The Malleability of TSP_{2Opt}

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Abstract

We prove that the local search optimization problem TSP_{2Opt}—though not known to be PLS-complete—shares an important infeasibility property with other PLS-complete sets.

1 Introduction

NP-Optimization problems form a central object of study in operations research. Since at present no technique is known to find optimal solutions for these problems in reasonable time, nor is such a technique likely to be uncovered, practical algorithms tend to give near optimal solutions, where "near optimal" means different things in different contexts.

One of the techniques, recently introduced by Johnson, Papadimitriou and Yannakakis, is called "Local Search." Instead of striving for a local optimimum for the problem, one is content with a solution that is the best solution in a region. The technique starts out with an initial solution as its current solution and then keeps repeating the following steps. It considers some neighbors of the current solution to see if one of these is a better solution than the current one. If it can find such a solution it makes this solution the current solution. Otherwise it decides that the current solution is the best solution among the neighbors, stops and outputs the current solution as its current optimum. Problems with the property that an initial solution can be construed in polynomial time, the neighborhood can be inspected for a better solution in polynomial time, and the conclusion that a given solution is locally optimal can be made in polynomial time, are called "polynomial local search problems." The class that captures these problems is called PLS.

Among the problems in PLS, some have the property that finding a local optimum is "as hard as" finding a local optimum for any problem in PLS. That is, for these problems a translation mechanism, the *PLS reduction* exists, that translates local optima back to local optima in the problem reduced from. These problems are called PLS-complete. In the paper [PSY90] also a stronger form of the PLS reduction is defined—the tight PLS reduction. This reduction again requires that a local optimum can be translated back. Furthermore, the longest augmenting path of the problem reduced from is related to an augmenting path in the problem reduced to. The related path in the second problem must be at least as long as the path in the first problem.

Hence if one can prove that for a certain problem the path leading to a local optimum is long or otherwise computationally hard to construct then so will the

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path be in a search problem to which this problem can be reduced using a tight PLS reductions. Papadimitriou, Shaeffer and Yannakakis demonstrated a local search problem for which the question whether a specified local optimum that is reachable from a given initial solution is complete for $PSPACE$. Since this was the only local optimum reachable from the initial solution, it follows that computing a local optimum from a given solution is $PSPACE$-hard for this problem. They finally showed tht for the problem CircuitFlip this question is $PSPACEF$-complete. A number of local search problems were proven PLS-complete by a tight PLS-reduction from CircuitFlip, see for instance [K89], [K90], [SY], [P90]. So for these problems the question about computing a local optimum reachable from an initial solution is also $PSPACEF$-complete

The traveling salesperson problem with the 2Opt neighborhood (more formally defined in the next section) is defined as an optimization problem on a complete weighted graph. An initial Hamiltonian circuit in the graph is given. In each step the local search algorithm may select two edges (x_1, x_2), (y_1, y_2), remove these from the tour, and add the edges (x_1, y_2) and (y_1, x_2). This problem/strategy is of great practical interest since in practice it can near optimally solve TSP for a large number of cities (See Yannakakis STACS/Rouen). It is presently unknown whether this problem is complete for PLS under PLS reductions.

The main result of this paper is to show that TSP_{2Opt} shares the computational improbability of determining a reachable local optimum with the PLS-complete problems, and we will do so in Section 4. In the next section we will give definitions and notations and in particular discuss the nature of TSP_{2Opt}.

2 Preliminaries

2.1 Introduction

In this section we give the definitions used throughout this paper. We also give some earlier results.

2.2 Local search

Definition 2.1 *[JPY88] A PLS problem A is a five tuple $\langle I_A, \mathcal{FS}_A, f_A, opt_A, N_A \rangle$, where I_A is the set of instances of A, for each $I \in I_A$ the set $\mathcal{FS}_A(I)$ is the set of feasible solutions of I, f_A assigns to every feasible solution an integer value, opt_A is either min or max depending on the optimization nature of the problem, and for each $I \in I_A, s \in \mathcal{FS}_A(I)$, $N_A(I, s)$ is the set of neighbors of s, i.e., a set of feasible solutions associated with solution s.*

It is assumed that $I_A \in P$. Also, it is required that for each $I \in I_A$ an initial solution in $\mathcal{FS}_A(I)$ can be computed in polynomial time. Finally, for all $s \in \mathcal{FS}_A(I)$ it must be possible to decide in polynomial time whether s is locally optimal, i.e., whether s has a better cost than every $s' \in N_A(I, s)$, and if s is not locally optimal it must also be possible to compute in polynomial time a neighbor of s with a better cost.

The set of feasible solutions of a local search problem, together with a neighborhood structure, can be interpreted as a directed graph. By (a, b) we denote the edge directed from a to b.

Definition 2.2 *[PSY90] Let $A = \langle I_A, \mathcal{FS}_A, f_A, opt_A, N_A \rangle \in PLS$. For $I \in I_A$, define the* local search graph *$G_A(I) = (V_A(I), E_A(I))$ as follows:*

$V_A(I) = \{s \mid s \in \mathcal{FS}_A(I)\}$.
$E_A(I) = \{(s, s') \mid s' \in N_A(I, s)$ *and* $f_A(I, s') > f_A(I, s)\}$.

Let $s, s' \in \mathcal{FS}_A(I)$. *We say that feasible solution* s' *is* reachable *from feasible solution* s, *if there is a directed path from* s *to* s' *in* $G_A(I)$. *Such a directed path is called an* augmenting *path.*

Let A be a problem in PLS. The *standard local search algorithm*, given $I \in I_A$, will first compute an initial solution $s \in \mathcal{FS}_A(I)$ in polynomial time. Then the following step is repeated, until a locally optimal solution is found.

The standard local search algorithm decides in polynomial time whether s *is locally optimal. If* s *is not locally optimal, it computes a solution* $s' \in N_A(I, s)$ *with a better cost than* s *and sets* s *to* s'.

The standard local search algorithm will only go from one feasible solution to another with a strictly better cost. Consider the local search graph G_A. The standard local search algorithm walks along the paths of this graph. One arc in G_A denotes one step of the standard local search algorithm.

2.3 Space Machines

Definition 2.3 *[BDG88] A Turing machine with one tape is a five-tuple* $M = \langle Q, \Sigma, \delta, q_0, F\rangle$, *where*

1. Q *is a finite set of internal states*
2. Σ *is the tape alphabet*
3. $q_0 \in Q$ *is the initial state*
4. F *is a set of accepting final states*
5. $\delta : Q \times \Sigma \to Q \times \Sigma \times \{R, L, N\}$ *is a partial function called the* transition function *of* M.

If a Turing machine M runs in polynomial space, there exists a polynomial p, such that each computation of M on a string x uses at most $p(|x|)$ tapecels of the tape.

Definition 2.4 *Let* M *be a Turing machine, a* configuration *of* M *is a description of the contents of the tape of* M, *the state of* M *and the tapecel that is scanned by the head of* M.

A computation of M is described by a sequence

$$c_0, c_1, \ldots, c_f$$

of configurations, where c_0 is the initial configuration of M on x, c_f is the final configuration of M on x and c_i can be obtained from c_{i-1} by applying the transition function of M once.

Without loss of generality, in this paper we assume that $F = \{q_f\}$ and $\Sigma = \{0, 1\}$. Furthermore, at the end of a computation, the first $p(|x|)$ tapecels on the tape of M contain the symbol 0 and the head of M scans the $(p(|x|) + 1)$'st tapecel.

2.4 Problems

In this paper we consider the local search problem TSP with the $2Opt$-neighborhood. Instances for TSP are complete, weighted graphs. The feasible solution set of an instance G contains all tours in G. The cost of a tour in G is the sum of all the weights on its edges. We try to find a tour in G with a minimal cost. Let T be a tour in G. The set of neighbors of T contains all tours T', that can be obtained by deleting two edges from T and inserting two new edges in T.

Based on the local search problem TSP_{2Opt} is the decision problem $RDV(TSP_{2Opt})$.

Definition 2.5 *Let G be a complete, weighted graph and T, T_f two tours in G. Assume that T_f is locally optimal. The decision problem $RDV(TSP_{2Opt})$ is defined as follows.*
Given *G, T and T_f*
Question *Is T_f reachable from T by a sequence of improving $2Opt$-changes.*

We also consider the following decision-problem on space bounded Turing machines.

Definition 2.6 Polynomial Space Acceptance
Given *A polynomial space bounded Turing machine M and string x.*
Question *Does M accept x?*

It is well-known that *Polynomial Space Acceptance* is $PSPACE$-complete. For instance, see [GJ79] for a proof that *Linear Space Acceptance* is $PSPACE$-complete.

3 Space-bounded Machines and Complete, Weighted Graphs

In this section we consider a deterministic Turing Machine M working in polynomial space. We accomplish in this section three things.

- We build from M a complete, weighted graph G.
- We associate with each configuration c of M and timestamp t, a tour $T_{\langle c,t\rangle}$ in G.
- Let c, c' be configurations of M and let $t > 0$ be a timestamp. We show that there is exactly one augmenting path leaving $T_{\langle c,t\rangle}$, and that $T_{\langle c',t-1\rangle}$ is on this path if and only if M can go in one computation step from c to c', or $c = c'$ is a final configuration.

3.1 Constructing G

Let p be the polynomial bounding the space-complexity of M. Let M have tape alphabet $\Sigma = \{0, 1\}$ and a set of states Q.

Graph G has three types of vertices, the *normal*-vertices, the *guide*-vertices and the *dummy*-vertices. The normal-vertices are used to represent configurations by tours in G. The guide-vertices are used to simulate computations of M by improving $2Opt$-changes. The dummy-vertices are used to prove that there do not exist different short augmenting paths. The normal, guide and dummy-vertices are grouped into subsets. The subsets of guide-vertices are divided into layers and the subsets of guide-vertices and normal-vertices are grouped in triples. In sequence we will define the subsets of normal, guide and dummy-vertices in G, the layer-division, the triples and the weights assigned to the edges of G.

3.1.1 The normal-vertices in G

For each tapecel i of M, G contains a subset V_i of normal-vertices. This subset V_i again can be divided into subsets, each having its own function.

1. Subset $H^i \subset V_i$ contains the vertices $h_1^i, h_2^i, h_3^i, h_4^i$. These vertices are used to indicate whether the head of M is on tapecel i or not.

