Hybrid Logics.
Characterization, Interpolation and Complexity

Carlos Areces
University of Amsterdam
The Netherlands
carlos@wins.uva.nl

Patrick Blackburn
Universität des Saarlandes
Germany
patrick@coli.uni-sb.de

Maarten Marx
University of Amsterdam
The Netherlands
marx@wins.uva.nl

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Abstract

Hybrid languages are extended modal languages which can refer to (or even quantify over) worlds. The use of strong hybrid languages dates back to at least [Pri67], but recent work (for example [BS98, BT99]) has focussed on a more constrained system called $\mathcal{H}(\downarrow, @)$. The purpose of the present paper is to show in detail that $\mathcal{H}(\downarrow, @)$ is a modally natural system. We begin by studying its expressivity, and provide both model theoretic characterizations (via a restricted notion of Ehrenfeucht-Fraïssé game, and an enriched notion of bisimulation) and a syntactic characterization (in terms of bounded formulas). The key result to emerge is that $\mathcal{H}(\downarrow, @)$ corresponds precisely to the first-order fragment which is invariant for generated submodels. We further establish that $\mathcal{H}(\downarrow, @)$ has (strong) interpolation, and provide failure results in the finite variable fragments. We also show that weak interpolation holds for the sublanguage $\mathcal{H}(@)$, and provide complexity results for $\mathcal{H}(@)$ and other fragments and variants (the full logic being undecidable).

1 Introduction

In their simplest form, hybrid languages are modal languages which use *formulas* to refer to worlds. To build a simple hybrid language, take an ordinary language of propositional modal logic (built over some collection of propositional variables p, q, r, and so on) and add a second type of atomic formula. These new atoms are called *nominals*, and are typically written i, j and k. Both types of atom can be freely combined to form more complex formulas in the usual way; for example,

$$\Diamond(i \wedge p) \wedge \Diamond(i \wedge q) \rightarrow \Diamond(p \wedge q)$$

is a well formed formula. And now for the key idea: insist that *each nominal must be true at exactly one world in any model.* Thus a nominal names a world by being true there and nowhere else. This simple idea gives rise to richer logics (note, for example, that the previous formula is valid: if the antecedent is satisfied at a world m, then the unique world named by i must be accessible from m, and both p and q must be true there) and enables us to define classes of frames that ordinary modal languages cannot (we'll see some examples later).

Once the idea of using "formulas as terms" has been noted (Arthur Prior [Pri67], influenced by unpublished work of C. A. Meredith, seems to have been the first to grasp its potential) the way lies open for further enrichments. The most obvious is to regard nominals not as names but as *variables over individual worlds*, and to add quantifiers. That is, we now allow expressions like

$$\forall x.\Diamond(x \wedge \exists y.\Diamond(y \wedge \Diamond y))$$

to be formed. This sentence is satisfied at a world m if and only if from every world x that is accessible from m we can reach at least one reflexive world y. No formula with this property exists in ordinary modal languages, or even in modal languages enriched with nominals. Unsurprisingly, if we are allowed to quantify over worlds in this manner, it is straightforward to define hybrid languages that offer first-order expressivity over models. Early work on hybrid languages (notably that of Bull [Bul70] and Passy and Tinchev [PT85, PT91]) was largely concerned with such systems.

The idea of binding variables to worlds underlies much current work on hybrid languages, but for many purposes the $\forall$ binder is arguably too strong: languages which employ $\forall$ obscure the *locality* intuition central to Kripke semantics. Fundamental to Kripke semantics is the relativization of semantic evaluation to worlds. That is, to evaluate a modal formula we need to specify some world m (the *current* world) and begin evaluation *there*. The function of the modalities is to scan the worlds accessible from m, the worlds accessible from those worlds, and so on; in short, m is the starting point for step-wise local exploration of the model. Languages which allow variables to be bound to *arbitrary* worlds (some of which may not be reachable from m by chaining through the accessibility relation) don't mesh well with this intuition.

For this reason, recent work on hybrid languages has focussed on a local language called $\mathcal{H}(\downarrow, @)$. This extends the simplest type of hybrid language (propositional modal logic plus nominals) with two new mechanisms, $\downarrow$ and $@$. Now $\downarrow$ does bind variables to worlds, but (unlike $\forall$) it does so in an intrinsically *local* way:

> *The $\downarrow$ binder binds variables to the* current *world. In essence it enables us to create a name for the here-and-now.*

The $@$ operator (which does *not* bind variables) is a natural counterpart to $\downarrow$. Whereas $\downarrow$ "stores" the current world (by binding a variable to it), $@$ enables us to "retrieve" worlds. More precisely, a formula of the form $@_x\varphi$ is an instruction to move to the world labeled by the variable x and evaluate φ there. Previous work on $\mathcal{H}(\downarrow, @)$ has concentrated on relating it to other hybrid languages [BS98], studying it axiomatically [BT99], and developing analytic proof techniques [Bla98, Tza98]. Taken together, this work suggests $\mathcal{H}(\downarrow, @)$ and certain of its sublanguages (notably $\mathcal{H}(@)$) are important systems. The purpose of the present paper is to demonstrate in detail that this impression is justified.

We do so as follows. After defining $\mathcal{H}(\downarrow, @)$ and noting some basic results in Section 2, we turn in Section 3 to the task of characterizing its expressivity. The key result to emerge is this: $\mathcal{H}(\downarrow, @)$ is not merely local, it is the language which *characterizes* locality. More precisely, $\mathcal{H}(\downarrow, @)$ corresponds to the fragment of first-order logic which is invariant under generated submodels. Previous discussions of $\mathcal{H}(\downarrow, @)$ have stressed that it is "modally natural"; our characterization confirms this impression and makes it precise. In Section 4 we discuss the consequences of this characterization for frame-definability, completeness, and tense logic. In Section 5, we show that $\mathcal{H}(\downarrow, @)$ is well-behaved in yet another way: it has the strong (arrow) interpolation property, and the sublanguage $\mathcal{H}(@)$ has weak interpolation

In Section 6 we turn to complexity. It is known that $\mathcal{H}(\downarrow, @)$ has an undecidable satisfaction problem (indeed, this is clear from the characterization result); but it is also known that $\mathcal{H}(@)$ is decidable, that this sublanguage can define more classes of frames than ordinary propositional modal logic can, and that it is proof-theoretically well-behaved. Intriguingly, these properties come at no computational cost: we shall show that $\mathcal{H}(@)$ has a PSPACE-complete satisfaction problem, just as ordinary propositional modal logic does. However we also show that there are limits on how far this result can be generalized: in particular we show that if we start with a language of tense logic, then adding even a single nominal moves the complexity of the local satisfaction problem from PSPACE to EXPTIME. We close the paper with a discussion of a key open problem.

The paper is largely self contained, but as the existing literature on hybrid languages is small and scattered it seems appropriate to give the reader a swift overview of what is available. First, two early papers on $\forall$-based hybrid languages (namely [Bul70] and [PT91]) deserve to be far more widely read: both contain important technical ideas and interesting motivation for the use of hybrid languages. Second, some work has been done on very basic hybrid languages (that is, modal or tense languages enriched with nominals, but with no additional mechanisms such as $\downarrow$, $@$, or $\forall$); the main references here are [GG93] and [Bla93]. Third, while [BS98] and [BT99] are the basic references for $\mathcal{H}(\downarrow, @)$, an interesting discussion of $\downarrow$ as part of a stronger system can be found in [Gor96]. Finally, in addition to the proof theoretical investigations of [Bla98] and [Tza98], there is also [Sel91, Sel97]. Seligman's work deals with stronger ($\forall$-based) systems, but many of the key ideas underlying hybrid deduction (in particular, the deductive significance of $@$) were first explored here, and they are still the only source we know of for discussion of natural deduction techniques for hybrid logics.

2 Preliminaries

In this section we define the syntax and semantics of $\mathcal{H}(\downarrow, @)$ and note some of its basic properties.

Definition 2.1 (Language) *Let $PROP = \{p_1, p_2, \ldots\}$ be a countable set of* propositional variables, *$NOM = \{i_1, i_2, \ldots\}$ a countable set of* nominals, *and $WVAR = \{x_1, x_2, \ldots\}$ a countable set of* world variables. *We assume that PROP, NOM and WVAR are pairwise disjoint. We call $WSYM = NOM \cup WVAR$ the set of* world symbols, *and $ATOM = PROP \cup NOM \cup WVAR$ the set of* atoms. *The* well-formed formulas *of the hybrid language (over the signature $\langle PROP, NOM, WVAR \rangle$) are*

$$\varphi := \top \mid \bot \mid a \mid \neg\varphi \mid \varphi \wedge \varphi' \mid \Box\varphi \mid \ \downarrow x_j.\varphi \mid @_s\varphi$$

where $a \in ATOM$, $x_j \in WVAR$ and $s \in WSYM$. Let $\mathcal{L}$ be the set of all well-formed formulas. For $\varphi \in \mathcal{L}$, $PROP(\varphi)$, $NOM(\varphi)$ and $WVAR(\varphi)$ denote, respectively, the set of propositional variables, nominals, and world variables which occur in φ. $\mathbb{P}(\varphi)$ will denote $PROP(\varphi) \cup NOM(\varphi)$, and will be called the language *of φ.*

In what follows we assume that a signature $\langle \mathrm{PROP}, \mathrm{NOM}, \mathrm{WVAR} \rangle$, and hence $\mathcal{L}$, has been fixed. We usually write p, q and r for propositional variables, i, j and k for nominals, and x, y and z for world variables. As usual, $\Diamond\varphi$ is defined to be $\neg\Box\neg\varphi$

Note that all three types of atomic symbol are *formulas*. Further, note that the above syntax is simply that of ordinary unimodal propositional modal logic extended by the clauses for $\downarrow x_j.\varphi$ and $@_s\varphi$. Finally, the difference between nominals and world variables is simply this: nominals cannot be bound by $\downarrow$, whereas world variables can. In fact, nominals could be dispensed with (it is always possible to make do with free world variables instead) but for some purposes it is useful to have a special kind of world symbol that can't be accidentally bound.

Definition 2.2 *The notions of* free *and* bound *world variable are defined in the manner familiar from first-order logic, with $\downarrow$ as the only binding operator. Similarly, other syntactic notions (such as* substitution, *and of a world symbol t* being substitutable for x in φ*) are defined as the corresponding notions in first-order logic. We use $\varphi[t/s]$ to denote the formula obtained by replacing all free instances of the world symbol t by the world symbol s.*

A sentence *is a formula containing no free world variables. Furthermore, a formula is* pure *if it contains no propositional variables, and* nominal-free *if it contains no nominals.*

Definition 2.3 (Semantics) *A (hybrid)* model *$\mathfrak{M}$ for $\mathcal{L}$ is a triple $\mathfrak{M} = \langle M, R, V\rangle$ such that M is a non-empty set, R a binary relation on M, and $V : PROP \cup NOM \to Pow(M)$ such that for all nominals $i \in NOM$, $V(i)$ is a singleton subset of M. (We use calligraphic letters $\mathfrak{M}$ for models, italic roman M for their domains. When necessary, we use sub or supra-indexes.) We usually call the elements of M worlds (though sometimes we call them times or states), and we call R the* accessibility relation *and V the* valuation. *A* frame *is a pair $\mathfrak{M} = \langle M, R\rangle$; that is, a frame is a model without a valuation.*

An assignment *g for $\mathfrak{M}$ is a mapping $g : WVAR \to M$. Given an assignment g, we define the assignment g^i_m by $g^i_m(x_j) = g(x_j)$ for $j \neq i$ and $g^i_m(x_i) = m$. We say that g^i_m is* an x_i-variant of g.

Let $\mathfrak{M} = \langle M, R, V\rangle$ be a model, $m \in M$, and g an assignment. For any atom a, let $[V, g](a) = \{g(a)\}$ if a is a world variable, and $V(a)$ otherwise. Then the forcing relation *is defined as follows*

$$\begin{array}{lll}
\mathfrak{M}, g, m \Vdash \top & & \\
\mathfrak{M}, g, m \nVdash \bot & & \\
\mathfrak{M}, g, m \Vdash a & \textit{iff} & m \in [V, g](a),\ a \in ATOM \\
\mathfrak{M}, g, m \Vdash \neg\varphi & \textit{iff} & \mathfrak{M}, g, m \nVdash \varphi \\
\mathfrak{M}, g, m \Vdash \varphi \wedge \psi & \textit{iff} & \mathfrak{M}, g, m \Vdash \varphi \textit{ and } \mathfrak{M}, g, m \Vdash \psi \\
\mathfrak{M}, g, m \Vdash \Box\varphi & \textit{iff} & \forall m'(Rmm' \Rightarrow \mathfrak{M}, g, m' \Vdash \varphi) \\
\mathfrak{M}, g, m \Vdash \downarrow x_i.\varphi & \textit{iff} & \mathfrak{M}, g^i_m, m \Vdash \varphi \\
\mathfrak{M}, g, m \Vdash @_s\varphi & \textit{iff} & \mathfrak{M}, g, m' \Vdash \varphi, \textit{ where } [V, g](s) = \{m'\},\ s \in WSYM.
\end{array}$$

When $\mathfrak{M}$ and g are understood by context we will simply write $m \Vdash \varphi$ for $\mathfrak{M}, g, m \Vdash \varphi$. We write $\mathfrak{M}, g \Vdash \varphi$ iff for all $m \in M$, $\mathfrak{M}, g, m \Vdash \varphi$. We write $\mathfrak{M} \models \varphi$ iff for all g, $\mathfrak{M}, g \Vdash \varphi$.

The first six clauses are essentially the standard Kripke forcing relation for propositional modal logic; in fact the only difference is that whereas the standard definition relativizes semantic evaluation to worlds m, we relativize to variable assignments g as well. Note that the clause for atoms covers all three types of symbol (propositional variables, nominals, and world variables) and that given any model $\mathfrak{M}$ and assignment g, any world symbol (whether it is a nominal or a world variable) will be forced at a *unique* world; this is an immediate consequence of the way we defined valuations and assignments. As promised in the introduction, $\downarrow$ binds world variables to the world where evaluation is being performed (the *current world*), and $@_s$ shifts evaluation to the world named by s. Just as in first-order logic, if φ is a *sentence* it is irrelevant which assignment g is used to perform evaluation: $\mathfrak{M}, g, m \Vdash \varphi$ for *some* assignment g iff $\mathfrak{M}, g, m \Vdash \varphi$ for *all* assignments g. Hence for sentences the relativization to assignments of the forcing relation can be dropped, and we simply write $\mathfrak{M}, m \Vdash \varphi$ instead of $(\forall g).\mathfrak{M}, g, m \Vdash \varphi$.

A formula φ is *satisfiable* if there is a model $\mathfrak{M}$, an assignment g on $\mathfrak{M}$, and a world $m \in M$ such that $\mathfrak{M}, g, m \Vdash \varphi$. A formula φ is *valid* if for all models $\mathfrak{M}$, $\mathfrak{M} \models \varphi$. A formula φ is a *local consequence* of a set of formulas T if for some finite subset $\{\varphi_1, \ldots, \varphi_n\}$ of T, $\varphi_1 \wedge \ldots \wedge \varphi_n \to \varphi$ is valid. A formula φ is a *global consequence* of a set of formulas T if for all models $\mathfrak{M}$, $\mathfrak{M} \models \varphi$ only if for all $\psi \in T$, $\mathfrak{M} \models \psi$. We denote local consequence by $T \models \varphi$ and global consequence by $T \models^{glo} \varphi$. As in ordinary propositional modal local, local consequence is strictly stronger than global consequence.

$\mathcal{H}(\downarrow, @)$ offers us considerable expressive power over models. For example we can define the *Until* operator:

$$Until(\varphi, \psi) \;:= \downarrow x.\Diamond \downarrow y.@_x(\Diamond(y \wedge \varphi) \wedge \Box(\Diamond y \to \psi)).$$

Note how this works: we name the current world x, use $\Diamond$ to move to an accessible world, which we name y, and then use $@$ to jump us back to x. We then use the modalities to insist that (1) φ holds at the world named y, and (2) ψ holds at all successors of the current world that precede this y-labeled world.

