Normal Gentzen deductions in the classical case

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1 Introduction

The idea that proofs are objects capable of being treated by a mathematical theory is due to Hilbert. Proofs are represented as *derivations*, or *deductions*, i.e. finite and labeled trees which are build up according to a set of rules; these rules are provided by a formal system and change from one system to another.

Structural Proof Theory studies the properties of deductions as combinatorial structures, and the relations between formal systems.

The best known formal systems studied in literature are the Natural Deduction systems **N** and the sequent calculi (Gentzen systems) **G** (see [8]).

There is a very interesting metamathematical feature of the rules of **G**-systems: besides their intended interpretation in the construction of the deduction trees of **G**, they can be seen as instructions for constructing derivations of **N**.

In this perspective, the rules in **G** systems might be seen as non-trivial reformulations of the rules of **N** systems, where the left rules are much more restricted than the corresponding elimination rules; for example the L$\rightarrow$ rule

$$\frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta}$$

corresponds to the $\rightarrow$E rule restricted to those cases in which the major premise is an open assumption.

The equivalence that holds between **G** and **N** systems is much easier to prove with the aid of the Cut rule (cf. [8] 3.3.1), which Gentzen [2] proved to be a derived rule of **G** (Cut elimination theorem, or *Hauptsatz*).

The proof of the Cut elimination theorem gives a procedure to obtain, in a finite number of steps, a Cut free deduction in sequent style from a deduction which may contain Cuts.

This procedure resembles the one used in the normal form theorem for **N** which transforms any deduction of **N** into a normal one.

The notion of normality is very natural, so to speak, for deductions in **N**: the rules of **N** allow the construction of prooftrees in which there might be the occurrence of some unnecessary detours; the normalization procedure identifies the unnecessary detours caused by the introduction of formulas which are immediately afterwards eliminated, and removes them.

Cut elimination could be seen as the corresponding notion of normality on the **G** systems, because by removing the Cut rule one avoids the possibility of introducing a formula at some node of the prooftree and eliminating it immediately afterwards.

These and other remarks on how a Cut free deduction in **G** can be translated into a *normal* deduction of **N** (see [5]) may lead to think that Cut free deductions in **G** are the perfect counterparts to normal deductions in **N**, but it turns out that it is not so: in the first place, there are many Cut free deductions which correspond to the same deduction in **G**+Cut, namely, when we run the Cut elimination procedure on a deduction, we might obtain different outcomes depending on the choice of the transformation rule by which an application of the Cut rule is eliminated (see [8], 4.1.3).

This does not happen to the normal form procedure in **N**: normal form is unique, i.e. the normalization procedure is deterministic.

Moreover, a normal natural deduction can be intuitively associated to several Cut free deductions in **G**, and they can be obtained one from the other by the permutation of some applications of the L$\rightarrow$ rule.

Finally, there exist strategies for Cut elimination for Gentzen systems which cannot be seen as the "translation" of a normalization strategy in the corresponding natural deduction systems (cf. [8], 5.1.9, 6.7.2, 6.8.3).

So this leaves open the problem of defining a suitable notion of normal deductions for the **G** systems, whose corresponding normal form theorem gives a procedure that transforms any deduction in the **G** system into a *unique* normal deduction with the same conclusion. This notion of normal form for **G** systems should also reflect the intuitive notion of normality for natural deductions (i.e. the straightforwardness of normal natural deductions); this requirement is guaranteed by the construction of a *natural correspondence* (a "good translation") between normal deductions in **G** and normal deductions in **N**.

As for the intuitionistic case, a notion of normality and a correspondence between normal Gentzen deductions and normal natural deductions has been proposed first by Howard [3] and then by Mints [4] and Troelstra [10]; in Mints' work the correspondence between normal Gentzen deductions and normal natural deductions is defined via the isomorphism between lambda terms and natural deductions. In the work of Troelstra, no term labels are involved and correspondingly a version of $\rightarrow$**Ni** obeying the Complete Discharge Convention is used; moreover, a direct one-to-one correspondence is constructed between normal Gentzen deductions and natural deductions in long normal form, i.e., natural deductions in normal form that also satisfy another condition (see definition 2.4).

In this thesis I define the notion of normality for deductions in a Gentzen system for the classical case (section 3); I prove the normalization theorem for this notion (section 3) and I build two direct correspondences (sections 6 and 7)

$$\mathbf{G_n} \longrightarrow \mathbf{N}^*$$
$$\mathbf{N}^* \longrightarrow \mathbf{G_n}$$

between normal Gentzen deductions and natural deductions *in $*$-normal form*, i.e., natural deductions in normal form that also satisfy other conditions (section 5). The definitions of normal deductions for Gentzen systems and of natural deductions in $*$-normal form are motivated by the well-definedness of the two correspondences: namely, the conditions stated in the definition of normal deductions for Gentzen systems are sufficient conditions for the well-definedness of the correspondence from **G** to **N**, and the conditions stated in the definition

$\rightarrow$**Nc**	$\dfrac{\begin{matrix}[A]^u \\ \mathcal{D} \\ B\end{matrix}}{A \rightarrow B} \rightarrow I, u$	$\dfrac{\begin{matrix}\mathcal{D}_1 \\ A \rightarrow B\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ A\end{matrix}}{B} \rightarrow E$	$\dfrac{\begin{matrix}[\neg A]^u \\ \mathcal{D} \\ \bot\end{matrix}}{A} \bot_c, u$
$\rightarrow$**Gc**	$Axiom\,1: \ \Gamma, A \Rightarrow A, \Delta$ $Axiom\,2: \ \Gamma, \bot \Rightarrow A, \Delta$	$R \rightarrow \dfrac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta}$	$L \rightarrow \dfrac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta}$

Table 1: The systems $\rightarrow$**Nc** and $\rightarrow$**Gc**

of natural deductions in $*$-normal form are sufficient conditions for the well-definedness of the correspondence from **N** to **G**. In section 8 I sketch the proof of a $*$-normalization theorem, i.e. I give a procedure for transforming a normal deduction of $\rightarrow$**Nc** into a suitable deduction in $*$-normal form.

I chose to restrict the presentation of these results to the implication fragment of classical logic, because the most interesting phenomena occur in this fragment, and still things remain easy to read.

I also chose to use the version of $\rightarrow$**Nc** that obeys to the CDC (Complete Discharge Convention, see [8], 2.1.5), because, from the point of view of the correspondence, deductions of $\rightarrow$**Gc** with the contraction rule absorbed seem to be more naturally translatable into natural deductions under CDC, and this is analogous to what Troelstra pointed out for the intuitionistic case in [10].

2 The intuitionistic case

Let us consider the following system $\rightarrow$**Gi** :

$$Axiom\,1: \quad \Gamma, A \Rightarrow A$$

$$Axiom\,2: \quad \Gamma, \bot \Rightarrow A$$

$$R \rightarrow \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}$$

$$L \rightarrow \frac{\Gamma, A \rightarrow B \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \rightarrow B \Rightarrow C}$$

Definition 2.1. *Let $\mathcal{D}$ be a deduction in $\rightarrow$**Gi** . $\mathcal{D}$ is* normal *iff the active antecedent formula of any application of the $L\rightarrow$ rule in $\mathcal{D}$ is a principal formula (either of an axiom or of another application of the $L\rightarrow$ rule).*

Proposition 2.2. *(Normalization theorem for* **Gi***) Let $\mathcal{D}$ be a deduction in $\rightarrow$**Gi** , whose conclusion is $\Gamma \Rightarrow X$; then there exists a procedure which transforms $\mathcal{D}$ into a normal derivation with the same conclusion.*

Proof. The proof is analogous to the one of proposition 3.3. ☐

Remark 2.3. *Let $\mathcal{D}$ be a normal deduction in $\rightarrow$**Gi** . We can transform $\mathcal{D}$ into a* pruned *deduction, i.e. a normal deduction whose conclusion is of the form $\Gamma \Rightarrow A$ with Γ a set (therefore there are no multiple occurrences of formulas in Γ), and such that there is no formula A that occurs in the antecedent of every sequent and is not active in any of the axioms.*

Definition 2.4. *Let $\mathcal{D}$ be a deduction in* $\rightarrow$**Ni** *. $\mathcal{D}$ is in* long normal form *if $\mathcal{D}$ is normal and all the formulas that occur at any minimal position of $\mathcal{D}$ are atomic.*

Remark 2.5. *The definition of formula occurrence* at minimal position *for the implication fragment is equivalent to the following one: a formula occurrence A is said to be in minimal position in a deduction $\mathcal{D}$ iff A is the conclusion of an application of $\rightarrow E$, and at least one of the following conditions hold:*

(i) A is the premise of an application of $\rightarrow I$, or

(ii) A is the conclusion of $\mathcal{D}$, or

(iii) A is the minor premise of an application of $\rightarrow E$.

Proposition 2.6. *(Correspondence from* **Gi** *to* **Ni***) There is a one-to-one correspondence between pruned deductions in* $\rightarrow$**Gi** *(with axioms restricted to atomic active formulas) and deductions in* $\rightarrow$**Ni** *under the Complete Discharge Convention in long normal form.*

Proof. The proof is straightforward and is done by induction on the size of the prooftree. □

3 Normal deductions in $\rightarrow$Gc

Definition 3.1. *Let $\mathcal{D}$ be a deduction in* $\rightarrow$**Gc**. *$\mathcal{D}$ is* normal *iff $\mathcal{D}$ satisfies the following two conditions:*

(a) the active antecedent formula of any application of the $L\rightarrow$ rule in $\mathcal{D}$ is a principal formula (either of an axiom or of another application of $L\rightarrow$ rule);

(b) the conclusion of $\mathcal{D}$ is not of the following kind[1]:

$$\Gamma, \neg A \Rightarrow A, \Delta.$$

Lemma 3.2. *(Inversion lemma) In* $\rightarrow$**Gc** *, with active formulas in axioms atomic, there exists an operation ϕ such that*

$$\text{if } \mathcal{D} \vdash_n \Gamma \Rightarrow A \rightarrow B, \Delta, \text{ then } \phi(\mathcal{D}) \vdash_n \Gamma, A \Rightarrow B, \Delta.$$

ϕ does not increase the number of violations of the condition (a) of the definition 3.1.

Proof. Let $\mathcal{D}$ be a deduction in $\rightarrow$**Gc** (with active formulas in axioms atomic), whose conclusion is $\Gamma \Rightarrow A \rightarrow B, \Delta$.

[1] The condition (a) of def. 3.1 is the only requirement stated in the definition of normal Gentzen deductions for the intuitionistic case in [10]. It is a sufficient condition to the well-definedness of the correspondence from $\rightarrow$**Gc** to $\rightarrow$**Nc** . Condition (b) will play a role in a further developement of this work, namely in the characterization of the class of deductions of $\rightarrow$**Gc** for which the composition of the two correspondences is the identity map. I included it in def. 3.1 because the normalization theorem for this definition has a non-standard formulation and a non-trivial extra step (cf. normalization theorem in [10]).

– if $A \to B$ is a side formula at every node of $\mathcal{D}$, then $\phi(\mathcal{D})$ is the deduction obtained by removing $A \to B$ at every node of $\mathcal{D}$ in which $A \to B$ occurs and weakening at those nodes with A in the antecedent and B in the succedent; if in this operation some new axioms are created, then we remove the subdeduction immediately above each new axiom.

– if $A \to B$ has been introduced at some node of $\mathcal{D}$ with an application of R→ , then $\mathcal{D}$ has the following shape:

$$\cfrac{\begin{array}{c}\mathcal{D}_1\\ \Pi, A \Rightarrow B, \Sigma\end{array}}{\begin{array}{c}\Pi \Rightarrow A \to B, \Sigma\\ \mathcal{D}_2\\ \Gamma \Rightarrow A \to B, \Delta\end{array}}$$

and $A \to B$ is side formula at every node of $\mathcal{D}_2$, so we can transform $\mathcal{D}_2$ into a deduction $\mathcal{D}_2{}^*$ following the procedure described in the previous case; then $\phi(\mathcal{D})$ is the following deduction:

$$\begin{array}{c}\mathcal{D}_1\\ \Pi, A \Rightarrow B, \Sigma\\ \mathcal{D}_2{}^*\\ \Gamma, A \Rightarrow B, \Delta\end{array}$$

This procedure does not increase the number of violations to the condition (a) of the definition 3.1(this is easily seen by inspection). □

Proposition 3.3. *(Normalization theorem for* →**Gc** *) Let $\mathcal{D}$ be a deduction in* →**Gc** *(with axioms restricted to atomic active formulas), whose conclusion is $\Gamma \Rightarrow \Delta$; then there exists a procedure which transforms $\mathcal{D}$ into a* unique *normal derivation with the same conclusion, if $\mathcal{D}$ already satisfies condition (b), and into a* unique *normal derivation whose conclusion is $\Gamma, \Rightarrow A, A, \Delta$, if $\mathcal{D}$ does not satisfy condition (b).*

Proof. We will prove the proposition in three steps: with the first step we will transform $\mathcal{D}$ into a deduction which satisfies condition (b); with the second step we will transform $\mathcal{D}$ into a deduction $\mathcal{D}$' which satisfies the following condition:

(iii) for every application of L→ in $\mathcal{D}$':

$$\cfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_1 \Rightarrow A, \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma_1, B \Rightarrow \Delta\end{array}}{\Gamma_1, A \to B \Rightarrow \Delta}$$

the last rule applied in $\mathcal{D}_2$ is not R→ ;

with the third step we remove the remaining violations to the condition (a), and we will transform $\mathcal{D}$' into a normal deduction.