2. Subset $R^i \subset V_i$ contains the vertices $r_1^i, r_2^i, r_3^i, r_4^i$. These vertices are used to indicate whether the symbol in tapecel i is 0.

3. Subset $S^i \subset V_i$ contains the vertices $s_1^i, s_2^i, s_3^i, s_4^i$. These vertices are used to indicate whether the symbol in tapecel i is 1.

For each state $q_j \in Q$, subset Q^j is a subset of normal-vertices in G and contains the vertices $q_1^j, q_2^j, q_3^j, q_4^j$. These vertices indicate whether M is in state q_j or not. Notice that, since M is deterministic, if M has $p(|x|)$ tapecels, each finite computation of M on x has at most $2^{p(|x|)}$ steps. We can represent this number by a bitstring σ of $p(|x|)$ bits. The bitstring σ is sometimes called the *time bitstring*, since it will keep track of at which point in the computation we are. For each bit i in σ, G contains a subset $T^i = \{t_1^i, t_2^i, t_3^i, t_4^i\}$ and a subset $U^i = \{u_1^i, u_2^i, u_3^i, u_4^i\}$ of normal-vertices. These vertices are used to represent that the value of the i'th bit in the time bitstring is 0 or 1, respectively.
Finally, for convenience, G contains a subset D of normal vertices, containing d_1, d_2, d_3, d_4.

3.1.2 The guide-vertices in G

We now turn to the guide-vertices. The guide-vertices are also grouped in subsets. Graph G contains subset H of guide-vertices. Subset H contains the guide-vertices η_0, η_1, η_2 and $\delta\eta_1, \delta\eta_2$. For each tapecel i, a subset A^i of guide-vertices is in G. Subset A^i contains the guide vertices $\alpha_0^i, \alpha_1^i, \alpha_2^i$ and $\delta\alpha_1^i, \delta\alpha_2^i$. For each symbol $b \in \Sigma$ and tapecel i, a subset $A^{i,b}$ of guide-vertices is in G. Subset $A^{i,b}$ contains the guide-vertices $\alpha_0^{i,b}, \alpha_1^{i,b}, \alpha_2^{i,b}$ and $\delta\alpha_1^{i,b}, \delta\alpha_2^{i,b}$. For each bit i in the time bitstring, subsets Θ^i, $\Lambda^{i,j}$ and $M^{i,j}$, $j \in \{1,2\}$ of guide-vertices are in G. Subset Θ^i contains the vertices $\vartheta_0^i, \vartheta_1^i, \vartheta_2^i$ and $\delta\vartheta_1^i, \delta\vartheta_2^i$. Subset $\Lambda^{i,j}$ contains the vertices $\lambda_0^{i,j}, \lambda_1^{i,j}, \lambda_2^{i,j}$ and $\delta\lambda_1^{i,j}, \delta\lambda_2^{i,j}$. And, finally, subset $M^{i,j}$ contains the vertices $\mu_0^{i,j}, \mu_1^{i,j}, \mu_2^{i,j}$ and $\delta\mu_1^{i,j}, \delta\mu_2^{i,j}$, $j \in \{1,2\}$.

The last sets of guide-vertices are used to simulate transitions of M. For each tapecel i, each symbol b, each state q_j and for each $k \in \{1,2,3,4,5,6\}$, we introduce the subsets $A^{i,b,q_j,k}$ of guide-vertices. A subset $A^{i,b,q_j,k}$ contains the vertices $\alpha_0^{i,b,q_j,k}, \alpha_1^{i,b,q_j,k}, \alpha_2^{i,b,q_j,k}$ and $\delta\alpha_1^{i,b,q_j,k}, \delta\alpha_2^{i,b,q_j,k}$.

3.1.3 The dummy-vertices in G

For each subset S of normal-vertices or guide-vertices we define a subset $\Delta(S)$ of dummy-vertices. Subset $\Delta(S)$ contains the vertices $\Delta_1(S)$ and $\Delta_2(S)$.

3.1.4 Layer division

To make the analysis of our reduction easier, we divide the subsets of vertices a number of different layers L_i, $1 \le i \le 5p(|x|) + 9$. Layer L_1 contains subset H. Layer L_2 contains the subsets A^i, $1 \le i \le p(|x|)$. Layer L_3 contains the subsets $A^{i,b}$, $1 \le i \le p(|x|)$ and $b \in \{0, 1\}$. Layer L_{3+k}, $1 \le k \le 6$, contains the subsets $A^{i,b,q_j,k}$, $1 \le i \le p(|x|)$, $b \in \{0, 1\}$ and $q_j \in Q$. Layer L_{9+i}, $1 \le i \le p(|x|)$, contains the subset Θ^i. Layer $L_{9+p(|x|)+2i-1}$, $1 \le i \le p(|x|)$, contains the subset $\Lambda^{i,1}$. Layer $L_{9+p(|x|)+2i}$, $1 \le i \le p(|x|)$, contains the subset $\Lambda^{i,2}$. Layer $L_{9+5p(|x|)-2(i+1)-1}$, $1 \le i \le p(|x|)$, contains the subset $M^{p(|x|)-i,1}$. Layer $L_{9+5p(|x|)-2(i+1)}$, $1 \le i \le p(|x|)$, contains the subset $M^{p(|x|)-i,2}$.

3.1.5 Triples

All subsets of guide and normal vertices are grouped in triples. Each triple consists of two subsets Ξ and Ψ of guide vertices and one subset X of normal vertices. We use $[\![\Xi, \Psi, X]\!]$, to denote that Ξ, Ψ and X belong to the same triple. Figure 1 gives all triples that exist among the subsets of vertices in G.

In the first part of the figure, the triples that always exist among the subset are denoted. In the second part, the triples that depend on the transition function of M are denoted.

We can prove that each triple is determined by exactly two of its members.

Property 1 *Let Ξ, Ψ, Ξ' and Ψ' be subsets of guide vertices and let X and X' be subsets of normal vertices. Suppose that $[\![\Xi, \Psi, X]\!]$ and $[\![\Xi', \Psi', X']\!]$ are two triples in Figure 1. Then*

$$|\{\Xi, \Psi, X\} \cap \{\Xi', \Psi', X'\}| \ne 2$$

Proof (Sketch)
This property can be proved by carefully checking every pair of triples. The intuition behind this property is as follows. Consider a triple $[\![\Xi, \Psi, X]\!]$ in Figure 1. There are two different kinds of triples. The first kind determines something, for instance which tapecel is scanned by the head, or whether a certain bit in the time bitstring has value 0. For each possible outcome, a different subset of guide vertices is involved. Translated to the triples this means that, given Ξ, for each possible X, a different subset Ψ is involved.

The second kind has to enforce an action. In our case, the actions will consist of inserting certain specified edges in a tour. Suppose that Ξ is in layer L_i, $4 \le i \le 9$. The edges that must be inserted are determined by the transition function of M and by which tapecel is scanned by M. For each possibility, G contains a unique subset of guide vertices $A^{i,b,q_j,1}$. This subset takes care that the correct action is performed. This action involves a subset of guide-vertices, different from $A^{i,b,q_j,1}$, and this subset is uniquely determined by the computation. The case that Ξ is in layer $L_{9+p(|x|)+i}$ or in layer $L_{9+3p(|x|)+i}$, $1 \le i \le 2p(|x|)$, is treated in a similar way.

Special precaution has been taken to prevent that there exists a triple $[\![\Xi, \Psi, X]\!]$ of the first kind and a triple $[\![\Psi, \Delta, X]\!]$ of the second kind. $\square$

There exist a relation between the layer division and the triple division of the guide subsets of G.

Property 2 *Consider any of the triples $[\![\Xi, \Psi, X]\!]$ of Figure 1. Suppose that Ξ belongs to layer L_i and Ψ to layer L_j. Then $i < j$. Furthermore, there are at most two possibilities for the value of j, given i, and there are at most two possibilities for the value of i, given j.*

Let $1 \le i \le p(|x|)$, $b \in \{0,1\}$ and $q_j \in Q$.

