# Definability and Changing Perspectives

The Beth Property for Three Extensions of Modal Logic

by

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# Introduction

Formulated as a slogan, a logic has the Beth property if every implicit definition can be matched by and explicit definition. Its significance was elegantly formulated by E. W. Beth in the following two quotes.

> Logic has always been concerned, not only with reasoning, but also with definition. Tarski especially emphasized the parallelism between deduction theory and definition theory: along with the question of whether a desired conclusion $B$ can be derived from given premises $A_1, A_2, \ldots$, one faces the question of whether a definiendum $a$ can be defined using the given terms $t_1, t_2, \ldots$. [1]

> A system will be *complete* from the standpoint of the theory of definition if, whenever within the system a notion $a$ is not definable in terms of certain notions $t_1, t_2, \ldots, t_k$, this can be proved by means of Padoa's method.[2]

Beth definability has been extensively investigated for many logics. Continuing in this tradition, we investigate this property for a number of extensions and variations of modal logic, each of which offers a slightly different perspective on relational structures.

Apart from defining basic concepts and giving some background, the first chapter introduces interpolation and the Beth property formally. It then recalls a connection between these two concepts, namely that, under certain natural conditions, interpolation implies Beth definability.

In the second chapter we formulate and prove some theorems enabling us to characterize the Beth property in a way that will be useful in subsequent proofs and counterexamples. In particular a connection is made between versions of bisimulation and the Beth property.

Chapter 3 begins by introducing the difference operator, an operator enabling one to talk about states other than the actual. We consider pure difference logic as well as the combinations obtained by adding the difference operator to basic modal logic and tense logic.

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[1] From Beth's address to the Royal Dutch Academy of Sciences on 8 October 1956. Quoted from [Van Ulsen, 2000], page 116. Freely translated by the author.

[2] Again quoted from [Van Ulsen, 2000].

Whereas the difference operator allows one to count states in a certain, very limited way, Chapter 4 takes the idea of counting states more seriously. We introduce three types of graded modalities. These enable one to make certain statements as to the number of successors a point has, the number of successors it has satisfying a certain condition as well as the number of points in a model satisfying a condition. We investigate Beth definability for various logics making using of these modalities.

Chapter 5 presents some Beth results for hybrid logic. Hybrid logic makes the state (or point) a first class citizen by enabling one to name states, and talk about these states explicitly, using their names. Once again we are able to do a little bit of counting as well as to make statements about states *different* from each other. We exhibit a hybrid logic which has the Beth property as well as one for which it fails.

# Chapter 1

# Beth Definability Defined

In this chapter the central theme of this thesis is introduced and put into some perspective. What is Beth definability? Why is it interesting? What does it tell us about a logical system and what is its relation to other properties of logical systems? These questions are explored here.

## 1.1 Preliminaries

Before we begin we have to fix some terminology and notation and make precise some concepts to be used in the exposition of what is to follow.

### 1.1.1 Systems and Logics

As Beth definability is a property that we will be attributing to *logics*, it would perhaps be wise to make clear just what we mean by a logic. This is what we now do. More on this way of looking at modal logics may be found in [Marx and Venema, 2001], for a more general perspective see [Gabbay, 1994]. Define a (modal) system to be a triple $\langle \mathcal{L}, \mathsf{K}, \Vdash \rangle$, where $\mathcal{L}$ is a (modal) language, $\mathsf{K}$ is a class of models, and $\Vdash$ is a function interpreting formulas from $\mathcal{L}$ on models in $\mathsf{K}$.

All languages we will consider are propositional. They will consist of a (possibly sorted) set of propositional variables, and a collection of connectives including the booleans. The formulas of the language will then be built up in a straightforward way. To be absolutely rigorous one would have to explicitly specify the set of proposition letters used. In this text, however, we shall only do so as the need arises.

A model in $\mathsf{K}$ will be a relational structure (i.e. a set, called the domain, together with relations on that set) to which we add a function from the proposition letters in $P$ to subsets of the domain, called a valuation. In symbols: if $\mathfrak{F} = (W, R_1, R_2, \ldots)$ is a relational structure, called a (Kripke)*frame*, then $\mathcal{M} = (\mathfrak{F}, V)$ is a model (also denoted $\mathcal{M} = (W, R_1, R_2, \ldots, V)$, where $V : P \to \mathcal{P}(W)$. Call $\mathcal{M} = (\mathfrak{F}, V)$ a model *based* on the frame $\mathfrak{F}$. We will consider only binary relations and only models with no more than one binary relation. The elements of the models will be called points, states or worlds. The size of a model will be the cardinality of the domain.

The function $\Vdash$ (also called a meaning function) takes as arguments a model $\mathcal{M} \in \mathsf{K}$, and a formula $\varphi$ from $\mathcal{L}$, yielding a set of points in $\mathcal{M}$, denoted by $[\![\varphi]\!]_{\mathcal{M}}$. This set will be thought of as the meaning of $\varphi$ in $\mathcal{M}$. We will impose a number of natural conditions on $\Vdash$, but we will be rather implicit about these conditions since explicating them all will require a long and involved discussion of abstract logic, which, although interesting for its own sake, is not pertinent to this thesis. Let us name but two examples: $\Vdash$ will be compositional and uniform, i.e. the meaning function will act like a homomorphism from the formula algebra to the powerset of the domain of a model, and it will respect isomorphisms.

Instead of writing $m \in (\Vdash (\mathcal{M}, \varphi))$ we will usually write $\mathcal{M}, m \Vdash \varphi$, and say that $\varphi$ is *true* at the state $m$ in the model $\mathcal{M}$. Note that it is now possible to define the usual concepts of truth and validity. We will say that a formula $\varphi$ is true on a model $\mathcal{M} = (W, R_1, R_2, \ldots, V)$ if $[\![\varphi]\!]_M = W$, and write $\mathcal{M} \Vdash \varphi$. Say that a formula $\varphi$ is valid (on the class $\mathsf{K}$) if $\mathcal{M} \Vdash \varphi$ for all $\mathcal{M} \in \mathsf{K}$, and write $\Vdash \varphi$. If $\Gamma$ is a set of formulas from $\mathcal{L}$ then we will say $\Gamma$ *entails* $\varphi$ and write $\Gamma \Vdash \varphi$, if $\mathcal{M} \Vdash \varphi$ for all $\mathcal{M} \in \mathsf{K}$ on which all formulas in $\Gamma$ are true. (Here we are using the meaning function to define a consequence relation denoted by the same symbol.) Note that this is a global (as opposed to local) perspective on entailment. Precisely how $\Vdash$ is defined will of course differ from system to system and clearly depend on the modal- and other operators present in the language $\mathcal{L}$.

We shall say that a formula $\varphi$ is satisfiable on a model $\mathcal{M}$, if there is some state $w$ in $\mathcal{M}$ such that $\mathcal{M}, w \Vdash \varphi$. More generally, a set of formulas $\Sigma$ is *satisfiable* on $\mathcal{M}$ if there is some state $w$ in $\mathcal{M}$ such that $\mathcal{M}, w \Vdash \psi$ for all $\psi \in \Sigma$. A formula $\varphi$ is satisfiable in class $\mathsf{K}$ if $\varphi$ is satisfiable on some $\mathcal{M} \in \mathsf{K}$, and similarly for sets of formulas. (The logic corresponding to $\Vdash$ will usually be implicit in the context.)

All our work will be semantical, so henceforth we will identify logics with systems. If $S = \langle \mathcal{L}, \mathsf{K}, \Vdash \rangle$ is a logic we shall often write $\Vdash_S \varphi$, $\Gamma \Vdash_S \psi$ etc., indexing the consequence relation with the name of the logic for clarity if potential for confusion exists.

A few properties which logics may have are given in the following definitions.

**Definition 1.1.1** Let $\mathsf{K}$ be as class of models. $\mathsf{K}$ is called an *elementary class* if there exists a set $\Sigma$ of first-order formulas such that $\mathcal{M} \in \mathsf{K}$ if and only if $\mathcal{M} \models \Sigma$. Let $S = \langle \mathcal{L}, \mathsf{K}, \Vdash \rangle$ be a logic with $\mathsf{K}$ an elementary class. $S$ is called *elementary* if for every formula $\varphi \in \mathcal{L}$ there is a first-order formula $\mathsf{T}_x(\varphi)$ in one free variable ($x$ in this case), called a first-order translation of $\varphi$, such that for all $\mathcal{M} \in \mathsf{K}$ and all states $m \in \mathcal{M}$

$$\mathcal{M}, m \Vdash \varphi \text{ iff } \mathcal{M} \models \mathsf{T}_x(\varphi)[x/m]$$

$\lhd$

All the logics considered in this text will be elementary. Note that we may think of the models of an elementary logic as first-order models. We shall often have occasion to do so, and hence be able to transfer many results from first-order model theory (see for example section 1.1.4).

**Definition 1.1.2** A logic $S = \langle \mathcal{L}, \mathsf{K}, \Vdash \rangle$ is called *compact*, if for every set of formulas $\Sigma \subseteq \mathcal{L}$, if every finite subset of $\Sigma$ is satisfiable in $S$, then $\Sigma$ is satisfiable in $S$. $\lhd$

Note that as first-order logic is compact (in the usual sense of the word), we obtain, via the completeness theorem, compactness (in the sense of definition 1.1.2) for all elementary logics.

### 1.1.2 Languages

Language-related issues are of cardinal importance in what follows. Whereas one usually does not need to be too explicit about the proposition letters in one's language, investigating Beth definability requires to practice careful bookkeeping as regards these matters. We therefore take the time to make a few things precise.

Let $\mathcal{L}$ be a language with some specified syntax. Let $P$ be a set of proposition letters. To emphasize that the proposition letters occurring in $\mathcal{L}$ are precisely the proposition letters in $P$, we write $\mathcal{L}(P)$. If $\Sigma$ is a set of formulas, let $\mathsf{Prop}(\Sigma)$ denote the set of proposition letters occurring in formulas in $\Sigma$. We write $\mathcal{L}(\Sigma)$ as short for $\mathcal{L}(\mathsf{Prop}(\Sigma))$. If $\Gamma$ is a set of formulas and $\overline{q} = (q_1, q_2, \ldots, q_n)$ a sequence of proposition letters, we may write $\Gamma(\overline{q})$ instead of $\Gamma$ to emphasize that all proposition letters occurring in $\Gamma$ are in $\{q_1, q_2, \ldots, q_n\}$. Note that this does not necessarily mean that all the proposition letters in $\{q_1, q_2, \ldots, q_n\}$ actually occur in formulas in $\Gamma$.

### 1.1.3 Notation

A few remarks on notation. For models $\mathcal{M} = (M, R, V)$ and $\mathcal{M}' = (M', R', V')$ and points $m \in M$, $m' \in M'$ we write $\mathcal{M} \cong \mathcal{M}'$ to indicate that $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic. We shall often subscript the $\cong$-symbol, eg. $\mathcal{M} \cong_{\mathcal{L}(\Phi)} \mathcal{M}'$ when it is necessary to be precise about the relations and proposition letters which the isomorphism respects. To indicate that there exists an isomorphism $f : M \to M'$, such that $f(m) = m'$, we shall write $\mathcal{M}, m \cong \mathcal{M}', m'$, also subscripting as the need arises.

In what is to follow, much will be made of points in models satisfying exactly the same formulas in some language. We therefore introduce the following bit of terminology and notation. Let $S = \langle \mathcal{L}, \mathsf{K}, \Vdash \rangle$ be a logic, $\mathcal{M}, \mathcal{N} \in \mathsf{K}$ and $m$ and $n$ points in $\mathcal{M}$ and $\mathcal{N}$ respectively. We say $m$ and $n$ are equivalent with respect to, or indistinguishable in, the language $\mathcal{L}$ if for all $\varphi \in \mathcal{L}$, $\mathcal{M}, m \Vdash \varphi$ if and only if $\mathcal{N}, n \Vdash \varphi$. For the sake of brevity, however, we shall usually say merely that $m$ and $n$ are $\mathcal{L}$-equivalent. We write $\mathcal{M}, m \leftrightsquigarrow_{\mathcal{L}} \mathcal{N}, n$. The particular set of proposition letters over which $\mathcal{L}$ is taken will usually be important, and we shall then subscript the $\leftrightsquigarrow$-sign accordingly.

### 1.1.4 Some Tools: Saturated Models and Ultrapowers

Let us briefly recall some concepts from first-order model theory. In particular we reiterate the definition of an ultrapower of a model, as well as some properties of ultrapowers. For a thorough discussion of these matters the reader is referred to any standard text on first-order model theory, eg. [Hodges, 1993]. The process sketched below for obtaining a saturated models from a given model is to be found in greater detail in section 2.6 of [Blackburn et al., 2001], where it is performed for basic modal logic to obtain the van Benthem Characterization Theorem.

Let $\mathfrak{M}$ be a model for a first-order language $\mathcal{L}$. If $W$ is the domain of $\mathfrak{M}$ and $A \subseteq W$, $\mathcal{L}[A]$ denotes the langauge obtained from $\mathcal{L}$ by adding a new constant $\underline{a}$ for each $a \in A$. $\mathfrak{M}_A$ denotes the model obtained by expanding $\mathfrak{M}$ to a model for the language $\mathcal{L}[A]$, where $\underline{a}$ is interpreted as $a$, for all $a \in A$.

**Definition 1.1.3** If $\lambda$ be a cardinal. A model $\mathfrak{M}$ with domain $W$ is said to be $\lambda$-*saturated* if for every $A \subseteq W$ with $|A| < \lambda$, $\mathfrak{M}_A$ realizes every set $\Sigma(x)$ of $\mathcal{L}[A]$-formulas consistent with the first-order theory of $\mathfrak{M}_A$. Of particular interest for our purposes shall be the case where $\lambda = \omega$. An $\omega$-saturated model is said to be *countably saturated.* $\lhd$

Saturated models are extremely handy objects to have in many situations. The question is, how does one obtain them? The next definition provides one answer: given a model, we can construct from it an $\omega$-saturated model resembling it, by taking a suitable ultrapower. We formulate the definition not for first-order models, but for the (modal) models we shall be working with.

**Definition 1.1.4** Let $\mathcal{M} = (W, R, V)$ be a model, $I$ a non-empty index set and $U$ an ultrafilter over $I$. Then the ultrapower of $\mathcal{M}^I/U = (W^I/U, V_U, R_U)$ is the model given as follows:

- The universe of $\mathcal{M}^I/U$ is $W^I/U$. i.e. the set of all equivalence classes of elements of $W^I$ modulo $U$, where for $a, b \in W^I$, $a \sim_U b$ if and only if $\{i \in I : a(i) = b(i)\} \in U$.
- The valuation is $V_U$, such that for all $a/U \in W^I/U$, $a/U \in V_U(p)$ if and only if $\{i \in I : a(i) \in V(p)\} \in U$.
- $R_U(a/U, b/U)$ if and only if $\{i \in I : Ra(i)b(i)\} \in U$.

$\lhd$

An ultrafilter is said to be *countably incomplete* if it is not closed under countable intersections.

**Proposition 1.1.5** *Let $\mathcal{M} = (W, R, V)$ be a model, $I$ a non-empty index set and $U$ a countably incomplete ultrafilter over $I$. Then the ultrapower of $\mathcal{M}^I/U = (W^I/U, V_U, R_U)$ is $\omega$-saturated.*

Now for every $w \in W$ we denote by $id_w$ the element of $W^I$ such that $id_w(i) = w$ for all $i \in I$. Thinking of $\mathcal{M}$ as a first-order model, it follows from Łoś's Theorem (see [Hodges, 1993]) that $w$ in $\mathcal{M}$ is elementary equivalent to $id_w/U$ in $\mathcal{M}^I/U$, i.e. for all first order formulas $\varphi(x)$ in one free variable $\mathcal{M} \models \varphi(x)[w]$ if and only if $\mathcal{M}^I/U \models \varphi(x)[id_w/U]$. Another consequence of Łoś's Theorem is that if $\Gamma$ is a set of formulas and $\mathcal{M} \Vdash \Gamma$, then also $\mathcal{M}^I/U \Vdash \Gamma$.

### 1.1.5 More Tools: Operations on Models

In this subsection some definitions concerning operations on models are briefly recalled. Only models having one binary relation are considered, as these are sufficient for our purposes. These concepts are however easily generalized to models with more relations.

**Definition 1.1.6** Let $\mathcal{M} = (M, R, V)$ be a model and $S$ any subset of $M$. Let $\mathcal{M}(S) = (M(S), R(S), V(S))$ where

- $M(S)$ is the set of all points $m$ in $M$, such that there is a finite $R$-path in $\mathcal{M}$ from some point in $S$ to $m$.
- $R(S) = R \upharpoonright M(S)$
- $V(S)(p) = V(p) \cap M(S)$ for all proposition letters $p$

$\mathcal{M}(S)$ is called the submodel of $\mathcal{M}$ *generated* by $S$. If $S = \{s\}$ then we also write $\mathcal{M}(s)$ instead of $\mathcal{M}(\{s\})$. $\lhd$

Call a model $\mathcal{M}$ *rooted* at $v \in \mathcal{M}$ if $\mathcal{M} = \mathcal{M}(v)$. We say $v$ is a root of $\mathcal{M}$.

