well, and all variables in ψ' are in P, we must have that $\mathcal{N}, v \Vdash \psi'$. However, an easy inductive proof shows that our definition of V' ensures that for every $\mathcal{L}_\diamond$-formula χ and every point w in N we have

$$\mathcal{N}, w \Vdash \chi' \text{ iff } \mathcal{N}', w \Vdash \chi.$$

From this it is immediate that $\mathcal{N}', v \Vdash \psi$, which proves the first part of the proposition.

For part 2, suppose that u and v are base points in $\mathcal{N}'$ such that $P_u = P_v$ and $Q_u = Q_v$; that is, u and v agree on the truth of all proposition letters. We prove by induction on the complexity of bnf-formulas that for each formula ψ in Θ we have that $\mathcal{N}', u \Vdash \psi$ iff $\mathcal{N}', v \Vdash \psi$. First suppose that ψ is a proposition letter; this means that ψ is an element of P. But then we obtain the desired result from the assumption that $P_u = P_v$. The Boolean case of the induction step is easy and left to the reader. Now assume that ψ is of the form $\Diamond_s(\pi \wedge \Diamond_1\psi_1 \wedge \Diamond_2\psi_2)$. It follows immediately that π is a propositional list and that each ψ_i belongs to Θ. By symmetry it suffices to prove that

$$\mathcal{N}', u \Vdash \psi \text{ only if } \mathcal{N}', v \Vdash \psi. \tag{9}$$

Assume that $\mathcal{N}', u \Vdash \psi$. Then there are a middle point m and base points u_1, u_2 such that Sum, $R_i m u_i$, $\mathcal{N}', m \Vdash \pi$ and $\mathcal{N}', u_i \Vdash \psi_i$. Recall that P_m denotes the set of propositional variables in P that are true at m, and that Θ_{u_i} is the set of formulas in Θ true of u_i. Obviously, we have that $\mathcal{N}', m \Vdash \Omega_{P_m|P} \wedge \Diamond_1\Omega_{\Theta_{u_1}|\Theta} \wedge \Diamond_2\Omega_{\Theta_{u_2}|\Theta}$. It follows by definition of V' that $\mathcal{N}', u \Vdash q_{(P_m,\Theta_{u_1},\Theta_{u_2})}$, so by assumption this proposition letter holds at v as well. Hence, there are a middle point k and base points v_1 and v_2 such that SvK, $R_i k v_i$, $\mathcal{N}', k \Vdash \Omega_{P_m|P}$ and $\mathcal{N}', v_i \Vdash \Omega_{\Theta_{u_i}|\Theta}$. It is straightforward to prove that this implies $\mathcal{N}', k \Vdash \pi$ and $\mathcal{N}', v_i \Vdash \psi_i$. From this it is immediate that $\mathcal{N}', v \Vdash \psi$. This proves (9) and thus finishes the proof of the second part of the proposition.

Finally, the last part of the proposition is immediate by the fact that θ only uses old proposition letters, and for these we have that $V(p) = V'(p)$. Thus the truth of θ is not affected by moving from $\mathcal{N}$ to $\mathcal{N}'$. QED

Now we can work towards the $\mathcal{L}_\nabla$-logic Λ. With each base point u in N^- we will associate a *characteristic* $\mathcal{L}_\nabla$-formula χ_u: this formula is supposed to give a description of u in terms of $\mathcal{L}_\nabla$-formulas. These formulas are defined by a *downward* induction on the height of u. The idea is that the closer we get to the root of $\mathcal{N}'$, the more information we need to gather; hence, the complexity (in terms of ∇-depth) of χ_u will increase as we move towards the root of $\mathcal{N}'$.

If u is on the edge of N^- we define

$$\chi_u := \Omega_{P_u|P} \wedge \Omega_{Q_u|Q};$$

that is, we only need propositional information for points on the edge.

Now assume that u is a base point in N^- of height less than $depth(\theta)$. We will define χ_u as the conjunction of four formulas, of which the third and fourth require some explanation and introduction. The well-definedness of these two conjuncts (and thus, of χ_u) will be based on the inductive assumption that for all points v of greater height than u, the formula χ_v has been defined and $depth(\chi_v) \leq depth(\theta) - height(v)$. Clearly this is true for any point v on the edge. Let $C(u) = \{(w_1, w_2) \mid \exists m(Swm \wedge R_1 m w_1 \wedge R_2 m w_2)\}$ be the set of pairs of base successors of w. Define

$$A_u = \{(\chi_{v_1}, \chi_{v_2}) \mid (v_1, v_2) \in C(u)\}.$$

It follows from the inductive assumption and the fact that the collection $P \cup Q$ of propositional variables is finite, that A_u is finite although $C(u)$ may be infinite. Now define, for any finite set $A = \{(\alpha_i, \beta_i) \mid i \in I\}$ of pairs of $\mathcal{L}_\nabla$-formulas, the formula σ_A as follows:

$$\sigma_A = \bigwedge_{J \subseteq I} \Delta(\bigvee_{j \in J} \alpha_j, \bigvee_{i \in I \setminus J} \beta_i).$$

The meaning of this formula is expressed by the following lemma.

Proposition 7.5 *Let $A = \{(\alpha_i, \beta_i) \mid i \in I\}$ be a finite set of pairs of $\mathcal{L}_\nabla$-formulas. Let $\mathcal{M}$ be an $\mathcal{L}_\nabla$-model, and let r be a point in $\mathcal{M}$. Then $\mathcal{M}, r \Vdash \sigma_A$ if and only if for all pairs of points s, t such that $Trst$, there is some $i \in I$ such that $\mathcal{M}, s \Vdash \alpha_i$ and $\mathcal{M}, t \Vdash \beta_i$.*

Proof. Assume that A, $\mathcal{M}$ and r are as in the statement of the proposition.

For the direction from left to right, assume that $\mathcal{M}, r \Vdash \sigma_A$ and that s and t are points such that $Trst$. First take $J = I$. Then $\mathcal{M}, r \Vdash \Delta(\bigvee_{j \in I} \alpha_j, \bigvee_{i \in \varnothing} \beta_i)$ implies (by the definition of Δ) that $\mathcal{M}, s \Vdash \alpha_j$ for some $j \in I$ or else $\mathcal{M}, t \Vdash \bot$. Thus we find that $\mathcal{M}, s \Vdash \alpha_j$ for some $j \in I$. Hence, if we define $J_0 = \{j \in I \mid \mathcal{M}, s \nVdash \alpha_j\}$, we clearly have that J_0 is distinct from I. But it follows from $\mathcal{M}, r \Vdash \Delta(\bigvee_{j \in J_0} \alpha_j, \bigvee_{i \in I \setminus J_0} \beta_i)$ and $\mathcal{M}, s \Vdash \neg \bigvee_{j \in J_0} \alpha_j$ that $\mathcal{M}, t \Vdash \bigvee_{i \in I \setminus J_0} \beta_i$. This implies that there is some $i \notin J_0$ such that $t \Vdash \beta_i$. By definition of J_0 we also have $s \Vdash \alpha_i$, so clearly this is the (kind of) i that we were looking for.

For the other direction, suppose that for all pairs of points s, t such that $Trst$, there is some $i \in I$ such that $\mathcal{M}, s \Vdash \alpha_i$ and $\mathcal{M}, t \Vdash \beta_i$. In order to prove that $\mathcal{M}, r \Vdash \sigma_A$, let J be an arbitrary subset of I, and let s and t be arbitrary points such that $Trst$. Let $i \in I$ be the index such that $\mathcal{M}, s \Vdash \alpha_i$ and $\mathcal{M}, t \Vdash \beta_i$. Hence, if i belongs to J we obtain that $s \Vdash \bigvee_{j \in J} \alpha_j$, while if $i \notin J$ we find that $t \Vdash \bigvee_{i \in I \setminus J} \beta_i$. In either case, we have shown that $s \Vdash \bigvee_{j \in J} \alpha_j$ or $t \Vdash \bigvee_{i \in I \setminus J} \beta_i$. This suffices to prove that $\mathcal{M}, r \Vdash \Delta(\bigvee_{j \in J} \alpha_j, \bigvee_{i \in I \setminus J} \beta_i)$, and thus, that $\mathcal{M}, r \Vdash \sigma_A$. QED

Now that we have established this, we are ready for the definition of χ_u. Define the following formulas:

$$\begin{array}{rcl} \chi_u^3 & = & \sigma_{A_u} \\ \chi_u^4 & = & \displaystyle\bigwedge_{(w_1, w_2) \in C(w)} \nabla(\chi_{w_1}, \chi_{w_2}) \\ \chi_u & = & \Omega_{P_w | P} \wedge \Omega_{Q_w | Q} \wedge \chi_u^3 \wedge \chi_u^4. \end{array}$$

It is clear from this definition that the ∇-depth of χ_u is at most one higher than the highest depth of the formula associated with any of its immediate successors; hence, we have that $depth(\chi_u) \leq depth(\theta) - height(u)$, so the inductive assumption will remain valid.