$[\![H, A^i, H^i]\!]$	H in L_1	A^i in L_2
$[\![A^i, A^{i,0}, R^i]\!]$	A^i in L_2	$A^{i,0}$ in L_3
$[\![A^i, A^{i,1}, S^i]\!]$	A^i in L_2	$A^{i,1}$ in L_3
$[\![A^{i,b}, A^{i,b,q_j,1}, Q^j]\!]$	$A^{i,b}$ in L_3	$A^{i,b,q_j,1}$ in L_4
$[\![\Theta^i, \Lambda^{i,1}, T^i]\!]$	Θ^i in L_{9+i}	$\Lambda^{i,1}$ in $L_{9+p(\|x\|)+2i-1}$
$[\![\Theta^i, \Theta^{i+1}, U^i]\!]$	Θ^i in L_{9+i}	Θ^{i+1} in L_{9+i+1}
$[\![\Lambda^{i,1}, \Lambda^{i,2}, U^i]\!]$	$\Lambda^{i,1}$ in $L_{9+p(\|x\|)+2i-1}$	$\Lambda^{i,2}$ in $L_{9+p(\|x\|)+2i}$
$[\![\Lambda^{i,2}, M^{i-1,1}, T^i]\!]$	$\Lambda^{i,2}$ in $L_{9+p(\|x\|)+2i}$	$M^{i-1,1}$ in $L_{9+5p(\|x\|)-2i-1}$
$[\![M^{i,1}, M^{i,2}, T^i]\!]$	$M^{i,1}$ in $L_{9+5p(\|x\|)-2(i+1)-1}$	$M^{i,2}$ in $L_{9+5p(\|x\|)-2(i+1)}$
$[\![M^{i,2}, M^{i-1,1}, U^i]\!]$	$M^{i,2}$ in $L_{9+5p(\|x\|)-2(i+1)}$	$M^{i-1,1}$ in $L_{9+5p(\|x\|)-2i-1}$
$[\![M^{1,2}, H, U^1]\!]$	$M^{1,2}$ in $L_{9+5p(\|x\|)}$	H in L_1
$[\![A^{i,b,q_j,1}, A^{i,b,q_j,2}, H^{i+1}]\!]$	if M moves on symbol b and state q_j its head one place to the right $A^{i,b,q_j,1}$ in L_4, $A^{i,b,q_j,2}$ in L_5	
$[\![A^{i,b,q_j,1}, A^{i,b,q_j,2}, H^{i-1}]\!]$	if M moves on symbol b and state q_j its head one place to the left $A^{i,b,q_j,1}$ in L_4, $A^{i,b,q_j,2}$ in L_5	
$[\![A^{i,b,q_j,1}, A^{i,b,q_j,3}, D]\!]$	if M does not move on symbol b and state q_j its head $A^{i,b,q_j,1}$ in L_4, $A^{i,b,q_j,3}$ in L_6	
$[\![A^{i,b,q_j,2}, A^{i,b,q_j,3}, H^i]\!]$	if M moves on symbol b and state q_j its head $A^{i,b,q_j,2}$ in L_5, $A^{i,b,q_j,3}$ in L_6	
$[\![A^{i,0,q_j,3}, A^{i,0,q_j,4}, R^i]\!]$	if M on symbol 0 and in state q_j changes the symbol $A^{i,b,q_j,3}$ in L_6, $A^{i,b,q_j,4}$ in L_7	
$[\![A^{i,1,q_j,3}, A^{i,1,q_j,4}, R^i]\!]$	if M on symbol 1 and in state q_j changes the symbol $A^{i,b,q_j,3}$ in L_6, $A^{i,b,q_j,4}$ in L_7	
$[\![A^{i,b,q_j,3}, A^{i,b,q_j,5}, D]\!]$	if M does not change on symbol b and state q_j the value of the symbol $A^{i,b,q_j,3}$ in L_6, $A^{i,b,q_j,5}$ in L_8	
$[\![A^{i,0,q_j,4}, A^{i,0,q_j,5}, S^i]\!]$	if M on symbol 0 and in state q_j changes the symbol $A^{i,b,q_j,4}$ in L_7, $A^{i,b,q_j,5}$ in L_8	
$[\![A^{i,1,q_j,4}, A^{i,1,q_j,5}, S^i]\!]$	if M on symbol 1 and in state q_j changes the symbol $A^{i,b,q_j,4}$ in L_7, $A^{i,b,q_j,5}$ in L_8	
$[\![A^{i,b,q_j,5}, A^{i,b,q_j,6}, Q^j]\!]$	if M changes on symbol b and in state q_j its state to $q_{j'}$ $A^{i,b,q_j,5}$ in L_8, $A^{i,b,q_j,6}$ in L_9	
$[\![A^{i,b,q_j,5}, \Theta^1, D]\!]$	if M does not change on symbol b and state q_j its state $A^{i,b,q_j,5}$ in L_8, Θ^1 in L_{9+1}	
$[\![A^{i,b,q_j,6}, \Theta^1, Q^j]\!]$	if M changes its state on symbol b and in state q_j $A^{i,b,q_j,6}$ in L_9, Θ^1 in L_{9+1}	

Figure 1: The triples among the subsets of vertices in G

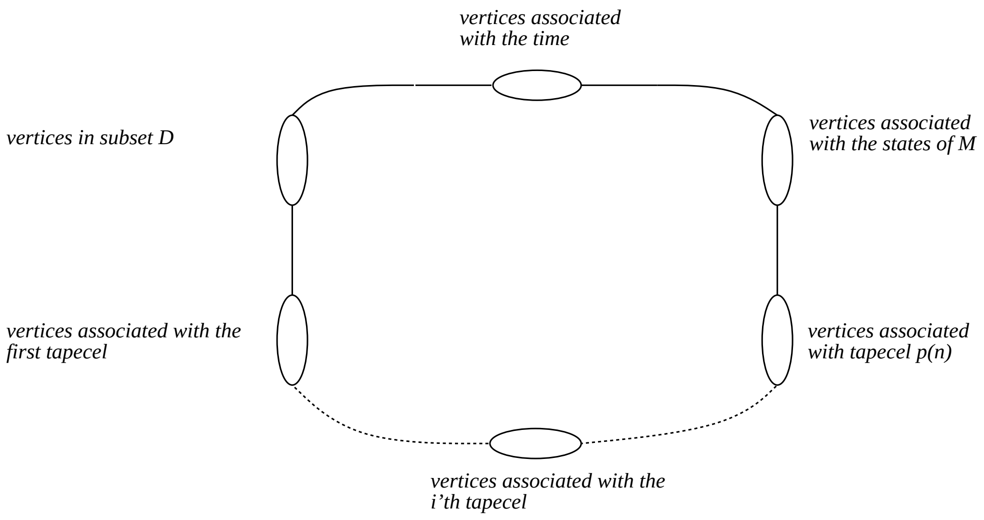

Figure 2: The order of traversing the subsets of normal vertices in G

Proof
This property can be checked using Figure 1. □

3.2 Tours in G

We have now defined G. In this section we associate with a configuration of M and a timestamp a tour in G. We will describe in which order $T_{\langle c,t\rangle}$ traverses the subsets of G by describing in which order the pairs S and $\Delta(S)$ are traversed by $T_{\langle c,t\rangle}$. Here S is a subset of normal or guide-vertices and $\Delta(S)$ is its associated dummy subset. Tour $T_{\langle c,t\rangle}$ alwawys first traverses the vertices in S and then in $\Delta(S)$. Suppose that $\Delta(S)$ contains the vertices $\Delta_1(S)$ and $\Delta_2(S)$. Then $T_{\langle c,t\rangle}$ first visits $\Delta_1(S)$ and then $\Delta_2(S)$.

The normal-vertices in G are used to represent configurations of M together with a timestamp. Let c be a configuration of M and t a timestamp, $1 \leq t \leq 2^{p(|x|)}$. Let $1 \leq i' \leq p(|x|)$ and $1 \leq i \leq 9 + 5p(|x|)$. We first show how tour $T_{\langle c,t\rangle}$ traverses the normal vertices. Tour $T_{\langle c,t\rangle}$ traverses all subsets $V_{i'}$ of normal vertices in order. So first the vertices in the subset belonging to tapecel 1 are traversed , then all vertices in the subset belonging to tapecel 2 and so on, until all vertices in the subset belonging to tapecel $p(|x|)$ have been traversed. Figure 2 shows how the subsets $V_{i'}$ are traversed by $T_{\langle c,t\rangle}$ (counter clockwise). We assume that the subsets in $V_{i'}$ are traversed in the following order. First the pair of subsets $H^{i'}, \Delta(H^{i'})$ is traversed, then the pair of subsets $R^{i'}, \Delta(R^{i'})$ and then the pair of subsets $S^{i'}, \Delta(S^{i'})$. After traversing all subsets $V_{i'}$, tour $T_{\langle c,t\rangle}$ traverses the pair of subsets $Q^j, \Delta(Q^j)$, $q_j \in Q$, in order of index j. To traverse the subsets of vertices associated with the bits of the time bitstring, we start traversing the subsets associated with the least significant bit and we end by traversing the subsets associated with the most significant bit. For each bit i, we first traverse the pair of subsets $T^{i'}, \Delta(T^{i'})$ and then the pair of subsets $U^{i'}, \Delta(U^{i'})$.

The order in which the vertices in a subset $V_{i'}$ are traversed indicates information about which tapecel the head of M scans and the contents of each tapecel of the worktape of M. The fact that the head of M is on tapecel i' is encoded in a tour by traversing the vertices in $H^{i'}$ in order $h_1^{i'}, h_3^{i'}, h_2^{i'}, h_4^{i'}$. Otherwise, the vertices are traversed in order $h_1^{i'}, h_2^{i'}, h_3^{i'}, h_4^{i'}$. If tapecel i' contains symbol 0, this is encoded by traversing the vertices in $R^{i'}$ and $S^{i'}$ in order $r_1^{i'}, r_3^{i'}, r_2^{i'}, r_4^{i'}$ and $s_1^{i'}, s_2^{i'}, s_3^{i'}, s_4^{i'}$. If tapecel i' contains symbol 1, this is encoded by traversing the vertices in $R^{i'}$ and $S^{i'}$ in order $r_1^{i'}, r_2^{i'}, r_3^{i'}, r_4^{i'}$ and $s_1^{i'}, s_3^{i'}, s_2^{i'}, s_4^{i'}$. The state of M is represented by the order in which the vertices of the subsets Q^j are traversed. If M is in state q_j, the vertices in subset Q^j are traversed in order $q_1^j, q_3^j, q_2^j, q_4^j$. Otherwise the vertices are traversed in order $q_1^j, q_2^j, q_3^j, q_4^j$.

We also encode a value of the time bitstring in tours of G. If bit i' has value 0 the vertices in $T^{i'}$ and $U^{i'}$ are traversed in order $t_1^{i'}, t_3^{i'}, t_2^{i'}, t_4^{i'}$ and $u_1^{i'}, u_2^{i'}, u_3^{i'}, u_4^{i'}$. If bit i' has value 1 the vertices in $T^{i'}$ and $U^{i'}$ are traversed in order $t_1^{i'}, t_2^{i'}, t_3^{i'}, t_4^{i'}$ and $u_1^{i'}, u_3^{i'}, u_2^{i'}, u_4^{i'}$.