First step: let us suppose that $\mathcal{D}$ is a deduction of →**Gc** whose conclusion is of the following kind:

$$\Gamma, \neg A \Rightarrow A, \Delta.$$

then two cases may occur:

(i) $\neg A$ is side formula at every node of $\mathcal{D}$;

(ii) $\neg A$ has been introduced with an application of L$\rightarrow$.

If (i), then we transform $\mathcal{D}$ into the deduction $\mathcal{D}$' which is obtained from $\mathcal{D}$ by removing every occurrences of $\neg A$ and by weakening with A at every node in the succedent.

If (ii), then $\mathcal{D}$ has the following shape:

$$\begin{array}{c}
\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_1 \Rightarrow A, \Sigma\end{array} \qquad \Gamma_1, \bot \Rightarrow \Sigma}{\Gamma_1, \neg A \Rightarrow \Sigma}\\
\mathcal{D}_3\\
\Gamma, \neg A \Rightarrow A, \Delta
\end{array}$$

where $\neg A$ is side formula at each node of $\mathcal{D}_3$. We transform $\mathcal{D}$ into the following deduction:

$$\begin{array}{c}
\mathcal{D}_1\\
\Gamma_1 \Rightarrow A, \Sigma\\
\mathcal{D}_3{}^*\\
\Gamma \Rightarrow A, A, \Delta
\end{array}$$

Where $\mathcal{D}_3{}^*$ is obtained from $\mathcal{D}_3$ by removing every occurrence of $\neg A$ in $\mathcal{D}_3$ and by weakening with A at every node in the succedent.

Second step: Let us suppose that $\mathcal{D}$ satisfies condition (b) and let us transform it into a deduction $\mathcal{D}$' with the same conclusion as $\mathcal{D}$ and which also satisfies condition (iii). By induction on the number n of violations to the condition (iii).

- **Basis:** If $n = 0$ trivial.
- **Induction:** let us suppose that $n \geq 1$; therefore $\mathcal{D}$ has the following shape:

$$\begin{array}{c}
\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_1 \Rightarrow A, C \rightarrow D, \Delta_1\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_3\\ \Gamma_1, B, C \Rightarrow D, \Delta_1\end{array}}{\Gamma_1, B \Rightarrow C \rightarrow D, \Delta_1}}{\Gamma_1, A \rightarrow B \Rightarrow C \rightarrow D, \Delta_1}\\
\mathcal{D}_4
\end{array}$$

then we transform it into the following deduction $\mathcal{D}$':

$$\begin{array}{c}
\dfrac{\dfrac{\begin{array}{c}\phi(\mathcal{D}_1)\\ \Gamma_1, C \Rightarrow A, D, \Delta_1\end{array} \qquad \begin{array}{c}\mathcal{D}_3\\ \Gamma_1, B, C \Rightarrow D, \Delta_1\end{array}}{\Gamma_1, A \rightarrow B, C \Rightarrow D, \Delta_1}}{\Gamma_1, A \rightarrow B \Rightarrow C \rightarrow D, \Delta_1}\\
\mathcal{D}_4
\end{array}$$

where $\phi(\mathcal{D}_1)$ is obtained applying the procedure described in the previous lemma to $\mathcal{D}_1$. We notice that the number of violations to the condition (iii) in $\mathcal{D}_1$ (and so in $\phi(\mathcal{D}_1)$) and in $\mathcal{D}_3$ is smaller than n, therefore we can apply induction hypothesis on $\mathcal{D}_1{}^*$ and on $\mathcal{D}_3$ and suppose that $\mathcal{D}$' contains no violation to the condition (iii).

Third step: let us suppose that there are no violations in $\mathcal{D}$ to the conditions of the first and of the second step; therefore $\mathcal{D}$ has the following shape: for every application of L$\rightarrow$ in $\mathcal{D}$:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_1 \Rightarrow A, \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma_1, B \Rightarrow \Delta\end{array}}{\Gamma_1, A \rightarrow B \Rightarrow \Delta}$$

either the conclusion of $\mathcal{D}_2$ is an axiom or it is the conclusion of an application of L$\rightarrow$.

Therefore we notice that the applications of L$\rightarrow$ always appear in $\mathcal{D}$ " in blocks ":

$$\dfrac{\mathcal{D}_n \qquad \dfrac{\mathcal{D}_{n-1} \qquad \dfrac{\mathcal{D}_1 \qquad \Gamma_0 \Rightarrow \Delta}{\Gamma_1 \Rightarrow \Delta}}{\Gamma_{n-1} \Rightarrow \Delta}\ n \geq 1 \text{ applications of L}\rightarrow}{\Gamma_n \Rightarrow \Delta}$$

where $\Gamma_0 \Rightarrow \Delta$ is an axiom.

We will transform $\mathcal{D}$ into a normal deduction by induction on the number n of violations to the normality conditions in $\mathcal{D}$.

• **Basis:** if $n = 0$ trivial.

• **Induction:** Let us suppose that $n \geq 1$; therefore $\mathcal{D}$ has at least an application of L$\rightarrow$ which violates the normality condition. Since $\mathcal{D}$ satisfies the condition of step one, the offending application of L$\rightarrow$ is placed in a rightmost branch of $\mathcal{D}$, at a certain distance from the top node of this branch, labelled with an axiom. We continue the proof by induction on the distance of the offending application of L$\rightarrow$ from the top node of the rightmost branch on which it is placed.

• **Basis:** Let us suppose that the offending application of L$\rightarrow$ is the one immediately under the axiom; then two cases may occur:

(a) the axiom is of the first kind;

(b) the axiom is of the second kind.

if (a), then the offending application of L$\rightarrow$ has the following shape:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_1, C \Rightarrow C, A, \Delta_1\end{array} \qquad \Gamma_1, B, C \Rightarrow C, \Delta_1}{\Gamma_1, A \rightarrow B, C \Rightarrow C, \Delta_1}$$

We notice that, either in the case that $C = A$ or in the case that $C \neq A$, the conclusion of this application of L$\rightarrow$ is an axiom, of which C is the principal formula, therefore the application of L$\rightarrow$ is redundant. So we can remove it and leave the conclusion:

$$\Gamma_1, A \rightarrow B, C \Rightarrow C, \Delta_1$$

and we obtain a deduction whose number of violations to the normality condition is $n - 1$, therefore we can apply induction hypothesis on it and suppose that it has already been transformed into a normal deduction.

If (b), then $\mathcal{D}$ has the following shape:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_1, \bot \Rightarrow A, \Delta\end{array} \qquad \Gamma_1, B, \bot \Rightarrow \Delta}{\Gamma_1, A \to B, \bot \Rightarrow \Delta}$$

We notice that the conclusion of $\mathcal{D}$ is an axiom, of which $\bot$ is the principal formula, therefore the application of L$\to$ is redundant. So we can remove it and leave the conclusion:

$$\Gamma_1, A \to B, \bot \Rightarrow \Delta$$

and we obtain a deduction whose number of violations to the normality condition is $n-1$, therefore we can apply induction hypothesis on it and suppose that it has already been transformed into a normal deduction.

• **Induction:** Let us suppose that we have a block of n applications of L$\to$ such that only the last one violates the normality condition:

$$\dfrac{\mathcal{D}_n \qquad \dfrac{\mathcal{D}_{n-1} \qquad \begin{array}{c}\dfrac{\mathcal{D}_1 \qquad \Gamma, C, B_1 \Rightarrow \Delta}{\Gamma, C, B_2 \Rightarrow \Delta}\\ \vdots \\ \Gamma, C, B_{n-1} \Rightarrow \Delta\end{array}}{\Gamma, C, B_n \Rightarrow \Delta}}{\Gamma, B_n, A_n \to C \Rightarrow \Delta} \tag{1}$$

where $\Gamma, C, B_1 \Rightarrow \Delta$ is an axiom, the conclusion of $\mathcal{D}_n$ is $\Gamma, B_n \Rightarrow A_n, \Delta$ and for $i = 1, \ldots, n-1$ the conclusion of $\mathcal{D}_i$ is $\Gamma, C \Rightarrow A_i, \Delta$, and $B_{i+1} = A_i \to B_i$. Since, for $j = 1, \ldots, n$, the number of violations to the normality condition in $\mathcal{D}_j$ is surely smaller than the number of violations to the normality condition in $\mathcal{D}$, we can apply induction hypothesis on $\mathcal{D}_j$ and suppose that they have been already transformed into normal deductions. We transform this block into the following deduction:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1{}^{**}\\ \Gamma \Rightarrow A_n, A_{n-1}, \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_{n-1}\\ \Gamma, C \Rightarrow A_{n-1}, \Delta\end{array}}{\Gamma, A_n \to C \Rightarrow A_{n-1}, \Delta} \qquad \dfrac{\begin{array}{c}\mathcal{D}_0{}^{**}\\ \Gamma, B_{n-1} \Rightarrow A_n, \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}"\\ \Gamma, C, B_{n-1} \Rightarrow \Delta\end{array}}{\Gamma, B_{n-1}, A_n \to C \Rightarrow \Delta}}{\Gamma, B_n, A_n \to C \Rightarrow \Delta}$$

where $\mathcal{D}_0{}^{**}$ and $\mathcal{D}_1{}^{**}$ are obtained from $\mathcal{D}_n$ in the following way:

- if B_n is a side formula at every node of $\mathcal{D}_n$, then $\mathcal{D}_0{}^{**}$ is the deduction obtained by removing B_n at every node of $\mathcal{D}_n$ in which B_n occurs and weakening at those nodes with B_{n-1} in the antecedent; if in this operation some new axioms are created, then we remove the subdeduction immediately above each new axiom; $\mathcal{D}_1{}^{**}$ is the deduction obtained by removing B_n at every node of $\mathcal{D}_n$ in which B_n occurs and weakening at those nodes with A_{n-1} in the succedent; if in this operation some new axioms are created, then we remove the subdeduction immediately above each new axiom.

- if $B_n := A_{n-1} \to B_{n-1}$ has been introduced at some node of $\mathcal{D}_n$ with an application of L$\to$, then $\mathcal{D}_n$ has the following shape:

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Pi \Rightarrow A_{n-1}, \Sigma\end{array} \qquad \begin{array}{c}\mathcal{D}_{0,0}\\ \Pi, B_{n-1} \Rightarrow \Sigma\end{array}}{\Pi, B_n \Rightarrow \Sigma}$$
$$\begin{array}{c}\mathcal{D}_2\\ \Gamma, B_n \Rightarrow A_n, \Delta\end{array}$$

and B_n is side formula at every node of $\mathcal{D}_2$, so $\mathcal{D}_0{}^{**}$ is the following deduction:

$$\begin{array}{c}\mathcal{D}_{0,0}\\ \Pi, B_{n-1} \Rightarrow \Sigma\\ \mathcal{D}_2{}^{*}\\ \Gamma, B_{n-1} \Rightarrow A_n, \Delta\end{array}$$

where $\mathcal{D}_2{}^{*}$ is obtained by $\mathcal{D}_2$ by removing B_n at every node of $\mathcal{D}_2$ in which B_n occurs and weakening at those nodes with B_{n-1} in the antecedent; if in this operation some new axioms are created, then we remove the subdeduction immediately above each new axiom. $\mathcal{D}_0{}^{**}$ is the following deduction:

$$\begin{array}{c}\mathcal{D}_0\\ \Pi \Rightarrow A_{n-1}, \Sigma\\ \mathcal{D}_2{}^{**}\\ \Gamma \Rightarrow A_n, A_{n-1}, \Delta\end{array}$$

where $\mathcal{D}_2{}^{**}$ is obtained by $\mathcal{D}_2$ by removing B_n at every node of $\mathcal{D}_2$ in which B_n occurs and weakening at those nodes with A_{n-1} in the succedent; if in this operation some new axioms are created, then we remove the subdeduction immediately above each new axiom.

Let us consider the following deduction:

$$\dfrac{\begin{array}{c}\mathcal{D}_0{}^{**}\\ \Gamma, B_{n-1} \Rightarrow A_n, \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}"\\ \Gamma C, B_{n-1} \Rightarrow \Delta\end{array}}{\Gamma, B_{n-1}, A_n \rightarrow C \Rightarrow \Delta}$$

the rightmost branch of this deduction consists of a block of $n-1$ applications of L$\rightarrow$ such that only the last one violates the normality condition, therefore we can apply induction hypothesis on it and transform it into a normal deduction $\mathcal{D}^{**}$ with the same conclusion. So we transform the block (1) into the following deduction:

$$\dfrac{\begin{array}{c}\mathcal{D}_{n-1}{}^{*}\\ \Gamma'_{n-1}, A_n \rightarrow C \Rightarrow A_{n-1}, \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}^{**}\\ \Gamma, B_{n-1}, A_n \rightarrow C \Rightarrow \Delta\end{array}}{\Gamma_{n+1}, B_n, A_n \rightarrow C \Rightarrow \Delta}$$

where $\mathcal{D}_{n-1}{}^{*}$ is the following deduction:

$$\dfrac{\begin{array}{c}\mathcal{D}_1{}^{**}\\ \Gamma \Rightarrow A_n, A_{n-1}, \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_{n-1}\\ \Gamma, C \Rightarrow A_{n-1}, \Delta\end{array}}{\Gamma, A_n \rightarrow C \Rightarrow A_{n-1}, \Delta}$$

$\square$

3.1 Example

Let us consider the following Gentzen deduction:

$$
\dfrac{(A \to D) \to C, B \Rightarrow B, C \qquad \dfrac{\dfrac{\dfrac{\dfrac{B, A \Rightarrow A, C, D \qquad \dfrac{A, B \Rightarrow B, C, D \qquad A, B, C \Rightarrow C, D}{A, B, B \to C \Rightarrow C, D}}{A, B, A \to (B \to C) \Rightarrow C, D}}{B, A \to (B \to C) \Rightarrow A \to D, C} \qquad B, A \to (B \to C), C \Rightarrow C}{B, A \to (B \to C), (A \to D) \to C \Rightarrow C}}{\dfrac{(A \to D) \to C, B, B \to [A \to (B \to C)] \Rightarrow C}{(A \to D) \to C, B \to [A \to (B \to C)] \Rightarrow B \to C}}
$$

This is not a normal deduction, because of the first application of L$\to$ going from bottom to top.