**Definition 1.1.7** Let $\mathcal{M} = (M, R, V)$ rooted at $r$. The *unravelling* of $\mathcal{M}$ around $r$ is the model $\vec{\mathcal{M}}(r) = (\vec{M}(r), \vec{R}, \vec{V})$ where

- $\vec{M}(r)$ is the set of all sequences of points in $M$, $\vec{s} = (w_0, w_1, \ldots, w_n)$ such that $r = w_0$ and $Rw_iw_{i+1}$ for all $0 \leq i < n$.
- For $\vec{s}_1, \vec{s}_2 \in \vec{M}(r)$ we have $\vec{R}\vec{s}_1\vec{s}_2$ if and only if there is $v \in M$ such that $\vec{s}_1 \circ (v) = \vec{s}_2$, where $\circ$ denotes sequence concatenation.
- $(w_0, w_1, \ldots, w_n) \in \vec{V}(p)$ if and only if $w_n \in V(p)$.

$\lhd$

## 1.2 Beth's Theorem

In 1953 the Dutch logician E. W. Beth proved what has become known as Beth's Theorem for first-order logic. The next proposition states this theorem, reformulated in more modern terms. Our chief source in this section and the next is [Hoogland, 2000], to which the interested reader is also referred for an in-depth discussion of these matters as well as many illuminating examples and counterexamples.

**Proposition 1.2.1** *Let $R$ be a relation symbol, $\Gamma$ a set of first-order formulas, and $R'$ a relation symbol (of the same arity as $R$) which does not occur in $\Gamma$. Let $\Gamma'$ be the result of uniformly substituting $R'$ for all occurrences of $R$ in $\Gamma$. Suppose $\Gamma, \Gamma' \models \forall\overline{x}(R(\overline{x}) \leftrightarrow R'(\overline{x}))$. Then there exists a first-order formula $\varphi(\overline{x})$ (in which the the symbol $R$ does not occur), such that $\Gamma \models \forall\overline{x}(R(\overline{x}) \leftrightarrow \varphi(\overline{x}))$.*

Roughly speaking, the Beth property says that implicit definitions can be matched by explicit definitions. Proposition 1.2.1 says that first order logic has the Beth property. We now proceed to formulate the notions of implicit- and and explicit definitions for propositional (modal) logics, and subsequently define the Beth property formally.

**Definition 1.2.2** Let $S = \langle \mathcal{L}, \mathsf{K}, \Vdash \rangle$ be a logic, $p$ be a proposition letter, $\Gamma(\overline{q}, p)$ a set of formulas in $\mathcal{L}$ and $p'$ a proposition letter not occurring in $\Gamma$. Let $\Gamma(\overline{q}, p')$ be the result of uniformly substituting $p'$ for every occurrence of $p$ in $\Gamma$. $\Gamma$ is called an *implicit definition* of $p$, if $\Gamma(\overline{q}, p), \Gamma(\overline{q}, p') \Vdash_S p \leftrightarrow p'$. $\lhd$

The intuitive meaning of this rather technical-looking definition is the following: Given an interpretation of the symbols in $\Gamma$ except $p$, the interpretation of $p$ is fixed. Or stated another way, given any model for the language $\mathcal{L}(\Gamma) - \{p\}$, this model can be extended in at most one way to a model of $\Gamma$. Showing that a set of formulas $\Gamma$ is *not* an implicit definition of $p$ then boils down to demonstrating the existence of two models of $\Gamma$, based on the same frame and having the same $\overline{q}$-valuation, but differing on their $p$-valuation. This was first observed by Padoa in 1900 and the method, which has become known as *Padoa's method*, will be used several times in this thesis.

**Definition 1.2.3** Let $S = \langle \mathcal{L}, \mathsf{K}, \Vdash \rangle$ be a logic, $p$ be a proposition letter and $\Gamma(\overline{q}, p)$ a set of formulas in $\mathcal{L}$. A formula $\varphi \in \mathcal{L}(\overline{q})$ is called an *explicit definition* of $p$ relative to $\Gamma(\overline{q}, p)$ if $\Gamma(\overline{q}, p) \Vdash_S p \leftrightarrow \varphi$. $\lhd$

**Definition 1.2.4** A logic $S = \langle \mathcal{L}, \mathsf{K}, \Vdash \rangle$ is said to have the *Beth property* (or simply Beth definability) if for any implicit definition $\Gamma$, of a proposition letter $p$, there exists an explicit definition $\varphi$ of $p$ relative to $\Gamma$. $\lhd$

The Beth property has been proven for quite a number of (propositional) modal logics, including the logic $\mathsf{K}$. One particularly interesting result here is that due to [Maksimova, 1993]. This states that all normal extensions of the logic $\mathsf{K4}$ have the Beth property.

## 1.3 Interpolation and its Relation to Beth Definability

Another important property of logics, related to the Beth property, is interpolation. A study of Beth definability should take note of interpolation, as under certain often-satisfied conditions interpolation implies definability.

**Definition 1.3.1** Let $S$ be a logic. $S$ is said to possess interpolation if, for any two formulas $\varphi$ and $\psi$ such that $\models_S \varphi \to \psi$, there exists a formula $\vartheta$ in the common language of $\varphi$ and $\psi$ (i.e. a formula whose non-logical constants occur both in $\varphi$ and $\psi$) such that $\models_S \varphi \to \vartheta$ and $\models_S \vartheta \to \psi$. $\lhd$

Since it was W. Craig that in 1957 first proved this property for first-order logic, it is often referred to as Craig interpolation. This also helps to distinguish it from other, similar properties, eg. turnstile interpolation. As Craig interpolation is the only kind of interpolation that we shall have occasion to consideration in this text, we shall henceforth refer to it simply as "interpolation". To be able to formulate the conditions under which interpolation implies Beth definability we first need another definition.

**Definition 1.3.2** A logic $S$ is said to have the *local deduction property* if for each pair of formulas $\varphi$ and $\psi$, there exists a formula $\chi_\varphi$ in the same language as $\varphi$ (called a *deduction term* for $\varphi$ and $\psi$) such that for all formulas $\xi$ the following holds

- $\xi, \varphi \Vdash_S \psi$ if and only if $\xi \Vdash_S \chi_\varphi \to \psi$, and
- $\varphi, \chi_\varphi \to \xi \Vdash_S \xi$

$\triangleleft$

Note that first-order logic has a strong form of the local deduction property: $\chi_\varphi$ can uniformly be chosen to be $\varphi$ and hence does not depend on the pair $\varphi$ and $\psi$. Basic modal logic (K) also has the local deduction property, but in this case the best we can say is that for each pair of formulas $\varphi$ and $\psi$, there exists an $n \in \omega$ such that $\varphi \wedge \Box\varphi \wedge \cdots \wedge \Box^n\varphi$ is a deduction term. (Recall that we are working with the *global* consequence relation.) We are now ready to formulate the connection between interpolation and definability. As it affords a nice opportunity to put the definitions of these concepts to work, we include the short and instructive proof, adapted from [Hoogland, 2001].

**Proposition 1.3.3** *Let $S$ be a compact logic with the local deduction property. If $S$ has interpolation then $S$ has the Beth property.*

Proof: Suppose $\Gamma(\overline{q}, p)$ implicitly defines $p$. As $S$ is compact we may assume $\Gamma$ to be a single formula, say $\gamma(\overline{q}, p)$. We have $\gamma(\overline{q}, p), \gamma(\overline{q}, p') \Vdash_S p \leftrightarrow p'$. Hence there are formulas $\chi_{\gamma(\overline{q},p)}$ and $\chi'_{\gamma(\overline{q},p')}$ in the same language as $\gamma(\overline{q}, p)$ and $\gamma(\overline{q}, p')$ respectively, such that $\gamma(\overline{q}, p) \Vdash_S \chi'_{\gamma(\overline{q},p')} \to (p \leftrightarrow p')$ and subsequently $\Vdash_S \chi_{\gamma(\overline{q},p)} \to (\chi'_{\gamma(\overline{q},p')} \to (p \leftrightarrow p'))$, from which it easily follows that

$$\Vdash_S (\chi_{\gamma(\overline{q},p)} \wedge p) \to (\chi'_{\gamma(\overline{q},p')} \to p') \tag{1.1}$$

Since $S$ has interpolation there exists a formula $\vartheta(\overline{q})$ such that

$$\Vdash_S (\chi_{\gamma(\overline{q},p)} \wedge p) \to \vartheta \tag{1.2}$$

and

$$\Vdash_S \vartheta \to (\chi'_{\gamma(\overline{q},p')} \to p') \tag{1.3}$$

From 1.2 it follows that $\Vdash_S \chi_{\gamma(\overline{q},p)} \to (p \to \vartheta)$, thus as $\chi_{\gamma(\overline{q},p)}$ is a deduction term for $\gamma(\overline{q}, p)$ we have

$$\chi_{\gamma(\overline{q},p)} \Vdash_S p \to \vartheta. \tag{1.4}$$

From 1.3 it follows that $\Vdash_S \chi'_{\gamma(\overline{q},p')} \to (\vartheta \to p')$, thus $\gamma(\overline{q}, p') \Vdash_S \vartheta \to p'$, now replacing $p'$ with $p$ we get

$$\gamma(\overline{q}, p) \Vdash_S \vartheta \to p \tag{1.5}$$

hence by combining 1.4 and 1.5 we obtain $\gamma(\overline{q}, p) \Vdash_S \vartheta \leftrightarrow p$, i.e. an explicit definition of $p$ relative to gamma, completing the proof. $\dashv$

# Chapter 2

# Characterizing Implicit and Explicit Definitions

In this chapter we explore the consequences of the definitions of implicit and explicit definitions a little further. We begin by giving a slight generalization of the notion of an implicit definition (which in fact amounts to little more than a reformulation of what we already had), by considering isomorphic models instead of "the same model". This gives one a little more room to manoeuver. Having an implicit definition and being able to construct isomorphic models from certain given models will turn out to be the key to quite a few proofs.

More importantly we provide the tools that will be used in the proofs of most of the results in subsequent chapters of this text. If one is to prove Beth definability for a logic, one has to have a way of proving the existence of explicit definitions relative to implicit definitions. Suppose we were to regards points in models as equivalent if they satisfy the same formulas in the language under consideration. It turns out that the existence of an explicit definition is equivalent to the extendability of these equivalences from the language without the proposition letters being implicitly defined, to the language including that proposition letter. This can also be reformulated in a more structural way, making use of similarity relations – a notion inspired by modal bisimulations.

## 2.1 Implicit Definitions: A Necessary and Sufficient Condition

Let us recall what was meant by an implicit definition. Let $S = \langle \mathcal{L}, \mathsf{K}, \Vdash \rangle$ be a logic. A set of formulas $\Gamma(\overline{q}, p)$ in the propositional variables $\overline{q} = q_1, q_2, \ldots, q_n$ and $p$ implicitly defines $p$ if, on any model of $\Gamma$, the valuation of $q_1, q_2, \ldots, q_{n-1}$ and $q_n$ determines the valuation of $p$. This is often expressed as follows: if $\Gamma(\overline{q}, p')$ is the set of formulas obtained from $\Gamma(\overline{q}, p)$ by uniformly replacing all occurrences of $p$ with $p'$, it is the case that $\Gamma(\overline{q}, p), \Gamma(\overline{q}, p') \Vdash_S p \leftrightarrow p'$. Another way of looking at this is saying that any two models for the language $\mathcal{L}(\overline{q})$ can be extended in at most one way to a model of $\Gamma(\overline{q}, p)$. Hence, in order to *disprove* that $\Gamma(\overline{q}, p)$ is an implicit definition of $p$, it is sufficient to find two models $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\Gamma$, such that $\mathcal{M}_1$ and $\mathcal{M}_2$ are *not* isomorphic with respect to the language $\mathcal{L}(\overline{q}, p)$ but are isomorphic with respect to the sublanguage $\mathcal{L}(\overline{q})$. So we have the following proposition:

**Proposition 2.1.1** *$\Gamma(\overline{q},p)$ implicitly defines $p$ if and only if for every pair of models $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\Gamma$, if $\mathcal{M}_1 \cong_{\mathcal{L}(\overline{q})} \mathcal{M}_2$ then $\mathcal{M}_1 \cong_{\mathcal{L}(\overline{q},p)} \mathcal{M}_2$.*

## 2.2 Explicit Definitions : A Necessary and Sufficient Condition

The following theorem will prove very useful in the subsequent chapters. Formulated as a slogan, theorem 2.2.1 says that the existence of an explicit definition of a proposition letter, relative to a set of formulas, is equivalent to the extendibility of $\mathcal{L}$-equivalences (from $\mathcal{L}(\overline{q})$-equivalences to $\mathcal{L}(\overline{q},p)$-equivalences).

**Theorem 2.2.1** *Let $S = \langle \mathcal{L}, \mathsf{K}, \Vdash \rangle$ be a compact logic, and $\Gamma(\overline{q},p)$ a (possibly empty) set of formulas. There exists an explicit definition of $p$ in the language $\mathcal{L}(\overline{q})$ relative to $\Gamma(\overline{q},p)$ if and only if for any models $\mathcal{M}_1 = (W_1, R_1, V_1)$ and $\mathcal{M}_2 = (W_2, R_2, V_2)$ of $\Gamma$ and points $m_1 \in W_1$ and $m_2 \in W_2$,*

$$m_1 \leftrightsquigarrow_{\mathcal{L}(\overline{q})} m_2 \text{ only if } m_1 \leftrightsquigarrow_{\mathcal{L}(\overline{q},p)} m_2.$$

PROOF: Suppose there exists an explicit definition of $p$ relative to $\Gamma(\overline{q},p)$, say $\varphi \in \mathcal{L}(\overline{q})$. Specifically $\Gamma \Vdash p \leftrightarrow \varphi$, hence if $\psi \in \mathcal{L}(\overline{q},p)$ and $\psi[p/\varphi]$ is the result of uniformly replacing $p$ with $\varphi$ in $\psi$, we have that $\Gamma \Vdash \psi \leftrightarrow \psi[p/\varphi]$. Let $\mathcal{M} = (W,R,V)$ be any model of $\Gamma$ and $m_1, m_2 \in W$. Suppose $m_1 \leftrightsquigarrow_{\mathcal{L}(\overline{q})} m_2$. If $\psi \in \mathcal{L}(\overline{q},p)$ then $\mathcal{M}, m_1 \Vdash \psi$ iff $\mathcal{M}, m_1 \Vdash \psi[p/\varphi]$ iff $\mathcal{M}, m_2 \Vdash \psi[p/\varphi]$ iff $\mathcal{M}, m_2 \Vdash \psi$. Hence $m_1 \leftrightsquigarrow_{\mathcal{L}(\overline{q},p)} m_2$.

To prove the right-to-left direction we proceed by contraposition. Suppose that there exists no explicit definition of $p$ relative to $\Gamma(\overline{q},p)$. Let $\psi_0, \psi_1, \psi_2, \ldots$ be an enumeration of the formulas in $\mathcal{L}(\overline{q})$. For each $\psi_i$ let $\psi_i^1 = \psi_i$ and $\psi_i^0 = \neg\psi_i$.

CLAIM 1: For each $n \in \omega$ there exists a function $\chi : \{0,1,\ldots,n\} \to \{0,1\}$ such that both $p \wedge \bigwedge_{i=1}^n \psi_i^{\chi(i)}$ and $\neg p \wedge \bigwedge_{i=1}^n \psi_i^{\chi(i)}$ are satisfiable on models of $\Gamma$.

PROOF OF CLAIM 1: Let $n \in \omega$. Note that there are $2^n$ functions $\chi : \{0,1,\ldots,n\} \to \{0,1\}$, call them $\chi_1, \chi_2, \ldots, \chi_{2^n}$. Suppose to the contrary that for no function $\chi : \{0,1,\ldots,n\} \to \{0,1\}$ both $p \wedge \bigwedge_{i=1}^n \psi_i^{\chi(i)}$ and $\neg p \wedge \bigwedge_{i=1}^n \psi_i^{\chi(i)}$ are $\Gamma$-satisfiable. Hence for each $\chi$, either $\Gamma \Vdash \bigwedge_{i=1}^n \psi_i^{\chi(i)} \to p$ or $\Gamma \Vdash \bigwedge_{i=1}^n \psi_i^{\chi(i)} \to \neg p$. Let $\Psi_j = \bigwedge_{i=1}^n \psi_i^{\chi_j(i)}$ for each $1 \leq j \leq 2^n$. Note that $\Vdash \bigvee_{j=1}^{2^n} \Psi_j \leftrightarrow \top$. Let $J_1 = \{j : \Gamma \Vdash \Psi_j \to p\}$ and $J_2 = \{j : \Gamma \Vdash \Psi_j \to \neg p\}$. Then clearly $\Gamma \Vdash \bigvee_{j \in J_1} \Psi_j \to p$ and $\Gamma \Vdash \bigvee_{j \in J_2} \Psi_j \to \neg p$, and $J_1 \cup J_2 = \{1, 2, \ldots, 2^n\}$. Hence $\Gamma \Vdash \bigvee_{j \in J_1} \Psi_j \leftrightarrow p$, yielding an explicit definition and a contradiction. ⊣

CLAIM 2: There exists a chain of functions $\rho_1 \subseteq \rho_2 \subseteq \rho_3 \subseteq \cdots$ where for each $i$, $\rho_i : \{1,2,\ldots,i\} \to \{0,1\}$, such that if $\Phi_i = \bigwedge_{j=1}^i \psi_j^{\rho_i(j)}$ then both $\Phi_i \wedge p$ and $\Phi_i \wedge \neg p$ are $\Gamma$-satisfiable.

