Duality for Lattices with Operators: a modal logic approach.

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October 5, 2000

Abstract

This thesis discusses a proposal for a duality theory for bounded lattices with operators, amalgamating work on distributive lattices and on lattices with operators.

Introduction

1.

This work belongs to the field of algebraic logic. Algebraic logic has its roots in the 19th century, starting with the work of Boole. Its main aim is to study logic from an algebraic perspective in order to translate problems in logic to (universal) algebraic questions. At the beginning of the twentieth century, Tarski introduced the connection between boolean algebras and classical propositional calculus.

Duality theory serves as a tool for translating questions from one perspective (logic) to another one (universal algebra): the existence of a duality between two categories means that the two categories are essentially the same; this makes it possible to use one category in order to understand the other one.

Duality theory between logic and universal algebra has its basis in Stone's duality for boolean algebras. In 1936, Stone proved his representation theorem, stating that every boolean algebra A can be embedded in a power set algebra A^σ (i.e. An algebra of the form $\langle \mathcal{P}(X), \cap, \cup, \_^c, \emptyset, X \rangle$ for some set X) that is called the canonical extension of A. His proof uses a duality result: a Boolean Space is a compact topological space satisfying some strong separation condition; Stone constructed a duality between boolean algebras and boolean spaces.

Stone's duality has been generalized in two senses: by adding operations to the boolean algebras and by weakening the basis of the algebras to the case without negation (distributive lattices) and later to the non-distributive case (lattices).

Concerning the generalization by adding operations, in 1952, Jónsson and Tarski ([JT]) introduced the notion of boolean algebra with operators: a boolean algebra with operators (BAO) is a boolean algebra A endowed with functions $f_i : A^{n_i} \to A$ that preserve joins in each coordinate. They extended Stone's Representation Theorem in order to construct the canonical extension of a Boolean Algebra with operators. Although a duality for BAO's is implicit in their work, it was Goldblatt, in 1989 ([G]) that described explicitly a duality for the category of boolean algebras with operators. In Goldblatt's duality, the additional operations in the boolean side are represented by relations in the boolean spaces, obtaining what it is called today descriptive general frames. The duality between the category of Boolean Algebras with Operators and the category of Descriptive General Frames makes explicit the connection between the theory of Boolean Algebras with Operators and Classical Modal Logic.

With respect to distributive lattices, Stone's duality has been generalized in 1970 by Priestley ([P]): she constructed a duality between the category of Distributive Lattices and the category of what it is known now as "Priestley's Spaces". A Priestley Space is a partially ordered set endowed with a compact topology and that verifies a weaker (but still) Hausdorff condition.

We will discuss the non distributive case later.

Concerning the generalization of Priestley's duality to the case with additional operations, Goldblatt introduced (also in [G]), a duality between the category $\mathcal{DLO}$ of Distributive Lattices with Operators and the category of what he calls "Relational Priestley Spaces". His duality is a generalization of Priestley's duality to the case with operators.

A modal logic approach to the theory of Distributive Lattices with Operators has been proposed by Celani and Jansana in their works on Positive Modal Algebras (cf. [CJ]).
In all cases, the dual structures associated with the algebras with operators involve a set endowed with relations and with a topology.
Two concepts that arise naturally in the study of such dualities are that of the canonical extension of an algebra, and that of canonical equations or inequalities.
The notion of canonical extension has been defined with purely abstract algebraic means, first by Jónsson and Tarski ([JT]) for boolean algebras and later by Gehrke and Harding for the general case of lattices ([GH]).
A natural question is whether equations or inequalities holding in A will hold in A^σ (equations or inequalities that are preserved under taking canonical extensions are called canonical). Two overlapping results concerning this question are well known: in 1952, Jónsson and Tarski ([JT]), proved, from an algebraic perspective, canonicity of what they called quasi-equations; in 1975, Sahlqvist ([S]) proved , from a pure modal logic perspective, the theorem known later as "Sahlqvist Theorem" that states a completeness and first-order correspondence result for Sahlqvist equations. In 1995, de Rijke and Venema [dRV] showed how Sahlqvist Theorem can be applied to boolean algebras with operators in order to obtain a canonicity result. In 1994, Jónsson ([J]) gave his algebraic proof.
The lack of negation in the distributive case leads to consider inequalities instead of equations. In this direction, canonicity of Sahlqvist inequalities has been proved also from the two approaches: in 1994, by Gehrke and Jónsson ([GJ1]); in 1997, Celani and Jansana ([CJ]) gave a proof within the framework of a positive modal logic, using also a first-order correspondence result.

Dual structures of BA's and DL's.
Let us concentrate on the case without operators.
The dual structure of a Boolean Algebra is the set of its maximal filters endowed with a particular compact topology given by a subbase of clopen subsets (open and closed) D. The natural generalization of the notion of a maximal filter to distributive lattices is the notion of prime filters; the prime filters of a distributive lattice L constitute the universe X of the dual Priestley Space associated with it: a Priestley Space is a set endowed with a partial order and a particular compact topology given by a subbase of clopen subsets $D \cup D^{co}$, where D is a set of increasing subsets of X and D^{co} is the set of the complements of the elements in D. In the particular case that the distributive lattice is boolean, we find that the partial order is trivial and that $D^{co} = D$. (This means that the duality for distributive lattices generalizes the boolean duality).
The dualities for BA and DL can be illustrated in the following scheme:

BA	maximal filters + topology
DL	prime filters + topology + partial order

The extension of this dualities to the case with operators is very natural and adds to the dual structures a $n+1$-ary decreasing relation for each n-ary join preserving operation. In the distributive case, lack of negation leads to consider also meet-preserving operations that will become, in the dual structure, $n+1$-ary increasing relations (cf. definition 3.1.1 of "increasing" and "decreasing" relations). These ideas form the basis Goldblatt's approach in [G].

Duality for arbitrary lattices.
We give now a report of what has been done for the general case, i.e. the case of lattices in which $\wedge$ and $\vee$ are only related by the fact that they induce the same order. Several dualities have been presented for lattices perse, i.e. without operators. In 1978, Urquhart ([U]) proved a Representation Theorem for lattices. Later, Hartung ([H]) in 1992, Hartonas and Dunn ([DH]) in 1997 gave alternatives representations. All these representations use the same idea: they look at a lattice $\langle L, \wedge, \vee, 0, 1\rangle$ as consisting of two semilattices $\langle L, \wedge, 0\rangle$ and $\langle L, \vee, 1\rangle$, linked by the fact that $\wedge$ and $\vee$ induce the same partial order on L, namely $a \leq b$ if and only if $a \wedge b = a$ if and only if $a \vee b = b$.
The approach of [U] has the advantage that it generalizes the distributive case; the approach of [H] and [DH] do not, but their representation reflects much more transparently, in our opinion, this way of looking at a lattice. They represent the "meet"-semilattice by a general frame whose universe consist of filters of L, the "join"-structure by a general frame whose universe consist of ideals of L; these frames are then linked by a Galois connection between the power set of this two universes.
In 1997, Hartonas ([Ha]) presented a duality for lattices with operators. This duality is inspired by those of Goldblatt and Urquhart but it does not generalize the distributive case.
Concerning the morphisms, $[U]$ could extend its representation only to surjective morphisms of lattices, while the duality in [DH] can be extended to all morphisms.

Canonicity for the general case.
With respect to correspondence and canonicity results, some positive answers have been given by Gehrke and Harding in [GH]: if $\mathcal{L} = \langle L, \wedge, \vee, 0, 1, (f_i)_{i\in I}$ is a lattice with only join-preserving operations $f_i : L^{n_i} \to L$, they defined its canonical extension L^σ and proved that every equation is canonical. They did not use any duality result in their proof. Their result is restrictive in the sense that they do not allow operators like $\Box$.

2.

When we started this work, we were pursuing basically three aims that we enumerate here in the order we have treated them:

1. To propose a duality for lattices that would generalize Goldblatt's duality and, at the same time, keep the transparency of the Galois connection used in [DH] and result "handly" in order to treat modal logic problems.

2. To extend this duality to lattices with operators and to morphisms of lattices with operators; that is to have a functor from the category of lattices with operators to some dual category of relational structures.

3. To use ideas from modal logic in order to prove canonicity results.

We will give now a report of our achievements concerning these 3 aims.

1. Duality for lattices.
Concerning our first aim, we succeeded in obtaining a duality for the objects of the category of lattices that generalizes Priestley's Representation Theorem and at the same time, keeps the transparency of the Galois connection used in [DH]. To this aim, we generalized the notion of a prime filter to that of an "optimal" filter: the optimal filters of a distributive lattice happen to be exactly its prime filters. Following the perspective taken in [DH], we look at a lattice as consisting of two semi-lattices linked by the fact that they induce the same partial order.
To be more precise, the lattice $\langle L, \wedge, \vee, 0, 1\rangle$ consist of two semilattices $\langle L, \wedge, 0\rangle$ and $\langle L, \vee, 1\rangle$, linked by the facts that they have the same universe and that $a \wedge b = a$ if and only if $a \vee b = b$.
We represent the meet semi-lattice L_1 by a partial order on the set of all optimal filters of L_1 and the join semi-lattice L_2 by the set of all optimal ideals of L_2 (the notion of an optimal ideal is defined dually to that of an optimal ideal).
We have then the following diagram:

$$\begin{tikzcd} \langle L,\wedge,0\rangle \arrow[r, rightsquigarrow] \arrow[d, no head, squiggly, "\leq"] & \text{optimal filters + partial order} \arrow[d, no head, squiggly, "\perp"] \\ \langle L,\vee,1\rangle \arrow[r, rightsquigarrow] & \text{optimal ideals + partial order} \end{tikzcd}$$

If $F_*(L)$ and $I_*(L)$ are respectively the set of optimal filters and optimal ideals of L, the link in the dual structure representing the link given by $\leq$ in the lattice, will be given by a relation $\perp \subseteq F_*(L) \times I_*(L)$. This relation will induce a Galois connection between $\mathcal{P}(F_*(L)$ and $\mathcal{P}(I_*(L))$. The role played by increasing subsets in the distributive case will be played here by the stable subsets of $F_*(L)$ and the stable subsets of $I_*(L)$ (this terminology will be defined in chapter 1).
Of course, we need to impose on our two-sorted structures some basic conditions in order to guarantee that they arise from a lattice. Some of these conditions are not relevant to make precise here, but there is one in particular that causes some problems: as in the distributive case, it is still necessary to introduce a topological notion in order to give a more precise description of the dual structure. Each of the two structures associated with a lattice has a distinguished set D of stable subsets (we have a set D_1 of stable subsets of optimal filters and a set D_2 of stable subsets of optimal ideals, but we refer to them as a set D without making precise in which of the structures we are working).
Because we wanted to use these dual structures to obtain canonicity results, we had to generalize the topological condition imposed to Priestley's Spaces to a topological condition C in such a way that:

1. the dual structure of a lattice has to verify C;

2. every two-sorted structure (meeting some additional constraints that we do not mention here) that verifies C has to arise from a lattice; that is, it has to be (up to isomorphisms) the dual structure associated with a certain lattice;

3. it is possible to prove canonicity results in a similar way as for the boolean and distributive case (cf. [BdRV] and [CJ] respectively).

Note that, while 1 asks condition C to be general enough to make every dual structure associated with a lattice verify C, 2 and 3 push exactly in the opposite direction: they require to the topological condition C to be strong enough in order to have that every two-sorted structure of our type verifying condition C comes from a lattice (2) and that the modal version of canonicity (namely d-persistence, cf. [BdRV]) holds (3).
This was the first big problem we faced: we succeeded in finding a topological condition that is weak enough to verify 1 and strong enough to verify 2, but unfortunately our definition seems to be too weak to use it for the third aim. Because of time constraints, we decided to use this condition, hoping that further research will find the way of using our duality for proving canonicity results.
We also wanted to generalize the representation proposed for lattices to morphisms of lattices. The natural way to extend this duality (between objects) to a functor (in order to have also a representation of morphisms of lattices) gave rise to a second problem: the natural way to construct a morphism in the dual category from a morphism of lattices did not work for arbitrary morphisms of lattices, but only for the surjective ones. We will be more specific about this while presenting our achievements concerning our second aim.

2. Extending the duality to the case with operators.
Concerning the extension of the functor between lattices and two-sorted structures to the case with operators, we did not encounter any problem, succeeding in our second aim. The absence of negation led us to consider both join and meet-preserving (in each coordinate) operations, instead of only join-preserving ones; a lattice endowed with this kind of operation is what we call a lattice with operators. In fact, we will work only with a basic similarity type, in which the lattices are endowed with 2 unary operations: one meet-preserving and the other one join-preserving.
In our work, we describe directly the partial duality for lattices with operators, without treating first the lattice case. The dual category for lattices with operators has as objects what we call "descriptive general lattice frames" and is denoted by $\mathcal{DGLF}$.
The only morphisms that can be represented under our functor are the surjective ones; that is, the epimorphisms of lattices.
To be more precise, we obtained a contravariant functor:

$$H = (H_O, H_M) : \mathcal{LWO}_e \to \mathcal{DGLF}_m,$$

that will be proved to be a "partial duality" in some sense that we will make explicit in chapter 2 (section 2.4). Here $\mathcal{LWO}_e$ is the category whose objects are the lattices with operators and whose morphisms are the surjective morphisms of lattices with operators and $\mathcal{DGLF}_m$ is the category of descriptive general lattice frames restricted to monomorphisms (cf. definitions 2.1.5 and 2.1.11).
We refer to this functor as a "partial" duality.
Our duality becomes a total functor when it is applied to distributive lattices (it can be defined for any morphism of distributive lattice), that is essentially Goldblatt's duality for distributive lattice with operators.
From this representation of lattices with operators follows in a very natural way how to construct the canonical extension of a lattice with operators. Our construction is the same as the one Gehrke and Harding used in [GH].

3. Applying the duality to Modal Logic.
Finally, concerning the application of this partial duality to non distributive modal logic, the difficulty comes from the following: a modal approach to correspondence and canonicity that understands lattices with operators in terms of descriptive general lattice frames has to be formulated syntactically but based on semantic ideas. When working on lattice-based algebras, one has the same syntax as for boolean algebras but given our duality, the semantics work out in a much more complicated way, because they involve the Galois connection.
This will be specified in the conclusions of our thesis. The thesis itself does not contain any result about this subject. We do believe that a correspondence result can be proved, but we did not have the time to work properly on it. With respect to canonicity, the problem mentioned before (i.e. concerning the strong topological condition imposed to the general lattice frames) appears as hard to deal with. We will also be more precise about this in the conclusions.

Organization of the thesis.
This thesis is divided in 3 chapters:

- Chapter 1 is about the concepts and results the reader is suppose to know in order to understand the other chapters.
- In Chapter 2, we present the obtained duality for lattices with operators and we construct the canonical extension of such a lattice. We prove some nice properties of the canonical extension that can be useful when treating modal logic problems.
- In Chapter 3, we restrict the results of Chapter 2 to the particular case of distributive lattices, showing that our duality is a generalization of the duality presented by Goldblatt in [G].

We list here the original results that appear as a consequence of our work:

1. the generalization of the notion of a prime filter to that of an optimal filter and the theory of optimal filters;
2. the generalization of the notion of atomicity for boolean algebras to arbitrary lattices;
3. the construction of a partial duality that generalizes Goldblatt's duality.

Chapter 1

Preliminaries

This thesis presupposes a basic knowledge in lattice theory. The reader can refer basically to [DP], where he will find the necessary background concerning Stone and Priestley dualities. In the Algebra Chapter (chapter 5) of [BdRV], a duality result for boolean algebras with operators is proved and can be very helpful in order to understand our work.

1.1 Notation and conventions.

Let $f : X \to Y$ be a function.
If $S \subseteq X$, $f[S]$ denotes the subset $\{f(s) \mid s \in S\}$ of Y.
If $S \subseteq Y$, $f^{-1}(S)$ denotes the subset $\{x \in X \mid f(x) \in S\}$ of X.

Let X be a set. The power set of X is denoted by $\mathcal{P}(X)$. If $S \subseteq \mathcal{P}(X)$, then $S^{co}_X = \{X \setminus A \mid A \in S\}$. In the case that X is clear from the context, we write S^{co}.

We use L, L' for lattices, $a, b, c \cdots$ for elements of a lattice, X, X' for the underlying sets of the lattice frames, $x, y, \cdots x', y', \cdots$ for its elements, F, G, H for filters, I, J, K for ideals, R, Q for relations, $\perp, \perp'$ for relations inducing Galois correspondences.

1.2 Background knowledge.

We give in this section the definitions and results necessary to understand this work and that are supposed to be known by the reader; proofs of the results in this section, if not given here, can be found in [DP].
None of the results in this section are new.

1.2.1 Lattices

Definition 1.2.1 *A* **lattice** *is a structure $\langle L, \wedge, \vee, 0, 1\rangle$ such that:*
L is a set, $0, 1 \in L$ and $\wedge, \vee : L \times L \to L$ are associative and commutative

operations verifying:

- *for every* $a, b \in L$: $a \wedge (a \vee b) = a$ *and* $a \vee (a \wedge b) = a$,
- *for every* $a \in L$: $a \vee a = a$ *and* $a \wedge a = a$,
- *for every* $a \in L$: $0 \vee a = a$ *and* $1 \wedge a = a$.

The operations $\wedge, \vee$ are called **meet** and **join** respectively.

One can find in the literature that a lattice need not have 0 and 1 and that what we call a lattice appears as "bounded lattice"; as we will work only with bounded lattices, we preferred to use the name "lattice" for them.
When there is no confusion possible, we will use L (the name of the underlying set) to denote the lattice.

The **dual** of a statement about lattices phrased in $0, 1, \wedge, \vee$ is obtained by interchanging 0 and 1 on one hand, $\wedge$ and $\vee$ on the other. We will talk very frequently about "dual result", "dual statement" and "dual proof".

Remark 1.2.2 *Every lattice induces a partial order* $\leq$ *in its underlying set, given by:*

$$a \leq b \text{ iff } a \wedge b = a$$

or equivalently,

$$a \leq b \text{ iff } a \vee b = b.$$

Definition 1.2.3 *A lattice is said to be* **complete** *if and only if any family* $\{a_i \mid i \in I\} \subseteq L$ *has a* $\leq$*-supremum (denoted by* $\bigvee_{i \in I} a_i$*) and a* $\leq$*-infimum (denoted by* $\bigwedge_{i \in I} a_i$*).*

Definition 1.2.4 *Given two lattices* $\langle L, \wedge, \vee, 0, 1\rangle$ *and* $\langle L', \wedge', \vee', 0', 1'\rangle$*, a function* $f : L \to L'$ *is a* **morphism of lattices** *if and only if:*

- $f(a \wedge b) = f(a) \wedge' f(b)$,
- $f(a \vee b) = f(a) \vee' f(b)$,
- $f(0) = 0'$,
- $f(1) = 1'$.

Definition 1.2.5 *Given two complete lattices* $\langle L, \wedge, \vee, 0, 1\rangle$ *and* $\langle L', \wedge', \vee', 0', 1'\rangle$*, a function* $f : L \to L'$ *is a* **morphism of complete lattices** *if and only if:*

- f *is a morphism of lattices,*
- $f(\bigwedge_{i \in I} a_i) = \bigwedge'_{i \in I} f(a_i)$,
- $f(\bigvee_{i \in I} a_i) = \bigvee'_{i \in I} f(a_i)$.

Definition 1.2.6 *Given a lattice* L *and a set* $F \subseteq L$*, we say that* F *is a* **filter** *of L if and only if:*

- *if* $a, b \in F$*, then* $a \wedge b \in F$,

- *if $a \in F$ and $b \in L$ are such that $a \leq b$, then $b \in F$.*

Dually, a set $I \subseteq L$ is called an **ideal** *of L if and only if:*

- *if $a, b \in I$, then $a \vee b \in I$,*
- *if $a \in L$ and $b \in I$ are such that $a \leq b$, then $a \in I$.*

Note that for every non-empty filter F: $1 \in F$ and, if $0 \in F$, then $F = L$. Dually, every non-empty ideal I is such that: $0 \in I$ and, if $1 \in I$, then $I = L$. Note as well that $\{1\}$ is a filter and $\{0\}$ is an ideal.

In the rest of the chapter, we will give definitions and results concerning filters, but everything can be done dually for ideals.

Definition 1.2.7 *A non-empty filter F of L is said to be:*

- **proper** *iff $F \neq L$, or equivalently iff $0 \notin F$;*
- **maximal** *iff F is proper and, if $F \subseteq G$ and G is a proper filter, then $F = G$;*
- **prime** *iff F is proper and for every $a, b \in L$, if $a \vee b \in F$ then $a \in F$ or $b \in F$.*

Definition 1.2.8 *Let L be a lattice and $E \subseteq L$. The* **filter of L generated by** *E is the minimal filter of L containing E as a subset. The* **ideal of L generated by** *E is the minimal ideal of L containing E as a subset.*

Proposition 1.2.9

- *The filter generated by E is the set*

$$\{x \in L / \exists a_1, a_2, \cdots, a_n \in E \text{ such that } a_1 \wedge a_2 \wedge \cdots \wedge a_n \leq x\}.$$

- *The filter generated by E is proper if and only if $E \neq \emptyset$ and for every $a_1, a_2, \cdots, a_n \in E : \bigwedge_{i=1}^{n} a_i \neq 0$.*
- *For the particular case in which $E = \{x\}$, $x\!\uparrow$ the filter generated by E and we have $x\!\uparrow = \{y \in L \mid x \leq y\}$.*

Definition 1.2.10 *Given two partially ordered sets $\langle X, \leq \rangle$ and $\langle X', \leq' \rangle$, a function $f : X \to X'$ is said to be*

- **order-preserving** *if and only if for any $a, b \in X$ such that $a \leq b$, we have $f(a) \leq' f(b)$,*
- **order-reversing** *if and only if for any $a, b \in X$ such that $a \leq b$, we have $f(b) \leq' f(a)$.*

Definition 1.2.11 *Let $\langle L, \wedge, \vee, 0, 1 \rangle$ and $\langle L', \wedge', \vee', 0', 1' \rangle$ be lattices; a function $f : L \to L'$ is said to be*

- **meet-preserving** *if and only if for any $a, b \in L$: $f(a \wedge b) = f(a) \wedge' f(b)$,*
- **join-preserving** *if and only if for any $a, b \in L$: $f(a \vee b) = f(a) \vee' f(b)$,*

- **completely meet-preserving** *if and only if for any* $(a_i)_{i\in I} \subseteq L$, *if* $\bigwedge_{i\in I} a_i$ *exists in* L, *then* $f(\bigwedge_{i\in I} a_i)$ *exists in* L' *and coincides with* $\bigwedge'_{i\in I} f(a_i)$.

- **completely join-preserving** *if and only if for any* $(a_i)_{i\in I} \subseteq L$, *if* $\bigvee_{i\in I} a_i$ *exists in* L, *then* $f(\bigvee_{i\in I} a_i)$ *exists in* L' *and coincides with* $\bigvee'_{i\in I} f(a_i)$.

Lemma 1.2.12

1. *If* $f : L \to L'$ *is meet-preserving, then it is order preserving.*
2. *If* $f : L \to L'$ *is join-preserving, then it is order preserving.*
3. *If* $f : L \to L'$ *is meet-preserving and* F *is a filter of* L', *then* $f^{-1}(F)$ *is a filter of* L.
4. *If* $f : L \to L'$ *is join-preserving and* I *is an ideal of* L', *then* $f^{-1}(I)$ *is an ideal of* L.

Proof: We give here only the proof of 1 and 3, the proofs of 2 and 4 being respectively dual.
1. Take $a, b \in L$ such that $a \leq b$. Then $a \wedge b = a$ and therefore $f(a \wedge b) = f(a)$. We use here that f is meet-preserving and we obtain: $f(a) \wedge' f(b) = f(a)$; that is, $f(a) \leq' f(b)$ and we are done.
3. Take $a, b \in f^{-1}(F)$; then $f(a \wedge b) = f(a) \wedge' f(b)$. As $f(a), f(b) \in F$ and F is a filter, we deduce $f(a) \wedge' f(b) \in F$, hence $f(a \wedge b) \in F$. We conclude that $a \wedge b \in f^{-1}(F)$.
Suppose now that $a \in f^{-1}(F)$ and that $b \in L$ is such that $a \leq b$. Then, by (a), we have $f(a) \leq' f(b)$. As $f(a) \in F$ and F is a filter, we deduce that $f(b) \in F$, hence $b \in f^{-1}(F)$ and we are done. □

Some particular meet-preserving and join-preserving functions:
In modal logic, when we want to interpret the symbols $\diamond$ and $\Box$ in a model, we do it by means of two type of functions, that depend on a binary relation R of the model. We present here this 2 functions:
Given a relation $R \subseteq X \times Y$, we have 2 induced functions:

- $m_R : \mathcal{P}(Y) \to \mathcal{P}(X)$, given by:
  if $S \subseteq Y$, then $m_R(S) = \{x \in X \mid \exists y \in S \text{ such that } Rxy\}$,
- $m_R^\delta : \mathcal{P}(Y) \to \mathcal{P}(X)$ given by:
  if $S \subseteq Y$, then $m_R^\delta(S) = \{x \in X \mid \text{ if } Rxy, \text{ then } y \in S\}$.