By the normalization procedure, the deduction above is transformed into the following normal deduction:

$$
\dfrac{\dfrac{\dfrac{A, B \Rightarrow B, C, D \qquad \dfrac{B, A \Rightarrow A, C, D \qquad \dfrac{A, B \Rightarrow B, C, D \qquad A, B, C \Rightarrow C, D}{A, B, B \to C \Rightarrow C, D}}{A, B, A \to (B \to C) \Rightarrow C, D}}{A, B, B \to [A \to (B \to C)] \Rightarrow C, D}}{B, B \to [A \to (B \to C)] \Rightarrow A \to D, C} \qquad B, B \to [A \to (B \to C)], C \Rightarrow C}{\dfrac{(A \to D) \to C, B, B \to [A \to (B \to C)] \Rightarrow C}{(A \to D) \to C, B \to [A \to (B \to C)] \Rightarrow B \to C}}
$$

By the correspondence **G**$\longrightarrow$**N** the normal deduction above is transformed into the following natural deduction in $*$-normal form:

$$
u\,\dfrac{v\,\dfrac{\dfrac{[\neg C]^v \qquad \dfrac{(A \to D) \to C \qquad z\,\dfrac{\dfrac{\dfrac{[\neg C]^v \qquad \dfrac{\dfrac{\dfrac{B \to [A \to (B \to C)] \qquad [B]^u}{A \to (B \to C)} \qquad [A]^z}{B \to C} \qquad [B]^u}{C}}{\bot}}{D}}{A \to D}}{C}}{\bot}}{C}}{B \to C}
$$

4 Basic ideas on the correspondences

Deductions in sequent calculus style in the classical case are characterized by the occurrence of a multi-succedent sequent at some node of the prooftree; deductions in natural deduction system in the classical case are characterized by the occurrence of an application of the $\bot_c$ rule.

It seems natural to try to connect these two things in some way.

As a consequence of the occurrence of multi-succedent sequents, one can "switch" from one formula to another in successive applications of R-rules; this will be better explained with an example:

$$
\dfrac{\dfrac{A, C \Rightarrow A, B}{C \Rightarrow A \to B, A}}{\Rightarrow A \to B, C \to A} \tag{2}
$$

Here, the main formula of the first application of R$\rightarrow$ is a side formula in the second application of R$\rightarrow$. In a sense, this is also what happens in the following deduction:

$$\frac{A \Rightarrow B, A}{\Rightarrow A \rightarrow B, A} \tag{3}$$

Indeed, if we think of an axiom as an R-rule with no premises, then we have that the right occurrence of A, which is the main formula in the application of the "Axiom"-rule, is side formula in the application of the successive R-rule. This suggests that these two things can be treated in a similar way.

The main idea for a correspondence in the classical case is described by the following way of associating 2 and 3: we associate 3 to the following deduction:

$$\cfrac{\cfrac{\cfrac{\neg A \qquad [A]^v}{\perp}}{B}\, u}{A \rightarrow B}\, v \tag{4}$$

and 2 to the following deduction:

$$\cfrac{\cfrac{\cfrac{\neg(A \rightarrow B) \qquad \cfrac{\cfrac{\cfrac{[\neg A]^w \qquad [A]^x}{\perp}}{B}\, v}{A \rightarrow B}\, x}{\perp}}{A}\, w}{C \rightarrow A}\, u \tag{5}$$

i.e., we translate a switch of principal formula (from X to Y, going down in the deduction) on the right side with an application of the $\perp_c$ rule whose conclusion is Y, and whose premise is the conclusion of an application of $\rightarrow$E , whose major premise is rule applied in 2.

We also notice that in the natural deductions that we have associated to 2 and to 3 there are some applications of $\rightarrow$E rule that do not correspond to any application of L$\rightarrow$ rule in 2 and 3. So if we try to formalize this empirical association into a general procedure, we have to take into account the fact that some applications of the $\rightarrow$E rule will correspond to no application of L$\rightarrow$, but to a "switch" of principal formula in two successive applications of R-rules.

This will be taken into account by the introduction of a special multiset $\neg\Sigma$ of all the open assumptions of the form $X = \neg Y$ which are the major premises of an application of $\rightarrow$E ; so we will associate the following deduction of $\rightarrow$**Nc** :

$$\begin{array}{c} \Gamma, \neg\Sigma \\ \mathcal{D} \\ A \end{array}$$

to the following deduction of $\rightarrow$**Gc** :

$$\begin{array}{c} \mathcal{D}' \\ \Gamma \Rightarrow A, \Sigma \end{array}$$

Γ being the multiset of the open assumptions of $\mathcal{D}$ that do not belong to $\neg\Sigma$.

5 Natural deductions in $*$-normal form

Let us consider the following system $\rightarrow$**Nc** :

$$\dfrac{\begin{array}{c}[A]^u\\ \mathcal{D}\\ B\end{array}}{A \rightarrow B} \rightarrow I, u$$

$$\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ A \rightarrow B & A\end{array}}{B} \rightarrow E$$

$$\dfrac{\begin{array}{c}[\neg A]^u\\ \mathcal{D}\\ \bot\end{array}}{A} \bot_c, u$$

Definition 5.1. *Let $\mathcal{D}$ be a normal deduction of* $\rightarrow$**Nc** *; $\mathcal{D}$ is* in $*$-normal form *iff:*

1. *If $\bot$ occurs in $\mathcal{D}$ as a nonempty class of open assumptions, then $\mathcal{D}$ has the following shape:*

$$\dfrac{\bot}{A} \bot_c, u$$

2. *the leftmost branch of $\mathcal{D}$ and the leftmost branch of every subdeduction $\mathcal{D}'$ of $\mathcal{D}$ such that the conclusion of $\mathcal{D}'$ is the* minor *premise of an application of $\rightarrow E$ rule in $\mathcal{D}$ have the following shape:*
   going from the root to the leaves there are
   - *$n \geq 0$ applications of $\rightarrow I$;*
   - *$a \leq 1$ applications of $\bot_c$;*
   - *$m \geq 0$ applications of $\rightarrow E$.*

3. *For every occurrence of the following subdeduction in $\mathcal{D}$:*

$$\dfrac{\dfrac{\begin{array}{cc} & \mathcal{D}_1\\ [\neg Y]^v & Y\end{array}}{\bot}}{A} \bot_c, u \qquad (6)$$

   such that $Y \neq A$, either the class of assumptions $[\neg A]^u$ is not an open class of assumptions of $\mathcal{D}_1$, or, if it is open, the assumption $[\neg A]^u$ always occurs in $\mathcal{D}_1$ as the major premise of an application of $\rightarrow E$.

4. *For every occurrence of the following subdeduction in $\mathcal{D}$:*

$$\dfrac{\dfrac{\begin{array}{cc} & \mathcal{D}_1\\ [\neg C]^u & C\end{array}}{\bot}}{\neg\neg C} \rightarrow I, u \qquad (7)$$

   the class of assumptions $[\neg C]^u$ is not an open class of assumptions of $\mathcal{D}_1$ (so, either it is discharged, or it does not occur as an assumption in $\mathcal{D}_1$).

5. *For every occurrence of the following subdeduction in $\mathcal{D}$:*

$$\dfrac{\dfrac{[\neg B]^u \qquad \begin{array}{c}[\neg B]^u\\ \mathcal{D}_1\\ B\end{array}}{\bot}}{B}\ \bot_c, u \tag{8}$$

then

   (a) *the class of assumptions $[\neg B]^u$ is open and nonempty in $\mathcal{D}_1$; moreover,*

   (b) *the class of assumptions $[\neg B]^u$ always occurs in $\mathcal{D}_1$ as the major premise of an application of $\to E$ rule:*

$$\dfrac{\dfrac{[\neg B]^u \qquad \begin{array}{c}\dfrac{\dfrac{[\neg B]^u \qquad \begin{array}{c}[\neg B]^u\\ \mathcal{D}_2\\ B\end{array}}{\bot}}{X}\ v \neq u, \textit{ and so } X \neq B \textit{ for (a)}\\ \mathcal{D}_3\\ B\end{array}}{\bot}}{B}\ u$$

   (c) *the last rule applied in $\mathcal{D}_3$ is $\to E$.*

Remark 5.2. *The conditions stated in the definition above are sufficient conditions to the well-definedness of the correspondence from deductions in* $\to$**Nc** *to deductions in* $\to$**Gc** *.*

6 Correspondence Nc$\longrightarrow$Gc

Proposition 6.1. *Let $\mathcal{D}$ be a deduction in* $\to$**Nc** *in $*$-normal form, whose conclusion is A; let $\neg\Delta'$ be an arbitrary (possibly empty) subset of the multiset $\neg\Sigma$ of all the* open *assumptions of $\mathcal{D}$ of the form $X = \neg Y$, which are the major premises of an application of $\to E$ in $\mathcal{D}$; let Γ be the multiset of the (nonempty) classes of open assumptions of $\mathcal{D}$ which do not belong to $\neg\Delta'$ and which are different from $\neg A$; let us suppose that if $\neg X$ occurs in $\neg\Delta'$ with multiplicity ≥ 1 then the multiplicity of $\neg X$ in Γ is 0, i.e., if an assumption belongs to $\neg\Delta'$, then all the occurrences of that assumptions in $\mathcal{D}$ belong to $\neg\Delta'$. Let Δ be a finite (possibly empty) multiset of formulas.*

There exists a procedure which associates – for every choice of Δ and $\neg\Delta'$ – $\mathcal{D}$ to a unique *normal deduction $\mathcal{D}$' of* $\to$**Gc** *whose conclusion is the sequent $\Gamma \Rightarrow A, \Delta', \Delta$.*

Proof. By induction on *depth* $(\mathcal{D})$.

- **Basis:** If *depth* $(\mathcal{D}) = 1$ then $\mathcal{D}$ reduces to the single-node deduction

$$A$$

So, A is the conclusion and an open assumption of $\mathcal{D}$. We notice that $A \notin \neg\Sigma = \emptyset$, therefore $A \in \Gamma$, and so we associate it to

$$A \Rightarrow A, \Delta$$

• **Induction:** Let us suppose that $depth\,(\mathcal{D}) > 1$. Our Induction Hypothesis says:

> for every deduction $\mathcal{D}_1$ in $*$-normal form s.t. $depth\,(\mathcal{D}_1) < depth\,(\mathcal{D})$, for every subset $\neg\Delta'_1$ of the multiset $\neg\Sigma_1$ of all the open assumptions of $\mathcal{D}_1$ of the form $X = \neg Y$, which are major premises of an application of $\rightarrow$E in $\mathcal{D}_1$ (we suppose that, if $\neg X \in \neg\Delta'_1$ then all the occurrences of $\neg X$ in Σ_1 belong to $\neg\Delta'_1$.), for every (possibly empty) set Δ_1 of formulas, we know how to associate to $\mathcal{D}_1$ a unique deduction $\mathcal{D}'_1$ of $\rightarrow$**Gc** whose conclusion is the sequent $\Gamma_1 \Rightarrow A_1, \Delta'_1, \Delta_1$. Here, A_1 is the conclusion of $\mathcal{D}_1$ and Γ_1 is the set of the open assumptions of $\mathcal{D}_1$ that are not in $\neg\Delta'_1$.

Let us suppose that $\rightarrow$I is the last rule applied in $\mathcal{D}$; therefore $\mathcal{D}$ has the following shape:

$$\cfrac{\begin{array}{c}\Gamma, [A]^u, \neg\Delta' \\ \mathcal{D}_1 \\ B\end{array}}{A \rightarrow B}\; u$$

We notice that all the occurrences of $[A]^u$ are discharged in $\mathcal{D}$, therefore $[A]^u$ will never belong to $\neg\Delta'$, whereas $[A]^u$ is open for $\mathcal{D}_1$ (it can be empty or not). Two cases may occur:

- $B \neq \bot$
- $B = \bot$

If $B \neq \bot$ then we associate $\mathcal{D}$ to the following deduction of $\rightarrow$**Gc** :

$$\cfrac{\begin{array}{c}\mathcal{D}'_1 \\ \Gamma, [A] \Rightarrow B, \Delta', \Delta\end{array}}{\Gamma \Rightarrow A \rightarrow B, \Delta', \Delta}$$

Here we have applied the Induction Hypothesis on $\mathcal{D}_1$ choosing:

$$\neg\Delta'_1 := \neg\Delta'$$
$$\Delta_1 := \Delta$$

therefore $\Gamma_1 = \Gamma$ if the assumption $[A]^u$ is empty in $\mathcal{D}_1$, and $\Gamma_1 = \Gamma + A$ if $[A]^u$ is nonempty in $\mathcal{D}_1$.