Proof of Claim 2: To see that this is the case, let as represent the set of functions $\mathcal{F} = \{\chi : \{0, 1, \ldots, n\} \to \{0, 1\} : n \in \omega\}$ as a binary tree (see figure 2.1). We call this tree $\mathcal{T}$. More precisely, a function $\chi : \{0, 1, \ldots, n\} \to \{0, 1\}$ is represented by a path from the root to level $n+1$, where, if the left branch is taken at level $i$, it means that $\chi(i) = 0$, and if the right is taken that $\chi(i) = 1$. Clearly there is a one-to-one correspondence between functions in $\mathcal{F}$ and finite paths in $\mathcal{T}$.

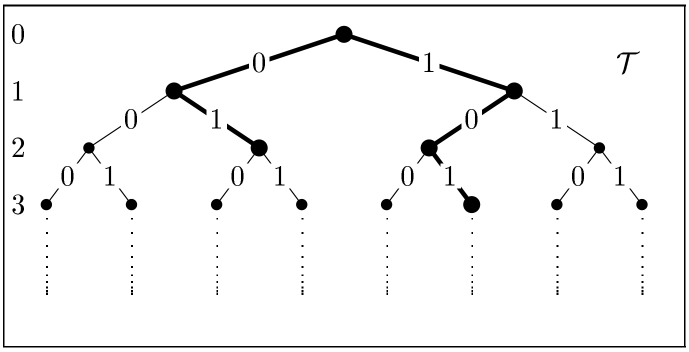


Figure 2.1: Acceptable Functions

We call a function $f \in \mathcal{F}$ acceptable for $n \in \omega$ if $f : \{0, 1, \ldots, m\} \to \{0, 1\}$ for some $m \geq n$, and both $p \wedge \bigwedge_{i=1}^{n} \psi_i^{f(i)}$ and $\neg p \wedge \bigwedge_{i=1}^{n} \psi_i^{f(i)}$ are satisfiable on models of $\Gamma$. Clearly, if $f$ is acceptable for $n$, then $f$ is acceptable for all $j \leq n$. We call a path of length $n$ acceptable, if it corresponds to some function in $\mathcal{F}$ which is acceptable for $n$. (In figure 2.1 the thick edges and big nodes represent the acceptable paths.)

By claim 1 there are arbitrarily long acceptable paths from the root of $\mathcal{T}$. Thus by König's Lemma there must be an infinitely long path from the root of $\mathcal{T}$, call it $P$, such that every finite sub-path of $P$, starting at the root is acceptable. The desired chain of functions is then the chain corresponding to the finite sub-paths of $P$, starting at the root. $\dashv$

Let $U = \{p\} \cup \{\Phi_i : i \in \omega\}$ and $T = \{\neg p\} \cup \{\Phi_i : i \in \omega\}$. Every finite subset of $U$ is $\Gamma$-satisfiable, as is every finite subset of $T$, hence $U$ and $T$ are both $\Gamma$-satisfiable, hence by compactness (recall that by assumption we are dealing with a compact modal logic) both $U$ and $T$ are satisfiable on models of $\Gamma$. Moreover $U$ and $T$ are maximal satisfiable sets in the language $\mathcal{L}(\overline{q})$, as for each $\psi_i$ either $\psi_i$ or $\neg\psi_i$ is a conjunct of $\Phi_i$. This means there are models of $\Gamma$, $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ say, and points $w \in W$ and $w' \in W'$ such that every formula in $U$ is satisfied at $w$ and every formula in $T$ is satisfied at $w'$. As $U$ and $T$ are maximal in $\mathcal{L}(\overline{q})$, $w$ and $w'$ agree on all formulas in $\mathcal{L}(\overline{q})$, but differ on their valuation of $p$, i.e $w \leftrightsquigarrow_{\mathcal{L}(\overline{q})} w'$ but $w \not\leftrightsquigarrow_{\mathcal{L}(\overline{q},p)} w'$. $\dashv$

Although being an easy corollary of theorem 2.2.1, the next result, when applicable, can streamline proofs considerably.

**Corollary 2.2.2** *Let $S$ be a compact logic for which the truth of formulas is invariant under* *the taking of disjoint unions of models, and $\Gamma(\overline{q},p)$ a set of formulas. There exists an explicit definition of $p$ in the language $\mathcal{L}(\overline{q})$ relative to $\Gamma(\overline{q},p)$ if and only if for any model $\mathcal{M} = (W,R,V)$ of $\Gamma$ and points $w,w' \in W$, if $w \leftrightsquigarrow_{\mathcal{L}(\overline{q})} w'$ then $w \leftrightsquigarrow_{\mathcal{L}(\overline{q},p)} w$.*

Proof: Simply note that at the end of the proof of theorem 2.2.1we can take the disjoint union of the models $\mathcal{M}$ and $\mathcal{M}'$ obtaining a new modal of $\Gamma$ and the desired offending points in *the same* model. $\dashv$

Note that the above corollary applies to many well known canonical logics in the basic modal language e.g, $\mathsf{K4}$ and $\mathsf{S5}$ to name only two. It notably does not apply to logics with the difference operator, the universal modality or nominals, as the truth of formulas in logics containing these, is *not* invariant under taking disjoint unions of models. (Some logics containing these operators will be discussed later in this text.)

The characterization of implicit definitions given by proposition 2.1.1, allows us to consider only the subclass of $\mathsf{K}$ consisting of models of $\Gamma$ and isomorphisms between these. It gives us the freedom, in a certain sense, to forget about the language, and work only with the structures in the class. (The notion of isomorphism still depending on the proposition letters in the language under consideration of course.)

On the other hand, theorem 2.2.1 and corollary 2.2.2 allow us to prove Beth definability by proving the extendibility of modal equivalences from the restricted language to the entire language (i.e. the language including the proposition letter being implicitly defined.) Unlike the characterization of implicit definitions, however, the language is still very much involved in this - we consider points satisfying the same *formulas.*

One would like a theorem which would allow one (as with implicit definitions) to distance oneself from the language to a certain degree, and characterize the existence of explicit definitions much more structurally. This will be satisfied by theorem 2.2.4. Theorem 2.2.4 is analogous to proposition 2.1.1, using instead of isomorphisms, the notion of similarity relations. These we now define.

**Definition 2.2.3** Let $S = \langle \mathcal{L}, \mathsf{K}, \Vdash \rangle$ be a logic. Let $\mathsf{R}$ be a class of binary relations between the domains of the models in $\mathsf{K}$. $\mathsf{R}$ is called a *similarity class for $S$* if it satisfies the following:

1. For any $\mathcal{M},\mathcal{N} \in \mathsf{K}$ and $m \in \mathcal{M}$ and $n \in \mathcal{N}$, and for any relation $Z \in \mathsf{R}$, if $mZn$, then $\mathcal{M}, m \leftrightsquigarrow_{\mathcal{L}} \mathcal{N}, n$.

2. There exists a non-empty subclass $\mathsf{K}'$ of $\mathsf{K}$, such that

   (a) for any models $\mathcal{M},\mathcal{N} \in \mathsf{K}'$ and points $m \in \mathcal{M}$ and $n \in \mathcal{N}$, if $m \leftrightsquigarrow_{\mathcal{L}} n$ then there exists a relation $Z \in \mathsf{R}$ such that $mZn$.

   (b) For any $\mathcal{M} \in \mathsf{K}$ and point $m \in \mathcal{M}$, there is a model $\mathcal{M}^* \in \mathsf{K}'$ and point $m^* \in \mathcal{M}^*$ such that (i) for all $\varphi \in \mathcal{L}$ such that $\mathcal{M} \Vdash \varphi$, it is the case that $\mathcal{M} \Vdash \varphi$, and (ii) $m \leftrightsquigarrow_{\mathcal{L}} m^*$.

If $m \in \mathcal{M}$ and $n \in \mathcal{N}$ are related by a relation $Z$ in a similarity class $\mathsf{R}$ for $S$, we write $Z : \mathcal{M}, m \mathrel{\underline{\leftrightarrow}}^{\mathsf{R}}_{\mathcal{L}} \mathcal{N}, n$, or if the particular relation is not important, simply $\mathcal{M}, m \mathrel{\underline{\leftrightarrow}}^{\mathsf{R}}_{\mathcal{L}} \mathcal{N}, n$.

The superscript R will usually be omitted where no potential for confusion exists. A relation from a similarity class for $S$ will be called a similarity relation for $S$. $\lhd$

Definition 2.2.3 is inspired by the notion of bisimulation for modal logic. (Note that the class of all bisimulations is a similarity class for basic modal logic.) The first clause demands that if two points are related by a similarity relation, then they satisfy the same formulas from $\mathcal{L}$, while the second is an analogue of the "bisimulation-implies-modal-equivalence-somewhere-else"-theorem from modal logic. If two points are $\mathcal{L}$-equivalent, then there are points in models in K, which are $\mathcal{L}$-equivalent to the first pair points, and are related by a similarity relation. Furthermore these two models are members of a subclass of K where $\mathcal{L}$-equivalence implies being related by a relation from R. The notion of similarity relations, and hence also the notation $\mathcal{M}, m \mathrel{\underline{\leftrightarrow}}_{\mathcal{L}} \mathcal{N}, n$, will usually be relativized to a fragment of the language over a particular set of proposition letters, eg. $\mathcal{M}, m \mathrel{\underline{\leftrightarrow}}_{\mathcal{L}(\Psi)} \mathcal{N}, n$. This means that the notion of $\mathcal{L}$-equivalence in the definition is in fact replaced by $\mathcal{L}(\Psi)$-equivalence.

All this is terribly abstract. In practice, however, these similarity relations will usually be such familiar things as bisimulations, or slight variations on the theme of bisimulation. The exact nature and extent of the variation will be dictated by the peculiarities of the different languages and logics for which these are intended.

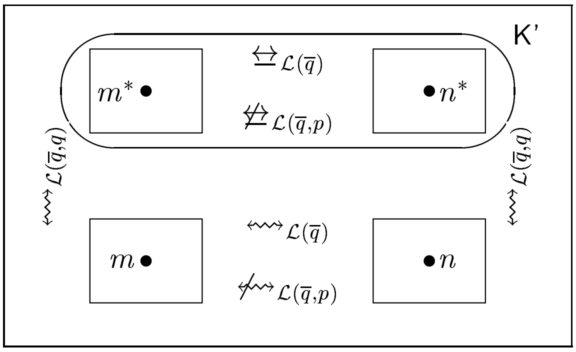


Figure 2.2: Theorem 2.2.4

**Theorem 2.2.4** *Let $S = \langle \mathcal{L}, K, \Vdash \rangle$ be a compact logic and $\Gamma(\overline{q}, p)$ a set of formulas. Let $R$ be a similarity class for $S$ (for $\underline{\leftrightarrow}_{\mathcal{L}}$ read $\underline{\leftrightarrow}^{R}_{\mathcal{L}}$). If for all models $\mathcal{M}_1, \mathcal{M}_2 \in K$ of $\Gamma$, and points $m_1 \in \mathcal{M}_1$ and $m_2 \in \mathcal{M}_2$, such that $m_1 \mathrel{\underline{\leftrightarrow}}_{\mathcal{L}(\overline{q})} m_2$, we have that $m_1 \mathrel{\underline{\leftrightarrow}}_{\mathcal{L}(\overline{q},p)} m_2$, then there exists an explicit definition of $p$ in the language $\mathcal{L}(\overline{q})$ relative to $\Gamma(\overline{q}, p)$*

PROOF: We proceed by contraposition. Suppose that there exists no explicit definition of $p$ relative $\Gamma$, then by theorem 2.2.1 we know that there exists models $\mathcal{M}$ and $\mathcal{N}$ of $\Gamma$ in K and points $m \in \mathcal{M}$ and $n \in \mathcal{N}$ such that $\mathcal{M}, m \leftrightsquigarrow_{\mathcal{L}(\overline{q})} \mathcal{N}, n$, but $\mathcal{M}, m \not\leftrightsquigarrow_{\mathcal{L}(\overline{q},p)} \mathcal{N}, n$. Now by definition 2.2.3 of a similarity class and the fact that there exists such a class R, for $S$, we know that there are models of $\Gamma$, $\mathcal{M}^*$ and $\mathcal{N}^*$, and points $m^* \in \mathcal{M}^*$ and $n^* \in \mathcal{N}^*$ such that $m \leftrightsquigarrow_{\mathcal{L}(\overline{q},p)} m^*$ and $n \leftrightsquigarrow_{\mathcal{L}(\overline{q},p)} n^*$ (see figure 2.2). As $\mathcal{M}, m \leftrightsquigarrow_{\mathcal{L}(\overline{q})} \mathcal{N}, n$, we know that

$$\mathcal{M}^*, m^* \leftrightsquigarrow_{\mathcal{L}(\overline{q})} \mathcal{N}^*, n^*$$

and hence that

$$\mathcal{M}^*, m^* \leftrightarroweq_{\mathcal{L}(\overline{q})} \mathcal{N}^*, n^*$$

As $\mathcal{M}, m \not\leftrightsquigarrow_{\mathcal{L}(\overline{q},p)} \mathcal{N}, n$ however, it can not be that $\mathcal{M}^*, m^* \leftrightsquigarrow_{\mathcal{L}(\overline{q},p)} \mathcal{N}^*, n^*$ and hence

$$\mathcal{M}^*, m^* \not\leftrightarroweq_{\mathcal{L}(\overline{q},p)} \mathcal{N}^*, n^*$$

As we have demonstrated the existence of the desired points, the proof is complete. $\dashv$

Recall that theorem 2.2.1 said that the existence of an explicit definition of a proposition letter, relative to a set of formulas, is equivalent to the extendibility of $\mathcal{L}$-equivalences (from $\mathcal{L}(\overline{q})$-equivalences to $\mathcal{L}(\overline{q}, p)$-equivalences). Theorem 2.2.4 now says that the extendibility of similarity relations (from the language without $p$ to the language with $p$) is a sufficient condition for the existence of an explicit definition of $p$ relative to a set of formulas.

Note in particular that theorem 2.2.4 holds for modal logic over any first-order definable class of models/frames: as such a logic is elementary it is compact, and by the well-known results on bisimulations on modally saturated classes of models (see [Blackburn et al,2001]), bisimulations do the work of the required similarity class.

# Chapter 3

# The Difference Operator

We now introduce one of the principal players in this text, namely the difference operator. The idea is very simple and natural, and can be explained as follows. It may often be desirable to express some fact regarding a state *different* from the present or actual, regardless of how this state might (not) be accessible. This obviously leads to a global perspective over models, also making it possible to define the so-called global modalities as well as to count states in a limited way. For a thorough discussion of the difference operator the interested reader is referred to [de Rijke, 1992].

We shall begin by introducing the logic of pure difference, which shall be referred to simply as difference logic. After showing that this logic does not enjoy interpolation, we shall prove that it nevertheless has the Beth property. We will proceed to investigate the consequences of adding the difference operator to basic modal logic and tense logic, as far as interpolation and definability are concerned.

## 3.1 Introducing the Difference Operator

The simplest language making use of the difference operator, $\mathcal{L}_{\mathsf{D}}$, is defined as follows.