$C(u)$ may be empty — this indicates that u is a blind world in the tree $\mathcal{N}$, that is, it does not have any successor. For such a blind world the third conjunct of χ_u simply boils down to $\Delta(\bot, \bot)$.

The following proposition shows that the characteristic formulas are at least *correct* descriptions.

Proposition 7.6 *For each base point w in N^- we have that $\mathcal{N}', w \Vdash \chi^{\diamond}_w$.*

Proof. By a straightforward downward induction on the height of w. QED

The last proposition takes us into an area where we may use the assumed completeness of Λ. For, it follows from Proposition 7.6 that $\mathcal{N}', r \Vdash \chi^{\diamond}_r$. But since $\mathcal{N}'$ is a model for Λ^{sim} this means that $\chi^{\diamond}_r$ is Λ^{sim}-consistent. Thus, it follows from the Simulation Theorem that χ_r is Λ-consistent. Hence, by completeness of Λ with respect to the frame class C, we may infer that χ_r is satisfiable in C — in fact, the same holds for all the χ_w's.

Let $\mathcal{F} = (F, T)$ in C be a frame in which χ_r is satisfiable; that is, for some $w_0 \in F$ and some valuation U we have that

$$\mathcal{F}, U, w_0 \Vdash \chi_r.$$

It follows from Proposition 3.5 that $\chi^{\diamond}_r$ is satisfiable in $\mathcal{F}^\bullet$, but this is of marginal interest only. What we are after is a $\mathsf{C}^\star_{mf}$-frame $\mathcal{F}'$ in which θ will turn out to be satisfiable. Analogous to our definition of $\mathcal{F}^\bullet$ we will obtain $\mathcal{F}'$ from $\mathcal{F}$ by adding middle points; the difference with $\mathcal{F}^\bullet$ will be that for a triple (s_0, s_1, s_2) in T we may now want to add *several* middle points. To determine the number and names of these middle points we turn to the points in N^- for information.

Let $t = (t_0, t_1, t_2)$ be a triple of points in $\mathcal{F}$ such that $Tt_0t_1t_2$, and let m be a middle point in N^-. We say that *t resembles m* if $\mathcal{F}, U, t_i \Vdash \chi_{r_i m}$ for each i (the partial maps r_i were given in definition 4.2). The point m is *minimal for t* if t resembles m while t resembles *no* m' with $P_m = P_{m'}$ that is closer to the root of $\mathcal{N}'$ (more formally, this condition can be expressed by requiring that there is no m' with $P_m = P_{m'}$ such that $\mathit{height}(r_0 m') < \mathit{height}(r_0 m)$).

In order to give a formal definition of $\mathcal{F}'$, we first associate a set X_t with each triple $t \in T$. For an arbitrary such triple $t = (t_0, t_1, t_2)$, distinguish the following cases.

- If t resembles no middle point in N^-, then define X_t to be some singleton set, say,

$$X_t := \{*\}.$$

- If on the other hand, t *does* resemble some middle point in N^-, then put

$$X_t := \quad \{(P_m, \Theta_{r_1 m}, \Theta_{r_2 m}) \mid \ m \text{ is minimal for } t\}.$$

The idea will be that each element a of X_t determines one middle point that we insert between t_0, t_1 and t_2, and that we will make a variable p true at such a middle point (t, a) if the first coordinate of a is a set of propositional variables that tells us to do so.

Formally, the frame $\mathcal{F}' = (W', S', R'_0, R'_1, R'_2)$ and the valuation U' will be obtained from the frame $\mathcal{F}$ and the valuation U as follows.

$$\begin{array}{rcl} W' &=& W \cup \{(t, a) \mid t \in T, a \in X_t\}, \\ R'_i &=& \{((t, a), t_i) \mid t \in T, a \in X_t\} \\ S' &=& (R'_0)^{-1} \\ U'(q) &=& U(q) \\ U'(p) &=& U(p) \cup \{(t, (I, \Theta_1, \Theta_2)) \mid t \in T, (I, \Theta_1, \Theta_2) \in X_t \text{ and } p \in I\}. \end{array}$$

The following proposition ensures that $\mathcal{F}'$ indeed belongs to the class $\mathsf{C}^\star_{mf}$ and that with respect to translated $\mathcal{L}_\nabla$-formulas, $\mathcal{F}'$ and U' display the right behaviour.

Proposition 7.7 *1. $(\mathcal{F}')_\bullet = \mathcal{F}$. Hence, $\mathcal{F}'$ belongs to the class $\mathsf{C}^\star_{mf}$.*

2. For each base point t_0 of $\mathcal{F}'$ and each $\mathcal{L}_\nabla$-formula ψ we have

(10)
$$\mathcal{F}, U, t_0 \Vdash \psi \text{ iff } \mathcal{F}', U', t_0 \Vdash \psi^\Diamond.$$

3. As a corollary, we have that $\mathcal{F}', U', w_0 \Vdash \chi_r^\Diamond$.

Proof. The first part is immediate from the definitions and the fact that, for each $t \in T$, the set X_t is finite. Part 2 then follows from Proposition 4.6, while the last part of the proposition is an immediate consequence of Proposition 7.6. QED

However, we already mentioned that it is not enough to prove that $\chi_r^\Diamond$ is satisfiable in $\mathcal{F}'$: we want to prove that our *original formula* $\mathsf{b} \wedge \theta$ is satisfied in $\mathcal{F}'$. It is here that the characteristic formulas χ_u are of crucial importance. Roughly speaking, we will show that whenever $\mathcal{F}', U', s \Vdash \chi_u$, then s (in $(\mathcal{F}', U')$) and u (in $\mathcal{N}'$) satisfy the same formulas in Θ, up to a certain depth. More precisely, we will establish the following result.

Proposition 7.8 *Let s be a point in $\mathcal{F}', U'$ and let u be a base point in N^- such that $\mathcal{F}, U, s \Vdash \chi_u$. Then for all formulas ψ in Θ we have that*

(11)
$$\mathcal{N}', u \Vdash \psi \text{ iff } \mathcal{F}', U', s \Vdash \psi,$$

provided that $height(u) + depth(\psi) \leq depth(\theta)$.

Proof. The proof is by induction on the complexity of ψ (as a bnf-formula). If ψ is propositional, then (11) is immediate by the fact that $\mathcal{F}, U, s \Vdash \chi_u$ implies that s in $(\mathcal{F}', U')$ and u in $\mathcal{N}$ satisfy the same P-variables. The Boolean cases of the inductive step are trivial. Hence, by our definition of Θ as being a set of bnf-formulas, we may restrict our attention to the case that ψ is of the form $\Diamond_s(\pi \wedge \Diamond_1\psi_1 \wedge \Diamond_2\psi_2)$. Assume that $height(u) + depth(\psi) \leq depth(\theta)$. Since $depth(\psi) > 0$, this means in particular that we may assume that $height(u) < depth(\theta)$; that is, u is off the edge of N^-. It is convenient for us to write t_0 instead of s. We will prove that

(12)
$$\mathcal{N}', u \Vdash \Diamond_s(\pi \wedge \Diamond_1\psi_1 \wedge \Diamond_2\psi_2) \text{ iff } \mathcal{F}', U', t_0 \Vdash \Diamond_s(\pi \wedge \Diamond_1\psi_1 \wedge \Diamond_2\psi_2).$$

First assume that $\mathcal{N}', u \Vdash \psi$. Then there are a middle point m and base points u_1 and u_2 in $\mathcal{N}'$ such that $Sum, \mathcal{N}', m \Vdash \pi$ and $R_i m u_i$, $\mathcal{N}', v_i \Vdash \psi_i$ for $i = 1, 2$.

Since u if off the edge, χ_u has a conjunct of the form $\chi_u^3 = \bigwedge_{(w_1,w_2)\in C(u)} \nabla(\chi_{w_1}, \chi_{w_2})$. Hence, it follows from $\mathcal{F}, U, t_0 \Vdash \chi_u$ that $\mathcal{F}, U, t_0 \Vdash \nabla(\chi_{v_1}, \chi_{v_2})$. This means that there must be points t_1 and t_2 in $\mathcal{F}$ such that $Tt_0t_1t_2$ and $\mathcal{F}, U, t_i \Vdash \chi_{v_i}$ for $i = 1, 2$. Thus $t = (t_0, t_1, t_2)$ resembles m. Also, for $i = 1, 2$ we may apply the inductive hypothesis to the formula ψ_i, with respect to the points t_i and v_i. For, while the height of v_i is one bigger than that of u, the depth of ψ_i is at least one smaller than that of ψ; this means that $height(v_i) + depth(\psi_i) \leq depth(\theta)$, as required for applying the inductive hypothesis. This yields that $\mathcal{F}', U', t_i \Vdash \psi_i$, for each i.