If $B = \bot$ then we continue the proof by induction on $depth(\mathcal{D}_1)$:

• **Basis:** if $depth(\mathcal{D}_1) = 1$ then $\bot$ is an open assumption of $\mathcal{D}$, then we associate $\mathcal{D}$to the following deduction:

$$\bot \Rightarrow \neg A$$

• **Induction:** if $depth(\mathcal{D}_1) > 1$ then $\perp$ is not an open assumption of $\mathcal{D}$, therefore $\perp$ is the conclusion of an application of $\rightarrow$E and so $\mathcal{D}$ has the following shape:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_3\\ \neg Y\end{array} \qquad \begin{array}{c}[A]^u\\ \mathcal{D}_2\\ Y\end{array}}{\perp}}{\neg A} \rightarrow I, u$$

By induction on $depth(\mathcal{D}_3)$:

• **Basis:** if $depth(\mathcal{D}_3) = 1$ then $\neg Y$ is an assumption of $\mathcal{D}$:

$$\dfrac{\dfrac{[\neg Y]^v \qquad \begin{array}{c}[A]^u\\ \mathcal{D}_2\\ Y\end{array}}{\perp}}{\neg A} \rightarrow I, u$$

then two cases may occur:

(a) $\neg Y \neq A$;

(b) $\neg Y = A$.

If $\neg Y \neq A$, then $[\neg Y]^v$ is an open assumption of $\mathcal{D}$, then two cases may occur:

(a1) $\neg Y \in \neg\Delta'$;

(a2) $\neg Y \notin \neg\Delta'$.

If $\neg Y \in \neg\Delta'$, then we will associate $\mathcal{D}$ to the following deduction:

$$\dfrac{\begin{array}{c}\mathcal{D}_2'\\ \Gamma, [A] \Rightarrow \perp, \Delta', \Delta\end{array}}{\Gamma \Rightarrow \neg A, \Delta', \Delta}$$

Here we have applied the Induction Hypothesis on $\mathcal{D}_2$ choosing:

$$\neg\Delta_2' := \neg\Delta' - \neg Y$$
$$\Delta_2 := \Delta + \perp$$

therefore $\Gamma_2 = \Gamma$ if the assumption $[A]^u$ is empty in $\mathcal{D}_2$, and $\Gamma_2 = \Gamma + A$ if $[A]^u$ is nonempty in $\mathcal{D}_1$.

If $\neg Y \notin \neg\Delta'$, then $\neg Y$ belongs to Γ, then we will associate $\mathcal{D}$ to the following deduction:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_2'\\ \Gamma_1, [A] \Rightarrow Y, \perp, \Delta', \Delta\end{array} \qquad \Gamma_1, [A], \perp \Rightarrow \perp, \Delta', \Delta}{\Gamma_1, \neg Y, [A] \Rightarrow \perp, \Delta', \Delta}}{\Gamma_1, \neg Y, \Rightarrow \neg A, \Delta', \Delta}$$

Here we have applied the Induction Hypothesis on $\mathcal{D}_2$ choosing:

$$\neg\Delta'_2 := \neg\Delta'$$
$$\Delta_2 := \Delta + \bot$$

therefore $\Gamma_2 = \Gamma_1$ if the assumption $[A]^u$ is empty in $\mathcal{D}_2$, and $\Gamma_2 = \Gamma_1 + A$ if $[A]^u$ is nonempty in $\mathcal{D}_2$, and $\Gamma = \Gamma_1 + \neg Y$.

If $\neg Y = A$, then any occurrence of $[\neg Y]^v$ is discharged in $\mathcal{D}$, therefore $\neg Y$ belongs neither to Γ nor to $\neg\Delta'$; but since $\mathcal{D}$ is in $*$-normal form, $\mathcal{D}$ satisfies condition 4. of definition 5.1, therefore $[\neg Y]^v$ is not an open assumption of $\mathcal{D}_2$, and so $\neg Y$ belongs neither to $\neg\Sigma_2$ nor to Γ_2, then we will associate $\mathcal{D}$ to the following deduction:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}'_2\\ \Gamma \Rightarrow Y, \bot, \Delta', \Delta\end{array} \qquad \Gamma, \bot \Rightarrow \bot, \Delta', \Delta}{\Gamma, \neg Y \Rightarrow \bot, \Delta', \Delta}}{\Gamma, \Rightarrow \neg\neg Y, \Delta', \Delta}$$

Here we have applied the Induction Hypothesis on $\mathcal{D}_2$ choosing:

$$\neg\Delta'_2 := \neg\Delta'$$
$$\Delta_2 := \Delta + \bot$$

- **Induction:** If $depth(\mathcal{D}_3) > 1$, then $\neg Y$ is the conclusion of some rule applied in $\mathcal{D}_3$;

  – $\neg Y$ is not the conclusion of an application of $\rightarrow$I , since $\mathcal{D}$, and so $\mathcal{D}_3$, is normal;

  – $\neg Y$ is not the conclusion of an application of $\bot_c$, since $\mathcal{D}$ is in $*$-normal form (see the condition 2. of definition 5.1);

therefore $\neg Y$ must be the conclusion of an application of $\rightarrow$E ; we can reason in the same fashion for the major premise of this other application of $\rightarrow$E and so on; after $m - 4$ steps ($m \geq 4$) we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B_m \qquad \begin{array}{c}\Gamma_m, \neg\Delta'_m\\ \mathcal{D}_m\\ A_m\end{array}}{B_{m-1}}}{B_4} \qquad \begin{array}{c}\Gamma_4, \neg\Delta'_4\\ \mathcal{D}_4\\ A_4\end{array}}{\neg Y} \qquad \begin{array}{c}\Gamma_2, \neg\Delta'_2\\ \mathcal{D}_2\\ Y\end{array}}{\bot}}{\neg A} \rightarrow I, u$$

we will associate $\mathcal{D}$ to the following deduction:

$$\dfrac{\dfrac{\mathcal{D}'_m \qquad \dfrac{\mathcal{D}'_{m-1} \qquad \dfrac{\mathcal{D}'_4 \qquad \dfrac{\mathcal{D}'_2 \qquad \bot \Rightarrow \bot, \Delta', \Delta}{\Gamma_2, \neg Y, [A] \Rightarrow \bot, \Delta', \Delta}}{\Gamma_{m-2}, \dots, \Gamma_2, B_{m-2}, [A] \Rightarrow \bot, \Delta', \Delta}}{\Gamma_{m-1}, \dots, \Gamma_2, B_{m-1}, [A] \Rightarrow \bot, \Delta', \Delta}}{\Gamma_m, \dots, \Gamma_2, B_m, [A] \Rightarrow \bot, \Delta', \Delta}}{\Gamma_m, \dots, \Gamma_2, B_m \Rightarrow \neg A, \Delta', \Delta}$$

where $\mathcal{D}'_i$ has as conclusion $\Gamma_i, [A] \Rightarrow A_i, \bot, \Delta', \Delta$, for $i = 2, \ldots, M$. Here we have applied the Induction Hypothesis on $\mathcal{D}_i$, $i = 2, 4, \ldots, m$, choosing:

$$\neg\Delta'_i := \neg\Delta' \cap \Sigma_i$$
$$\Delta_i := \Delta + \bot + (\neg\Delta' - \neg\Delta'_i)$$

therefore $\Gamma_i = \Gamma$ if the assumption $[A]^u$ is empty in $\mathcal{D}_i$, and $\Gamma_i = \Gamma + A$ if $[A]^u$ is nonempty in $\mathcal{D}_i$.

Let us suppose that $\to$E is the last rule applied in $\mathcal{D}$; therefore $\mathcal{D}$ has the following shape:

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ A \to B\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ A\end{matrix}}{B}$$

by induction on *depth* $(\mathcal{D}_1)$:

• **Basis:** If *depth* $(\mathcal{D}_1) = 1$, then $A \to B$ is an open assumption of $\mathcal{D}$, and $\mathcal{D}$ has the following shape:

$$\dfrac{A \to B \qquad \begin{matrix}\mathcal{D}_2 \\ A\end{matrix}}{B}$$

two cases may occur:

– $B \neq \bot$;

– $B = \bot$;

if $B \neq \bot$, we associate $\mathcal{D}$ to the following deduction:

$$\dfrac{\begin{matrix}\mathcal{D}'_2 \\ \Gamma \Rightarrow A, B, \Delta', \Delta\end{matrix} \qquad B \Rightarrow B, \Delta', \Delta}{\Gamma, A \to B \Rightarrow B, \Delta', \Delta}$$

Here we have applied the Induction Hypothesis on $\mathcal{D}_2$ choosing:

$$\neg\Delta'_2 := \neg\Delta'$$
$$\Delta_2 := \Delta + B.$$

If $B = \bot$, then $A \to B = \neg A$ is an open assumption of $\mathcal{D}$, and is the major premise of an application of $\to$E , therefore two cases may occur:

– $\neg A \in \neg\Delta'$;

– $\neg A \notin \neg\Delta'$.

If $\neg A \in \neg\Delta'$, then we associate $\mathcal{D}$ to the following deduction:

$$\begin{matrix}\mathcal{D}'_2 \\ \Gamma \Rightarrow \bot, \Delta', \Delta\end{matrix}$$

Here we have applied the Induction Hypothesis on $\mathcal{D}_2$ choosing:

$$\neg\Delta'_2 := \neg\Delta' - \neg A$$
$$\Delta_2 := \Delta + \bot.$$

If $\neg A \notin \neg\Delta'$, then, since $\neg A$ is an open assumption of $\mathcal{D}$, $\neg A \in \Gamma$, therefore we associate to $\mathcal{D}$ the following deduction:

$$\dfrac{\begin{array}{c}\mathcal{D}_2' \\ \Gamma_1 \Rightarrow A, \bot, \Delta', \Delta\end{array} \qquad \bot \Rightarrow \bot, \Delta', \Delta}{\Gamma_1, \neg A \Rightarrow \bot, \Delta', \Delta}$$

Here we have applied the Induction Hypothesis on $\mathcal{D}_2$ choosing:

$$\begin{array}{c}\neg\Delta_2' := \neg\Delta' \\ \Delta_2 := \Delta + \bot.\end{array}$$

• **Induction:** Let us suppose that *depth* $(\mathcal{D}_1) > 1$; therefore $A \rightarrow B$ is the conclusion of an application of a rule.

- $A \rightarrow B$ is not the conclusion of an application of $\rightarrow$I , since $\mathcal{D}$ is normal;
- $A \rightarrow B$ is not the conclusion of an application of $\bot_c$, since $\mathcal{D}$ is in $*$-normal form (see the condition 2. of definition 5.1);

therefore $A \rightarrow B$ must be the conclusion of an application of $\rightarrow$E ; we can reason in the same fashion for the major premise of this other application of $\rightarrow$E and so on; in the general case we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{B_m \qquad \begin{array}{c}\Gamma_m, \neg\Delta_m' \\ \mathcal{D}_m \\ A_m\end{array}}{B_{m-1}}}{\dfrac{}{B_3}} \qquad \begin{array}{c}\Gamma_3, \neg\Delta_3' \\ \mathcal{D}_3 \\ A_3\end{array}}{B_2} \qquad \begin{array}{c}\Gamma_2, \neg\Delta_2' \\ \mathcal{D}_2 \\ A_2\end{array}}{A_1}$$

where $B_1 = A_1$, $B_i = A_i \rightarrow B_{i-1}$, $i = 2 \ldots m$;
we associate it to the following deduction:

$$\dfrac{\mathcal{D}_m' \qquad \dfrac{\mathcal{D}_{m-1}' \qquad \dfrac{\mathcal{D}_3' \qquad \dfrac{\mathcal{D}_2' \qquad A_1 \Rightarrow A_1\Delta', \Delta}{\Gamma_2, B_2 \Rightarrow A_1, \Delta', \Delta}}{\Gamma_{m-2}, \ldots, \Gamma_2, B_{m-2} \Rightarrow A_1, \Delta', \Delta}}{\Gamma_{m-1}, \ldots, \Gamma_2, B_{m-1} \Rightarrow A_1, \Delta', \Delta}}{\Gamma_m, \ldots, \Gamma_2, B_m \Rightarrow A_1, \Delta', \Delta}$$

where $\mathcal{D}_i'$ has as conclusion $\Gamma_i \Rightarrow A_i, A_1, \Delta', \Delta$ for $i = 2, \ldots, m$. Here we have applied the Induction Hypothesis on $\mathcal{D}_i$, $i = 2, \ldots, m$, choosing:

$$\begin{array}{c}\neg\Delta_i' := \neg\Delta' \cap \Sigma_i \\ \Delta_i := \Delta + A_1 + (\neg\Delta - \neg\Delta_i)\end{array}$$

Let us suppose that $\bot_c$ is the last rule applied in $\mathcal{D}$; therefore $\mathcal{D}$ has the following shape:

$$\dfrac{\begin{array}{c}\mathcal{D}_3 \\ \bot\end{array}}{A}\, u$$

by induction on $depth\,(\mathcal{D}_3)$:

• **Basis:** if $depth\,(\mathcal{D}_3) = 1$, then $\mathcal{D}$ reduces to the following deduction:

$$\frac{\perp}{A}$$

then we associate it to the following deduction:

$$\perp \Rightarrow A, \Delta$$

• **Induction:** If $depth\,(\mathcal{D}_3) > 1$, then $\perp$ is not an assumption of $\mathcal{D}$, therefore $\perp$ must be the conclusion of an application of $\rightarrow$E , so $\mathcal{D}$ has the following shape:

$$\cfrac{\cfrac{\begin{matrix}\Gamma_1, \neg\Delta_1' \\ \mathcal{D}_1 \\ \neg Y\end{matrix} \qquad \begin{matrix}\Gamma_2, \neg\Delta_2' \\ \mathcal{D}_2 \\ Y\end{matrix}}{\perp}}{A}\,u$$

by induction on $\mathcal{D}_1$:

• **Basis:** If $depth\,(\mathcal{D}_1) = 1$, then $\mathcal{D}$ has the following shape:

$$\cfrac{\cfrac{\neg Y \qquad \begin{matrix}[\neg A]^u, \Gamma, \neg\Delta' \\ \mathcal{D}_2 \\ Y\end{matrix}}{\perp}}{A}\,u$$

Two cases may occur:

(c) $Y \neq A$;

(d) $Y = A$.