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \vee \psi \mid \mathsf{D}\varphi$$

The semantics we shall give to this language holds no surprises, the only interesting clause being that for the difference operator. $\mathcal{L}_{\mathsf{D}}$ will be interpreted on models with no structure, that is to say sets with valuations defined on them. If $\mathcal{M} = (W, V)$ is such a model and $m \in W$, then

- $\mathcal{M}, m \Vdash p$ iff $m \in V(p)$
- $\mathcal{M}, m \Vdash \bot$ iff never
- $\mathcal{M}, m \Vdash \neg\varphi$ iff $\mathcal{M}, m \nVdash \varphi$
- $\mathcal{M}, m \Vdash \varphi \vee \psi$ iff $\mathcal{M}, m \Vdash \varphi$ or $\mathcal{M}, m \Vdash \psi$
- $\mathcal{M}, m \Vdash \mathsf{D}\varphi$ iff there exists a point $n \in M$ such that $n \neq m$ and $\mathcal{M}, n \Vdash \varphi$

**Definition 3.1.1** Let $\mathsf{Dif} = \langle \mathcal{L}_{\mathsf{D}}, \mathsf{S}, \Vdash \rangle$. Where $\mathsf{S}$ is the class of all sets and $\Vdash$ as given above. Call $\mathsf{Dif}$ the *logic of pure difference* or simply *difference logic.* $\lhd$

The global modality (cf. [Blackburn et al., 2001], p. 415 ) is now definable:

$$\mathsf{E}\varphi \equiv \varphi \vee \mathsf{D}\varphi$$

and

$$\mathsf{A}\varphi \equiv \neg(\mathsf{E}\neg\varphi)$$

Obviously $\mathcal{M}, m \Vdash \mathsf{E}\varphi$ if and only if there is some point $n \in \mathcal{M}$ such that $\mathcal{M}, n \Vdash \varphi$. From this it should be clear that the truth of formulas in $\mathsf{Dif}$ is *not* invariant under the taking of disjoint unions of models.

## 3.2 Interpolation and Beth Definability for Difference Logic

**Definition 3.2.1** Let $\mathcal{M} = (M, V)$ and $\mathcal{M}' = (M', V')$ be models. A relation $Z \subseteq M \times M'$ is called a *d-relation* if the following hold:

**local harmony** if $mZm'$ then $\mathcal{M}, m \Vdash p$ iff $\mathcal{M}', m' \Vdash p$, for all $p$.

**d-forth** if $mZm'$ and $n \in M$ such that $m \neq n$, then there exists a point $n' \in M'$ such that $m' \neq n'$ and $nZn'$.

**d-back** if $mZm'$ and $n' \in M'$ such that $m' \neq n'$, then there exists a point $n \in M$ such that $m \neq n$ and $nZn'$.

We say two models are d-related if there exists a d-relation between them, specifically that $m \in \mathcal{M}$ and $m' \in \mathcal{M}'$ are d-related if there is a d-relation between $\mathcal{M}$ and $\mathcal{M}'$ linking $m$ and $m'$. We write $\mathcal{M}, m \leftrightarroweq_{\mathcal{L}_{\mathsf{D}}} \mathcal{M}', m'$. If the set of proposition letters with respect to which the local harmony clause is taken is to be made explicit, we appropriately parameterize the subscript, for example $\mathcal{M}, m \leftrightarroweq_{\mathcal{L}_{\mathsf{D}}(\Pi)} \mathcal{M}', m'$, where $\Pi$ is some set of proposition letters. $\lhd$

**Lemma 3.2.2** *Let $\mathcal{M} = (M, V)$ and $\mathcal{M}' = (M', V')$ be two models with $m \in M$ and $m' \in M'$. Let $\Pi$ be a set of proposition letters. If $\mathcal{M}, m \leftrightarroweq_{\mathcal{L}_D(\Pi)} \mathcal{M}', m'$ then $\mathcal{M}, m \leftrightsquigarrow_{\mathcal{L}_D(\Pi)} \mathcal{M}', m'$.*

Proof: Let $\xi \in \mathcal{L}_{\mathsf{D}}(\Pi)$. We proceed by induction on $\xi$ and show that $\mathcal{M}, m \Vdash \xi$ if and only if $\mathcal{M}', m' \Vdash \xi$. The base case and induction steps for the boolean connectives are trivial, so suppose $\xi$ is of the form $\mathsf{D}\psi$. If $\mathcal{M}, m \Vdash \mathsf{D}\psi$, then exists a $n \in \mathcal{M}$, $n \neq m$ such that $\mathcal{M}, n \Vdash \psi$. By the d-forth condition there exists a $n' \in \mathcal{M}'$, $n' \neq m'$ and $nZn'$. As $\psi$ falls under the inductive hypothesis we get that $\mathcal{M}', n' \Vdash \psi$, so $\mathcal{M}', m' \Vdash \mathsf{D}\psi$. The converse is symmetric, using the d-back clause. $\dashv$

**Theorem 3.2.3** *Difference logic does not have interpolation.*

PROOF: Let $p$ and $r$ be proposition letters and let

$$\varphi = \mathsf{D}p \wedge \mathsf{D}\neg p$$

and

$$\psi = (\mathsf{E}(r \wedge \neg \mathsf{D}r)) \to \mathsf{E}(\neg r \wedge \mathsf{D}\neg r)$$

It is not difficult to check that $\Vdash \varphi \to \psi$. Let $\mathcal{M} = (M, R, V)$ be a model and $m \in M$. If $\mathcal{M}, m \Vdash \varphi$, then $\mathcal{M}$ must have at least three points. $\mathsf{E}(r \wedge \neg \mathsf{D}r)$ says that $r$ holds at precisely one point of the model (i.e. we define $r$ to be a nominal). So suppose $\mathcal{M}, m \Vdash \mathsf{E}(r \wedge \neg \mathsf{D}r)$. We need to show that $\mathcal{M}, m \Vdash \mathsf{E}(\neg r \wedge \mathsf{D}\neg r)$. But $\mathsf{E}(\neg r \wedge \mathsf{D}\neg r)$ says that there are at least two $\neg r$-points in $\mathcal{M}$. This is obviously the case as $\mathcal{M}$ has only one $r$-point and at least three points in total. Hence $\mathcal{M}, m \Vdash \varphi \to \psi$. As $\mathcal{M}$ and $m$ were arbitrary we have that $\Vdash \varphi \to \psi$. The common language of $\varphi$ and $\psi$ is $\mathcal{L}_{\mathsf{D}}(\emptyset)$. Note that

$$\neg\psi \equiv \mathsf{E}(r \wedge \neg \mathsf{D}r) \wedge \mathsf{A}(r \vee \neg \mathsf{D}\neg r)$$

Now consider the models $\mathcal{N}$ and $\mathcal{N}'$ in figure 3.1. It is easy to check that $\mathcal{N}, n \Vdash \varphi$, $\mathcal{N}, n \Vdash \psi$ and that $\mathcal{N}', n' \Vdash \neg\psi$. But note that $\mathcal{N}, n \,\underline{\leftrightarrow}_{\mathcal{L}_{\mathsf{D}}(\emptyset)}\, \mathcal{N}', n'$, so by lemma 3.2.2 $\mathcal{N}, n \leftrightsquigarrow_{\mathcal{L}_{\mathsf{D}}(\emptyset)} \mathcal{N}', n'$. Hence every formula $\vartheta \in \mathcal{L}_{\mathsf{D}}(\emptyset)$ for which it holds that $\Vdash \varphi \to \vartheta$, is consistent with $\neg\psi$. Thus there can be no formula $\vartheta \in \mathcal{L}_{\mathsf{D}}(\emptyset)$ such that $\Vdash \varphi \to \vartheta$ and $\Vdash \vartheta \to \psi$. $\dashv$

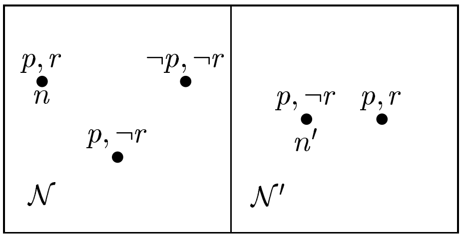


Figure 3.1: Theorem 3.2.3

We make the following convention: Let $\overline{q} = (q_1, q_2, \ldots, q_n)$ be a (possibly empty) sequence of proposition letters and let

- $\pi_0 = \neg q_n \wedge \neg q_{n-1} \wedge \cdots \neg q_2 \wedge \neg q_1$
- $\pi_1 = \neg q_n \wedge \neg q_{n-1} \wedge \cdots \wedge \neg q_2 \wedge q_1$
- $\cdots$
- $\pi_{2^n-1} = q_n \wedge q_{n-1} \wedge \cdots \wedge q_2 \wedge q_1$

Notice that we are counting in binary. Such a conjunction will be referred to loosely as a "conjunction of $q$'s". For the case where $n = 0$, we take the empty conjunction to be $\top$, as usual.

**Definition 3.2.4** Let $\overline{q} = (q_1, q_2, \ldots, q_n)$ be a sequence of proposition letters, and let $\mathcal{M}$ be a model. Let $\pi_j$ be one of the conjunctions as in the above convention. We say $\mathcal{M}$ has

- no $\pi_j$'s if $|[\![\pi_j]\!]_{\mathcal{M}}| = 0$,
- one $\pi_j$ if $|[\![\pi_j]\!]_{\mathcal{M}}| = 1$,
- many $\pi_j$'s if $|[\![\pi_j]\!]_{\mathcal{M}}| > 1$.

We say models $\mathcal{M}$ and $\mathcal{M}'$ have the same $\overline{q}$-profile if they have the same number (in the above sense) of $\pi_i$'s for all $0 \leq i \leq 2^n - 1$. $\lhd$

Note that a model's $\overline{q}$-profile is definable in $\mathcal{L}_{\mathsf{D}}(\overline{q})$: for example, $\neg\mathsf{E}\pi$, $\mathsf{E}(\pi\wedge\neg\mathsf{D}\pi)$ and $\mathsf{E}(\pi\wedge\mathsf{D}\pi)$ are true precisely on models with no $\pi$'s, one $\pi$ and many $\pi$'s respectively. Hence a suitable conjunction of formulas of this form can precisely define a model's $\overline{q}$-profile (assuming $\overline{q}$ is finite, of course). Let us make this precise. For $\pi_i$ a conjunction of $q$'s, let

$$\begin{array}{rcl} \pi_i^0 & := & \neg\mathsf{E}\pi_i \\ \pi_i^1 & := & \mathsf{E}(\pi_i \wedge \neg\mathsf{D}\pi_i) \\ \pi_i^{\mathsf{m}} & := & \mathsf{E}(\pi_i \wedge \mathsf{D}\pi_i) \end{array}$$

Now if $\mathcal{M}$ is any model, a formula of the form $\pi_0^{t_0} \wedge \pi_1^{t_1} \wedge \cdots \wedge \pi_{2^n-1}^{t_{2^n-1}}$ with $t_i \in \{0, 1, \mathsf{m}\}$ for $0 \leq i \leq 2^n - 1$, such that there is at least one $i$ such that $t_i \neq 0$ (if we are to exclude empty models), defines the $\overline{q}$-profile of $\mathcal{M}$.

**Lemma 3.2.5** *Let $\mathcal{M}$ and $\mathcal{M}'$ be models that have the same $\overline{q}$-profile. Let*

$$Z = \{(m, m') : m \in \mathcal{M},\ \textit{and } m' \in \mathcal{M}' \textit{ and for some } i,\ \mathcal{M}, m \Vdash \pi_i \textit{ and } \mathcal{M}', m' \Vdash \pi_i\}$$

*Then $Z$ is a d-relation with respect to $\overline{q}$, such that every point in $\mathcal{M}$ is related to some point in $\mathcal{M}'$, and vice versa.*

Proof: Notice that $Z$ relates all those points that agree on the valuation of the proposition letters in $\overline{q}$. Checking the details is trivial. $\dashv$

**Lemma 3.2.6** *Let $\Gamma(\overline{q}, p)$ be a set of formulas. If there is a model $\mathcal{M}$ of $\Gamma$, points $u, v \in \mathcal{M}$ and a conjunction of $q$'s, say $\pi$, such that $\mathcal{M}, u \Vdash p \wedge \pi$ and $\mathcal{M}, v \Vdash \neg p \wedge \pi$, then $\Gamma$ does not define $p$ implicitly.*

Proof: Suppose there exists such a model $\mathcal{M} = (M, V)$, points $u, v \in \mathcal{M}$ and a conjunction $\pi$ of $q$'s, such that $\mathcal{M}, u \Vdash p \wedge \pi$ and $\mathcal{M}, v \Vdash \neg p \wedge \pi$ . Let $\mathcal{M}^* = (M, V^*)$ with $V^*(q) = V(q)$ for all $q \in \overline{q}$, but $V^*(p) = V(p) - \{u\} + \{v\}$. Then $\mathcal{M}$ and $\mathcal{M}^*$ are d-related: Let $Z = \{(w, w) : w \in M - \{u, v\}\} \cup \{(u, v), (v, u)\}$, i.e. we relate all points in $\mathcal{M}$ except $u$ and $v$ to the corresponding points in $\mathcal{M}^*$, and 'swap' $u$ and $v$. It is trivial to check that $Z$ is a d-relation with respect to $\{\overline{q}, p\}$. Then by lemma 3.2.2 (taking $\Pi = \{\overline{q}, p\}$) $\mathcal{M}^*$ is also a model of $\Gamma$. But now, using Padoa's method, we have proved that $\Gamma$ does not define $p$ implicitly. $\dashv$

**Theorem 3.2.7** *Dif has the Beth property.*

PROOF: Let $\Gamma(\overline{q}, p)$ be a set of formulas that implicitly defines $p$, and suppose $\overline{q} = (q_1, q_2, \ldots, q_n)$. We will show that it is possible to find an explicit definition of $p$, relative to $\Gamma$. It follows from lemma 3.2.6 that for any model of $\mathcal{M}$ of $\Gamma$, $[\![p]\!]_{\mathcal{M}} = \bigcup_{i \in I} [\![\pi_i]\!]_{\mathcal{M}}$ for some $I \subseteq \{0, 1, \ldots, 2^n - 1\}$.

CLAIM: For any models $\mathcal{M}$ and $\mathcal{N}$ of the same $\overline{q}$-profile, if $\mathcal{M} \Vdash \Gamma$ and $\mathcal{N} \Vdash \Gamma$, then $[\![p]\!]_{\mathcal{M}} = \bigcup_{i \in I} [\![\pi_i]\!]_{\mathcal{M}}$ and $[\![p]\!]_{\mathcal{N}} = \bigcup_{i \in I} [\![\pi_i]\!]_{\mathcal{N}}$ for *the same* $I \subseteq \{0, 1, \ldots, 2^n - 1\}$.

PROOF OF CLAIM: Suppose to the contrary that $[\![p]\!]_{\mathcal{M}} = \bigcup_{j \in J} [\![\pi_j]\!]_{\mathcal{M}}$ and $[\![p]\!]_{\mathcal{N}} = \bigcup_{k \in K} [\![\pi_k]\!]_{\mathcal{N}}$ for $J, K \subseteq \{0, 1, \ldots, 2^n - 1\}$ and $J \neq K$. Now, alter the valuation on $\mathcal{M}$ to obtain $\mathcal{M}'$, in such a way that $[\![q]\!]_{\mathcal{M}'} = [\![q]\!]_{\mathcal{M}}$ for all $q \in \overline{q}$, but $[\![p]\!]_{\mathcal{M}'} = \bigcup_{k \in K} [\![\pi_i]\!]_{\mathcal{M}'}$. As $\mathcal{M}$ and $\mathcal{N}$ have the same $\overline{q}$-profile to start with, $\mathcal{M}'$ and $\mathcal{N}$ have the same $(\overline{q}, p)$-profile: To see this, note that $\mathcal{M}'$ and $\mathcal{N}$ have the same number of $\pi_k$'s for each $k \in K$ in the sense of definition 3.2.4 (remember $\mathcal{M}$ and $\mathcal{N}$ have the same $\overline{q}$-profile). Hence by the construction of $\mathcal{M}'$, $\mathcal{M}'$ and $\mathcal{N}$ have the same $(\overline{q}, p)$-profile. By lemma 3.2.5 there is a d-relation (respecting the proposition letters in $\{\overline{q}, p\}$) between $\mathcal{M}'$ and $\mathcal{N}$ relating every point in $\mathcal{M}'$ to some point in $\mathcal{N}$. As $\mathcal{N} \Vdash \Gamma$, $\mathcal{M}' \Vdash \Gamma$. But now, using Padoa's method, we have shown that $\Gamma$ does not implicitly define $p$. Contradiction. $\dashv$

We see that for any given $\overline{q}$-profile, defined by a formula $C$, that there is a set $I \subseteq \{0, 1, \ldots, 2^n - 1\}$, say $I = \{i_1, i_2, \ldots, i_k\}$, such that $\Gamma \Vdash C \to [(\pi_{i_1} \vee \pi_{i_2} \cdots \vee \pi_{i_k}) \leftrightarrow p]$. Let $C_1, C_2, \ldots, C_k$ be a list of all the different $\overline{q}$-profiles of models of $\Gamma$, and $I_1, I_2, \ldots, I_k$ the corresponding subsets of $\{0, 1, \ldots, 2^n - 1\}$. Let $\psi_i = \bigvee_{j \in I_i} \pi_j$. Then

$$\Gamma \Vdash [(C_1 \wedge \psi_1) \vee (C_2 \wedge \psi_2) \vee \cdots \vee (C_k \wedge \psi_k)] \leftrightarrow p,$$

yielding the desired explicit definition. $\dashv$

## 3.3 Adding the Difference Operator to Basic Modal Logic

In the previous section the difference operator was considered in isolation, and the language $\mathcal{L}_{\mathsf{D}}$ was interpreted over models without structure. In this section we combine the local perspective offered by the modal diamond with the global perspective yielded by the difference operator. As we shall see, doing so leads to the loss of Beth definability.