It is easy to check the following

Lemma 1.2.13 *If we impose the usual lattice structure to* $\mathcal{P}(X)$ *and* $\mathcal{P}(Y)$, *we get that:*

- m_R *is join-preserving,*
- m_R^δ *is meet-preserving.*

1.2.2 Boolean Algebras

Definition 1.2.14 *A* **boolean algebra** *is a structure* $\langle A, \wedge, \vee, \neg, 0, 1\rangle$ *such that:*

1. $\langle A, \wedge, \vee, 0, 1\rangle$ *is a distributive lattice (cf definition 1.2.21),*
2. $\neg : A \to A$ *is such that for any* $a \in A$*:*
   - $a \wedge \neg a = 0$,
   - $a \vee \neg a = 1$.

Maximal filters play a fundamental role in the theory of boolean algebras. We will state here some essential facts, the proofs of which can be found in [DP].

Theorem 1.2.15 *(Ultrafilter Theorem.)*
Let $\langle A, \wedge, \vee, \neg, 0, 1\rangle$ *be a boolean algebra. Given a proper filter* F *and an ideal* I *of* A *such that* $F \cap I = \emptyset$ *there is a maximal filter* U *of* L *such that* $F \subseteq U$ *and* $U \cap L = \emptyset$.

Definition 1.2.16 *Let* $\langle A, \wedge, \vee, \neg, 0, 1\rangle$ *be a boolean algebra. An element* $a \in A$ *is called an* **atom** *of* A *if and only if*

- $0 < a$,
- *if* $0 < x \leq a$*, then* $x = a$.

Theorem 1.2.17 *Let* $\langle A, \wedge, \vee, \neg, 0, 1\rangle$ *be a boolean algebra and let* $a \in A$*. Then:* a *is an atom if and only if the filter* $a\uparrow$ *is maximal.*

Definition 1.2.18 *A boolean algebra is said to be* **atomic** *if and only if for any* $x \in A$ *such that* $x \neq 0$*, there is an atom* $a \in A$ *such that* $a \leq x$.

Proposition 1.2.19 *A complete boolean algebra* A *is atomic if and only if for every* $x \in A$*:* $x = \bigvee\{a \in A \mid a \leq x \text{ and } a \text{ is an atom of } A\}$.

Given a set X, the structure $\langle \mathcal{P}(X), \cap, \cup, \_^c, \emptyset, X\rangle$ is a boolean algebra called the **power set algebra of X** and has two basic nice properties: it is complete and it is atomic.

The following theorem guarantees that every boolean algebra can be embedded in a power set algebra.

Theorem 1.2.20 *(Stone's Representation Theorem)*
Given a boolean algebra $\langle A, \wedge, \vee, \neg, 0, 1\rangle$*, let* $Uf A$ *be the set of maximal filters of* A*. Then the function:*

$$\nu : A \to \mathcal{P}(Uf\ A)$$

given by $\nu(a) = \hat{a} = \{U \in Uf A \mid a \in U\}$*, is an injective morphism of lattices.*

1.2.3 Distributive lattices

Definition 1.2.21 *A lattice $\langle L, \wedge, \vee, 0, 1\rangle$ is* **distributive** *if and only if:*

$$\forall a, b, c \in L : a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Proposition 1.2.22 *A lattice L is distributive if and only if the following dual property holds:*

$$\forall a, b, c \in L : a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

In order to state the following theorem we need one more definition:

Definition 1.2.23 *We say that a lattice L has the* **Prime Filter Property** *(denoted* **PFP***) if and only if: given a proper filter F and an ideal I of L such that $F \cap I = \emptyset$ there is a prime filter P of L such that $F \subseteq P$ and $P \cap L = \emptyset$.*

The following theorem is known as the Prime Filter Theorem and is crucial for our work; we will consequently give the proof, but it can be found in [DP] under the name DPI.

Theorem 1.2.24 *A lattice L has the PFP if and only if L is distributive.*

Proof: Suppose first that the lattice L has the PFP; we will prove that L is distributive. Suppose not: then for some $a, b, c \in L$, we have:

$$a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c).$$

It is easy to check that in every lattice the left side is smaller or equal than the right side, so we have:

$$(a \vee b) \wedge (a \vee c) \nleq a \vee (b \wedge c).$$

Let $x = (a \vee b) \wedge (a \vee c)$ and $y = a \vee (b \wedge c)$ and let $F = x\uparrow$ and $I = y\downarrow$. As $x \nleq y$, we have $x \neq 0$ and therefore F is proper; also because $x \nleq y$, we have that $F \cap I = \emptyset$ and therefore there is a prime filter P such that $F \subseteq P$ and $P \cap I = \emptyset$. But $(a \vee b) \wedge (a \vee c) = x \in F \subseteq P$. As P is a filter, we have $a \vee b \in P$ and $a \vee c \in P$.
One of the following holds:
1) $a \notin P$. As $a \vee b \in P$ and P is prime, we have $b \in P$. Similarly, we have $c \in P$ and therefore $b \wedge c \in P$. But $b \wedge c \leq a \vee (b \wedge c) = y$, so $y \in P$.
2) $a \in P$. As $a \leq a \vee (b \wedge c) = y$, we conclude again $y \in P$.
In both cases we arrive at a contradiction, since $y \in I$ and $P \cap I = \emptyset$. We conclude that L is distributive.

The converse direction is an application of Zorn's Lemma. Take a proper filter F and an ideal I of L such that $F \cap I = \emptyset$. Consider the family

$$\mathcal{F} = \{G \text{ proper filter of } L \mid F \subseteq G, G \cap I = \emptyset\},$$

ordered by inclusion.
We have that $F \in \mathcal{F}$ and therefore $\mathcal{F}$ is non-empty. The union of a chain of filters is a filter. Moreover, if each filter of the chain contains F and is disjoint from I , then the union contains F and is disjoint from I.

The family $\mathcal{F}$ verifies then the hypothesis of Zorn's Lemma (namely that it is non-empty and its chains have an upper-bound in the family). $\mathcal{F}$ has therefore a maximal element P. We will see that the filter P is prime.
Suppose not. Then for some $a, b \in L$, we have $a \vee b \in P$, $a \notin P$ and $b \notin P$. Consider the filter P_a generated by $P \cup \{a\}$. As $P \subsetneq P_a$ (since $a \notin P$) and $F \subseteq P_a$, by maximality of P in $\mathcal{F}$, we have that $P_a \cap I \neq \emptyset$. That is, there is $x_a \in P$ such that $x_a \wedge a \in I$. Similarly, as $b \notin P$, we have that there is $x_b \in P$ such that $x_b \wedge b \in I$.
Therefore, we have:

$$(x_a \wedge a) \vee (x_b \wedge b) \in I$$

and using that L is distributive, we get:

$$x = (x_a \vee x_b) \wedge (x_a \vee b) \wedge (x_b \vee a) \wedge (a \vee b) \in I.$$

But, every filter being increasing, the inequalities $x_a \leq x_a \vee x_b$, $x_a \leq x_a \vee b$ and $x_b \leq x_b \vee a$ and the fact that $x_a, x_b \in P$ let us conclude that the elements $x_a \vee x_b$, $x_a \vee b$ and $x_b \vee a$ are in P. On the other hand, $a \vee b \in P$, by assumption. We get then $x \in P$, contradicting the fact that $P \cap I = \emptyset$.
We have proved then that P is prime. □

The following definition generalizes the notion of atom to the non-boolean case. What we call here **join-prime** appears in [DP] as **join-irreducible**. We prefer in this case to follow the terminology used in [GJ1].

Definition 1.2.25 *Given a lattice L, we say that $a \in L$ is* **join prime** *(***completely join prime***) if and only if for any $b, c \in L$ (resp. for any $b_i \in L, i \in I$), if $a \leq b \vee c$ (resp. $a \leq \bigvee_{i \in I} b_i$), then $a \leq b$ or $a \leq c$ (resp. $a \leq b_i$ for some $i \in I$).*

Note that the completely join-prime elements of a complete boolean algebra are exactly its atoms.

By definition of join prime element and prime filter, we have the following

Proposition 1.2.26 *Let L be a distributive lattice and let $a \in L$. Then: a is join prime if and only if the filter $a\uparrow$ is prime.*

Of course, we can define dually the notion of **meet-prime** and **completely meet-prime** element of L and prove a dual result.

We will need later one more result concerning prime filters and ideals.

Lemma 1.2.27 *Let L be a lattice.*

- *If F is a prime filter of L, then F^c is a prime ideal of L.*
- *If I is a prime ideal of L, then I^c is a prime filter of L.*

Proof: We prove only the first item, the second one being dual.
We prove first that F^c is an ideal of L. Let $a, b \in F^c$. We have to prove that $a \vee b \in F^c$. Suppose not; then, $a \vee b \in F$ and therefore, F being prime, we have $a \in F$ or $b \in F$, contradicting the assumption.
Suppose now that $a \in F^c$ and that $b \leq a$. We have to prove that $b \in F^c$. If

not, then $b \in F$ and, as every filter is increasing, we have $a \in F$, contradicting the assumption.
We have proved that F^c is an ideal. We proceed now to prove that it is prime. First note that $F^c \neq L$, since $F \neq \emptyset$. Suppose now that $a \wedge b \in F^c$. We have to prove that $a \in F^c$ or $b \in F^c$. In order to get a contradiction, suppose $a \notin F^c$ and $b \notin F^c$; then, $a \in F$ and $b \in F$ and, F being a filter, we conclude $a \wedge b \in F$, contradicting the assumption. Hence, F^c is prime and we are done. □

1.2.4 Some topological background

We give here some topological notions that will be needed later and can be found in [K].

Definition 1.2.28 *A topological space $\langle X, \tau \rangle$ is said to be* **compact** *if and only if for any family $\{A_i \mid i \in I\}$ of open subsets of X such that $X \subset \bigcup_{i \in I} A_i$, there are $i_1, i_2, \cdots i_n \in I$ such that $X \subseteq \bigcup_{j=1}^{n} A_{i_j}$.*

Definition 1.2.29 *Given a topological space $\langle X, \tau \rangle$, a family D of open subsets of X is called a* **subbase** *of the space if and only if every open subset of X can be written as an union of finite intersections of elements in D.*

We give now some characterizations of compactness that will be used later.

Theorem 1.2.30 *(Alexandrov Theorem)*
Let $\langle X, \tau \rangle$ be a topological space and D a subbase for it. Then $\langle X, \tau \rangle$ is compact if and only if for any family $\{A_i \mid i \in I\}$ of elements of D such that $X \subset \bigcup_{i \in I} A_i$, there are $i_1, i_2, \cdots i_n \in I$ such that $X \subseteq \bigcup_{j=1}^{n} A_{i_j}$.

Theorem 1.2.31 *A topological space $\langle X, \tau \rangle$ is compact if and only if for any family C closed subset of X and for any family $\{A_i \mid i \in I\}$ of open elements of X, if $C \subseteq \bigcup_{i \in I} A_i$, then there are $i_1, i_2, \cdots i_n \in I$ such that $C \subseteq \bigcup_{j=1}^{n} A_{i_j}$.*

The following theorem is just Alexandrov Theorem stated in terms of closed subsets. We use D^{co} for $D^{co}_X = \{X \setminus a \mid a \in D\}$.

Theorem 1.2.32 *Let $\langle X, \tau \rangle$ be a topological space and D a subbase for it. Then $\langle X, \tau \rangle$ is compact if and only if for any family $\{C_i \mid i \in I\}$ of elements of D^{co} such that $\bigcap_{i \in I} C_i = \emptyset$, there are $i_1, i_2, \cdots i_n \in I$ such that $\bigcap_{j=1}^{n} C_{i_j} = \emptyset$.*

Definition 1.2.33 *Given a topological space $\langle X, \tau \rangle$, a set $a \subset X$ is said to be* **clopen** *if and only if a is open and closed.*

In chapter 2, we will generalize the notion of compactness to a structure that is more complex than a topological space. The following characterization of compactness will help to see that generalization as "natural".

Theorem 1.2.34 *Let $\langle X, \tau \rangle$ be a topological space and let D be a set of clopen subsets of X such that $D \cup D^{co}$ is a subbase of X. Then $\langle X, \tau \rangle$ is compact if and only if for any $S, T \subseteq D$: if $\bigcap S \subseteq \bigcup T$, then there are $S' \subseteq S$ and $T' \subseteq T$ finite such that $\bigcap S' \subseteq \bigcup T'$.*

1.2.5 Priestley Duality.

We state here some fundamental results concerning the generalization of Stone's duality to the non boolean but still distributive case, whose proofs can be found in chapter 10 of [DP].

Definition 1.2.35 *Given a partial order $\langle X, \leq \rangle$, and a given $S \subseteq X$, we say that: S is* **increasing** *if and only if for any $x, y \in X$, if $x \in S$ and $x \leq y$, then $y \in S$.*
We write $\mathcal{O}(X)$ for the set of increasing subsets of X.

In particular, if $x \in X$, the set $x\uparrow = \{y \in X \mid x \leq y\}$ is increasing. Sets of this form are called **principal increasing** sets.

Definition 1.2.36 *A* **Priestley Space** *is a structure $\langle X, \leq, D \rangle$ such that: $\leq$ is a partial order on the set X and D is a collection of subsets of X satisfying:*

- *D is closed under binary (and therefore under finite) unions and intersections,*
- *for every $x, y \in X$ with $x \not\leq y$, there is $a \in D$ such that $x \in a, y \notin a$,*
- *if we consider the topology τ of X having $D \cup D^{co}$ as a subbase, then $\langle X, \tau \rangle$ is compact.*

Note that $\emptyset \in D$ and that $X \in D$, since D is closed under empty unions and empty intersections (the union of an empty family of sets is the empty set and the intersection of an empty family of sets is the hole universe).
In fact, we can derive that $\langle D, \cap, \cup, \emptyset, X \rangle$ is a sublattice of the distributive lattice $\langle \mathcal{O}(X), \cap, \cup, \emptyset, X \rangle$.

What we call a Priestley Space here appears in [DP] under the name of **compact totally ordered disconnected space** (CTOD). The definition of a CTOD that Davey and Priestley give in [DP] coincides with the standard definition of a Priestley Space used for example in [G]; this definition is not in terms of the set D: they define what means for a structure $\langle X, \leq, \tau \rangle$ (where τ is a topology in X and $\leq$ is a partial order on X) to be a Priestley Space. In particular, they require the following: if $x \not\leq y$, there exist a clopen increasing subset $a \subseteq X$ such that: $x \in a$ and $y \notin a$.
The following proposition shows that the condition in D of our definition is equivalent to that one, stating that the set D is exactly the set of clopen increasing subsets of $\langle X, \leq, \tau \rangle$.

Proposition 1.2.37 *If $\langle X, \leq, D \rangle$ is a Priestley Space, then D is the set of all clopen (that is, open and closed) increasing subsets of X.*

Proof: Every element in D is increasing, by assumption. As $D \cup D^{co}$ is a subbase, we have that every element of D is open and has open complement, i.e. every element in D is clopen.

We show now that every clopen increasing subset of X is an element of D.
Suppose we have $S \subseteq X$ clopen and increasing. We want to show that $S \in D$.

Take an arbitrary $y \notin S$. For any $x \in S$, as S is increasing, we have $x \not\leq y$; there is then $a_{xy} \in D$ such that $x \in a_{xy}$ and $y \notin a_{xy}$. Hence,

$$S \subseteq \bigcup_{x \in S} a_{xy},$$

for every $y \notin X$.
As S is compact and the a_{xy} are open, there are $x_1, x_2, \cdots, x_{n_y}$ such that

$$S \subseteq a_{x_1 y} \cup a_{x_2 y} \cup \cdots \cup a_{x_{n_y} y}.$$

For every $y \notin S$, let $a_y = \bigcup_{i=1}^{n_y} a_{xy}$. Then, $y \notin a_y$ and, as D is closed under finite unions, we have that for every $y \notin S$: $a_y \in D$. Hence, we have that for every $y \notin S$, there is an $a_y \in D$ such that $y \notin a_y$ and $S \subseteq a_y$. From this we may infer that $S \subseteq \bigcap_{y \notin S} a_y$, and therefore

$$\bigcup_{y \notin S} a_y^c \subseteq S^c.$$

Furthermore, for every $y \in S^c$, we have $y \notin a_y$ and therefore $y \in a_y^c$. Therefore

$$S^c \subseteq \bigcup_{y \notin S} a_y^c.$$

But, from the facts that S is open (that is, S^c is closed) and that every a_y^c is open, we infer, by the characterization of compactness given by theorem 1.2.31, that there are $y_1, y_2, \cdots, y_n$ such that

$$S^c \subseteq \bigcup_{i=1}^{n} a_y^c.$$

Hence,

$$\bigcup_{i=1}^{n} a_{y_i}^c \subseteq \bigcup_{y \notin S} a_y^c \subseteq S^c \subseteq \bigcup_{i=1}^{n} a_{y_i}^c$$

and therefore $S^c = \bigcup_{i=1}^{n} a_y^c$; that is, $S = \bigcap_{i=1}^{n} a_y$. As D is closed under finite intersections, we conclude that $S \in D$. □

The topological duality for Boolean Algebras of Stone that associates with a Boolean Algebra A a general frame whose universe is the set of maximal filters of A is described in [DP]. Its generalization to distributive lattices, also presented in [DP], associates with each distributive lattice the Priestley Space described in the following proposition.

Definition 1.2.38 *Given a distributive lattice L, define $F_p(L)$ to be the set of all prime filters of L.*

Proposition 1.2.39 *Let L be a distributive lattice and $D = \{\hat{a} \mid a \in L\}$, with $\hat{a} = \{F \in F_p(L) \mid a \in F\}$.*
The structure $g(L) = \langle F_p(L), \subseteq, D \rangle$ is a Priestley Space called the dual space of L.

The next theorem completes in some sense proposition 1.2.39, by stating that the set D has a natural lattice structure that makes it isomorphic to the lattice L. The lattice L is then, by proposition 1.2.37, isomorphic to the clopen increasing subsets of $F_P(L)$.

Theorem 1.2.40 Priestley's Representation Theorem. *Let L be a distributive lattice. Then the map*

$$\nu : L \to \mathcal{O}(F_p(L)),$$

given by $\nu(a) = \hat{a}$, is an injective morphism of lattices and has as image the lattice of all clopen increasing subsets of $F_p(L)$.

We have seen that the notion of atom for boolean algebras generalizes in the non boolean case to the notion of join prime; similarly, the notion of atomicity generalizes to the following.

Definition 1.2.41 *A complete distributive lattice is* **atomic** *if and only for every $x \in L$ such that $x \neq 0$, there is a completely join-prime element $a \in L$ such that $a \leq x$.*

Proposition 1.2.42 *A complete distributive lattice L is atomic if and only if for every $x \in L$:*

$$x = \bigvee\{a \in L \mid a \leq x \textit{ and } a \textit{ is a completely join prime element of } L\}.$$

We did not find in the litterature any name for this kind of distributive lattices; we only find the word "atomic" referring to boolean algebras.

Given a partial order $\langle X, \leq \rangle$, the structure $\langle \mathcal{O}(X), \cap, \cup, \emptyset, X \rangle$ is a distributive lattice and has two basic nice properties: it is complete and it is atomic.

1.2.6 Categories.

We give here some definitions in the framework of category theory that are used in our work. For more detail we refer to [B].

Definition 1.2.43 *A* **category** *$\mathcal{C}$ consists of a class of objects $\mathcal{O}\lfloor(\mathcal{C})$ and, for every pair of objects X, Y, a class of morphisms $Hom_{\mathcal{C}}(X, Y)$ (whose elements we denote by $f : X \to Y$), such that:*

- *for X, Y, Z objects of $\mathcal{C}$, and $f : X \to Y$, $g : Y \to Z$ morphisms of $\mathcal{C}$, there is a morphism $g \circ f : X \to Z$ called the composition of f and g,*
- *this composition is associative, i.e.: given X, Y, Z, T objects of $\mathcal{C}$ and $f : X \to Y$, $g : Y \to Z$, $h : Z \to T$ morphisms of $\mathcal{C}$, we have that $(f \circ g) \circ h = f \circ (g \circ h)$,*
- *for every object X of $\mathcal{C}$, there is a distinguished morphism $id_X : X \to X$, called the identity on X verifying for every object Y of $\mathcal{C}$, every morphism $f : X \to Y$ and every morphism $g : Y \to X$: $f \circ id_X = f$ and $id_x \circ g = g$.*

We denote by $M(\mathcal{C})$ the class of all morphisms of $\mathcal{C}$.

Example: the category *Set* whose objects are the sets and whose morphisms are the functions with usual composition.

Definition 1.2.44 *Given a category $\mathcal{C}$, its* **opposite category** *$\mathcal{C}^{op}$ is the category given by:*

- *$Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$,*
- *$Hom_{\mathcal{C}^{op}}(X,Y)$ corresponds bijectively to $Hom_{\mathcal{C}}(Y,X)$, where f^{op} is the morphism in $\mathcal{C}^{op}$ that is image of the morphism f in $\mathcal{C}$.*
- *if $\circ_{op}$ is the composition in $\mathcal{C}^{op}$, then $f^{op} \circ_{op} g^{op} = (g \circ f)^{op}$.*

Definition 1.2.45 *Given two categories $\mathcal{C}$ and $\mathcal{D}$. We say that $\mathcal{C}$ is a* **subcategory** *of $\mathcal{D}$ if and only if $Ob(\mathcal{C}) \subseteq Ob(\mathcal{D})$ and for every $X, Y \in Ob(\mathcal{C})$, we have $Hom_{\mathcal{C}}(X,Y) \subseteq Hom_{\mathcal{D}}(X,Y)$.*
In case the last inclusion is an equality for every $X, Y \in Ob(\mathcal{C})$, the subcategory is said to be **full**.

Definition 1.2.46 *A category is said to be* **concrete** *iff it is a subcategory of Set.*

Definition 1.2.47 *Given a category $\mathcal{C}$, X, Y objects of $\mathcal{C}$, a morphism $f : X \to Y$ is called:*

- **epimorphism** *if and only if for every object Z and every morphisms $g, h : Y \to Z$: if $g \circ f = h \circ f$, then $g = h$,*
- **monomorphism** *if and only if for every object Z and every morphisms $g, h : Z \to X$: if $f \circ g = f \circ h$, then $g = h$,*
- **isomorphism** *if and only if it is an epimorphism and a monomorphism. When there is an isomorphism between two objects X and Y, we write $X \cong Y$.*

Proposition 1.2.48 *If $\mathcal{C}$ is a concrete category, then f is an epimorphism (monomorphism) if and only if f is surjective (injective).*

Definition 1.2.49 *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A covariant (contravariant)* **functor** *F from $\mathcal{C}$ to $\mathcal{D}$ is a pair of functions:*

$$F_O : Ob(\mathcal{C}) \to Ob(\mathcal{D}), \qquad F_M : M(\mathcal{C}) \to M(\mathcal{D}),$$

such that

- *if $f : X \to Y$, then $F_M(f) : F_O(X) \to F_O(Y)$ (resp. $F_M(f) : F_O(Y) \to F_O(X)$),*
- *$F_M(f \circ g) = F_M(f) \circ F_M(g)$ (resp. $F_M(f \circ g) = F_M(g) \circ F_M(f)$),*
- *$F_M(id_X) = id_{F_O(X)}$.*

In order to define what will be the natural notion of "isomorphism" **between** categories, i.e. a functor that has an inverse in some sense, we need the following definition:

Definition 1.2.50 *A covariant (contravariant) functor $F : \mathcal{C} \to \mathcal{D}$ is said to be:*

- **faithful** *if and only if F_M is injective,*
- **full** *if and only if for any $s : F_O(X) \to F_O(Y)$, there is $f : X \to Y$ (resp. $f : Y \to X$) such that $s = F_M(f)$,*
- **dense** *if and only if for any $U \in Ob(\mathcal{D})$, there is $X \in Ob(\mathcal{C})$ such that $F_O(X) \cong U$.*

In words, a functor is faithful if it is injective on the morphisms; it is full if it satisfies on the morphisms a weak surjectivity condition, namely the function:

$$F_M : Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F_O(X), F_O(Y))$$

is surjective for every objects X, Y of $\mathcal{C}$.
A functor is dense if it is surjective on the objects up to isomorphisms.