If $Y \neq A$, then $[\neg Y]^v$ is an open assumption of $\mathcal{D}$, and it is the major premise of an application of $\rightarrow$E ; so two cases may occur:

(c1) $\neg Y \notin \neg\Delta'$;

(c2) $\neg Y \in \neg\Delta'$;

If $\neg Y \notin \neg\Delta'$, then $\neg Y \in \Gamma$, therefore we associate $\mathcal{D}$ to the following deduction:

$$\cfrac{\begin{matrix}\mathcal{D}_2' \\ \Gamma_1 \Rightarrow Y, A, \Delta', \Delta\end{matrix} \qquad \perp \Rightarrow A, \Delta', \Delta}{\Gamma_1, \neg Y \Rightarrow A, \Delta', \Delta}$$

Here we have applied Induction Hypothesis on $\mathcal{D}_2$ choosing:

$$\neg\Delta_2' := \neg\Delta'$$
$$\Delta_2 := \Delta + A$$

If $\neg Y \in \neg\Delta'$ then $\neg Y \notin \Gamma$ and by condition 3. of definition 5.1 either the assumption $\neg A$ is not open for $\mathcal{D}_2$ or it belongs to $\neg\Sigma_2$, therefore we associate $\mathcal{D}$ to the following deduction:

$$\begin{matrix}\mathcal{D}'_2 \\ \Gamma \Rightarrow A, \Delta', \Delta\end{matrix}$$

Here we have applied Induction Hypothesis on $\mathcal{D}_2$ choosing:

$$\neg\Delta'_2 := \neg\Delta' \cap \neg\Sigma_2$$
$$\Delta_2 := \Delta + A + (\neg\Delta' - \neg\Delta'_2)$$

if the assumption $\neg A$ is not open for $\mathcal{D}_2$ or

$$\neg\Delta'_2 := \neg\Delta' + \neg A - \neg Y$$
$$\Delta_2 := \Delta$$

if the assumption $\neg A$ is open for $\mathcal{D}_2$. If $Y = A$, then every occurrence of the assumption $[\neg Y]^v = [\neg A]^u$ is discharged in $\mathcal{D}$, therefore $\neg A$ cannot occur neither in $\neg\Delta'$ nor in Γ, whereas, since $\mathcal{D}$ is in $*$-normal form, $\mathcal{D}$ meets the conditions 5(a) and 5(b) of definition 5.1, therefore $[\neg A]^u$ occurs in $\mathcal{D}_2$ as an open assumption and the major premise of an application of $\rightarrow$E rule; moreover it also satisfies condition 5(c) of definition 5.1, therefore it has the following shape:

$$\dfrac{\dfrac{[\neg A]^u \qquad \dfrac{\dfrac{\dfrac{B_m \qquad \begin{matrix}\mathcal{D}_m \\ A_m\end{matrix}}{B_{m-1}}}{B_4} \qquad \begin{matrix}\mathcal{D}_4 \\ A_4\end{matrix}}{A}}{\perp}}{A}\, u$$

and there exists at least one index $j \in \{4, \ldots, m\}$ such that $[\neg A]^u$ occurs in $\mathcal{D}_j$ as an open assumption which is the major premise of an application of the $\rightarrow$E rule; we will associate $\mathcal{D}$ to the following deduction:

$$\dfrac{\begin{matrix}\mathcal{D}'_m \\ \Gamma_m \Rightarrow A_m, A, \Delta', \Delta\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\mathcal{D}'_4 \\ \Gamma_4 \Rightarrow A_4, A, \Delta', \Delta\end{matrix} \qquad A \Rightarrow A, \Delta', \Delta}{\Gamma_2, B_4, \Rightarrow A, \Delta', \Delta}}{\Gamma_{m-1}, \ldots, \Gamma_2, B_{m-1} \Rightarrow A, \Delta', \Delta}}{\Gamma_m, \ldots, \Gamma_2, B_m \Rightarrow A, \Delta', \Delta}$$

Here we have applied the Induction Hypothesis on $\mathcal{D}_i$, $i = 4, \ldots, m$, choosing:

$$\neg\Delta'_h := \neg\Delta' \cap \neg\Sigma_h$$
$$\Delta_h := \Delta + A + (\neg\Delta' - \neg\Delta'_h)$$

for every index $h \in \{4, \ldots, m\}$ such that $[\neg A]^u$ is an empty class of assumptions for $\mathcal{D}_h$, and

$$\neg\Delta'_j := (\neg\Delta' \cap \neg\Sigma_j) + \neg A$$
$$\Delta_j := \Delta + (\neg\Delta' - \neg\Delta'_j)$$

for every index $j \in \{4, \ldots, m\}$ such that $[\neg A]^u$ is a nonempty class of assumptions for $\mathcal{D}_j$.

• **Induction:** Let us suppose that *depth* $(\mathcal{D}_1) > 1$; therefore $\neg Y$ is the conclusion of an application of a rule.

– $\neg Y$ is not the conclusion of an application of $I \rightarrow$,since $\mathcal{D}$ is normal;

– $\neg Y$ is not the conclusion of an application of $\perp_c$, since $\mathcal{D}$ is in $*$-normal form, therefore it meets the condition 2. of definition 5.1;

therefore $\neg Y$ must be the conclusion of an application of $\rightarrow$E ; we can reason in the same fashion for the major premise of this other application of $\rightarrow$E and so on; after m steps ($m \geq 1$) we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{B_m \qquad \begin{matrix}\Gamma_m, \neg\Delta'_m\\ \mathcal{D}_m\\ A_m\end{matrix}}{B_{m-1}}}{A_3 \rightarrow \neg B} \qquad \begin{matrix}\Gamma_3, \neg\Delta'_3\\ \mathcal{D}_3\\ A_3\end{matrix}}{\neg B} \qquad \begin{matrix}\Gamma_2, \neg\Delta'_2\\ \mathcal{D}_2\\ B\end{matrix}}{\dfrac{\perp}{A}\, u}$$

We associate it to the following deduction:

$$\dfrac{\begin{matrix}\mathcal{D}'_m\\ \Gamma_m \Rightarrow A_m, A, \Delta', \Delta\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\mathcal{D}'_3\\ \Gamma_3 \Rightarrow A_3, A, \Delta', \Delta\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}'_2\\ \Gamma_2 \Rightarrow B, A, \Delta', \Delta\end{matrix} \qquad \perp \Rightarrow A, \Delta', \Delta}{\Gamma_2, \neg B, \Rightarrow A, \Delta', \Delta}}{\Gamma_3, \Gamma_2, A_3 \rightarrow \neg B \Rightarrow A, \Delta', \Delta}}{\Gamma_{m-1}, \ldots, \Gamma_2, B_{m-1} \Rightarrow A, \Delta', \Delta}}{\Gamma_m, \ldots, \Gamma_2, B_m \Rightarrow A, \Delta', \Delta}$$

Here we have applied the Induction Hypothesis on $\mathcal{D}_i$, $i = 2, \ldots, m$, choosing:

$$\begin{aligned}\neg\Delta'_i &:= \neg\Delta' \cap \neg\Sigma_i\\ \Delta_i &:= \Delta + A + (\neg\Delta' - \neg\Delta'_i).\end{aligned}$$

□

Corollary 6.2. *Let $\mathcal{D}$ be a deduction in* $\rightarrow$**Nc** *whose conclusion is A; let $\neg\Sigma$ be the multiset of the open assumptions of $\mathcal{D}$ of the form $X = \neg Y$ such that X is the major premise of an application of $E \rightarrow$; let Γ be the multiset of the open assumptions of $\mathcal{D}$ which do not belong to $\neg\Sigma$.*

There exists a procedure which associates $\mathcal{D}$ to a unique *deduction $\mathcal{D}'$ of* $\rightarrow$**Gc** *whose conclusion is the sequent $\Gamma \Rightarrow A, \Sigma$.*

Proof. The procedure is the one described in the proposition above for $\Delta = \emptyset, \neg\Delta' = \neg\Sigma$. □

7 Correspondence Gc $\longrightarrow$ Nc

Proposition 7.1. *Let $\mathcal{D}$ be a normal deduction in* $\rightarrow$**Gc** *whose conclusion is $\Gamma \Rightarrow \Delta$.*

There exists a non-trivial procedure which associates – for every $X \in \Delta$ – $\mathcal{D}$ to a deduction $\mathcal{D}$' in $\rightarrow$**Nc** in $*$-normal form, *whose conclusion is X and whose set of open assumptions is a subset of* $\Gamma \cup (\neg\Delta \setminus \{\neg X\})$.

Proof. By induction on $depth(\mathcal{D})$.

• **Basis:** If $depth(\mathcal{D}) = 1$, then the prooftree of $\mathcal{D}$ is a single-node labelled with an axiom.

We associate

$$\Gamma_1, A \Rightarrow A, \Delta_1$$

to the single-node natural deduction

$$A$$

if $X = A$, or to

$$\dfrac{\dfrac{[\neg A]^u \qquad A}{\perp}}{X}\ \perp_c, x$$

if $X \neq A$. We associate

$$\Gamma_1, \perp \Rightarrow \Delta$$

to the following natural deduction:

$$\dfrac{\perp}{X}\ \perp_c, x$$

for every $X \in \Delta$.

All these deductions are in $*$-normal form and the sets of their open assumptions are subsets of $\Gamma \cup (\neg\Delta \setminus \{\neg X\})$.

• **Induction:** Let us suppose that $depth(\mathcal{D}) > 1$, and that R$\rightarrow$ is the last rule applied in $\mathcal{D}$. Then $\mathcal{D}$ will have the following shape:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_1, A \Rightarrow B, \Delta_1\end{array}}{\Gamma_1 \Rightarrow A \rightarrow B, \Delta_1}$$

We associate $\mathcal{D}$ to the following deduction:

$$\dfrac{\begin{array}{c}\Pi\\ \mathcal{D}'_1\\ B\end{array}}{A \rightarrow B}$$

if $X = A \rightarrow B$ (Here, by induction hypothesis, $\Pi \subseteq \Gamma_1 \cup \{A\} \cup \neg\Delta_1$), or to the following deduction:

$$\dfrac{\dfrac{\neg(A \rightarrow B) \qquad \dfrac{\begin{array}{c}\Pi\\ \mathcal{D}'_1\\ B\end{array}}{A \rightarrow B}}{\perp}}{X}\ \perp_c, x$$

if $X \neq A \rightarrow B$.

All these deductions are in $*$-normal form and the sets of their open assumptions are subsets of $\Gamma \cup (\neg\Delta \setminus \{\neg X\})$.

Let us suppose that L$\rightarrow$ is the last rule applied in $\mathcal{D}$. Then $\mathcal{D}$ has the following shape:

$$\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ \Gamma_1 \Rightarrow A, \Delta & \Gamma_1, B \Rightarrow \Delta\end{array}}{\Gamma_1, A \to B \Rightarrow \Delta}$$

Since $\mathcal{D}$ is normal, $\mathcal{D}$ meets condition (a) of the definition 3.1, therefore the antecedent active formula B is itself principal formula (of an axiom or of another application of the L$\to$ rule.).

Let us suppose that B is the principal formula of an axiom Ax (so $depth(\mathcal{D}_2) = 1$); then two cases may occur:

(a) Ax is $\Gamma_1, B \Rightarrow B, \Delta_1$;

(b) Ax is $\Gamma_1, \perp \Rightarrow \Delta$ (so $B = \perp$).

If (a), then we associate $\mathcal{D}$ to the following deduction:

$$\dfrac{A \to B \qquad \begin{array}{c}\Pi\\ \mathcal{D}'_1\\ A\end{array}}{B}$$

if $X = B$, and $\neg B$ is not an open assumption of $\mathcal{D}'_1$, or to the following deduction:

$$\dfrac{\dfrac{\dfrac{\neg B \qquad \dfrac{A \to B \qquad \begin{array}{c}\Pi\\ \mathcal{D}'_1\\ A\end{array}}{B}}{\perp}}{X}\ \perp_c, x}{}$$

in any other cases.

If (b), then we associate $\mathcal{D}$ to the following deduction $\mathcal{D}$':

$$\dfrac{\dfrac{\neg A \qquad \begin{array}{c}\Pi\\ \mathcal{D}'_1\\ A\end{array}}{\perp}}{X}\ \perp_c, x$$

for every $X \in \Delta$. We notice that, since $\mathcal{D}$ is normal, $\mathcal{D}$ meets the condition (b) of definition 3.1, therefore $A \notin \Delta$, and so in this case the assumption $\neg A$ is open in $\mathcal{D}$'.

Let us suppose that B is the principal formula of an application of L$\to$; so $\mathcal{D}$ has the following shape:

$$\dfrac{\mathcal{D}_{n-1} \qquad \dfrac{\mathcal{D}_2 \qquad Ax}{\Gamma_{n-1}, \ldots, \Gamma_2, B_{n-1} \Rightarrow \Delta}\ n \geq 2 \text{ applications of L}\to}{\Gamma_n, \ldots, \Gamma_2, B_n \Rightarrow \Delta}$$

where the conclusion of $\mathcal{D}_i$ is $\Gamma_i \Rightarrow A_i, \Delta$ for $i = 2, \ldots, n$; moreover, $A_1 = B_1$ is the principal antecedent formula of Ax, and $B_{i+1} = A_{i+1} \to B_i$ for $i = 1, \ldots, n$.