To begin with let us formally define the language and logic we shall be working with. The language $\mathcal{L}_{\Diamond,\mathsf{D}}$ which combines the modal diamond and the difference operator, is built up as follows

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \vee \psi \mid \Diamond\varphi \mid \mathsf{D}\varphi$$

We give semantics to $\mathcal{L}_{\Diamond,\mathsf{D}}$ in the obvious way, interpreting formulas on models with a single binary relation.

**Definition 3.3.1** Let $\mathsf{ML}_{\mathsf{D}} = \langle \mathcal{L}_{\Diamond,\mathsf{D}}, \mathsf{K}, \Vdash \rangle$, where $\mathsf{K}$ is the class of all models. $\lhd$

**Definition 3.3.2** Let $\mathcal{M} = (M, R, V)$ and $\mathcal{M}' = (M', R', V')$ be models. A relation $Z \subseteq M \times M'$ is called a *difference bisimulation* if it satisfies the following.

**bisimulation** $Z$ is a bisimulation

**difference forth** if $uZu'$, $v \in M$ and $v \neq u$, then there exists a point $v' \in M'$ such that $v' \neq u'$ and $vZv'$

**difference back** if $uZu'$, $v' \in M$ and $v' \neq u'$, then there exists a point $v \in M$ such that $v \neq u$ and $vZv'$

We write $\mathcal{M}, m \leftrightarroweq_{\mathcal{L}_{\diamond,\mathsf{D}}} \mathcal{M}', m'$ to indicate that there exists a difference bisimulation between $m$ and $m'$, or $\mathcal{M}, m \leftrightarroweq_{\mathcal{L}_{\diamond,\mathsf{D}}(\Pi)} \mathcal{M}', m'$ if $\Pi$ is the set of proposition letters with respect to which the local harmony clause of the bisimulation is taken. $\lhd$

The proof of the next lemma is standard.

**Lemma 3.3.3** *Let $\mathcal{M} = (M, R, V)$ and $\mathcal{M}' = (M', R', V')$ be models, $m \in M$ and $m' \in M'$. If $\mathcal{M}, m \leftrightarroweq \mathcal{L}_{\diamond,\mathsf{D}}\mathcal{M}', m'$ then $\mathcal{M}, m \leftrightsquigarrow \mathcal{L}_{\diamond,\mathsf{D}}\mathcal{M}', m'$.*

**Theorem 3.3.4** *$\mathsf{ML}_\mathsf{D}$ does not have the Beth property.*

PROOF: Let $p$,$q$ and $r$ be proposition letters and let

$$\Gamma(q, r, p) = \left\{ \begin{array}{l} \mathsf{E}(p \wedge q \wedge \neg\mathsf{D}(p \wedge q)) \\ \neg q \to p \\ \mathsf{E}r \\ r \to \Diamond\top \\ r \to \Box(p \wedge q) \end{array} \right\}$$

The intuitive content of this proof is the following. On any model of $\Gamma$ all the $\neg q$-points are $p$-points, and further there is precisely one $(p \wedge q)$-point. But at which $q$-point must $p$ hold? This is pointed out by some $r$-point: the unique point which is the successor of all the $r$-points. The death-blow to any aspiring explicit definition however, is the fact that a $p$-points might not be able to "see" if it is being pointed at by an $r$-point. Let us make this precise.

CLAIM: If $\mathcal{M} = (M, R, V)$ and $\mathcal{M} \Vdash \Gamma$, then

$$[\![p]\!]_\mathcal{M} = [\![\neg q]\!]_\mathcal{M} \cup \{m \in M : \mathcal{M}, m \Vdash q \text{ and there is a } w \in M \text{ such that } \mathcal{M}, w \Vdash r \text{ and } Rwm\}$$

PROOF OF CLAIM: For convenience of notation put $S = \{m \in M : \mathcal{M}, m \Vdash q$ and there is a $w \in M$ such that $\mathcal{M}, w \Vdash r$ and $Rwm\}$. Firstly note that as $\mathcal{M} \Vdash \mathsf{E}(p \wedge q \wedge \neg\mathsf{D}(p \wedge q))$, there is precisely one $(p \wedge q)$-point in $\mathcal{M}$. Suppose that $x \in [\![p]\!]_\mathcal{M}$. If $x \notin [\![\neg q]\!]_\mathcal{M}$, then $x$ is the unique $(p \wedge q)$-point in $\mathcal{M}$. There exists at least one $r$-point in $\mathcal{M}$, say $y$. As $\mathcal{M} \Vdash (r \to \Diamond\top) \wedge (r \to \Box(p \wedge q))$ it must be the case that $Ryx$. Hence $x \in S$. Thus $[\![p]\!]_\mathcal{M} \subseteq [\![\neg q]\!]_\mathcal{M} \cup S$.

To prove the other inclusion assume that $x \in [\![\neg q]\!]_{\mathcal{M}} \cup S$. If $x \in [\![\neg q]\!]_{\mathcal{M}}$ then as $\mathcal{M} \Vdash \neg q \to p$ we have $x \in [\![p]\!]_{\mathcal{M}}$. So suppose $x \in S$. Then there is a point $y$ such that $y \Vdash r$ and $Ryx$. As $\mathcal{M} \Vdash r \to \Box(p \wedge q)$ it must be the case that $x \in [\![p]\!]_{\mathcal{M}}$. Thus $[\![\neg q]\!]_{\mathcal{M}} \cup S \subseteq [\![p]\!]_{\mathcal{M}}$. Combining the two inclusions we have $[\![p]\!]_{\mathcal{M}} = [\![\neg q]\!]_{\mathcal{M}} \cup S$. ⊣

So $\Gamma$ implicitly defines $p$. But now consider the model in figure 3.2. It is easy to check that it indeed models $\Gamma$. Note that $\alpha \mathrel{\underline{\leftrightarrow}}_{\mathcal{L}_{\Diamond,\mathsf{D}}(q)} \beta$ (the relation relating the top point to itself and $\alpha$ and $\beta$ to each other will suffice.) This means $\alpha \leftrightsquigarrow_{\mathcal{L}_{\Diamond,\mathsf{D}}(q)} \beta$. Hence there can be no explicit definition of $p$ relative to $\Gamma$. ⊣

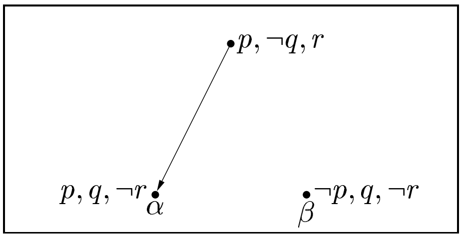


Figure 3.2: Theorem 3.3.4

The most natural response to the counterexample given in the proof of theorem 3.3.4 would be to insist that points should indeed be able to "look back" along the relation $R$, i.e. one would feel that this situation cannot arise in tense logic. Following up on this hunch in the next section, we investigate what happens when we add the difference operator to tense logic.

## 3.4 Adding the Difference Operator to Tense Logic

Recall that the language of basic tense logic, which we shall denote with $\mathcal{L}_t$, is recursively defined as follows.

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \vee \psi \mid \mathsf{F}\varphi \mid \mathsf{P}\varphi$$

This language is interpreted over bidirectional frames, i.e. the relations used for interpreting the $\mathsf{F}$ (future) and $\mathsf{P}$ (past) modalities are each other's converses. Denote the converse of a relation $R$ with $\breve{R}$. We will not go into any details here, but the interested reader is referred to [Blackburn et al., 2001].

Following up on the remarks made at the end of the previous section, we now add the difference operator to tense logic. Define the language $\mathcal{L}_{t,\mathsf{D}}$ as follows.

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \vee \psi \mid \mathsf{F}\varphi \mid \mathsf{P}\varphi \mid \mathsf{D}\varphi$$

The duals of $\mathsf{F}$ and $\mathsf{P}$ are $\mathsf{G}$ and $\mathsf{H}$ respectively. This language will be interpreted over models with one binary relation. If $\mathcal{M} = (M, R, V)$ is a model and $m \in M$ then

- $\bot$, proposition letters and boolean combinations are given the usual semantics

- $\mathcal{M}, m \Vdash \mathsf{F}\varphi$ iff there exists a point $n \in M$ such that $Rmn$ and $\mathcal{M}, n \Vdash \varphi$
- $\mathcal{M}, m \Vdash \mathsf{P}\varphi$ iff there exists a point $n \in M$ such that $Rnm$ and $\mathcal{M}, n \Vdash \varphi$
- $\mathcal{M}, m \Vdash \mathsf{D}\varphi$ iff there exists a point $n \in M$ such that $n \neq m$ and $\mathcal{M}, n \Vdash \varphi$

**Definition 3.4.1** Let $\mathsf{TL}_\mathsf{D} = \langle \mathcal{L}_{t,\mathsf{D}}, \mathsf{K}, \Vdash \rangle$, where $\mathsf{K}$ is the class of all models. $\lhd$

**Definition 3.4.2** Let $\mathcal{M} = (M, R, V)$ and $\mathcal{M}' = (M', R', V')$ be models. A relation $Z \subseteq M \times M'$ is called a *temporal difference bisimulation* if the following hold

**bisimulation** $Z$ is a (multi-modal) bisimulation with the back- and forth clauses taken with respect to the relations $R$ and $\breve{R}$.

**difference forth** if $uZu'$, $v \in M$ and $v \neq u$, then there exists a point $v' \in M'$ such that $v' \neq u'$ and $vZv'$

**difference back** if $uZu'$, $v' \in M$ and $v' \neq u'$, then there exists a point $v \in M$ such that $v \neq u$ and $vZv'$

We write $\mathcal{M}, m \leftrightarrows_{\mathcal{L}_{t,\mathsf{D}}} \mathcal{M}', m'$ to indicate that there exists a temporal difference bisimulation linking $m$ and $m'$, parameterizing with sets of proposition letters as the need arises. $\lhd$

The proof of the following lemma is routine.

**Lemma 3.4.3** *Let $\mathcal{M}$ and $\mathcal{M}'$ be models, $m \in \mathcal{M}$ and $m' \in \mathcal{M}'$. If $\mathcal{M}, m \leftrightarrows_{\mathcal{L}_{t,D}} \mathcal{M}', m'$ then $\mathcal{M}, m \leftrightsquigarrow_{\mathcal{L}_{t,D}} \mathcal{M}', m'$.*

**Theorem 3.4.4** *$\mathsf{TL}_\mathsf{D}$ does not have the Beth property.*

PROOF: Let $p$ and $q$ be proposition letters and let

$$\Gamma(q,p) = \left\{ \begin{array}{l} \mathsf{E}((p \wedge \neg q) \wedge \neg \mathsf{D}(p \wedge \neg q)) \\ q \to p \\ (p \wedge \neg q) \to \mathsf{F}\top \\ (p \wedge \neg q) \to \mathsf{G}q \\ q \to \mathsf{P}(p \wedge \neg q) \\ \mathsf{E}((q \wedge \mathsf{P}(\neg p)) \wedge \neg \mathsf{D}(q \wedge \mathsf{P}(\neg p))) \\ \mathsf{E}(q \wedge \mathsf{D}q) \end{array} \right\}$$

CLAIM: If $\mathcal{M}$ is any model and $\mathcal{M} \Vdash \Gamma$ then

$$[\![p]\!]_\mathcal{M} = [\![q]\!]_\mathcal{M} \cup \{m \in M : \mathcal{M}, m \Vdash \neg q \text{ and for all } n \in M,\ Rmn \text{ iff } \mathcal{M}, n \Vdash q\}$$

PROOF OF CLAIM: Let $S = [\![q]\!]_\mathcal{M} \cup \{m \in M : \mathcal{M}, m \Vdash \neg q$ and for all $n \in M,\ Rmn$ iff $\mathcal{M}, n \Vdash q\}$. Suppose $x \in [\![p]\!]_\mathcal{M}$. If $\mathcal{M}, x \Vdash q$, then $x \in S$, so suppose $\mathcal{M}, x \Vdash \neg q$ (note that this means $x$ is the unique point in $M$ satisfying $p \wedge \neg q$). Let $n \in M$ arbitrarily. If $Rxn$ then, as $\mathcal{M} \Vdash (p \wedge \neg q) \to \mathsf{G}q$ and $\mathcal{M}, x \Vdash p \wedge \neg q$ we must have that $\mathcal{M}, n \Vdash q$. Also, if $\mathcal{M}, n \Vdash q$, as $\mathcal{M} \Vdash q \to \mathsf{P}(p \wedge \neg q)$ and $x$ is the unique point in $M$ where both $p$ and $\neg q$ hold, we must have that $Rxn$. So for all $n \in M$, $Rxn$ iff $\mathcal{M}, n \Vdash q$. So also in this case $x \in S$. Hence we have $[\![p]\!]_{\mathcal{M}} \subseteq S$.

Now suppose $x \in S$. If $x \in [\![q]\!]_{\mathcal{M}}$, then as $\mathcal{M} \Vdash q \rightarrow p$, we have $x \in [\![p]\!]_{\mathcal{M}}$. So suppose $x \in \{m \in M : \mathcal{M}, m \Vdash \neg q$ and for all $n \in M$, $Rmn$ iff $\mathcal{M}, n \Vdash q\}$. Note that $\mathcal{M}, x \Vdash \neg q$, and that $x$ "sees" all the $q$-points. There are at least two $q$-points in $M$ ($\mathcal{M} \Vdash \mathsf{E}(q \wedge \mathsf{D}q)$), say $n_1$ and $n_2$, and $Rxn_1$ and $Rxn_2$. As $\mathcal{M} \Vdash \mathsf{E}((q \wedge \mathsf{P}(\neg p)) \wedge \neg\mathsf{D}(q \wedge \mathsf{P}(\neg p)))$, it can not be the case that $\mathcal{M}, x \Vdash \neg p$ (lest we are to have two $q$-points seen by a $\neg p$-point), so $x$ must be a $p$-point, i.e. $x \in [\![p]\!]_{\mathcal{M}}$. Hence $S \subseteq [\![p]\!]_{\mathcal{M}}$, and $S = [\![p]\!]_{\mathcal{M}}$ ⊣

So $\Gamma$ implicitly defines $p$. Now consider the model depicted in figure 3.3. It is easy to check that this is a model of $\Gamma$. But points $\alpha$ and $\beta$ are indistinguishable in the language $\mathcal{L}_{t,\mathsf{D}}(q)$. This can be seen by defining an appropriate temporal difference bisimulation, linking $\alpha$ and $\beta$, as is done in the figure (as indicated by the dotted lines). Note that the bisimulation relates each node to itself but that, in order to avoid clutter, these loops are not drawn. Hence there can be no explicit definition of $p$ relative to $\Gamma$. ⊣

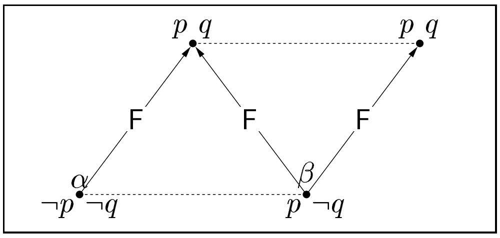


Figure 3.3: Theorem 3.4.4

In the next chapter we consider languages which have much more powerful "counting abilities". A number of the (counter)examples given in the present chapter will be used there with very little (or even no) change.

# Chapter 4

# Graded Modalities

## 4.1 Counting Successors

Modal logic only allows one to express the *existence* a successor satisfying a certain condition, or that *all* successors satisfy a condition. We cannot count: we are not ably to say that a point has, say, *five* successors at which $p$ is true (unless there are enough other formulas on which they differ, of course). From the perspective of modal logic, if all the "blind" points in figure 4.1 have the same valuation, points $a$, $b$ and $c$ are indistinguishable. This may certainly be an advantage in many settings and is, one might even argue, part of the beauty and simplicity of modal logic. There are cases however, where this is a drawback.