Definition 1.2.51 *A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is called an* **equivalence** *if and only if there is a functor $G : \mathcal{D} \to \mathcal{C}$, and two families $(\phi_X)_{X \in Ob(\mathcal{C})}$, $(\psi_U)_{U \in Ob(\mathcal{D})}$, such that:*

$$\phi_X : X \to GF(X) \text{ is an isomorphism in } \mathcal{C} \text{ for each } X \in Ob\mathcal{C},$$

$$\psi_U : U \to FG(U) \text{ is an isomorphism in } \mathcal{D} \text{ for each } U \in Ob\mathcal{D},$$

and such that the following diagrams commute:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\phi_X"'] & Y \arrow[d, "\phi_Y"'] \\
GF(X) \arrow[r, "GF(f)"] & GF(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, "\psi_U"'] & U \arrow[d, "\psi_V"'] \\
FG(U) \arrow[r, "FG(f)"] & FG(V)
\end{tikzcd}$$

Definition 1.2.52 *A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is called a* **duality** *if and only if the covariant functor $F^{op} : \mathcal{C} \to \mathcal{D}^{op}$ given by: $(F^{op})_O(X) = F_O(X)$ and $(F^{op})_M(f) = (F_M(f))^{op}$ is an equivalence.*

There is a nice characterization of equivalence and duality that we will use; we state it in the following theorem:

Theorem 1.2.53 *A functor is an equivalence (duality) if and only if it is covariant (resp. contravariant), faithful, full and dense.*

Theorem 1.2.54

Every equivalence takes monomorphisms to monomorphisms and epimorphisms to epimorphisms.

Every duality takes monomorphisms to epimorphisms and epimorphisms to monomorphisms.

Definition 1.2.55 *If $(F_O, F_M) : \mathcal{C} \to \mathcal{D}$ is a functor, the* **image** *of F is the structure $Im(F) = \langle O(F), M(F) \rangle$, where $O(F)$ is the class of objects of $\mathcal{D}$ of the form $F_O(X)$ for some object X of $\mathcal{C}$ and $M(F)$ is the class of morphisms of $\mathcal{D}$ of the form $F_M(\phi)$ for some morphism ϕ of $\mathcal{C}$.*

Proposition 1.2.56 *The image of a functor is a category.*

1.3 Basic concepts.

The present section introduces the concepts and terminology we will use along this work, proving every stated result.
The notion of maximal filter-ideal pair is not new, it can be found in [U]. The notions of "optimal" filter and "optimal" ideal appearing in section 1.3.2 and the theory concerning them that we develop here are ours.
The rest of the results in this section are not original of this work.

1.3.1 Galois correspondences.

The notion of Galois correspondence is crucial for our work. Our representation of a lattice will consist of two structures of the form $\langle X, \leq, D\rangle$ (where $\leq$ is a partial order on X and $D \subseteq \mathcal{O}(X)$) linked by a Galois correspondence. The needed background on this subject is presented here.

Definition 1.3.1 *Given partially ordered sets $(X_1, \leq_1)$ and $(X_2, \leq_2)$, a* **Galois correspondence** *or* **Galois connection** *is a pair of functions $\lambda : X_1 \to X_2$ and $\rho : X_2 \to X_1$ satisfying:*

$$x \leq_1 \rho(y) \text{ if and only if } y \leq_2 \lambda(x) \tag{1.1}$$

We say that $x \in X_1$ is **stable** *if and only if $\rho\lambda(x) = x$.*

We list and prove here some basic properties of a Galois connection:

Proposition 1.3.2

1. *If $x \in X_1$ then $x \leq_1 \rho\lambda(x)$.*
2. *$\rho\lambda\rho = \rho$.*
3. *An element $x \in X_1$ is stable if and only if there exists $y \in X_2$ such that $x = \rho(y)$.*

Proof:

1. Take $y = \lambda(x)$ and apply the definition of Galois connection.
2. Take any $y \in X_2$ and observe that defining $x = \rho(y)$, the inequality $\rho(y) \leq_1 \rho\lambda\rho(y)$ follows from 1.1.
On the other hand, by symmetry with 1, we know that $y \leq_2 \lambda\rho(y)$; from this it follows that $y \leq_2 \lambda\rho\lambda\rho y$, which is, by definition, equivalent to $\rho\lambda\rho(y) \leq_1 \rho(y)$.
3. If $x = \rho(y)$, then $\rho\lambda(x) = \rho\lambda\rho(y) = y$, the last equality justified by 2.
To prove the part of 3 that is not obvious, suppose that there exists $y \in X_2$ such that $x = \rho(y)$. Then x is stable by property 2.

□

<u>Remark:</u> As the definition is symmetric, we can also define stable elements of X_2 and the properties above have its obvious counterparts.

The standard example of a Galois connection is the following.

Definition 1.3.3 *Given two sets A_1 and A_2, and a relation $\perp \subseteq A_1 \times A_2$, the Galois correspondence* **induced** *by $\perp$ is defined between the power set of A_1 and the power set of A_2 as follows: for $i = 1, 2$, let $X_i = \mathcal{P}(A_i)$ endowed with the usual orders of inclusion ($\subseteq$), and define:*

$$\lambda_\perp : X_1 \to X_2 \text{ and } \rho_\perp : X_2 \to X_1 \text{ as}:$$
$$\lambda_\perp(C) = \{y \in A_2 \ : \ \forall x \in C, (x, y) \in \perp\};$$
$$\rho_\perp(D) = \{x \in A_1 \ : \ \forall y \in D, (x, y) \in \perp\}$$

To prove that this is in fact a Galois correspondence, take $C \in X_1$, $D \in X_2$, and observe that the properties $C \subseteq \rho_\perp(D)$ and $D \subseteq \lambda_\perp(C)$ are both equivalent to the property: for every $x \in C$ and $y \in D$ it holds that $(x, y) \in \perp$.

Using the definition of $\lambda_\perp$ and $\rho_\perp$, it is easy to check the following.

Proposition 1.3.4

1. *$\lambda_\perp$ and $\rho_\perp$ are order-reversing.*
2. *$\lambda_\perp(\bigcup_{i\in I} C_i) = \bigcap_{i\in I} \lambda_\perp(C_i)$ and $\rho_\perp(\bigcup_{i\in I} D_i) = \bigcap_{i\in I} \rho_\perp(D_i)$.*
3. *The functions $\rho_\perp\lambda_\perp : X_1 \to X_1$ and $\lambda_\perp\rho_\perp : X_2 \to X_2$ are both order-preserving.*

We have a dual result that applies only for stable sets:

Lemma 1.3.5 *In the previous context, if for every $i \in I$, $a_i \in X_i$ is stable, then $\rho_\perp(\bigcap_{i\in I} a_i) = \rho_\perp\lambda_\perp\left(\bigcup_{i\in I} \rho_\perp(a_i)\right)$.*

Proof: Let $b_i = \rho_\perp a_i$; we have

$$\rho_\perp\left(\bigcap_{i\in I} a_i\right) = \rho_\perp\left(\bigcap_{i\in I} \lambda_\perp(b_i)\right) = \rho_\perp\lambda_\perp\left(\bigcup_{i\in I} b_i\right) = \rho_\perp\lambda_\perp\left(\bigcup \rho_\perp(a_i)\right)$$

□

1.3.2 Optimal filters and ideals.

In the duality theory of Boolean Algebras, maximal filters play a crucial part. They constitute the points of the dual space of a Boolean Algebra. In the case of distributive lattices, this role is played by the prime filters. Note that the notion of a prime filter generalizes that of a maximal filter.
Similarly, we generalize here the notion of a prime filter to that of an **optimal filter**. We will prove that the optimal filters of a distributive lattice are exactly its prime filters.
As we mentioned before, the dual space of a lattice will be a two-sorted space, i.e. it will consist basically of two sets, each one of them endowed with a partial order and a set of increasing subsets: one of them will have as points the optimal filters of L, the other will have as points the optimal ideals of L.

We present here the definition of these notions and useful properties pertaining to them.

Definition 1.3.6 *Let F, I be respectively a proper filter and a proper ideal of a lattice L.*

We say that F is I**-maximal** *iff it is maximal with respect to being disjoint from I; that is:*

1. *$F \cap I = \emptyset$,*
2. *if F' is a filter such that $F \subsetneq F'$, then $F' \cap I \neq \emptyset$.*

Dually, we define the notion of I being F**-maximal**.

We say that the pair $\langle F, I\rangle$ is a **maximal filter-ideal pair** *iff F is I-maximal and I is F-maximal. We say also in this case that F and I are* **companions** *of each other.*

Finally, we say that a filter F is **optimal** *iff there is an ideal I such that the pair $\langle F, I\rangle$ is a maximal filter-ideal pair. We define similarly an* **optimal ideal**.

The following theorem is the generalization of (one direction of) the Prime Filter Theorem.

Theorem 1.3.7 *Given a proper filter F and a proper ideal I such that $F \cap I = \emptyset$, there is a filter G that contains F and is I-maximal.*

Of course, the dual result also holds, namely that there is an ideal J that contains I and is F-maximal.

Proof: F and I being disjoint, we want to find a filter G containing F that is I-maximal. This follows by applying Zorn's lemma to the family:

$$\mathcal{F} = \{G \text{ proper filter of } L \mid G \cap I = \emptyset \text{ and } F \subseteq G\}$$

ordered with inclusion. $\mathcal{F}$ is non empty because $F \in \mathcal{F}$. The union of a chain of filters disjoint from I and containing F is a filter disjoint from I and containing F; hence, every chain of $\mathcal{F}$ is bounded in $\mathcal{F}$ and therefore $\mathcal{F}$ has a maximal element F_0. This F_0 is, by definition, I-maximal. $\square$

Corollary 1.3.8 *Let L be a lattice.*

1. *If F is a filter of L and $a \in L \backslash F$, then there is an ideal I that is F-maximal and such that $a \in I$.*
2. *If I is an ideal of L and $a \in L \backslash I$, then there is a filter F that is I-maximal and such that $a \in F$.*

Proof: We prove only the first item, the other being dual.
It is enough to consider the ideal $a{\downarrow}$ of L. As $a \notin F$, we have that if $b \in L$ is such that $b \leq a$, then $b \notin F$. Therefore $F \cap a{\downarrow} = \emptyset$ and we can apply theorem 1.3.7 in order to obtain I. $\square$

Corollary 1.3.9 *Let L be a lattice. If F is a proper filter of L and I is a proper ideal of L such that $F \cap I = \emptyset$, then there are a filter G and an ideal J of L such that $F \subseteq G$, $I \subseteq J$ and $\langle G, J\rangle$ is a maximal filter-ideal pair.*

Proof: We use theorem 1.3.7 for constructing a filter G containing F and I-maximal. As $G \cap I = \emptyset$, we use then the dual version of theorem 1.3.7 for constructing an ideal J containing I and G-maximal. It is clear that G is J-maximal, since if we have a filter H such that $G \subsetneq H$, then $H \cap I \neq \emptyset$ and then $H \cap J \neq \emptyset$. The pair $\langle G, J \rangle$ is then a maximal filter-ideal pair. □

By definition, every optimal filter is I-maximal for some ideal I. The converse also holds, that is:

Proposition 1.3.10 *If there is an ideal I of L such that the filter F is I-maximal, then F is an optimal filter.*

Proof: This follows by a direct application of theorem 1.3.7. Suppose that we have a filter F and an ideal I such that F is I-maximal. Then $F \cap I = \emptyset$. We can then apply corollary 1.3.9 and obtain an ideal J containing I that is F-maximal.

We show now that the pair $\langle F, J \rangle$ is then a maximal ideal-filter pair. We know that J is F-maximal. In order to prove that F is J-maximal, assume that G is a filter properly containing F; then $G \cap I \neq \emptyset$ and consequently ($I \subseteq J$) we have $G \cap J \neq \emptyset$ and we are done. □

Concerning prime filters, we have the following two propositions:

Proposition 1.3.11

1. *Every prime filter is optimal. Moreover, if F is a prime filter, then there is exactly one ideal I that is F- maximal.*

2. *As a corollary, we have: if F is a prime filter, then there is exactly one ideal I companion of F.*

Proof:

1. By lemma 1.2.27, if F is a prime filter of L, then F^c is a prime ideal of L. It is then trivial to show that $\langle F, F^c \rangle$ is a maximal filter-ideal pair; hence, the ideal F^c is F-maximal.
   We have to show that F^c is the only ideal that is F-maximal: suppose I is F-maximal, then $F \cap I = \emptyset$ and therefore $I \subseteq F^c$. By a maximality argument, it follows that $I = F^c$ and we are done.

2. Suppose now that I is F-maximal. Then $I \cap F = \emptyset$, that is $I \subseteq F^c$. If I was strictly contained in F^c, then I would not be F-maximal (since F^c is an ideal disjoint from F). Then $I = F^c$ and we are done.

□

With respect to the converse of the last assertions, we have the following

Proposition 1.3.12 *Let L be a lattice and F be a proper filter of L.*

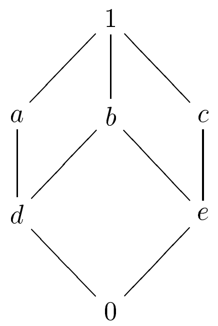
1. *If there is exactly one ideal I that is F-maximal, then F is prime.*
   But:

2. *On the other hand, the fact that there is exactly one ideal I companion of F does not imply that F is prime.*

Proof:

1. Let F be a filter such that there is only one F-maximal ideal. Suppose F is not prime. Then there are $a, b \in F$ such that $a \vee b \in F$, $a \notin F$ and $b \notin F$. Clearly, the ideal $a\!\downarrow$ is disjoint from F and therefore can be extended to an F-maximal ideal I. If $b \in I$, then $a \vee b \in I$ and $F \cap I \neq \emptyset$, contradicting the fact that I is F-maximal. Hence $b \notin I$.
   Similarly, the ideal $b\!\downarrow$ can be extended to an F-maximal ideal J. As $b \in J$ and $b \notin I$, we have $I \neq J$, contradicting the assumption.
   Hence we have proved that F is prime.

2. We give here an example of a filter (actually, an optimal filter) that belongs to a **unique** maximal filter-ideal pair and is not prime:
   Consider the following lattice L:



   The filter $F = \{1, a\}$ verifies:

   - F is non prime: $1 = d \vee c \in F$, but $d \notin F$ and $c \notin F$,
   - F is optimal: if $I = \{0, b, d, e\}$, then the pair $\langle F, I\rangle$ is maximal (if we consider F' to be the filter generated by $F \cup \{c\}$ then, as $a \in F$, we have $0 = a \wedge c \in F'$ then $F' \cap I \neq \emptyset$; similarly, if we take I' to be the ideal generated by $I \cup \{c\}$, we get $F \cap I' \neq \emptyset$),
   - the only companion of F is I: suppose we have J ideal companion of F, $J \neq I$; then by maximality of the pair $\langle F, J\rangle$, we have that $J \not\subseteq I$ and we also have $J \cap F = \emptyset$. Then $c \in J$. The only ideal disjoint from F and containing c is $J = \{c, e, 0\}$, but F can be extended to the filter $G = \{1, a, b, d\}$, disjoint from J, hence the pair $\langle F, J\rangle$ is not maximal and therefore J is not companion of F, contradicting the assumption. Hence the only companion of F is the ideal I.

□

Note that the filter F used in the last proof is an example of an optimal filter that has exactly one companion but such that there is more than one ideal that is F-maximal: the ideal I mentioned in the proof is F-maximal, but also the ideal $\{0, e, c\}$ is F-maximal.

Of course, we have the dual results of propositions 1.3.11 and 1.3.12, concerning ideals.

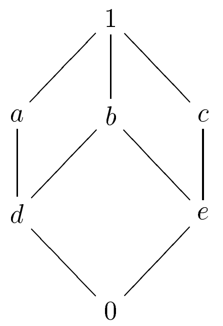
Concerning distributive lattices, we get, as a consequence of the previous results, the following

Proposition 1.3.13

1. *The optimal filters of a distributive lattice are exactly the prime filters of it.*
2. *Conversely, if every optimal filter of a lattice L is prime, then L is distributive.*
3. *The fact that every optimal filter F of a lattice L has a unique companion does not imply that L is distributive.*

Proof:

1. Let L be a distributive lattice. We have already shown that any prime filter is optimal (we do not need distributivity for this).
   Suppose now that F is optimal. Then there is an ideal I such that $\langle F, I\rangle$ is a maximal ideal-filter pair. By the Prime Filter Theorem for distributive lattices, we know that there is a prime filter G containing F and disjoint from I. But, we have that F is I-maximal, therefore $F = G$ and we conclude that F is prime.
2. In the particular case that every optimal filter is prime, theorem 1.3.7 corresponds to the fact of L having the PFP. Then, applying theorem 1.2.24, we deduce that L is distributive.
3. We give an example of a non distributive lattice in which every optimal filter has a unique companion:

   First observe that L is not distributive: $a \wedge (c \vee d) = a \wedge 1 = a$; $(a \wedge c) \vee (a \wedge d) = 0 \vee d = d$.
   We see now that every optimal filter has a unique companion:

   - The set $\{1, a\}$ is an optimal filter that has a unique companion (cf. proof of 1.3.12.2), namely the ideal $\{0, d, b, e\}$.
   - Similarly we can prove that the filter $\{1, c\}$ is optimal with a unique companion.
   - The filter $\{1, a, b, d\}$ has as its unique companion the ideal $\{0, c, e\}$.
   - Similarly, the filter $\{1, b, c, e\}$ has as unique companion the ideal $\{0, d, a\}$.

- The filter $F = \{1, b\}$ is non optimal: suppose we have an ideal I disjoint from F. We will see that then F can be properly extended to a filter G disjoint from I; that is, we will prove, for any ideal I of L, that F is not I-maximal.
  - If $I = \{0\}$ then clearly the filter F is not I-maximal, since F can be extended to the filter $G = \{1, b, c, e\}$ disjoint from I.
  - If $d \in I$, then $c \notin I$ (for $c \vee d = 1 \notin I$) and similarly, $e \notin I$. So we have necessarily $I \subseteq \{0, d, a\}$ and again the same G is a filter properly extending F and disjoint from I.
  - If $I \neq \{0\}$ and $d \notin I$, then we have $e \in I$. A similar argument than in the previous item shows that the filter $H = \{1, a, b, d\}$ extends properly F and is disjoint from I.

$\square$

For finishing this chapter, we generalize the notion of join prime to the non distributive case, obtaining a generalization of proposition 1.2.26.

Definition 1.3.14 *$a \in L$ is called* **join irreducible** *(***completely join irreducible***) iff:*

$$a = b \vee c \Rightarrow a = b \text{ or } a = c$$

(resp. $a = \bigvee_{i \in I} b_i \Rightarrow a = b_i$ for some $i \in I$).

We define dually **meet irreducible** and **completely meet irreducible**.

Concerning the relation between join-irreducible elements of L and optimal filters of L we have the following generalization of proposition 1.2.26:

Proposition 1.3.15 *$a \in L$ is join irreducible if and only if the filter $a\uparrow$ is optimal.*

Proof: Direction from left to right:
Let I be the set of points strictly smaller than a, i.e. $I = \{x \in L \mid x < a\}$. We first see that I is an ideal of L: take $b, c \in I$; then $b < a$ and $c < a$. We have then $b \vee c \leq a$. Moreover $b \neq a$ and $c \neq a$, therefore, using that a is join-irreducible, we have that $b \vee c \neq a$, that is $b \vee c < a$, and therefore $b \vee c \in I$.
We will see now that $a\uparrow$ is I-maximal (and therefore optimal). Take a filter $F \supseteq a\uparrow$ and consider $b \in F \setminus a\uparrow$. Then $a \not\leq b$.
We have that $b \wedge a \leq a$, moreover $b \wedge a < a$ (since if $b \wedge a = a$ we would have $a \leq b$). Then we have $b \wedge a \in I$ and we can conclude, from $b \wedge a \in F$ that $F \cap I \neq \emptyset$; hence, $a\uparrow$ is I-maximal and therefore optimal.
For the converse direction, suppose $a = b \vee c$, with $b \neq a$ and $c \neq a$.
Then $b < a$ and $c < a$, and therefore $a\uparrow \subsetneq b\uparrow$ and $a\uparrow \subsetneq c\uparrow$. As $a\uparrow$ is optimal, there is an ideal I of L such that the pair $\langle a\uparrow, I\rangle$ is maximal. Therefore $b\uparrow \cap I \neq \emptyset$ and $c\uparrow \cap I \neq \emptyset$, that is $b \in I$ and $c \in I$, leading to the contradiction $a \in I$. We have proved then that a is join-irreducible.

$\square$

Chapter 2

"Partial" duality for lattices with operators

Priestley duality for distributive lattices has been extended to the non distributive case: in 1978, Urquhart ([U]) proved a Representation Theorem for lattices; later, Hartung ([H]) in 1992, Hartonas and Dunn ([DH]) in 1997, gave alternative ways of representing lattices.
The aim of this chapter is to present a duality for lattices endowed with additional operations $f : L^n \to L$. None of the works mentioned above consider lattices with additional operations.
The representation we propose in this chapter, although it has been inspired by this works, is not a generalization of any of the representations given by them.
The lattices we will consider are endowed with two unary maps, one join-preserving and the other meet-preserving. What is done here can be extended to a more general similarity type, but in order to simplify the writing and clarify the reading, we treat only this particular similarity type.
We consider then the category $\mathcal{LWO}$ given by:

- objects of the form $\mathcal{L} = \langle L, \wedge, \vee, 0, 1, g, f \rangle$, where $\langle L, \wedge, \vee, 0, 1 \rangle$ is a lattice, $g, f : L \to L$ are respectively meet and join-preserving (cf section 1.1),
- if $\mathcal{L} = \langle L, \wedge, \vee, 0, 1, f, g \rangle$ and $\mathcal{L}' = \langle L', \wedge', \vee', 0', 1', f', g' \rangle$, $\phi : \mathcal{L} \to \mathcal{L}'$ is a **morphism of lattices with operators** iff it is a function from L to L' such that:
  - $\phi(a \wedge b) = \phi(a) \wedge' \phi(b)$, $\phi(a \vee b) = \phi(a) \vee' \phi(b)$,
  - $\phi(0) = 0'$, $\phi(1) = 1'$,
  - $\phi(fa) = f'(\phi a)$, $\phi(ga) = g'(\phi a)$,

When there is no possible confusion, we will call L the lattice with operators $\mathcal{L}$; that is, we will use the same name for a lattice with operators and its underlying set.
The dual spaces of the objects of $\mathcal{LWO}$ will be what we call descriptive general lattice frames (defined in 2.1.5); they consist of two structures of the form $\langle X, \leq, R, D \rangle$ (where X is called the universe of the structure) linked by a Galois connection. A lattice with operators will be represented by a descriptive general lattice frame as follows:

1. the "semi-lattice with operators" $\langle L, \wedge, 0, g\rangle$ will be represented by a structure having as universe X the set of all optimal filters of L denoted by $F_*(L)$;

2. the "semilattice with operators" $\langle L, \vee, 1, f\rangle$ will be represented by a structure having as universe the set of all optimal ideals of L denoted by $I_*(L)$;

3. the link between this two semi-lattices, given by the fact that they define the same order on L, will be represented by a Galois correspondence induced by a relation $\perp \subseteq F_*(L) \times I_*(L)$.

The descriptive general lattice frames will be considered as objects of a category that we call $\mathcal{GDLF}$.
Unfortunately, we can not give a duality between the hole categories $\mathcal{LWO}$ and $\mathcal{GDLF}$. The natural functor to consider can only be defined for epimorphisms in $\mathcal{LWO}$. If:

1. $\mathcal{LWO}_e$ is the subcategory of $\mathcal{LWO}$ whose objects are all lattices with operators and whose morphisms are epimorphisms of lattices with operators,

2. $\mathcal{GDLF}_m$ is the subcategory of $\mathcal{GDLF}$ whose objects are the objects of $\mathcal{GDLF}$ and whose morphisms are the monomorphisms of $\mathcal{GDLF}$,

(it is easy to check that $\mathcal{L}WO_e$ and $\mathcal{DGLF}_m$ are categories) we will define a functor:

$$H : \mathcal{LWO}_e \to \mathcal{GDLF}_m.$$

that will be, in some sense that will be clear in section 2.4, a "partial duality". This chapter is divided in five sections:

- In section 2.1, we present the category $\mathcal{GDLF}$.
- In section 2.2, we restrict ourselves to the objects in $\mathcal{LWO}$ and define their dual descriptive general lattice frames. We will have a function H_O from the objects of $\mathcal{LWO}$ to the objects of $\mathcal{GDLF}$.
- In section 2.3, we define a function H_M on epimorphisms of lattices with operators in order to obtain a functor:

  $$H = (H_O, H_M) : \mathcal{LWO}_s \to \mathcal{GDLF}_m.$$

  We prove that this functor can not be extended in general to all the morphisms of $\mathcal{LWO}$.
- In section 2.4, we show that the functor described in section 2.3 is a "partial duality".
- In section 2.5, we show how to construct the canonical extension of a lattice, by using its dual relational structure.