The pairs of subsets of guide-vertices and their associated dummy subset are traversed in the following order. Let Ξ be a subset of guide-vertices. All pairs of subsets $\Xi, \Delta(\Xi)$, where Ξ is in layer L_i, are traversed before the pairs of subsets $\Psi, \Delta(\Psi)$, where Ψ belongs to layer L_{i+1}, $1 \leq i \leq 9 + 5p(|x|) - 1$. In L_2, the pairs are traversed in the order of their index. So the subsets $A^{i'}, \Delta(A^{i'})$ are traversed before the subsets $A^{i'+1}, \Delta(A^{i'+1})$, $1 \leq i' \leq p(|x|) - 1$. In L_3, the pairs $A^{i',b}\Delta(A^{i',b})$ are traversed in lexicographical order on $\langle i', b\rangle$. In L_k, $4 \leq k \leq 9$, the pairs $A^{i',b,q_j,k}\Delta(A^{i',b,q_j,k})$ are traversed in lexicographical order on $\langle i', b, j\rangle$. For each subset $\Xi \neq H$ of guide-vertices, the vertices are traversed in order

$$\delta\xi_1, \delta\xi_2, \xi_0, \xi_1, \xi_2$$

The vertices of subset H are traversed in order

$$\delta\eta_1, \eta_1, \delta\eta_2, \eta_0, \eta_2$$

3.2.1 Weights in G

The graph G is weighted and complete. This means that there exists a funcion w that assigns to each edge in G an integer value. Weights of the edges in G are denoted by 2-tuples. The tuple-number $\langle x, y\rangle$ indicates the value $x\mathcal{R}+y$, where $\mathcal{R}$ is a large constant. The value of the first component of a weight is sometimes denoted by a quinary string of $p(|x|)$ symbols of the set $\{0,1,2,3,4\}$. The value $val(\sigma)$ of quinary string $\sigma = \sigma_{p(|x|)}\sigma_{p(|x|)-1}\sigma_{p(|x|)-2}\ldots\sigma_0$ is defined as

$$val(\sigma) = \sum_{i=0}^{p(|x|)} \sigma_i 5^i$$

To describe which weights are assigned to which edges, we use guide subsets Ξ and Ψ, normal subset X and dummy subset $\Delta(S)$. Let $\Xi = \{\delta\xi_1, \delta\xi_2, \xi_0, \xi_1, \xi_2\}$, $\Psi = \{\delta\psi_1, \delta\psi_2, \psi_0, \psi_1, \psi_2\}$, $X = \{x_1, x_2, x_3, x_4\}$ and $\Delta(S) = \{\Delta_1(S), \Delta_2(S)\}$. Let $1 \leq i \leq 9+5p(|x|)$, and let $1 \leq i' \leq p(|x|)$. Figure 3 shows all edges between vertices in $\Xi \cup \Psi \cup X$ with a weight smaller than ∞. The weights to the edges of G are defined as follows.

1. The edges (x_1, x_2), (x_2, x_3), (x_3, x_4), (x_1, x_3) and (x_2, x_4) have weight $\langle 0, 1\rangle$, unless $X = U^{i'}$. Then edge $(u_1^{i'}, u_3^{i'})$ has weight $\langle 0^{p(|x|)-i'}20^{i'-1}, 0\rangle$.

2. The edges $(\delta\xi_1, \delta\xi_2)$, $(\delta\xi_2, \xi_0)$, $(\delta\xi_1, \xi_1)$, $(\xi_1, \delta\xi_2)$, (ξ_0, ξ_1) and (ξ_1, ξ_2) have weight $\langle 0, 1\rangle$.

3. Edge $(\Delta_1(S), \Delta_2)$ has weight $\langle 0, 1\rangle$.

4. Edge $(x_4, \Delta_1(X))$ has weight $\langle 0, 1\rangle$. Edge $(\xi 2, \Delta_1(\Xi))$ has weight $\langle 0, 1\rangle$. Suppose that after subset $\Delta(S)$, tour T_0 traverses subset $\tilde{S}$. If $\tilde{S}$ is a normal subset, edge $(\Delta_2(S), \tilde{s}_1)$ has weight $\langle 0, 1\rangle$. If $\tilde{S}$ is a guide subset, edge $(\Delta_2(S), \delta\tilde{s}_1)$ has weight $\langle 0, 1\rangle$.

5. Suppose that Ξ belongs to layer L_i. Then the second component of the weight of (ξ_0, ξ_2) is equal to $(11 + 5p(|x|) - i)9$. If $1 \leq i \leq 9 + p(|x|)$ or if $\Xi = \Lambda^{i',1}$, the first component of the weight of (ξ_0, ξ_2) is equal to 0. If $\Xi = \Lambda^{i',2}$, then the first component of the weight of (ξ_0, ξ_2) is equal to $0^{p(|x|)-i'}20^{i'-1}$. If $\Xi = M^{i',1}$ or $\Xi = M^{i',2}$, then the first component of the weight of (ξ_0, ξ_2) is equal to $0^{p(|x|)-(i'+1)}20^{i'}$.

6. To enforce sequences of improving $2Opt$-changes, assign to edge (ξ_0, x_2) weight $w(\xi_0, \xi_2) - \langle 0, 1\rangle$, unless $\Xi = \Lambda^{i',1}$. Then assign to edge (ξ_0, x_2) weight $w(\xi_0, \xi_2) - \langle 0, 1\rangle + \langle 0^{p(|x|)-i'}20^{i'-1}, 0\rangle$.

   If Ξ is in layer L_k, $4 \leq k \leq 9$, in layer $L_{9+p(|x|)+2i'-1}$ or in layer $L_{9+3p(|x|)+2i'}$, then assign to

edge	weight
(ξ_0, x_3)	$w(\xi_0, x_2) - \langle 0, 1\rangle$
(ξ_0, ψ_1)	$w(\xi_0, x_2) - \langle 0, 2\rangle$
$(\psi_1, \delta\xi_2)$	$w(\xi_0, x_2) - \langle 0, 3\rangle$
$(\delta\xi_2, \delta\psi_2)$	$w(\xi_0, x_2) - \langle 0, 4\rangle$
$(\delta\psi_2, \xi_1)$	$w(\xi_0, x_2) - \langle 0, 5\rangle$
$(\delta\psi_2, x_1)$	$w(\xi_0, x_2) - \langle 0, 6\rangle$
(ψ_2, x_1)	$w(\xi_0, x_2) - \langle 0, 7\rangle$
(ξ_2, x_1)	$\langle 0, 1\rangle$
(ψ_0, x_3)	$\langle 0, 1\rangle$

   Otherwise, assign to

edge	weight
(ξ_0, ψ_1)	$w(\xi_0, x_2) - \langle 0, 2\rangle$
$(\psi_1, \delta\xi_2)$	$w(\xi_0, x_2) - \langle 0, 3\rangle$
$(\delta\xi_2, \delta\psi_2)$	$w(\xi_0, x_2) - \langle 0, 4\rangle$
$(\delta\psi_2, \xi_1)$	$w(\xi_0, x_2) - \langle 0, 5\rangle$
$(\delta\psi_2, x_1)$	$w(\xi_0, x_2) - \langle 0, 6\rangle$
(ψ_2, x_1)	$w(\xi_0, x_2) - \langle 0, 7\rangle$
(ξ_2, x_1)	$\langle 0, 1\rangle$
(ψ_0, x_2)	$\langle 0, 1\rangle$

7. All edges not mentioned so far have a huge weight, which we will denote with ∞. Such edges can never exist on a tour reachable from T_0, since all edges on T_0 have a weight smaller than ∞.

Consider set Ξ. If the edges (ξ_0, ξ_1), (ξ_1, ξ_2), $(\delta\xi_2, \xi_0)$ and $(\delta\xi_1, \delta\xi_2)$ are on a tour, we say that the vertices in subset Ξ are traversed in their *restorder*. If the edges (ξ_0, ξ_2), $(\delta\xi_1, \xi_1)$, $(\delta\xi_2, \xi_0)$ and $(\xi_1, \delta\xi_2)$ are on a tour, we say that the restorder of the vertices in subset Ξ is disturbed.

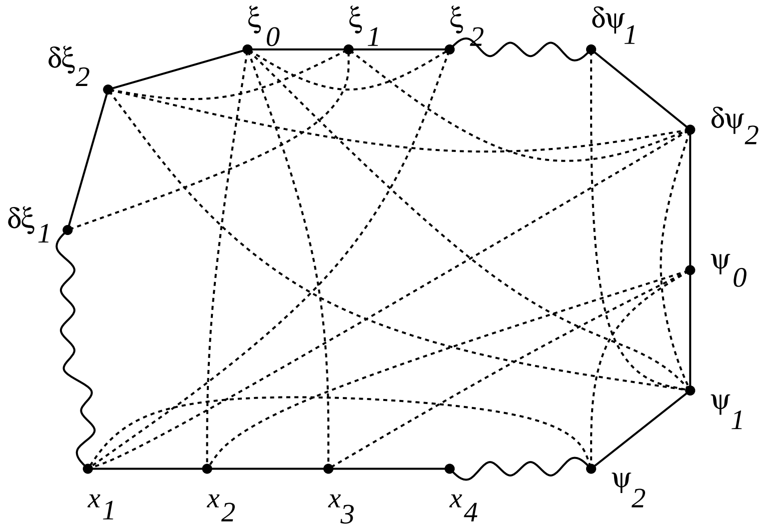

Figure 3: Edges that have a weight smaller than ∞

Consider the sets Ξ, Ψ and X, such that $[\![\Xi, \Psi, X]\!]$ is a triple in Figure 1. Figure 3 shows the edges between vertices in $\Xi \cup \Psi \cup X$ with a weight smaller than ∞.

We have one important property on the weights of the edges in G.

Property 3 *Let Ξ, Ψ be two different sets of guide-vertices. Let X be a set of normal vertices. All edges between X and Ψ or all edges between Ξ and Ψ have weight ∞ if and only if $[\![\Xi, \Psi, X]\!]$ is not a triple in Figure 1.*

3.3 Augmenting paths in $G_{TSP_{2Opt}}(G)$

3.3.1 Introduction

We first prove which short augmenting paths leave tours in G. These augmenting paths have length 8 or 9. We prove also prove that these are the only possible paths leaving certain tours in G. Then we describe the paths leaving tours $T_{\langle c,t\rangle}$ and finally we prove that there is an augmenting path from $T_{\langle c,t\rangle}$ to $T_{\langle c',t-1\rangle}$ if and only if M can go in one step from c to c'. This fact is used in the proof of our main result.

3.3.2 Facts

Definition 3.1 *Let G be the complete, weighted graph as defined in the previous section. Let T be a tour in G. Let $\Xi = \{\delta\xi_1, \delta\xi_2, \xi_0, \xi_1, \xi_2\}$ be a subset of guide-vertices in G. Assume that*

1. *T traverses all vertices in one subset of normal, guide or dummy-vertices consecutively, i.e. one subset is traversed after the other.*

2. *T traverses all vertices in the guide-subsets in their restorder, except for guide-subset Ξ, which is traversed in disturbed order.*

3. *all edges on T, except (ξ_0, ξ_2) have weight $\langle 0,1\rangle$.*

Then T is called a useful *tour.*

Notice that we do not indicate the exact order in which the vertices of a subset of normal vertices are traversed. We only state that they are traversed consecutively, i.e. one subset is traversed after the other.