Moreover, by definition of $\mathcal{F}'$, between t_0, t_1 and t_2 there must be a middle point $k = (t, a)$ such that $a \in X_t$ is of the form $(P_{m'}, \Theta_{v_1'}, \Theta_{v_2'})$ with $P_{m'} = P_m$. (Note that a need not necessarily be of the form $(P_m, \Theta_{v_1}, \Theta_{v_2})$; it may be of the form $a = (P_{m'}, \Theta_{r_1 m'}, \Theta_{r_2 m'})$ such that m' is closer to the root r of $\mathcal{N}'$ than m. However, there must be at least one $a \in X_t$ of the described form, i.e., with $P_{m'} = P_m$.) Then by definition of U' we know that t in $\mathcal{F}'$ and m in $\mathcal{N}'$ make precisely the same P-variables true, so from $\mathcal{N}', m \Vdash \pi$ if is immediate that $\mathcal{F}', U', t \Vdash \pi$. Hence we may conclude that indeed $\mathcal{F}', U', t_0 \Vdash \psi$.

For the other direction, assume that $\mathcal{F}', U', t_0 \Vdash \psi$. Then there are a middle point k and base points t_1 and t_2 such that St_0k, R_ikt_i, $\mathcal{F}', U', k \Vdash \pi$ and $\mathcal{F}', U', t_i \Vdash \psi_i$ for $i = 1, 2$. From the definition of $\mathcal{F}'$ it follows that in $\mathcal{F}$ we must have $Tt_0t_1t_2$.

By Proposition 7.5 it follows from $\mathcal{F}', U', t_0 \Vdash \chi_u^3$ that there are points v_1 and v_2 in $\mathcal{N}'$ such that $(v_1, v_2) \in C(u)$, $\mathcal{F}', U', t_1 \Vdash \chi_{v_1}$ and $\mathcal{F}', U', t_2 \Vdash \chi_{v_2}$. Hence, $t = (t_0, t_1, t_2)$ resembles the (unique) middle point m of u, v_1 and v_2.

Hence, by definition of $\mathcal{F}'$, the point k must be of the form $k = (t, (P_{m'}, \Theta_{r_1 m'}, \Theta_{r_2 m'}))$ where m' is such that t resembles m', $P_{m'} = P_m$ and $\mathit{height}(r_0 m') \leq \mathit{height}(u)$. Abbreviate $u' = r_0 m'$; since t resembles m' we have that $\mathcal{F}', U', t_0 \Vdash \chi_{u'}$. By our assumption that $\mathcal{F}', U', t_0 \Vdash \chi_u$ and the definition of the characteristic formulas, it follows that u and u' agree on the truth of all proposition letters. Hence, by part 2 of Proposition 7.4, u and u' agree on the truth of all formulas in Θ. Thus, it suffices to prove that $\mathcal{N}', u' \Vdash \psi$.

From the fact that u and u' agree on all proposition letters we may draw another interesting conclusion, namely, that $\mathcal{N}', u' \Vdash q_{(P_m, \Theta_{v_1}, \Theta_{v_2})}$. By definition of the valuation V' in $\mathcal{N}'$, this implies that there are base points w_1, w_2 and a middle point n in $\mathcal{N}'$ such that $Su'n$, R_inw_i, $P_n = P_m$ and $\Theta_{w_i} = \Theta_{v_i}$. Since for each i, we have that $\mathit{height}(w_i) + \mathit{depth}(\psi_i) \leq (\mathit{height}(u')+1)+(\mathit{depth}(\psi)-1) = \mathit{height}(u')+\mathit{depth}(\psi) \leq \mathit{height}(u)+\mathit{depth}(\psi) \leq \mathit{depth}(\theta)$, we may apply, for $i = 1, 2$, the inductive hypothesis to ψ_i, for the points s_i and w_i, respectively. This gives that $\mathcal{N}', w_i \Vdash \psi_i$.

Also, from the fact that $P_n = P_m = P_{m'}$ and the definition of U' we may infer that k and m agree on the truth of all proposition letters in P; hence, we find that $\mathcal{N}', n \Vdash \pi$.

From this it is immediate that $\mathcal{N}', u' \Vdash \psi$, and from our earlier observation this gives that $\mathcal{N}', u \Vdash \psi$, as required. This proves (11) and thus finishes the proof of the proposition. QED

From the above propositions it is immediate that $\mathcal{F}', U', w_0 \Vdash \mathsf{b} \wedge \theta$, which shows that indeed, the formula $\mathsf{b} \wedge \theta$ is satisfiable in the class $\mathsf{C}^\star_{mf}$. This finishes the proof of Proposition 7.2.

7.2 Preservation of general completeness

In this subsection we discuss a preservation result concerning completeness with respect to classes of descriptive general frames. Recall that we needed this result in order to show that $(\cdot)^{sim}$ distributes over meets of logics, cf. the proof of Theorem 5.3.

Proposition 7.9 *Let Λ be a $\mathcal{L}_\nabla$-logic which is complete with respect to a class C of descriptive general $\mathcal{L}_\nabla$-frames. Then Λ^{sim} is complete for $\mathsf{C}^\star$.*

Proof. Since the proof of this proposition is a variation on the proof of Proposition 7.2, we allow ourselves to be fairly sketchy. We will show that for an arbitrary bnf-formula θ such that $\mathsf{b} \wedge \theta$ is Λ^{sim}-consistent, we can satisfy θ in a descriptive general frame that belongs to $\mathsf{C}^{\star}$.

Copying the proof of Proposition 7.2 and using its definitions and notation, we arrive at a descriptive general frame $\mathcal{G} = (\mathcal{F}, A)$ which belongs to C and in which the formula χ_r can be satisfied under some admissible valuation U. We will define a descriptive general *Sim*-frame $\mathcal{G}''$ such that $\mathcal{G}''_{\bullet} = \mathcal{G}$ and in which θ is satisfiable under an admissible valuation U''. The definition of $\mathcal{G}''$ will be a 'multiple middle point' version of $\mathcal{G}^{\bullet}$, in much the same way that we defined $\mathcal{F}'$ as a variation on $\mathcal{F}^{\bullet}$. The frame $\mathcal{F}''$ underlying $\mathcal{G}''$ will indeed look very much like $\mathcal{F}'$, but it will not be exactly the same: in $\mathcal{F}''$ we cannot have two distinct middle points of the form (t,x) and (t,x') that make the same proposition letters true.

Formally, we will define $\mathcal{F}''$ as follows. Let T_1 be the set of triples $t \in T$ that resemble some middle point m in N^-, and define $T_2 = T \setminus T_1$. Observe that a triple $t \in T$ belongs to T_1 if and only if for some middle point m in N^- we have that $\mathcal{F}, U, t_i \Vdash \chi_{r_i m}$ for each i. Since there are only finitely many formulas of the form χ_u, and since U is an admissible valuation in $\mathcal{G}$, this shows that in $\mathcal{G}^{\bullet}$ the set T_1 (and hence, also T_2) is admissible.

Define, for any $t \in T$:

$$X''_t = \begin{cases} \{*\} & \text{if } t \in T_2, \\ \{I \subseteq P \mid \ t \text{ resembles some } m \text{ with } P_m = I\} & \text{if } t \in T_1. \end{cases}$$

and put $T''_j = \{(t,x) \mid x \in X_t\}$ for $j = 1, 2$.

Now we can define the frame $\mathcal{F}'' = (W'', S'', R''_0, R''_1, R''_2)$ and the valuation U'' as follows.

$$\begin{array}{rcl} W'' & = & W \cup T''_1 \cup T''_2, \\ R''_i & = & \{((t,x), t_i) \mid t \in T, x \in X_t\}, \\ S'' & = & (R''_0)^{-1}, \\ U''(q) & = & U(q), \\ U''(p) & = & U(p) \cup \{(t,I) \mid t \in T_1,\ I \in X_t \text{ and } p \in I\}. \end{array}$$

It is easy to see that the model $(\mathcal{F}'', U'')$ is in fact a bounded morphic image of $(\mathcal{F}', U')$ under the map that identifies middle points which make the same proposition letters true. From this it is immediate that the formula $\mathsf{b} \wedge \theta$ is satisfiable in $(\mathcal{F}'', U'')$.

Now, to complete the definition of $\mathcal{G}''$, we call a subset of W'' *admissible* if it is of the form $a \cup b \cup c$ with: $a \subseteq W$ is an admissible set of $\mathcal{G}$; $c \subseteq T''_2$ is such that $\{t \in T \mid (t,*) \in c\}$ is admissible in $\mathcal{G}^{\bullet}$; and b is of the form $T''_1 \cap U''(\pi)$ for some propositional formula π.