2.1 The category $\mathcal{GDLF}$.

In this section we describe the category $\mathcal{GDLF}$, whose objects, the descriptive general lattice frames, will be the dual objects associated with lattices with operators.

Definition 2.1.1 *A* **semi-lattice frame** *is a structure of the form $\langle X, \leq, R\rangle$, where $\leq$ is a partial order on X and R is a binary relation on X satisfying $\leq \circ R \circ \leq \subseteq R$.*
Relations satisfying this condition will be called increasing.

Definition 2.1.2 *A* **general semi-lattice frame** *is a structure $< X, \leq, D, R >$ such that:*

1. *$\langle X, \leq R\rangle$ is a semi-lattice frame,*
2. *$D \subseteq \mathcal{O}(X)$,*
3. *D is closed under binary intersections,*
4. *$\emptyset, X \in D$,*
5. *The fact that R is increasing guarantees that $m_R^\delta(a) \in O(X)$ for all $a \in D$; we require more of a general semillatice-frame, namely: $m_R^\delta(a) \in D$ for every $a \in D$.*
   The elements of the set D are called the **admissible** *subsets of the general semi-lattice frame.*

The set X is called the **universe** *of the general semi-lattice frame.*

Definition 2.1.3 *A semi-lattice frame $\langle X, \leq, D, R\rangle$ is* **tight** *if and only if:*

- *if $x, y \in X$ are such that $x \not\leq y$, then there is $a \in D$ such that $x \in a$, $y \notin a$,*
- *if $x, y \in X$ are such that $\neg(R\ xy)$, then there is $a \in D$ such that $x \in m_R^\delta(a)$ and $y \notin a$.*

Definition 2.1.4 *Given two partially ordered sets $\langle X_1, \leq_1\rangle$ and $\langle X_2, \leq_s\rangle$ and a relation $\perp \subseteq X_1 \times X_2$, let $A_\perp = \{\langle x, y\rangle \in \perp^c \mid \text{if } x <_1 x' \text{ or } y <_2 y' \text{ then } x' \perp y'\}$ be the set of all maximal $\perp^c$-pairs of H.*

Definition 2.1.5 *A* **descriptive general lattice frame** *is a structure*

$$H =< g_1, g_2, \perp >$$

such that:

1. *$g_1 = \langle X_1, \leq_1, D_1, R_1\rangle$ and $g_2 = \langle X_2, \leq_2, D_2, R_2\rangle$ are tight general semi-lattice frames.*
2. *If $x_1 \perp x_2$, $x_1 \leq_1 y_1$ and $x_2 \leq_2 y_2$, then $y_1 \perp y_2$ (where $\leq_i$ is the partial order of F_i, for $i \in \{1, 2\}$).*
3. *The Galois connection $\langle \rho_\perp, \lambda_\perp\rangle$ between $\mathcal{P}(X_1)$ and $\mathcal{P}(X_2)$ induced by $\perp$ is such that: $\lambda_\perp : D_1 \to D_2$ is a bijection with inverse $\rho_\perp$,*

4. *If $S \subseteq X_i$ is stable, then $m^\delta_{R_i}(S)$ is stable, for $i \in \{1,2\}$.*

5. *If $x_1 \in X_1$ and $x_2 \in X_2$ are such that $x_1 \perp y_2$, then there is $a \in D_1$ such that $x_1 \in a$ and $x_2 \in \lambda a$ (note that this is a a tightness condition for $\perp^c$).*

6. *For all $x \in X_1$ there is $y \in X_2$ such that $< x, y > \in A_\perp$.*
   For all $y \in X_2$ there is $x \in X_1$ such that $< x, y > \in A_\perp$.
   For all $\langle z, t\rangle \in \perp^c$, there is $x \in X_1, y \in X_2$ such that $x \leq_1 x$, $t \leq_2 y$ and $\langle x, y \in A_\perp$.

7. *If $S, T \subseteq D_1$ are such that for every finite S', T', subsets of S and T respectively:*
$$\bigcap S' \not\subseteq \rho_\perp \lambda_\perp (\bigcup T'),$$
   then
$$\bigcap S \not\subseteq \rho_\perp \lambda_\perp (\bigcup T),$$

Condition 3 in definition 2.1.5 guarantees that the elements of D_1 are stable subsets of X_1: let $a \in D_1$, then $\rho_\perp \lambda_\perp (a) = a \in D_1$.

Note also that $\lambda_\perp : D_1 \to D_2$ is order-reversing (cf proposition 1.3.4).

When there is no confusion possible, we will use λ and ρ to denote $\lambda_\perp$ and $\rho_\perp$ respectively.

Condition 7 in definition 2.1.5 will be called **compactness**.
This name is due to the fact that it generalizes the compactness condition of a Priestley Space. We will explain this in detail, in order to clarify what this abstract condition means.
The set D_1 is closed under intersections but not under unions; in order to have a join in D_1 (a structure of lattice for D_1 will be shown later, in lemma 2.1.6), we can define, if $a_i \in D_1$ for every $i \in I$: $\bigsqcup_{\imath \in I} a_i = \rho\lambda \left(\bigcup_{i \in I} a_i\right)$ (that $\bigsqcup_{i \in I} a_i \in D_1$ will be also proved in lemma 2.1.6).
What condition 7 is stating is the following: if $S, T \subseteq D_1$ are such that $\bigcap S \subseteq \bigsqcup T$, then there are $S' \subseteq S$ and $T' \subseteq T$ finite such that $\bigcap S' \subseteq \bigsqcup T'$.
The characterization of compactness of a topological space given in theorem 1.2.34 shows that condition 7 in definition 2.1.5 is a generalization of compactness for descriptive general lattice frames.

Note also that this compactness condition for D_1 implies an analogous one for D_2, namely:
if $S, T \subseteq D_2$ are such that for every finite S', T', subsets of S and T respectively:
$$\bigcap S' \not\subseteq \lambda_\perp \rho_\perp (\bigcup T'),$$
then there is $y \in \bigcap S$ and $x \in \rho(\bigcup T')$ such that $\langle x, y\rangle \in A_\perp$. (This follows straightforward from general properties of Galois connections and the fact that the restricted Galois connection $\lambda : D_1 \leftrightarrow D_2 : \rho$ is an isomorphism.)

Lemma 2.1.6 *Given a descriptive general lattice frame as above, the structure:*
$$\langle D_1, \cap, \sqcup, \emptyset, X_1, m^\delta_{R_1}, \rho m^\delta_{R_2} \lambda\rangle,$$

with $a \sqcup b = \rho\lambda(a \cup b)$ is a lattice with operators.

Remark: The apparent asymmetry of this lemma can be surprising, since definition 2.1.5 is symmetric with respect to g_1 and g_2. In fact, there is no asymmetry: this lemma has of course its dual version stating that the structure $\langle D_2, \cap, \sqcup_2, \emptyset, X_2, m^{\delta}_{R_2}, \lambda m^{\delta}_{R_1}\rho\rangle$, where $a \sqcup_2 b = \lambda\rho(a \cup b)$, is a lattice with operators. For our purposes, we only need the results stated by the lemma.

Proof: We will first prove that $\langle D_1, \cap, \sqcup, \emptyset, X_1\rangle$ is a lattice.
Note that $\rho\lambda(a \cup b) = \rho(\lambda a \cap \lambda b) \in \rho(D_2) = D_1$.
We show now that $\langle D_1, \cap, \sqcup, \emptyset, X_1\rangle$ is a lattice. Take $a, b \in D_1$. We claim that $a \cap (a \sqcup b) = a$. This is because $a \subseteq (a \cup b) \subseteq \rho\lambda(a \cup b) = a \sqcup b$. Moreover, $a \sqcup (a \cap b) = \rho\lambda(a \cup (a \cap b)) = \rho\lambda a = a$, since $a \in D_1$.
It is clear that $a \cap a = a$. We prove now that $a \sqcup a = a$: $a \sqcup a = \rho\lambda(a \cup a) = \rho\lambda a = a$.
Moreover, for any $a \in D_1$, it is clear that $\emptyset \sqcup a = \rho\lambda(\emptyset \cup a) = \rho\lambda(a) = a$ and that $X_1 \cap a = a$.
We have proved then that we have a lattice.
Now we show that D_1 is closed under the operations $m^{\delta}_{R_1}$ and $\rho m^{\delta}_{R_2}\lambda$.
We know that for any $a \in D_1$, $m^{\delta}_{R_1}(a) \in D_1$. We know as well that for any $b \in D_2$, $m^{\delta}_{R_2}(b) \in D_2$; therefore if we take $a \in D_1$, as $\lambda(a) \in D_2$, we have $m^{\delta}_{R_2}(\lambda a) \in D_2$, hence $\rho m^{\delta}_{R_2}\lambda(a) \in D_1$.
We still have to prove that $m^{\delta}_{R_1}$ is $\cap$-preserving and that $\rho m^{\delta}_{R_2}\lambda$ is $\sqcup$-preserving. The first assertion follows from general facts (cf 1.2.13). For the second one, we use the fact that $m^{\delta}_{R_2}$ is $\cap$-preserving (also justified by 1.2.13) and we argue as follows:

$$\begin{aligned}
\rho m^{\delta}_{R_2}\lambda(a \sqcup a') &= \rho m^{\delta}_{R_2}\lambda\rho\lambda(a \cup a') \\
&= \rho m^{\delta}_{R_2}\lambda(a \cup a') \\
&= \rho m^{\delta}_{R_2}(\lambda a \cap \lambda a') \\
&= \rho\left(m^{\delta}_{R_2}\lambda a \cap m^{\delta}_{R_2}\lambda a'\right) \\
&= \rho m^{\delta}_{R_2}\lambda a \sqcup \rho m^{\delta}_{R_2}\lambda a'
\end{aligned}$$

the last equality justified by lemma 1.3.5: using the fact that if $b \in D_1$, then $m^{\delta}_{R_2}\lambda(b) \in D_2$, we get in particular $m^{\delta}_{R_2}\lambda a$ and $m_{R_2}\lambda a'$ are stable. □

The following proposition can be helpful in order to understand the compactness condition imposed to a general lattice-frame. It states that the set D_1 of a descriptive general lattice frame has not "too many" elements but only enough in order to make $\langle X_1, \leq_1\rangle$ tight. This follows essentially from the compactness condition of the frame. Naturally, a similar result can be proved for the set D_2. We need one more definition in order to state the lemma.

Definition 2.1.7 *Let $\langle X, \leq\rangle$ be a partial order. A family D of subsets of X is called a* **separating family** *of $\langle X, \leq\rangle$ if and only if for every $x, y \in X$ such that $x \not\leq y$, there is $a \in D$ such that $x \in a$ and $y \notin a$.*

Proposition 2.1.8 *Let $H = \langle\langle X_1, \leq_1, R_1, D_1\rangle, \langle X_2, \leq_2, R_2, D_2\rangle, \perp\rangle$ is a descriptive general lattice frame. Let $D \subseteq D_1$ be a separating family of the partial order $\langle X_1, \leq_1\rangle$ that is closed under $\cap$ and $\sqcup$, then $D = D_1$.*

Proof: Take an arbitrary $a \in D_1$. We have to show that $a \in D$. Consider x to be an element of a.
For every $y \in a^c$, we have, from the fact that a is increasing, that $x \not\leq_1 y$. There is then $a_{xy} \in D$ such that $x \in a_{xy}$ and $y \notin a_{xy}$.
As for every $y \in a^c$, we have that $y \in a^c_{xy}$, we get $a^c \subseteq \bigcup a^c_{xy}$; that is, $\bigcap_{y\notin a} a_{xy} \subseteq a = \rho\lambda a$ (the last equality comes from the fact that $a \in D_1$).
There are then, from the compactness condition, $y_1, y_2, \cdots y_{n_x} \notin a$ such that $\bigcap_{i=1}^{n_x} a_{xy_i} \subseteq \rho\lambda a = a$.
For each $x \in a$, let $a_x = \bigcap_{i=1}^{n_x} a_{xy_i}$. As D is closed under intersections, we have that for every $x \in a$, $a_x \in D$.
As $x \in a_{xy}$ for every $y \notin a$, we have $x \in a_x$.
Therefore, we get $a \subseteq \bigcup_{x\in a} a_x$ and then $a \subseteq \rho\lambda(\bigcup_{x\in a} a_x)$.
Using again the compactness condition, we obtain $x_1, x_2, \cdots, x_t \in a$ such that $a \subseteq \rho\lambda(a_{x_1} \cup \cdots \cup a_{x_t}) = a_{x_1} \sqcup \cdots \sqcup a_{x_t}$. Let $a' = a_{x_1} \sqcup \cdots \sqcup a_{x_t}$. As D is closed under $\sqcup$ and every $a_x \in D$, we have that $a' \in D$.
We will show that $a = a'$. We already have that $a \subseteq a'$.
On the other hand, we have that for every $x \in a$, $a_x \subseteq a$ and therefore that $a_{x_1} \cup \cdots \cup a_{x_t} \subseteq a$.
As ρ and λ are order reversing (lemma 1.3.4), we have that $\rho\lambda$ is order preserving and therefore $\rho\lambda(a_{x_1} \cup \cdots \cup a_{x_t}) \subseteq \rho\lambda(a)$; that is $a' = a_{x_1} \sqcup \cdots \sqcup a_{x_t} \subseteq a$ and we are done. □

Proposition 2.1.9 *Let $\langle X_1, \leq_1\rangle$ and $\langle X_2, \leq_2\rangle$ be partially ordered sets and let $\perp \subseteq X_1 \times X_2$ such that: if $x_1 \perp x_2$, $x_1 \leq_1 y_1$ and $x_2 \leq_2 y_2$, then $y_1 \perp y_2$. Then the stable subsets of X_1 and of X_2 under the Galois connection induced by $\perp$ are increasing.*

Proof: Suppose $S \subseteq X_1$ is stable. Then, by proposition 1.3.2, $S = \rho(T)$ for some $T \subseteq X_2$. Suppose $s \in S$ and $s \leq_1 s'$. We will show that $s' \in S$. For this, it is enough to take an arbitrary $t \in T$ and prove that $s' \perp t$: given $t \in T$, we have that $s \perp t$ and by the second condition in definition 2.1.5, since $s \leq_1 s'$, we have $s' \perp t$.
We have $s' \perp t$ for every $t \in T$, in other words $s' \in \rho(T) = S$ and we are done.
Of course the proof for stable subsets of X_2 is similar. □

Proposition 2.1.10 *In the context of proposition 2.1.9, principal increasing subsets of X_1 and principal increasing subsets of X_2 are stable (cf. definition 1.2.35).*

Proof: We claim that:

- if $x \in X_1$, then $x\uparrow = \rho\lambda(\{x\})$,
- if $y \in X_2$, then $y\uparrow = \lambda\rho(\{y\})$.

We will prove only the result concerning $x \in X_1$, the other proof being dual.
We want to show that $x\uparrow = \rho\lambda(\{x\})$.

For the inclusion from left to right, we argue as follows: take $y \in X_1$ such that $x \leq y$; by proposition 1.3.2, we have that $x \in \rho\lambda(\{x\})$ and that $\rho\lambda(\{x\})$ is stable; but then, by the previous lemma, $\rho\lambda(\{x\})$ is increasing, and therefore $y \in \rho\lambda(\{x\})$.
For the converse inclusion, suppose we have $y \in \rho\lambda(\{x\})$. We want to show that $x \leq_1 y$. Suppose not; then, by tightness, there is $a \in D_1$ such that $x \in a$ and $y \notin a$. But then $\{x\} \subseteq a$, hence, by corollary **??** and the fact that admissible sets are stable, we have that $\rho\lambda(\{x\}) \subseteq \rho\lambda a = a$ and therefore, as we assume $y \in \rho\lambda(\{x\})$, we get $y \in a$, contradicting the choice of a. We conclude $x \leq_1 y$, that is $y \in x\uparrow$. $\square$

We have defined the objects of the category $\mathcal{DGLF}$. It remains to define its morphisms:

Definition 2.1.11 *Given two descriptive general lattice frames* $H = \langle\langle X_1, \leq_1, D_1, R_1\rangle, \langle X_2, \leq_2, D_2, R_2\rangle, \perp\rangle$ *and* $H' = \langle\langle X_1', \leq_1', D_1', R_1'\rangle, \langle X_2', \leq_2', D_2', R_2'\rangle, \perp'\rangle$,

$(\sigma_1, \sigma_2) : H \to H'$ *is a* **descriptive morphism** *if and only if:*

1. $\sigma_1 : X_1 \to X_1'$ *and* $\sigma_2 : X_2 \to X_2'$ *are order-preserving functions,*
2.
   - $\forall x, y \in X_1$*: if* xR_1y*, then* $\sigma_1(x)R_1'\sigma_1(y)$*,*
   - $\forall x \in X_1, \forall y' \in X_1'$ *if* $\sigma_1(x)R_1'y'$*, then:* $\forall z' \in X_2'$ *such that* $\neg(y'\perp'z')$ *there is* $y \in X_1$ *such that* xR_1y *and* $\neg(\sigma_1(y)\perp'z')$*.*
3.
   - $\forall x, y \in X_2$*: if* xR_2y*, then* $\sigma_2(x)R_2'\sigma_2(y)$*,*
   - $\forall x \in X_2, \forall y' \in X_2'$ *if* $\sigma_2(x)R_2'y'$*, then:* $\forall z' \in X_1'$ *such that* $\neg(z'\perp'y')$ *there is* $y \in X_2$ *such that* xR_2y *and* $\neg(z'\perp'\sigma_2(y))$*.*
4. (a) $\forall x \in x_1, \forall y \in X_2$*, if* $\sigma_1(x)\perp'\sigma_2(y)$*, then* $x\perp y$*,*
   (b) $\forall x \in X_1, \forall t \in X_2'$*, if* $\neg(\sigma_1(x)\perp't)$*, then there is* $y \in X_2$ *such that* $\neg(x\perp y)$ *and* $t \leq_2' \sigma_2(y)$*,*
   (c) $\forall s \in X_1', \forall y \in X_2$*, if* $\neg(s\perp'\sigma_2(y))$*, then there is* $x \in X_1$ *such that* $\neg(x\perp y)$ *and* $s \leq_1' \sigma_1(x)$*,*
5.
   - *for any* $a \in D_1'$*,* $\sigma_1^{-1}(a) \in D_1$*,*
   - *for any* $b \in D_2'$*,* $\sigma_2^{-1}(b) \in D_2$*.*

It is easy to check that if $H = \langle\langle X_1, \leq_1, R_1, D_1\rangle, \langle X_2, \leq_2, R_2, D_2\rangle, \perp\rangle$ is a descriptive general lattice frame, then (id_{X_1}, id_{X_2}) is a descriptive morphism and that it is the identity morphism id_H.

Definition 2.1.12 *The category* $\mathcal{DGLF}$ *has as objects the descriptive general lattice frames and as morphisms the descriptive morphisms.*

Condition 2 is a generalization of a bounded morphism condition of the function σ_1 for the relations R_1 and R_1' and that condition 3 is a generalization of a bounded morphism condition of the function σ_2 for the relations R_2 and R_2' (cf. [G]).
Similarly, condition 4 is a generalization of a bounded morphism condition of the pair (σ_1, σ_2) for $\perp^c$ and $\perp'^c$. We will refer to it as "the bounded morphisms

condition for $\perp^c$ and $\perp'^c$. We will prove here that this condition is equivalent to the fact that the following diagram commutes:

$$\begin{tikzcd}
St(X_1) \arrow[r, shift left, "\lambda"] & St(X_2) \arrow[l, shift left, "\rho"] \\
St(X'_1) \arrow[u, "\sigma_1^{-1}"] \arrow[r, shift left, "\lambda'"] & St(X'_2) \arrow[u, "\sigma_2^{-1}"'] \arrow[l, shift left, "\rho'"]
\end{tikzcd}$$

that is, to the following equalities:

$$\lambda\left(\sigma_1^{-1}(X')\right) = \sigma_2^{-1}\left(\lambda'(X')\right), \quad \forall X' \in St(X'_1)$$

and

$$\rho\left(\sigma_2^{-1}(Y')\right) = \sigma_1^{-1}\left(\rho'(Y')\right), \quad \forall Y' \in St(X'_2),$$

where we denote by $St(Z)$ the set of all stable (by the corresponding Galois connection) subsets of Z and where λ', ρ' denote respectively $\lambda_{\perp'}$ and $\rho_{\perp'}$.

In order to prove the equivalence between the last condition in definition **??** and the commutation of the diagram, we will show the following proposition.

Proposition 2.1.13

1. *Condition 1 in definition 2.1.11 is equivalent to*

$$\sigma_2^{-1}\left(\lambda'(X')\right) \subseteq \lambda\left(\sigma_1^{-1}(X')\right), \textit{ for every } X' \textit{ stable subset of } X'_1$$

   and is also equivalent to

$$\sigma_1^{-1}\left(\rho'(Y')\right) \subseteq \rho\left(\sigma_2^{-1}(X')\right), \textit{ for every } Y' \textit{ stable subset of } X'_2,$$

2. *condition 2 in definition 2.1.11 is equivalent to*

$$\rho\left(\sigma_2^{-1}(Y')\right) \subseteq \sigma_1^{-1}\left(\rho'(Y')\right), \textit{ for every } Y' \textit{ stable subset of } X'_2,$$

3. *condition 3 in definition 2.1.11 is equivalent to*

$$\lambda\left(\sigma_1^{-1}(X')\right) \subseteq \sigma_2^{-1}\left(\lambda'(X')\right) \textit{ for every } X' \textit{ stable subset of } X'_1.$$

Proof:

1. Suppose that condition 1 holds. Let X' be a stable subset of X'_1 and $y \in X_2$ such that $\sigma_2(y) \in \lambda'(X')$. We have to prove that for every $x \in X_1$ such that $\sigma_1(x) \in X'$ we have $x \perp y$: suppose then that we have $x \in X_1$ with $\sigma_1(x) \in X'$; then, as $\sigma_2(y) \in \lambda'(X')$, we have $\sigma_1(x) \perp \sigma_2(y)$ and by condition 1 we deduce $x \perp y$ and we are done.
   For the other direction, suppose now that for any X' stable subset of X'_1, we have

$$\sigma_2^{-1}\left(\lambda'(X')\right) \subseteq \lambda\left(\sigma_1^{-1}(X')\right), \text{ for every } X' \text{ stable subset of } X'_1,$$

   and suppose we have $x \in X_1$ and $y \in X_2$ such that $\sigma_1(x) \perp' \sigma_2(y)$.
   Consider $X' = \sigma_1(x)\uparrow = \{x' \in X'_1 \mid \sigma_1(x) \leq'_1 x'\}$.

By lemma 2.1.10, we have that X' is a stable subset of X_1'.
Moreover we have $x' \perp \sigma_2(y)$ for every $x' \in X'$; in other words $\sigma_2(y) \in \lambda'(X')$, that is, $y \in \sigma_2^{-1}(\lambda'(X'))$. By assumption, we deduce $y \in \lambda\left(\sigma_1^{-1}(X')\right) \subseteq \lambda(\{x\})$, the last inclusion due to the facts that $x \in \sigma_1^{-1}(X')$ and that λ is order-reversing. We have then $y \in \lambda(\{x\})$, that is $x \perp y$ as we wanted.
Similarly we prove that the first condition in **??** is equivalent to

$$\sigma_1^{-1}(\rho'(Y')) \subseteq \rho\left(\sigma_2^{-1}(Y')\right), \text{ for every } Y' \text{ stable subset of } X_2'.$$

2. Suppose now that condition 2 holds and that we have Y' a stable subset of X_2' and $x \in X_1$ such that $x \notin \sigma_1^{-1}(\rho(Y'))$; there is then some $y' \in Y'$ such that $\neg(\sigma_1(x) \perp' y')$. By assumption, there is some $y \in X_2$ such that $\neg(x \perp y)$ and $t \leq \sigma_2(y)$. By proposition 2.1.9, Y' is increasing; hence, we have, since $y' \in Y'$, that $\sigma_2(y) \in Y'$, that is $y \in \sigma_2^{-1}(Y')$. As $\neg(x \perp y)$, we conclude $x \notin \rho\left(\sigma_2^{-1}(Y')\right)$.
   Conversely, suppose that $\rho\left(\sigma_2^{-1}(Y')\right) \subseteq \sigma_1^{-1}(\rho'(Y'))$ for every Y' stable subset of X_2', and that we have $x \in X_1$ and $y' \in X_2'$ such that $\neg(\sigma_1(x) \perp y')$. Take $Y' = \{t' \in X_2' \mid t \leq_2 t'\}$. By lemma 2.1.10, Y' is stable. As $\sigma_1(x) \notin \rho'(Y')$, that is $x \notin \sigma_1^{-1}(\rho'(X'))$, we have $x \notin \rho\left(\sigma_2^{-1}(Y')\right)$; then there is $y \in \sigma_2^{-1}(Y')$ such that $\neg(x \perp y)$. We have then $\neg(x \perp y)$ and $y' \leq_2' \sigma_2(y)$ as we wanted.