Definition 3.2 *Let Ξ, Ψ be subsets of guide vertices and let X be a set of normal vertices, such that $[\![\Xi, \Psi, X]\!]$ is a triple in Figure 1. Suppose that tour $T \in \mathcal{T}$ contains edge (x_1, x_2). Subset X is* active *wrt Ξ, if (ξ_0, x_2) has a weight smaller than ∞. Otherwise, X is* passive *wrt Ξ.*

In the next lemma we prove the existence of certain specified short augmenting paths from one usefu tour to another. The intermediate tours on such a short augmenting path need not to be useful. They satisfy the property that the second component of the weights of all edges, except for one, has value 1.

Lemma 3.1 *Let Ξ, Ψ be subsets of guide vertices and let X be a subset of normal vertices, such that $[\![\Xi, \Psi, X]\!]$ is a triple in Figure 1. Let T be a useful tour in G. Suppose that T traverses all guide-subsets in their restorder, except for Ξ. Suppose that X is active wrt Ξ in T. Then there is a useful tour T' reachable from T with*

1. *The restorder of Ξ is restored and the restorder of Ψ is disturbed on T'.*

2. *If*
$$\Xi \in \{H, A^i, A^{i,b}, \Theta^i | 1 \leq i \leq p(|x|), b \in \{0,1\}\}$$
the length of the augmenting path from T to T' is 8 and the traversal-order of X is not changed.

3. *If*
$$\Xi \in \{A^{i,b,q_j,k}, \Lambda^{i,j'}, M^{i,j'} | 1 \leq i \leq p(|x|), 1 \leq k \leq 6, b \in \{0,1\}, j' \in \{1,2\}\}$$
the length of the augmenting path from T to T' is 9 and the traversal-order of X is changed.

4. *There is at most one augmenting path from T to T' of length 8 or 9.*

Proof The proof of this lemma is given in three parts. In the first two parts we prove the lemma true under the condition that each edge not between two vertices in $X \cup \Xi \cup \Psi$ and not on T has weight ∞. In the first part we assume $\Xi \in \{H, A^i, A^{i,b}, \Theta^i | 1 \leq i \leq p(|x|), b \in \{0,1\}\}$ In the second part we assume $\Xi \in \{A^{i,b,q_j,k}, \Lambda^{i,j'}, M^{i,j'} | 1 \leq i \leq p(|x|), 1 \leq k \leq 6, b \in \{0,1\}, j' \in \{1,2\}\}$. In the third part we show that we can remove the condition.

Part 1

Figure 4 depicts all the $2Opt$-changes. Let $T = T_1$. All edges on T_1 have weight $\langle 0, 1\rangle$, except for the edge (ξ_0, ξ_2). If the condition holds, there are two edges incident with ξ_2, not already on T, with a weight smaller than ∞. These are the edges (ξ_2, x_1) and (ξ_1, ξ_2). Since the edge $(\xi_0, \delta\xi_1)$ has weight ∞, there is exactly one improving $2Opt$-change. This $2Opt$-change deletes (ξ_0, ξ_2) and (x_1, x_2) and inserts the edges (ξ_2, x_1) and (ξ_0, x_2). Let T_2 be the tour in G obtained from T_1 by performing this improving $2Opt$-change.

Consider tour T_2. All edges on T_2 have weight $\langle 0, 1\rangle$, except for the edge (ξ_0, x_2). When the condition holds, there are three edges incident with x_2, not already on T, with a weight smaller than ∞. These are the edges (x_1, x_2), (x_2, x_4) and (x_2, ψ_0). Since the weight of (ξ_0, ξ_2) is larger than the weight of (ξ_0, x_2) and since the weight of

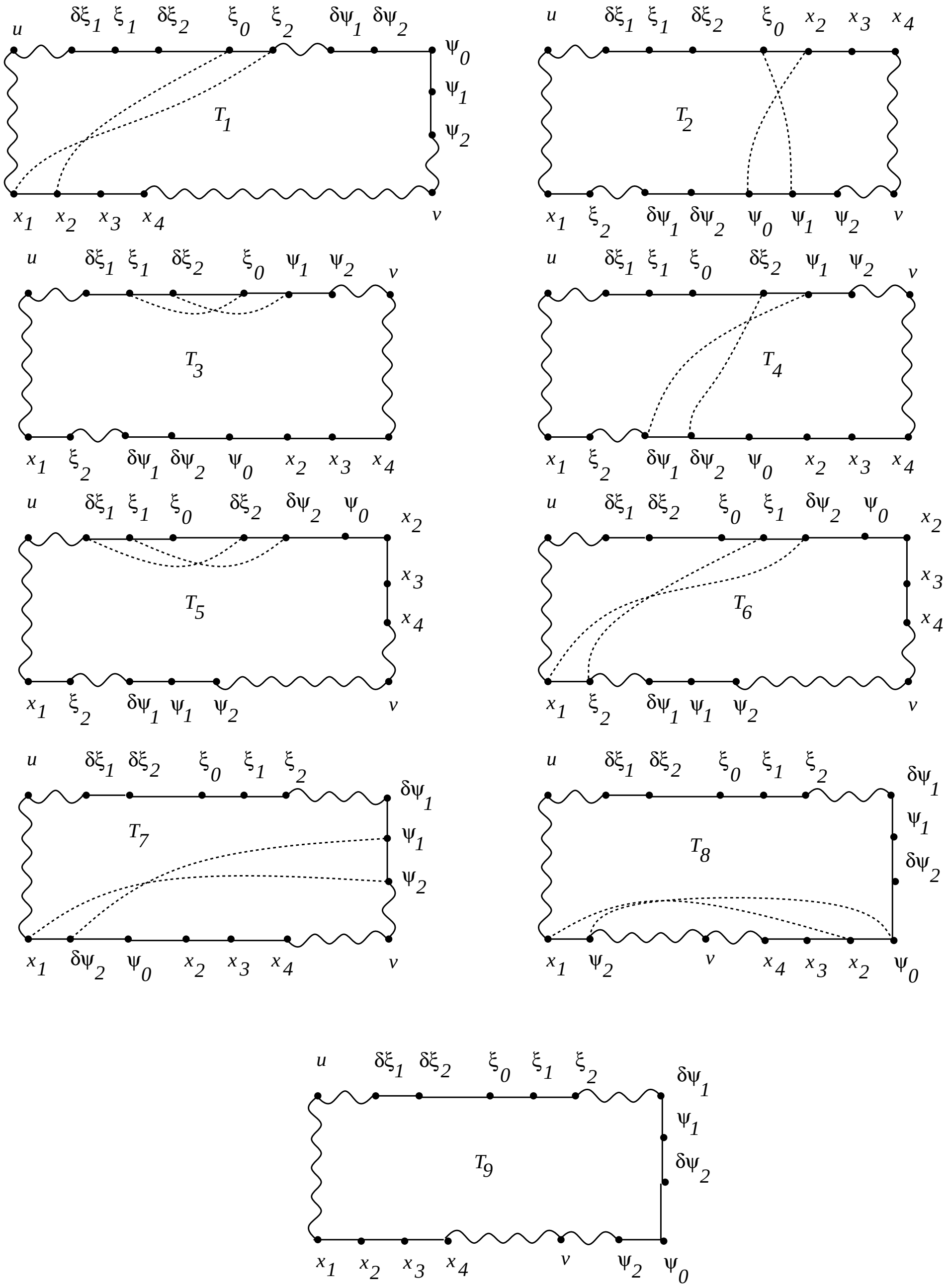

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Figure 4: A sequence of 8 unique $2Opt$-changes

(ξ_0, x_3) is ∞, there is exactly one improving $2Opt$-change. This $2Opt$-change deletes (ξ_0, x_2) and (ψ_0, ψ_1) and inserts the edges (ψ_0, x_2) and (ξ_0, ψ_1). Let T_3 be the tour in G obtained from T_2 by performing this improving $2Opt$-change.

Consider tour T_3. All edges on T_3 have weight $\langle 0, 1\rangle$, except for the edge (ξ_0, ψ_1). When the condition holds, there are three edges incident with ξ_0, not already on T, with a weight smaller than ∞. These are the edges (ξ_0, ξ_1), (ξ_0, ξ_2) and (ξ_0, x_2). Since $\Xi \in \{H, A^i, A^{i,b}, \Theta^i\}$, edge (ξ_0, x_3) has weight ∞. Since the weights of (ξ_0, ξ_2) and (ξ_0, x_2) are larger than the weight of (ξ_0, ψ_1), there is exactly one improving $2Opt$-change. This $2Opt$-change deletes (ξ_0, ψ_1) and $(\xi_1, \delta\xi_2)$ and inserts the edges (ξ_0, ξ_1) and $(\psi_1, \delta\xi_2)$. Let T_4 be the tour in G obtained from T_3 by performing this improving $2Opt$-change.

Consider tour T_4. All edges on T_4 have weight $\langle 0, 1\rangle$, except for the edge $(\psi_1, \delta\xi_2)$. When the condition holds, there are three edges incident with ψ_1, not already on T, with a weight smaller than ∞. These are the edges (ψ_0, ψ_1), $(\delta\psi_1, \psi_1)$ and $(\psi_1, \delta\psi_2)$. Since the edges $(\delta\xi_2, x_2)$ and $(\delta\xi_2, \psi_0)$ have weight ∞, there is exactly one improving $2Opt$-change. This $2Opt$-change deletes $(\delta\xi_2, \psi_1)$ and $(\delta\psi_1, \delta\psi_2)$ and inserts the edges $(\delta\xi_2, \delta\psi_2)$ and $(\psi_1, \delta\psi_1)$. Let T_5 be the tour in G obtained from T_4 by performing this improving $2Opt$-change.

Consider tour T_5. All edges on T_5 have weight $\langle 0, 1\rangle$, except for the edge $(\delta\xi_2, \delta\psi_2)$. When the condition holds, there are three edges incident with $\delta\xi_2$, not already on T, with a weight smaller than ∞. These are the edges $(\delta\xi_1, \delta\xi_2)$, $(\xi_1, \delta\xi_2)$ and $(\delta\xi_2, \psi_1)$. Since the weight of $(\delta\xi_2, \psi_1)$ is larger than the weight of $(\delta\xi_2, \delta\psi_2)$ and $(\xi_0, \delta\psi_2)$ has weight ∞, there is exactly one improving $2Opt$-change. This $2Opt$-change deletes $(\delta\xi_2, \delta\psi_2)$ and (ξ_0, ξ_1) and inserts the edges $(\delta\xi_1, \delta\xi_2)$ and $(\delta\psi_2, \xi_1)$. Let T_6 be the tour in G obtained from T_5 by performing this improving $2Opt$-change.

