Cut-Elimination in the implicative fragment $\to$G3mi of an intuitionistic G3-Gentzen-System and its Computational Meaning

(Abstract)

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This thesis consists of two chapters, the first of which is centered around a very recent article [Vest99] by R. Vestergaard, whereas the second is concerned with the topic of strong cut-elimination in the Gentzen-systems $\to$**G3mi** and $\to\perp$**G3i**.

R. Vestergaard in [Vest99] presented a "computational anomaly" in a typed Gentzen-system (here called:) $\boldsymbol{\mathcal{G}}_v^+$, which is based on the formulation of the intuitionistic and minimal **G3**-systems **G3[mi]** without explicit weakening and contraction-rules in the book [TS96] by A.S. Troelstra and H. Schwichtenberg. $\boldsymbol{\mathcal{G}}_v^+$ is a typed version of the implicative fragment $\to$**G3mi** of these systems only; the antecedents of sequents in $\boldsymbol{\mathcal{G}}_v^+$ consist of multisets of annotated formulas. $\boldsymbol{\mathcal{G}}_v^+$ is moreover tailormade to the purposes of (1) allowing to view a typed derivation-term t^C in the succedent of the conclusion $\Gamma \Rightarrow t : C$ of a $\boldsymbol{\mathcal{G}}_v^+$-derivation $\mathcal{D}$ very directly as the "computational meaning" of $\mathcal{D}$ (which corresponds to the "image" $\mathcal{D}^*$ of $\mathcal{D}$ as a natural-deduction-derivation with the conclusion C from marked open assumption-classes occuring among the annotated formulas in Γ), (2) facilitating to carry through cut-elimination as a stepwise process of local transformations by the use of explicit rules of $\boldsymbol{\mathcal{G}}_v^+$ for contraction and inversion during the procedure (the effects of weakening during cut-elimination are taken to be trivial and ignored in [Vest99]), (3) making it furthermore possible, that cut-elimination for $\boldsymbol{\mathcal{G}}_v^+$ can still be done in quite the same way as for (here only the special case of the implicative fragment $\to$**G3mi** of) the systems **G3[mi]**, which is described implicitly in [TS96].

Vestergaard then essentially presents a sequence $\{\mathcal{D}_n\}_{n\in\mathbb{N}}$ of distinct derivations $\mathcal{D}_n$ in $\boldsymbol{\mathcal{G}}_v^+$+Cut, which by the cut-elimination procedure related to that for **G3[mi]** in [TS96] all reduce to the same derivation $\mathcal{D}'$, but where the "computational meanings" of $\mathcal{D}_n$ are mutually different. This is what in [Vest99] is called a "computational anomaly".

Here a variant-system $\boldsymbol{\mathcal{G}}^+$ of $\boldsymbol{\mathcal{G}}_v^+$ is considered, in which the effects of applications of a (multiple-) weakening-rule are also accounted for in the derivation-terms forming the succedents of sequents. The relation between $\boldsymbol{\mathcal{G}}_0^+$-derivations[1] $\mathcal{D}$ and their "computational

[1]The system $\boldsymbol{\mathcal{G}}_0^+$ is defined from $\boldsymbol{\mathcal{G}}^+$ by restricting the rules of the system to its "logical rules" R$\to$ and L$\to$ (hence by dropping the structural rules weakening and contraction as well as the additional rule inversion of $\boldsymbol{\mathcal{G}}^+$).

meanings" $\Phi(\mathcal{D})$ is made more visual by the definition and use of a map Φ (similar to one that was essentially first described in [Pra65] by D. Prawitz) from derivations in $(\boldsymbol{\mathcal{G}}_\mathbf{0}^+ + \mathrm{Cut})$ to derivations in $\rightarrow$**N[mi]***, a typed version of a natural-deduction calculus for the implicative fragment of intuitionistic or minimal logic. Vestergaard's example leads to basically the same "anomaly" also in the setting of the system $\boldsymbol{\mathcal{G}}^+$, but it does not look quite convincing there any more. In fact, if one step during cut-elimination for upwards-permutation of contraction is altered slightly to a more careful form, this "anomaly" disappears completely. Nevertheless, a modified, but different example of an "anomaly" is given here for the situation arising from the reformulation of this step. (A certain possibility of a still more refined version of this contraction permutation-step is not covered by the new example.)