Then two cases may occur:

(a) Ax is $\Gamma_1, B_1 \Rightarrow B_1, \Delta_1$;

(b) Ax is $\Gamma_1, \perp \Rightarrow \Delta$.

If (a), then we associate $\mathcal{D}$ to the following deduction:

$$\dfrac{\begin{matrix}\dfrac{B_n \qquad \begin{matrix}\Pi_n\\ \mathcal{D}'_n\\ A_n\end{matrix}}{B_{n-1}}\\ \vdots\\ B_2\end{matrix} \qquad \begin{matrix}\Pi_2\\ \mathcal{D}'_2\\ A_2\end{matrix}}{B_1}$$

if $X = B_1$, or to the following deduction:

$$\dfrac{\dfrac{\neg B_1 \qquad \dfrac{\begin{matrix}\dfrac{B_n \qquad \begin{matrix}\Pi_n\\ \mathcal{D}'_n\\ A_n\end{matrix}}{B_{n-1}}\\ \vdots\\ B_2\end{matrix} \qquad \begin{matrix}\Pi_2\\ \mathcal{D}'_2\\ A_2\end{matrix}}{B_1}}{\bot}}{X}\ \bot_c, x$$

if $X \neq B_1$. In both cases, by induction hypothesis, $\Pi_i \subseteq \Gamma_i \cup \neg\Delta$ for $i = 2, \ldots, n$.

If (b), then $B_1 = \bot$ and $B_2 = \neg A_2$; we associate $\mathcal{D}$ to the following deduction:

$$\dfrac{\dfrac{\begin{matrix}\dfrac{B_n \qquad \begin{matrix}\Pi_n\\ \mathcal{D}'_n\\ A_n\end{matrix}}{B_{n-1}}\\ \vdots\\ \neg A_2\end{matrix} \qquad \begin{matrix}\Pi_2\\ \mathcal{D}'_2\\ A_2\end{matrix}}{\bot}}{X}\ \bot_c, x$$

for every $X \in \Delta$.

All these deductions are in $*$-normal form and the sets of their open assumptions are subsets of $\Gamma \cup (\neg\Delta \setminus \{\neg X\})$.

$\square$

7.1 Some examples

7.1.1

$$\dfrac{\dfrac{\dfrac{[\neg\neg A]^v \qquad \dfrac{\dfrac{[\neg A]^u \qquad [A]^z}{\bot}}{\neg A}\ I\rightarrow, z}{\bot}}{A}\ u}{\neg\neg A \rightarrow A}\ v$$

is associated to

$$\dfrac{\dfrac{\dfrac{A \Rightarrow \bot, A}{\Rightarrow \neg A, A} \qquad \bot \Rightarrow A}{\neg\neg A \Rightarrow A}}{\Rightarrow \neg\neg A \to A}$$

7.1.2

$$\dfrac{\dfrac{\dfrac{[\neg A]^v \qquad \dfrac{[(A \to B) \to A]^u \qquad \dfrac{\dfrac{\dfrac{[\neg A]^v \qquad [A]^y}{\bot}}{B}\,w}{A \to B}\,y}{A}}{\bot}}{A}\,v}{((A \to B) \to A) \to A}\,u$$

is associated to

$$\dfrac{\dfrac{\dfrac{A \Rightarrow B, A}{\Rightarrow A \to B, A} \qquad A \Rightarrow A}{(A \to B) \to A \Rightarrow A}}{\Rightarrow ((A \to B) \to A) \to A}$$

7.1.3

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg(A \to \neg B)]^u \qquad \dfrac{\dfrac{\dfrac{[\neg A]^z \qquad [A]^y}{\bot}}{\neg B}\,x}{A \to \neg B}\,y}{\bot}}{A}\,z}{B \to A}\,v}{\neg(A \to \neg B) \to (B \to A)}\,u$$

is associated to

$$\dfrac{\dfrac{\dfrac{\dfrac{B, A \Rightarrow \neg B, A}{B \Rightarrow A \to \neg B, A} \qquad B\bot \Rightarrow A}{\neg(A \to \neg B), B \Rightarrow A}}{\neg(A \to \neg B) \Rightarrow B \to A}}{\Rightarrow \neg(A \to \neg B) \to (B \to A)}$$

7.1.4

$$
\begin{prooftree}
\AxiomC{$[\neg A]^w$}
\AxiomC{$[(A \to C) \to (B \to A)]^u$}
\AxiomC{$[\neg A]^w$}
\AxiomC{$[A]^z$}
\BinaryInfC{$\bot$}
\RightLabel{x}
\UnaryInfC{C}
\RightLabel{z}
\UnaryInfC{$A \to C$}
\BinaryInfC{$B \to A$}
\AxiomC{$[\neg B]^y$}
\AxiomC{$[B]^v$}
\BinaryInfC{$\bot$}
\RightLabel{y}
\UnaryInfC{B}
\BinaryInfC{A}
\BinaryInfC{$\bot$}
\RightLabel{w}
\UnaryInfC{A}
\RightLabel{v}
\UnaryInfC{$B \to A$}
\RightLabel{u}
\UnaryInfC{$[(A \to C) \to (B \to A)] \to (B \to A)$}
\end{prooftree}
$$

is associated to

$$
\begin{prooftree}
\AxiomC{$A, B \Rightarrow A, C$}
\UnaryInfC{$B \Rightarrow A \to C, A$}
\AxiomC{$B \Rightarrow B, A$}
\AxiomC{$A, B \Rightarrow A$}
\BinaryInfC{$B \to A, B \Rightarrow A$}
\BinaryInfC{$(A \to C) \to (B \to A), B \Rightarrow A$}
\UnaryInfC{$(A \to C) \to (B \to A) \Rightarrow B \to A$}
\UnaryInfC{$\Rightarrow [(A \to C) \to (B \to A)] \to (B \to A)$}
\end{prooftree}
$$

8 The $*$-normalization theorem

In this section I give the procedure for transforming a normal deduction in $\to$**Nc** into a deduction in $*$-normal form. The procedure consists of seven transformation steps: at each step, $\mathcal{D}$ is transformed into a deduction which satisfies all the conditions of def. 5.1 satisfied by $\mathcal{D}$, plus a new one.

Definition 8.1. *Let $\mathcal{D}$ be a deduction of $\to$**Nc** . A* $*$-cut *in $\mathcal{D}$ is a sequence of occurrences $B_0 \dots B_n$ of formulas at nodes of $\mathcal{D}$ such that:*

1. *B_i is the major premise of an application of $\to E$, $i = 1 \dots n$;*
2. *B_0 is not the major premise of an application of $\to E$;*
3. *B_i is the conclusion of the application of $\to E$ of which B_i is the major premise,$i = 0 \dots n-1$;*
4. *B_n is introduced with an application of $\bot_c$.*

The $*$-cutformula *of a $*$-cut $B_0 \dots B_n$ is B_n.*
The $*$-cutrank *of a $*$-cut $B_0 \dots B_n$ is the complexity of the formula B_n.*

Lemma 8.2. *Let $\mathcal{D}$ be a normal deduction of $\to$**Nc** whose set of open assumptions is Γ and whose conclusion is A.*
There exists a normal deduction $\mathcal{D}'$ which satisfies condition 2. of definition 5.1, whose set of open assumptions is $\Gamma' \subseteq \Gamma$ and whose conclusion is A.

Proof. By induction on $(a(\mathcal{D}), b(\mathcal{D}), c(\mathcal{D}))$, where:

- $a(\mathcal{D})$ is the maximum $*$-cutrank of a $*$-cut of $\mathcal{D}$;
- $b(\mathcal{D})$ is is the number of $*$-cuts of maximum $*$-cutrank;

- $c(\mathcal{D})$ is the maximum depth of a subdeduction of $\mathcal{D}$ whose conclusion is a $*$-cut formula of maximum $*$-cutrank.

• **Basis:** If $(a(\mathcal{D}) = 0$, then there are no $*$-cuts in $\mathcal{D}$, and so $\mathcal{D}$ satisfies condition 2.
• **Induction:** Suppose $(a(\mathcal{D}) \geq 1$: then $\mathcal{D}$ has the following shape:

$$n \geq 0 \text{ applications of } \to\mathrm{E} \quad \dfrac{\begin{matrix}\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0\\ \bot\end{matrix}}{B_n}\,\bot_c, u \qquad \begin{matrix}\mathcal{D}_n\\ A_n\end{matrix}}{\vdots}\\ B_1\end{matrix} \qquad \begin{matrix}\mathcal{D}_1\\ A_1\end{matrix}}{\begin{matrix}B_0\\ \vdots\end{matrix}}$$

where $B_0 \dots B_n$ is a $*$-cut of maximum $*$-cutrank, and such that $depth(\mathcal{D}_{0,0}) = c(\mathcal{D})$ [2], therefore we have that $(a(\mathcal{D}_i) < (a(\mathcal{D})\ i = 0 \dots n$, and so $(a(\mathcal{D}_i), b(\mathcal{D}_i), c(\mathcal{D}_i)) < (a(\mathcal{D}), b(\mathcal{D}), c(\mathcal{D}))\ i = 0 \dots n$, and so we can suppose by induction hypothesis that $\mathcal{D}_i$ satisfies condition 2. $i = 0 \dots n$.
Two cases may occur:

(a) $B_0 \neq \bot$;

(b) $B_0 = \bot$.

Case (a): We transform $\mathcal{D}$ into the following deduction $\mathcal{D}^*$:

$$\dfrac{\begin{matrix}\mathcal{D}'_0\\ \bot\end{matrix}}{\begin{matrix}B_0\\ \vdots\end{matrix}}\,\bot_c, u$$

where $\mathcal{D}_0 = \mathcal{D}'_0$ if $[\neg B_n]^u$ is not an open assumption of $\mathcal{D}_0$, otherwise $\mathcal{D}'_0$ is obtained from $\mathcal{D}_0$ by substituting every occurrence of $[\neg B_n]^u$ which is not the major premise of an application of $\to$E with the following deduction of $\neg B_n$:

$$\dfrac{\dfrac{[\neg B_0]^w \qquad \dfrac{\begin{matrix}\dfrac{[B_n]^z \qquad \begin{matrix}\mathcal{D}_n\\ A_n\end{matrix}}{\vdots}\\ B_1\end{matrix} \qquad \begin{matrix}\mathcal{D}_1\\ A_1\end{matrix}}{B_0}}{\bot}}{\neg B_n}\,\to\mathrm{I}\ ,\ z$$

(we notice that we are not creating any new $*$-cut with this substitution), and every occurrence of $[\neg B_n]^u$ as the major premise of an application of $\to$E :

[2] $\mathcal{D}_{0,0}$ is the following deduction:

$$\dfrac{\begin{matrix}\mathcal{D}_0\\ \bot\end{matrix}}{B_n}\,\bot_c, u$$

[3]
$$\dfrac{\dfrac{[\neg B_n]^u \qquad \begin{matrix}\mathcal{D}''\\ B_n\end{matrix}}{\bot}}{X}\,R$$

with the following deduction of X:

$$\dfrac{\dfrac{[\neg B_0]^w \qquad \dfrac{\begin{matrix}\dfrac{\begin{matrix}\mathcal{D}''\\ B_n\end{matrix}\quad\begin{matrix}\mathcal{D}_n\\ A_n\end{matrix}}{}\\ \vdots\\ B_1\end{matrix}\qquad \begin{matrix}\mathcal{D}_1\\ A_1\end{matrix}}{B_0}}{\bot}}{X}\,R$$

(in this case the substitution process will be progressive and will start from a subdeduction 8 such that in its $\mathcal{D}''$ there are no open occurrences of the assumption $\neg B_n$.) Since $\mathcal{D}''$ is a subdeduction of $\mathcal{D}_0$, $\mathcal{D}''$ is normal and satisfies condition 2. If $depth(\mathcal{D}'') = 1$ or $\rightarrow$E is the last rule applied in $\mathcal{D}''$, it holds that the subdeduction 8 is normal and no other $*$-cuts are created, therefore $(b(\mathcal{D}^*) < (b(\mathcal{D})$, and so $(a(\mathcal{D}^*), b(\mathcal{D}^*), c(\mathcal{D}^*)) < (a(\mathcal{D}), b(\mathcal{D}), c(\mathcal{D}))$, and so, by induction hypothesis, $\mathcal{D}^*$ satisfies condition 2. Let us suppose that $\rightarrow$I is the last rule applied in $\mathcal{D}''$: then deduction 8 has the following shape:

$$\dfrac{\dfrac{[\neg B_0]^w \qquad \dfrac{\begin{matrix}\dfrac{\dfrac{\begin{matrix}\mathcal{D}'''\\ B_{n-1}\end{matrix}}{B_n}\rightarrow\mathrm{I}\quad\begin{matrix}\mathcal{D}_n\\ A_n\end{matrix}}{}\\ \vdots\\ B_1\end{matrix}\qquad \begin{matrix}\mathcal{D}_1\\ A_1\end{matrix}}{B_0}}{\bot}}{X}\,R$$

where $B_n = A_n \rightarrow B_{n-1}$. Let us consider the following subdeduction $\tilde{\mathcal{D}}$ of 8:

$$\dfrac{\begin{matrix}\dfrac{\dfrac{\begin{matrix}\mathcal{D}'''\\ B_{n-1}\end{matrix}}{B_n}\rightarrow\mathrm{I}\quad\begin{matrix}\mathcal{D}_n\\ A_n\end{matrix}}{}\\ \vdots\\ B_1\end{matrix}\qquad \begin{matrix}\mathcal{D}_1\\ A_1\end{matrix}}{B_0}$$

(Since $\mathcal{D}'''$ is a subdeduction of $\mathcal{D}''$, $\mathcal{D}'''$ is normal and satisfies condition 2.) $\tilde{\mathcal{D}}$ is not normal. The normalization process applied on $\tilde{\mathcal{D}}$ can generate new $*$-cuts [4], but all the $*$-cut formulas of the possible new $*$-cuts generated by the normalization process are subformulas of A_n, which is a *proper* subformula of B_n, therefore the $*$-cutranks of the new $*$-cuts will be strictly less than $a(\mathcal{D})$,

[3] R can either be an application of $\rightarrow$I or of $\bot_c$.