We first show that $\mathcal{G}'' = (\mathcal{F}'', A'')$ is a general frame. The only non-trivial part of this proof is to show that A'' is closed under the operation m_s. Consider an arbitrary element $a \cup b \cup c$ of A'', with a, b and c as described above. Clearly we have that $m_s(a \cup b \cup c) = m_s b \cup m_s c$. It is not hard to see that $m_s c$ belongs to A, whence it is admissible in $\mathcal{G}''$. In order to prove that $b = T''_1 \cap U''(\pi)$ is admissible we have to work harder. The crucial observation, which we leave for the reader to prove, is that a state w belongs to $m_s b$ if and only if there is some point u off the edge of N^- such that $\mathcal{F}, U, w \Vdash \chi_u$ and $\mathcal{N}', u \Vdash \Diamond_s \pi$. From this observation it follows that $m_s b$ is a (finite!) union of sets $U''(\chi_u)$ and since U is admissible in $\mathcal{G}$ this gives that $m_s b$ belongs to A''.

We also leave it for the reader to verify that $\mathcal{G}''$ is descriptive; note that for differentiation we need the new definition of X''_t, in order to show that any two distinct points (t, I) and (t, I') can be separated by an admissible set of the form $T''_1 \cap U''(p)$ for some propositional letter p.

Since it is obvious that $\mathcal{G}''_\bullet = \mathcal{G}$ we find by Proposition 4.17 that $\mathcal{G}'' \Vdash \Lambda^{sim}$. And since U'' is admissible, it follows from the satisfiability of $\mathsf{b} \wedge \theta$ in $\mathcal{G}$ that $\mathsf{b} \wedge \theta$ is satisfiable in $\mathcal{G}''$. This proves that indeed Λ^{sim} is complete with respect to $\mathsf{C}^\star$. QED

8 Unsimulating a logic

In order to prove reflection results concerning our simulation map, it is useful to define an *unsimulation*, that is, a function mapping $\mathcal{L}_\diamond$-logics extending **Sim** to $\mathcal{L}_\nabla$-logics.

Definition 8.1 Let Θ be an $\mathcal{L}_\diamond$-logic extending **Sim**. Define Θ_{sim} to be the $\mathcal{L}_\nabla$-logic consisting of those $\mathcal{L}_\nabla$-formulas that are valid on the class of unsimulations of descriptive general Θ-frames: $\Theta_{sim} := \{\varphi \in \mathcal{L}_\nabla \mid (\mathsf{D}_\Theta)_\bullet \Vdash \varphi\}$. $\lhd$

The connection between the maps $(\cdot)^{sim}$ and $(\cdot)_{sim}$ is concisely summarized by the following proposition.

Proposition 8.2 *Let Λ be an $\mathcal{L}_\nabla$-logic, and Θ an $\mathcal{L}_\diamond$-logic extending* **Sim**. *Then*

1. $\Theta_{sim} = \{\varphi \mid \varphi^\diamond \in \Theta\}$;
2. $\Lambda \subseteq \Theta_{sim}$ *iff* $\Lambda^{sim} \subseteq \Theta$;
3. $(\Theta_{sim})^{sim} \subseteq \Theta$;
4. $(\Lambda^{sim})_{sim} = \Lambda$.

Proof. Part 1 of the proposition is immediate from the following chain of equivalences: $\varphi \in \Theta_{sim}$ iff (by definition) $\mathcal{G}_\bullet \Vdash \varphi$ for all $\mathcal{G}$ in D_Θ iff (by Proposition 3.8) $\mathcal{G} \Vdash \varphi^\diamond$ for all $\mathcal{G}$ in D_Θ iff (by completeness) $\varphi^\diamond \in \Theta$.

Now consider part two of the proposition. First assume that $\Lambda \subseteq \Theta_{sim}$. Since Λ^{sim} is axiomatized by $\Lambda^\diamond \cup$ **Sim** and **Sim** $\subseteq \Theta$ by assumption, it suffices to prove that $\Lambda^\diamond \subseteq \Theta$. Hence, take an arbitrary $\varphi \in \Lambda$. By assumption, φ belongs to Θ. It follows from part 1 of this proposition that $\varphi^\diamond \in \Theta$. Since φ was arbitrary this shows that $\Lambda^\diamond \subseteq \Theta$.

For the other direction, assume that $\Lambda^{sim} \subseteq \Theta$. In order to prove that $\Lambda \subseteq \Theta_{sim}$, consider an arbitrary formula φ in Λ. Since $\varphi^\diamond \in \Lambda^{sim}$ by Theorem 5.2, we have that $\varphi^\diamond \in \Theta$ by our assumption. Thus φ belongs to Θ_{sim} by the first part of the Proposition.

Part 3 of the proposition is immediate by part 2 and the observation that $\Theta_{sim} \subseteq \Theta_{sim}$.

Likewise, the right to left inclusion of part 4 is immediate by part 2 and the fact that $\Lambda^{sim} \subseteq \Lambda^{sim}$. For the other, left to right, inclusion, suppose that φ is a theorem of $(\Lambda^{sim})_{sim}$. It follows from part 1 that $\varphi^\diamond \in \Lambda^{sim}$, whence $\varphi \in \Lambda$ by Theorem 5.2. QED

8.1 Reflection of axiomatizability

In this subsection we will see how to find an axiomatization for an arbitrary $\mathcal{L}_\nabla$-logic Λ on the basis of a given axiomatization for Λ^{sim}. Since Λ is the unsimulation of Λ^{sim}, it will pay off to look at the more general question how we can obtain an axiomatization for the unsimulation Θ_{sim} of an arbitrary $\mathcal{L}_\diamond$-logic Θ, once we know that Θ is axiomatized by a set of $\mathcal{L}_\diamond$-formulas Γ. The basic idea underlying our approach is that given an $\mathcal{L}_\diamond$-formula γ we want to encode its 'effect' on the language $\mathcal{L}_\nabla$; this effect, or *residual* as we will call it, (notation: $\mathrm{Res}(\gamma)$) will be defined as a set of $\mathcal{L}_\nabla$-formulas that can be obtained effectively from γ. The main technical result of this section (Proposition 8.9) states that if an $\mathcal{L}_\diamond$-logic Θ is axiomatized by a set of formulas Γ, then the collection $\bigcup_{\gamma\in\Gamma}\mathrm{Res}(\gamma)$ axiomatizes Θ_{sim}. From this it is not very difficult to derive the reflection under $(\cdot)^{sim}$ of finite and recursive axiomatizability.

The formal definition of residuals is fairly complicated; let us provide some intuitions first. To start with, when dealing with $\mathcal{L}_\diamond$-formulas we are hindered by the fact that although semantically there is a clear and useful distinction between base points and middle points, the proposition letters of our language may be interpreted at base points and middle points alike. For some purposes it would be much more convenient if we were working in a *sorted* modal language in which the proposition letters were separated into two sorts, of base variables and middle variables, respectively.

Remark 8.3 There is no need for formalizing such a sorted version of $\mathcal{L}_\diamond$, but it might be useful to look at the following definition and observation, which, strictly speaking, are not used further on.

Given a set *Var* of propositional variables for $\mathcal{L}_\diamond$, let Var^s be the set $\{p^m, p^b \mid p \in Var\}$. For a propositional formula π, let π^m (π^b) denote the formula we obtain from π by replacing all occurrences of a proposition letter p by the variable p^m (by p^b, respectively). For an $\mathcal{L}_\diamond$-formula in base normal form, say θ, we define its *sorted variant* θ^s by the following induction:

$$\begin{array}{rcl} p^s &=& p^b, \\ (\neg\theta)^s &=& \neg\theta^s, \\ (\theta_1 \wedge \theta_2)^s &=& \theta_1^s \wedge \theta_2^s, \\ \diamond_s(\pi \wedge \diamond_1\theta_1 \wedge \diamond_2\theta_2)^s &=& \diamond_s(\pi^m \wedge \diamond_1\theta_1^s \wedge \diamond_2\theta_2^s). \end{array}$$

For an $\mathcal{L}_\diamond$-formula θ in normal form, say, θ is of the form (7), we define θ^s to be the formula

$$(\mathsf{b} \wedge \pi_0^b \wedge \theta_0^s) \ \vee \ (\mathsf{m} \wedge \bigvee_{j\in J}(\pi_j^m \wedge \diamond_0\theta_{0,j}^s \wedge \diamond_1\theta_{1,j}^s \wedge \diamond_2\theta_{2,j}^s)). \tag{13}$$

Clearly, the idea underlying these definitions is that we replace each 'base occurrence' of a proposition letter p by p^b and each 'middle occurrence', by p^m. Obviously, $\mathcal{L}_\diamond$-formulas are not equivalent to their sorted versions, but nevertheless, we can prove that the sorted, normalized version of a collection of formulas axiomatizes the same logic as the set itself. That is, for any set Γ of $\mathcal{L}_\diamond$-formulas Γ we have that

$$\mathbf{Sim}.\Gamma = \mathbf{Sim}.\Gamma^s.$$

(Here $\Gamma^s = \{(\gamma^n)^s \mid \gamma \in \Gamma\}$ denotes the collection of sorted variants of the normalized formulas in Γ.)