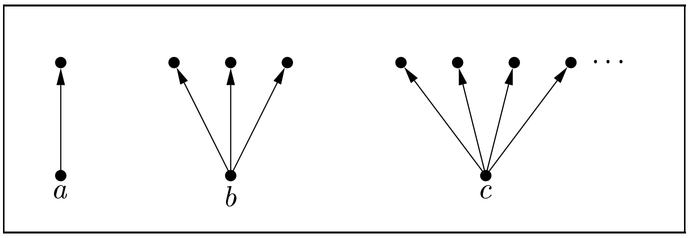


Figure 4.1: Indistinguishable Points

So-called graded modalities have been studied in a number of settings, see for example [D'Agostino, 1998], [Andréka et al., 1995], [Fine, 1972] and [Van der Hoek and de Rijke, 1995]. As an example, Description logics (see [Baader and Nutt, 2002]), provide tools to make statements concerning numbers of successors. This is not the place for a discussion of these logics, nor shall we be working with description logics as such. It is enough to note that they have a very strong modal flavour, and in many cases constitute nothing more than modal logics in disguise. For these latter cases [Baader and Nutt, 2002] formulates a nice version of Beth definability, as a direct corollary of the Beth definability theorem for (multi-) modal logic. As description logics routinely deal with predicates such as MAN, WOMAN and ANIMAL, and relations such as HAS-CHILD and IS-MARRIED, the notions of implicit- and explicit definitions we have been dealing with take on very pretty concrete forms in these logics.

It might also sometimes be desirable to take a more global perspective and have the expressive capability to name the exact number of states in a model satisfying a certain condition. Eg., still thinking in the jargon of description logic, we might like to assert that there are exactly 314 women resident in Pofadder, or at least 5000 elderly widowers with dentures idling away their days in Port Edward.

Of importance for our purposes is the notion of "graded modalities" or "counting modalities". Our treatment will be in the framework of the modal systems as introduced in section 1.1.1, but the results are easily reformulated in the terminology of description logics, or whatever other formalism one prefers.

Before going into the details, we should mention the following. In [D'Agostino, 1998] it is claimed that the logic $\mathcal{L}^{\Diamond<\aleph_0}$, corresponding to our system $\mathsf{G}_2$ below, has interpolation, and that this has been proved in [Andréka et al., 1995]. Our copy of the latter paper however, mentions what we call $\mathsf{G}_3$ instead of $\mathsf{G}_2$, and states (without proof) that $\mathsf{G}_3$ has interpolation. From this our theorem 4.5.2 would easily follow, using proposition 1.3.3 and an easy local deduction theorem. Due to lack of time we were unfortunately not able to settle this issue yet, but correspondence with the cited authors promises to clear this up soon.

## 4.2 Three Languages with Graded Modalities

We shall only study models that have at most a single, binary relation. We now introduce three types of graded modalities, with three basic corresponding languages, and semantics for these. Combinations and extensions of these are subsequently considered.

### 4.2.1 Type 1

We define the language $\mathcal{L}_1$ inductively as follows, where $n$ ranges over $\omega$.[1]

$$\varphi ::= p \mid \top \mid \neg\psi \mid \psi \vee \varphi \mid \ \lhd_{\leq n} \top$$

Let $\mathcal{M} = (M, R, V)$ be a model and $m \in M$. Proposition letters and boolean connectives are interpreted with the usual semantics. If $n \in \omega$ then

$$\mathcal{M}, m \Vdash \lhd_{\leq n}\top \text{ if and only if } |\{y \in M : Rmy\}| \leq n$$

Thus $\lhd_{\leq n}\top$ holds at precisely those points that have at most $n$ successors. Note that, as defined, this language does not have the expressive capability to make any assertions as to the nature of these successors, other than their existence. We can now define formulas that hold at precisely those points that have, respectively, strictly more that $n$ successors, at least $n$ successors, or precisely $n$ successors:

- $\lhd_{>n}\top \equiv \neg \lhd_{\leq n} \top$

[1] Note that one might be tempted to use, instead of the $\lhd$-symbol, the $\Diamond$-symbol to denote the operator (and one does indeed find this usage in the literature), especially in the light of the resemblance of the subsequently defined language $\mathcal{L}_2$ to an amplified version of basic modal logic. As $\lhd_{\leq n}$ does not distribute over disjunction, however, it is not a "diamond" in the usual sense of the word. We choose to reserve the $\Diamond$-notation only for operators that are indeed diamonds in this sense.

- $\lhd_{\geq n}\top \equiv \lhd_{>n-1}\top$
- $\lhd_{=n}\top \equiv (\lhd_{\leq n}\top \wedge \lhd_{\geq n}\top)$.

**Definition 4.2.1** Let $\mathsf{G}_1$ be the logic $\langle \mathcal{L}_1, \mathsf{K}, \Vdash_{\mathsf{G}_1}\rangle$, where $\mathcal{L}_1$ is as defined above, $\mathsf{K}$ is the class of all models with one, binary relation, and $\Vdash_{\mathsf{G}_1}$ is the consequence relation for the semantics of $\mathcal{L}_1$ as defined above. $\lhd$

**Definition 4.2.2** The standard translation, $\mathsf{ST}_x(\cdot)$, of $\mathcal{L}_1$ is defined inductively as follows:

- $\mathsf{ST}_x(\top) = \top$
- $\mathsf{ST}_x(p) = P(x)$
- $\mathsf{ST}_x(\neg\varphi) = \neg\mathsf{ST}_x(\varphi)$
- $\mathsf{ST}_x(\varphi \vee \psi) = \mathsf{ST}_x(\varphi) \vee \mathsf{ST}_x(\psi)$
- $\mathsf{ST}_x(\lhd_{\leq n}\top) = \neg(\exists y_1, \ldots, \exists y_n, \exists y_{n+1}((\bigwedge_{i\neq j} y_i \neq y_j) \wedge (\bigwedge_{i=1}^{n+1} R(x, y_i))))$

$\lhd$

As no confusion ought to arise in practice, and in favour of lightness of notation, we refrain from sub- or superscription the standard translation function to indicate the particular logic for which it is intended. It is easy to check that, for any formula $\varphi \in \mathcal{L}_1$, any model $\mathcal{M}$ and $m \in \mathcal{M}$,

$$\mathcal{M}, m \Vdash_{\mathsf{G}_1} \varphi \text{ if and only if } \mathcal{M} \models \mathsf{ST}_x(\varphi)[x/m]$$

**Remark 4.2.3** *The existence of the standard translation, together with the fact that $\mathsf{K}$ is the class of all models, means that $\mathsf{G}_1$ is elementary.*

### 4.2.2 Type 2

The language we shall now define is quite similar to $\mathcal{L}_1$, but yet much more expressive. Whereas $\mathcal{L}_1$ allowed only the constant $\top$ to appear in the scope of the operator $\lhd_{\leq n}$, $\mathcal{L}_2$ will allow *any* formula in the scope of $\lhd_{\leq n}$. Let $n$ range over $\omega$. Then we recursively define $\mathcal{L}_2$ as follows:

$$\varphi ::= p \mid \top \mid \neg\psi \mid \psi \vee \varphi \mid \lhd_{\leq n}\varphi$$

Let $\mathcal{M} = (M, R, V)$ be a model and $m \in M$. Once again proposition letters and boolean connectives are interpreted with the usual semantics. If $n \in \omega$ then

$$\mathcal{M}, m \Vdash \lhd_{\leq n}\varphi \text{ if and only if } |\{y \in M : Rmy \text{ and } \mathcal{M}, y \Vdash \varphi\}| \leq n$$

Whereas with $\mathcal{L}_1$ we were unable to "look" further than the immediate successors of a point, $\mathcal{L}_2$ clearly subsumes ordinary modal logic in terms of expressivity. The ordinary modal diamond is definable in $\mathcal{L}_2$, along with the following analogues of what we were able to define for $\mathcal{L}_1$.

- $\lhd_{>n}\varphi \equiv \neg \lhd_{\leq n} \varphi$
- $\lhd_{\geq n}\varphi \equiv \lhd_{>n-1}\varphi$
- $\lhd_{=n}\varphi \equiv (\lhd_{\leq n}\varphi \wedge \lhd_{\geq n}\varphi)$.
- $\Diamond\varphi \equiv \lhd_{\geq 1}\varphi$

**Definition 4.2.4** Let $\mathsf{G}_2 = \langle \mathcal{L}_2, \mathsf{K}, \Vdash_{\mathsf{G}_2}\rangle$, where $\mathcal{L}_2$ is as defined above, $\mathsf{K}$ is the class of all models with one, binary relation, and $\Vdash_{\mathsf{G}_2}$ is the consequence relation for the semantics of $\mathcal{L}_2$ as defined above. $\lhd$

**Definition 4.2.5** The standard translation, $\mathsf{ST}_x(\cdot)$, of $\mathcal{L}_2$ is defined inductively as follows:

- $\mathsf{ST}_x(\top) = \top$
- $\mathsf{ST}_x(p) = P(x)$
- $\mathsf{ST}_x(\neg\varphi) = \neg\mathsf{ST}_x(\varphi)$
- $\mathsf{ST}_x(\varphi \vee \psi) = \mathsf{ST}_x(\varphi) \vee \mathsf{ST}_x(\psi)$
- $\mathsf{ST}_x(\lhd_{\leq n}\varphi) = \neg(\exists y_1, \ldots, \exists y_n, \exists y_{n+1}((\bigwedge_{i\neq j} y_i \neq y_j) \wedge (\bigwedge_{i=1}^{n+1} \mathsf{ST}_{y_i}(\varphi)) \wedge (\bigwedge_{i=1}^{n+1} R(x, y_i)))$

$\lhd$

Once again it is easy to check that this translation works as intended, i.e. if $\varphi$ is any formula in $\mathcal{L}_2$, $\mathcal{M}$ a model and $m \in \mathcal{M}$, then

$$\mathcal{M}, m \Vdash_{\mathsf{G}_2} \varphi \text{ if and only if } \mathcal{M} \models \mathsf{ST}_x(\varphi)[x/m]$$

**Remark 4.2.6** *The standard translation, together with the fact that $\mathsf{K}$ is the class of all models, means that $\mathsf{G}_2$ is elementary.*

### 4.2.3 Type 3

Now we take a global perspective. The language $\mathcal{L}_3$ introduced now will be used to express facts about the number of points in the *entire modal* satisfying a certain formula. In its basic form however, it will be interpreted over models with no relations, i.e. merely sets together with valuations defined on them. Clearly this is a simple language intended for talking about simple structures. The recursive definition of $\mathcal{L}_3$ is as follows, where once again $n$ ranges over $\omega$.

$$\varphi ::= p \mid \top \mid \neg\psi \mid \psi \vee \varphi \mid \mathsf{U}_{\leq n}\varphi$$

The expected abbreviations, as well as the difference operator and universal modality, are defined as follows.

- $\mathsf{U}_{>n}\varphi \equiv \neg\mathsf{U}_{\leq n}\varphi$
- $\mathsf{U}_{\geq n}\varphi \equiv \mathsf{U}_{>n-1}\varphi$

- $\mathsf{U}_{=n}\varphi \equiv (\mathsf{U}_{\leq n}\varphi \wedge \mathsf{U}_{\geq n}\varphi)$.
- $\mathsf{D}\varphi \equiv (\varphi \wedge \mathsf{U}_{\geq 2}\varphi) \vee (\neg\varphi \wedge \mathsf{U}_{\geq 1}\varphi)$
- $\mathsf{E}\varphi \equiv \varphi \vee \mathsf{D}\varphi$

**Definition 4.2.7** Let $\mathsf{G}_3 = \langle \mathcal{L}_3, \mathsf{K}, \Vdash_{\mathsf{G}_3}\rangle$, where $\mathcal{L}_3$ is as defined above, $\mathsf{K}$ is the class of all models with no relations (i.e the class of all sets), and $\Vdash_{\mathsf{G}_3}$ is the consequence relation for the semantics of $\mathcal{L}_3$ as defined above. $\lhd$

**Definition 4.2.8** The standard translation, $\mathsf{ST}_x(\cdot)$, of $\mathcal{L}_3$ is defined inductively as follows:

- $\mathsf{ST}_x(\top) = \top$
- $\mathsf{ST}_x(p) = P(x)$
- $\mathsf{ST}_x(\neg\varphi) = \neg\mathsf{ST}_x(\varphi)$
- $\mathsf{ST}_x(\varphi \vee \psi) = \mathsf{ST}_x(\varphi) \vee \mathsf{ST}_x(\psi)$
- $\mathsf{ST}_x(\mathsf{U}_{\leq n}\varphi) = \neg(\exists y_1, \ldots, \exists y_n, \exists y_{n+1}((\bigwedge_{i\neq j} y_i \neq y_j) \wedge (\bigwedge_{i=1}^{n+1} \mathsf{ST}_{y_i}(\varphi))))$

$\lhd$

As expected we have that for all models $\mathcal{M} = (M, V)$ and $m \in M$

$$\mathcal{M}, m \Vdash_{\mathsf{G}_3} \varphi \text{ if and only if } \mathcal{M} \models \mathsf{ST}_x(\varphi)[x/m]$$

**Remark 4.2.9** *$G_3$ is elementary.*

### 4.2.4 Extensions and Combinations

We can of course add other operators (eg. the modal diamond and the difference operator) to the languages defined in sections 4.2.1, 4.2.2 and 4.2.3, as well as take combinations of these languages. The resulting languages have the obvious semantics, standard translations and logics associated with them (we indicate the latter with $\mathsf{G}_{1,\mathsf{D}}$, $\mathsf{G}_{2,\mathsf{D},\Diamond}$ etc.). In the following we investigate definability for those listed in table 4.1.

Note that in fact $\mathcal{L}_{2,\Diamond} = \mathcal{L}_2$ and that $\mathcal{L}_{3,\mathsf{D}} = \mathcal{L}_3$, as the additional operators are already definable in the smaller languages. Similarly, as $\mathcal{L}_{1,2} = \mathcal{L}_2$, we do not include the former in table 4.1.

## 4.3 Beth definability for $\mathbf{G}_1$ and extensions

As the following lemma shows, the $\mathcal{L}_1$-formulas satisfied at a point, are determined entirely by the valuation at that point and the number of its successors. If $\mathcal{M}$ is a model, we shall say that two points $m, m' \in \mathcal{M}$ "see" the same number of points in $\mathcal{M}$, if $\{x \in M : Rmx\}$ and $\{x \in M : Rm'x\}$ are either both finite and have the same cardinality, or are both infinite (but not necessarily of the same infinite cardinality).

| | | |
|---|---|---|
| $\mathcal{L}_{1,\mathsf{D}}$ | : | $\varphi ::= p \mid \top \mid \neg\psi \mid \psi \vee \varphi \mid \ \lhd_{\leq n} \top \mid \mathsf{D}\varphi$ |
| $\mathcal{L}_{1,\Diamond}$ | : | $\varphi ::= p \mid \top \mid \neg\psi \mid \psi \vee \varphi \mid \ \lhd_{\leq n} \top \mid \Diamond\varphi$ |
| $\mathcal{L}_{1,\Diamond,\mathsf{D}}$ | : | $\varphi ::= p \mid \top \mid \neg\psi \mid \psi \vee \varphi \mid \ \lhd_{\leq n} \top \mid \Diamond\varphi \mid \mathsf{D}\varphi$ |
| $\mathcal{L}_{2,\mathsf{D}}$ | : | $\varphi ::= p \mid \top \mid \neg\psi \mid \psi \vee \varphi \mid \ \lhd_{\leq n} \varphi \mid \mathsf{D}\varphi$ |
| $\mathcal{L}_{2,\Diamond}$ | : | $\varphi ::= p \mid \top \mid \neg\psi \mid \psi \vee \varphi \mid \ \lhd_{\leq n} \varphi \mid \Diamond\varphi$ |
| $\mathcal{L}_{2,\Diamond,\mathsf{D}}$ | : | $\varphi ::= p \mid \top \mid \neg\psi \mid \psi \vee \varphi \mid \ \lhd_{\leq n} \varphi \mid \Diamond\varphi \mid \mathsf{D}\varphi$ |
| $\mathcal{L}_{3,\mathsf{D}}$ | : | $\varphi ::= p \mid \top \mid \neg\psi \mid \psi \vee \varphi \mid \mathsf{U}_{\leq n}\varphi \mid \mathsf{D}\varphi$ |
| $\mathcal{L}_{3,\Diamond}$ | : | $\varphi ::= p \mid \top \mid \neg\psi \mid \psi \vee \varphi \mid \mathsf{U}_{\leq n}\varphi \mid \Diamond\varphi$ |
| $\mathcal{L}_{3,\Diamond,\mathsf{D}}$ | : | $\varphi ::= p \mid \top \mid \neg\psi \mid \psi \vee \varphi \mid \mathsf{U}_{\leq n}\varphi \mid \Diamond\varphi \mid \mathsf{D}\varphi$ |
| $\mathcal{L}_{2,3}$ | : | $\varphi ::= p \mid \top \mid \neg\psi \mid \psi \vee \varphi \mid \mathsf{U}_{\leq n}\varphi \mid \ \lhd_{\leq n} \varphi$ |
| $\mathcal{L}_{1,3}$ | : | $\varphi ::= p \mid \top \mid \neg\psi \mid \psi \vee \varphi \mid \mathsf{U}_{\leq n}\varphi \mid \ \lhd_{\leq n} \top$ |

Table 4.1: Languages with Graded Modalities

**Lemma 4.3.1** *Let $\Phi$ be a set of proposition letters. Let $n \in \mathcal{N}$ and $n' \in \mathcal{N}'$, where $\mathcal{N}$ and $\mathcal{N}'$ are any models. If $n$ and $n'$ have the same $\Phi$-valuation and see the same number of points, then $n \leftrightsquigarrow_{\mathcal{L}_1(\Phi)} n'$.*

PROOF: Let $\varphi \in \mathcal{L}_1(\Phi)$. We show by induction on $\varphi$ that $\mathcal{N}, n \Vdash \varphi$ iff $\mathcal{N}', n' \Vdash \varphi$. The cases for proposition letters and the boolean connectives are trivial, so suppose $\varphi$ is of the form $\lhd_{\leq n}\top$ for some $n \in \omega$. Then $\mathcal{N}, m \Vdash \lhd_{\leq n}\top$ if and only if $m$ sees at most $n$ points in $\mathcal{N}$ if and only if $m'$ sees at most $n$ points in $\mathcal{N}'$ if and only if $\mathcal{N}', m' \Vdash \lhd_{\leq n}$. ⊣

**Lemma 4.3.2** *Let $\mathcal{M}$ and $\mathcal{N}$ be models, and $\mathcal{M} \uplus \mathcal{N}$ their disjoint union. For any points $m \in \mathcal{M}$ and $n \in \mathcal{N}$ it is the case that*

$$\mathcal{M}, m \leftrightsquigarrow_{\mathcal{L}_1} \mathcal{M} \uplus \mathcal{N}, m$$

*and*

$$\mathcal{N}, n \leftrightsquigarrow_{\mathcal{L}_1} \mathcal{M} \uplus \mathcal{N}, n.$$

PROOF: A direct consequence of lemma 4.3.1. ⊣

**Theorem 4.3.3** *$\mathsf{G}_1$ has the Beth property.*

PROOF: Let $\Gamma(\overline{q}, p)$ be a set of formulas that implicitly defines $p$. Note that it follows from remark 4.2.3 that $\mathsf{G}_1$ is compact. This, together with lemma 4.3.2, allows us to apply corollary 2.2.2. Let $\mathcal{M} = (M, R, V)$ be any model of $\Gamma$ and $m_1, m_2 \in M$ such that $m_1 \leftrightsquigarrow_{\mathcal{L}_1(\overline{q})} m_2$. Hence, by corollary 2.2.2 it is sufficient to show that $m_1 \leftrightsquigarrow_{\mathcal{L}_1(\overline{q},p)} m_2$. This is what we will now proceed to do.