But there is an obvious (and modally natural) limit to the expressive power $\mathcal{H}(\downarrow, @)$ gives us: any nominal-free sentence is preserved under the formation of *point-generated* (or *rooted*) submodels. That is, if a sentence φ is satisfied at a world m in a model $\mathfrak{M}$, and we form a submodel $\mathfrak{M}_m$ by discarding from $\mathfrak{M}$ all the worlds that are *not* reachable by making a finite (possibly empty) sequence of transitions from m, then $\mathfrak{M}_m$ also satisfies φ at m. (The key point to observe is that in any subformula of φ of the form $@_t\psi$, t must be a world variable bound by some previous occurrence of $\downarrow$. As $\downarrow$ binds to the current world, t is bound to some world in the submodel generated by m, thus φ is unaffected by the restriction to $\mathfrak{M}_m$.) That is, $\mathcal{H}(\downarrow, @)$ is genuinely local: only reachable worlds are relevant to semantic evaluation. In the following section we shall generalize this observation (we have not merely preservation, but *invariance*) and show that it characterizes the expressivity of $\mathcal{H}(\downarrow, @)$.

$\mathcal{H}(\downarrow, @)$ also offers us considerable expressive power with respect to *frames*. Modal logicians like to view modal languages as tools for talking about frames, and they do so via the concept of frame validity:

Definition 2.4 *A formula φ is* valid on a frame $\mathfrak{F} = \langle M, R\rangle$ *if for every valuation V on $\mathfrak{F}$, and every assignment g on $\mathfrak{F}$, and every $m \in M$, $\langle \mathfrak{F}, V\rangle, g, m \Vdash \varphi$. A formula is* valid on a class of frames F *if it is valid on every frame $\mathfrak{F}$ in F. A formula φ* defines a class of frames *if it is valid on precisely the frames in F, and it defines a* property *of frames (for example, transitivity) if it defines the class of frames with that property.*

Many interesting properties are definable using pure, nominal-free, sentences:

$\downarrow x.\Box\neg x$	*Irreflexivity*
$\downarrow x.\Box\Box\neg x$	*Asymmetry*
$\downarrow x.\Box(\Diamond x \to x)$	*Antisymmetry*
$\downarrow x.\Box \downarrow y.@_x\Diamond\Diamond y$	*Density*
$\downarrow x.\Box\Box \downarrow y.@_x\Diamond y$	*Transitivity*
$\downarrow x.\Diamond \downarrow y.@_x(\Box\Box\neg y \wedge \Box \downarrow z.@_y(z \vee \Diamond z))$	*Right-Discreteness*

With the exception of transitivity and density, none of these properties are definable in ordinary modal logic. In Section 4 we shall characterize the classes of frames that pure, nominal-free, sentences can define.

[BT99] provides the following complete axiom system for $\mathcal{H}(\downarrow, @)$:

Definition 2.5 (Axiomatization) *Let φ, ψ be formulas, v a metavariable over world variables and s, t metavariables over world symbols. The hybrid logic $\mathsf{K}[\mathcal{H}(\downarrow, @)]$ is the smallest subset of $\mathcal{L}$ containing all instances of propositional tautologies, all instances of the following axiom schemes and closed under the following deduction rules*

K. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.
N. $\vdash \varphi \Rightarrow \vdash \Box\varphi$.
MP. $\vdash \varphi \to \psi, \vdash \varphi \Rightarrow \vdash \psi$.

Q1. $\downarrow v.(\varphi \to \psi) \to (\varphi \to \downarrow v.\psi)$, *$\varphi$ without free occurrences of v.*
Q2. $\downarrow v.\varphi \to (s \to \varphi[v/s])$, *$s$ substitutable for v in φ.*
Q3. $\downarrow v.(v \to \varphi) \to \downarrow v.\varphi$.
Self Dual$_\downarrow$. $\downarrow v.\varphi \leftrightarrow \neg \downarrow v.\neg\varphi$.
$N_\downarrow$. $\vdash \varphi \Rightarrow \vdash \downarrow v.\varphi$.

$K_@$. $@_s(\varphi \to \psi) \to (@_s\varphi \to @_s\psi)$.
Self Dual$_@$. $@_s\varphi \leftrightarrow \neg @_s\neg\varphi$.
Introduction. $(s \wedge \varphi) \to @_s\varphi$.
Label. $@_s s$.
Nom. $@_s t \to (@_t\varphi \to @_s\varphi)$.
Swap. $@_s t \leftrightarrow @_t s$.
Scope. $@_t @_s\varphi \leftrightarrow @_s\varphi$.
$N_@$. $\vdash \varphi \Rightarrow \vdash @_s\varphi$.

Back. $\Diamond @_s\varphi \to @_s\varphi$.
Bridge. $(\Diamond s \wedge @_s\varphi) \to \Diamond\varphi$.

Paste-0. $\vdash @_s(t \wedge \varphi) \to \psi \Rightarrow \vdash @_s\varphi \to \psi$, $t \in WSYM \setminus WSYM(\varphi, \psi, s)$.
Paste-1. $\vdash @_s\Diamond(t \wedge \varphi) \to \psi \Rightarrow \vdash @_s\Diamond\varphi \to \psi$, $t \in WSYM \setminus WSYM(\varphi, \psi, s)$.

We use this axiomatization to help prove strong interpolation in Section 5. For this purpose, the following derived theorems will be useful (their derivation from the above axiomatization is a simple exercise; see [BT99] (Lemma 4.1) and [BT98] (Lemma 7) for details).

Proposition 2.6 (Derived Theorems and Properties)

i. $K_{\downarrow}$. $\downarrow v.(\varphi \to \psi) \to (\downarrow v.\varphi \to \downarrow v.\psi)$.

ii. $Distr_{@}$. $@_s(\varphi \wedge \psi) \leftrightarrow (@_s\varphi \wedge @_s\psi)$.

iii. Free variables and nominals. If $\vdash \varphi$ *and* i *is a nominal in* φ*, then for some world variable* x *not occurring in* φ*,* $\vdash \downarrow x.\varphi[i/x]$*. If* $\vdash \varphi$ *and* x *is a free variable in* φ*, then for some nominal* i *not occurring in* φ*,* $\vdash \varphi[x/i]$*.*

But we are interested in this axiomatization for another reason. The completeness result proved in [BT99] is very general: not only does this axiomatization generate all valid formulas, but it automatically extends to many stronger logics. In particular, if we add a pure, nominal-free, sentence φ as an additional axiom, the resulting system is strongly complete with respect to the class of frames that φ defines. In Section 4 we shall characterize the completeness results covered by such extensions.

Before starting our investigations, one final remark. We have defined $\mathcal{H}(\downarrow, @)$ to be an extension of unimodal propositional modal logic. But of course, it would have been equally straightforward to extend a multimodal language (that is, a language containing an indexed collection of modalities, each interpreted by a separate relation) or a language of tense logic (that is, propositional modal logic enriched with modalities which scan the converse of the relation R) with nominals, variables, @ and $\downarrow$ in exactly the same way. For the most part we will work with the above version of $\mathcal{H}(\downarrow, @)$ (most results go through for other formulations essentially unchanged) but sometimes it will be interesting to switch to a richer underlying modal language, especially when we discuss computational complexity in Section 6.

3 Characterizing $\mathcal{H}(\downarrow, @)$

In this section we characterize (the first-order language corresponding to) $\mathcal{H}(\downarrow, @)$. We begin by providing a *syntactic* characterization. In particular, we shall first extend the *standard translation ST* of modal logic into first-order logic (cf. [Ben83]) to $\mathcal{H}(\downarrow, @)$. It will be clear that the range of our translation lies in a certain *bounded fragment*, and we shall define a reverse translation HT which maps the bounded fragment back into the the hybrid language. Thus we are free to think either in terms of $\mathcal{H}(\downarrow, @)$ or the corresponding bounded fragment.

But how are these languages characterized *semantically*? It should be clear that $\mathcal{H}(\downarrow, @)$ is a genuine hybrid of modal and first-order ideas (after all, it combines Kripke semantics with the idea of binding variables to worlds) thus there are two obvious ways to proceed. The first is essentially first-order: we could look for a weaker notion of Ehrenfeucht-Fraïssé game. The second is essentially modal: we could try looking for a stronger notion of bisimulation. We shall pursue *both* options. As we shall see, both yield natural notions of equivalence between models, and by relating them (and drawing on our syntactic characterization) we can provide a detailed picture of what $\mathcal{H}(\downarrow, @)$ offers.

3.1 Translations

We focus on two kinds of signature for first-order logic with equality. First we have *modal* signatures (familiar from modal correspondence theory [Ben83]) which consist of one binary predicate R, countably many unary predicates, and no constant symbols. It will be convenient to make the set of first-order variables at our disposal explicit in the signature (just as we did when we defined hybrid signatures in Definition 2.1) thus, a modal signature has the form $\langle\{R\} \cup \text{UN-REL}, \{\}, \text{VAR}\rangle$. A *hybrid* signature is an expansion of the modal signature with countably many constant symbols, thus hybrid signatures have the form $\langle\{R\} \cup \text{UN-REL}, \text{CONS}, \text{VAR}\rangle$. Note that any hybrid model $\mathfrak{M} = \langle M, R, V\rangle$ can be regarded as a first-order model with domain M, for the accessibility relation R can be used to interpret the binary predicate R, unary predicates can be interpreted by the subsets V assigns to propositional variables, and constants (if any) can be interpreted by the worlds that nominals name. So we let the context determine whether we are thinking of first-order or hybrid models, and continue to use the notation $\mathfrak{M} = \langle M, R, V\rangle$.

We first extend the well-known *standard translation* to $\mathcal{H}(\downarrow, @)$. The translation ST from the hybrid language over $\langle\text{PROP}, \text{NOM}, \text{WVAR}\rangle$ into first-order logic over the signature $\langle\{R\} \cup \{P_j \mid p_j \in \text{PROP}\}, \text{NOM}, \text{WVAR} \cup \{x, y\}\rangle$ is defined by mutual recursion between two functions ST_x and ST_y. Recall that $\varphi[x/y]$ means "replace all free instances of x by y".

$$
\begin{array}{lcl|lcl}
ST_x(p_j) & = & P_j(x),\ p_j \in \text{PROP}. & ST_y(p_j) & = & P_j(y),\ p_j \in \text{PROP}. \\
ST_x(i_j) & = & x = i_j,\ i_j \in \text{NOM}. & ST_y(i_j) & = & y = i_j,\ i_j \in \text{NOM}. \\
ST_x(x_j) & = & x = x_j,\ x_j \in \text{WVAR}. & ST_y(x_j) & = & y = x_j,\ x_j \in \text{WVAR}. \\
ST_x(\neg\varphi) & = & \neg ST_x(\varphi). & ST_y(\neg\varphi) & = & \neg ST_y(\varphi). \\
ST_x(\varphi \wedge \psi) & = & ST_x(\varphi) \wedge ST_x(\psi). & ST_y(\varphi \wedge \psi) & = & ST_y(\varphi) \wedge ST_y(\psi). \\
ST_x(\Diamond\varphi) & = & \exists y.(Rxy \wedge ST_y(\varphi)). & ST_y(\Diamond\varphi) & = & \exists x.(Ryx \wedge ST_x(\varphi)). \\
ST_x(\downarrow x_j.\varphi) & = & (ST_x(\varphi))[x_j/x]. & ST_y(\downarrow x_j.\varphi) & = & (ST_y(\varphi))[x_j/y]. \\
ST_x(@_s\varphi) & = & (ST_x(\varphi))[x/s]. & ST_y(@_s\varphi) & = & (ST_y(\varphi))[y/s].
\end{array}
$$

For m an element in the domain of a given model $\mathfrak{M}$ we will often write $ST_m(\varphi)$ as shorthand for $ST_x(\varphi)[m]$. This translation differs from the one given in [BS98]; these authors handle $\downarrow$ and @ as follows:

$$
\begin{array}{lcl}
ST_x(\downarrow x_j.\varphi) & = & \exists x_j.(x = x_j \wedge ST_x(\varphi)). \\
ST_x(@_s\varphi) & = & ST_s(\varphi),\ s \in \text{NOM} \cup \text{VAR}.
\end{array}
$$

Now the [BS98] translation makes the quantificational effect of $\downarrow$ clear, but our translation draws attention to another perspective: in adding $\downarrow$ and @ we have in effect enriched the modal language with an *explicit substitution operator*. Such operators are used in the study of cylindric algebras, and were added to cylindric modal logic in [Ven94].

The link between $\downarrow$ and explicit substitution can be made even more clear if we expand the first-order language with an explicit substitution operator (like s^i_j in the theory of cylindric algebras) and adjust our definition of ST to take advantage of it. We do this as follows. Add the following clause to the grammar generating the first-order language:

if φ is a formula and x, y are variables, then $S^x_y\varphi$ is a formula.

Interpret S^x_y as follows:

$$
\mathfrak{M} \models S^x_y\varphi[g] \Leftrightarrow \begin{cases} \mathfrak{M} \models \varphi[g] & \text{for } x = y \\ \mathfrak{M} \models \varphi[g^x_y] & \text{for } x \neq y. \end{cases}
$$

Clearly $S^x_y\varphi$ and $\varphi[x/y]$. This enrichment of first-order logic can be axiomatized by adding the following axiom schemata to a complete axiomatization of first-order logic with equality:

$$
\begin{array}{lcl}
S^x_x\varphi & \leftrightarrow & \varphi \\
S^x_y\varphi & \leftrightarrow & \exists x.(x = y \wedge \varphi) \quad \text{for } x \neq y.
\end{array}
$$

And now we can give transparent translations of $\downarrow$ and @:

$$\begin{array}{lcl} ST_x(\downarrow x_j.\varphi) & = & S_x^{x_j} ST_x(\varphi) \\ ST_x(@_s\varphi) & = & S_s^x ST_x(\varphi). \end{array}$$

Note that theorems like $\downarrow v.@_v\varphi \leftrightarrow \downarrow v.\varphi$ follow more or less immediately, for $ST_x(\downarrow v.@_v\varphi) = S_x^v S_v^x ST_x(\varphi)$, but $S_x^v S_v^x \varphi \equiv S_x^v S_x^x \varphi = S_x^v \varphi$, which is equivalent to $S_x^v ST_x(\varphi)$. However we shall stick with our original formulation of ST in what follows.

Proposition 3.1 (*ST* Preserves Truth) *Let φ be a hybrid formula, then for all hybrid models $\mathfrak{M}$, $m \in M$ and assignments g, $\mathfrak{M}, g, m \Vdash \varphi$ iff $\mathfrak{M} \models ST_m(\varphi)[g]$.*

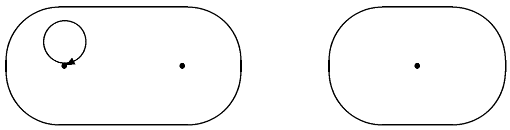
PROOF. A straightforward extension of the induction familiar from ordinary modal logic. The only cases which are new are $ST_x(\downarrow x_j.\varphi)$ and $ST_x(@_s\varphi)$. But $\mathfrak{M}, g, m \Vdash \downarrow x_j.\varphi$, iff $\mathfrak{M}, g_m^{x_j}, m \Vdash \varphi$, by IH iff, $\mathfrak{M} \models ST_m(\varphi)[g_m^{x_j}]$, iff $\mathfrak{M} \models (ST_m(\varphi))[x_j/x]\ \ [g]$. The argument for $ST_x(@_s\varphi)$ is similar. QED

Now for the interesting question: what is the *range* of ST? In fact it belongs to a *bounded* fragment of first-order logic. Given a first-order signature $\langle\{R\} \cup \text{UN-REL}, \text{CONS}, \text{VAR}\rangle$ we define the *bounded* fragment as the set of formulas generated by the following grammar:

$$\varphi = Rtt' \mid P_jt \mid t = t' \mid \neg\varphi \mid \varphi \wedge \varphi' \mid \exists x_i.(Rtx_i \wedge \varphi) \text{ (for } x_i \neq t).$$

where $x_i \in \text{VAR}$ and $t, t' \in \text{VAR} \cup \text{CONS}$.