The second crucial intuition underlying our definition of residuals is based on the requirement that we want $\mathrm{Res}(\gamma)$ to be a set of $\mathcal{L}_\nabla$-formulas. This means that we somehow have to get rid of the propositional lists that have 'middle occurrences' within an $\mathcal{L}_\Diamond$-formula in normal form. Our solution to this problem is fairly straightforward, but technically rather involved: the best way to explain it is by saying that first, we simply replace such middle occurrences with suitable substitution instances, and that second, we show that the resulting formulas are equivalent to translated $\mathcal{L}_\nabla$-formulas. For each suitable substitution τ we thus obtain a function $\mathrm{res}_\tau(\cdot)$ mapping (normal forms of) $\mathcal{L}_\Diamond$-formulas to $\mathcal{L}_\nabla$-formulas. Let us now turn to the technical details.

Definition 8.4 A *sorting substitution* is an $\mathcal{L}_\Diamond$-substitution replacing each propositional variable p with a formula of the form

$$(\mathsf{b} \wedge p^b) \vee (\mathsf{m} \wedge \bigvee_{j \in J^\tau_p} (\Diamond_0 p^{0,j} \wedge \Diamond_1 p^{1,j} \wedge \Diamond_2 p^{2,j})),$$

where J^τ_p is some finite index set. Given a (sorted) substitution τ and an $\mathcal{L}_\Diamond$-formula γ, let $\tau(\gamma)$ denote the result of applying τ to γ. $\lhd$

As will become more clear when we prove Proposition 8.8, sorted substitutions are the syntactic counterpart of the definition of the collection of admissible sets in a general simulation frame, cf. Definition 3.6.

The following rather technical lemma states that on 'middle occurrences' of a propositional list π, a sorting substitution τ has the effect of 'spreading' the information of π over 'neighboring' base occurrences.

Proposition 8.5 *There is an effective procedure which on an input consisting of a propositional list π and a sorting substitution τ, returns a finite set $\{\pi^\tau_{i,k} \mid k \in K_{\tau,\pi}\}$ of propositional lists such that*

$$\vdash_{\mathbf{Sim}} \mathsf{m} \rightarrow (\tau(\pi) \leftrightarrow \bigvee_{k \in K_{\tau,\pi}} (\Diamond_0 \pi^\tau_{0,k} \wedge \Diamond_1 \pi^\tau_{1,k} \wedge \Diamond_2 \pi^\tau_{2,k})).$$

Note that we do not require the formulas $\pi^\tau_{i,k}$ to be substitution instances of π.

Proof. Assume that π is the propositional list $p_1 \wedge \ldots \wedge p_n \wedge \neg q_1 \wedge \ldots \wedge \neg q_m$, then $\tau(\pi)$ is the formula

$$\tau(p_1) \wedge \ldots \wedge \tau(p_n) \wedge \neg\tau(q_1) \wedge \ldots \wedge \neg\tau(q_m)$$

We first consider the positive literals. Suppose that $\tau(p)$ is the formula $(\mathsf{b} \wedge p^b) \vee (\mathsf{m} \wedge \bigvee_j (\Diamond_0 p^{0,j} \wedge \Diamond_1 p^{1,j} \wedge \Diamond_2 p^{2,j}))$. Obviously then, at middle points, $\tau(\pi)$ is equivalent to the formula $\bigvee_j (\Diamond_0 p^{0,j} \wedge \Diamond_1 p^{1,j} \wedge \Diamond_2 p^{2,j})$.

Our second observations concerns the negative literals of π. Let $\neg q$ be an arbitrary example, and suppose that $\tau(q)$ is the formula $(\mathsf{b} \wedge q^b) \vee (\mathsf{m} \wedge \bigvee_j (\diamond_0 q^{0,j} \wedge \diamond_1 q^{1,j} \wedge \diamond_2 q^{2,j}))$. At middle points, the negation of this formula is equivalent to $\bigwedge_j (\diamond_0 \neg q^{0,j} \vee \diamond_1 \neg q^{1,j} \vee \diamond_2 \neg q^{2,j})$.

Hence, using the Boolean distributive laws we can show that at middle points, the formula $\tau(\pi)$ is equivalent to a disjunction $\bigvee_k \gamma_k$ of conjunctions γ_k of formulas of the form $\diamond_i p$ and $\diamond_i \neg q$. Since each diamond $\diamond_i$ distributes over conjunctions (modulo **Sim**), we can rewrite each of these conjunctions into the form $\diamond_0 \pi_{0,k} \wedge \diamond_1 \pi_{1,k} \wedge \diamond_2 \pi_{2,k}$ with each $\pi_{i,k}$ a propositional list. This proves the proposition. QED

Now we have sufficient material to define the residuals.

Definition 8.6 Let τ be a sorting substitution. We first define, by induction to the complexity of bnf-formulas, the untranslation θ^τ of an $\mathcal{L}_\diamond$-formula θ in base normal form:

$$\begin{array}{rcl} p^\tau & = & p^b, \\ (\neg\theta)^\tau & = & \neg\theta^\tau, \\ (\theta_1 \wedge \theta_2)^\tau & = & \theta_1^\tau \wedge \theta_2^\tau, \\ (\diamond_s(\pi \wedge \diamond_1\theta_1 \wedge \diamond_2\theta_2))^\tau & = & \bigvee_{k \in K_{\tau,\pi}} (\pi^\tau_{0,k} \wedge (\pi^\tau_{1,k} \wedge \theta_1^\tau) \nabla (\pi^\tau_{2,k} \wedge \theta_2^\tau)). \end{array}$$

Of a formula θ in normal form, the τ*-residual* $\mathrm{res}_\tau(\theta)$ is defined as follows. Assume that θ has the form (7):

$$(\mathsf{b} \wedge \theta_0) \;\vee\; (\mathsf{m} \wedge \bigvee_{j \in J} (\pi_j \wedge \diamond_0\theta_{0,j} \wedge \diamond_1\theta_{1,j} \wedge \diamond_2\theta_{2,j})),$$

where all the π formulas are propositional lists and all the θ formulas are in base normal form. Let $\eta_{i,j,k}$ abbreviate the formula $(\pi_j)^\tau_{i,k} \wedge \theta^\tau_{i,j}$, and let $J_\tau(\theta)$ denote the set $\{(j,k) \mid k \in K_{\tau,\pi_j}\}$. The *residual* $\mathrm{res}_\tau(\theta)$ is defined as the following formula:

$$(14) \qquad \theta_0^\tau \;\wedge\; (\top\nabla\top \to \bigvee_{L \subseteq J_\tau(\theta)} (\bigwedge_{(j,k)\in L} \eta_{0,j,k} \wedge \bigwedge_{M \subseteq L} \Delta(\bigvee_{(j,k)\in M} \eta_{1,j,k}, \bigvee_{(j,k)\notin M} \eta_{2,j,k}))).$$

For an arbitrary $\mathcal{L}_\diamond$-formula γ we define the τ-residual $\mathrm{res}_\tau(\gamma)$ as the $\mathcal{L}_\nabla$-formula $\mathrm{res}_\tau(\gamma^n)$; that is, as the τ-residual of the normal form of γ.

Finally, we define the *residual set* of an $\mathcal{L}_\diamond$-formula γ as the set of $\mathcal{L}_\nabla$-formulas $\mathrm{Res}(\gamma) = \{\mathrm{res}_\tau(\gamma^n) \mid \tau \text{ a sorting substitution }\}$. If Γ is a set of $\mathcal{L}_\diamond$-formulas, we write $\mathrm{Res}(\Gamma)$ to denote the set $\bigcup_{\gamma\in\Gamma} \mathrm{Res}(\gamma))$. $\triangleleft$

Forbidding as these definitions may look, it is not hard to check syntactically that the formula $\mathrm{res}_\tau(\theta)$ is at least a well-formed $\mathcal{L}_\nabla$-formula. Its meaning should become clear from the (proofs of the) following propositions.

Proposition 8.7 *Let τ be a sorting substitution, let $\mathcal{F}$ be a Sim-frame and let V and V' be valuations on $\mathcal{F}$ and $\mathcal{F}_\bullet$, respectively, such that $V(p) = V'(\tau(p))$.*

1. *for any middle point m and for any propositional list π we have that $\mathcal{F}, V, m \Vdash \pi$ if and only if for some $k \in K_{\tau,\pi_j}$ it holds for each $i \in \{0,1,2\}$ that $\mathcal{F}_\bullet, V', r_i m \Vdash \pi^\tau_{i,k}$.*

2. *for any base point w and for any $\mathcal{L}_\Diamond$-formula in base normal form: $\mathcal{F}, V, w \Vdash \theta$ if and only if $\mathcal{F}_\bullet, V', w \Vdash \theta^\tau$.*

3. *for any $\mathcal{L}_\Diamond$-formula γ: $\mathcal{F}, V \Vdash \gamma$ if and only if $\mathcal{F}_\bullet, V' \Vdash \mathrm{res}_\tau(\gamma)$.*

Proof. The first part of the proposition follows almost immediately from Proposition 8.5; the details are left to the reader.