3. The third equivalence is proved similarly to the second one.

□

2.2 Dual object of a lattice with operators.

We will now see how to associate with a lattice with operators L a descriptive general lattice frame $H_O(L)$; that is, we will define a function

$$H_O : Ob(\mathcal{LWO}) \to Ob(\mathcal{GDLF}).$$

As we mentioned in the introduction to this chapter, the main idea of this representation of a lattice with operators by means of a descriptive general lattice frame is the following:

1. the "meet-structure" $\langle L, 0, \wedge, g \rangle$ will be represented by a tight general semi-lattice frame $g_1(L)$, whose universe is the set of all optimal filters of L, denoted by $F_*(L)$;

2. the "join-structure" $\langle L, 1, \vee, f \rangle$ will be represented by a tight general semi-lattice frame $g_2(L)$, whose universe is the set of all optimal ideals of L, denoted by $I_*(L)$;

3. there is a link between the meet and the join structures of the lattice with operators given by the fact that for every $a, b \in L$, $a \wedge b = a$ if and only if $a \vee b = b$, i.e. $\wedge$ and $\vee$ induce the same partial order in L. This link is represented by a Galois correspondence induced by the relation $\perp \subseteq F_*(L) \times I_*(L)$ given by $F \perp I$ if and only if $F \cap I \neq \emptyset$;

4. the hole lattice with operators $\langle L, \wedge, \vee, 0, 1, g, f\rangle$ will be therefore represented by the structure $\langle g_1(L), g_2(L), \perp\rangle$, that will be proved to be a descriptive general lattice frame.

This section is about describing this representation in full detail.

Definition 2.2.1 *Given a lattice L, $F_*(L)$ is the set of all optimal filters of L and $I*(L)$ is the set of all optimal ideals of L.*

Definition 2.2.2 *Given a lattice L, a meet-preserving function $g : L \to L$ and a join-preserving function $f : L \to L$, we define the following binary relations:*

- *$Q_g \subseteq F_*(L) \times F_*(L)$ is such that $Q_g FG$ if and only if $g^{-1}(F) \subseteq G$,*
- *$R_f \subseteq I_*(L) \times I_*(L)$ is such that $R_f IJ$ if and only if $f^{-1}(I) \subseteq J$.*

Definition 2.2.3 *Given a lattice with operators L, we define*

1. *$g_1(L) = \langle F_*(L), \subseteq, Q_g, D_1\rangle$, where $D_1 = \{\hat{a}/a \in L\}$ (with $\hat{a} = \{F \in F_*(L)/a \in F\}$)*
2. *$g_2(L) = \langle I_*(L), \subseteq, R_f, D_2\rangle$, where $D_2 = \{\check{a}/a \in L\}$ (with $\check{a} = \{I \in I_*(L)/a \in I\}$),*
3. *$\perp \subseteq F_*(L) \times I*(L)$ such that $F \perp I$ if and only if $F \cap I \neq \emptyset$.*

Lemma 2.2.4 *Let L be a lattice. For any $a, b \in L$:*

1. *$(a \wedge b)\hat{} = \hat{a} \cap \hat{b}$,*
2. *$(a \vee b)\check{} = \check{a} \cap \check{b}$,*
3. *$(g(a))\hat{} = m^{\delta}_{R_g}(\hat{a})$,*
4. *$(f(a))\check{} = m^{\delta}_{R_f}(\check{a})$,*
5. *$\hat{0} = \emptyset$,*
6. *$\hat{1} = F_*(L)$.*

Proof: For 5, we use the fact that optimal filters are proper (and therefore do not contain 0 as an element).
For 6, we use that every filter contains 1 as an element.
We prove now 1 and 3, 2 and 4 being dual.
1. For the inclusion from left to right, take $F \in F_*(L)$ such that $a \wedge b \in F$; as $a \wedge b \leq a$ and $a \wedge b \leq b$, we have $a, b \in F$ and therefore $F \in \hat{a} \cap \hat{b}$.
For the converse inclusion, note that if $a, b \in F$ then $a \wedge b \in F$.
3. For the inclusion from left to right, take $F \in F_*(L)$ such that $g(a) \in F$ and assume that FR_gG. We want to show that $a \in G$. It is clear, since $a \in g^{-1}(F) \subseteq G$. Hence, $F \in m^{\delta}_{R_f}(\hat{a})$.
For the converse inclusion, suppose $F \in m^{\delta}_{R_g}(\hat{a})$. We want to show that $g(a) \in F$. Suppose not; then, $a \notin g^{-1}(F)$. But, as g is meet-preserving, we have (by lemma 1.2.12) that $G = g^{-1}(F)$ is a filter of L and then, applying corollary 1.3.8, we obtain an ideal J that is G-maximal and such that $a \in J$. We can extend the pair $\langle G, J\rangle$ to a maximal filter-ideal pair, $\langle H, J\rangle$. We have then $g^{-1}(F) = G \subseteq H$, that is FR_gH and $a \notin H$, contradicting the assumption. $\square$

Lemma 2.2.5 *Let L be a lattice with operators. If Q_g and R_f are the relations defined in 2.2.2 and $\perp$ is the relation defined in 2.2.3, we have:*

- *if $S \subseteq F_*(L)$ is stable, then $m^{\delta}_{Q_g}(S) \subseteq F_*(L)$ is stable,*
- *if $T \subseteq I_*(L)$ is stable, then $m^{\delta}_{R_f}(T) \subseteq I_*(L)$ is stable.*

Proof: We prove only the first assertion, the other one being dual.
Simplifying notation, we will use in this proof m for $m^{\delta}_{Q_g}$ and λ and ρ for denoting $\lambda_\perp$ and $\rho_\perp$.
Let S be a stable subset of $F_*(L)$. It suffices to prove that $\rho\lambda m(S) \subseteq m(S)$.
Suppose that $F \notin m(S)$. We will find an optimal ideal J of L such that:

- $F \cap J = \emptyset$,
- for every $H \in m(S)$, $H \cap J \neq \emptyset$.

This will show that $F \notin \rho\lambda m(S)$ and we will be done.

As $F \notin m(S)$, there is an optimal filter $F' \notin S$ such that $Q_g FF'$, i.e. such that $g^{-1}(F) \subseteq F'$.
As S is stable, we have that $S = \rho\lambda(S)$ and therefore $F' \notin \rho\lambda(S)$. Hence, there is an ideal $I' \in \lambda(S)$ such that $F' \cap I' = \emptyset$ and therefore $g^{-1}(F) \cap I' = \emptyset$.
Let K be the ideal generated by $g[I']$.
We claim that $F \cap K = \emptyset$. Suppose not, then there is $b \in F$ such that $b \leq g(a_1) \vee g(a_2) \vee \cdots \vee g(a_n)$, for some $a_1, a_2, \cdots, a_n \in I'$. As g is meet-preserving, we have, by lemma 1.2.12, that it is order preserving and therefore $g(a_i) \leq g(a_1 \vee a_2 \vee \cdots \vee a_n)$, for every $i \in \{1, 2, \cdots, n\}$; hence, $g(a_1) \vee g(a_2) \vee \cdots \vee g(a_n) \leq g(a_1 \vee a_2 \vee \cdots \vee a_n)$. Then we have that $b \leq g(a_1 \vee a_2 \vee \cdots \vee a_n)$ and, as $b \in F$ and every filter is increasing, we have that $g(a_1 \vee a_2 \vee \cdots \vee a_n) \in F$, i.e. $a_1 \vee a_2 \vee \cdots \vee a_n \in g^{-1}(F)$. On the other hand, as every $a_i \in I'$, we have that $a_1 \vee a_2 \vee \cdots \vee a_n \in I'$. But this is a contradiction with the fact that $g^{-1}(F) \cap I' = \emptyset$. This shows that $F \cap K = \emptyset$.
Then, by theorem 1.3.7, there is an ideal J that is F-maximal and such that $K \subseteq J$. We will prove that J is the optimal ideal we are looking for.
We have already that $F \cap J = \emptyset$.
We have to prove that for every $H \in m(S)$, $H \cap J \neq \emptyset$. In other words, we have to prove that if $H \cap J = \emptyset$, then $H \notin m(S)$. Suppose then that H is an optimal filter of L such that $H \cap J = \emptyset$. Then $g^{-1}(H) \cap g^{-1}(J) = \emptyset$. But $I' \subseteq g^{-1}(J)$, hence $g^{-1}(H) \cap I' = \emptyset$. As g is meet-preserving and H is a filter, we have (by lemma 1.2.12) that $g^{-1}(H)$ is a filter. Applying once again theorem 1.3.7, we obtain an optimal filter H' containing $g^{-1}(H)$ as a subset and disjoint from I'. But $I' \in \lambda(S)$; we have then, from $H \cap I' = \emptyset$, that $H' \notin S$. On the other hand, as $g^{-1}(H) \subseteq H'$, we have that $Q_g HH'$, so we can conclude that $H \notin m(S)$ as we wanted.
Hence, we have proved that $F \notin \rho\lambda m(S)$. □

Lemma 2.2.6 *Let L be a lattice and let $\perp \subseteq F_*(L) \times I_*(L)$ be defined by $F \perp I$ if and only if $F \cap I \neq \emptyset$. Then, for every $a \in L$, we have:*

$$\lambda_\perp(\hat{a}) = \check{a} \text{ and } \rho_\perp(\check{a}) = \hat{a}.$$

Proof: We prove only the first equality, the second one being dual. We will use λ and ρ to denote $\lambda_\perp$ and $\rho_\perp$ respectively.
For the inclusion from left to right, take $I \in \lambda(\hat{a})$. We have to prove that $I \in \check{a}$, in other words that $a \in I$. Suppose not; then, by corollary 1.3.8, there is an optimal filter F such that $a \in F$ and $F \cap I = \emptyset$. But from $a \in F$, we have $F \in \hat{a}$, and from $F \cap I = \emptyset$, we have $\neg(F \perp I)$; hence $I \notin \lambda(\hat{a})$, contradicting the assumption.
For the converse inclusion, suppose $I \in \check{a}$, i.e. $a \in I$. We have to prove that $I \in \lambda(\hat{a})$. Take an arbitrary $F \in \hat{a}$; then $a \in F$ and, as $a \in I$, we get $F \cap I \neq \emptyset$. For any $F \in \hat{a}$, we got $F \cap I \neq \emptyset$; hence, $I \in \lambda(\hat{a})$. $\square$

Lemma 2.2.7 *If $a \neq a'$, then $\hat{a} \neq \hat{a}'$.*

Proof: Without loss of generality, we can assume that $a \not\leq b$ and therefore $b \notin a\uparrow$. By corollary 1.3.8, there is an optimal filter F of L such that $a \in F$ and $b \notin F$; that is, $F \in \hat{a}$ and $F \notin \hat{b}$; hence, $\hat{a} \neq \hat{b}$ and we are done. $\square$

The following proposition states that the lattice structure impose to D_1 in proposition 2.1.6 makes D_1 isomorphic to L.

Proposition 2.2.8 *Let L be a lattice with operators and $D_1 = \{\hat{a} \mid a \in L\}$ endowed with the lattice structure given in lemma 2.1.6.*
Then the map $\nu : L \to D_1$, given by $\nu(a) = \hat{a}$, is an isomorphism of lattices.

Proof: We prove first that ν is a morphism of lattices. In each item of the proof we use the corresponding item of lemma 2.2.4.

1. $\nu(a \wedge b) = \widehat{a \wedge b} = \hat{a} \cap \hat{b} = \nu(a) \cap \nu(b)$,
2. $\nu(a \vee b) = \widehat{a \vee b} = \rho\left((a \vee b)\check{}\right) = \rho(\check{a} \cap \check{b}) = \rho(\check{a}) \sqcup \rho(\check{b}) = \hat{a} \sqcup \hat{b}$ (for the second and the last equalities we use lemma 2.2.6, for the third one we use lemma 2.2.4 and for the fourth one we use lemma 1.3.5),
3. $\nu\left(g(a)\right) = \widehat{g(a)} = m^{\delta}_{R_g}(\hat{a})$,
4. $\nu\left(f(a)\right) = \widehat{f(a)} = \rho\left((f(a)\check{}\right) = \rho\left(m^{\delta}_{R_f}(\check{a})\right) = \rho m^{\delta}_{R_f}\lambda(\hat{a})$, (here we use again lemma 2.2.6 for the second and the last equalities, lemma 2.2.4 for the third one),
5. $\nu(0) = \hat{0} = \emptyset$,
6. $\nu(1) = \hat{1} = F_*(L)$.

We prove now that ν is bijective. Surjectivity is clear, since every element in D_1 is, by definition, of the form $\hat{a} = \nu(a)$.
Injectivity follows immediately from lemma 2.2.7. $\square$

Proposition 2.2.9 *The structure $H_O(L) = \langle g_1(L), g_2(L), \perp\rangle$ is a descriptive general lattice frame.*

Proof: We follow the same enumeration than in definition 2.1.5.We use λ and ρ for $\lambda_\perp$ and $\rho_\perp$.

1. We prove only that $g_1(L)$ is a tight general semi-lattice frame, the proof for $g_2(L)$ being similar.
We prove first that $g_1(L)$ is a general semi-lattice frame.

   - It is clear that $\subseteq$ is a partial order on $F_p(L)$. In order to prove that R_g is increasing, suppose that $F \subseteq F' R_g G \subseteq G'$. Then $g^{-1}(F) \subseteq g^{-1}(F') \subseteq G \subseteq G'$ and therefore $F R_g G'$ and we are done.
   We have proved that $\langle F_*(L), \subseteq, R\rangle$ is a semi-lattice frame.
   - We prove here that every element of D_1 is increasing: for $a \in L$, consider $F \in \hat{a}$ and G an optimal filter such that $F \subseteq G$. Then we have that $a \in F \subseteq G$, hence $a \in G$, that is $G \in \hat{a}$. We have proved for every $a \in L$ that $\hat{a}$ is increasing.
   - By lemma 2.2.4 we have $\hat{a} \cap \hat{b} = \widehat{a \wedge b} \in D_1$. We have proved that D_1 is closed under taking binary intersections.
   - Also by lemma 2.2.4, $\emptyset = \hat{0} \in D_1$, $F_*(L) = \hat{1} \in D_1$.
   - Using once again lemma 2.2.4, we have that $m^{\delta}_{R_g}(\hat{a}) = \widehat{g(a)} \in D_1$.

   This proves that $g_1(L)$ is a general semi-lattice frame. We will show now that it is tight.
   We prove first the tightness condition for $\subseteq$. Let F, G be optimal filters of L such that $F \not\subseteq G$. Then there is $a \in L$ such that $a \in F$, $a \notin G$, and therefore $F \in \hat{a}$, $G \notin \hat{a}$ as we wanted.
   For the tightness condition for R_g, consider optimal filters F, G of L such that $\neg(F R_g G)$. Then $g^{-1}(F) \not\subseteq G$; that is, there is $a \in L$ such that $g(a) \in F$ and $a \notin G$. But $m^{\delta}_{R_g}(\hat{a}) = \widehat{g(a)}$; hence, we have:

   $$F \in \widehat{g(a)} = m^{\delta}_{R_g}(\hat{a}); \qquad G \notin a,$$

   as we needed.

2. Properties of $\perp$:
Let F, G be optimal filters and I, J be optimal ideals of L such that $F \perp I$, $F \subseteq G$ and $I \subseteq J$. From $F \perp I$, we have $F \cap I \neq \emptyset$. From $F \subseteq G$ and $I \subseteq J$, we have $F \cap I \subseteq G \cap J$ and therefore $G \cap J \neq \emptyset$, that is $G \perp J$.

3. Properties referring to D_1 and D_2:
From lemma 2.2.6, we have $\lambda(\hat{a}) = \check{a}$ and $\rho(\check{a}) = \hat{a}$. We deduce that λ restricted to D_1 is a bijection with inverse ρ.

4. By lemma 2.2.5, we have that $m^{\delta}_{Q_g}$ and $m^{\delta}_{R_f}$ take stable sets into stable sets.

5. Tightness condition for $\perp^c$:
Suppose we have F and I optimal filter and ideal of L respectively, such that $F \perp I$. Then $F \cap I \neq \emptyset$, i.e. there is $a \in L$ such that $a \in F \cap I$. Then $F \in \hat{a}$ and $I \in \check{a} = \lambda(\hat{a})$ and we are done.

6. The set $A_\perp$ is here the set of all maximal filter-ideal pairs.
The properties referring to $A_\perp$ follow straightforward from the definition of a maximal filter-ideal pair.
Moreover

7. Compactness condition:
We will prove only the part concerning D_1, since the other part can be proved dually. Consider then $S, T \subseteq D_1$ such that for any finite $\bar{S}, \bar{T}$ subsets of S and T respectively, the following holds:
$$\bigcap \bar{S} \not\subseteq \rho\lambda\left(\bigcup \bar{T}\right).$$
Let $S^* = \{a \in L \mid \hat{a} \in S\}$ and $T^* = \{a \in L \mid \hat{a} \in T\}$.
Consider the filter F of L generated by S^* and the ideal I of L generated by T^*.
We claim that condition $(*)$ implies that $F \cap I = \emptyset$. For suppose not; then there exist $a_1, \cdots, a_n \in S^*$ and $b_1, \cdots, b_m \in T^*$ such that $a_1 \wedge \cdots \wedge a_n \leq b_1 \vee \cdots \vee b_m$.
From this it follows that every filter containing $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ also contains $b_1 \vee b_2 \vee \cdots \vee b_m$, in other words:
$$(a_1 \wedge \cdots \wedge a_n)\hat{} \subseteq (b_1 \vee \cdots \vee b_m)\hat{}$$
and, applying proposition 2.2.8, we get
$$\hat{a_1} \cap \cdots \cap \hat{a_n} \subseteq \hat{b_1} \sqcup \cdots \sqcup \hat{b_m},$$
contradicting $(*)$ for $\bar{S} = \{\hat{a_1}, \cdots, \hat{a_n}\}$ and $\bar{T} = \{\hat{b_1}, \cdots, \hat{b_m}\}$.
We have proved that the filter F of L generated by S^* and the ideal I of L generated by T^* are disjoint. There exist then, by theorem 1.3.7, $G \in F_*(L)$ and $J \in I_*(L)$ such that $\langle G, J\rangle$ is a maximal ideal pair, and $F \subseteq G$, $I \subseteq J$.
We will prove that $G \in \bigcap S$ and that $G \notin \bigsqcup T$.
We know that for every $\hat{a} \in S$, we have $a \in F$, then $a \in G$, that is $G \in \hat{a}$. So $G \in \bigcap S$.
On the other hand, we know that for $\hat{b} \in T$, we have $b \in I$, then $b \in J$, that is $J \in \check{b} = \lambda(\hat{b})$ and therefore $J \in \bigcap \lambda[T]$. Then, as $G \cap J = \emptyset$, $G \notin \rho(\bigcap \lambda[T]) = \rho\lambda(\bigcup T) = \bigsqcup T$. and we are done.

Hence, we have proved that $H(L)$ is a descriptive general lattice frame. □

2.3 Dual arrow of an arrow in $\mathcal{LWO}$.

In section 2.2, we have associated, with every lattice with operators L, an object $H_O(L)$ of the category $\mathcal{DGLF}$.
In this section, we will associate a descriptive morphism $H_M(\phi)$ with every surjective morphism of lattices ϕ. Unfortunately, as we mentioned before, we can not do this for every morphism of lattices, but only for the surjective ones.
We will prove that
$$H = (H_O, H_M) : \mathcal{LWO}_e \to \mathcal{DGLF}_m$$
is a contravariant functor, where $\mathcal{LWO}_e$ is the category of lattices with operators with epimorphisms of lattices with operators and $\mathcal{DGLF}_m$ is the category of descriptive general lattice frames with descriptive monomorphisms.

Definition 2.3.1 *An* **epimorphism of lattices with operators** *is an epimorphism in $\mathcal{LWO}$.*
A **descriptive monomorphism** *is a monomorphism in $\mathcal{DGLF}$.*

As $\mathcal{LWO}$ is a concrete category, we have, by proposition 1.2.48, that the epimorphisms of lattices with operators are the surjective morphisms of lattices with operators.
It can be checked that the descriptive monomorphisms are the descriptive morphisms (ϕ_1^*, ϕ_2^*) such that both ϕ_1^* and ϕ_2^* are injective.
The following proposition shows how to associate with an epimorphism of lattice a descriptive monomorphism.
The following lemma follows trivially from the definition of a morphism of lattices; we skip the proof.

Lemma 2.3.2 *Let $\langle L, \wedge, \vee, 0, 1, g, f\rangle$ and $\langle L', \wedge', \vee', 0', 1', g', f'\rangle$ be lattices with operators and let $\phi : L \to L'$ be a morphism of lattices.*
Then, for any $X' subseteq L'$, we have:

$$g^{-1}(\phi^{-1}(X')) = \phi^{-1}(g'^{-1}(X')) \text{ and } f^{-1}(\phi^{-1}(X')) = \phi^{-1}(f'^{-1}(X'))$$

Proposition 2.3.3 *Given an epimorphism of lattices with operators*

$$\phi : L \to L',$$

the map

$$\phi^* = (\phi_1^*, \phi_2^*) : H(L') \to H(L),$$

is a descriptive monomorphism, where

$$\phi_1^* : F_*(L') \to F_*(L), \text{ is given by } \phi_1^*(F') = \phi^{-1}(F')$$

and

$$\phi_2^* : I_*(L') \to I_*(L), \text{ is given by } \phi_2^*(I') = \phi^{-1}(I').$$

Proof: Let $\phi : L \to L'$ be an epimorphism of lattices.
We first prove that ϕ^* is well-defined.
Let F' be an optimal filter of L'. Then there is an ideal I' such that $\langle F', I'\rangle$ is a maximal filter-ideal pair of L'. We will show that $\langle \phi^{-1}(F), \phi^{-1}(I)\rangle$ is a maximal filter-ideal pair of L.
By 1.2.12, the fact that ϕ is $\wedge$-preserving guarantees that $F = \phi^{-1}(F')$ is a filter of L and the fact that ϕ is $\vee$-preserving guarantees that $I = \phi^{-1}(I)$ is an ideal of L. Moreover, suppose $x \in F \cap I$ then $\phi(x) \in F' \cap I'$, contradicting the fact that F' and I' are disjoint. Hence, $F \cap I = \emptyset$.
We will prove that F is I-maximal.
As ϕ is surjective and $I' \neq \emptyset$, we have $I \neq \emptyset$ and therefore $F \neq L$; moreover, as $F' \neq \emptyset$, we have $F \neq \emptyset$; that is, F is proper.
Consider now an optimal filter G of L such that $F \subsetneq G$. We will show that $G \cap I \neq \emptyset$. Take $a \in L, a \in G \setminus F$; then $\phi(a) \notin F'$. Let G' be the filter generated by $F' \cup \{\phi(a)\}$. As $F' \subsetneq G'$, by maximality of the pair $\langle F', I'\rangle$, we have that $G' \cap I' \neq \emptyset$. There is then an element $b' \in F'$ such that $b' \wedge \phi(a) \in I'$. But ϕ is surjective, therefore there is $b \in L$ such that $\phi(b) = b'$. By definition of F, we

have $b \in F$. As $\phi(b \wedge a) = \phi(b) \wedge \phi(a) = b' \wedge \phi(a) \in I$, we conclude $b \wedge a \in I'$. But $b \in F$, hence $b \wedge a \in G$ and therefore $G \cap I \neq \emptyset$. Hence F is I-maximal.
It can be proved dually that I is F-maximal. Therefore $\langle F, I \rangle$ is a maximal filter-ideal pair of L and the map ϕ^* is well-defined.

We prove now that ϕ^* is a descriptive morphism, following the enumeration of definition 2.1.11.

1. Clearly, $\phi_1^* : F_*(L') \to F_*(L)$ and $\phi_2^* : I_*(L') \to I_*(L)$ are order-preserving maps.

2. We prove now the bounded morphism condition of σ_2 for the relations Q_g and $Q_{g'}$.
   First observe that from the fact that for every $a \in L$, $\phi(g(a)) = g'(\phi(a))$, it is easy to infer that for any optimal filter F' of L' we have that
   $$g^{-1}\left(\phi_1^*(F')\right) = \phi_1^*\left(g'^{-1}(F')\right).$$
   - Let F', G' be optimal filters of L' such that $Q_{g'}F'G'$. Then $g'^{-1}(F') \subseteq G'$. We have to show that $Q_g\phi_1^*(F')\phi_1^*(G')$, i.e. that $g^{-1}\left(\phi^{-1}(F')\right) \subseteq \phi^{-1}(G')$.
     Let $a \in g^{-1}\left(\phi^{-1}(F')\right)$. As $g^{-1}\left(\phi_1^*(F')\right) = \phi_1^*\left(g'^{-1}(F')\right) \subseteq \phi^{-1}(G')$, we have that $a \in \phi^{-1}(G')$ as we wanted.