The side-condition on the generation of existentially quantified formulas is crucial: it prevents sentences like $\exists x.(Rxx \wedge x = x)$ from falling into the fragment. The sentence $\exists x.Rxx$ is probably the simplest example of a first-order sentence which is *not invariant for generated submodels* (or subframes). In fact it is not even *preserved* under the formation of generated submodels, for it is true in the model $\mathfrak{M}_1$ but not in the generated submodel $\mathfrak{M}_2$:



$\mathfrak{M}_1$ $\qquad\qquad$ $\mathfrak{M}_2$

Clearly ST generates formulas in the bounded fragment. Crucially, however, we can also translate the bounded fragment into $\mathcal{H}(\downarrow, @)$. The translation HT from the bounded fragment over $\langle\{R\} \cup \text{UN-REL, CONS, VAR}\rangle$ into the hybrid language over $\langle\text{UN-REL}, \text{CONS}, \text{VAR}\rangle$ is defined as follows:

$$\begin{array}{lcl} HT(Rtt') & = & @_t\Diamond t'. \\ HT(P_jt) & = & @_tp_j. \\ HT(t = t') & = & @_tt'. \\ HT(\neg\varphi) & = & \neg HT(\varphi). \\ HT(\varphi \wedge \psi) & = & HT(\varphi) \wedge HT(\psi). \\ HT(\exists v.(Rtv \wedge \varphi)) & = & @_t\Diamond \downarrow v.HT(\varphi). \end{array}$$

(Here $t, t' \in \text{VAR} \cup \text{CONS}$.)

By construction, $HT(\varphi)$ is a hybrid formula, but furthermore it is a boolean combination of @-formulas (formulas whose main operator is @). So, because @ commutes with all the booleans (recall the Self Dual$_@$ axiom and Proposition 2.6), it is equivalent to a formula whose main operator is @. We can now prove the following strong truth preservation result. Recall that $\mathfrak{M}, g \Vdash \phi$, abbreviates "(for all $m \in M$) $\mathfrak{M}, g, m \Vdash \phi$".

Proposition 3.2 (HT Preserves Truth) *Let φ be a bounded formula. Then for every first-order model $\mathfrak{M}$ and for every assignment g, $\mathfrak{M} \models \varphi[g]$ iff $\mathfrak{M}, g \Vdash HT(\varphi)$.*

PROOF. The proof uses the following fact about formulas whose main operator is @:

$$\text{(there exists an } m\text{)}, \mathfrak{M}, g, m \Vdash @_s\varphi \text{ if and only if } \mathfrak{M}, g \Vdash @_s\varphi.$$

Again there is only one interesting case: $HT(\exists v.(Rtv \wedge \varphi))$. Now $\mathfrak{M} \models \exists v.(Rtv \wedge \varphi)[g]$ iff $\mathfrak{M} \models (Rtv \wedge \varphi)[g^v_{\mathbf{m}}]$ for some $\mathbf{m} \in M$. Let $\mathbf{t}$ be the interpretation of t in $\mathfrak{M}$ under $g^v_{\mathbf{m}}$. Because of the restriction on variables in bounded quantification, $t \neq v$, whence $\mathbf{t}$ is also the interpretation of t in $\mathfrak{M}$ under g. So $R\mathbf{tm}$ holds in $\mathfrak{M}$ and $\mathfrak{M} \models \varphi[g^v_{\mathbf{m}}]$. By the inductive hypothesis, $\mathfrak{M}, g^v_{\mathbf{m}} \Vdash HT(\varphi)$ iff $\mathfrak{M}, g, \mathbf{m} \Vdash \downarrow v.HT(\varphi)$, iff $\mathfrak{M}, g, \mathbf{t} \Vdash \Diamond \downarrow v.HT(\varphi)$ iff $\mathfrak{M}, g, \mathbf{t} \Vdash @_t\Diamond \downarrow v.HT(\varphi)$ iff $\mathfrak{M}, g \Vdash @_t\Diamond \downarrow v.HT(\varphi)$. QED

As simple corollaries we have:

Corollary 3.3 *Let $\varphi(x)$ be a bounded formula with only x free, then for all models $\mathfrak{M}$ and for all $m \in M$, $\mathfrak{M} \models \varphi(m)$ iff $\mathfrak{M}, m \Vdash \downarrow x.HT(\varphi)$.*

Corollary 3.4 *Let φ be a first-order formula in the hybrid signature. Then the following are equivalent.*

i. φ is equivalent to the standard translation of a hybrid formula.

ii. φ is equivalent to a formula in the bounded fragment.

Moreover, there are effective translations between $\mathcal{H}(\downarrow, @)$ and the bounded fragment.

3.2 Generated back-and-forth systems

We now turn to the problem of providing semantic characterizations of $\mathcal{H}(\downarrow, @)$ (or equivalently, of the bounded fragment). In this section we shall adopt an essentially first-order approach: we define *generated back-and-forth systems*, basically a restricted form of Ehrenfeucht-Fraïssé game, and link it to the concept of *generated submodels*.

Definition 3.5 (Partial Isomorphism) *Let $\mathfrak{M}$ and $\mathfrak{N}$ be two hybrid models. A function h from a subset of M to a subset of N is called a* partial isomorphism *if*

1. *h is a bijection;*
2. *for all x in the domain of h, for all $a \in PROP \cup NOM$, $x \in V^{\mathfrak{M}}(a)$ iff $h(x) \in V^{\mathfrak{N}}(a)$;*
3. *for all x, y in the domain of h, $R^{\mathfrak{M}}xy$ iff $R^{\mathfrak{N}}h(x)h(y)$.*

Generated back-and-forth systems Let $\mathfrak{M}$ and $\mathfrak{N}$ be two first-order models in the hybrid signature. A generated back-and-forth system between $\mathfrak{M}$ and $\mathfrak{N}$ is a non-empty family F of finite partial isomorphisms between $\mathfrak{M}$ and $\mathfrak{N}$ satisfying the following two extension rules:

($\Diamond$-extension)

- (forth) if $h \in F$, x in its domain and $R^{\mathfrak{M}}xy$, then $h \cup \{\langle y, y' \rangle\} \in F$ for some $y' \in N$.
- (back) if $h \in F$, x in its range and $R^{\mathfrak{N}}xy$, then $h \cup \{\langle y', y \rangle\} \in F$ for some $y' \in M$.

(nominal extension)

- (forth) if $h \in F$ and there exists an $x \in M$ such that $V^{\mathfrak{M}}(i) = \{x\}$ for some nominal i, then there exists an $x' \in N$ such that $h \cup \{\langle x, x' \rangle\} \in F$.
- (back) a similar condition backwards.

Let $m \in M$ and $n \in N$. Then $(\mathfrak{M}, m) \equiv_R (\mathfrak{N}, n)$ means that there exists a generated back-and-forth system between $\mathfrak{M}$ and $\mathfrak{N}$ which connects m and n. More generally, let $\bar{m}$ ($\bar{n}$) be a finite M-tuple (N-tuple). Then $(\mathfrak{M}, \bar{m}) \equiv_R (\mathfrak{N}, \bar{n})$ means that there exists a generated back-and-forth system between $\mathfrak{M}$ and $\mathfrak{N}$, containing a partial isomorphism which sends $m(i)$ to $n(i)$.

Note how closely this definition follows the familiar one from first-order logic (cf. e.g., [Hod93]). In fact, if we think of such a system as describing an Ehrenfeucht-Fraïssé game, then the sole difference is that in the "generated back-and-forth game" the universal player ($\forall$belard) must choose his moves from R-successors or worlds named by a nominal, whereas he can play whatever he likes in the full first-order game. Unsurprisingly, restricting the play to *accessible* worlds means that generated back-and-forth systems and *generated submodels* are closely connected.

Definition 3.6 (Generated Submodel) *Let $\mathfrak{M} = \langle M, R, V \rangle$ be a hybrid model and $S \subseteq M$. Let NOM denote the subset of M whose elements are in the interpretation of some nominal. The* submodel of $\mathfrak{M}$ generated by S *is the substructure of $\mathfrak{M}$ with domain $\{m \in M \mid \exists s \in S \cup NOM(R^* sm)\}$ (R^* is the reflexive and transitive closure of R). This is also called the S-generated submodel of $\mathfrak{M}$.*

Note that if $NOM = \emptyset$, we obtain the familiar modal notion of a generated submodel; and that if in addition S is a singleton set, we have the usual modal notion of a point-generated (or rooted) submodel.

We now define two notions of invariance. The first is taken from [Ben83]. A first-order formula $\varphi(\bar{x})$ in free variables $\bar{x}$ in a signature with one binary relation R, unary predicates and constants (and equality) is *invariant for generated submodels* if for all models $(\mathfrak{M}, \bar{m})$ and $(\mathfrak{M}', \bar{m})$ such that $\mathfrak{M}'$ is the $\bar{m}$-generated submodel of $\mathfrak{M}$,

$$\mathfrak{M} \models \varphi(\bar{m}) \text{ if and only if } \mathfrak{M}' \models \varphi(\bar{m}).$$

In a similar spirit, we shall say that a first-order formula $\varphi(\bar{x})$ in the same signature is invariant for generated back-and-forth systems if for all models $(\mathfrak{M}, \bar{m})$ and $(\mathfrak{N}, \bar{n})$, $(\mathfrak{M}, \bar{m}) \equiv_R (\mathfrak{N}, \bar{n})$ implies

$$\mathfrak{M} \models \varphi(\bar{m}) \text{ if and only if } \mathfrak{N} \models \varphi(\bar{n}).$$

Theorem 3.7 *Let $\varphi(\bar{x})$ be a first-order formula in the hybrid signature. Then the following are equivalent.*

i. $\varphi(\bar{x})$ is equivalent to a formula in the bounded fragment.
ii. $\varphi(\bar{x})$ is invariant for generated submodels.
iii. $\varphi(\bar{x})$ is invariant for generated back-and-forth systems.

PROOF.
i. $\Rightarrow$ *ii.* is obvious.

ii. $\Rightarrow$ *iii.* First note that $\varphi(\bar{x})$ is invariant for generated submodels if and only if $\neg\varphi(\bar{x})$ is. Now suppose $\varphi(\bar{x})$ is invariant for generated submodels but *not* preserved under generated back-and-forth systems. Then we have models $(\mathfrak{M}, \bar{m})$ and $(\mathfrak{N}, \bar{n})$, a generated back-and-forth system linking $\bar{m}$ and $\bar{n}$, and $\mathfrak{M} \models \varphi(\bar{m})$ while $\mathfrak{N} \models \neg\varphi(\bar{n})$.

Let $\mathfrak{M}'$ ($\mathfrak{N}'$) be the $\bar{m}$- ($\bar{n}$) generated submodel of $\mathfrak{M}$ ($\mathfrak{N}$). Then still $\mathfrak{M}' \models \varphi(\bar{m})$ and $\mathfrak{N}' \models \neg\varphi(\bar{n})$, by invariance. Clearly $(\mathfrak{M}', m) \equiv_R (\mathfrak{N}', n)$. But then $(\mathfrak{M}', m)$ and $(\mathfrak{N}', n)$ have the same first-order theory by the following argument. Because $(\mathfrak{M}', m) \equiv_R (\mathfrak{N}', n)$ holds, $\exists$loise has a winning strategy in all games where $\forall$belard only plays *immediate* R-successors or points named by a nominal. But since the models are generated, in the first-order back-and-forth game he can only play worlds which are accessible by a finite R-transition from either the root or one of the named worlds. But then she can compute a winning answer for the classic Ehrenfeucht-Fraïssé from her winning generated back-and-forth strategy. This contradicts the claim that $\mathfrak{M}' \models \varphi(\bar{m})$ and $\mathfrak{N}' \models \neg\varphi(\bar{n})$.

iii. $\Rightarrow$ *i.* We use a van Benthem style diagram-chasing argument [Ben96]. We only provide the outline. Let $\varphi(\bar{x})$ be as in the hypothesis and $BC(\varphi(\bar{x}))$ the bounded consequences of $\varphi(\bar{x})$ (that is, the consequences of $\varphi(\bar{x})$ that belong to the bounded fragment). We will show that $BC(\varphi(\bar{x})) \models \varphi(\bar{x})$, from which the result follows by compactness. (In this notation we interpret the $\bar{x}$ as constants, or equivalently we use the local version of first-order consequence, cf. [End72].)

If $BC(\varphi(\bar{x}))$ is inconsistent we are done. Otherwise, let $\mathfrak{M}, \bar{m}$ be a model of $BC(\varphi(\bar{x}))$ and $\mathfrak{N}, \bar{n}$ be a model of $\varphi(\bar{x})$ together with the bounded theory of $\mathfrak{M}, \bar{m}$. (Such a model can easily be shown to exist.) Take ω-saturated extensions $\mathfrak{M}^+, \bar{m}$ and $\mathfrak{N}^+, \bar{n}$. Create a family F of finite functions between M^+ and N^+ as follows: $f : \bar{x} \to \bar{y}$ is in F iff $\mathfrak{M}^+, \bar{x}$ and $\mathfrak{N}^+, \bar{y}$ make the same bounded formulas true. It is easy to show that F is a generated back and forth system linking $\bar{m}$ and $\bar{n}$. Now we can start diagram chasing: $\mathfrak{N} \models \varphi(\bar{n})$ then (by elementary extension) $\mathfrak{N}^+ \models \varphi(\bar{n})$, then (by invariance) $\mathfrak{M}^+ \models \varphi(\bar{m})$, then (passing to an elementary submodel) $\mathfrak{M} \models \varphi(\bar{m})$ as desired. QED

3.3 Hybrid bisimulations

We have just seen that by weakening the notion of an Ehrenfeucht-Fraïssé game we can link the bounded fragment (and hence $\mathcal{H}(\downarrow, @)$) with generated submodels. But in spite of its binding apparatus, $\mathcal{H}(\downarrow, @)$ has a distinctly modal flavor. Is it not also possible to strengthen the notion of *bisimulation* (the standard notion of equivalence between models used in modal logic) with clauses for $\downarrow$ and $@$, and so characterize $\mathcal{H}(\downarrow, @)$ in intrinsically modal terms? That's what we will do in this section. The approach has an advantage over the use of generated back-and-forth systems: preservation results can be easily obtained for reducts as well.

Recall that for ordinary propositional modal logics, bisimulations are non-empty binary relations linking the domains of models, with the restriction that only worlds with identical atomic information and matching accessibility relations should be connected (see Definition 3.7 [Ben83]; here bisimulations are called p-relations). Now, if we want to extend this notion to $\mathcal{H}(\downarrow, @)$, we need to take care of assignments to world variables as well. To this end, hybrid bisimulations will not simply link worlds, rather they will link pairs $(\bar{m}, m)$, where m is a world and $\bar{m}$ is an assignment. We start by defining k-bisimulations, which are the correct notion of bisimulation for formulas φ such that $\mathrm{WVAR}(\varphi) \subseteq \{x_1, \ldots, x_k\}$.

k-bisimulation. Let $\mathfrak{M}$ and $\mathfrak{N}$ be two hybrid models. Let $\overset{k}{\sim}$ be a binary relation between ${}^kM \times M$ and ${}^kN \times N$. So $\overset{k}{\sim}$ relates tuples $((m_1, \ldots, m_k), m)$ with tuples $((n_1, \ldots, n_k), n)$. We write these tuples as $(\bar{m}, m)$. Notice that $\bar{m}$ can be seen as an assignment over $(x_1, \ldots, x_k)$. A non-empty relation $\overset{k}{\sim}$ is called a *k-bisimulation* if it satisfies the following properties

(prop) If $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$, then $m \in V^{\mathfrak{M}}(a)$ iff $n \in V^{\mathfrak{N}}(a)$, for $a \in \mathrm{PROP} \cup \mathrm{NOM}$.

(var) If $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$, then for all $j \leq k$, $m_j = m$ iff $n_j = n$.

(forth) If $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$ and $R^{\mathfrak{M}}mm'$, then there exists an $n' \in N$ such that $R^{\mathfrak{N}}nn'$ and $(\bar{m}, m') \overset{k}{\sim} (\bar{n}, n')$.

(back) A similar condition from $\mathfrak{N}$ to $\mathfrak{M}$.

(@) If $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$, then for every nominal $i \in \mathrm{NOM}$, if $m' \in V^{\mathfrak{M}}(i)$ and $n' \in V^{\mathfrak{N}}(i)$ then $(\bar{m}, m') \overset{k}{\sim} (\bar{n}, n')$, and for every $j \leq k$, $(\bar{m}, m_j) \overset{k}{\sim} (\bar{n}, n_j)$

($\downarrow$) If $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$, then for every $j \leq k$, $(\bar{m}^j_m, m) \overset{k}{\sim} (\bar{n}^j_n, n)$.