CLAIM: Let $m, m' \in M$. If $m$ and $m'$ have the same $\overline{q}$-valuation and see the same number of points, then $m$ and $m'$ have the same $p$-valuation.

PROOF OF CLAIM: Suppose to the contrary that $m$ and $m'$ have the same $q$-valuation and see the same number of points, but that $\mathcal{M}, m \Vdash p$ and $\mathcal{M}, m' \Vdash \neg p$. We will create a new model, $\mathcal{M}' = (M, R, V')$ of $\Gamma$ on the frame $(M, R)$, having the same $\overline{q}$-valuation as $\mathcal{M}$ but a different $p$-valuation. Let $V'(q) = V(q)$ for all $q \in \overline{q}$, let the $V'(p) = (V(p) - \{m\}) \cup \{m'\}$. This is to say, $V'$ "swaps" the $p$-valuations of $m$ and $m'$. By lemma 4.3.1 we have that for each $x \in M - \{m, m'\}$, $\mathcal{M}, x \leftrightsquigarrow_{\mathcal{L}_1(\overline{q},p)} \mathcal{M}', x$. Further $\mathcal{M}, m \leftrightsquigarrow_{\mathcal{L}_1(\overline{q},p)} \mathcal{M}', m'$ and $\mathcal{M}, m' \leftrightsquigarrow_{\mathcal{L}_1(\overline{q},p)} \mathcal{M}', m$. Hence every point in $\mathcal{M}'$ is modally equivalent to a point in $\mathcal{M}$, so $\mathcal{M}' \Vdash \Gamma$. This contradicts the fact that $\Gamma$ implicitly defines $p$. $\dashv$

So as $m_1 \leftrightsquigarrow_{\mathcal{L}_1(\overline{q})} m_2$, we know that $m_1$ and $m_2$ have the same $\overline{q}$-valuation and see the same number of points, so by the above claim $m_1$ and $m_2$ have the same $p$-valuation also. Hence by lemma 4.3.1 $m_1 \leftrightsquigarrow_{\mathcal{L}_1(\overline{q},p)} m_2$, and the proof is complete. $\dashv$

Now let us add the difference operator to $\mathcal{L}_1$ to obtain $\mathcal{L}_{1,\mathsf{D}}$ and see what happens. The following lemma shows that, within a model, states having the same valuation and seeing the same number of points, satisfy the same $\mathcal{L}_{1,\mathsf{D}}$-formulas.

**Lemma 4.3.4** *Let $\Phi$ be a set of proposition letters. Let $\mathcal{N}$ be any model and $n, n' \in \mathcal{N}$ such that $n$ and $n'$ have the same $\Phi$-valuation and see the same number of points. Then $\mathcal{N}, n \leftrightsquigarrow_{\mathcal{L}_{1,D}(\Phi)} \mathcal{N}, n'$.*

PROOF: We may assume that $n \neq n'$. Let $\varphi \in \mathcal{L}_{1,\mathsf{D}}(\Phi)$ and proceed by induction to show that $n \Vdash \varphi$ if and only if $n' \Vdash \varphi$. The cases for proposition letters and the boolean connectives are trivial and the case for $\varphi$ of the form $\lhd_{\leq n}\top$ is the same as in the proof of lemma 4.3.1. So it remains to check the case for $\varphi$ of the form $\mathsf{D}\psi$. Note that $n \Vdash \mathsf{D}\psi$ iff there exists a point $w \neq n$ such that $w \Vdash \psi$. Now there are two cases, either $w = n'$ or $w \neq n'$. Clearly in the second case it follows that $n' \Vdash \mathsf{D}\psi$. In the first case we may apply the inductive hypothesis and conclude that since $n' \Vdash \psi$, also $n \Vdash \psi$ and hence $n' \Vdash \mathsf{D}\psi$. Showing the other direction is precisely symmetrical. $\dashv$

**Theorem 4.3.5** *$G_{1,D}$ has the Beth property.*

PROOF: Let $\Gamma(\overline{q}, p)$ be a set of formulas that implicitly defines $p$. Note that $\mathsf{G}_{1,\mathsf{D}}$ is elementary and hence compact. Let $\mathcal{M}_1 = (M_1, R_1, V_1)$ and $\mathcal{M}_2 = (M_2, R_2, V_2)$ be any models of $\Gamma$ and $m_1 \in M_1$ and $m_2 \in M_2$ such that $\mathcal{M}_1, m_1 \leftrightsquigarrow_{\mathcal{L}_{1,\mathsf{D}}(\overline{q})} \mathcal{M}_2, m_2$. Hence, by theorem 2.2.1 it is sufficient to show that $\mathcal{M}_1, m_1 \leftrightsquigarrow_{\mathcal{L}_{1,\mathsf{D}}(\overline{q},p)} \mathcal{M}_2, m_2$. This is what we will now proceed to do. Note that as the truth of formulas in this logic is *not* invariant under the taking of disjoint unions of models (global perspective!), we are not able to apply corollary 2.2.2 as in the proof of theorem 4.3.3.

CLAIM 1: Let $\mathcal{N} = (N, R, V)$ be a model of $\Gamma$ and $n, n' \in N$. If $n$ and $n'$ have the same

$\overline{q}$-valuation and see the same number of points, then $n$ and $n'$ have the same $p$-valuation.

Proof of Claim 1: Suppose to the contrary that $n$ and $n'$ have the same $\overline{q}$-valuation and see the same number of points, but that $\mathcal{N}, n \Vdash p$ and $\mathcal{N}, n' \Vdash \neg p$. Proceeding in much the same way as in the proof of theorem 4.3.3, we will create a new model, $\mathcal{N}^* = (N, R, V^*)$ of $\Gamma$ on the frame $(N, R)$, having the same $\overline{q}$-valuation as $\mathcal{N}$ but a different $p$-valuation. Let $V^*(q) = V(q)$ for all $q \in \overline{q}$, and let $V^*(p) = (V(p) \cap (N - \{n\})) \cup \{n'\}$. This is to say, $V^*$ "swaps" the $p$-valuations of $n$ and $n'$. We have to show that $\mathcal{N}^*$ is a model of $\Gamma$. Call points $w$ and $w^*$ in $\mathcal{N}$ and $\mathcal{N}^*$ respectively, "corresponding points" if either $w = w^* \in M_1 - \{n, n'\}$, or $\{w, w^*\} = \{n, n'\}$. Note that the relation of "being corresponding points" is a bijection between the domains of $\mathcal{N}$ and $\mathcal{N}^*$. We will show that for each pair of corresponding points $w$ and $w^*$, it is the case that $\mathcal{N}, w \leftrightsquigarrow_{\mathcal{L}_{1,\mathsf{D}}(\overline{q},p)} \mathcal{N}^*, w^*$.

Let $\varphi \in \mathcal{L}_{1,\mathsf{D}}(\overline{q}, p)$. We proceed by induction on $\varphi$. The cases for proposition letters, boolean connectives and the graded modality are trivial, so suppose $\varphi$ is of the form $\mathsf{D}\psi$. $\mathcal{N}, w \Vdash \mathsf{D}\psi$ iff there is a point $w \neq v \in N$ such that $\mathcal{N}, v \Vdash \psi$ iff (by the induction hypothesis) for the corresponding point $v^* \in N$, $\mathcal{N}^*, v^* \Vdash \psi$ iff $\mathcal{N}^*, w^* \Vdash \mathsf{D}\psi$. (Note that as the relation of being corresponding points is a bijection, $v^* \neq w^*$.) This completes the induction.

Hence every point in $\mathcal{N}^*$ is $\mathcal{L}_{1,\mathsf{D}}(\overline{q}, p)$-equivalent to a point in $\mathcal{N}$, so $\mathcal{N}^* \Vdash \Gamma$. This contradicts the fact the $\Gamma$ implicitly defines $p$. ⊣

Now consider $\omega$-saturated models of $\Gamma$, $\mathcal{S}_1 = (S_1, R_1^S, V_1^S)$ and $\mathcal{S}_2 = (S_2, R_2^S, V_2^S)$ with $s_1 \in \mathcal{S}_1$ and $s_2 \in \mathcal{S}_2$ such that

$$\mathcal{M}_1, m_1 \leftrightsquigarrow_{\mathcal{L}_{1,\mathsf{D}}(\overline{q},p)} \mathcal{S}_1, s_1 \tag{4.1}$$

and

$$\mathcal{M}_2, m_2 \leftrightsquigarrow_{\mathcal{L}_{1,\mathsf{D}}(\overline{q},p)} \mathcal{S}_2, s_2 \tag{4.2}$$

These exist by proposition 1.1.5 and the comments following it (we may take suitable ultrapowers of $\mathcal{M}_1$ and $\mathcal{M}_2$). As $\mathcal{M}_1, m_1 \leftrightsquigarrow_{\mathcal{L}_{1,\mathsf{D}}(\overline{q})} \mathcal{M}_2, m_2$ we have

$$\mathcal{S}_1, s_1 \leftrightsquigarrow_{\mathcal{L}_{1,\mathsf{D}}(\overline{q})} \mathcal{S}_2, s_2 \tag{4.3}$$

Now define a relation $Z \subseteq S_1 \times S_2$ between the domains of $\mathcal{S}_1$ and $\mathcal{S}_2$ as follows

$$Z = \{(x, y) \in S_1 \times S_2 : x \leftrightsquigarrow_{\mathcal{L}_{1,\mathsf{D}}(\overline{q})} y\}$$

Claim 2: For each $u \in S_1$ there exists a $v \in S_2$ such that $uZv$, and vice versa.

Proof of Claim 2: Let $u \in \mathcal{S}_1$ and $\Sigma = \{\varphi \in \mathcal{L}_{1,\mathsf{D}}(\overline{q}) : \mathcal{S}_1, u \Vdash \varphi\}$ and let $\mathsf{ST}_x(\Sigma) = \{\mathsf{ST}_x(\varphi) : \varphi \in \Sigma\}$. Then for every for every finite subset $\Delta$ of $\Sigma$ we have, as $\mathcal{S}_1, s_1 \Vdash \mathsf{E} \bigwedge \Delta$, that $\mathcal{S}_2, s_2 \Vdash \mathsf{E} \bigwedge \Delta$, and hence that $\mathcal{S}_2 \models \exists x(\mathsf{ST}_x(\bigwedge \Delta))$. (Recall that $\mathsf{E}\varphi$ is an abbreviation for $\varphi \vee \mathsf{D}\varphi$.) So every finite subset of $\Sigma$ is satisfiable on $\mathcal{S}_2$. Hence the type $\mathsf{ST}_x(\Sigma)$ is finitely realized in $\mathcal{S}_2$, and as $\mathcal{S}_2$ is countably saturated, it realizes $\mathsf{ST}_x(\Sigma)$. Hence there is a point $v \in S_2$ such that $\mathcal{S}_2, v \Vdash \varphi$ for all $\varphi$ in $\Sigma$. Thus $u \leftrightsquigarrow_{\mathcal{L}_{1,\mathsf{D}}(\overline{q})} v$, i.e. $uZv$. ⊣

Note that $s_1 Z s_2$. We now construct a new model of $\Gamma$, $\mathcal{S}^* = (S_1, R_1^S, V^*)$ on the same frame as $\mathcal{S}_1$, as follows. Let $V^*(r) = \{x \in S_1 : xZy \text{ for some } y \in V_1^S(r)\}$ for all $r \in (\overline{q}, p)$. This agrees with the $\overline{q}$-valuation of $\mathcal{S}_1$, and by claim 1 the $p$-valuation is indeed well-defined. Note that

$$\text{if } xZy, \text{ then } \mathcal{S}^*, x \leftrightsquigarrow_{\mathcal{L}_{1,\mathsf{D}}(\overline{q})} \mathcal{S}_2, y$$

We will now show that

$$\text{if } xZy \text{ then } \mathcal{S}^*, x \leftrightsquigarrow_{\mathcal{L}_{1,\mathsf{D}}(\overline{q},p)} \mathcal{S}_2, y$$

Let $\xi \in \mathcal{L}_{1,\mathsf{D}}(\overline{q}, p)$. We proceed by induction on $\xi$ to show that $\mathcal{S}^*, x \Vdash \xi$ if and only if $\mathcal{S}_2, y \Vdash \xi$, for all $x$ and $y$ such that $xZy$. The case for proposition letters follows form the construction of $\mathcal{S}^*$ and the cases for the boolean connectives and counting modalities are trivial. We check the case where $\xi$ is of the form $\mathsf{D}\psi$. If $\mathcal{S}^*, x \Vdash \mathsf{D}\psi$, then there is a point $x' \neq x$ in $\mathcal{S}^*$ such that $\mathcal{S}^*, x' \Vdash \psi$. Now by claim 2 there is a point $y'$ in $\mathcal{S}_2$ such that $x'Zy'$. But the question is now - is $y'$ *different* form $y$? Let $\Sigma$ be the set of all $\mathcal{L}_{1,\mathsf{D}}(\overline{q})$-formulas that hold at $x'$. Then, using compactness we may assume that $\Sigma$ is a single formula, say $\sigma$. Then as $\mathcal{S}^*, x \Vdash \mathsf{D}\sigma$ it follows that $\mathcal{S}_2, y \Vdash \mathsf{D}\sigma$ (note that $\sigma$ is a $\mathcal{L}_{1,\mathsf{D}}(\overline{q})$-formula). Hence there is a point in $\mathcal{S}_2$, different from $y$, say $y'$ (!), such that $x'Zy'$. As $\psi$ falls under the inductive hypothesis it follows that $\mathcal{S}_2, y' \Vdash \psi$, and hence $\mathcal{S}_2, y \Vdash \mathsf{D}\psi$. The other direction is symmetrical.