   - Suppose now that F' is an optimal filter of L' and G is an optimal filter of L such that $Q_g\phi_1^*(F')G$. For every ideal J such that $G \cap J = \emptyset$, we have to find an optimal filter G' of L' such that $Q_{g'}F'G'$ and $\phi_1^*(G') \cap J = \emptyset$.
     to complete.
     Suppose now that F' is an optimal filter of L' and G is an optimal filter of L such that $Q_g\phi_1^*(F')G$. Take an ideal J such that $G \cap J = \emptyset$. Consider the ideal J' of L' generated by $\phi[J]$. We will show first that $g'^{-1}(F') \cap J' = \emptyset$.
     In order to get a contradiction, suppose there are $a_1, a_2, \cdots a_n \in J$ and $b \in g'^{-1}(F')$ such that $b \leq \phi(a_1) \vee \cdots \vee \phi(a_n)$. As ϕ is order preserving, we have that $b \leq \phi(a_1 \vee \cdots \vee a_n)$. But g' being meet-preserving, we now that $g'^{-1}(F')$ is a filter and hence increasing. We have then that $\phi(a_1 \vee \cdots \vee a_n) \in g^{-1}(F')$; that is, $a_1 \vee \cdots \vee a_n \in \phi^{-1}(g'^{-1}(F')) = g^{-1}(\phi^{-1}(F'))$ by lemma 2.3.2. But we know, by the fact that $Q_g\phi_1^*(F')G$, that $g^{-1}(\phi^{-1}(F')) \subseteq G$ and therefore we have $J \cap G \neq \emptyset$, contradicting the assumption.
     We have proved that $g'^{-1}(F') \cap J' = \emptyset$. There is then an optimal ideal G' containing $g'^{-1}(F')$ and an optimal filter K' containing J' such that $G' \cap K' = \emptyset$. We have then $Q_{g'}F'G'$.
     It remains to prove that $\phi^{-1}(G') \cap J = \emptyset$. From $G' \cap K' = \emptyset$, we get $\phi^{-1}(G') \cap \phi^{-1}(K') = \emptyset$. As $J \subseteq \phi^{-1}(J') \subseteq \phi^{-1}(K')$, we conclude that $\phi^{-1}(G') \cap J = \emptyset$ and we are done.

3. The bounded morphism condition of σ_2 for the relations R_f and R_f' has a dual proof that we skip.

4. We prove now the bounded morphism condition for $\perp^c$ and $\perp'^c$, namely that the following diagram commutes:

$$\begin{tikzcd} St\left(F_*(L')\right) \arrow[r, leftrightarrow, "\lambda", "\rho"'] & St\left(I_*(L')\right) \\ St(F_*(L) \arrow[u, "(\phi_1^*)^{-1}"] \arrow[r, leftrightarrow, "\lambda'", "\rho'"'] & St(I_*(L)). \arrow[u, "(\phi_2^*)^{-1}"'] \end{tikzcd}$$

We will check only that

$$\lambda\left((\phi_1^*)^{-1}(X)\right) = (\phi_2^*)^{-1}\left(\lambda'(X)\right), \quad \forall X \in St(F_*(L);$$

the other equality can be proved similarly.
Inclusion from left to right: take X stable subset of $F_*(L)$ and take an optimal ideal I' of L' such that for every F': if $\phi_1^*(F') \in X$, then $F' \cap I' \neq \emptyset$. We have to prove that $\phi_2^*(I') \in \lambda'(X)$.
Take $F \in X$ and let F' be the filter generated by $\phi[F]$. We will show that $F \cap \phi^{-1}(I') \neq \emptyset$. First note that $F \subseteq \phi^{-1}(F')$. As X is stable, by proposition 2.1.9 it is increasing, and as $F \in X$ we have that $\phi^{-1}(F') = \phi_1^*(F') \in X$. Hence, by assumption, $F' \cap I' = \emptyset$ and therefore $\phi^{-1}(F') \cap \phi^{-1}(I') = \emptyset$. As $F \subseteq \phi^{-1}(F')$, we have also $F \cap \phi^{-1}(I') = \emptyset$.
Inclusion from right to left: take again X stable subset of $F_*(L)$ and take an optimal ideal I' such that $F \cap \phi_2^*(I') \neq \emptyset$ for every $F \in X$. We have to prove that if F' is such that $\phi_1^*(F') \in X$ then $F' \cap I' \neq \emptyset$.
Take such an F'. As $\phi_1^*(F') \in X$, we have $\phi_1^*(F') \cap \phi_2^*(I') \neq \emptyset$, that is there is $x \in L$ such that $\phi(x) \in F' \cap I'$ and therefore $F' \cap I' \neq \emptyset$ as we wanted.

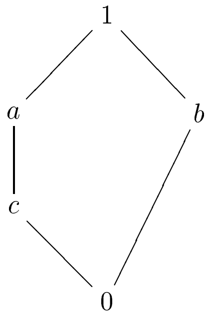
5. Finally, we have to show that if $\hat{a} \in D_1$, then there is some $a' \in L'$ such that $(\phi_1^*)^{-1}(\hat{a}) = \hat{a'}$ (the analogous condition for D_2 and D_2' has a similar proof).
Consider $a \in L$, we claim that $(\phi_1^*)^{-1}(\hat{a}) = \widehat{\phi(a)} \in D_1'$ and if $\check{a} \in D_2$, then $(\phi_2^*)^{-1}(\check{a}) = (\phi(a))\check{} \in D_2'$. We show only the first equality, the second one having a dual proof.
Inclusion from left to right: Take an optimal filter F' of L' such that $\phi_1^*(F') \in \hat{a}$; then $a \in \phi^{-1}(F')$. Hence, we have that $\phi(a) \in F'$, in other words $F' \in \widehat{\phi(a)}$.
For the converse inclusion, suppose that we have an optimal filter F' of L' with $F' \notin (\phi_1^*)^{-1}(\hat{a})$; then $a \notin \phi_1^*(F') = \phi^{-1}(F')$. Hence, $\phi(a) \notin F'$ and we are done.

We have proved that (ϕ_1^*, ϕ_2^*) is a descriptive morphism. It remains to prove that ϕ_1^* and ϕ_2^* are injective: we prove it only for ϕ_1^*, the proof for ϕ_2^* being similar.
For this aim, we will show that if $F', G' \in F_*(L')$ are such that $F' \nsubseteq G'$, then $\phi_1^*(F') \nsubseteq \phi_1^*(G')$.
Suppose then $F' \nsubseteq G'$. Then, there is $a' \in F'$ such that $a' \notin G'$. But $\phi : L \to L'$ is surjective, hence we have some $a \in L$ such that $\phi(a) = a'$. But $a \in \phi^{-1}(F') = \phi_1^*(F')$ and also $\phi(a) \notin G'$, i.e. $a \notin \phi^{-1}(G') = \phi_1^*(G')$ and

we are done. $\square$

It would be more satisfying to drop the condition of surjectivity for the arrows in $\mathcal{LWO}$. Unfortunately, this is not possible in order to define our dual arrow in a natural way, since the set of preimages of an optimal filter under a morphism of lattices need not be an optimal filter. We present here a counterexample.

Consider the following lattice L.



and the morphism $h : L \to L$ given by:

$$h(c) = h(a) = c; \quad h(b) = b; \quad h(1) = 1; \quad h(0) = 0.$$

It can be readily checked that h is a morphism of lattices and that, if

$$I = \{0, c\} \quad ; \quad F = \{1, a\}.$$

then $\langle F, I\rangle$ is a maximal filter-ideal pair of L and therefore F is an optimal filter. But $h^{-1}(F) = \{1\}$. We will see that $h^{-1}(F)$ is not an optimal filter of L.
Take an arbitrary ideal J such that $\{1\} \cap J = \emptyset$, i.e. $1 \notin J$. We will prove that there is a proper filter G such that $F \subsetneq G$ and $G \cap J$. This will show that F is not J-maximal, for any J and therefore not optimal.
First suppose $a \in J$. As $1 = a \vee b \notin J$, we can deduce that $b \notin J$. We can then extend $\{1\}$ to the filter $G = \{1, b\}$ disjoint from J.
Suppose now that $a \notin J$. Then, similarly, the filter $G = \{1, a\}$ is a proper extension of $\{1\}$ disjoint from J and we are done. $\square$

Definition 2.3.4 *The category $\mathcal{LWO}_e$ has as objects the lattices with operators and as morphisms the distributive lattices with operators.*
The category $\mathcal{DGLF}_m$ has as objects the descriptive general lattice frames and as morphisms the descriptive monomorphisms.

It is easy to check that if $\phi : L \to L'$ and $\psi : L' \to L''$ are epimorphisms of distributive lattices, then:

$$\phi_1^* \circ \psi_1^* = (\psi \circ \phi)_1^*, \qquad \phi_2^* \circ \psi_2^* = (\psi \circ \phi)_2^*$$

and

$$(id_L)^* = \left(id_{F_*(L)}, id_{I*(L)}\right);$$

that is, the following proposition:

Proposition 2.3.5

$$\langle H_O, H_M \rangle : \mathcal{LWO}_e \to \mathcal{DGLF}_m$$

is a partial **contravariant** *functor.*

We will refer often to H as a "partial" functor in $\mathcal{LWO}$.

2.4 Partial duality.

In this section we will prove that the functor

$$H = (H_O, H_M) : \mathcal{LWO}_e \to \mathcal{DGLF}_m$$

defined in sections 2.2 and 2.3 is faithful and dense. Moreover, we prove some "fullness": every descriptive morphism arises from a morphism of lattices with operators. In order to have a duality, it is necessary to prove that in case that the descriptive morphism is a monomorphism, the morphism of lattices with operators giving rise to it is surjective; we could not prove this fact.
That is why we decided to call H a "partial duality".

Theorem 2.4.1 *H is faithful; in other words, if $\phi, \psi : L \to L'$ are morphisms of lattice with operators such that $\phi \neq \psi$, then $\phi^* \neq \psi^*$.*

Proof: By assumption, there is $a \in L$ such that $\phi(a) \neq \psi(a)$. Without loss of generalization, we can assume that $\phi(a) \not\leq \psi(a)$. We have then that $\psi(a) \notin \phi(a)\!\downarrow$. Hence, by corollary 1.3.8, there is a filter F that is $\phi(a)\!\downarrow$-maximal and such that $\psi(a) \in F$. F is then an optimal filter such that $\phi(a) \in F$ and $\psi(a) \notin F$; that is, $a \in \phi^{-1}(F) = \phi_1^*(F)$ and $a \notin \psi^{-1}(F) = \psi_1^*(F)$. We conclude $\phi_1^* \neq \phi_2^*$ and therefore $(\phi_1^*, \phi_2^*) \neq (\psi_1^*, \psi_2^*)$, i.e. $\phi^* \neq \psi^*$ and we are done. □

Theorem 2.4.2 *H is dense; in other words, for every descriptive general lattice frame H, there is a lattice with operators L such that $H \cong H_O(L)$.*

Proof: Let $H = \langle\langle X_1, \leq_1, D_1, R_1\rangle, \langle X_2, \leq_2, D_2, R_2\rangle, \perp\rangle$ be a general lattice frame.
Consider the structure

$$\mathcal{D}_1 = \langle D_1, \cap, \sqcup, \emptyset, X_1, m^{\delta}_{R_1}, \rho m^{\delta}_{R_2} \lambda\rangle.$$

We know, by lemma 2.1.6, that $\mathcal{D}_1$ is a lattice with operators. From now on, we will denote it by D_1.
We will prove that $H \cong H_O(D_1)$.
In order to do this, consider the maps $h_1 : X_1 \to \mathcal{P}(D_1)$ and $h_2 : X_2 \to \mathcal{P}(D_1)$, given by:

$$h_1(x) = \{a \in D_1 \mid x \in a\} \text{ and } h_2(y) = \{b \in D_1 \mid y \in \lambda(b)\}.$$

We claim that $(h_1, h_2) : H \to H_O(D_1)$ is a descriptive isomorphism; we have show this in the following order:

- first we prove that $F_*(L) \subseteq Im(h_1)$ and that $I_*(L) \subseteq Im(h_2)$,

- then we prove that $Im(h_1) \subseteq F_*(L)$ and that $Im(h_2) \subseteq I_*(L)$,
- (h_1, h_2) is a descriptive morphism,
- h_1 and h_2 are injective,
- h_1 and h_2 are surjective.

We will prove only the results concerning h_1, the results concerning h_2 having a dual proof.

- Consider F to be an optimal filter of D_1. Consider also an optimal ideal I such that $\langle F, I\rangle$ is a maximal filter-ideal pair.
  Then we have that for any finite $\bar{F} \subseteq F$, $\bar{I} \subseteq I$, the following holds:
  $$\bigcap \bar{F} \not\subseteq \bigsqcup \bar{I}.$$
  By the compactness condition for H we have that there are $x \in \bigcap F$ and $y \in \bigcap \lambda[I]$ such that $\langle x, y\rangle \in A_\perp$, where A is the set of maximal $\perp^c$-pairs of H.
  We claim that $F = h_1(x)$ and $I = h_2(y)$. The inclusion $F \subseteq h_1(x)$ is because $a \in F$, implies $x \in a$ and therefore $a \in h_1(x)$. For showing that $I \subseteq h_2(y)$, take $b \in I$; we have then $y \in \lambda(b)$ and therefore $\rho\lambda(b) = b \in h_2(y)$ and we are done.
  We claim that $h_1(x)$ is a filter and that $h_2(y)$ is an ideal. We prove only the second assertion, the first one being trivial. Take $b, b' \in h_2(y)$; then $y \in \lambda(b) \cap \lambda(b') = \lambda(b \cup b')$ by proposition 1.3.4; but $\lambda(b \cup b') = \lambda\rho\lambda(b \cup b') = \lambda(b \sqcup b')$ and therefore $b \sqcup b' \in I$. Suppose now that $b \in h_2(y)$ and that $a \in D_1$ is such that $a \subseteq b$. By proposition 1.3.4, $\lambda(b) \subseteq \lambda(a)$. As $b \in h_2(y)$, we have $y \in \lambda(b)$ and therefore $y \in \lambda(a)$. We conclude $a \in h_2(y)$.
  Now, we have that $\langle F, I\rangle$ is a maximal pair. Suppose that $F \subsetneq h_1(x)$. As $h_1(x)$ is a filter we have $h_1(x) \cap I \neq \emptyset$ that is, there is $a \in D_1$ such that $x \in a$ and $y \in \lambda(a)$. We deduce that $x \perp y$ and this contradicts the fact that $\langle x, y\rangle \in A_\perp$. So $F = h_1(x)$ and in a dual way we can proved that $I = h_2(x)$.

- Let us see now that $h_1(x)$ is in effect an optimal filter of D_1.
  We know that $h_1(x)$ is a filter for every $x \in X_1$ and $h_2(y)$ is an ideal for every $y \in X_2$. We have to prove that they are optimal.
  Given $x \in X_1$, take $y \in X_2$ such that $\langle x, y\rangle \in A_\perp$. We will show that $h_1(x)$ is $h_2(y)$- maximal (and therefore it is optimal).
  We claim that $h_1(x) \cap h_2(y) = \emptyset$: suppose not, then there is $a \in D_1$ such that $x \in a$ and $y \in \lambda a$: hence $x \perp y$ contradicting the assumption on x, y.
  Suppose $h_1(x)$ is not $h_2(y)$-maximal. Then, by theorem 1.3.7, it is strictly contained in an $h_2(y)$-maximal filter F. From the fact that F is optima and from the previous item, we conclude that for some $z \in X_1$ we have $F = h_1(z)$.
  That is, we are assuming $h_1(x) \subseteq h_1(z)$.
  From this inclusion we get that $x \leq z$ (suppose $x \not\leq z$; then for some $a \in D_1$ $x \in a, z \notin a$, that is $a \in h_1(x), a \notin h_1(z)$).
  From the fact that the inclusion is strict we get of course $x \neq z$.
  We have then $x < z$ and therefore $z \perp y$. But then, there is $a \in D_1$ such

that $z \in a$ and $y \in \lambda(a)$. Hence, we have $a \in h_1(z) \cap h_2(y)$, contradicting the fact that $h_1(z)$ is I-maximal.
Note that we have proved something stronger than what we need: not only for every $x \in X_1$ and every $y \in X_2$, we have that $h_1(x)$ is an optimal filter of D_1 and $h_2(y)$ is an optimal ideal of D_1 but moreover, if $\langle x, y\rangle \in A_\perp$ then $\langle h_1(x), h_2(y)\rangle$ is a maximal filter-ideal pair of D_1.

- We show now that (h_1, h_2) is a descriptive morphism, by following the order in definition 2.1.11.

  1. We show first that h_1 is order-preserving, the proof for h_2 being dual: suppose $x \leq_1 y$, and that $a \in h_1(x)$; then $x \in a$ and, by proposition 1.3.4, a is increasing; hence, $y \in a$. We conclude $a \in h_1(y)$.
  2. Let Q_1 and Q_2 be the relations of $H_O(D_1)$. The bounded morphism condition of h_1' for the relations R_1 and Q_1 has a similar but easier proof that the bounded condition proved in item 3, so we skip the proof.
  3. We show now the bounded morphism condition of h_2 for the relations R_2 and Q_2.
     Note that we have, for $F, G \in F_*(D_1)$
     - FQ_1G if and only if $\left(m^\delta_{R_1}\right)^{-1}(F) \subseteq G$,
     - IQ_2J if and only if $\left(\rho m^\delta_{R_2}\lambda\right)^{-1}(I) \subseteq J$.

     Suppose $y, y' \in X_2$ such that yR_2y'. We have to prove that:
     $$\left(\rho\ m^\delta_{R_2}\ \lambda\right)^{-1}(h_2(y)) \subseteq h_2(y').$$
     Take $b \in D_2$ such that $\left(\rho m^\delta_{R_2}\lambda\right)(b) \in h_2(y)$, that is $y \in \lambda\rho\left(m^\delta_{R_2}\lambda(b)\right)$. But, as $\lambda(b) \in D_2$, we have, by lemma 2.1.6 that $c := m^\delta_{R_2}\lambda(b) \in D_2$ and therefore $\lambda\rho c = c$. Hence, we have $y \in m^\delta_{R_2}\lambda(b)$ and yR_2y', therefore $y' \in \lambda(b)$ and we conclude $b \in h_2(y')$.
     Now we have to prove the hard part of the bounded morphism condition. Suppose we have $x \in X_2$ and I an optimal filter of D_1 such that $Q_2h_2(x)I$. Take an optimal ideal F of D_1 disjoint from I. There is then an F-maximal ideal J containing I. We will use here that h_2 is surjective; we did not prove this result yet but the reader can find it in the last item of this long proof of proposition 2.4.2. We have then that $J = h_2(x')$ for some $x' \in X_2$.
     We have that $F \cap h_2(x') = \emptyset$; it remains to prove that R_2xx'.
     Note that from $Q_2h_2(x)I$, we get that $(\rho m^\delta_{R_2}\lambda)^{-1}(h_2(x)) \subseteq I$. As $I \subseteq J = h_2(x')$, we have the following:
     for every stable set $S \subseteq X_1$, if $\rho m^\delta_{R_2}\lambda(S) \in h_2(x)$, then $S \in h_2(x')$. In other words, if $x \in \lambda\rho m^\delta_{R_2}\lambda(S) = m^\delta_{R_2}\lambda(S)$ (the equality due to lemma 2.2.5), then $x' \in \lambda(S)$.
     In order to get a contradiction, suppose that $\neg(R_2xx')$. Then, by tightness for R_2, there is $a \in D_2$ such that $x \in m^\delta_{R_2}(a)$ and $x' \notin a$. Taking $S_0 = \rho a$, we have that $x \in m^\delta_{R_2}(\lambda S_0)$ and $x' \notin \lambda(S_0)$. We arrived to a contradiction; hence, R_2xx' and we are done.

4. We prove now the bounded morphism condition for $\perp^c$ and $\perp'^c$, where $\perp' \subseteq F_*(D_1) \times I_*(D_1)$ is given by:

$$F \perp' I \text{ if and only if } F \cap I = \emptyset.$$

   - Suppose $x \in X_1$ and $y \in X_2$ are such that $h_1(x) \perp' h_2(y)$. Then there is $a \in h_1(x) \cap h_2(y)$, that is there is $a \in D_1$ such that $x \in a$ and $y \in \lambda a$. We conclude $x \perp y$.
   - Suppose now $x \in X_1$ and $I \in I_*(D_1)$ are such that $\neg(h_1(x) \perp' I)$. We know that there is $y \in X_2$ such that $I = h_2(y)$. We have then $h_1(x) \cap h_2(y) = \emptyset$ that is, for any $a \in D_1$, if $x \in a$, then $y \notin \lambda a$; by the "tightness" condition for $\perp^c$, we have $\neg(x \perp y)$. Moreover $I \subseteq h_2(y)$ and we are done.
   - The condition dual to the one above is proved dually.

5. Finally, we prove that the preimages under h_1 and h_2 of admissible sets are admissible. We do it only for h_1, the proof for h_2 being similar.
   The admissible subsets of the general semi-lattice frame $g_1(D_1)$ are of the form $\epsilon_a = \{F \in F_*(D_1) \mid a \in F\}$ for some $a \in D_1$. We will show that $h_1^{-1}(\epsilon_a) = a \in D_1$. Dually, it can be proved that the image under h_2^{-1} of an admissible subset of $g_2(D_2)$ is in D_2.
   In the proof of

$$h_1^{-1}(\epsilon_a) = a \in D_1,$$

   the inclusion from left to right goes at follows: take $x \in X_1$ such that $h_1(x) \in \epsilon_a$; then $a \in h_1(x)$ and, by definition of $h_1(x)$, we get $x \in a$. For the inclusion from right to left, assume that $x \in X_1$ is such that $x \notin \epsilon_a$; then $a \notin h_1(x)$, i.e. $x \notin a$ and we are done.

- We prove now that h_1 is injective.
  Suppose we have $h_1(x) = h_1(y)$ for some $x, y \in X_1$. We will show that $x = y$. If not, then we can assume, without loss of generality, that $x \not\leq_1 y$, and then, by tightness, there is $a \in D_1$ such that $x \in a$ and $y \notin a$; but then $a \in h_1(x)$ and $a \notin h_1(y)$, contradicting the assumption.

- We prove now that h_1 is surjective. Take an optimal filter F of D_1. There is then an optimal ideal I of D_1 that is a companion of F. As $F \cap I = \emptyset$, we have that for every $a_1, a_2, \cdots a_n \in F$, $b_1, b_2, \cdots b_t \in I$, $a_1 \cap a_2 \cdots \cap a_n \not\subseteq b_1 \sqcup b_2 \sqcup b_t$ and therefore, by the compactness condition for $H_O(L)$, we have that $\bigcap F \not\subseteq \bigsqcup I = \rho\lambda(\bigcup I) = \rho(\bigcap \lambda[I])$. There are then $z \in \bigcap F$ and $t \in \bigcap \lambda[I]$ such that $z \perp^c t$; there are then, by the conditions referring to $A_\perp$, $x \in F$ and $y \in I$ such that $z \leq_1 x$, $t \leq_2 y$ and $\langle x, y \rangle \in A_\perp$. As every element in F is increasing and $z \in \bigcap F$, we have that $x \in \bigcap F$; as every stable set is increasing, and $t \in \bigcap \lambda[T]$, we have that $y \in \bigcap \lambda[T]$. We will prove that $F = h_1(x)$. For any $a \in F$, we have that $x \in a$ and therefore $a \in h_1(x)$. We have then that $F \subseteq h_1(x)$. In order to get a contradiction, suppose the inclusion is strict. Then, as $\langle F, I \rangle$ is a maximal pair, we have that $h_1(x) \cap I \neq \emptyset$; that is, there is $a \in I$ such that $x \in a$. But if $a \in I$, we have $y \in \lambda a$ and therefore $x \perp y$, contradicting the fact that $\langle x, y \rangle \in A_\perp$. We conclude $F = h_1(x)$.
  The prove that h_2 is surjective is dual.

$\square$

Theorem 2.4.3 *Let $(\sigma_1, \sigma_2) : H_O(L') \to H_O(L)$ be a descriptive morphism. there is a morphism of lattices with operators $\phi : L \to L'$ such that $\phi^* = (\sigma_1, \sigma_2)$.*

Proof: Let $(\sigma_1, \sigma_2) : H_O(L') \to H_O(L)$ be a descriptive morphism. We have to find an morphism of lattices with operators $\phi : L \to L'$ such that $\phi_1^* = \sigma_1$ and $\phi_2^* = \sigma_2$.
Since $\hat{a} \in D_1$, we have, by condition 5 in definition 2.1.11, that for every $a \in L$: $\sigma_1^{-1}(\hat{a}) \in D_1'$, i.e. there is a (unique, by lemma 2.2.7) $a' \in L'$ such that $\sigma_1^{-1}(\hat{a}) = \hat{a'}$.
Let $\phi : L \to L'$ such that $\phi(a) = a'$.