Note that since $\downarrow$ and $@$ are self-dual, we can collapse the back and forth clauses for these modalities into one. We write $\mathfrak{M} \overset{k}{\sim} \mathfrak{N}$ if there exists a k-bisimulation between the two models.

To extend the notion to the full language we need to add only one further condition.

ω-bisimulation. Let $\mathfrak{M}$ and $\mathfrak{N}$ be two hybrid models. An ω-bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$ is a non-empty family of k-bisimulations satisfying the following *storage rule*:

(sto) If $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$, then $(\bar{m} * m, m) \overset{k+1}{\sim} (\bar{n} * n, n)$.

Here and elsewhere, $\bar{m} * m$ denotes the tuple obtained from concatenating $\bar{m}$ and m. Let $\bar{m}$ ($\bar{n}$) be an M-tuple (N-tuple). Then $(\mathfrak{M}, \bar{m}) \overset{\omega}{\sim} (\mathfrak{N}, \bar{n})$ means that there exists an ω-bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$ such that $(\bar{m}, m(0)) \overset{k}{\sim} (\bar{n}, n(0))$, for k the length of $\bar{m}$.

Some remarks. First, k and ω-bisimulations can be restricted to a given set of propositional variables and nominals $\mathrm{PROP} \cup \mathrm{NOM}$ by restricting **(prop)** and **(@)** accordingly. Second, the modular definition of k and ω-bisimulation will lead to results for reducts of the language as well. For instance if we delete $\downarrow$ from the language, we just delete the **($\downarrow$)** clause from the definition of bisimulation and we obtain the appropriate notion for $\mathcal{H}(@)$. Of course, if we delete the variables from the language, we don't need the assignment tuples anymore, and the bisimulation becomes just a relation between worlds, as usual. Then for the language without $\downarrow, @$ and variables, the standard definition of bisimulation applies (the condition **(prop)** takes care of the nominals). If we add @ to this language, we just have to add the following clause

(@′) For all nominals i, if $V^{\mathfrak{M}}(i) = \{m\}$ and $V^{\mathfrak{N}}(i) = \{n\}$, then $m \sim n$.

Preservation results for all these alternatives can be given (the required proofs follow much the same lines as the proofs below) and we shall prove one such result in Section 7.

The first important fact about hybrid bisimulations is that they preserve truth:

Proposition 3.8 *Let $\mathfrak{M}$ and $\mathfrak{N}$ be two hybrid models, $m \in M$, $n \in N$. Then,*

i. If $\mathfrak{M} \overset{k}{\sim} \mathfrak{N}$, with $\overset{k}{\sim}$ over a given set $PROP \cup NOM$, then for all formulas φ over the signature $\langle PROP, NOM, \{x_1, \ldots, x_k\}\rangle$, $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$ implies $\mathfrak{M}, \bar{m}, m \Vdash \varphi \Leftrightarrow \mathfrak{N}, \bar{n}, n \Vdash \varphi$.

ii. If $(\mathfrak{M}, m) \overset{\omega}{\sim} (\mathfrak{N}, n)$, with $\overset{\omega}{\sim}$ over a given set $PROP \cup NOM$, then for all sentences *φ over the signature $\langle PROP, NOM, WVAR\rangle$, $\mathfrak{M}, m \Vdash \varphi \Leftrightarrow \mathfrak{N}, n \Vdash \varphi$. (Recall that for sentences the choice of assignment is irrelevant.)*

Proof.

i. By a straightforward inductive argument.

ii. Let $(\mathfrak{M}, m) \overset{\omega}{\sim} (\mathfrak{N}, n)$ and let φ be a hybrid sentence. Then it contains variables (after renaming) say $\{x_1, \ldots, x_k\}$. We have $(\langle m\rangle, m) \overset{1}{\sim} (\langle n\rangle, n)$, so $k-1$ applications of the storage rule give us $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$, where $\bar{m}$ is a k-tuple consisting of m's and similarly for $\bar{n}$. But then, by *i.*, $\mathfrak{M}, \bar{m}, m \Vdash \varphi \Leftrightarrow \mathfrak{N}, \bar{n}, n \Vdash \varphi$, whence since φ is a sentence $\mathfrak{M}, m \Vdash \varphi \Leftrightarrow \mathfrak{N}, n \Vdash \varphi$.
QED

The notion of k-bisimulation has a distinct modal flavor. But a very first-order notion is hidden inside: partial isomorphism.

Proposition 3.9 *Let $k \geq 2$, and let $\mathfrak{M} \overset{k}{\sim} \mathfrak{N}$. If $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$, then the function f defined as $f(m) = n$ and $f(m(i)) = n(i)$ is a partial isomorphism between $\{m(1), \ldots, m(k), m\}$ and $\{n(1), \ldots, n(k), n\}$.*

Proof. The map f is a bijection by **(var)** and **(@)**. By **(prop)** and **(@)**, f preserves nominals and propositional variables. To see that it preserves the accessibility relation suppose $R^{\mathfrak{M}}xy$. There are three cases.

(Case 1: $x = m$, $y = m_i$.) Then by **(forth)** there exists an n' such that $R^{\mathfrak{N}}nn'$ and $(\bar{m}, m_i) \overset{k}{\sim} (\bar{n}, n')$. But $\bar{m}(i) = m_i$, so by **(var)**, $n' = \bar{n}(i)$, whence $R^{\mathfrak{N}}nf(m(i))$.

(Case 2: $x = m_i$, $y = m$.) Let $j \neq i$. Such a j exists because we assumed that $k \geq 2$. By **($\downarrow$)**, $(\bar{m}^j_m, m) \overset{k}{\sim} (\bar{n}^j_n, n)$. Then by **(@)**, $(\bar{m}^j_m, m_i) \overset{k}{\sim} (\bar{n}^j_n, n_i)$. Now continue as in case 1.

(Case 3: $x = m_i$, $y = m_j$.) By **(@)**, $(\bar{m}, m_i) \overset{k}{\sim} (\bar{n}, n_i)$. Now continue as in case 1.
Thus $R^{\mathfrak{M}}xy$ implies $R^{\mathfrak{N}}f(x)f(y)$. For the other direction use **(back)** in the same way. QED

Note that the condition $k \geq 2$ is crucial. We use it together with **($\downarrow$)** to store the information about m. Of course in a model where Rm_im holds, we have $\bar{m}, m \Vdash \downarrow x_j.@_{x_i}\Diamond x_j$.

Thus there is a clear link between our earlier work on generated back-and-forth systems, and the next theorem shouldn't come as a surprise:

Theorem 3.10 *Let $(\mathfrak{M}, \bar{m})$ and $(\mathfrak{N}, \bar{n})$ be two models. Then the following are equivalent:*
i. $(\mathfrak{M}, \bar{m}) \overset{\omega}{\sim} (\mathfrak{N}, \bar{n})$.
ii. $(\mathfrak{M}, \bar{m}) \equiv_R (\mathfrak{N}, \bar{n})$.

Proof.
i. $\Rightarrow$ *ii.* Let $(\mathfrak{M}, \bar{m}) \overset{\omega}{\sim} (\mathfrak{N}, \bar{n})$. Define a family F of maps as follows: $f \in F$ if there exists $(\bar{x}, x') \overset{k}{\sim} (\bar{y}, y')$ and f is defined as in Proposition 3.9.

Clearly $\bar{m}$ and $\bar{n}$ are connected by a map. By Proposition 3.9 all maps are partial isomorphisms. We show the forth side of **(nominal extension)**; all other conditions have a similar proof. Suppose $f \in F$ and $\mathbf{z} \in M$ and $V^{\mathfrak{M}}(i) = \{\mathbf{z}\}$, for some nominal i. Then for some $\bar{x}$, x, $\bar{y}$, y, $(\bar{x}, x') \overset{k}{\sim} (\bar{y}, y')$ by definition of F. Then $(\bar{x} * x', x') \overset{k+1}{\sim} (\bar{y} * y', y')$ by **(sto)**. But then by **(@)**, $(\bar{x} * x', \mathbf{z}) \overset{k+1}{\sim} (\bar{y} * y', \mathbf{z}')$ for $V^{\mathfrak{N}}(i) = \{\mathbf{z}'\}$. Thus the wanted extension is in F.

ii. $\Rightarrow$ *i.* Let $(\mathfrak{M}, \bar{m}) \equiv_R (\mathfrak{N}, \bar{n})$. We define an ω-bisimulation in the obvious way: for any $f \in F$, for any k, for any tuple $\bar{m}$ in the k-th power of the domain of f and for any m in the domain of f, we set $(\bar{m}, m) \overset{k}{\sim} (f\bar{m}, f(m))$. It is easy to check that this is an ω-bisimulation.
QED

It is possible to prove a direct characterization result for $\mathcal{H}(\downarrow, @)$ in terms of invariance for k-bisimulations, using again a diagram chasing argument. We're not going to do this here since in the next section we shall take a detour via the bounded fragment to reach the same result. It is also possible to develop k-pebble versions of generated back-and-forth systems; this notion takes the exact number of variables used in formulas into account. It is not difficult to see that $k+1$-pebble generated back-and-forth systems correspond to k-bisimulations, but we won't follow up these matters here.

4 Harvest

It is time to draw together the threads developed in the previous section. First we note their consequences for $\mathcal{H}(\downarrow, @)$ expressivity over models. Then we note the consequences for frames and what this tells us about hybrid completeness. Finally we note the consequences for hybrid *tense* logic.

4.1 Expressivity over models

We have the following five-fold characterization of $\mathcal{H}(\downarrow, @)$:

Theorem 4.1 *Let $\varphi(\bar{x})$ be a first-order formula in the hybrid signature (with equality). Then the following are equivalent.*
i. $\varphi(\bar{x})$ is equivalent to the standard translation of a $\mathcal{H}(\downarrow, @)$ formula.
ii. $\varphi(\bar{x})$ is invariant for generated submodels.
iii. $\varphi(\bar{x})$ is invariant for generated back-and-forth systems.
iv. $\varphi(\bar{x})$ is invariant for ω-bisimulation.
v. $\varphi(\bar{x})$ is equivalent to a formula in the bounded fragment of first-order logic.

Proof. By Corollary 3.4, Theorem 3.7 and Proposition 3.8. QED

But these have obvious consequences for the ordinary modal correspondence language. In particular, if we consider *nominal-free* hybrid sentences, then we obtain a five-fold characterization of the fragment of first-order logic in the classical modal signature which is invariant for generated submodels:

Corollary 4.2 *Let $\varphi(x)$ be a first-order formula in the modal signature with equality. Then the following are equivalent.*

i. $\varphi(x)$ is equivalent to the standard translation of a nominal-free $\mathcal{H}(\downarrow, @)$ sentence.

ii. $\varphi(x)$ is invariant for generated submodels (now in the standard modal sense).

iii. $\varphi(x)$ is invariant for R-generated back-and-forth systems, where an R-generated back-and-forth system is a back-and-forth system satisfying only the **$\Diamond$-extension** *rule.*

iv. $\varphi(x)$ is invariant for ω-bisimulation.

v. $\varphi(x)$ is equivalent to a formula in the bounded fragment of first-order logic without constants.

4.2 Frames and completeness

Recall that a frame $\mathfrak{F}$ is a pair $\langle W, R\rangle$ (that is, a model without a valuation). Since the late 1950s, one of the central topics in modal logic has been linking modal formulas to properties of frames and investigating when they give rise to complete axiomatizations for the frame classes they define. The work of the previous section easily yields a characterization of the frame-defining abilities of pure nominal-free sentences. Moreover, the axiomatic investigations of [BT99] (and indeed, the tableaux-based investigations of [Bla98]) show that there is a perfect match between definability and completeness for pure nominal-free sentences. By combining these results we obtain matching definability and completeness results for a wide range of first-order definable frame classes.

In modal correspondence theory, the first-order language (with equality) over the signature consisting simply of a binary symbol R is called the (first-order) *frame language*. We shall call a formula φ in the frame language containing exactly one free variable a *frame condition*. The class of frames defined by a frame condition $\varphi(x)$ is the class in which the universal closure $\forall x\varphi(x)$ is true; we call this class $\textsc{frames}(\forall x.\varphi(x))$.

Before proceeding further, two simple observations are in order. First, note that if we apply the standard translation ST to a pure nominal-free sentence α , then $ST(\alpha)$ is a frame condition with free-variable x. Furthermore, note that for any frame $\mathfrak{F} = \langle W, R\rangle$ we have that $\mathfrak{F} \Vdash \alpha$ iff $\mathfrak{F} \Vdash \forall x.ST(\alpha)$; this is an immediate consequence of the definition of frame validity (see Definition 2.4).

Theorem 4.3 *Let $\mathsf{K}[\mathcal{H}(\downarrow, @)]$ be the axiomatization given in Section 2, and for any hybrid sentence α let $\mathsf{K}[\mathcal{H}(\downarrow, @)] + \alpha$ be the system obtained by adding α as an additional axiom. Then, if $\varphi(x)$ is a frame condition and $\varphi(x)$ is invariant under generated submodels (in the usual modal sense) we have that:*

i. If $\varphi(x)$ is in the bounded fragment, then the pure nominal free sentence $\downarrow x.HT(\varphi(x))$ defines $\textsc{frames}(\forall x.\varphi(x))$. Moreover, $\mathsf{K}[\mathcal{H}(\downarrow, @)]+ \downarrow x.HT(\varphi(x))$ is strongly complete with respect to $\textsc{frames}(\forall x.\varphi(x))$.

ii. If $\varphi(x)$ is not in the bounded fragment, there is a pure nominal free sentence α such that α defines $\textsc{frames}(\forall x.\varphi(x))$, and $ST(\alpha)$ is equivalent to $\varphi(x)$. Moreover, $\mathsf{K}[\mathcal{H}(\downarrow, @)] + \alpha$ is strongly complete with respect to $\textsc{frames}(\forall x.\varphi(x))$.

Conversely, if α is pure nominal-free sentence, then α defines FRAMES$(\forall x.ST(\alpha(x)))$, *and* $\mathsf{K}[\mathcal{H}(\downarrow,@)] + \alpha$ *is complete with respect to* FRAMES$(\forall x.ST(\alpha(x)))$.

PROOF. The converse condition was proved in [BT98], so let's examine the other direction.

For item *i.*, we first remark that as $\varphi(x)$ belongs to the *frame* language, it contains no unary predicate symbols, hence $HT(\varphi(x))$ is a *pure* formula; that $\downarrow x.HT(\varphi(x))$ is a pure nominal-free sentence is thus clear. Now, by Corollary 3.3, for any model $\mathfrak{M} = (\mathfrak{F}, V)$ and any $m \in M$,

$$(\mathfrak{F}, V) \models \varphi(m) \text{ iff } (\mathfrak{F}, V), m \Vdash \downarrow x.HT(\varphi).$$

But this means that

$$(\mathfrak{F}, V) \models \forall x.\varphi \text{ iff } (\mathfrak{F}, V) \Vdash \downarrow x.HT(\varphi).$$

But as $\varphi(x)$ contains no unary predicate symbols (and $\downarrow x.HT(\varphi)$ no propositional variables) V is irrelevant, and hence

$$\mathfrak{F} \models \forall x.\varphi(x) \text{ iff } \mathfrak{F} \Vdash \downarrow x.HT(\varphi).$$

But this means that $\downarrow x.HT(\varphi(x))$ defines FRAMES$(\forall x.\varphi(x))$. Completeness follows using the arguments of [BT98].