Part 2 can be proved by a fairly straightforward induction on the complexity of bnf-formulas. We will only consider one case of the inductive step, namely, in which θ is of the form $\Diamond_s(\pi \wedge \Diamond_1\theta_1 \wedge \Diamond_2\theta_2)$. For the sake of a more concise formulation we write w_0 instead of w.

First suppose that $\mathcal{M}, w_0 \Vdash \theta$. Then there must be a middle point m and base points w_1 and w_2 such that $r_i m = w_i$ for $i = 0,1,2$, $\mathcal{M}, m \Vdash \pi$ and $\mathcal{M}, w_i \Vdash \theta_i$ for $i = 1,2$. It follows from part 1 of this proposition that there is a k such that for $i = 0,1,2$ we have that $\mathcal{M}, w_i \Vdash \pi^\tau_{i,k}$. Also, by the induction hypothesis we have that $\mathcal{M}, w_i \Vdash \theta^\tau_i$ for $i = 1,2$. From these facts it is straightforward to derive that $\mathcal{M}, w_0 \Vdash \pi^\tau_{0,k} \wedge (\pi^\tau_{1,k} \wedge \theta^\tau_1)\nabla(\pi^\tau_{2,k} \wedge \theta^\tau_2)$, showing that $\mathcal{M}, w_0 \Vdash \theta^\tau$. The other direction of the equivalence, which can be proved just as easily, is left to the reader.

For the last part of the proposition, we may assume without loss of generality that γ is in normal form, say γ is of the form (7). First assume that $\mathcal{F}, V \Vdash \gamma$. In order to prove that $\mathcal{F}_\bullet, V' \Vdash \mathrm{res}_\tau(\gamma)$, take an arbitrary point w in $W_\bullet$; we will show that both conjuncts of θ^τ, cf. (14), hold at w.

First, it follows from $\mathcal{F}, V \Vdash \gamma$ and the fact that w is a base point that $\mathcal{F}, V, w \Vdash \theta_0$. Hence, by the second part of the proposition we have that $\mathcal{F}_\bullet, V', w \Vdash \theta^\tau_0$. This takes care of the first conjunct of θ^τ.

In order to show that the second conjunct of θ^τ holds at w as well, assume that $\mathcal{F}_\bullet, V', w \Vdash \top\nabla\top$. That is, w has T-successors in $\mathcal{F}_\bullet$; hence, w has S-successors in $\mathcal{F}$. For each S-successor m of w, by our assumption on the shape of γ there is some j such that $\mathcal{F}, V, m \Vdash \pi_j \wedge \Diamond_0\theta_{0,j} \wedge \Diamond_1\theta_{1,j} \wedge \Diamond_2\theta_{2,j}$. By part 1 of this proposition, we may derive from $\mathcal{F}, V, m \Vdash \pi_j$ the existence of a $k \in K_{\tau,\pi_j}$ such that for each $i = 0,1,2$ we have that $\mathcal{F}, V, r_i m \Vdash (\pi_j)^\tau_{i,k}$. Likewise, we may use part 2 of the proposition to infer from $\mathcal{F}, V, m \Vdash \Diamond_i\theta_{i,j}$ that $\mathcal{F}_\bullet, V', r_i m \Vdash \theta^\tau_{i,j}$. Now suppose that we use the same notation as in Definition 8.6 and define L as the set of pairs (j,k) in $J_\tau(\theta)$ such that for each $i = 0,1,2$ we have that $\mathcal{F}_\bullet, V', r_i m \Vdash \eta_{i,j,k}$. It follows from our earlier remarks that L is not empty. Now we claim that

$$\mathcal{F}_\bullet, V', w \Vdash \bigwedge_{(j,k)\in L} \eta_{0,j,k} \wedge \bigwedge_{M\subseteq L} \Delta\Big(\bigvee_{(j,k)\in M} \eta_{1,j,k}, \bigvee_{(j,k)\notin M} \eta_{2,j,k}\Big). \tag{15}$$

First, take an arbitrary pair $(j,k) \in L$. By definition of L, there is some S-successor m of w such that $\mathcal{F}_\bullet, V', r_0 m \Vdash \eta_{0,j,k}$. However, since m is an S-successor of w we have that $r_i m = w$; and since (j,k) was an arbitrary element of L this gives that $\mathcal{F}_\bullet, V', w \Vdash \bigwedge_{(j,k)\in L} \eta_{0,j,k}$.

Now let v_1 and v_2 be arbitrary base points such that Twv_1v_2. By definition of T there must be a middle point m between w, v_1 and v_2. Thus by definition of L there must be some pair $(j,k) \in L$ such that for each $i = 1,2$ we have $\mathcal{F}_\bullet, V', v_i \Vdash \eta_{i,j,k}$. But then it follows by Proposition 7.5 that $\mathcal{F}_\bullet, V', w \Vdash \bigwedge_{M \subseteq L} \Delta(\bigvee_{(j,k)\in M} \eta_{1,j,k}, \bigvee_{(j,k)\notin M} \eta_{2,j,k})$.

Thus we proved (15) and hence, we established that the second conjunct of $\mathrm{res}_\tau(\gamma)$ holds at w as well. This shows that $\mathcal{F}_\bullet, V', w \Vdash \mathrm{res}_\tau(\gamma)$ and thus finishes the proof of the left to right direction of the last part of the proposition.

In order to finish our proof of part 3, now suppose that $\mathcal{F}_\bullet, V' \Vdash \mathrm{res}_\tau(\gamma)$. We will show that $\mathcal{F}, V \Vdash \gamma$. To do so, take an arbitrary point in $\mathcal{F}$. Distinguish cases according to the nature of this point.

If the point is a base point, say w, then it follows from $\mathcal{F}_\bullet, V', w \Vdash \mathrm{res}_\tau(\gamma)$ that $\mathcal{F}_\bullet, V', w \Vdash \theta_0^\tau$, so using the second part of this proposition we obtain that $\mathcal{F}, V, w \Vdash \theta_0$ which immediately gives that $\mathcal{F}, V, w \Vdash \gamma$.

If, on the other hand, we are dealing with a middle point, say m, there is more work to do. We first turn to the (unique!) R_0-successor r_0m of m; note that it follows from the assumption that $\mathcal{F}_\bullet, V', r_0m \Vdash \mathrm{res}_\tau(\gamma)$. In particular, this gives that there is some subset L of $J_\tau(\theta)$ such that $\mathcal{F}_\bullet, V', r_0m \Vdash \bigwedge_{(j,k)\in L} \eta_{0,j,k}$ and $\mathcal{F}_\bullet, V', r_0m \Vdash \bigwedge_{M \subseteq L} \Delta(\bigvee_{(j,k)\in M} \eta_{1,j,k}, \bigvee_{(j,k)\notin M} \eta_{2,j,k})$. From the latter fact we may infer, using Proposition 7.5, that there is some pair $(j,k) \in L$ such that $\mathcal{F}_\bullet, V', r_1m \Vdash \eta_{1,j,k}$ and $\mathcal{F}_\bullet, V', r_2m \Vdash \eta_{2,j,k}$. Taking this together with the first fact we may certainly conclude that for some pair $(j,k) \in L$ it holds that

$$\mathcal{F}_\bullet, V', r_im \Vdash \eta_{i,j,k}, \text{ for each } i \in \{0,1,2\}.$$

Unraveling the definition of $\eta_{i,j,k}$ we may draw two conclusions. First, for each $i \in \{0,1,2\}$ we have $\mathcal{F}_\bullet, V', r_im \Vdash (\pi_j)^\tau_{i,k}$, so by the first part of this proposition we may conclude that $\mathcal{F}, V, m \Vdash \pi_j$. And second, for each $i \in \{0,1,2\}$ we have $\mathcal{F}_\bullet, V', r_im \Vdash \theta^\tau_{i,j}$, whence by the second part of this proposition we may infer that for each $i = 0,1,2$, it holds that $\mathcal{F}, V, r_im \Vdash \theta_{i,j}$. All in all, we find that $\mathcal{F}, V, m \Vdash \pi_j \wedge \Diamond_0\theta_{0,j} \wedge \Diamond_1\theta_{1,j} \wedge \Diamond_2\theta_{2,j}$, which immediately gives that $\mathcal{F}, V, m \Vdash \theta$. QED

Proposition 8.8 *1. Let $\mathcal{G}$ be a descriptive general $\mathcal{L}_\nabla$-frame. If $\mathcal{G} \Vdash \mathrm{Res}(\gamma)$ then $\mathcal{G}^\bullet \Vdash \gamma$.*

2. Conversely, let $\mathcal{H}$ be a descriptive general Sim-frame. If $\mathcal{H} \Vdash \gamma$ then $\mathcal{H}_\bullet \Vdash \mathrm{Res}(\gamma)$.