Consider tour T_6. All edges on T_6 have weight $\langle 0, 1\rangle$, except for the edge $(\xi_1, \delta\psi_2)$. When the condition holds, there are three edges incident with ξ_1, not already on T, with a weight smaller than ∞. These are the edges (ξ_1, ξ_2), $(\xi_1, \delta\xi_1)$ and $(\delta\xi_2, \xi_1)$. Since the weight of $(\delta\xi_2, \delta\psi_2)$ is larger than the weight of $(\xi_1, \delta\psi_2)$ and the edge $(\xi_0, \delta\psi_2)$ has weight ∞, there is exactly one improving $2Opt$-change. This $2Opt$-change deletes $(\xi_1, \delta\psi_2)$ and (x_1, ξ_2) and inserts the edges $(\delta\psi_2, x_1)$ and (ξ_2, ξ_1). Let T_7 be the tour in G obtained from T_6 by performing this improving $2Opt$-change.

Consider tour T_7. All edges on T_7 have weight $\langle 0, 1\rangle$, except for the edge $(x_1, \delta\psi_2)$. When the condition holds, there are four edges incident with x_1, not already on T, with a weight smaller than ∞. These are the edges (x_1, x_2), (x_1, x_3), (x_1, ξ_2) and (x_1, ψ_2). Since the weight of $(\delta\psi_2, \xi_1)$ is larger than the weight of $(x_1, \delta\psi_2)$ and the edges $(\delta\psi_2, x_3)$ and $(\delta\psi_2, x_4)$ have weight ∞, there is exactly one improving $2Opt$-change. This $2Opt$-change deletes $(x_1, \delta\psi_2)$ and (ψ_1, ψ_2) and inserts the edges (x_1, ψ_2) and $(\delta\psi_2, \psi_1)$. Let T_8 be the tour in G obtained from T_7 by performing this improving $2Opt$-change.

Consider tour T_8. All edges on T_8 have weight $\langle 0, 1\rangle$, except for the edge (x_1, ψ_2). When the condition holds, there are two edges incident with ψ_2, not already on T, with a weight smaller than ∞. These are the edges (ψ_1, ψ_2) and (ψ_0, ψ_2). Since the weight of $(\delta\psi_2, x_1)$ is larger than the weight of (x_1, ψ_2), there is exactly one improving $2Opt$-change. This $2Opt$-change deletes (x_1, ψ_2) and (ψ_0, x_2) and inserts the edges (x_1, x_2) and $(\delta\psi_0, \delta\psi_2)$. Let T_9 be the tour in G obtained from T_8 by performing this improving $2Opt$-change.

Part 2

The proof of this second part is analogous to the proof of the first part. We leave it to the reader.

Part 3

We will prove that we can remove the condition for the first case. We leave the second case for the reader. Consider the tours $T_1, \ldots, T_9$ as defined in the first part of this proof. On each tour T_i, $1 \leq i \leq 9$, there is exactly one edge (χ_0, χ_2) with a weight not equal to $\langle 0, 1 \rangle$. On the tours T_2, T_7 and T_8, the edge (χ_0, χ_2) equals respectively (ξ_0, x_2), $(x_1, \delta\psi_2)$ and (x_1, ψ_2). Suppose that for any of these tours, if we remove the condition, there exists an improving *2Opt*-change that deletes the edges (χ_0, χ_2) and (y, z), y and z vertices in G. Then there are several possibilities.

1. Vertices $y, z \in X$ or $y, z \in \Psi$. This contradicts the fact that the lemma with the condition is true.

2. Vertices $y, z \in Y$, where Y is any subset of normal vertices, $X \neq Y$. This case is not possible, since all edges between a vertex in X and a vertex in Y have weight ∞.

3. Vertices $y, z \in \Psi'$, where Ψ' is any subset of guide-vertices, $\Psi \neq \Psi'$. Then, according to Property 3, $[\![\Xi, \Psi', X]\!]$ is a triple in Figure 1. This contradicts Property 1.

On tours T_3, T_4, T_5 and T_6, (χ_0, χ_2) equals respectively (ξ_0, ψ_1), $(\delta\xi_2, \psi_1)$, $(\delta\xi_2, \delta\psi_2)$ or $(\xi_1, \delta\psi_2)$. Suppose that for any of these tours, if we remove the condition, there exists an improving *2Opt*-change that deletes the edges (χ_0, χ_2) and (y, z), y and z vertices in G. Then there are several possibilities.

1. Vertices $y, z \in X$ or $y, z \in \Psi$. This contradicts the fact that the lemma with the condition is true.

2. Vertices $y, z \in Y$. This case is not possible, since in that case $[\![\Xi, \Psi, Y]\!]$ is also a triple in Figure 1, contradicting Property 1.

3. Vertices $y, z \in \Psi'$, $\Psi \neq \Psi'$. Notice that if y, z are guide vertices, then $\Xi = \Lambda^{i,2}$, $\Psi = M^{i,2}$ and $y, z \in M^{i-1,1}$. This is not possible, since Property 2 states that all edges between a set $\Lambda^{i,2}$ and a set $M^{i,2}$ have weight ∞.

We finish this part by considering edge (ξ_0, ξ_2). The vertices y, z can not be guide-vertices, since all edges between ξ_2 and a guide-vertex not in Ξ have weight ∞. Suppose that y and z are normal vertices in subset Y. If there is no triple $[\![\Xi, \Psi', Y]\!]$ in Figure 1, the *2Opt*-change that deletes (ξ_0, ξ_2) and (y, z) introduces an edge of weight ∞. So surely T' is not reachable from T by an augmenting path that first performs this *2Opt*-change. So suppose that there is a triple $[\![\Xi, \Psi', Y]\!]$ in Figure 1. Then there is a tour T'' reachable from T satisfying the properties of the lemma. Specificially, $T'' \neq T$. Let T_2' be the tour obtained from T by the *2Opt*-change that deletes (ξ_0, ξ_2) and (y, z). We depict the situation in Figure 5. We know that there is a unique augmenting paths from T_2 to T', and that all tours on this path, except for T' have exactly one better neighbor. Since $[\![\Xi, \Psi', Y]\!]$ satisfies the conditions given in the lemma, it follows by the same reasoning that there is an augmenting path from T_2' to a tour T'', and that all tours on this path, except for T'' have exactly one better neighbor. Notice that T' can not be obtained from T'' by one improving *2Opt*-change. So also in this case T' is not reachable from T_2'. $\square$

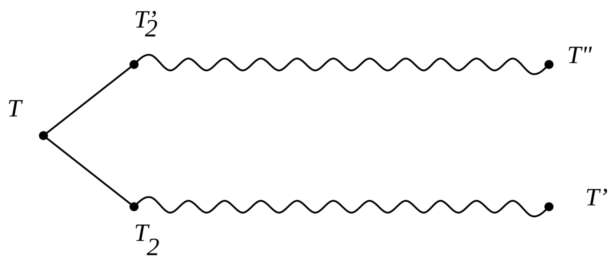

Figure 5: Two tours rechable from T

3.3.3 Paths

In this section we describe the augmenting paths leaving a tour $T_{\langle c,t\rangle}$. We make use of Lemma 3.1.

Let c be a configuration of M, let $1 \leq t \leq 2^{p(|x|)}$ be a timestamp and let $T_{\langle c,t\rangle}$ be the tour in G associated with c and t. Assume w.l.o.g. that the least significant bit in the time bitstring representing value 1 is bit k, $1 \leq k \leq p(|x|)$. Assume furthermore that the head of M scans tapecel i, $1 \leq i \leq p(|x|)$, the symbol in the i'th tapecel has value $b \in \{0,1\}$, and that M is in state $q_j \in Q$.

Determining head, symbol and state

Tour $T_{\langle c,t\rangle}$ traverses all subsets with guide vertices in their restorder, except for subset H. The only subset of normal vertices active wrt H on $T_{\langle c,t\rangle}$ is subset H^i. Notice that $[\![H, A^i, H^i]\!]$ is a triple in Figure 1. According to Lemma 3.1, there is a tour $T_{\langle c,t\rangle,2}$ reachable from $T_{\langle c,t\rangle}$ by a sequence of 8 improving $2Opt$-changes. On this tour the restorder of H is restored and the restorder of A^i is disturbed. The traversal-order of H^i is not changed.

Tour $T_{\langle c,t\rangle,2}$ traverses all subsets with guide vertices in their restorder, except for subset A^i. If $b = 0$, the only active subset wrt to A^i is R^i. If $b = 1$, the only active subset wrt to A^i is S^i. Notice that both triples $[\![A^i, A^{i,0}, R^i]\!]$ and $[\![A^i, A^{i,1}, S^i]\!]$ are triples in Figure 1. According to Lemma 3.1, there is a tour $T_{\langle c,t\rangle,3}$ reachable from $T_{\langle c,t\rangle,2}$ by a sequence of 8 improving $2Opt$-changes. On this tour the restorder of A^i is restored and the restorder of $A^{i,b}$ is disturbed. The traversal-order of neither R^i or S^i is changed.

Tour $T_{\langle c,t\rangle,3}$ traverses all subsets with guide vertices in their restorder, except for subset $A^{i,b}$. The only active subset wrt $A^{i,b}$ on $T_{\langle c,t\rangle,3}$ is Q^j. Notice that $[\![A^{i,b}, A^{i,b,q_j,1}, Q^j]\!]$ is a triple in Figure 1. According to Lemma 3.1, there is a tour $T_{\langle c,t\rangle,4}$ reachable from $T_{\langle c,t\rangle,3}$ by a sequence of 8 improving $2Opt$-changes. On this tour the restorder of $A^{i,b}$ is restored and the restorder of $A^{i,b,q_j,1}$ is disturbed. The traversal-order of Q^j is not changed.

Changing the head

Tour $T_{\langle c,t\rangle,4}$ traverses all subsets with guide vertices in their restorder, except for subset $A^{i,b,q_j,1}$. There is at most one subset of the subsets of normal vertices H^{i-1}, D, H^{i+1} active wrt $A^{i,b,q_j,1}$. Which subset depends on whether M moves its head one place to the left, does not move its head, or moves its head one place to the right, respectively. This is determined by the transition function of M.