For item *ii.*, we know that $\varphi(x)$ being invariant under generated submodels is equivalent to a formula in the bounded fragment — but is it equivalent to a *frame condition* $\varphi'(x)$? In fact, this can be established by modifying the diagram chasing argument used in the proof of Theorem 3.7. The key point to observe is that instead of showing that $BC(\varphi(x)) \models \varphi(x)$, we can show by the same method that $FC(\varphi(x)) \models \varphi(x)$, where FC are all the frame conditions implied by $\varphi(x)$. Thus there is an equivalent frame condition $\varphi'(x)$, and we can take α to be $\downarrow x.HT(\varphi'(x))$. The remainder of the proof is as for item *i.* QED

4.3 Hybrid tense logic

The characterization results have a particularly natural interpretation in the setting of hybrid *tense* logic. Recall that in tense logic we write $\Box$ as G, $\Diamond$ as F, and that we also have at our disposal an operator H (a $\Box$-operator that scans the *converse* of the accessibility relation) and its dual P (a $\Diamond$-operator that scans the converse of the accessibility relation). It is straightforward to hybridize tense logic by adding nominals, @ and $\downarrow$ (though now it is more natural to talk of *time* variables rather than world variables) thus forming the language $\mathcal{H}_t(\downarrow,@)$. To cope with the backward looking operators, tense logic requires a slightly more liberal notion of generated submodel: a point t belongs to the submodel *temporally generated* by a subset S if t is reachable from some point $s \in S$ by making a finite sequence of moves through the accessibility relation, where both forward *and backward* steps are allowed. The characterization results we have proved hold for $\mathcal{H}_t(\downarrow,@)$ under this notion of generated submodel.

But let's press matters a little further. Note that in *nominal-free sentences* of $\mathcal{H}_t(\downarrow,@)$, all occurrences of @ are eliminable. As a simple example, consider the definition of the *Until* operator:

$$\mathit{Until}(\varphi,\psi) \ := \downarrow x.\mathsf{F} \downarrow y.@_x(\mathsf{F}(y \wedge \varphi) \wedge \mathsf{G}(\mathsf{F}y \to \psi)).$$

(This is simply the definition given in Section 2 written in tense logical notation.) But observe that the following nominal-free sentence has the same effect:

$$\mathit{Until}(\varphi,\psi) \ := \downarrow x.\mathsf{F} \downarrow y.\mathsf{P}(x \wedge (\mathsf{F}(y \wedge \varphi) \wedge \mathsf{G}(\mathsf{F}y \to \psi))).$$

That is, instead of retrieving the point named x using the @ operator, we can "reach back" for this point using P.

This observation (first made in [BS98]) is completely general. As long as a $\mathcal{H}_t(\downarrow, @)$ formula doesn't contain nominals or free time variables, it will always be possible to simulate @ by zig-zagging back to the binding point using the tense operators. More precisely, suppose a nominal free sentence φ has a temporal depth of n (that is, the maximal depth of embedding of tense operators is n) and that φ is satisfied at a time m. Then when we evaluate a subformula of φ of the form $@_x\psi$ at some point m' (note that m' cannot be more than n forward and back steps from m) then we know that x must bound to a point m'' (which is also not more than n forward and back steps from m). Hence m' and m'' are separated by at most $2n$ (forward and back) steps. We can define an operator $@^{2n}$ that allows us to zig-zag to a named time lying within $2n$ steps as follows. Let $\mathrm{ZZ}2n$ be the set of all non-empty finite sequences of F and P operators of length at most $2n$. Then for any formula ψ and any variable x we define:

$$@_x^{2n}\psi \;:=\; (x \wedge \psi) \vee \bigvee_{Z \in \mathrm{ZZ}2n} Z(x \wedge \psi).$$

Hence, given a nominal free sentence φ of temporal depth n, we eliminate all occurrences of @ as follows. Let $@_x\psi$ be a subformula of φ where ψ contains no occurrences of @. Replace $@_x\psi$ by $@_x^{2n}\psi$ to form φ'. Repeating this procedure (starting with φ') produces an equivalent nominal-free sentence containing no occurrences of @. Thus, in the setting of tense logic, our characterization results for nominal free sentences go through without the help of @.

This is a pleasing result, for there are also non-technical reasons for viewing $\mathcal{H}_t(\downarrow)$ as a key system: this language can be viewed as a marriage between the ideas of Arthur Prior and Hans Reichenbach.

That $\mathcal{H}_t(\downarrow)$ captures the idea of Arthur Prior is clear: tense logic was invented by Prior precisely to capture the "internal" perspective on time which underlies temporal discourse in natural language (see in particular [Pri67]). But while the Priorean perspective gets a lot right, it misses a crucial fact about temporal discourse: tenses are very often *referential*. That is, tenses in natural language often achieve their effect by referring to specific points of time, and many semantic distinctions between natural language tenses cannot be drawn without taking referential effects into account. The importance of temporal reference was first made clear in the work of Hans Reichenbach, and many modern theories of tense (for example, [Com85]) adopt a fundamentally Reichenbachian stance.

It should be clear where this is heading: $\downarrow$ can be seen as the Reichenbachian device *par excellence*. In a sense, $\downarrow$ gives us a sort of *generalized present tense*; it enables us to "store" an evaluation point, thereby making it possible to insist later that certain events happened at *that* time, or that certain other events must be viewed from that particular perspective. This is precisely the kind of expressive power we need to encode Reichenbach's ideas. And crucially, the use of $\downarrow$ does not in any sense conflict with Prior's use of tense operators. Quite the reverse: F, P, and $\downarrow$ work together beautifully. No auxiliary apparatus (not even @) is required to blend the two approaches, and the result is a language which exactly captures first-order temporal reachability.

5 Interpolation

In this section we show that $\mathcal{H}(\downarrow, @)$ is well behaved in yet another sense: it has the interpolation property. Interpolation is a much studied notion. Originally considered a property of deductive systems, it was proved for first-order logic in [Cra57] using a proof theoretic argument. We shall view interpolation as a property of *consequence relations* and will prove it using semantic arguments (as is done, for example, in [CK90]).[1]

Before plunging into the details, note that in modal logic we can distinguish between *strong arrow interpolation* (AIP) and *weak turnstile interpolation* (TIP). AIP implies TIP, but not conversely. We will prove AIP for $\mathcal{H}(\downarrow, @)$, disprove AIP and TIP for its finite variable fragments (our earlier work on k-bisimulations will enable us to construct counterexamples fairly straightforwardly) and show that TIP interpolation holds for the sublanguage $\mathcal{H}(@)$. Here are the definitions of these concepts:

AIP A logic L has the *Arrow Interpolation Property* (AIP) if, whenever $\models_L \varphi \to \psi$, there exists a formula θ such that $\models_L \varphi \to \theta$, $\models_L \theta \to \psi$ and $I\!P(\theta) \subseteq I\!P(\varphi) \cap I\!P(\psi)$.

TIP A logic L has the *Turnstile Interpolation Property* (TIP) if, whenever $\varphi \models_L \psi$, there exists a formula θ such that $\varphi \models_L \theta$, $\theta \models_L \psi$ and $I\!P(\theta) \subseteq I\!P(\varphi) \cap I\!P(\psi)$.

For first-order logic these notions are equivalent, but in modal logic this is not the case (as we see below, equivalence depends on both compactness and the availability of a deduction theorem; cf. also [Cze82]). Further, note that the meaning of TIP depends on the way we define the consequence relation $\varphi \models \psi$. In Section 2 we introduced two consequence relations: local consequence and global consequence (see [Ben83] or [MV97] for a discussion of their relative merits). So there are two plausible definitions of the turnstile interpolation property, TIP^{loc} and TIP^{glo}.

In modal logic these different notions of interpolation are related as follows:

Proposition 5.1

i. AIP and TIP^{loc} are equivalent.

ii. If the local consequence relation is compact, then AIP implies TIP^{glo}.

For this reason, from now on we take TIP to be defined using the *global* consequence relation. AIP and TIP are often referred to as the strong and weak interpolation properties respectively, and we shall sometimes use this terminology.

We turn to the technicalities of the interpolation result. As is usual in interpolation proofs, where language related issues require special care, we replace the standard notion of consistency

Definition 5.2 (Consistency) *Let T be a set of formulas in $\mathcal{L}$. Then T is* consistent *iff there is a model $\mathfrak{M}$, an $m \in M$, and an assignment g such that for all $\varphi \in T$, $\mathfrak{M}, g, m \Vdash \varphi$.*

by the finer-grained notion of separability

Definition 5.3 (Separability) *Let T, U, L be sets of formulas in $\mathcal{L}$. We say that the pair* $\langle T, U\rangle$ is separable with respect to L *if there exists a formula $\theta \in L$ such that $T \models \theta$ and $U \models \neg\theta$.* $\langle T, U\rangle$ is inseparable with respect to L *if it is not separable with respect to L.*

We are ready to prove the main result of this section.

[1] Jerry Seligman has recently announced a proof-theoretic proof of interpolation for $\mathcal{H}(\downarrow, @)$. However his proof has not yet been made available.

Theorem 5.4 (Arrow Interpolation for $\mathcal{H}(\downarrow,@)$) *Let φ and ψ be formulas in $\mathcal{L}$ such that $\models \varphi \to \psi$. Then there exists a formula θ such that*

i. $\models \varphi \to \theta$ *and* $\models \theta \to \psi$.

ii. $I\!P(\theta) \subseteq I\!P(\varphi) \cap I\!P(\psi)$.

PROOF. Suppose we are given formulas φ_0 and ψ_0 such that there is no interpolant for $\varphi_0 \to \psi_0$. We will prove that $\not\models \varphi_0 \to \psi_0$ by producing a model $\mathfrak{M} = \langle M, R, V\rangle$ and an assignment g such that for some $m \in M$, $\mathfrak{M}, g, m \Vdash \varphi_0 \wedge \neg\psi_0$ (the proof follows the method of [CK90] in which two connected models are simultaneously built using fresh constants). Hence we will have shown the contrapositive.

We can assume that $\{\varphi_0\}$ and $\{\neg\psi_0\}$ are consistent (for if they are not, then either $\bot$ or $\top$ is an interpolant). Furthermore they must be inseparable over the formulas in $\mathcal{L}$ with propositional variables and nominals in $I\!P(\varphi_0) \cap I\!P(\psi_0)$.

Let $\mathcal{L}'$ be the hybrid language over the signature $\langle$PROP, NOM $\cup$ N, WVAR $\rangle$, where $\mathrm{N} = \{n_0, \ldots, n_k, \ldots\}$ is an countably infinite set of new nominals. For any formula φ define the restricted language $\mathcal{L}_\varphi$ as $\{\xi \in \mathcal{L} \mid I\!P(\xi) \subseteq I\!P(\varphi)\}$ and similarly for $\mathcal{L}'$. Let $\mathcal{L}'_{\varphi_0\psi_0} = \mathcal{L}'_{\varphi_0} \cap \mathcal{L}'_{\psi_0}$.

Let $\varphi_1, \ldots, \varphi_k, \ldots$ be an enumeration of all formulas in $\mathcal{L}'_{\varphi_0}$,
$\psi_1, \ldots, \psi_k, \ldots$ be an enumeration of all formulas in $\mathcal{L}'_{\psi_0}$.

We define the sequences, $\{n_0\} \cup \{\varphi_0\} = T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots$
$\{n_0\} \cup \{\neg\psi_0\} = U_0 \subseteq U_1 \subseteq U_2 \subseteq \ldots$

as follows:

- If $T_j \cup \{\varphi_j\}$ and U_j are separable over $\mathcal{L}'_{\varphi_0\psi_0}$ then $T_{j+1} = T_j$, else
  - if $\varphi_j \neq @_s s$ and $\varphi_j \neq @_s\Diamond\varphi'$ for $s \in$ WSYM, then $T_{j+1} = T_j \cup \{\varphi_j\}$,
  - if $\varphi_j = @_s s$, then $T_{j+1} = T_j \cup \{\varphi_j\} \cup \{@_s(n_k \wedge s)\}$, for $n_k \in \mathrm{N}\backslash\mathrm{NOM}(T_j \cup U_j)$,
  - if $\varphi_j = @_s\Diamond\varphi'$, then $T_{j+1} = T_j \cup \{\varphi_j\} \cup \{@_s\Diamond(n_k \wedge \varphi')\}$, for $n_k \in \mathrm{N}\backslash\mathrm{NOM}(T_j \cup U_j \cup \{\varphi_j\})$.
- If T_{j+1} and $U_j \cup \{\psi_j\}$ are separable over $\mathcal{L}'_{\varphi_0\psi_0}$ then $U_{j+1} = U_j$, else
  - if $\psi_j \neq @_s s$ and $\psi_j \neq @_s\Diamond\psi'$ for $s \in$ WSYM, then $U_{j+1} = U_j \cup \{\psi_j\}$,
  - if $\psi_j = @_s s$, then $U_{j+1} = U_j \cup \{\psi_j\} \cup \{@_s(n_k \wedge s)\}$, for $n_k \in \mathrm{N}\backslash\mathrm{NOM}(T_{j+1} \cup U_j)$,
  - if $\psi_j = @_s\Diamond\psi'$, then $T_{j+1} = T_j \cup \{\psi_j\} \cup \{@_s\Diamond(n_k \wedge \psi')\}$, for $n_k \in \mathrm{N}\backslash\mathrm{NOM}(T_{j+1} \cup U_j \cup \{\psi_j\})$.

The fresh nominals play the same role as *Henkin witnesses* in first-order proofs: they ensure that we obtain models in which every world has a name. Define

$$T_\omega = \bigcup_{j\in\omega} T_j \quad \text{and} \quad U_\omega = \bigcup_{j\in\omega} U_j.$$

Claim 1 *For all $j \in \omega$, $\langle T_j, U_j\rangle$ is inseparable with respect to $\mathcal{L}'_{\varphi_0\psi_0}$. Whence $\langle T_\omega, U_\omega\rangle$ is an inseparable pair with respect to $\mathcal{L}'_{\varphi_0\psi_0}$. Furthermore T_ω (resp. U_ω) is maximal consistent in $\mathcal{L}'_{\varphi_0}$ (resp. $\mathcal{L}'_{\psi_0}$). In particular, for all $\theta \in \mathcal{L}'_{\varphi_0\psi_0}$: $\theta \in T_\omega \Leftrightarrow \theta \in U_\omega$.*

Proof of Claim. The proof is by induction on j. Separability/inseparability below is with respect to $\mathcal{L}'_{\varphi_0\psi_0}$ except when otherwise mentioned.

Base Case $j=0$. Suppose $\langle T_0, U_0\rangle$ is separable. Then there is a formula $\theta \in \mathcal{L}'_{\varphi_0\psi_0}$ such that $\models n_0 \wedge \varphi_0 \to \theta$ and $\models n_0 \wedge \neg\psi_0 \to \neg\theta$. θ might contain some nominals of N, say $\{n_{i_1}, \dots, n_{i_k}\}, k \geq 0$. Let $x_0, x_{i_1}, \dots, x_{i_k} \in$ WVAR which don't occur in $\varphi_0, \psi_0, \theta$. We will write $\theta[x_0 x_{i_1} \dots x_{i_k}]$ for the formula obtained from θ by replacing n_{i_j} by x_{i_j}, and n_0 by x_0. Then, making use of the complete axiomatization given in Section 2, we have:

$\models \varphi_0 \to (n_0 \to \theta)$	
$\models \downarrow x_{i_k}.(\varphi_0 \to (n_0 \to \theta[x_{i_k}]))$	Proposition 2.6
$\models \varphi_0 \to (n_0 \to \downarrow x_{i_k}.\theta[x_{i_k}])$	*Q1* twice
$\models \varphi_0 \to (n_0 \to \downarrow x_{i_1}.\dots\downarrow x_{i_k}.\theta[x_{i_1}\dots, x_{i_k}])$	Similarly
$\models \downarrow x_0.(\varphi_0 \to (x_0 \to \downarrow x_{i_1}.\dots\downarrow x_{i_k}.\theta[x_0 x_{i_1}\dots, x_{i_k}]))$	Proposition 2.6
$\models \varphi_0 \to \downarrow x_0.(x_0 \to \downarrow x_{i_1}.\dots\downarrow x_{i_k}.\theta[x_0 x_{i_1}\dots, x_{i_k}]))$	*Q1*
$\models \varphi_0 \to \downarrow x_0.\downarrow x_{i_1}.\dots\downarrow x_{i_k}.\theta[x_0 x_{i_1}\dots, x_{i_k}]$	*Q3*
$\models \neg\psi_0 \to (n_0 \to \neg\theta)$	
$\models \neg\psi_0 \to \downarrow x_0.\downarrow x_{i_1}.\dots\downarrow x_{i_k}.\neg\theta[x_0 x_{i_1}\dots, x_{i_k}]$	As before
$\models \neg\psi_0 \to \neg\downarrow x_0.\downarrow x_{i_1}.\dots\downarrow x_{i_k}.\theta[x_0 x_{i_1}\dots, x_{i_k}]$	Self Dual$_\downarrow$

$\theta[x_0 x_{i_1}\dots, x_{i_k}]$ is a formula in $\mathcal{L}_{\varphi_0} \cap \mathcal{L}_{\psi_0}$ and thus $\langle\{\varphi_0\}, \{\neg\psi_0\}\rangle$ is separable over $\mathcal{L}_{\varphi_0} \cap \mathcal{L}_{\psi_0}$. Contradiction.