Proof. For the first part of the proposition, let $\mathcal{G}$ be a descriptive general $\mathcal{L}_\nabla$-frame such that $\mathcal{G} \Vdash \mathrm{Res}(\gamma)$. Let V be an arbitrary admissible valuation on $\mathcal{G}^\bullet$. By definition of $\mathcal{G}^\bullet$, for each p there are admissible subsets a^p, $a^p_{i,k}$ ($i \in \{0,1,2\}$, k in some finite set K) such that $V(p) = a^p \cup \bigcup_k (a^p_{0,k} \times a^p_{1,k} \times a^p_{2,k})$. Let τ be the substitution given by

$$\tau(p) = p^b \wedge \bigvee_k (\Diamond_0 p_{0,k} \wedge \Diamond_1 p_{1,k} \wedge \Diamond_2 p_{2,k}),$$

and consider the valuation V' given by $V'(p^b) = a^p$ and $V'(p_{i,k}) = a^p_{i,k}$.

Clearly, τ is a sorted substitution and V and V' satisfy the conditions of Proposition 8.7. Thus we may conclude that $\mathcal{G}^\bullet, V \Vdash \gamma$. Since V was arbitrary this implies that $\mathcal{G}^\bullet \Vdash \gamma$.

For the other direction, suppose that γ is valid on the descriptive general *Sim*-frame $\mathcal{H}$. In order to show that $\mathcal{H}_\bullet \Vdash \mathrm{Res}(\gamma)$, take an arbitrary sorted substitution τ, and an arbitrary $\mathcal{H}_\bullet$-admissible valuation V'. Let V be the valuation on the underlying frame of $\mathcal{H}$ which is given by $V(p) = V'(\tau(p))$. Then obviously, V is admissible on $\mathcal{H}$, whence by assumption we have that $\mathcal{H}, V \Vdash \gamma$. It follows by the definition of V that we may apply Proposition 8.7 which yields $\mathcal{H}_\bullet, V' \Vdash \mathrm{res}_\tau(\gamma)$. Since both V' and τ were arbitrary this shows that $\mathcal{H}_\bullet \Vdash \mathrm{Res}(\gamma)$. QED

Proposition 8.9 *Let Θ be an $\mathcal{L}_\nabla$-logic extending* **Sim**. *If Θ is axiomatized by Γ, then the set $\bigcup_{\gamma\in\Gamma} \mathrm{Res}(\gamma)$ axiomatizes Θ_{sim}.*

Proof. By Proposition 8.2 it suffices to prove that for all $\mathcal{L}_\nabla$-formulas φ we have the following equivalence:

$$\varphi \in \mathbf{K}_\nabla.\mathrm{Res}(\Gamma) \text{ iff } \varphi^\diamond \in \mathbf{Sim}.\Gamma.$$

For the direction from left to right, suppose that $\varphi^\diamond$ does not belong to $\mathbf{Sim}.\Gamma$. Then for some descriptive general *Sim*-frame $\mathcal{G}$ we have that $\mathcal{G} \Vdash \Gamma$, $\mathcal{G} \not\Vdash \varphi^\diamond$. It follows from Proposition 8.8 that $\mathcal{G}_\bullet \Vdash \mathrm{Res}(\Gamma)$ and from Proposition 4.17 that $\mathcal{G}_\bullet \not\Vdash \varphi$. But then it is immediate that φ does not belong to $\mathbf{K}_\nabla.\mathrm{Res}(\Gamma)$.

For the other direction, suppose that $\varphi \notin \mathbf{K}_\nabla.\mathrm{Res}(\Gamma)$. It follows that there must be a descriptive general $\mathcal{L}_\nabla$-frame $\mathcal{G}$ such that $\mathrm{Res}(\Gamma)$ is valid on $\mathcal{G}$ but φ is not. Then by Proposition 8.8 we have that $\mathcal{G}^\bullet \Vdash \Gamma$ while Proposition 3.8 implies that $\mathcal{G} \not\Vdash \varphi^\diamond$. From this it is immediate that $\varphi^\diamond$ does not belong to $\mathbf{Sim}.\Gamma$. QED

At first sight, Proposition 8.9 does not look very promising if one wants to show that $(\cdot)^{sim}$ reflects finite axiomatizability. First appearances may be deceptive however, as the following proposition shows.

Proposition 8.10 *Let Λ be an $\mathcal{L}_\nabla$-logic.*

1. *If Λ^{sim} is recursively axiomatizable, then so is Λ.*
2. *If Λ^{sim} is finitely axiomatizable, then so is Λ.*

Proof. It is fairly easy to derive from Proposition 8.9 that if Λ^{sim} has a recursive axiomatization, then Λ has a recursively enumerable set of axioms; however, a well known result sometimes called Craig's Lemma implies that then Λ must have a recursive axiomatization as well.

Now suppose that Λ^{sim} is finitely axiomatizable; then we may assume that Λ^{sim} is axiomatized over **Sim** by a *single* axiom, say, γ. It follows by the previous proposition that $\Lambda = (\Lambda^{sim})_{sim}$ is axiomatized by the set $\mathrm{Res}(\gamma)$. But then by Proposition 5.4, Λ^{sim} is axiomatized by the set $(\mathrm{Res}(\gamma))^\diamond = \{\mathrm{res}_\tau(\gamma)^\diamond \mid \tau \text{ a sorting substitution}\}$. Since γ belongs to Λ^{sim}, there must be a derivation of γ from the axioms in $(\mathrm{Res}(\gamma))^\diamond$. Since derivations are finite, only finitely many axioms, say, $(\mathrm{res}_{\tau_1}(\gamma))^\diamond, \ldots, (\mathrm{res}_{\tau_n}(\gamma))^\diamond$ can be involved in this derivation. From this it is not difficult to prove that the set $\{\mathrm{res}_{\tau_1}(\gamma), \ldots, \mathrm{res}_{\tau_n}(\gamma)\}$ axiomatizes Λ. QED

8.2 Reflection of first-order definability

In this subsection we will show that if Λ^{sim} defines an elementary class of frames, then so does Λ. That is, we will prove that $(\cdot)^{sim}$ reflects first-order definability. Roughly speaking, the proof of this result is similar to the proof of the corresponding preservation result: we will define a translation $(\cdot)_e$ mapping first order $\mathcal{L}^{fo}_{\Diamond}$-sentences to $\mathcal{L}^{fo}_{\nabla}$-sentences, and prove that if the set $\Sigma \subseteq \mathcal{L}^{fo}_{\Diamond}$ axiomatizes the class of frames for Λ^{sim}, then the set Σ_e axiomatizes the class of frames for Λ. There are some technical complications however, mainly due to the fact that in the direction from $\mathcal{L}^{fo}_{\Diamond}$ to $\mathcal{L}^{fo}_{\nabla}$, it is not immediately clear what to do with information pertaining to middle points. Our approach will be to make a distinction in $\mathcal{L}^{fo}_{\Diamond}$ between base variables and middle variables (obviously, this distinction cannot be made absolute since it is based on a semantic notion). The basic idea underlying our translation from $\mathcal{L}^{fo}_{\Diamond}$ to $\mathcal{L}^{fo}_{\nabla}$ will be to replace each middle variable y in $\mathcal{L}^{fo}_{\Diamond}$ with *three* variables y_0, y_1 and y_2 in $\mathcal{L}^{fo}_{\nabla}$; the semantic intuition is that these variables will refer to the respective (unique) R_0-, R_1- and R_2-successor of the middle point that y refers to. Now for the technical details.

Definition 8.11 Call a first order formula *clean* if no variable occurs both free and bound and no two distinct quantifier occurrences bind the same variable.

Suppose that X and Y are two disjoint sets of variables; a clean formula α is called X, Y-*translatable* if all of its free variables and none of its bound variables are taken from $X \cup Y$. $\lhd$

In the above definition, X and Y are the base and middle variables, respectively. Note that sentences are $\varnothing, \varnothing$-translatable, that each subformula of an X, Y-translatable formula is X, Y-translatable, and that if $\exists x\, \alpha$ is X, Y-translatable, then α is both $X \cup \{x\}, Y$ and $X, Y \cup \{x\}$-translatable. These observations show that the following definition is correct.