[4] Consider for example the case in which A_n occurs in $\mathcal{D}'''$ as an open assumption which is also the major premise of an application of $\rightarrow$E , and moreover the last rule applied in $\mathcal{D}_n$ is $\bot_c$.

therefore $(a(\tilde{\mathcal{D}}) < (a(\mathcal{D})$, and so $(a(\tilde{\mathcal{D}}), b(\tilde{\mathcal{D}}), c(\tilde{\mathcal{D}})) < (a(\mathcal{D}), b(\mathcal{D}), c(\mathcal{D}))$, and so, by induction hypothesis applied on $\tilde{\mathcal{D}}$, $\mathcal{D}^*$ satisfies condition 2.

Let us suppose that $\bot_c$ is the last rule applied in $\mathcal{D}''$: then deduction 8 has the following shape:

$$\dfrac{\dfrac{[\neg B_0]^w \qquad \dfrac{\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\mathcal{D}'''\\ \bot\end{array}}{B_n}\bot_c \quad \begin{array}{c}\mathcal{D}_n\\ A_n\end{array}}{\vdots}\\ B_1\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ A_1\end{array}}{B_0}}{\bot}}{X}R$$

Let us consider the following subdeduction $\tilde{\mathcal{D}}$ of 8:

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\mathcal{D}'''\\ \bot\end{array}}{B_n}\bot_c \quad \begin{array}{c}\mathcal{D}_n\\ A_n\end{array}}{\vdots}\\ B_1\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ A_1\end{array}}{B_0}$$

(Since $\mathcal{D}'''$ is a subdeduction of $\mathcal{D}''$, $\mathcal{D}'''$ is normal and satisfies condition 2.) It holds that $(a(\tilde{\mathcal{D}}) \leq (a(\mathcal{D})$ and $(b(\tilde{\mathcal{D}}) \leq (b(\mathcal{D})$, but we have that the following deduction:

$$\dfrac{[\neg B_n]^u \qquad \dfrac{\begin{array}{c}\mathcal{D}'''\\ \bot\end{array}}{B_n}}{X}R$$

is a subdeduction of $\mathcal{D}_{0,0}$, and so $c(\tilde{\mathcal{D}}) = \mathit{depth}(\mathcal{D}''') \leq \mathit{depth}(\mathcal{D}_{0,0}) - 2 < \mathit{depth}(\mathcal{D}_{0,0}) = c(\mathcal{D})$ therefore $(a(\tilde{\mathcal{D}}), b(\tilde{\mathcal{D}}), c(\tilde{\mathcal{D}})) < (a(\mathcal{D}), b(\mathcal{D}), c(\mathcal{D}))$, and so, by induction hypothesis, $\mathcal{D}^*$ satisfies condition 2.

Case (b): same fashion. □

Lemma 8.3. *Let $\mathcal{D}$ be a normal deduction of* $\to$**Nc** *whose set of open assumptions is Γ and whose conclusion is C; let us suppose that $\bot \notin \Gamma$ and that $\mathcal{D}$ satisfies condition 2. of definition 5.1.*

There exists a normal deduction $\mathcal{D}'$ whose set of open assumptions is $\Gamma' = \Gamma$, whose conclusion is C and that satisfies conditions 2. and 3. of definition 5.1.

Proof. By induction on the number n of violations to the condition 3. in $\mathcal{D}$.

- **Basis:** If $n = 0$ trivial.
- **Induction:** Let us suppose that $n \geq 1$; then $\mathcal{D}$ has the following shape:

$$\dfrac{[\neg Y]^v \qquad \begin{array}{c}[\neg A]^u\\ \mathcal{D}_1\\ Y\end{array}}{\begin{array}{c}\dfrac{\perp}{A}\perp_c, u\\ \mathcal{D}_2\end{array}}$$

where $Y \neq A$ and $[\neg A]^u$ is an open assumption of $\mathcal{D}_1$ and there is at least one occurrence of the assumption $[\neg A]^u$ as the minor premise of an application of $\rightarrow$E in $\mathcal{D}_1$ or as the premise of an application of $\rightarrow$I in $\mathcal{D}_1$. Then we will transform $\mathcal{D}$ substituting every occurrence of the assumption $[\neg A]^u$ as the minor premise of an application of $\rightarrow$E in $\mathcal{D}_1$ or as the premise of an application of $\rightarrow$I in $\mathcal{D}_1$ with the following deduction whose conclusion is $\neg A$:

$$\dfrac{[\neg A]^u \qquad [A]^v}{\dfrac{\perp}{\neg A} \rightarrow I, v} \tag{9}$$

and we obtain a deduction $\mathcal{D}'$ whose conclusion is C and whose set of open assumptions is Γ; moreover $\mathcal{D}'$ is normal, satisfies condition 2. and the number of violations to condition 3. in $\mathcal{D}'$ is $n-1$, therefore we can apply induction hypothesis to $\mathcal{D}'$, and suppose that $\mathcal{D}'$ has already been transformed into the deduction which we were looking for. □

Lemma 8.4. *Let $\mathcal{D}$ be a normal deduction of $\rightarrow$**Nc** whose set of open assumptions is Γ and whose conclusion is A; let us suppose that $\perp \notin \Gamma$ and that $\mathcal{D}$ satisfies conditions 2. and 3. of definition 5.1.*

There exists a normal deduction $\mathcal{D}'$ whose set of open assumptions is $\Gamma' = \Gamma$ and whose conclusion is A, which satisfies conditions 2., 3. and 4. of definition 5.1.

Proof. By induction on the number n of violations to the condition 4. in $\mathcal{D}$.

- **Basis:** If $n = 0$ trivial.
- **Induction:** Let us suppose that $n \geq 1$; then $\mathcal{D}$ has the following shape:

$$\dfrac{[\neg C]^u \qquad \begin{array}{c}[\neg C]^u\\ \mathcal{D}_1\\ C\end{array}}{\begin{array}{c}\dfrac{\perp}{\neg\neg C} \rightarrow I, u\\ \mathcal{D}_2\end{array}}$$

where $[\neg C]^u$ is an open assumption of $\mathcal{D}_1$. Then we will transform $\mathcal{D}$ into the following deduction:

$$\begin{array}{c}
\dfrac{\dfrac{[\neg C]^u \qquad \dfrac{\dfrac{[\neg C]^u \qquad \begin{array}{c}[\neg C]^u\\ \mathcal{D}_1\\ C\end{array}}{\bot}}{C}\ \bot_c, u}{\bot}}{\neg\neg C}\ \to I, u \\
\mathcal{D}_2
\end{array}$$

and we obtain a deduction $\mathcal{D}'$ whose conclusion is A and whose set of open assumptions is Γ; moreover $\mathcal{D}'$ is normal, satisfies condition 2. (because we join pieces of deductions together putting them on the minor premise of →E); it satisfies condition 3. (because with this transformation we are not introducing new occurrences of 6) and the number of violations to condition 4. in $\mathcal{D}'$ is $n-1$, therefore we can apply induction hypothesis to $\mathcal{D}'$, and suppose that $\mathcal{D}'$ has already been transformed into the deduction which we were looking for.

□

Lemma 8.5. *Let $\mathcal{D}$ be a normal deduction of* →**Nc** *whose set of open assumptions is Γ and whose conclusion is A; let us suppose that $\bot \notin \Gamma$ and that $\mathcal{D}$ satisfies conditions 2., 3. and 4. of definition 5.1.*

There exists a normal deduction $\mathcal{D}'$ whose set of open assumptions is $\Gamma' = \Gamma$ and whose conclusion is A, which satisfies conditions 2., 3.,4. and 5(a) of definition 5.1.

Proof. By induction on the number n of violations to the condition 5(a) in $\mathcal{D}$.

- **Basis:** If $n = 0$ trivial.
- **Induction:** Let us suppose that $n \geq 1$; then $\mathcal{D}$ has the following shape:

$$\begin{array}{c}
\dfrac{\dfrac{[\neg B]^u \qquad \begin{array}{c}\mathcal{D}_1\\ B\end{array}}{\bot}}{B}\ \bot_c, u \\
\mathcal{D}_2
\end{array}$$

where $[\neg B]^u$ is not an open assumption of $\mathcal{D}_1$.
We notice that $B \neq \bot$, because B is the conclusion of an application of a $\bot_c$ rule.
We will transform $\mathcal{D}$ into the following deduction:

$$\begin{array}{c}
\mathcal{D}_1\\
B\\
\mathcal{D}_2
\end{array}$$

and we obtain a deduction $\mathcal{D}'$ whose conclusion is A and whose set of open assumptions is Γ; moreover $\mathcal{D}'$ is normal,it satisfies condition 2. (because we join pieces of deductions together putting them on the minor premise of →E or on the premise of →I); it satisfies condition 3. (because, since $B \neq \bot$, with this transformation we are not introducing new occurrences of 6), it satisfies condition 4. (because, since $B \neq \bot$, with this transformation we are not introducing new occurrences of 7), and the number of violations to condition 5(a) in $\mathcal{D}'$ is

$n-1$, therefore we can apply induction hypothesis to $\mathcal{D}'$, and suppose that $\mathcal{D}'$ has already been transformed into the deduction which we were looking for.

□

Lemma 8.6. *Let $\mathcal{D}$ be a normal deduction of* $\rightarrow$**Nc** *whose set of open assumptions is Γ and whose conclusion is A; let us suppose that $\perp \notin \Gamma$ and that $\mathcal{D}$ satisfies conditions 2., 3., 4. and 5(a) of definition 5.1. There exists a procedure which transforms $\mathcal{D}$ into a normal deduction $\mathcal{D}'$ whose set of open assumptions is $\Gamma' = \Gamma$ and whose conclusion is A, which satisfies conditions 2., 3., 4., 5(a) and 5(b) of definition 5.1.*

Proof. By induction on the number n of violations to the condition 5(b) in $\mathcal{D}$.

- **Basis:** If $n = 0$ trivial.
- **Induction:** Let us suppose that $n \geq 1$; then $\mathcal{D}$ has the following shape:

$$\cfrac{\cfrac{[\neg B]^u \qquad \begin{array}{c}[\neg B]^u\\ \mathcal{D}_1\\ B\end{array}}{\perp}}{\begin{array}{c}B\\ \mathcal{D}_2\end{array}}\ \perp_c, u$$

where $[\neg B]^u$ is an open assumption of $\mathcal{D}_1$, and there is at least one occurrence of the assumption $[\neg B]^u$ as the minor premise of an application of $\rightarrow$E in $\mathcal{D}_1$ or as the premise of an application of $\rightarrow$I in $\mathcal{D}_1$.
We notice that $B \neq \perp$, because B is the conclusion of an application of a $\perp_c$ rule.

Then we will transform $\mathcal{D}$ substituting every occurrence of the assumption $[\neg B]^u$ as the minor premise of an application of $\rightarrow$E in $\mathcal{D}_1$ or as the premise of an application of $\rightarrow$I in $\mathcal{D}_1$ with the following deduction whose conclusion is $\neg B$:

$$\cfrac{\cfrac{[\neg B]^u \qquad [B]^v}{\perp}}{\neg B}\ \rightarrow I, v \tag{10}$$

and we obtain a deduction $\mathcal{D}'$ whose conclusion is A and whose set of open assumptions is Γ; moreover $\mathcal{D}'$ is normal, it satisfies condition 2. (because we join pieces of deductions together putting them on the minor premise of $\rightarrow$E or on the premise of $\rightarrow$I); it satisfies condition 3. (because with this transformation we are not introducing new occurrences of 6), it satisfies condition 4. (because with this transformation we are not introducing new occurrences of 7), it satisfies condition 5(a) (because with this transformation we are not introducing new occurrences of 8) and the number of violations to condition 5(b) in $\mathcal{D}'$ is $n-1$, therefore we can apply induction hypothesis to $\mathcal{D}'$, and suppose that $\mathcal{D}'$ has already been transformed into the deduction which we were looking for. □

Lemma 8.7. *Let $\mathcal{D}$ be a normal deduction of* $\rightarrow$**Nc** *whose set of open assumptions is Γ and whose conclusion is A; let us suppose that $\perp \notin \Gamma$ and that $\mathcal{D}$ satisfies conditions 2., 3., 4., 5(a) and 5(b) of definition 5.1.*

There exists a procedure which transforms $\mathcal{D}$ into a normal deduction $\mathcal{D}'$ whose set of open assumptions is $\Gamma' = \Gamma$ and whose conclusion is A, which satisfies conditions 2., 3., 4., 5(a), 5(b) and 5(c) of definition 5.1.

Proof. By induction on the number n of occurrences in $\mathcal{D}$ of a piece of deduction of the following kind:

$$\dfrac{\dfrac{[\neg B]^u \qquad B}{\perp}}{B}\, u$$

• **Basis:** If $n = 0$ trivial.

• **Induction:** Let us suppose that $n \geq 1$; then $\mathcal{D}$ has the following shape:

$$\begin{array}{c} \dfrac{\dfrac{[\neg B]^u \qquad \begin{array}{c}[\neg B]^u \\ \mathcal{D}_1 \\ B\end{array}}{\perp}}{B}\, \perp_c, u \\ \mathcal{D}_2 \end{array}$$

and, as $\mathcal{D}$ satisfies conditions 5(a) and 5(b), $[\neg B]^u$ is an open assumption of $\mathcal{D}_1$, and the assumption $[\neg B]^u$ always occurs in$\mathcal{D}_1$ as the major premise of an application of $E \rightarrow$ rule.