By using the inductive hypothesis "$\langle T_j, U_j\rangle$ is an inseparable pair" and going step by step through the construction, the inseparability of $\langle T_{j+1}, U_{j+1}\rangle$ is easily established. ◀

To start the construction of our model, we recall the notion of pasted maximal consistent sets (MCS) and labeled models from [BT99]. A maximal consistent set Γ is *pasted* if

$@_s\varphi \in \Gamma$ implies for some nominal i, $@_s(i \wedge \varphi) \in \Gamma$ and

$@_s\Diamond\varphi \in \Gamma$ implies for some nominal i, $@_s\Diamond(i \wedge \varphi) \in \Gamma$.

A pasted MCS Γ is *labeled* by a nominal i precisely when $i \in \Gamma$. Let Γ be a pasted MCS labeled by a nominal, then for all world symbols s appearing in Γ, let $\Delta_s \stackrel{def}{=} \{\varphi \mid @_s\varphi \in \Gamma\}$. Then the *labeled model* yielded by Γ is $\mathfrak{M} = \langle M, R, V\rangle$, where

$M = \{\Delta_s \mid s$ is a world symbol in $\Gamma\}$,

$R\Delta\Delta'$ iff $\{\varphi \mid \Box\varphi \in \Delta\} \subseteq \Delta'$,

$\Delta \in V(p)$ iff $p \in \Delta$, for p a propositional variable or nominal.

We define the natural assignment $g : \text{WVAR} \to M$ by $g(x) = \{m \in M \mid x \in m\}$.

By construction T_ω and U_ω are pasted MCSs labeled by the nominal $n_0 \in N$. Let $\mathfrak{M}_{\varphi_0} = \langle M_{\varphi_0}, R_{\varphi_0}, V_{\varphi_0}\rangle$ be the labeled model obtained from T_ω and $\mathfrak{M}_{\psi_0} = \langle M_{\psi_0}, R_{\psi_0}, V_{\psi_0}\rangle$ the one obtained from U_ω. We denote by $\Delta_s^{\varphi_0}$ (resp. $\Delta_s^{\psi_0}$) the elements of M_{φ_0} (resp. M_{ψ_0}).

Claim 2

i. $\Delta_{n_0}^{\varphi_0} = T_\omega$ *and* $\Delta_{n_0}^{\psi_0} = U_\omega$.

ii. For $\Delta_s^{\varphi_0} \in M_{\varphi_0}$ *there is* $n \in N$ *such that* $n \in \Delta_s^{\varphi_0}$ *(or equivalently* $\Delta_s^{\varphi_0} = \Delta_n^{\varphi_0}$*). Similarly for* $\Delta_s^{\psi_0} \in \mathfrak{M}_{\psi_0}$.

PROOF OF CLAIM.
i. We show that $\Delta^{\varphi_0}_{n_0} = T_\omega$; the other case is similar. $\Delta^{\varphi_0}_{n_0}$ is an MCS because $@_{n_0}$ is self-dual. So it is sufficient to show that $\Delta^{\varphi_0}_{n_0} \supseteq T_\omega$. Let $\varphi \in T_\omega$, by *Introduction* $\models n_0 \wedge \varphi \to @_{n_0}\varphi$. Because $@_{n_0}\varphi \in \mathcal{L}'_{\varphi_0}$, $n_0 \in T_\omega$ and T_ω is maximal in this language, $@_{n_0}\varphi \in T_\omega$. By definition $\varphi \in \Delta^{\varphi_0}_{n_0}$.
ii. By [BT99] Lemma 4.3.5 we have $n \in \Delta^{\varphi_0}_s \Rightarrow \Delta^{\varphi_0}_s = \Delta^{\varphi_0}_n$. We prove the case for $\mathfrak{M}_{\varphi_0}$. $@_s s$ is a formula in $\mathcal{L}'_{\varphi_0}$, hence $@_s s = \varphi_j$ for some j, and furthermore a theorem. Hence $\{@_s s\}$ will be added to T_{j+1} together with $@_s(s \wedge n_k)$ for a new nominal n_k. It follows that $n_k \in \Delta^{\varphi_0}_s$ and hence $\Delta^{\varphi_0}_{n_k} = \Delta^{\varphi_0}_s$. ◀

From Claim 2 using results in [BT99] it follows that $\mathfrak{M}_{\varphi_0}$ and $\mathfrak{M}_{\psi_0}$ are hybrid models satisfying a Truth Lemma, and the natural assignment g defined as $g(x) = \{m \in M \mid x \in m\}$ is an assignment. Thus we obtain

$$\mathfrak{M}_{\varphi_0}, g_{\varphi_0}, \Delta^{\varphi_0}_{n_0} \Vdash \varphi \text{ and } \mathfrak{M}_{\psi_0}, g_{\psi_0}, \Delta^{\psi_0}_{n_0} \Vdash \neg\psi. \tag{1}$$

Furthermore the two models are very closely related.

Claim 3 *Let a function $h : M_{\varphi_0} \to M_{\psi_0}$ be defined as $h(\Delta^{\varphi_0}_n) = \Delta^{\psi_0}_n$, for $n \in N$. Then h is a bijection which respects the accessibility relation and the propositional variables and nominals in the common language $\mathcal{L}'_{\varphi_0\psi_0}$. Moreover, $g^{\psi_0} = h \circ g^{\varphi_0}$, for g the natural assignment.*

PROOF OF CLAIM. h is defined at every member of the domain of $\mathfrak{M}_{\varphi_0}$ by Claim 2.*ii* and the fact that for any $n \in \mathrm{N}$, both $\Delta^{\varphi_0}_n$ and $\Delta^{\psi_0}_n$ are uniquely defined.

Moreover, h is a bijection because $@_n\top \in T_\omega$ iff $@_n\top \in U_\omega$ and in $\mathfrak{M}_{\varphi_0}$ and $\mathfrak{M}_{\psi_0}$ nominals are interpreted as singletons.

For all proposition variables p in $\mathcal{L}'_{\varphi_0\psi_0}$ we have $\Delta^{\varphi_0}_n \in V^{\varphi_0}(p)$ iff $@_n p \in T_\omega$ iff $@_n p \in U_\omega$ iff $h(\Delta^{\varphi_0}_n) \in V^{\psi_0}(p)$. For the relation R, $(\Delta^{\varphi_0}_n, \Delta^{\varphi_0}_{n'}) \in R^{\varphi_0}$ iff $@_n\Diamond n' \in T_\omega$ iff $@_n\Diamond n' \in U_\omega$ iff $(h(\Delta^{\varphi_0}_n), h(\Delta^{\varphi_0}_{n'})) \in R^{\psi_0}$. A similar argument shows $g^{\psi_0} = h \circ g^{\varphi_0}$. ◀

Since the two models share the same frame, and agree on the common language, there is a model $\mathfrak{M}$ for the union of the two languages which has $\mathfrak{M}_{\varphi_0}$ and $\mathfrak{M}_{\psi_0}$ as reducts. But then by (1), $\mathfrak{M}, g, \Delta_{n_0} \Vdash \varphi_0 \wedge \neg\psi_0$, and we have proved the contrapositive. QED

Actually, we can prove a stronger result: we can restrict the free variables occurring in the interpolant θ to only those appearing both in φ_0 and ψ_0. Reason as follows. Assume $\models \varphi_0(\bar{x}, \bar{z}) \to \psi_0(\bar{y}, \bar{z})$; here we explicitly show the world variables free in φ_0 and ψ_0 and indicate which are shared. Now by Proposition 2.6 we can replace free variables by new nominals $\bar{i_x}$, $\bar{i_y}$ and $\bar{i_z}$ such that $\models \varphi_0(\bar{i_x}, \bar{i_z}) \to \psi_0(\bar{i_y}, \bar{i_z})$. Use Theorem 5.4 to find an interpolant $\theta(\bar{s}, \bar{i_z})$. The $\bar{s}$ are free variables that might appear in θ; we can assume $\bar{s}$ disjoint from $\bar{x}, \bar{y}, \bar{z}$. By $Nec_\downarrow$ and Q_1 we obtain $\models \varphi_0(\bar{i_x}, \bar{i_z}) \to \downarrow\bar{s}.\theta(\bar{i_z})$ and $\models \downarrow\bar{s}.\theta(\bar{i_z}) \to \psi_0(\bar{i_y}, \bar{i_z})$. We now rename again the formulas to the original $\bar{x}, \bar{y}, \bar{z}$ and the interpolant will only contain free world variables common to φ_0 and ψ_0.

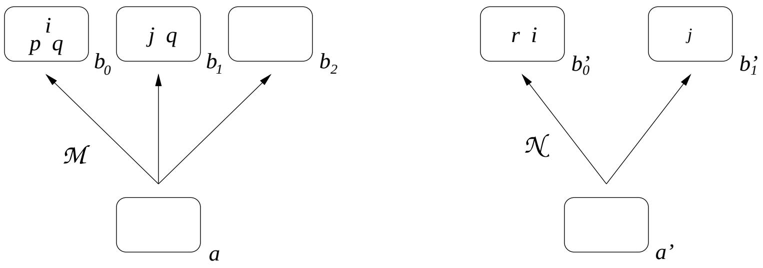
Furthermore, note that nothing in the proof is intrinsically tied to the number of modalities in the language. In other words, arrow interpolation also holds for the multi-modal versions of $\mathcal{H}(\downarrow, @)$ if modalities are allowed freely in the interpolant.

Corollary 5.5 *Multi-modal $\mathcal{H}(\downarrow, @)$ has arrow interpolation if no restriction on occurrences of modalities is imposed for the interpolant.*

However when we restrict the interpolant to contain only the modalities in the common language, then interpolation does *not* follows immediately, especially if the modalities interact (for example, if the theory contains axioms involving more than one modality; see [Mar99] for examples of this type). Nonetheless, we advance the following conjecture:

Conjecture 5.6 *Multi-modal $\mathcal{H}(\downarrow, @)$ has arrow interpolation even if the interpolant's modalities are restricted to the common language.*

On the other hand, it is clear from the proof that the number of world variables needed cannot be bounded (they are used to quantify away the nominals in the proof of Claim 1). Indeed, if we restrict $\mathcal{H}(\downarrow, @)$ to only a finite number of variables, then arrow interpolation *fails*. Because we have the notion of a k-bisimulations at our disposal, it is fairly straightforward to provide counterexamples. Let's consider first the case of $\mathcal{H}(\downarrow, @)$ restricted to only one world variable. Take the models



and the formulas

$$\varphi = \Diamond(p \wedge q) \wedge \Diamond(\neg p \wedge q) \wedge \Diamond(\neg p \wedge \neg q)$$

$$\psi = (\Diamond r \wedge \Box(r \to i)) \to (\Diamond(\neg r \wedge j) \to \Diamond(\neg r \wedge \neg j)).$$

It is easy to prove that $\varphi \to \psi$ is valid: in any model having a world with at least three different accessible worlds, if there is a unique accessible r–world and one of the accessible $\neg r$–worlds is named by the nominal j, then the second accessible $\neg r$–world is named $\neg j$. Furthermore, $\mathfrak{M}$ and $\mathfrak{N}$ 1-bisimulate in the common (empty) language via the relation

$$(m, m') \overset{1}{\sim} (n, n') \text{ iff } \mathrm{depth}(m) = \mathrm{depth}(n) \ \&\ \mathrm{depth}(m') = \mathrm{depth}(n') \ \&\ (m = m' \Leftrightarrow n = n'),$$
where depth is the distance from the root.

Finally, φ is true in $\mathfrak{M}, a$, while ψ is false at $\mathfrak{N}, a'$ which proves that an interpolant on only one variable does not exist[2]. Actually, not even an interpolant in two variables exists as also a 2-bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$ can be defined.

Notice that the heart of the counterexample is just a counting argument, which can be reproduced for the other finite variable fragments of $\mathcal{H}(\downarrow, @)$ by taking bigger and bigger models $\mathfrak{M}$ and $\mathfrak{N}$ exhibiting the same basic pattern. Hence:

Theorem 5.7 *Strong interpolation (AIP) fails in all finite world variable fragments of $\mathcal{H}(\downarrow, @)$.*

[2]The formulas $\varphi = \Diamond p \wedge \Diamond \neg p$ and $\psi = \Diamond i \to \Diamond \neg i$ provide a simpler counterexample to strong interpolation in the one-variable fragment, but they have $\Diamond \top \wedge (\Diamond x \to \Diamond \neg x)$ as weak interpolant. The more complex example proves failure in the weak case also.

A more complex counterexample based on the same idea can be set up to prove failure of weak (turnstile) interpolation. Take the same formulas φ and ψ as before; clearly $\varphi \models^{glo} \psi$. Take now the model $\mathfrak{M}$ and define $\mathfrak{M}'$ by linking new copies of b_0, b_1 and b_2 to each terminal world in $\mathfrak{M}$. Let $\mathfrak{M}_\omega$ be the infinite model obtained by iterating this operation ω times and similarly for $\mathfrak{N}_\omega$. Now $\mathfrak{M}_\omega$ makes φ globally true. Suppose θ is an interpolant on one variable. Then as $\varphi \models^{glo} \theta$, θ is globally true at $\mathfrak{M}_\omega$.

We need something stronger than a mere 1-bisimulations linking $\mathfrak{M}_\omega$ and $\mathfrak{N}_\omega$, as we want to transfer *global* truth. With ordinary modal languages, requiring $\sim$ to be total and surjective is enough, but we have to take care of assignments as well. We shall say that a k-bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$ is *full* if for every pair $\langle \bar{m}, m\rangle \in {}^kM \times M$ there exists $\langle \bar{n}, n\rangle \in {}^kN \times N$ such that $\langle \bar{m}, m\rangle \overset{k}{\sim} \langle \bar{n}, n\rangle$ and vice versa. If we can define a full 1-bisimulation between $\mathfrak{M}_\omega$ and $\mathfrak{N}_\omega$ then $\mathfrak{N}_\omega \models^{glo} \theta$. But $\overset{1}{\sim}$ defined as in the previous case is indeed full. Hence, as $\theta \models^{glo} \psi$, ψ should be globally true in $\mathfrak{N}_\omega$ — but it is not.

Theorem 5.8 *Weak interpolation (TIP) fails in all finite world variable fragments of $\mathcal{H}(\downarrow, @)$.*

It is not difficult to see that in both cases (arrow and turnstile)

$$\theta = \downarrow x.\Diamond(\downarrow y.@_x(\Diamond(\downarrow y'.@_x(\Diamond(y \wedge \neg y') \wedge \Diamond(\neg y \wedge y') \wedge \Diamond(\neg y \wedge \neg y')))))$$

(a sentence in 3 variables) is an interpolant.

Finally, we see from the proof of our main result that the $\downarrow$ binder is needed in Claim 1. So what about interpolation in the sublanguage $\mathcal{H}(@)$? We can again use models $\mathfrak{M}$ and $\mathfrak{N}$ to prove that arrow interpolation fails. We use the restricted version of k-bisimulation which leaves out condition **($\downarrow$)**. In this framework we can define for any k, a k-bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$ such that for any $\bar{m} \in {}^kM$ and any $\bar{n} \in {}^kN$, $(\bar{m}, a) \overset{k}{\sim} (\bar{n}, a')$. This proves that there is no arrow interpolant for $\varphi \to \psi$ in $\mathcal{H}(@)$.

Theorem 5.9 *Strong interpolation (AIP) fails in $\mathcal{H}(@)$.*

But *weak* interpolation holds for $\mathcal{H}(@)$ because the role of $\downarrow$ is played by the implicit quantification in the definition of $\varphi \models^{glo} \psi$. (Another way of looking at it is to say that the definition of $\models^{glo}$ sneaks in implicit quantification using the global hybrid binder $\forall$ mentioned in the introduction.)