Definition 8.12 For a set of variables Y, let $Y^{\Diamond}$ denote the set $\{y_0, y_1, y_2 \mid y \in Y\}$. We define the translation $(\cdot)_{X,Y}$ from X, Y-translatable $\mathcal{L}^{fo}_{\Diamond}$-formulas to $\mathcal{L}_{\nabla}$-formulas.

$$
\begin{array}{rcl}
(x = y)_{X,Y} & = & \left\{ \begin{array}{ll} x = y & \text{if } x, y \in X, \\ x_0 = y_0 \wedge x_1 = y_1 \wedge x_2 = y_2 & \text{if } x, y \in Y, \\ \bot & \text{otherwise.} \end{array} \right. \\
(R_i xy)_{X,Y} & = & \left\{ \begin{array}{ll} x = y_0 & \text{if } x \in X \text{ and } y \in Y, \\ \bot & \text{otherwise.} \end{array} \right. \\
(Sxy)_{X,Y} & = & (R_0 yx)_{X,Y} \\
(\neg\alpha)_{X,Y} & = & \neg\alpha_{X,Y} \\
(\alpha \wedge \beta)_{X,Y} & = & \alpha_{X,Y} \wedge \beta_{X,Y}, \\
(\exists z\, \alpha)_{X,Y} & = & \exists z\, \alpha_{X \cup \{z\}, Y} \vee \exists z_0 z_1 z_2\, (T z_0 z_1 z_2 \wedge \alpha_{X, Y \cup \{z\}}).
\end{array}
$$

Given a sentence α, let α_e denote the sentence $\alpha_{\varnothing,\varnothing}$. $\lhd$

It is our aim to show that for every $\mathcal{L}_{\nabla}$-frame $\mathcal{F}$ it holds that $\mathcal{F}^{\bullet} \models \alpha_e$ iff $\mathcal{F} \models \alpha$. Since the proof of this statement is based on a formula induction, we need means to compare assignments on $\mathcal{F}$ with those on $\mathcal{F}^{\bullet}$.

Definition 8.13 Let $\mathcal{F}$ be an $\mathcal{L}_\nabla$-frame, and let X and Y be two disjoint sets of variables. An assignment $b : (X \cup Y) \to W^\bullet$ is *well-sorted with respect to* X *and* Y, or briefly: *well-sorted*, if it maps variables in X and Y to base points and middle points, respectively. Given a well-sorted assignment $b : (X \cup Y) \to W^\bullet$, let $b_{X,Y} : (X \cup Y^\diamond) \to W$ be the assignment given by

$$\begin{array}{rcll} b_{X,Y}(x) & = & b(x) & \text{if } x \in X, \\ b_{X,Y}(y_i) & = & r_i(b(y)) & \text{if } y \in Y. \end{array}$$

$\lhd$

We can now state and prove the main technical lemma of this subsection.

Proposition 8.14 *Let $\mathcal{F}$ be an $\mathcal{L}_\nabla$-frame, and let X and Y be two disjoint sets of variables. Let α be an $\mathcal{L}^{fo}_\diamond$-formula, and let $b : (X \cup Y) \to W^\bullet$ be a well-sorted assignment. Then we have*

$$\mathcal{F}^\bullet \models \alpha[b] \text{ iff } \mathcal{F} \models \alpha_{X,Y}[b_{X,Y}].$$

In particular, if α is a sentence, we may take X and Y to be empty and obtain

$$\mathcal{F}^\bullet \models \alpha \text{ iff } \mathcal{F} \models \alpha_e.$$

Proof. We prove this proposition by induction on the complexity of α.

For the base case of the induction we consider the various atomic formulas separately. First assume that α is of the form $x = y$. Distinguish cases:

1. If x and y both belong to X, then $b_{X,Y}(x) = b(x)$ and $b_{X,Y}(y) = b(y)$, so indeed we find that $\mathcal{F}^\bullet \models x = y[b]$ iff $\mathcal{F} \models x = y[b_{X,Y}]$.

2. If only one of the two variables, say x, belongs to X, then by well-sortedness, b maps x to a base point and y to a middle point. Hence, we have $\mathcal{F}^\bullet \not\models x = y[b]$ and $\mathcal{F} \not\models \bot[b_{X,Y}]$, as required.

3. If both x and y belong to Y, then b maps both of them to middle points of $\mathcal{F}^\bullet$. Since two middle points in a simulation frame are identical if and only if their R_i-successors are identical, we have $\mathcal{F}^\bullet \models x = y[b]$ iff $b(x) = b(y)$ iff for all i, $r_i(b(x)) = r_i(b(y))$ iff for all i, $b_{X,Y}(x_i) = b_{X,Y}(y_i)$ iff $\mathcal{F} \models x_0 = y_0 \land x_1 = y_1 \land x_2 = y_2[b_{X,Y}]$, as required.

If α is an atomic formula of the form R_ixy, then again distinguish cases:

1. If x belongs to X and y belongs to Y, then we have that $\mathcal{F}^\bullet \models R_ixy[b]$ iff $b(x) = r_i(b(y))$ iff $b_{X,Y}(x) = b_{X,Y}(y_i)$ iff $\mathcal{F} \models x = y_i[b_{X,Y}]$.

2. In all other cases, it follows from well-sortedness of b that $\mathcal{F}^\bullet \not\models R_ixy[b]$. This is fine, since we also have $\mathcal{F} \not\models \bot[b_{X,Y}]$

The case that α is an atomic formula of the form Sxy is immediate.

The boolean cases of the induction step have a straightforward proof, so we concentrate on the case that α is of the form $\exists z\, \beta$.

First suppose that $\mathcal{F}^\bullet \models \exists z\, \beta[b]$. Then there is a point u in $\mathcal{F}^\bullet$ such that $\mathcal{F}^\bullet \models \beta[b']$, where b' is the assignment which may differ from β only in mapping z to u. Distinguish cases as to the nature of u:

1. If u is a base point, then b' is well-sorted with respect to $X \cup \{z\}$ and Y. Hence, it follows from the induction hypothesis that $\mathcal{F} \models \beta_{X\cup\{z\},Y}[b'_{X\cup\{z\},Y}]$. Clearly, $b'_{X\cup\{z\},Y}$ differs from $b_{X,Y}$ at most in what it maps z to. Hence, we have that $\mathcal{F} \models \exists z\, \beta_{X\cup\{z\},Y}[b_{X,Y}]$. That is, the left conjunct of $\alpha_{X,Y}$ holds in $\mathcal{F}$ under $b_{X,Y}$; but then we clearly have $\mathcal{F} \models \alpha_{X,Y}[b_{X,Y}]$.

2. If, on the other hand, u is a middle point, then b' is well-sorted with respect to X and $Y \cup \{z\}$. Note that by definition of $b'_{X,Y\cup\{z\}}$, we have that $\mathcal{F} \models Tz_0z_1z_2[b'_{X,Y\cup\{z\}}]$, and that it follows from the induction hypothesis that $\mathcal{F} \models \beta_{X,Y\cup\{z\}}[b'_{X,Y\cup\{z\}}]$. Since $b'_{X,Y\cup\{z\}}$ differs from $b_{X,Y}$ at most in what it maps z_0, z_1 and z_2 to, this means that $\mathcal{F} \models \exists z_0z_1z_2\,(Tz_0z_1z_2 \wedge \beta_{X,Y\cup\{z\}})[b_{X,Y}]$, so the right disjunct of $\alpha_{X,Y}$ holds in $\mathcal{F}$ under $b_{X,Y}$. Again, we find that $\mathcal{F} \models \alpha_{X,Y}[b_{X,Y}]$.

For the other direction, suppose that $\mathcal{F} \models (\exists z\, \beta)_{X,Y}[b_{X,Y}]$. By the definition of $(\exists z\, \beta)_{X,Y}$ we have either $\mathcal{F} \models \exists z\, \beta_{X\cup\{z\},Y}$ or $\mathcal{F} \models \exists z_0z_1z_2\,(Tz_0z_1z_2 \wedge \beta_{X,Y\cup\{z\}})[b_{X,Y}]$. We leave it for the reader that in the first case we may reverse the reasoning in item 1 above, and similarly for the second case and item 2. In both cases we find a (base, respectively middle) point u in $\mathcal{F}$ such that $\mathcal{F} \models \beta[b']$ where b' is the assignment which may differ from β only in mapping z to u. This shows that $\mathcal{F} \models \alpha[b]$. QED

Proposition 8.15 *Let Λ be an $\mathcal{L}_\nabla$-logic. If the class of $\mathcal{L}_\Diamond$-frames for Λ^{sim} is axiomatized by the set Σ of $\mathcal{L}_\Diamond^{fo}$-formulas, then the set Σ_e axiomatizes the class of $\mathcal{L}_\nabla$-frames for Λ.*

Hence, if Λ^{sim} is elementary (Δ-elementary), then so is Λ.

Proof. Assume that Λ and Σ are as in the statement of the proposition. The proposition follows from the following chain of equivalences:

$\mathcal{F} \Vdash \Lambda$
iff (Prop's 3.5 and 5.4) $\mathcal{F}^\bullet \Vdash \Lambda^{sim}$,
iff (assumption) $\mathcal{F}^\bullet \models \Sigma$,
iff (Proposition 8.14)) $\mathcal{F} \models \Sigma_e$. QED

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