T	T'	Disturbed	Active	Newly Disturbed
$T_{\langle c,t\rangle,4}$	$T_{\langle c,t\rangle,5}$	$A^{i,b,q_j,1}$	H^{i-1}	$A^{i,b,q_j,2}$
$T_{\langle c,t\rangle,4}$	$T_{\langle c,t\rangle,5}$	$A^{i,b,q_j,1}$	H^{i+1}	$A^{i,b,q_j,2}$
$T_{\langle c,t\rangle,4}$	$T_{\langle c,t\rangle,6}$	$A^{i,b,q_j,1}$	D	$A^{i,b,q_j,3}$

Figure 6: Augmenting paths leaving $T_{\langle c,t\rangle,4}$

Notice that one of the triples $[\![A^{i,b,q_j,1}, A^{i,b,q_j,2}, H^{i-1}]\!]$, $[\![A^{i,b,q_j,1}, A^{i,b,q_j,2}, H^{i+1}]\!]$ or $[\![A^{i,b,q_j,1}, A^{i,b,q_j,3}, D]\!]$ is a triple in Figure 1. According to Lemma 3.1, in the first and in the last case there is a tour $T_{\langle c,t\rangle,5}$ reachable from $T_{\langle c,t\rangle,4}$ by a sequence of 9 improving $2Opt$-changes. On this tour the restorder of $A^{i,b,q_j,1}$ is restored and the restorder of $A^{i,b,q_j,2}$ is disturbed. The traversal-order of the active subset is changed. According to Lemma 3.1, in the second case there is a tour $T_{\langle c,t\rangle,6}$ reachable from $T_{\langle c,t\rangle,4}$ by a sequence of 8 improving $2Opt$-changes. On this tour the restorder of $A^{i,b,q_j,1}$ is restored and the restorder of $A^{i,b,q_j,3}$ is disturbed. The traversal-order of D is not changed. Figure 6 represents the different possibilities for sequences of improving sequences leaving $T_{\langle c,t\rangle,4}$.

Tour $T_{\langle c,t\rangle,5}$ traverses all subsets with guide vertices in their restorder, except for subset $A^{i,b,q_j,2}$. If $T_{\langle c,t\rangle,5}$ is reachable from $T_{\langle c,t\rangle,4}$, M moves its head on symbol b and in state q_j. The only active subset wrt $A^{i,b,q_j,2}$ on $T_{\langle c,t\rangle,5}$ is H^i. Notice that $[\![A^{i,b,q_j,2}, A^{i,b,q_j,3}, H^i]\!]$ is a triple in Figure 1. According to Lemma 3.1, there is a tour $T_{\langle c,t\rangle,6}$ reachable from $T_{\langle c,t\rangle,5}$ by a sequence of 9 improving $2Opt$-changes. On this tour the restorder of $A^{i,b,q_j,2}$ is restored and the restorder of $A^{i,b,q_j,3}$ is disturbed. The traversal-order of H^i is changed.

Changing the symbol

Tour $T_{\langle c,t\rangle,6}$ traverses all subsets with guide vertices in their restorder, except for subset $A^{i,b,q_j,3}$. There is at most one subset of the subsets of normal vertices R^i, S^i, D active wrt $A^{i,b,q_j,3}$. Which subset depends on whether M changes its symbol respectively from 0 to 1, from 1 to 0 or not at all on symbol b and in state q_j. Notice that one of the triples $[\![A^{i,b,q_j,3}, A^{i,b,q_j,4}, R^i]\!]$, $[\![A^{i,b,q_j,3}, A^{i,b,q_j,4}, S^i]\!]$ or $[\![A^{i,b,q_j,3}, A^{i,b,q_j,5}, D]\!]$ is a triple in Figure 1. According to Lemma 3.1, in the first two cases there is a tour $T_{\langle c,t\rangle,7}$ reachable from $T_{\langle c,t\rangle,6}$ by a sequence of 9 improving $2Opt$-changes. On this tour the restorder of $A^{i,b,q_j,3}$ is restored and the restorder of $A^{i,b,q_j,4}$ is disturbed. The traversal-order of the active subset is changed. According to Lemma 3.1, in the last case there is a tour $T_{\langle c,t\rangle,8}$ reachable from $T_{\langle c,t\rangle,6}$ by a sequence of 8 improving $2Opt$-changes. On this tour the restorder of $A^{i,b,q_j,3}$ is restored and the restorder of $A^{i,b,q_j,5}$ is disturbed. The traversal-order of D is not changed. Figure 7 represents the different possibilities for sequences of improving sequences leaving $T_{\langle c,t\rangle,6}$.

Tour $T_{\langle c,t\rangle,7}$ traverses all subsets with guide vertices in their restorder, except for subset $A^{i,b,q_j,4}$. If $T_{\langle c,t\rangle,7}$ is reachable from $T_{\langle c,t\rangle,6}$, M changes the symbol on symbol b and in state q_j. If M changes its symbol from 1 to 0, the only active

T	T'	Disturbed	Active	Newly Disturbed
$T_{\langle c,t\rangle,6}$	$T_{\langle c,t\rangle,7}$	$A^{i,b,q_j,3}$	R^i	$A^{i,b,q_j,4}$
$T_{\langle c,t\rangle,6}$	$T_{\langle c,t\rangle,7}$	$A^{i,b,q_j,3}$	S^i	$A^{i,b,q_j,4}$
$T_{\langle c,t\rangle,6}$	$T_{\langle c,t\rangle,8}$	$A^{i,b,q_j,3}$	D	$A^{i,b,q_j,5}$

Figure 7: Augmenting paths leaving $T_{\langle c,t\rangle,6}$

T	T'	Disturbed	Active	Newly Disturbed
$T_{\langle c,t\rangle,8}$	$T_{\langle c,t\rangle,9}$	$A^{i,b,q_j,5}$	Q^j	$A^{i,b,q_j,6}$
$T_{\langle c,t\rangle,8}$	$T_{\langle c,t\rangle,9+1}$	$A^{i,b,q_j,5}$	D	Θ^1

Figure 8: Augmenting paths leaving $T_{\langle c,t\rangle,8}$

subset wrt $A^{i,b,q_j,4}$ on $T_{\langle c,t\rangle,7}$ is S^i. If M changes its symbol from 0 to 1, the only active subset wrt $A^{i,b,q_j,4}$ on $T_{\langle c,t\rangle,7}$ is R^i. Notice that $[\![A^{i,b,q_j,4}, A^{i,b,q_j,5}, S^i]\!]$ and $[\![A^{i,b,q_j,4}, A^{i,b,q_j,5}, R^i]\!]$ are triples in Figure 1. According to Lemma 3.1, there is a tour $T_{\langle c,t\rangle,8}$ reachable from $T_{\langle c,t\rangle,7}$ by a sequence of 9 improving $2Opt$-changes. On this tour the restorder of $A^{i,b,q_j,4}$ is restored and the restorder of $A^{i,b,q_j,5}$ is disturbed. The traversal-order of the active subset is changed.

Changing the state

Tour $T_{\langle c,t\rangle,8}$ traverses all subsets with guide vertices in their restorder, except for subset $A^{i,b,q_j,5}$. There is at most one subset of the subsets of normal vertices $Q^{j'}, D$ active wrt $A^{i,b,q_j,5}$. Which subset depends on whether M changes its state or not on symbol b and in state q_j. Notice that one of the triples $[\![A^{i,b,q_j,5}, A^{i,b,q_j,6}, Q^{j'}]\!]$, or $[\![A^{i,b,q_j,5}, \Theta^1, D]\!]$ is a triple in Figure 1. According to Lemma 3.1, in the first case there is a tour $T_{\langle c,t\rangle,9}$ reachable from $T_{\langle c,t\rangle,8}$ by a sequence of 9 improving $2Opt$-changes. On this tour the restorder of $A^{i,b,q_j,5}$ is restored and the restorder of $A^{i,b,q_j,6}$ is disturbed. The traversal-order of $Q^{j'}$ is changed. According to Lemma 3.1, in the last case there is a tour $T_{\langle c,t\rangle,9+1}$ reachable from $T_{\langle c,t\rangle,8}$ by a sequence of 8 improving $2Opt$-changes. On this tour the restorder of $A^{i,b,q_j,5}$ is restored and the restorder of Θ^1 is disturbed. The traversal-order of D is not changed. Figure 8 represents the different possibilities for sequences of improving sequences leaving $T_{\langle c,t\rangle,8}$.

Tour $T_{\langle c,t\rangle,9}$ traverses all subsets with guide vertices in their restorder, except for subset $A^{i,b,q_j,6}$. If $T_{\langle c,t\rangle,9}$ is reachable from $T_{\langle c,t\rangle,8}$, M changes its state on symbol b and in state q_j. The only active subset wrt $A^{i,b,q_j,6}$ on $T_{\langle c,t\rangle,9}$ is Q^j. Notice that $[\![A^{i,b,q_j,6}, \Theta^1, Q^j]\!]$ is a triple in Figure 1. According to Lemma 3.1, there is a tour $T_{\langle c,t\rangle,9+1}$ reachable from $T_{\langle c,t\rangle,9}$ by a sequence of 9 improving $2Opt$-changes. On this tour the restorder of $A^{i,b,q_j,6}$ is restored and the restorder of Θ^1 is disturbed. The traversal-order of Q^j is changed.

Determining least significant 1-bit

Tour $T_{\langle c,t\rangle,9+i}$, $1 \leq i < k$, traverses all subsets with guide vertices in their restorder, except for subset Θ^i. The only active subset wrt Θ^i on $T_{\langle c,t\rangle,9+i}$ is U^i. Notice that $[\![\Theta^i, \Theta^{i+1}, U^i]\!]$ is a triple in Figure 1. According to Lemma 3.1, there is a tour $T_{\langle c,t\rangle,9+i+1}$ reachable from $T_{\langle c,t\rangle,9+i}$ by a sequence of 9 improving $2Opt$-changes. On this tour the restorder of Θ^{i+1} is restored and the restorder of Θ^i is disturbed. The traversal-order of Q^j is changed.