Theorem 5.10 *Let φ and ψ be sentences of $\mathcal{H}(@)$ such that $\varphi \models^{glo} \psi$. Then there is a formula θ, which may contain additional free variables, such that*

i. $\varphi \models^{glo} \theta$ *and* $\theta \models^{glo} \psi$.

ii. $\mathbb{P}(\theta) \subseteq \mathbb{P}(\varphi) \cap \mathbb{P}(\psi)$.

Outline of Proof. We outline how the proof of arrow interpolation for $\mathcal{H}(\downarrow, @)$ should be modified to obtain the result.

First, the construction of the pasted sets T_ω and U_ω needs to be adjusted as we need to ensure that the labeled models obtained from them make φ_0 and $\neg\psi_0$ *globally* true. To that end, whenever we run into a formula of the form $@_s s$ or $@_s\Diamond\xi$ we paste not only a new nominal n_k but also the formulas we want to make globally true. For example one clause in the definition of T_{j+1} would read

– if $\varphi_j = @_s s$, then $T_{j+1} = T_j \cup \{\varphi_j\} \cup \{@_s(n_k \wedge s \wedge \varphi_0)\}$, for $n_k \in \mathrm{N}\backslash\mathrm{NOM}(T_j \cup U_j)$.

We will need to show that for all $j \in \omega$, $\langle T_j, U_j\rangle$ is (globally) inseparable with respect to $\mathcal{L}'_{\varphi_0\psi_0}$. The base case is simple: if θ (including perhaps some new nominals $\{n_{i_i}, \ldots, n_{i_k}\}$) separates $\langle T_0, U_0\rangle$ on $\mathcal{L}'_{\varphi_0\psi_0}$, then $\theta[x_{i_1} \ldots x_{i_k}]$ separates $\langle\{\varphi_0\}, \{\neg\psi_0\}\rangle$, for new variables $\{x_{i_i}, \ldots, x_{i_k}\}$; this is precisely where the free variables in the interpolant are needed.

What about the inductive step? Consider, for example, the case of $\varphi_j = @_s s$. Assume that $\langle T_j \cup \{\varphi_j\}, U_j\rangle$ is inseparable in $\mathcal{L}'_{\varphi_0\psi_0}$; we want to prove that $\langle T_j \cup \{\varphi_j, @_s(n_k \wedge s \wedge \varphi_0)\}, U_j\rangle$ is inseparable. Suppose θ separates this last pair. Then $U_j \models^{glo} \neg\theta$ while $T_j \cup \{@_s s, @_s(n_k \wedge s \wedge \varphi_0)\} \models^{glo} \theta$. Because $@_s(n_k \wedge s \wedge \varphi_0)$ is an @-formula, this is the case iff $T_j \cup \{@_s s\} \models^{glo} @_s(n_k \wedge s \wedge \varphi_0) \rightarrow \theta$. Furthermore, as $\varphi_0 \in T_j$ and n_k is a new nominal by definition, for all $\mathfrak{M}$, $\mathfrak{M} \models T_j$ implies $\mathfrak{M} \models @_s(n_k \wedge s \wedge \varphi_0)$. Hence $T_j \cup \{@_s s\} \models^{glo} \theta$. Contradiction.

From now on the proof follows the same lines as before. We obtain labeled models such that $\mathfrak{M}_{\varphi_0} \models \varphi_0$ and $\mathfrak{M}_{\psi_0} \models \neg\psi_0$ "sharing" the same frame, from which we build a model $\mathfrak{M}$ where $\varphi_0 \wedge \neg\psi_0$ holds globally. QED

We will prove in the next section that the $\mathcal{H}(@)$ is well behaved with respect to complexity: like ordinary unimodal logic it has a PSPACE-complete satisfaction problem. On the other hand, $\mathcal{H}(\downarrow, @)$ is known to be undecidable. Thus we have a decidable system with TIP, an undecidable system with AIP, and it is natural to ask:

Open Question 5.11 *Is there any decidable hybrid language extending $\mathcal{H}(@)$ that enjoys arrow interpolation?*

To close this section, some remarks on Beth Definability. The Beth Definability Property [Bet53] is commonly studied together with interpolation (and in many cases interpolation is considered just a step in the proof of the Beth Property). Loosely speaking, a logic has the Beth property if any implicit definition has also an explicit definition. More precisely, for hybrid languages we define:

Definition 5.12 (Beth Definability) *A hybrid logic has the* Beth Definability Property *if for all formulas $\varphi(\bar{a}, a)$ whose propositional letters, free world variables and nominals occur among $\bar{a}, a$, if $\models \varphi(\bar{a}, a/b_1) \wedge \varphi(\bar{a}, a/b_2) \rightarrow (b_1 \leftrightarrow b_2)$ then there is a formula $\psi(\bar{a})$ such that $\models \varphi(\bar{a}, a) \rightarrow (\psi(\bar{a}) \leftrightarrow a)$.*

Using a standard argument [Cra57] we can easily derive the Beth definability property for $\mathcal{H}(\downarrow, @)$ from Theorem 5.4.

Theorem 5.13 *$\mathcal{H}(\downarrow, @)$ has the Beth definability property.*

A careful analysis of weaker versions of the Beth property can be carried out for fragments of $\mathcal{H}(\downarrow, @)$ as we did with the interpolation property.

6 Complexity

Beautiful as $\mathcal{H}(\downarrow, @)$ is, it has a drawback: its satisfaction problem is undecidable (see [BS95]). The culprit is the binder $\downarrow$; @ is not needed. However $\mathcal{H}(\downarrow, @)$ has an interesting sublanguage, namely $\mathcal{H}(@)$. As we have just seen, although $\mathcal{H}(@)$ does not enjoy strong interpolation, it does have weak interpolation. Moreover, as is shown in [Bla98], simple tableaux and sequent systems for $\mathcal{H}(@)$ can be defined by exploiting the interplay between nominals (or free variables) and @; the underlying idea traces back to [Sel91]. Furthermore, while $\mathcal{H}(@)$ doesn't offer any exciting new expressivity at the level of *models* (for example, without the $\downarrow$ binder we can't define *Until*) it does provide new expressivity at the level of *frames*: we can define many properties that are not definable in ordinary propositional modal logic, including irreflexivity ($@_i\Box\neg i$), asymmetry ($@_i\Diamond j \to @_j\neg\Diamond i$) and antisymmetry ($@_i\Diamond j \wedge @_j\Diamond i \to @_i j$); these correspondences are easy to check using the standard translation. Moreover, pure formulas such as these automatically yield complete axiomatizations for the frame classes they define (see [BT99, Bla98]).

Thus there are many reasons for being interested in $\mathcal{H}(@)$, and a natural question to ask is: how high a computational price do we pay for these benefits? It turns out that (up to a polynomial) there are no extra computational costs when expanding unimodal logic (or even multimodal logic) with @ and nominals and/or free variables, but expanding tense logic with even a single nominal (or free variable) leads to an increase in complexity (assuming PSPACE $\neq$ EXPTIME).

For the most part we will study (local) K-*satisfaction problems*: given a formula φ, does there exist a model $\mathfrak{M}$, a world m and an assignment g, such that $\mathfrak{M}, g, m \Vdash \varphi$? (The K reflects the fact that we place no restrictions on the satisfying models; thus in effect we're measuring the complexity of the minimal logic K in whatever language we're working with.) Note that as far as local satisfaction problems are concerned, if we replace all variables in φ by nominals, obtaining φ', then φ is satisfiable if and only if φ' is satisfiable. For this reason we can restrict ourselves in the proofs to formulas without variables, and we won't need to mention variable assignments. However at the end of this section we examine the complexity of the *global* K-satisfaction problem and an interesting difference between nominals and world variables emerges.

We will consider expansions with @ and nominals of the following modal languages:

1. the unimodal language; that is, the language with one operator $\Box$ and its dual $\Diamond$;

2. the basic tense language; that is, the language with two operators G and H and their respective duals F and P, where the accessibility relation for H and P is the converse of the relation for G and F;

3. their multimodal versions: frames come with a family of accessibility relations $\{R_i\}_{i\in I}$, each has its own forward looking modality $[i]$ and dual $\langle i\rangle$, and in the tense case, also their backward looking counterparts $[i]^{\smile}$ and $\langle i\rangle^{\smile}$.

The complexity of the K-satisfaction problem for all these modal languages is known:

Theorem 6.1
i. [Lad77] The K*-satisfaction problem in the unimodal language is* PSPACE*-hard.*
ii. [Spa93b] The K*-satisfaction problem for the tensed multimodal languages is in* PSPACE.

That is, for ordinary modal languages the additional expressive power obtained by adding a backwards looking operator does not lead to an increase in complexity, and neither does the addition of more modalities, including multiple backwards looking operators. However if we hybridize these languages, the situation is rather different. For a start, adding even just one nominal to tense logic causes a jump in complexity from PSPACE to EXPTIME:

Theorem 6.2 *The* K*-satisfaction problem for a basic tense language containing at least one nominal is* EXPTIME*-hard.*

PROOF. We shall reduce the EXPTIME-complete *global* K-satisfaction problem for unimodal languages to the (local) K-satisfaction problem for a basic tense language that contains at least one nominal. The global K-satisfaction problem for unimodal languages is this: given a formula φ in the unimodal language, does there exist a Kripke model $\mathfrak{M}$ such that $\mathfrak{M} \models \varphi$ (in other words, where φ is true in *all* worlds)? The EXPTIME-completeness of this problem is an easy consequence of (the proof of) the EXPTIME-completeness of modal logic K expanded with the universal modality in [Spa93a].

Our reduction uses the *spy-point* technique from [BS95]. Define the following translation function $(\cdot)^t$ from ordinary unimodal formulas to formulas in a tense language that contains at least one nominal i:

$$\begin{array}{rcl} p^t & = & p \\ (\neg\varphi)^t & = & \neg\varphi^t \\ (\varphi \wedge \psi)^t & = & \varphi^t \wedge \psi^t \\ (\Diamond\varphi)^t & = & \mathsf{F}(\mathsf{P}i \wedge \varphi^t), \end{array}$$

Note that i is a *fixed* nominal in this translation. Clearly $(\cdot)^t$ is a linear reduction. We claim that for any formula φ, φ is globally K-satisfiable if and only if $i \wedge \mathsf{G}(\mathsf{P}i \rightarrow \varphi^t)$ is K-satisfiable.

For the left to right direction, let $\mathfrak{M} \models \varphi$, where $\mathfrak{M} = \langle M, R, V\rangle$ is an ordinary Kripke model. Define $\mathfrak{M}^*$ as follows:

$$\begin{array}{rcl} M^* & = & M \cup \{i\}, \\ R^* & = & R \cup \{(i, m) \mid m \in M\}, \\ V^* & = & V \cup \{(n, \{i\}) \mid \text{ for all nominals } n\}. \end{array}$$

$\mathfrak{M}^*$ is a hybrid model, for all nominals (including i) are interpreted by the singleton set $\{i\}$, our spy-point. We claim that for all $m \in M$, for all ψ, we have $\mathfrak{M}, m \Vdash \psi$ if and only if $\mathfrak{M}^*, m \Vdash \psi^t$. This follows by a simple induction. The only interesting step is for $\Diamond$:

$$\begin{array}{cl} & \mathfrak{M}, m \Vdash \Diamond\psi \\ \Leftrightarrow & (\exists m' \in M) : Rmm' \;\&\; \mathfrak{M}, m' \Vdash \psi \\ \Leftrightarrow & (\exists m' \in M^*) : R^*mm' \;\&\; \mathfrak{M}^*, m' \Vdash \psi^t \;\&\; R^*im' \text{ (by IH and definition of } R^*) \\ \Leftrightarrow & \mathfrak{M}^*, m \Vdash \mathsf{F}(\mathsf{P}i \wedge \psi^t) \\ \Leftrightarrow & \mathfrak{M}^*, m \Vdash (\Diamond\psi)^t \end{array}$$

It follows that $\mathfrak{M}^*, i \Vdash i \wedge \mathsf{G}(\mathsf{P}i \rightarrow \varphi^t)$, as desired.

For the other direction, let $\mathfrak{M}, w \Vdash i \wedge \mathsf{G}(\mathsf{P}i \rightarrow \varphi^t)$, where $\mathfrak{M} = \langle M, R, V\rangle$ is a hybrid model. Define $\mathfrak{M}^*$ as follows

$$\begin{array}{rcl} M^* & = & \{m \in M \mid Rwm\}, \\ R^* & = & R_{\restriction M^*}, \\ V^* & = & V_{\restriction M^*}. \end{array}$$

We claim that for all $m \in M^*$, for all ψ, $\mathfrak{M}, m \Vdash \psi^t$ if and only if $\mathfrak{M}^*, m \Vdash \psi$. Again we only present the inductive step for $\Diamond$:

$$
\begin{aligned}
& \mathfrak{M}, m \Vdash \mathsf{F}(\mathsf{P}i \wedge \psi^t) \\
\Leftrightarrow\ & (\exists m' \in M) : Rmm' \;\&\; Rwm' \;\&\; \mathfrak{M}, m' \Vdash \psi^t \\
\Leftrightarrow\ & (\exists m' \in M^*) : Rmm' \;\&\; Rwm' \;\&\; \mathfrak{M}, m' \Vdash \psi^t \\
\Leftrightarrow\ & (\exists m' \in M^*) : R^*mm' \;\&\; \mathfrak{M}^*, m' \Vdash \psi \text{ (by IH and definition of } M^*) \\
\Leftrightarrow\ & \mathfrak{M}^*, m \Vdash \Diamond\psi.
\end{aligned}
$$

For all $m \in M^*$, Rwm holds, whence for all $m \in M^*$, $\mathfrak{M}, m \Vdash \mathsf{P}i$. So, since $\mathfrak{M}, w \Vdash \mathsf{G}(\mathsf{P}i \to \varphi^t)$, for all $m \in M^*$, $\mathfrak{M}, m \Vdash \varphi^t$. Hence by our last claim $\mathfrak{M}^* \models \varphi$, which is what we needed to show. QED

We obtain the corresponding upper bound by translating into first-order logic. It turns out that we end up in (a version of) the guarded fragment with two variables (see [AvBN98]). The satisfaction problem of this fragment is known to be EXPTIME-complete, whence the desired result.

In fact we will prove a stronger result. The *universal modality* E is a modality with the following interpretation: in any model $\mathfrak{M}$, at any $m \in M$, $\mathfrak{M}, m \Vdash E\varphi$ iff for some $m' \in M$, $\mathfrak{M}, m' \Vdash \varphi$. (That is, with the aid of the universal modality we can scan the entire model for information.) The universal modality is very strong (see [GP92] for a detailed discussion) but for present purposes we need merely observe that, given E, we can define $@_i\varphi$ to be $E(i \wedge \varphi)$. In spite of its strength, we have the following upper bound:

Theorem 6.3 *The* K*-satisfaction problem for a multimodal tense language expanded with both nominals and the universal modality is* EXPTIME*-complete.*

PROOF. The lower bound was proved in the previous theorem. For the upper bound, use the standard translation to first-order logic with the following additional clause for the universal modality:

$$ST_x(E\varphi) \quad = \quad \exists x.ST_x(\varphi).$$

The translation only uses 2 variables. Moreover, inspection shows that the range of the translation belongs to the packed fragment defined in [Mar97]. (It does not fall in the guarded fragment as defined in [AvBN98] because the translation of E is not guarded, though a translation into the guarded fragment could be obtained by relativizing with a unary predicate $M(x)$ uniformly in all clauses of the translation.) But for a fixed number of variables, the satisfaction problem of the packed fragment is EXPTIME-complete; see [Mar97]. QED

Summing up: the addition of even one nominal to a basic tense language yields an EXPTIME-hard local K-satisfaction problem. However, adding further nominals, multiple forwards and backwards looking modalities, and even the universal modality, doesn't take us any higher in the complexity hierarchy.

This completes our complexity analysis of tense languages, so let's turn to a positive result: (multimodal) $\mathcal{H}(@)$ has the same complexity as ordinary propositional modal logic.