- First we show that ϕ is a morphism of lattices with operators.
  It is clear that $\sigma_1^{-1}(\hat{0}) = \sigma^{-1}(\emptyset) = \emptyset = \hat{0'}$ and then $\phi(0) = 0'$.
  Similarly, we prove that $\phi(1) = 1'$.

  As $\sigma^{-1}(\widehat{a \wedge b}) = \sigma^{-1}(\hat{a} \cap \hat{b}) = \sigma^{-1}(\hat{a}) \cap \sigma^{-1}(\hat{b}) = \hat{a'} \cap \hat{b'} = \widehat{a' \wedge' b'}$, we have that $\phi(a \wedge b) = a' \wedge' b'$.

  Recall that, by proposition 2.1.13, we have

  $$\sigma_2^{-1} \circ \lambda = \lambda' \circ \sigma_1^{-1} \text{ and } \sigma_1^{-1} \circ \rho = \rho' \circ \sigma_2^{-1}.$$

  From this, it follows that $\sigma_1^{-1} \circ \rho \circ \lambda = \rho' \circ \sigma_2^{-1} \circ \lambda = \rho' \circ \lambda' \circ \sigma_1^{-1}$. We have then

  $$\sigma_1^{-1}\left(\rho\lambda(\hat{a} \cup \hat{b})\right) = \rho'\lambda'\left(\sigma_1^{-1}(\hat{a} \cup \hat{b})\right) = \rho'\lambda'\left(\sigma_1^{-1}(\hat{a}) \cup \sigma_1^{-1}(\hat{b})\right),$$

  i.e.

  $$\sigma^{-1}(\hat{a} \sqcup \hat{b}) = \hat{a'} \sqcup \hat{b'}$$

  and therefore

  $$\sigma_1^{-1}(\widehat{a \vee b} = \sigma^{-1}(\hat{a} \sqcup \hat{b}) = \hat{a'} \sqcup \hat{b'} = \widehat{a' \vee' b'}.$$

  We conclude that $\phi(a \vee b) = a' \vee' b'$.

  We still have to prove that ϕ preserves the operators. We prove it only for g and g', the proof for f and f' being similar.
  We have to prove that

  $$\sigma_1^{-1}\left(g(\hat{a})\right) = g'(\hat{a'}).$$

  In other words (by lemma 2.2.4), we have to prove that

  $$\sigma_1^{-1}\left(m^{\delta}_{Q_g}(\hat{a})\right) = m^{\delta}_{Q_{g'}}(\hat{a'}).$$

  For the inclusion from left to right, take an optimal filter F' of L' such that $\sigma_1(F') \in m^{\delta}_{Q_g}(\hat{a})$ and suppose G' is an optimal filter of L' such that

$Q_{g'}F'G'$. We will show that $G' \in \hat{a'}$.
As $Q_{g'}F'G'$, using the bounded morphism condition of σ_1 for Q_g and $Q_{g'}$, we have that $Q_g\sigma_1(F')\sigma_1(G')$. Using now that $\sigma_1(F') \in m^\delta_{Q_g}(\hat{a})$, we get that $\sigma_1(G') \in \hat{a}$. In other words, $a \in \sigma_1(G')$ and therefore $\sigma_1(G') \in \hat{a}$. We conclude that $G' \in \sigma_1^{-1}(\hat{a}) = \hat{a'}$ and we are done.

For the converse inclusion, take an optimal filter $F' \in m^\delta_{Q_{g'}}(\hat{a'})$. We have to prove that $\sigma_1(F') \in m^\delta_{Q_g}(\hat{a})$. Suppose then that G is an optimal filter of L such that $Q_g\sigma_1^(F')G$. We have to show $G \in \hat{a}$.
Suppose not; then $a \notin G$ and there is an optimal ideal I disjoint from G and such that $a \in I$. By the other part of the bounded morphism condition, there is a filter G' of L' such that $Q_{g'}F'G'$ and $\sigma_1(G') \cap I = \emptyset$. From $a \in I$, we get $a \notin \sigma_1(G')$; that is, $\sigma_1(G') \notin \hat{a}$, i.e. $G' \notin \sigma_1^{-1}(\hat{a}) = \hat{a'}$. But, as $F' \in m^\delta_{Q_{g'}}(\hat{a'})$, we have that $G' \in \hat{a}'$; we arrived to a contradiction. We conclude that $a \in G$; that is, $G \in \hat{a}$ and we are done.

- Finally, we have to prove that $\phi^* = (\sigma_1, \sigma_2)$.
  We prove first that $\phi^*_1 = \sigma_1$.
  By definition, $\phi^* - 1 = \phi^{-1}$. Let F' be an optimal filter of L'. Then $\phi^*_1(F') = \{a \in L \mid \phi(a) \in F'\} = \{a \in L \mid F' \in \sigma_1^{-1}(\hat{a})\}$. We have then $\phi^*_1(F') = \{a \in L \mid \sigma_1(F') \in \hat{a}\} = \{a \in L \mid a \in \sigma_1(F')\} = \sigma_1(F')$ and we are done.
  In order to prove that $\phi^*_2 = \sigma_2$, we will use the characterization of the bounded morphism condition with respect to $\perp^c$ and $\perp'^c$ given in proposition 2.1.13, namely that $\rho' \circ \sigma_1^{-1} = \sigma_2^{-1} \circ \rho$.
  For an arbitrary optimal ideal I' of L', we want to show that

  $$\phi^{-1}(I') = \sigma_2(I').$$

  For the inclusion from left to right, take $a \in L$ such that $\phi(a) \in I'$. Then

  $$I' \in \check{\phi(a)} = \check{a}' = \rho'\hat{a'} = \rho'\sigma_1^{-1}(\hat{a}) = \sigma_2^{-1}(\rho\hat{a}).$$

  We have then that $\sigma_2(I') \in \hat{a}$ and therefore $a \in \sigma_2(I')$.
  For the converse inclusion, take $a \in \sigma_2(I')$. Then $\sigma_2(I') \in \check{a}$; that is

  $$I' \in \sigma_2^{-1}\check{a} = \sigma_2^{-1}\rho\hat{a} = \rho'\sigma_1{-}1(\hat{a}) = \rho'\hat{a'} = \check{a}'.$$

  We conclude $a' \in I'$; that is, $\phi(a) \in I'$ and therefore $a \in \phi^{-1}(I')$ and we are done.

□

The characterization of a duality given in theorem 1.2.53 induced us to call H a "partial duality".

2.5 The canonical extension of a lattice.

The canonical extension of a boolean algebra with operators $\langle A, \wedge, \vee, 0, 1, \neg, f\rangle$ (where f is join-preserving) is defined to be the power set algebra with operators of the set of maximal filters of A, namely $A^\sigma = \langle \mathcal{P}(UfA), \cap, \cup, \emptyset, \mathcal{U}\{A, \_^c, f^\sigma\rangle$

(cf. [BdRV]).
This has been extended to the non boolean case by Goldblatt's ([G]): he defines the canonical extension of a distributive lattice with operators $\langle L, \wedge, \vee, 0, 1, g, f\rangle$ (where g is meet-preserving and f is join-preserving) to be the distributive lattice with operators of increasing subsets of the set of prime filters of L, namely $L^\sigma = \langle \mathcal{O}(F_p(L)), \cap, \cup, \emptyset, F_p(L), g^\sigma, f^\sigma\rangle$.
(It is not relevant here to make explicit the definitions of f^σ and g^σ.)
Both extensions are complete lattices containing (up to isomorphisms) the original one and atomic.

Both canonical extensions are constructed by means of a duality result.
Abstracting the basic links between a distributive lattice and the complete and atomic distributive lattice in which it can be embedded, Gehrke and Harding [GH] arrived to the following definition of a canonical extension, that does not need any duality result.

Definition 2.5.1 *Given a complete lattice C, we say that a monomorphism of lattices $e : L \to C$ is:*

- **compact** *if and only if given $S, T \subseteq L$ such that $\bigwedge e[S] \leq \bigvee e[T]$ there are $\bar{S} \subseteq S$ and $\bar{T} \subseteq T$ finite, such that $\bigwedge e[\bar{S}] \leq \bigvee e[\bar{T}]$,*
- **dense** *if and only if every element in C can be expressed as a join of meets of elements in $e[L]$ and as a meet of joins of elements in $e[L]$.*

Definition 2.5.2 *Given a lattice L and a complete lattice C, we say that C is a* **canonical extension of** *L if and only if there is a monomorphism of lattices $e : L \to C$ that is compact and dense.*

Note that for the boolean and the distributive case the canonical extensions are given by the map ν of the Stone's and Priestley's Representation Theorems respectively.

Gehrke and Harding proved in [GH] the unicity (up to isomorphisms) of the canonical extension. In this section we will show its existence, by imitating the construction for the distributive and boolean case: we will use the dual structure associated to a lattice with operators in order to construct its canonical extension.
This construction appears in [GH]; we present it in more detail. We describe some nice properties of the canonical extension, in particular that it is atomic in some sense that we will define.

The following theorem ([GH]) states the unicity (up to isomorphisms) of the canonical extension of a lattice L.

Theorem 2.5.3 *Let L be a lattice. If $e : L \to C$ and $e' : L \to C'$ are monomorphisms making C and C' canonical extensions of L, then there is an isomorphism of complete lattices $\phi : C \to C'$ such that $\phi \circ e = e'$.*

The following concerns the existence of the canonical extension of a lattice.

Definition 2.5.4 *Let $H = \langle\langle X_1, \leq_1, R_1, D_1\rangle, \langle X_2, \leq_2, R_2, D_2\rangle, \perp\rangle$. Then*

$$St(H) = \langle St(X_1), \cap, \sqcup, \emptyset, X_1, m^\delta_{R_1}, \rho m^\delta_{R_2}\lambda\rangle,$$

with for $C_1, C_2 \in St(X_1) : C_1 \sqcup C_2 = \rho\lambda(C_1 \cup C_2)$.

Proposition 2.5.5 *If* H *is a descriptive general lattice frame,* $St(H)$ *is a complete lattice with operators. Moreover the operators* $m^{\delta}_{R_1}$ *and* $m^{\delta}_{R_2}$ *are respectively completely meet-preserving and completely join-preserving.*

Proof: The proof of this proposition iv very similar to the proof of proposition 2.1.6, so we skip it here.

Proposition 2.5.6 *If* L *is a lattice with operators,* $St(H_O(L))$ *is a canonical extension of* L*, given by the monomorphism*

$$\nu : L \to St(H_O(L)),$$

defined by $\nu(a) = \hat{a} = \{F \in F_*(L) \mid a \in F\}$.

We need some preliminary results in order to prove proposition 2.5.6.

Lemma 2.5.7

1. *Let* X *be a stable subset of* X_1 *and* Y *be a stable subset of* X_2*; then:*

$$\lambda(X) = \bigcap_{x \in X} \lambda(\{x\}) \text{ and } \rho(Y) = \bigcap_{y \in Y} \rho(\{y\}).$$

2. *Let* $x \in X_1$ *and* $y \in X_2$*; then:*

$$\lambda(\{x\}) = \lambda(x\uparrow) \text{ and } \rho(\{y\}) = \rho(y\uparrow).$$

3. *Let* $x \in X_1$ *and* $y \in X_2$*; then*

$$x\uparrow = \bigcap_{x \in a \in D_1} a$$

and

$$y\uparrow = \bigcap_{y \in b \in D_2} b.$$

Proof: In each case, we prove only the first equalities, the second ones being similar.

1. This follows from the fact that $\lambda = \lambda_\perp$ arises from a relation, by using lemma 1.3.4:
$$\lambda(X) = \lambda\left(\bigcup_{x \in X} x\right) = \bigcap_{x \in X} \lambda(\{x\}).$$

2. The inclusion $\lambda(x\uparrow) \subseteq \lambda(\{x\})$ follows trivially from the fact that λ is order-reversing and from $\{x\} \subseteq x\uparrow$.
The converse inclusion follows from the condition linking $\perp$ and $\leq_1$: if $x \perp y$ and $x \leq_1 x'$, then $x' \perp y$.

3. The inclusion from left to right is clear, since $x \in \bigcap_{x \in a \in D_1} a$ and every $a \in D_1$ is $\leq_1$ -increasing.
The converse inclusion is nothing but a reformulation of the tightness condition. Let $x' \in X_1$ such that $x \not\leq_1 x'$. Then, by tightness, there exist $a \in D_1$ such that $x \in a$, $x' \notin a$ and then $x' \notin \bigcap_{x \in a \in D_1} a$.

$\square$

Proof of proposition 2.5.6

- Injectivity: cf proof of proposition 2.2.8.
- $St(H_O(L))$ is complete: we know that an arbitrary intersection of stable sets is stable and that for any $X \subseteq X_1$, $\bigsqcup X$ is defined and stable ($\bigsqcup X = \rho\lambda(\bigcup X)$).
- Compactness follow directly from the fact that $H_O(L)$ verifies the compactness condition in definition 2.1.5.
- Density:

  We prove first that every stable subset of $F_*(L)$ can be written as a meet of joins of elements in $\nu[L]$.
  Let S be a stable subset of $F_*(L)$. Then

$$\begin{array}{rcll} S & = & \rho\lambda(S) & \\ & = & \bigcap_{I\in\lambda(S)} \rho(\{I\}) & \text{by lemma 2.5.7.1} \\ & = & \bigcap_{I\in\lambda(S)} \rho(I\uparrow)) & \text{by lemma 2.5.7.2} \\ & = & \bigcap_{I\in\lambda(S)} \rho(\bigcap_{b\in I} \check{b}\ ) & \text{by lemma 2.5.7.3} \\ & = & \bigcap_{I\in\lambda(S)} \rho(\bigcap_{b\in I} \lambda(\hat{b})) & \\ & = & \bigcap_{I\in\lambda(S)} \rho\lambda(\bigcup_{b\in I} \hat{b}) & \text{by lemma 1.3.4} \\ & = & \bigcap_{x\in\lambda(X)} (\bigsqcup_{b\in I} \hat{b}). & \end{array}$$

  We prove now that every stable subset of $F_*(L)$ can be written as a join of meets elements in $\nu[L]$.
  Let S be a stable subset of $F_*(L)$. Then

$$\begin{array}{rcll} S & = & \rho\lambda(S) & \\ & = & \rho(\bigcap_{F\in S} \lambda(\{F\})\ ) & \text{by lemma 2.5.7.1} \\ & = & \rho(\bigcap_{F\in S} \lambda(F\uparrow)\ ) & \text{by lemma 2.5.7.2} \\ & = & \rho\ (\bigcap_{F\in S} \lambda(\bigcap_{a\in F} \hat{a}) & \text{by lemma 2.5.7.3} \\ & = & \rho\lambda(\bigcup_{F\in S} (\bigcap_{a\in F} \hat{a}\ ) & \text{by lemma 1.3.4} \\ & = & \bigsqcup_{a\in S} \bigcap_{a\in F} \hat{a}. & \end{array}$$

$\square$

In the remaining of this section, w prove some nice properties of the canonical extensions.

L^σ is not distributive in general (otherwise, L itself would be distributive). However, "some" distributivity is verified in L^σ as has been proved in [GH] under the name of "restricted distributive laws". We give the proof in proposition 2.5.11. In order to state this proposition, we need some preliminary definitions and lemmas.

Definition 2.5.8 *Let $\langle X, \leq \rangle$ be a partial order. $D \subseteq X$ is said to be* **downwardly** *(***upwardly***) directed if and only if for every $x, y \in D$, there is $z \in D$ such that $z \leq x, z \leq y$ (resp. $z \geq x, z \geq y$).*

Definition 2.5.9 *Let $e : L \to L^\sigma$ be a canonical extension, where $\sqcap$ and $\sqcup$ are the meet and join operations in L^σ. An element in L^σ is said to be a* **$\sqcap$-element** *(***$\sqcup$-element***) of the extension if and only if it is an intersection of elements in $e[L]$ (resp. a union of elements in $e[L]$).*

Lemma 2.5.10 *Let $e : L \to L^\sigma$ be a canonical extension, where $\sqcap$ and $\sqcup$ are the meet and join operations in L^σ. Then:*

- *if c is a $\sqcap$-element, then $c = \sqcap\{a \in e[L] \mid c \leq a\}$,*
- *if c is a $\sqcup$-element, then $c = \sqcup\{a \in e[L] \mid a \leq c\}$.*

Proof: We prove only the first statement, the second one being dual.
The left side of the equality is trivially less than or equal to the right side.
For the converse inclusion, take a $\sqcap$-element c; we have that $c = \sqcap e[K]$ for some $K \subseteq L$.
Let $d = \sqcap\{a \in e[L] \mid c \leq a\}$. We have to prove that $d \leq c$.
As we have $c = \sqcap e[K]$, we get, for every $k \in K$, $c \leq e(k)$. As $K \subseteq L$, we have that $e[K] \subseteq \{a \in e[L] \mid c \leq a\}$; hence $d \leq \sqcap e[K]$ and we are done. $\square$

As we mentioned before, the proof of the following proposition is due to [GH].

Proposition 2.5.11 *Let $e : L \to L^\sigma$ be a canonical extension. Let X and Y be collections of subsets of L^σ such that:*

- *every element in X is a downwardly directed set of $\sqcap$-elements,*
- *every element in Y is an upwardly directed set of $\sqcup$-elements.*

We let $\Phi(X)/$ denote the family of choice functions for X. Then:

1. $\bigsqcup\{\sqcap A \mid A \in X\} = \sqcap\{\bigsqcup Im(\alpha) \mid \alpha \in \Phi(X)\}$,
2. $\sqcap\{\bigsqcup B \mid B \in Y\} = \bigsqcup\{\sqcap Im(\alpha) \mid \alpha \in \Phi(Y)\}$.

Proof: We prove only the first statement, the second one being dual.
For the inclusion from left to right, suppose $A \in X$ and $\alpha \in \Phi(X)$. As $\alpha(A) \in A$, it follows that $\sqcap A \leq \alpha(A) \leq \bigsqcup Im(\alpha)$ as we wanted.
For the converse inclusion, note that, by density, every element in L^σ can be expressed as a meet of $\sqcup$-elements. Let now $x = \bigsqcup\{\sqcap A \mid A \in X\}$ and $y = \sqcap\{\bigsqcup Im(\alpha) \mid \alpha \in \Phi(X)\}$. As x is a meet of $\sqcup$-elements, in order to prove that $y \leq x$, it is enough to show, for every $\sqcup$-element $z \in L^\sigma$, that:

$$x \leq z \text{ implies } y \leq z.$$

Consider then a $\sqcup$-element $z \in L^\sigma$ such that $x \leq z$. Then, for every $A \in X$, we have $\sqcap A \leq z$, and then, by compactness, using that z is a $\sqcup$-element, we have that there are $a_1, a_2, \cdots, a_n \in A$ such that $a_1 \cap \cdots a_n \leq z$. Using now that A is downwardly directed, we have that there is $a \in A$ such that $a \leq a_1 \cdots a_n$ and hence $a \leq z$. Thus, taking a choice function α such that for each $A \in X$, $\alpha(A) = a$, we have $\bigsqcup Im(\alpha) \leq z$, showing that $y \leq z$ and we are done. $\square$

The following proposition and corollary can be deduced from [GH] but do not appear there explicitly.

Proposition 2.5.12 *The completely join-irreducible elements of $L^\sigma = St(H_O(L))$ are the sets of the form $F\uparrow = \{G \in F_*(L) \mid F \subseteq G\}$ for some $F \in F_*(L)$.*

Proof: We prove first that every set of the form $F\uparrow$, with $F \in F_*(L)$ is completely join-irreducible: suppose $F\uparrow = \bigsqcup X$, with $X \subseteq St(X_1)$. then, for every $x \in X$, we have $x \subseteq \bigsqcup X = F\uparrow$. We need to prove that, there is $x \in X$ such that $F\uparrow \subseteq x$. As every element in X can be expressed as a join of $\sqcap$-elements, we can and will assume, without loss of generality that every element in X is a $\sqcap$-element.

Consider, for each $x \in X$, the set $F_x = \{a \in L \mid x \subseteq \hat{a}\}$. It is easy to see that F_x is a filter of L. Moreover, as $x \subseteq F\uparrow$, we have: if $a \in F$, then $F \in \hat{a}$, and therefore ($\hat{a}$ being increasing) $F\uparrow \subseteq \hat{a}$, concluding $x \subseteq \hat{a}$. Hence, $F \subseteq F_x$, for every $x \in X$.

In order to get a contradiction, suppose that for every $x \in X$: $F\uparrow \not\subseteq x$; then (since x is increasing for being stable) $F \notin x$. But, by lemma 2.5.10, we have that $x = \bigcap\{\hat{a} \in L \mid x \subseteq \hat{a}\}$ and therefore $F \notin \bigcap\{\hat{a} \in L \mid x \subseteq \hat{a}\}$; then, there is $a \in L$ such that $x \subseteq \hat{a}$ and $F \notin \hat{a}$. As $x \subseteq \hat{a}$, we have $a \in F_x$; as $F \notin \hat{a}$, we have $a \notin F$. We conclude that $F \subsetneq F_x$ for every $x \in X$.

Let $I \in I_*(L)$ such that $\langle F, I\rangle$ is a maximal filter-ideal pair. Then, for every $x \in X$, we have that $F_x \cap I \neq \emptyset$, i.e. there is $b_x \in F_x \cap I$. It follows that $\bigcap\{\hat{a} \in D_1 \mid a \in F\} = F\uparrow = \bigsqcup X \subseteq \bigsqcup_{x \in X} \hat{b_x}$ and therefore, by compactness, we have:

$$F\uparrow \subseteq \hat{b_1} \sqcup \hat{b_2} \sqcup \cdots \sqcup \hat{b_n},$$

for some $b_1, b_2, \cdots, b_n \in I$.

But then we have:

- $b_1 \vee b_2 \vee \cdots \vee b_n \in I$,
- $F \in (b_1 \vee b_2 \vee \cdots \vee b_n)\hat{}$, that is $b_1 \vee b_2 \vee \cdots \vee b_n \in F$,

contradicting the fact that $F \cap I = \emptyset$.

So we have proved that for some $x \in X$, $F\uparrow = x$ and therefore that $F\uparrow$ is completely join-irreducible.

We will show now that every completely join-irreducible element of $St(H_O(L))$ is of the form $F\uparrow$ for some $F \in F_*(L)$.

Let $S \in St(X_1)$ be completely join-irreducible. From the proof of density in proposition 2.5.6, we can infer that every stable subset of $F_*(L)$ can be expressed as a join of elements of the form $F\uparrow$ for some $F \in F_*(L)$. But then, S being completely join-irreducible, we have that $S = F\uparrow$ for some $F \in F_*(L)$ and we are done. $\square$

The following corollary states an "atomicity" result for lattices.

Corollary 2.5.13 *Let $e : L \to L^\sigma$ be a canonical extension. Then every element in L^σ can be expressed as a join of completely join-irreducibles.*

Proof: We can assume also here, by theorem 2.5.3, that we work with the canonical extension $\nu : L \to St(H_O(L))$ described in proposition 2.5.6. We have seen in the proof of proposition 2.5.6 that every element of $St(H_O(L))$ is a join of principal increasing subsets of $F_*(L)$.
By proposition 2.5.12, this subsets are completely join-irreducible elements of L^σ and we are done. $\square$

Chapter 3

The distributive case.

In [G], Goldblatt presents a ("total") duality for distributive lattices with operators: given a distributive lattice with operators $\langle L, \wedge, \vee, 0, 1, g, f\rangle$, he associates with it a structure (that he calls Relational Priestley Space) of the form

$$\langle X, \leq, Q, R, D\rangle,$$

where Q and R are binary relations in X, $\leq$ is a partial order on X and D is a subset of $\mathcal{O}(X)$, the set of increasing subsets of X.
This chapter is about the link between the "partial" duality that we presented in chapter 2, when applied to the particular case of distributive lattices with operators, and Goldblatt's duality.
We will see that our "partial" duality generalizes Goldblatt's duality in a sense to be made precise below. We divided the chapter in 4 sections:

- In section 3.1, we define the category $\mathcal{RPS}$ of Relational Priestley Spaces as in [G] and we describe Goldblatt's duality.
- In section 3.2, we describe the special features of this particular case.
- In section 3.3, we show that our "partial" duality is in fact a generalization of Golblatt's duality to the distributive case.
- In section 3.4, we show that the canonical extension we constructed in chapter 2, in the particular case of a distributive lattice, is exactly the canonical extension given by Priestley's Representation Theorem.

3.1 Goldblatt's duality.

In this section, we present, without proofs, the duality proposed by Golblatt's for distributive lattices with operators. For the proofs, we refer to [G].