Also a closer analysis of the obviously problematic cut-elimination-step in $\boldsymbol{\mathcal{G}}^+$, upwards-permutation of contraction, that needs an additional application of inversion, is given in the case of the underlying untyped system $\rightarrow$**G3mi**. This analysis leads to the interpretation, that while cut-elimination in a $\rightarrow$**G3mi**-derivation $\mathcal{D}$ (done in the way as given implicitly in [TS96]) does not correspond to normalization on the related natural-deduction derivation $\Phi(\mathcal{D})$, this procedure might nevertheless have a more special property of allowing to extract from a given derivation in $\boldsymbol{\mathcal{G}}^+$—in a certain specifiable sense—"just what is needed for its particular conclusion" ([Fef75]) in this system. It might therefore be a special feature of this system, that cut-elimination allows to do *more* simplifications than are possible by normalization on the natural-deduction images.

Furthermore a remark of Vestergaard is discussed, that his example of an "anomaly" would be avoided, if a variant Gentzen-system of $\boldsymbol{\mathcal{G}}^+$ with the antecedents of sequents consisting of *sets* instead of multisets of formulas were considered. This seems not directly possible. Instead a related typed system $\rightarrow\mathbf{G2'mi}^{e*}$ is presented here, which looks to be a good candidate of a typed $\rightarrow$**G3mi**-like system, in which "anomalies" like Vestergaard's are avoided.

In the second chapter of the thesis the topic of *strong cut-elimination* is investigated for the systems $\rightarrow$**G3mi** and $\rightarrow\perp$**G3i**. That is the question, whether cut-elimination steps from a fixed list L can be applied to a given derivation $\mathcal{D}$ containing Cut in one of these systems with respect to arbitrary cuts occuring in $\mathcal{D}$ (cuts that need not be topmost in $\mathcal{D}$), such that despite this freedome in the choice of an applicable cut-elimination step from L a cut-free derivation $\mathcal{D}'$ is always reached after the execution of finitely many such steps, when starting from $\mathcal{D}$.

Strong cut-elimination theorems for *LJ*-like systems sometimes follow from results about strong-normalization (like that in [Pra71] by D. Prawitz) with the help of the results of J. Zucker in [Zu74] and G. Pottinger in [Pott77] about the precise nature of the connection between cut-elimination in such sequent-calculi and normalization for the respective natural-deduction systems. But Vestergaard's result may be looked upon as a strong indication for the possibility, that the very smooth and direct connection described in [Zu74] and [Pott77] between normalization on the side of the natural-deduction calculi and cut-elimination for Gentzen-systems does not exist with respect to the **G3[mi]**-Gentzen-systems for perhaps substantial reasons. It is therefore at least not in an obvious manner possible to arrive at a strong cut-elimination result for **→G3mi** and **→⊥G3i** in the way mentioned before. Instead, ideas from a proof by A.G. Dragalin in [Drag79] for a Strong Form of a Cut-Elimination Theorem for *LJ and LK* (with respect to a list of exclusively such cut-elimination steps that have already been presented by G. Gentzen in [Gen35]) are used here to prove a strong cut-elimination theorem for **→G3mi** and **→⊥G3i**.

Additionally a generalization of this result with respect to a larger list of cut-elimination steps, where now limited permutations of the structural rules weakening and contraction and also of inversion and Cut over each other are permitted, is discussed and a proof for an in this respect more general version of the before shown strong cut-elimination theorem for **→G3mi** and **→⊥G3i** is sketched.

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