Theorem 6.4 *The* K*-satisfaction problem of multimodal language enriched with nominals and @ is* PSPACE*-complete.*

PROOF. The lower bound follows from Fact 6.1. We show the upper bound by defining the notion of a ξ-game between two players. We will show that the existential player has a winning strategy for the ξ-game if and only if ξ is satisfiable. Moreover every ξ-game stops after at most as many rounds as the modal depth of ξ and the information on the playing board is polynomial in the length of ξ. Using the close correspondence between Alternating Turing Machines (ATM's) and two player games [Chl86], it is straightforward to implement the problem of whether the existential player has a winning strategy in the ξ-game on a PTIME ATM. Because any PTIME ATM algorithm can be turned into a PSPACE Turing Machine program, we obtain our desired result. We present the proof only for unimodal $\mathcal{H}(@)$; it can be straightforwardly extended to the multimodal case.

Fix a formula ξ. A ξ-HS (for Hintikka Set) is a maximal consistent set of subformulas of ξ. We denote the set of subformulas of ξ by $SF(\xi)$. The ξ-game is played as follows. There are two players, $\forall$belard (male) and $\exists$loise (female). She starts the game by playing a collection $\{X_0, \ldots, X_k\}$ of Hintikka Sets, and she specifies a relation R on them.

$\exists$loise loses immediately if one of the following conditions is false:

1. X_0 contains ξ, and all the other X_l contain at least one nominal occurring in ξ.
2. no nominal occurs in two different Hintikka Sets.
3. for all X_l, for all $@_i\varphi \in SF(\xi)$, $@_i\varphi \in X_l$ iff $\{i, \varphi\} \subseteq X_k$, for some k.
4. for all $\Diamond\varphi \in SF(\xi)$, if RX_lX_k and $\Diamond\varphi \notin X_l$, then $\varphi \notin X_k$.

Now $\forall$belard may choose an X_l and a "defect-formula" $\Diamond\varphi \in X_l$. $\exists$loise must respond with a Hintikka Set Y such that

1. $\varphi \in Y$ and for all $\Diamond\psi \in SF(\xi)$, $\Diamond\psi \notin X_l$ implies that $\psi \notin Y$.
2. for all $@_i\varphi \in SF(\xi)$, $@_i\varphi \in Y$ iff $\{i, \varphi\} \subseteq X_k$, for some k.
3. if $i \in Y$ for some nominal i, then Y is one of the Hintikka Sets she played at the start. In this case the game stops and $\exists$loise wins.

If $\exists$loise cannot find a suitable Y, the game stops and $\forall$belard wins. If $\exists$loise does find a suitable Y (one that is not covered by the halting clause in item 3 above) then Y is added to the list of played sets, and play continues.

$\forall$belard must now choose a defect $\Diamond\varphi$ from the last played Hintikka Set. He can't choose any defect though: in round k he can only choose defects $\Diamond\varphi$ such that the modal depth of $\Diamond\varphi$ is less than or equal to the modal depth of ξ minus k. $\exists$loise must respond as before. She wins if she can survive all his challenges (in other words, he loses if he reaches a situation where he can't choose any more defects).

It is clear that the ξ-game stops after at most modal depth of ξ many rounds. The size of the information on the board is at any stage of the game polynomial in the length of ξ, as Hintikka Sets are polynomial in the length of ξ and ξ can only contain polynomially many nominals. We claim that $\exists$loise has a winning strategy iff ξ is satisfiable.

Now the right-to-left direction is clear: $\exists$loise has a winning strategy if ξ is satisfiable, for she need simply play by reading the required Hintikka sets off the model. The other direction requires more work. Suppose $\exists$loise has a winning strategy for the ξ-game. We shall create a model $\mathfrak{M}$ for ξ as follows. The domain M is build in steps by following her winning strategy.

M_0 consists of her initial move $\{X_0, \ldots, X_n\}$. Suppose M_j is defined. Then M_{j+1} consists of a copy of those Hintikka Sets she plays when using her winning strategy for each of ∀belard's possible moves played in the Hintikka Sets from M_j (except when she plays a HS from her initial move, then of course we do not make a copy). Let M be the disjoint union of all M_j for j smaller than the modal depth of ξ. Set Rmm' iff for all $\Diamond\varphi \in SF(\xi)$, $\Diamond\varphi \notin m \Rightarrow \varphi \notin m'$ holds, and set $V(p) = \{m \in M \mid p \in m\}$. Note that the rules of the game guarantee that nominals are interpreted as singletons.

We claim that the following truth-lemma holds.

Claim 1 *For all $m \in M$ which she plays in round j (i.e., $m \in M_j$), for all φ of modal depth less than or equal to the modal depth of ξ minus j, $\mathfrak{M}, m \Vdash \varphi$ if and only if $\varphi \in m$.*

Proof of Claim. By induction on the structure of formulas. For atoms, the booleans and @ the proof is easy. For $\Diamond$, if $\Diamond\varphi \in m$, then ∀belard challenged this defect, so ∃loise could respond with an m' containing φ. Since for all $\Diamond\varphi \in SF(\xi)$, $\Diamond\varphi \notin m \Rightarrow \varphi \notin m'$ holds, we have Rmm' and by induction hypothesis $\mathfrak{M}, m \Vdash \Diamond\varphi$. If $\Diamond\varphi \notin m$ but Rmm' holds, then by our definition of R, $\varphi \notin m'$, so again $\mathfrak{M}, m \nVdash \Diamond\varphi$. ◀

Since she plays a Hintikka Set containing ξ in the first round, $\mathfrak{M}$ satisfies ξ. QED

To close this section we discuss the *global* K-*satisfaction* problem; that is, the problem of deciding whether $\mathfrak{M} \models \varphi$. In the proof of Theorem 6.2 we made use of the fact that the global K-satisfaction problem for ordinary unimodal logic is EXPTIME complete. Indeed, for ordinary unimodal logic we have that the global K-*consequence* problem —does $\varphi \models^{glo} \psi$ hold?— is EXPTIME complete. It follows from Theorem 6.3 above that the same result holds when we expand even a multimodal tense language with *nominals*, @, and the universal modality. This is because the universal modality allows us to capture $\models^{glo}$ for this language. In particular, note that $\varphi \models^{glo} \psi$ iff $\models A\varphi \to \psi$, where A is the dual of E.

But this is not the end of the matter: what if we were to extend the unimodal language with *variables* instead of nominals? This *does* make a difference. Recall from Section 2 that $\mathfrak{M} \models \varphi$ is defined to hold iff for all g, $\mathfrak{M}, g \Vdash \varphi$. That is, g is *not* held constant. Now, if there are no world variables in φ, g is irrelevant and (as we have just observed) global satisfaction is EXPTIME-complete. But if φ is allowed to contain free variables, the implicit universal quantification in the definition certainly does change matters. In effect, we are surreptitiously using the global hybrid binder ∀ mentioned in the introduction, and we wind up with an undecidable global satisfaction problem.

This can be shown as follows. Let φ be the formula $(\Diamond x_1 \wedge \Diamond x_2 \wedge \Diamond x_3) \to \Box(x_1 \leftrightarrow x_2 \vee x_1 \leftrightarrow x_3 \vee x_2 \leftrightarrow x_3)$. Then for every model $\mathfrak{M} = \langle M, R, V\rangle$, $\mathfrak{M} \models \varphi$ if and only if $\mathfrak{M} \models \forall y x_1 x_2 x_3.((Ryx_1 \wedge Ryx_2 \wedge Ryx_3) \to (x_1 = x_2 \vee x_1 = x_3 \vee x_2 = x_3))$. In other words φ expresses that every world has at most two R-successors. Similarly we can create a formula without propositional variables expressing that every world has at most three two-step R-successors. But in [Spa93b] it is shown that the global satisfaction problem of the uni-modal logic of the class of frames satisfying these two conditions is undecidable (see the proof of Theorem 4.2.1 and its preliminaries). Note that this argument makes no use of @.

Pressing this line of thought further, we can show that if we have at least three modalities in the language, then we obtain an undecidable global consequence problem for the sublanguage containing only one variable; again, no use will be made of @. Suppose we are working in a language with three modalities. Consider the normal modal logic obtained by adding the following axioms to the minimal logic K:

$$\langle r\rangle\langle u\rangle p \to \langle d\rangle p \qquad\qquad \langle d\rangle p \to [u]\langle r\rangle p.$$

These (Sahlqvist) axioms correspond to the first-order frame conditions $\forall xyz.(Rxy \wedge Uyz) \to Dxz$ and $\forall xy.(Dxy \to \forall z.(Uxz \to Rzy))$, respectively. [Mar97] shows that the global satisfaction problem of this logic is undecidable. But then the problem of deciding whether a modal formula ϕ in the language with diamonds $\{\langle r\rangle, \langle u\rangle, \langle d\rangle\}$ globally follows from $\langle r\rangle\langle u\rangle x \to \langle d\rangle x$ and $\langle d\rangle x \to [u]\langle r\rangle x$ (that is, the axioms with the *propositional* variable p replaced by the *world* variable x) must also be undecidable.

It is at present unknown precisely how many world variables are needed to obtain an undecidable global satisfaction problem for the unimodal case.

7 Further Work

In their long (if sparse) history, hybrid languages have attracted a number of enthusiastic advocates. Some have claimed that hybridization is a natural way to "power-up" the expressive power of modal languages, others have been impressed by the proof theoretical options they open up, or the ease with which general completeness can be proved. And underlying most of this work lies a simple (and seductive) idea: that by exploiting the notion of formulas as terms to the full, it should be possible to define systems which in some sense combine the best of modal and classical techniques.

We believe that the results of this paper confirm the interest of hybridization. For a start, our characterization of $\mathcal{H}(\downarrow, @)$ shows that relatively simple tools are capable of capturing first-order fragments that are central from a modal perspective; invariance under generated submodels mirrors the key notion of locality, and it is pleasing that it can be pinned down so simply. Furthermore, the results on interpolation and complexity tend to confirm that we are dealing with a natural collection of ideas, ideas that are well behaved even in relatively weak sublanguages. Finally in writing this paper it has become very clear to us that working with hybrid languages involves a genuine interplay of modal and classical methods (for example, both Ehrenfeucht-Fraïssé games and bisimulations were involved in the expressivity result, and interpolation was proved by blending the modal notion of canonical models with the classical idea of Henkin-models). This is something previous writers on hybrid languages have emphasized (see for example [PT91]) and the natural way these methods blend bodes well for further developments.

Nonetheless, to close this paper it seems more appropriate to emphasize what remains to be done; for the blunt fact remains that compared with orthodox modal languages, the study of hybrid languages is in its infancy. Many fundamental questions have not been satisfactorily resolved, and to close the paper we're going to discuss a very obvious one that we regard as particularly important, namely:

> *Which classes of frames are definable using $\mathcal{H}(@)$ formulas whose only atoms are world variables?*

In a sense, the standard translation ST already gives us an answer to this question. Let F be a class of frames defined by a sentence φ of the first-order frame language. Then F is definable by a formula of $\mathcal{H}(@)$ whose only atoms are variables iff there is some formula α in this fragment such that φ is equivalent to the universal closure of $ST(\alpha)$.

Unfortunately, this is not very helpful. Ideally we would like a syntactic characterization of the range of ST when restricted to $\mathcal{H}(@)$ formulas whose only atoms are world variables, together with a reverse translation (like our earlier HT). But at present we only have partial results in this direction.

What about a semantic characterization? Here we can do a little better by introducing the concept of an *@-bisimulation.* Let $\mathfrak{M}$ and $\mathfrak{N}$ be models and $\bar{m} \in {}^kM$ and $\bar{n} \in {}^kN$. A relation $B \subseteq M \times N$ is called a *@-k-bisimulation* between $(\mathfrak{M}, \bar{m})$ and $(\mathfrak{N}, \bar{n})$ if it satisfies the following conditions:

1. B is a modal bisimulation;
2. $(\forall 0 \leq i < k) : (m_i, n_i) \in B$;
3. $(\forall x \in N) : (m_i, x) \in B \Rightarrow x = n_i$;
4. $(\forall x \in M) : (x, n_i) \in B \Rightarrow x = m_i$.

We denote this by $(\mathfrak{M}, \bar{m}) \overset{B}{\sim} (\mathfrak{N}, \bar{n})$. A first-order formula $\varphi(\bar{x}, y)$ is called invariant for @-k-bisimulation, if for all models $\mathfrak{M}$, $\mathfrak{N}$, for all k-tuples $\bar{m}$ and $\bar{n}$, for all B such that $(\mathfrak{M}, \bar{m}) \overset{B}{\sim} (\mathfrak{N}, \bar{n})$,

$$(m, n) \in B \Rightarrow \mathfrak{M} \models \varphi(\bar{m}, m) \Leftrightarrow \mathfrak{N} \models \varphi(\bar{n}, n).$$

Theorem 7.1 *A first-order formula $\varphi(\bar{x}, y)$ is invariant for @-k-bisimulation if and only if it is equivalent to the standard translation of an $\mathcal{H}(@)$-formula containing the variables $\bar{x}$.*

Proof. The proof follows the standard modal lines. We only sketch the differences due to the new operator and the variables. For the preservation part, suppose $(\mathfrak{M}, \bar{m}) \overset{B}{\sim} (\mathfrak{N}, \bar{n})$ and $(m, n) \in B$. Then $\mathfrak{M}, \bar{m}, m \Vdash v_i \Leftrightarrow m = m_i$; but this implies that $n = n_i$, whence $\mathfrak{N}, \bar{n}, n \Vdash v_i$. For @-formulas, $\mathfrak{M}, \bar{m}, m \Vdash @_{v_i}\varphi \Leftrightarrow \mathfrak{M}, \bar{m}, m_i \Vdash \varphi$; but then since $(m_i, n_i) \in B$ and inductive hypothesis, $\mathfrak{N}, \bar{n}, n_i \Vdash \varphi \Leftrightarrow \mathfrak{N}, \bar{n}, n \Vdash @_{v_i}\varphi$.

For the characterization part we also copy the standard diagram-chasing proof. Let $\mathfrak{M}, \bar{m}, m$ and $\mathfrak{N}, \bar{n}, n$ have the same hybrid theory. Define a relation B between elements in the ω-saturated extensions $\mathfrak{M}^+$ and $\mathfrak{N}^+$ as follows:

$$(x, y) \in B \Leftrightarrow (\forall \varphi) : \mathfrak{M}^+, \bar{m}, x \Vdash \varphi \Leftrightarrow \mathfrak{N}^+, \bar{n}, y \Vdash \varphi.$$

The standard proof shows that B is a modal bisimulation. We check the extra conditions. For all i, $(m_i, n_i) \in B$ holds by the following argument. $\mathfrak{M}^+, \bar{m}, m_i \Vdash \varphi \Leftrightarrow \mathfrak{M}^+, \bar{m}, m \Vdash @_{v_i}\varphi \Leftrightarrow \mathfrak{N}^+, \bar{n}, n \Vdash @_{v_i}\varphi \Leftrightarrow \mathfrak{N}^+, \bar{n}, n_i \Vdash \varphi$. The other two conditions are satisfied because of the following. Let $(m_i, y) \in B$. Since $\mathfrak{M}^+, \bar{m}, m_i \Vdash v_i$, also $\mathfrak{N}^+, \bar{n}, y \Vdash v_i$. But then $y = n_i$. This finishes the outline of the proof. QED

Using this notion it is easy to show that $\exists y.(Rxy \wedge Ryy)$ is not equivalent to an @-formula with one free variable, and that $Rxy \wedge Rxz \rightarrow \exists w.(Ryw \wedge Rzw)$ is not equivalent to an @-formula in three free variables. And it does tell us something about frame definability:

Corollary 7.2 *Let* F *is a class of frames defined by a sentence* φ *of the first-order frame language. Then* F *is definable by a formula of* $\mathcal{H}(@)$ *whose only atoms are variables iff* φ *is equivalent to the universal closure of a formula that is invariant under @-*k*-bisimulations.*

But in practice this characterization does not seem to be particularly helpful. One of the nicer aspects of our characterization of $\mathcal{H}(\downarrow, @)$ was the way the notion of k-bisimulation linked with the notion of generated submodels to yield a natural "geometric" characterization of definable frame classes. It is hard to see what the definition of @-k-bisimulation invariance tells us about frame geometry. Clearly new ideas are called for here.

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