Let us prove that the last rule applied in $\mathcal{D}_1$ is $E \rightarrow$ by induction on the complexity of B.

• **Basis:** If B is atomic, the last rule applied in $\mathcal{D}_1$ is $\rightarrow$E : indeed,

- B cannot be the conclusion of an application of $I \rightarrow$, because B is atomic;
- B cannot be the conclusion of an application of $\perp_c$, because otherwise $[\neg B]^u$ would have been discharged in $\mathcal{D}_1$.

• **Induction:** Let us suppose that B is not atomic (so $B \neq \perp$) and that the last rule applied in $\mathcal{D}$ is not $\rightarrow$E ; then the last rule applied in $\mathcal{D}$ is $\rightarrow$I , because otherwise $[\neg B]^u$ would have been discharged in $\mathcal{D}_1$. Therefore, $B = A_1 \rightarrow B_1$ and $\mathcal{D}$ has the following shape:

$$\begin{array}{c} \dfrac{\dfrac{[\neg B]^u \qquad \dfrac{\begin{array}{c}[\neg B]^u \\ \mathcal{D}_4 \\ B_1\end{array}}{B} \rightarrow I, w}{\perp}}{B}\, \perp_c, u \\ \mathcal{D}_2 \end{array}$$

Two cases may occur:

- $B_1 \neq \perp$;
- $B_1 = \perp$.

If $B_1 \neq \bot$, then we will consider every occurrence of the assumption $[\neg B]^u$ in $\mathcal{D}_4$ as the major premise of an application of →E :

$$\dfrac{\dfrac{[\neg B]^u \qquad \begin{array}{c}\mathcal{D}''\\ B\end{array}}{\bot}}{X}\,R \tag{11}$$

and substitute $[\neg B]^u$ with the following deduction whose conclusion is $\neg B$:

$$\dfrac{\dfrac{[\neg B_1]^z \qquad \dfrac{[B]^v \qquad [A_1]^w}{B_1}}{\bot}}{\neg B} \rightarrow I, v \tag{12}$$

and we obtain the following deduction $\mathcal{D}_0$:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg B_1]^z \qquad \dfrac{[B]^w \qquad [A_1]^v}{B_1}}{\bot}}{\neg B} \rightarrow I, w \qquad \begin{array}{c}\mathcal{D}''\\ B\end{array}}{\bot}}{X}\,R$$

which in general is not even normal; we apply the normalization procedure to $\mathcal{D}_0$ and we obtain a deduction $\mathcal{D}_{0,N}$ whose conclusion is X and whose open assumptions are among the open assumptions of $\mathcal{D}_0$ (therefore $\neg B_1$ might be an open assumption of $\mathcal{D}_{0,N}$, but surely $\neg B$ is not an open assumption of $\mathcal{D}_{0,N}$.).

We apply – one by one – all the above lemmas to $\mathcal{D}_{0,N}$ and we obtain a normal deduction $\mathcal{D}'_0$ whose conclusion is X and whose open assumptions are among the open assumptions of $\mathcal{D}_{0,N}$; moreover $\mathcal{D}'_0$ satisfies conditions 2., 3., 4., 5(a) and 5(b).

We will transform $\mathcal{D}$ substituting every occurrence of a subdeduction of the kind 11 in $\mathcal{D}_4$ with the corresponding deduction $\mathcal{D}'_0$ obtained applying the procedure described here above, and we obtain the following deduction $\mathcal{D}*$:

$$\begin{array}{c}\dfrac{\dfrac{[\neg B]^u \qquad \dfrac{\begin{array}{c}[\neg B_1]^z\\ \mathcal{D}'_4\\ B_1\end{array}}{B} \rightarrow I, w}{\bot}}{B}\,\bot_c, u\\ \mathcal{D}_2\end{array}$$

which is normal, its conclusion is A, its open assumptions (with maybe the exception of $\neg B_1$) are among the open assumptions of $\mathcal{D}$ (so $\bot$ is not an open assumption of $\mathcal{D}*$), and satisfies conditions 2., 3. and 4.

Since $[\neg B]^u$ is not an open assumption of $\mathcal{D}'_4$, $\mathcal{D}*$ has a violation of condition 5(a); we apply lemma 8.5 to $\mathcal{D}*$ and we obtain the following deduction $\mathcal{D}'$:

$$\begin{array}{c}\dfrac{\begin{array}{c}\mathcal{D}'_4\\ B_1\end{array}}{B} \rightarrow I, v\\ \mathcal{D}_2\end{array}$$

Two cases may occur:

- $[\neg B_1]^z$ is not an open assumption of $\mathcal{D}'_4$;
- $[\neg B_1]^z$ is an open assumption of $\mathcal{D}'_4$.

If $[\neg B_1]^z$ is not an open assumption of $\mathcal{D}'_4$, then $\mathcal{D}'$ has the same conclusion as $\mathcal{D}'$, its open assumptions are among the open assumptions of $\mathcal{D}$, it satisfies conditions 2., 3., 4., 5(a) and 5(b) and the number of occurrences of 8 in $\mathcal{D}'$ is $n-1$. Therefore we can apply our induction hypothesis to $\mathcal{D}'$ and suppose that $\mathcal{D}'$ has already been transformed into the deduction we were looking for.

Let us suppose that $[\neg B_1]^z$ is an open assumption of $\mathcal{D}'_4$; then we will transform $\mathcal{D}'$ into the following deduction $\mathcal{D}'*$:

$$\begin{array}{c}\dfrac{\dfrac{\dfrac{[\neg B_1]^z \qquad \begin{array}{c}[\neg B_1]^z\\ \mathcal{D}'_4\\ B_1\end{array}}{\bot}}{B_1}\,\bot_c, z}{B}\,\to I, v\\ \mathcal{D}_2\end{array}$$

Since the complexity of B_1 is smaller than the complexity of B, we can apply our induction hypothesis to $\mathcal{D}*'$, and suppose that the last rule applied in $\mathcal{D}'_4$ is $\to$E .

If $B_1 = \bot$, then $\mathcal{D}$ has the following shape:

$$\begin{array}{c}\dfrac{\dfrac{[\neg\neg A_1]^u \qquad \dfrac{\begin{array}{c}[\neg\neg A_1]^u\\ \mathcal{D}_3\\ \bot\end{array}}{\neg A_1}\,\to I, v}{\bot}}{\neg A_1}\,\bot_c, u\\ \mathcal{D}_2\end{array}$$

where $[\neg\neg A_1]^u$ is an open assumption of $\mathcal{D}_3$ and the assumption $[\neg\neg A_1]^u$ always occurs in $\mathcal{D}_3$ as the major premise of an application of $\to$E .

Since $\bot \not\in \Gamma$, we have that $depth(\mathcal{D}_3) > 1$, therefore $\mathcal{D}$ has the following shape:

$$\begin{array}{c}\dfrac{\dfrac{[\neg\neg A_1]^u \qquad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_5\\ \neg Y\end{array} \quad \begin{array}{c}\mathcal{D}_6\\ Y\end{array}}{\bot}}{\neg A_1}\,\to I, v}{\bot}}{\neg A_1}\,\bot_c, u\\ \mathcal{D}_2\end{array}$$

and we have that $[\neg\neg A_1]^u$ is an open assumption of $\mathcal{D}_5$ or of $\mathcal{D}_6$ and the assumption $[\neg\neg A_1]^u$ always occurs in $\mathcal{D}_5$ and in $\mathcal{D}_6$ as the major premise of an application of $\to$E .

Then we will consider every occurrence of the assumption $[\neg\neg A_1]^u$ in $\mathcal{D}_5$ and in $\mathcal{D}_6$ as the major premise of an application of $\rightarrow$E :

$$\cfrac{\cfrac{[\neg\neg A_1]^u \qquad \begin{matrix}\mathcal{D}'' \\ \neg A_1\end{matrix}}{\bot}}{X}\ R \tag{13}$$

and substitute $[\neg\neg A_1]^u$ with the following deduction whose conclusion is $\neg\neg A_1$:

$$\cfrac{\cfrac{[\neg A_1]^w \qquad [A_1]^v}{\bot}}{\neg\neg A_1}\ I \rightarrow, w \tag{14}$$

and we obtain the following deduction $\mathcal{D}_{0,0}$:

$$\cfrac{\cfrac{\cfrac{\cfrac{[\neg A_1]^w \qquad [A_1]^v}{\bot}}{\neg\neg A_1} \rightarrow I, w \qquad \begin{matrix}\mathcal{D}'' \\ \neg A_1\end{matrix}}{\bot}}{X}\ R$$

which in general is not even normal; we apply the normalization procedure to $\mathcal{D}_{0,0}$ and we obtain a deduction $\mathcal{D}_{1,N}$ whose conclusion is X and whose open assumptions are among the open assumptions of $\mathcal{D}_{0,0}$ (therefore $\neg\neg A_1$ is not an open assumption of $\mathcal{D}_{1,N}$.).

We apply – one by one – all the above lemmas to $\mathcal{D}_{1,N}$ and we obtain a normal deduction $\mathcal{D}'_{0,0}$ whose conclusion is X and whose open assumptions are among the open assumptions of $\mathcal{D}_{1,N}$; moreover $\mathcal{D}'_{0,0}$ satisfies conditions 2., 3., 4., 5(a) and 5(b).

We will transform $\mathcal{D}$ substituting every occurrence of a subdeduction of the kind 13 in $\mathcal{D}_5$ and in $\mathcal{D}_6$ with the corresponding deduction $\mathcal{D}'_{0,0}$ obtained applying the procedure described here above, and we obtain the following deduction $\mathcal{D}**$:

$$\begin{matrix}\cfrac{\cfrac{[\neg\neg A_1]^u \qquad \cfrac{\cfrac{\begin{matrix}\mathcal{D}'_5 \\ \neg Y\end{matrix} \qquad \begin{matrix}\mathcal{D}'_6 \\ Y\end{matrix}}{\bot}}{\neg A_1} \rightarrow I, v}{\bot}}{\neg A_1}\ \bot_c, u \\ \mathcal{D}_2\end{matrix}$$

which is normal, its conclusion is A, its open assumptions are among the open assumptions of $\mathcal{D}$ (so $\bot$ is not an open assumption of $\mathcal{D}**$), and satisfies conditions 2., 3. and 4.

Since $[\neg\neg A_1]^u$ is neither an open assumption of $\mathcal{D}'_5$ nor an open assumption of $\mathcal{D}'_6$, $\mathcal{D}**$ has a violation of condition 5(a); we apply lemma 8.5 to $\mathcal{D}**$ and we obtain the following deduction $\mathcal{D}'$:

$$\cfrac{\cfrac{\begin{matrix}\mathcal{D}'_5 & \mathcal{D}'_6 \\ \neg Y & Y\end{matrix}}{\bot}}{\neg A_1} \to I, v$$
$$\mathcal{D}_2$$

$\mathcal{D}'$ has the same conclusion as $\mathcal{D}$, its open assumptions are among the open assumptions of $\mathcal{D}$, it satisfies conditions 2., 3., 4., 5(a) and 5(b) and the number of occurrences of 8 in $\mathcal{D}'$ is $n-1$. Therefore we can apply our induction hypothesis to $\mathcal{D}'$ and suppose that $\mathcal{D}'$ has already been transformed into the deduction we were looking for.

□

Proposition 8.8. *Let $\mathcal{D}$ be a normal deduction of* $\to$**Nc** *whose set of open assumptions is Γ and whose conclusion is A.*

There exists a procedure which transforms $\mathcal{D}$ into a deduction $\mathcal{D}'$ in $*$-normal form whose set of open assumptions is $\Gamma' \subseteq \Gamma$ and whose conclusion is A.

Proof. If $\bot \in \Gamma$ then we transform $\mathcal{D}$ into the following deduction:

$$\frac{\bot}{A} \bot_c, u$$

which is a deduction in $*$-normal form whose set of open assumptions is $\{\bot\} \subseteq \Gamma$.

Therefore we can assume that $\bot \notin \Gamma$ (i.e., the condition 2. is always satisfied).

We apply – one by one – all the lemmas here above to $\mathcal{D}$ and we obtain the deduction $\mathcal{D}'$ we were looking for. □

9 Conclusions and further developments

The main results in this thesis are:

- The definition of normality for deductions in a Gentzen system for the classical case.
- The proof of the normalization theorem for this definition.
- The definition of natural deductions in $*$-normal form.
- The construction of two direct correspondences

$$\mathbf{G_n} \longrightarrow \mathbf{N}^*$$
$$\mathbf{N}^* \longrightarrow \mathbf{G_n}$$

  between normal Gentzen deductions and natural deductions in $*$-normal form.

I recall that the conditions stated in the definitions of normal deductions for Gentzen systems and of natural deductions in $*$-normal form are sufficient conditions for the well-definedness of the correspondence from **G** to **N** and of the correspondence from **N** to **G**, respectively. I also sketched the proof of the

"$*$-normalization theorem", i.e. I gave a procedure for transforming a normal deduction of $\rightarrow$**Nc** into a unique deduction in $*$-normal form.

Further developments of this work will be a deeper investigation of the class of $*$-normal deductions, the characterization of the classes of natural deductions and of Gentzen deductions for which the composition of these two maps is the identity map, and the discussion about similar questions for some other proof systems such as the one described by Cellucci in [1] and the one-sided Gentzen systems **GS** (see [8], 3.5.2).

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