Definition 3.1.1 *Given a partially ordered set $\langle X, \leq\rangle$ and given a relation $R \subseteq X \times X$, we say that:*

- *R is* **increasing** *if and only if $(\leq \circ\, R \circ\, \leq) \subseteq R$.*
- *R is* **decreasing** *if and only if $(\geq \circ\, R \circ\, \geq) \subseteq R$.*

Definition 3.1.2 *A* **Relational Priestley Space** *is a structure of the form* $\langle X, \leq, Q, R, D\rangle$ *such that:*

- $\langle X, \leq, D\rangle$ *is a Priestley Space,*
- R *is increasing and* Q *is decreasing,*
- *For all* $x, y \in X$*, if* $\neg(R\ xy)$*, then there is* $a \in D$ *such that* $x \notin m_R(a)$ *and* $y \in a$*;*
  For all $x, y \in X$*, if* $\neg(Q\ xy)$*, then there is* $a \in D$ *such that* $x \in m^{\delta}_Q(a)$*,* $y \notin a$*.*
- D *is closed under* m_R *and* m^{δ}_Q*.*

Definition 3.1.3 *Given two relational Priestley Structures* $\langle X, \leq, Q, R, D\rangle$ *and* $\langle X', \leq', Q', R', D'\rangle$*, a morphism of Relational Priestley Spaces is a function:*

$$\sigma : X \to X',$$

that is order-preserving and verifies:

- *Bounded-morphism conditions for* Q *and* Q'*:*
  - *for any* $x, y \in X$ *such that* Qxy*, we have* $Q'\sigma(x)\sigma(y)$*,*
  - *if* $Q'\sigma(x)y'$ *for some* $x \in X$ *and some* $y' \in X'$*, then there is* $y \in X$ *such that* Qxy *and* $\sigma(y) \leq y'$*.*
- *Bounded-morphism conditions for* R *and* R'*:*
  - *for any* $x, y \in X$ *such that* Rxy*, we have* $R'\sigma(x)\sigma(y)$*,*
  - *if* $R'\sigma(x)y'$ *for some* $x \in X$ *and some* $y' \in X'$*, then there is* $y \in X$ *such that* Rxy *and* $y' \leq \sigma(y)$
- *If* $a' \in D'$*, then* $\sigma^{-1}(a') \in D$*.*

Definition 3.1.4 *The category* $\mathcal{DLWO}$ *is the full subcategory of* $\mathcal{LWO}$ *whose objects are distributive lattices with operators.*

Definition 3.1.5 *The category* $\mathcal{RPS}$ *has: as objects, the Relational Priestley Spaces; as morphisms, the morphisms of Relational Priestley Spaces.*

Goldblatt's presents in [G] a duality between $\mathcal{DLWO}$ and $\mathcal{RPS}$, that we describe in the rest of the section. Before stating Goldblatt's results, we will need some more definitions.

Definition 3.1.6 *Given a distributive lattice with operators* $\langle L, \wedge, \vee, 0, 1, g, f\rangle$*,*

- $F_p(L)$ *is the set of all prime filters of* L*,*
- Q_g *is the relation in* $F_p(L) \times F_p(L)$ *given by:*

$$Q_g\ FG \text{ if and only if } g^{-1}(F) \subseteq G,$$

- R'_f *is the relation in* $F_p(L) \times F_p(L)$ *given by:*

$$R'_f\ FG \text{ if and only if } G \subseteq f^{-1}(F),$$

- *D is the subset of $\mathcal{O}(F_p(L))$ whose elements are of the form* $\hat{a} = \{F \in F_p(L) \mid a \in F\}$, *for every* $a \in L$.
- *$g(L)$ is the structure* $\langle F_p(L), \subseteq, Q_g, R'_f, D\rangle$.

The following proposition and theorem are not proved. For their proofs, we refer to [G].

Proposition 3.1.7

- *$g(L)$ is a Relational Priestley Space,*
- *if $\phi : L \to L'$ is a morphism in $\mathcal{DLWO}$, then the map $\phi^* : g(L') \to g(L)$ given by $\phi^*(F') = \phi^{-1}(F)$ is a morphism in $\mathcal{RPS}$.*

Theorem 3.1.8 *The maps*

$$g_O : Ob(DLWO) \to Ob(RPS), \quad g_M : M(DLWO) \to M(RPS)$$

given by $g_O(L) = g(L)$ and $g_M(\phi) = \phi^*$ define a duality

$$g = (g_0, g_M) : \mathcal{DLWO} \to \mathcal{RPS}.$$

3.2 Dual objects and morphisms for the distributive case.

In this section, we come back to the "partial" functor (H_O, H_M) presented in 2.2 and we restricted it to the category $\mathcal{DLWO}$.

3.2.1 Objects.

We describe here the image of a distributive lattice with operators under the function H_O.

By Proposition 1.3.13, we have that the optimal filters of L are exactly its prime filters and similarly for ideals. In fact, if L is distributive, its maximal filter-ideal pairs are of the form $\langle F, F^c\rangle$ for some prime filter F.
If we denote by $F_p(L)$ the set of all prime filters of L and by $I_p(L)$ the set of all prime ideals of L, we have:

$$F_*(L) = F_p(L) \text{ and } I_*(L) = I_p(L).$$

We deduce the following proposition.

Proposition 3.2.1 *If $\langle L, \wedge, \vee, 0, 1, g, f\rangle$ is a distributive lattice with operators, then*

$$H_O(L) = \langle \langle F_p(L), \subseteq, Q_g, D_1\rangle, \langle I_p(L), \supseteq, R_f, D_2\rangle, \perp\rangle\,,$$

where $Q_g, R_f, D_1, D_2, \perp$ are defined as in section 2.2, namely:

$$\begin{array}{l} Q_g \subseteq F_*(L) \times F_*(L), Q_gFG \text{ if and only if } g^{-1}(F) \subseteq G, \\ R_f \subseteq I*(L) \times I*(L), R_fIJ \text{ if and only if } f^{-1}(I) \subseteq J, \\ D_1 = \{\hat{a} \mid a \in L\}, D_2 = \{\check{a} \mid a \in L\}, \\ \perp \subseteq F_*(L) \times I*(L), F \perp I \text{ if and only if } F \cap I = \emptyset. \end{array}$$

3.2.2 Morphisms.

We have seen in chapter 2, that the function H_M was defined only for surjective morphisms of lattices. We will show here that in the particular case of distributive lattices we can extend H_M to all the morphisms. This will lead us to a ("total") functor from $\mathcal{DLWO}$ to $\mathcal{DGLF}$.
The crucial point here is not the distributivity of the lattices itself, but the fact of being in a context in which every optimal filter is prime and every optimal ideal is prime: we can prove that the preimage under a morphism of lattices of a prime filter is a prime filter and the analogous result for ideals. We do it in the following proposition.

Proposition 3.2.2 *Let $\phi : L \to L'$ be a morphism of lattices and let F' be a prime filter of L' and I' be a prime ideal of L'. Then $\phi^{-1}(F')$ is a prime filter of L and $\phi^{-1}(I')$ is a prime ideal of L.*

Proof: We give the proof only for filters, its ideal counterpart having a dual proof.
ϕ is a morphism of lattices; it is then meet-preserving. Hence we know, by lemma 1.2.12, that $F = \phi^{-1}(F')$ is a filter.
We have to show that F is prime. Clearly $1_L \in F$ and $0_L \notin F$. Then F is proper. Suppose, in order to get a contradiction, that we have $a, b \in L \setminus F$ such that $a \vee b \in F$. But as $a, b \notin F$, then $\phi(a), \phi(b) \notin F'$. But, since F' is prime, this gives that $\phi(a \vee b) = \phi(a) \vee' \phi(b) \notin F'$, contradicting $a \vee b \in F$. □

We define, given a morphism $\phi : L \to L'$ of distributive lattices, $\phi_1^* : F_p(L') \to F_p(L)$ and $\phi_2^* : I_p(L') \to I_p(L)$ as in section 2.3.
It is easy to check that if $\phi : L \to L'$ and $\psi : L' \to L''$ are morphisms of distributive lattices, then:

$$\phi_1^* \circ \psi_1^* = (\psi \circ \phi)_1^*, \qquad \phi_2^* \circ \psi_2^* = (\psi \circ \phi)_2^*.$$

We have then a contravariant functor

$$H : \mathcal{DLWO} \to \mathcal{DGLF},$$

given by $H_O(L) = \langle\langle F_p(L), \subseteq, Q_g, D_1\rangle, \langle I_p(L), \subseteq, R_f, D_2\rangle, \perp\rangle$; $H_M(\phi) = (\phi_1^*, \phi_2^*)$.

3.3 Linking the dualities.

We will see here the link between the image $Im(H)$ (cf definition 1.2.55) of the functor H defined at the end of the last section and the category $\mathcal{RPS}$. We will show that for a distributive lattice, all the information in the ideal part is already contained in the filter part: the dual two-sorted structure associated with a lattice can be simplified, in the distributive case, to a one-sorted structure that is in fact the dual Relational Priestley Space associated with a distributive lattice by Goldblatt.
This will prove that our duality is in fact a generalization of Golblatt's duality to the non-distributive case.

3.3.1 Image of the objects.

Given a distributive lattice L, consider the following structures:

$$H_O(L) = \langle\langle F_p(L), \subseteq, Q_g, D_1\rangle, \langle I_p(L), \supseteq, R_f, D_2\rangle, \perp\rangle$$

and

$$H'_O(L) = \left\langle\langle F_p(L), \subseteq, Q_g, D_1\rangle, \langle F_p(L), \supseteq, R'_f, D_1^{co}\rangle, \perp'\right\rangle,$$

where $R'_f \subseteq F_p(L) \times F_p(L)$ is defined by: $R'_f FG$ if and only if $G \subseteq f^{-1}(F)$, and $\perp' \subseteq F_p(L) \times F_p(L)$ is defined by: $F \perp' G$ if and only if $F \nsubseteq G$.
We know that $H_O(L)$ is a descriptive general lattice frame. Our next goal is to prove that $H'_O(L)$ is also a descriptive general lattice frame and that it is isomorphic to $H_O(L)$.

Lemma 3.3.1 *If L is a distributive lattice, then $\hat{a} \cup \hat{b} = \widehat{a \vee b}$.*

Proof: The inclusion from left to right follows directly from the fact that every filter is increasing: if $F \in \hat{a} \cup \hat{b}$, then $F \in \hat{a}$ or $R \in \hat{b}$, that is $a \in F$ or $b \in F$; in both cases we conclude that $a \vee b \in F$, that is $F \in \widehat{a \vee b}$.
For the other inclusion, the crucial point is that every optimal filter is prime: if $F \in \widehat{a \vee b}$, then F is prime and $a \vee b \in F$; therefore $a \in F$ or $b \in F$, that is, $F \in \hat{a}$ or $F \in \hat{b}$; in other words $F \in \hat{a} \cup \hat{b}$. □

Lemma 3.3.2 *If L is a distributive lattice, then*

$$\langle F_p(L), \supseteq, R'_f, D_1^{co}\rangle$$

is a tight general semi-lattice frame.

Proof: We have to prove that:

1. $\langle F_p(L), \supseteq, R'_f\rangle$ is a semilattice frame,
2. $D_1^{co} \subseteq \mathcal{O}(X)$,
3. D_1^{co} is closed under binary intersections,
4. $\emptyset, F_p(L) \in D_1^{co}$,
5. D_1^{co} is closed under the operation $m^{\delta}_{R'_f}$,
6. tightness holds for $\supseteq$ and for R'_f.

We follow the proof in the order stated above.

1. $\langle F_p(L), \supseteq, R'_f\rangle$ is a semilattice frame: it is clear that $\supseteq$ is partial order on $F_p(L)$; the fact that R'_f is $\supseteq$-increasing follows straightforwardly from the definition of R'_f.
2. As every element of D_1 is $\subseteq$-increasing, their complements are $\subseteq$-decreasing, that is $\supseteq$-increasing.
3. That D_1^{co} is closed under binary intersections follows directly from lemma 3.3.1: let $a_1, a_2 \in D_1$; then:

$$a_1^c \cap a_2^c = (a_1 \cup a_2)^c = \widehat{a_1 \vee a_2}^{\,c} \in D_1^{co}.$$

4. As $\emptyset, F_p(L) \in D_1$, then $\emptyset^c = F_p(L)$ and $F_p(L)^c = \emptyset$ are elements of D_1^{co}.
5. We prove now that D_1^{co} is closed under $m^\delta_{R'_f}$. In fact we will prove that

$$m^\delta_{R'_f}(\hat{a}^c) = \left(\widehat{f(a)}\right)^c.$$

For the inclusion from left to right, take an arbitrary prime filter $F \notin \left(\widehat{f(a)}\right)^c$. We will show that $F \notin m^\delta_{R'_f}(\hat{a}^c)$. We have that $F \in \widehat{f(a)}$, that is $f(a) \in F$. Consider $I = F^c$; then $f(a) \notin I$. By lemma 1.2.27, we know that I is an ideal. Then, by lemma 1.2.12, that $J = f^{-1}(I)$ is an ideal; moreover $a \notin J$. Therefore, $J \cap a\uparrow = \emptyset$. By applying now the Prime Filter Theorem, we get a prime filter G such that $a \in G$ and $G \cap J = \emptyset$. But then $G \subseteq J^c = \left(f^{-1}(I)\right)^c = f^{-1}(I^c) = f^{-1}(F)$. From $G \subseteq f^{-1}(F)$, we get $R'_f FG$; as $a \in G$, we have $G \in \hat{a}$, that is $G \notin \hat{a}^c$. We conclude that $F \notin m^\delta_{R'_f}(\hat{a}^c)$ as we wanted.

For the converse inclusion, take an arbitrary $F \in \left(\widehat{f(a)}\right)^c$ and assume that $R'_f FG$ for some prime filter G. We want to show that $G \in \hat{a}^c$, that is that $a \notin G$. Suppose $a \in G$. Then, from $R'_f FG$, we get $G \subseteq f^{-1}(F)$ and therefore $f(a) \in F$, contradicting the assumption and we are done.

In order to see that $\langle F_p(L), \supseteq, R'_f, D_1^{co}\rangle$ is tight, we prove first the tightness condition for $\supseteq$. Suppose we have $F \not\supseteq G$; then, there is $a \in G$ such that $a \notin F$; that is, there is $a \in L$ such that $F \notin \hat{a}$ and $G \in \hat{a}$, in other words, $f \in \hat{a}^c$ and $G \notin \hat{a}^c$ and we are done.
We prove now tightness of R'_f. Suppose not $R'_f FG$, then $G \not\subseteq f^{-1}(F)$, that is there is $a \in G$ such that $f(a) \notin F$; in other words, we have $a \in L$ such that $G \in \hat{a}$ and $F \notin \widehat{f(a)}$, that is $F \in \left(\widehat{f(a)}\right)^c = m^\delta_{R'_f}(\hat{a}^c)$ and we are done.

□

Proposition 3.3.3 *If L is a distributive lattice, then*

$$H'_O(L) = \langle\langle F_p(L), \subseteq, Q_g, D_1\rangle, \langle F_p(L), \supseteq, R'_f, D_1^{co}\rangle, \perp'\rangle$$

is a descriptive general lattice frame.

Proof: The proof follows from lemma 3.3.2, the definition of $\perp'$ and the fact that the compactness condition here is equivalent to the compactness condition of the Priestley Space $\langle F_p(L), \subseteq, D_1\rangle$. We skip the details. □

The following proposition has a straightforward but too long proof that we also skip here.

Proposition 3.3.4 *Given a distributive lattice L, there is a descriptive isomorphism*

$$(\sigma_1, \sigma_2) : H_O(L) \to H'_O(L),$$

defined by $\sigma_1(F) = F$ and $\sigma_2(I) = I^c$.

Proposition 3.3.4 tell us that all the information contained in $H_O(L)$ is kept in $H'_O(L)$. The following proposition, allow us to simplify even more the dual structure.

Proposition 3.3.5 *If L is a distributive lattice, then the $\perp'$-stable subsets of $F_p(L)$ are exactly its $\subseteq$-increasing subsets.*

Proof: We know, by proposition 2.1.9, that every stable set is increasing.
For the converse, we will prove, for $X \subseteq F_p(L)$, that:

$$\rho_{\perp'}\lambda_{\perp'}(X) = \{G \in F_p(L) \mid \exists F \in X \text{ such that } F \subseteq G\}.$$

This is sufficient to show that every increasing subset of $F_p(L)$ is stable, because of the following: let $X \subseteq F_p(L)$ be increasing; then it is easy to check that $X = \{G \in F_p(L) \mid \exists F \in X \text{ such that } F \subseteq G\}$; hence, $X = \rho_{\perp'}\lambda_{\perp'}(X)$; that is, X is stable.
We proceed now to prove $(*)$. The inclusion from right to left follows easily by the facts that $X \subseteq \rho_{\perp'}\lambda_{\perp'}(X)$ and that every stable set is increasing.
For the converse inclusion, take $G \in \rho_{\perp'}\lambda_{\perp'}(X)$. We have to prove that there is $F \in X$ such that $F \subseteq G$. Suppose not; then, for any $F \in X$, we have $F \nsubseteq G$, that is $G \in \lambda_{\perp'}(X)$. But $G \subseteq G \in \lambda_{\perp'}(X)$; hence, by definition of $\perp'$, we have $G \notin \rho_{\perp'}\lambda_{\perp'}(X)$, contradicting the assumption. We conclude that $G \in m_{\supseteq}(X)$ and we are done. □

From propositions 3.3.4 and 3.3.5, we get that all the information of $H_O(L)$ is contained in the structure

$$\langle F_p(L), \subseteq, Q_f, R'_f, D_1\rangle,$$

which is in fact the Relational Priestley Space $g_O(L)$ associated with L by Goldblatt's duality.

3.3.2 Image of the arrows.

The arrows that are image of a morphism $\phi : L \to L'$ between distributive lattices are of the form (ϕ_1^*, ϕ_2^*), where:

$$\phi_1^* : F_p(L') \to F_p(L'), \quad \phi_2^* : I_p(L) \to I_p(L)$$

and $\phi_2^*(I) = \phi^{-1}(I) = \left(\phi^{-1}(I^c)\right)^c = (\phi_1^*(I^c))$.
Again, the information in (ϕ_1^*, ϕ_2^*) is contained in ϕ_1^*; moreover, ϕ_1^* is the morphism of relational structures associated with ϕ by Goldblatt's duality, i.e. $\phi_1^* = g_M(\phi)$.

3.4 Canonical Extension of a distributive lattice.

The results of section 3.3 show that the canonical extension given in chapter 2, for the particular case of a distributive lattice L with g meet-preserving and f join-preserving, is given by the following monomorphism of lattices:

$$\nu : \langle L, \wedge, \vee, 0, 1, g, f\rangle \to \langle \mathcal{O}(F_p(L)), \cap, \cup, \emptyset, F_p(L), m^{\delta}_{Q_g}, m_{R'_f}\rangle,$$

with $\nu(a) = \hat{a}$.
In this case $S \sqcup T = \rho\lambda(S \cup T) = m_{\geq}(S \cup T) = S \cup T$, the last equality justified by the fact that the union of increasing sets is increasing.
This is exactly the canonical extension given by Priestley's Representation Theorem.

Conclusions

For a summary of what we have done in this thesis we refer to the introduction. Here we focus on potential further research.
Historically, logicians have been paying attention to the preservation of equations and, more generally, inequalities under taking canonical extensions. Equations that are preserved in this sense are called canonical
For lattices with only join-preserving operations, a positive answer has been given by Gehrke and Harding in [GH]. From the modal logic perspective, this seems very restrictive: no meet-preserving operators are allowed and no inequalities are consider. In analogy with Sahlqvist theorem, we expect that equations and inequalities of a more general similarity type is preserved under taking canonical extensions: our first aim when we started this work, was to develop some Sahlqvist theory for general lattices. We encountered the problem of dealing with a two-sorted frame (that is, a frame that consist of two relational structures with an additional relation linking them); it is possible to deal with this problem by interpreting formulas with valuations in one of the semi-lattice frames.
We propose here a way of treating canonicity:
A lattice frame is the underlying structure of a descriptive general lattice-frame when the sets D_1 and D_2 are not consider. It is a structure of the form $\langle X_1 \leq_1, Q, X_2, \leq_2, R, \perp\rangle$.
A **stable valuation** V in a lattice frame $\langle X_1 \leq_1, Q, X_2, \leq_2, R, \perp\rangle$ is a function from the set of propositional letters into the family of stable subsets of X_1, extended to the set of positive formulas as follows:

- $V(\phi_1 \wedge \phi_2) = V(\phi_1) \cap V(\phi_2)$,
- $V(\phi_1 \vee \phi_2) = \rho\lambda\,(V(\phi_1) \cup V(\phi_2))$
- $V(\Diamond\phi) = \rho m_R^{\delta}\lambda\,(V(\phi))$,
- $V(\Box\phi) = m_Q^{\delta}(V(\phi))$.

If $H = \langle\langle X_1, \leq_1, D_1, Q\rangle, \langle X_2, \leq_2, D_2, R\rangle, \perp\rangle$, is a descriptive general lattice frame, and V is a stable valuation in $F = \langle\langle X_1, \leq_1, Q\rangle, \langle X_2, \leq_2, R\rangle, \perp, \rangle$, we say that V is **admissible** if and only if for every propositional letter p, we have: $V(p) \in D$ (and therefore, for every positive formula ϕ we will have, by remark 2.1.6, that $V(\phi) \in D$).
An inequality $\phi \preceq \psi$ is valid in a lattice-frame if every stable valuation V of the lattice frame verifies $V(\phi) \subseteq V(\psi)$ and the inequality is valid in a descriptive general lattice frame if every admissible valuation verifies that inclusion.
We know that canonicity is equivalent to the modal notion of d-persistence. In our case, an inequality I in the modal similarity type of lattices with operators is canonical if and only if the validity of I in a descriptive general lattice frame implies the validity of I in its underlying lattice frame.
The reader can be surprised with the fact that we choose in an arbitrary way to take valuations on the structure with universe X_1. But this becomes symmetric when we want to dualise results. In order to do this in an easy way, we can also consider valuations as stable subsets of X_2 and prove that an inequality $\phi \preceq \psi$ is valid if and only if $V(\phi) \supseteq V(\psi)$ for every stable (or admissible) valuation

in X_2. This is not a problem, since all the results concerning stable subsets of X_1 proved in this work can be proved for stable subsets of X_2 and even the canonical extension of a lattice with operators can be constructed from stable subsets of X_2.

Our way of interpreting formulas is quite natural from the perspective of duality theory that we have developed. However, if we want to develop (correspondence and) canonicity theory, we face two fundamental difficulties:
1) We work within a tight general semi-lattice frame (i.e. $\langle X_1, \leq_1 R_1, D_1\rangle$) that does not verify a compactness condition analogous to the Priestley's compactness condition: the compactness condition we have is slightly more weak.
More precisely, we have that if S and T are subsets of D such that $\bigcap S \subseteq \bigsqcup T$, then there are $S' \subseteq S$, $T' \subseteq T$ finite such that $\bigcap S' \subseteq \bigsqcup T'$.
The condition that would be more helpful in order to prove canonicity is the following: if S and T are subsets of D such that $\bigcap S \subseteq \bigcup T$, then there are $S' \subseteq S$, $T' \subseteq T$ finite such that $\bigcap S' \subseteq \bigcup T'$. We could not derive this condition from the compactness condition we have.
2) Our truth definition of the operator $\Diamond$ involves the Galois connection. A possible way to deal with this in the framework of modal correspondence theory would be to interpret $\perp^c$ as an accessibility relation of a classical $\Diamond_{\perp^c}$. In such a perspective $\Diamond p$ has the same interpretation as a classical $\Box_{\perp^c}\Diamond_{R_2}\Diamond_{\perp^c}p$: unfolding the definition of $V(\Diamond\phi)$ and of ρ and λ we would have:

$$ST_x(\Diamond\phi) = \forall y \in X_2[x\perp^c y \to \exists z \in X_2(\exists u \in X_1(u\perp^c z \wedge ST_u(\phi))]$$

i.e. the standard translation of the classical formula $\Box_{\perp^c}\Diamond_{R_2}\Diamond_{\perp^c}ST_x(\phi)$. This appears as not nice in order to do Sahlqvist Theory, not only in what concerns canonicity, but also in order to prove first-order correspondence results: it seems very restrictive in the kind of Sahlqvist inequalities that can be proved to have a first order correspondent: the problem is that diamond subformulas of the form $\Diamond\phi$ in the antecedent get translated as not allowed subformulas of the form $\Box_{\perp^c}\Diamond_{\perp^c}\phi$. To give an example, the Sahlqvist formula $\Diamond p \to \Box p$ would be translated by a formula that has $\Box_{\perp^c}\Diamond_{R_2}\Diamond_{\perp^c}\Box p$ as antecedent. A dual approach to this formula by considering valuations in X_2, would have the same kind of problem in the consequent.

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