Changing least significant 1-bit

Tour $T_{\langle c,t\rangle,9+k}$ traverses all subsets with guide vertices in their restorder, except for subset Θ^k. The only active subset wrt Θ^k on $T_{\langle c,t\rangle,9+k}$ is T^k. Notice that $[\![\Theta^k, \Lambda^{k,1}, T^k]\!]$ is a triple in Figure 1. According to Lemma 3.1, there is a tour $T_{\langle c,t\rangle,9+p(|x|)+2k-1}$ reachable from $T_{\langle c,t\rangle,9+k}$ by a sequence of 9 improving $2Opt$-changes. On this tour the restorder of Θ^k is restored and the restorder of $\Lambda^{k,1}$ is disturbed. The traversal-order of T^k is unchanged.

Changing lesser significant 0-bits

Tour $T_{\langle c,t\rangle,9+p(|x|)+2k-1}$ traverses all subsets with guide vertices in their restorder, except for subset $\Lambda^{k,1}$. The only active subset wrt $\Lambda^{k,1}$ on $T_{\langle c,t\rangle,9+p(|x|)+2k-1}$ is U^k. Notice that $[\![\Lambda^{k,1}, \Lambda^{k,2}, U^k]\!]$ is a triple in Figure 1. According to Lemma 3.1, there is a tour $T_{\langle c,t\rangle,9+p(|x|)+2k}$ reachable from $T_{\langle c,t\rangle,9+p(|x|)+2k-1}$ by a sequence of 9 improving $2Opt$-changes. On this tour the restorder of $\Lambda^{k,1}$ is restored and the restorder of $\Lambda^{k,2}$ is disturbed. The traversal-order of U^k is changed.

Tour $T_{\langle c,t\rangle,9+p(|x|)+2k}$ traverses all subsets with guide vertices in their restorder, except for subset $\Lambda^{k,2}$. The only active subset wrt $\Lambda^{k,2}$ on $T_{\langle c,t\rangle,9+p(|x|)+2k}$ is T^k. Notice that $[\![\Lambda^{k,2}, M^{k-1,1}, T^k]\!]$ is a triple in Figure 1. According to Lemma 3.1, there is a tour $T_{\langle c,t\rangle,9+5p(|x|)-2k+3}$ reachable from $T_{\langle c,t\rangle,9+p(|x|)+2k}$ by a sequence of 9 improving $2Opt$-changes. On this tour the restorder of $\Lambda^{k,2}$ is restored and the restorder of $M^{k-1,1}$ is disturbed. The traversal-order of T^k is changed.

Tour $T_{\langle c,t\rangle,9+5p(|x|)-2i-1}$, $1 \leq i < k$ traverses all subsets with guide vertices in their restorder, except for subset $M^{i,1}$. The only active subset wrt $M^{i,1}$ on $T_{\langle c,t\rangle,9+5p(|x|)-2i-1}$ is T^i. Notice that $[\![M^{i,1}, M^{i,2}, T^i]\!]$ is a triple in Figure 1. According to Lemma 3.1, there is a tour $T_{\langle c,t\rangle,9+5p(|x|)-2i}$ reachable from $T_{\langle c,t\rangle,9+5p(|x|)-2i-1}$ by a sequence of 9 improving $2Opt$-changes. On this tour the restorder of $M^{i,1}$ is restored and the restorder of $M^{i,2}$ is disturbed. The traversal-order of T^i is changed.

Tour $T_{\langle c,t\rangle,9+5p(|x|)-2i}$, $1 \leq i < k$ traverses all subsets with guide vertices in their restorder, except for subset $M^{i,2}$. The only active subset wrt $M^{i,2}$ on $T_{\langle c,t\rangle,9+5p(|x|)-2i}$ is U^i. Notice that $[\![M^{i,2}, M^{i-1,1}, T^i]\!]$ is a triple in Figure 1. According to Lemma 3.1, there is a tour $T_{\langle c,t\rangle,9+5p(|x|)-2(i-1)-1}$ reachable from $T_{\langle c,t\rangle,9+5p(|x|)-2i}$ by a sequence of 9 improving $2Opt$-changes. On this tour the restorder of $M^{i,2}$ is restored and the restorder of $M^{i-1,1}$ is disturbed. The traversal-order of U^i is changed. We assume that $M^{0,1} = H$.

3.3.4 Simulating a computation step

We are now ready to prove the following theorem.

Theorem 3.1 *Let c, c' be two configurations of M, and let $1 \leq t \leq 2^{p(|x|)}$. There is an augmenting path from $T_{\langle c,t \rangle}$ to $T_{\langle c',t-1 \rangle}$ if and only if M can go in one step from c to c'. Furthermore, if there is an augmenting path from $T_{\langle c,t \rangle}$ to $T_{\langle c',t-1 \rangle}$, then this augmenting path is unique.*

Proof In the previous section we described which augmenting paths can leave $T_{\langle c,t \rangle}$. From this the theorem follows. □

Corollary 1 *There is at most one augmenting path leaving $T_{\langle c,t \rangle}$.*

Proof Again by the description of the augmenting paths in the previous section. □

4 Main Result

With all the precomputation we did in the previoius section we come to our main theorem.

Theorem 4.1 *Polynomial Space Acceptance $\leq_m^p$ RDV(TSP_{2Opt}).*

Proof The many one reduction computes from M and x, the graph G, and the tours $T_{\langle c_0, 2^{p(|x|)} \rangle}$ and $T_{\langle c_f, 0 \rangle}$, where G, $T_{\langle c_0, 2^{p(|x|)} \rangle}$ and $T_{\langle c_f, 0 \rangle}$ are as in the previous section. Suppose that the computation of M on x consists of the configurations $c_0, c_1, \ldots, c_{2^{p(|x|)}}$, where M can go in one step from c_{i-1} to c_i, $1 \leq i \leq 2^{p(|x|)}$.

By Theorem 3.1, there exists a unique path from $T_{\langle c_{i-1}, 2^{p(|x|)}-i+1 \rangle}$ to $T_{\langle c_i, 2^{p(|x|)}-i \rangle}$ By Corrolary 1, this is the only augmenting path leaving $T_{\langle c_{i-1}, 2^{p(|x|)}-i+1 \rangle}$. So there is a unique augmenting path from $T_{\langle c_0, 2^{p(|x|)} \rangle}$ to $T_{\langle c_{2^{p(|x|)}}, 0 \rangle}$, and this is the only augmenting path leaving $T_{\langle c_0, 2^{p(|x|)} \rangle}$. So $c_{2^{p(|x|)}} = c_f$ if and only if $\langle G, T_{\langle c_0, 2^{p(|x|)} \rangle}, T_{\langle c_f, 0 \rangle} \rangle \in$ $RDV(TSP_{2Opt})$. □

Corollary 2 *The standard local optimum version of TSP_{2Opt} is $PSPACEF$-complete.*

Proof As we have shown in the preliminaries, $TSP_{2Opt} \in PSPACEF$. Notice that in the instances constructed by the reduction there is only one locally optimal tour reachable from the initially given tour. And this locally optimal tour indicates whether M accepts x or not. □

5 Conclusions

A very interesting question in the theory of local search problems is whether TSP with the $2Opt$-neighborhood is PLS-complete or not. To show this one must PLS-reduce a PLS-complete problem to TSP_{2Opt}. Krentel, see [K89], proved that $MaxSat$ with the Flip-neighborhood is PLS-complete. In the $MaxSat$ problem Boolean formulas in CNF are given, with on each clause a weight. Goal is to find an assignment that maximizes the sum of the weights of the satisfied clauses. The set

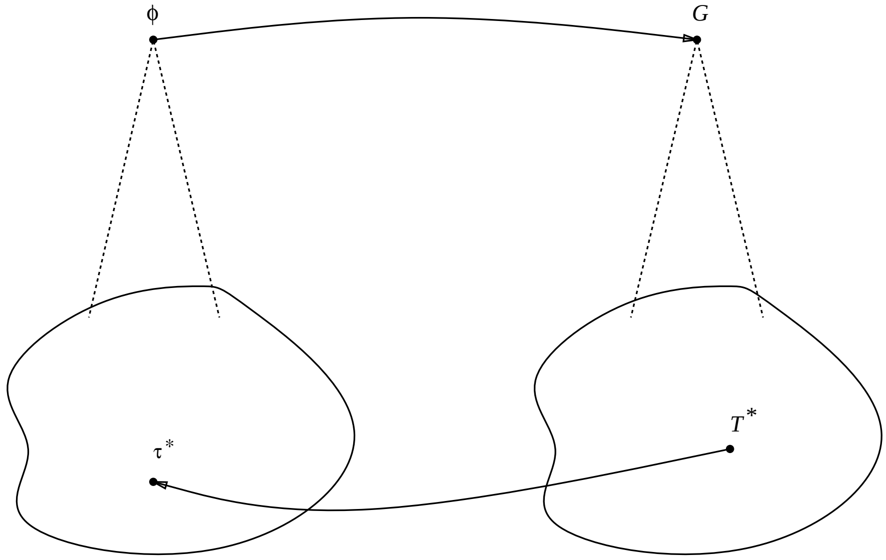

Figure 9: PLS-reduction from $MaxSat$ to TSP_{2Opt}

of neigbors of an assignment τ contains all assignments that assign to all variables but one, the same value as τ.

To PLS-reduce $MaxSat$ to TSP_{2Opt}, one must map an instance ϕ of $MaxSat$ to a complete graph G. Furthermore, each locally optimal tour of G must be mapped in polynomial time to a locally optimal assignment of ϕ. Figure 9 depicts this situation.

With the techniques developped in this paper, we can only partially achieve this goal. We can map ϕ to a complete graph G, such that each locally optimal solution not containing an edge of weight ∞ can be mapped to a locally optimal assignment to ϕ. Just use a local search variant of a polynomial space bounded Turing machine problem, such that for each assignment to ϕ, there is exactly one locally optimal solution reachable. Then simulate a computation of the Turing machine by an augmenting path in $G_{TSP_{2Opt}}(G)$. (Johnson et al. used this technique to prove $CircuitFlip$ PLS-complete, see [JPY88].) Unfortunately, we are neither able to prove that tours containing an edge of weight ∞ can not be locally optimal nor are we able to map a locally optimal tour containing an edge of weight ∞ to a locally optimal assignment.

With the presently available tools, the best result we can get is the $PSPACE$-hardness proof given in this paper. With the techniques presented, we hope to further interrelate TSP_{2Opt} with other PLS-complete problems, with a completeness proof as the ultimate goal.

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