Combining this with claim 2 it follows that $\mathcal{S}^* \Vdash \Gamma$. Specifically

$$\mathcal{S}^*, s_1 \leftrightsquigarrow_{\mathcal{L}_{1,\mathsf{D}}(\overline{q},p)} \mathcal{S}_2, s_2 \tag{4.4}$$

Now we have

$$\mathcal{S}_1, s_1 \cong_{\mathcal{L}_{1,\mathsf{D}}(\overline{q})} \mathcal{S}^*, s_1$$

and hence, as $\Gamma$ implicitly defines $p$, $\mathcal{S}_1 \Vdash \Gamma$ and $\mathcal{S}^* \Vdash \Gamma$, we have by proposition 2.1.1

$$\mathcal{S}_1, s_1 \cong_{\mathcal{L}_{1,\mathsf{D}}(\overline{q},p)} \mathcal{S}^*, s_1$$

hence

$$\mathcal{S}_1, s_1 \leftrightsquigarrow_{\mathcal{L}_{1,\mathsf{D}}(\overline{q},p)} \mathcal{S}^*, s_1 \tag{4.5}$$

Combining 4.4 and 4.5 we obtain

$$\mathcal{S}_1, s_1 \leftrightsquigarrow_{\mathcal{L}_{1,\mathsf{D}}(\overline{q},p)} \mathcal{S}_2, s_2 \tag{4.6}$$

Now applying 4.1 and 4.2 we have, as desired that

$$\mathcal{M}_1, m_1 \leftrightsquigarrow_{\mathcal{L}_{1,\mathsf{D}}(\overline{q},p)} \mathcal{M}_2, m_2 \tag{4.7}$$

and the proof is complete. $\dashv$

Adding the ordinary unary diamond ($\Diamond$) to the language $\mathcal{L}_{1,\mathsf{D}}$ to obtain $\mathcal{L}_{1,\mathsf{D},\Diamond}$ we lose Beth definability, as the following theorem shows.

**Theorem 4.3.6** $\mathsf{G}_{1,\mathsf{D},\Diamond}$ *does not have the Beth property.*

PROOF: Let $p$, $q$ and $r$ be proposition letters and let

$$\Gamma(q,r,p) = \left\{ \begin{array}{l} \mathsf{E}(p \wedge q \wedge \neg \mathsf{D}(p \wedge q)) \\ \neg q \to p \\ \mathsf{E}r \\ r \to \Diamond\top \\ r \to \Box(p \wedge q) \end{array} \right\}$$

We show that $\Gamma$ implicitly defines $p$. Let $\mathcal{M} = (M, R, V)$ be any model of $\Gamma$. Let $S_1 = [\![\neg q]\!]_{\mathcal{M}}$ and $S_2 = \{m \in M : \mathcal{M}, m \Vdash q$ and the exists a point $w \in M$ s.t. $\mathcal{M}, w \Vdash r$ and $Rwm\}$. We claim that $[\![p]\!]_{\mathcal{M}} = S_1 \cup S_2$.

Suppose $x \in [\![p]\!]_{\mathcal{M}}$. Now either $\mathcal{M}, x \Vdash q$ or $\mathcal{M}, x \Vdash \neg q$. In the second case $x \in S_1$ and we are done. In the first case $x$ is the unique $(p \wedge \neg q)$-point in $\mathcal{M}$. There is at least one $r$-point, say $y$, in $\mathcal{M}$. As $\mathcal{M} \Vdash (r \to \Diamond\top) \wedge (r \to \Box(p \wedge q))$, it must be the case that $Ryx$. Hence $x \in S_2$. Thus $[\![p]\!]_{\mathcal{M}} \subseteq S_1 \cup S_2$.

To prove the other inclusion assume that $x \in S_1 \cup S_2$. If $x \in S_1$ then $\mathcal{M}, x \Vdash \neg q$. As $\mathcal{M} \Vdash \neg q \to p$ it must be the case that $x \in [\![p]\!]_{\mathcal{M}}$. Otherwise, if $x \in S_2$, there exists an $r$-point, say $y$, such that $Ryx$. As $\mathcal{M} \Vdash r \to \Box(p \wedge q)$ it must be the case that $x \in [\![p]\!]_{\mathcal{M}}$. Hence $S_1 \cup S_2 \subseteq [\![p]\!]_{\mathcal{M}}$. Combining the two inclusions we have $[\![p]\!]_{\mathcal{M}} = S_1 \cup S_2$.

So $\Gamma$ implicitly defines $p$. But now consider the model in figure 4.2. It is easy to check that it is indeed a model of $\Gamma$. We may take $\Phi = \{q, r\}$ and apply lemma 4.3.4 to see that the

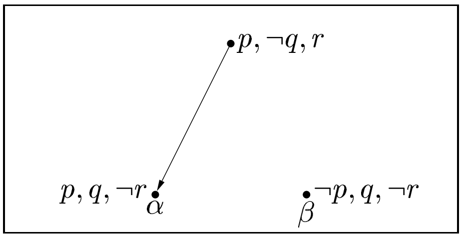


Figure 4.2: Theorem 4.3.6

points $\alpha$ and $\beta$ are indistinguishable by the language $\mathcal{L}_{1,\mathsf{D},\Diamond}(q,r)$. So there can be no explicit definition of $p$ relative to $\Gamma$. $\dashv$

The only remaining question now, as far as extensions of $\mathsf{G}_1$ are concerned, is whether $\mathsf{G}_{1,\Diamond}$ has the Beth property. The following theorem answers this in the affirmative. But first we need to introduce a new version of bisimulation and prove a few lemmas.

**Definition 4.3.7** Let $\mathcal{M} = (M, R, V)$ and $\mathcal{M}' = (M', R', V')$. A relation $Z \subseteq M \times M'$ is called a *weak counting bisimulation* if it satisfies the following

**bisimulation** $Z$ is a bisimulation

**successors forth** if $xZy$ and $S$ is a set of successors of $x$ of finite or countably infinite cardinality, then there exists a set $S'$ of successors of $y$, such that $|S| = |S'|$

**successors back** a similar condition for points is $M'$ and sets of their successors.

To indicate that $m \in \mathcal{M}$ and $m' \in \mathcal{M}'$ are linked by a weak counting bisimulation we write $\mathcal{M}, m \leftrightarroweq_{\mathcal{L}_{1,\diamond}} \mathcal{M}, m'$. $\lhd$

**Lemma 4.3.8** *Let $\mathcal{M} = (M, R, V)$ and $\mathcal{M}' = (M', R', V')$ be models and $Z \subseteq M \times M'$ a weak counting bisimulation. If $mZm'$, then $\mathcal{M}, m \leftrightsquigarrow_{\mathcal{L}_{1,\diamond}} \mathcal{M}', m'$*

PROOF: Let $\varphi \in \mathcal{L}_{1,\diamond}$. We proceed by induction to show that $\mathcal{M}, m \Vdash \varphi$ if and only if $\mathcal{M}', m' \Vdash \varphi$. As usual the cases for proposition letters and boolean connective are trivial. Suppose $\varphi$ is of the form $\lhd_{\leq n}\top$ for $n \in \omega$. Then $\mathcal{M}, m \Vdash \lhd_{\leq n}\top$ iff $m$ has at most $n$ successors iff $m'$ has at most $n$ successors (for if it had at least $n+1$, this would violate the successors back/forth clause) iff $\mathcal{M}', m' \Vdash \lhd_{\leq n}\top$. The case for $\varphi$ of the form $\Diamond\psi$ is standard.$\dashv$

**Lemma 4.3.9** *Let $\mathcal{M} = (M, R, V)$ and $\mathcal{M}' = (M', R', V')$ be models, $m \in M$ and $m' \in M'$. If $\mathcal{M}, m \leftrightsquigarrow_{\mathcal{L}_{1,\diamond}} \mathcal{M}', m'$, then there exist models $\mathcal{S}$ and $\mathcal{S}'$ and points $s \in \mathcal{S}$ and $s' \in \mathcal{S}'$ such that*

1. *$\mathcal{M}, m \leftrightsquigarrow_{\mathcal{L}_{1,\diamond}} \mathcal{S}, s$*
2. *$\mathcal{M}', m' \leftrightsquigarrow_{\mathcal{L}_{1,\diamond}} \mathcal{S}', s'$*
3. *for any points $u \in \mathcal{S}$ and $v \in \mathcal{S}'$, if $\mathcal{S}, u \leftrightsquigarrow_{\mathcal{L}_{1,\diamond}} \mathcal{S}', v$, then $\mathcal{S}, u \leftrightarroweq_{\mathcal{L}_{1,\diamond}} \mathcal{S}', v$.*
4. *for any set of formulas $\Gamma$ such that $\mathcal{M} \Vdash \Gamma$ ($\mathcal{M}' \Vdash \Gamma$) it is the case that $\mathcal{S} \Vdash \Gamma$ ($\mathcal{S}' \Vdash \Gamma$).*

PROOF: Let $\mathcal{S}$ and $\mathcal{S}'$ be suitable ($\omega$-saturated) ultrapowers of $\mathcal{M}$ and $\mathcal{M}'$ respectively, and $s$ and $s'$ be the equivalence classes of $id_m$ and $id'_m$ respectively. This satisfies conditions 1 and 2 and hence it follows that $\mathcal{S}, s \leftrightsquigarrow_{\mathcal{L}_{1,\diamond}} \mathcal{S}', s'$.

We need to show that on these $\omega$-saturated models, the relation $\leftrightsquigarrow_{\mathcal{L}_{1,\diamond}}$ is itself a weak counting bisimulation. Suppose $x \in \mathcal{S}$, $y \in \mathcal{S}'$ and $x \leftrightsquigarrow_{\mathcal{L}_{1,\diamond}} y$. Verifying the bisimulation clauses are standard, and the counting clauses don't cause any difficulty either, in fact they don't even use the $\omega$-saturation. We distinguish two cases. Case 1: $x$ has finitely many successors, say $n$. Thus $\mathcal{S}, x \Vdash \lhd_{=n}\top$ and $\mathcal{S}', y \Vdash \lhd_{=n}\top$. Obviously the counting clauses are satisfied. Case 2: $x$ has infinitely many successors. Thus $\mathcal{S}, x \Vdash \lhd_{\geq n}\top$ and $\mathcal{S}', y \Vdash \lhd_{\geq n}\top$ for all $n \in \omega$. Once again the counting clauses cannot be violated. $\dashv$

Note that this means that the class of weak counting bisimulations is a similarity class. We have proved all the facts about $\mathsf{G}_{1,\diamond}$ that are necessary to apply theorem 2.2.4. For the next lemma, recall the definition of the unravelling of a model (definition 1.1.7). We introduce the following notation. If $\overline{v} = (v_0, v_1, \ldots, v_n)$ is a sequence, let $\mathsf{last}(\overline{v}) = v_n$.

**Lemma 4.3.10** *Let $\mathcal{M} = (M, R, V)$ be a model and $m \in \mathcal{M}$. Then, $\mathcal{M}, m \leftrightarroweq_{\mathcal{L}_{1,\diamond}} \vec{\mathcal{M}}, (m)$. More generally, $\mathcal{M}, m \leftrightarroweq_{\mathcal{L}_{1,\diamond}} \vec{\mathcal{M}}, \overline{v}$ for each $m \in M$ and $\overline{v} \in \vec{M}(m)$ such that $\mathsf{last}(\overline{v}) = m$.*

PROOF: Let $m \in M$ and $\overline{v} \in \vec{M}(m)$ such that $\mathsf{last}(\overline{v}) = m$. The bisimulation clauses are standard. Let $S$ is the set of successors of $m$ in $\mathcal{M}$. Then if $\circ$ denotes sequence concatenation, $\{\overline{v} \circ (s) : s \in S\}$ is precisely the set of successors of $\overline{v}$ in $\vec{\mathcal{M}}, (m)$. Note that $|S| = |\{\overline{v} \circ (s) : s \in S\}|$. Hence any set of successors of $m$ can be matched with a set of successors of $\overline{v}$ with the same cardinality and vice versa, and the counting clauses are satisfied.⊣

Before stating the following lemma, let us give some intuitions behind it. We are working in a tree model of at most countable cardinality. Suppose $s_1$, $s_2$ and $s_3$ are siblings, say they are all children of a $t$, such that $s_1 \leftrightarroweq_{\mathcal{L}_{1,\diamond}} s_2$. We want to show that we can replace the subtree rooted at $s_2$ with a copy of the subtree rooted at $s_3$ (no typo!) without affecting truth at $t$. Specifically, we want to show that $t$ in the original tree is weakly counting bisimilar to $t$ in the tree obtained by doing the replacement. This makes sense, since $t$ in the obtained tree still has the same number of successors, and modulo the sets of $\mathcal{L}_{1,\diamond}$ formulas holding at successors of $t$ we have neither added nor taken away anything.

**Lemma 4.3.11** *Let $\mathcal{T} = (T, R, V)$ and $\mathcal{T}' = (T', R', V')$ be tree models rooted at $r$ and $r'$ respectively. Suppose there exists a weak counting bisimulation linking $r$ and $r'$. Let $t \in T$, $t' \in T'$ such that $t \leftrightarroweq_{\mathcal{L}_{1,\diamond}} t'$. Let $s_1$, $s_2$ and $s_3$ be successors of $t$ such that $s_1 \leftrightarroweq_{\mathcal{L}_{1,\diamond}} s_2$. Let $F^{s_2}$ and $F^{s_3}$ be the subtrees generated by $s_2$ and $s_3$ respectively (we will be sloppy and denote both the subtrees and their domains with this notation). Let $\mathcal{T}[F^{s_2}/F^{s_3}] = \mathcal{T}^* = (T^*, R^*, V^*)$ be the tree obtained from $\mathcal{T}$ by replacing $F^{s_2}$ with a (suitably renamed by superscripting with stars, say) copy of $F^{s_3}$. Formally:*

$$T^* = (T - F^{s_2}) \cup \{x^* : x \in F^{s_3}\}$$

$$R^* = \begin{array}{l} (R - \{(x, y) : x, y \in T^{s_2} \text{ and } (x, y) \in R\} - (t, s_2)) \\ \cup(\{(x^*, y^*) : x, y \in F^{s_3} \text{ and } (x, y) \in R\} \cup (t, s_3^*)) \end{array}$$

*and*

$$V^*(p) = (V(p) - F^{s_3}) \cup \{x^* : x \in V(p) \cap F^{s_3}\}.$$

*Then $\mathcal{T}^*, r \leftrightarroweq_{\mathcal{L}_{1,\diamond}} \mathcal{T}', r'$.*[2]

PROOF: All we need to show is that it is the case that $\mathcal{T}^*, t \leftrightarroweq_{\mathcal{L}_{1,\diamond}} \mathcal{T}', t'$. The local harmony clause is obviously satisfied. In $\mathcal{T}^*$, $t$ has the same number of successors as it had in $\mathcal{T}$, so the counting clauses are satisfied. Now for the back- and forth clauses. Suppose $Rts$ if $s \neq s_3^*$ then clearly the required weakly counting bisimilar successor of $t'$ exists in $\mathcal{T}'$, as $s$ was also a successor of $t$ in the original tree $\mathcal{T}$. If, on the other hand $s = s_3^*$ then, as there exists a successor of $t'$, weakly counting bisimilar to $s_3$, this successor is also weakly counting bisimilar to $s_3^*$. Now let $s'$ be a successor of $t'$ in $\mathcal{T}'$. There was a successor, say $s$, of $t$ in $\mathcal{T}$ weakly counting bisimilar to $s'$ in $\mathcal{T}$. If $s \neq s_2$ it is still in $\mathcal{T}^*$, otherwise $s_1$ is a suitable successor of $t$, as it is weakly counting bisimilar to $s_2$. ⊣

[2]A similar construction may be found in [Rosen, 1997], where various operations on (finite) models are considered.

**Theorem 4.3.12** *$G_{1,\diamond}$ has the Beth property.*

Proof: Let $\Gamma(\overline{q}, p)$ be a set of formulas that implicity defines $p$. Let $\mathcal{M}$ and $\mathcal{M}'$ be countable models of $\Gamma$, $m \in \mathcal{M}$ and $m' \in \mathcal{M}'$ such that $\mathcal{M}, m \leftrightarroweq_{\mathcal{L}_{1,\diamond}(\overline{q})} \mathcal{M}', m'$. (We may assume that $\mathcal{M}$ and $\mathcal{M}$ are rooted at $m$ and $m'$ respectively, since $\mathcal{M}, m \leftrightarroweq_{\mathcal{L}_{1,\diamond}(\overline{q},p)} \mathcal{M}(m), m$, and similarly for $\mathcal{M}', m'$.) By theorem 2.2.4 it is sufficient to show that $\mathcal{M}, m \leftrightarroweq_{\mathcal{L}_{1,\diamond}(\overline{q},p)} \mathcal{M}', m'$.

Let $\mathcal{T} = (T, R, V)$ and $\mathcal{T}' = (T', R', V')$ be the unravellings of $\mathcal{M}$ and $\mathcal{M}'$ around $m$ and $m'$ respectively. Recall that these are *trees* rooted at $m$ and $m'$ respectively. It follows from lemmas 4.3.10 and 4.3.8 that these are models of $\Gamma$, and that

$$\mathcal{M}, m \leftrightarroweq_{\mathcal{L}_{1,\diamond}(\overline{q},p)} \mathcal{T}, m$$

$$\mathcal{M}', m' \leftrightarroweq_{\mathcal{L}_{1,\diamond}(\overline{q},p)} \mathcal{T}', m'$$

and hence that

$$\mathcal{T}, m \leftrightarroweq_{\mathcal{L}_{1,\diamond}(\overline{q})} \mathcal{T}', m'$$

Our aim is now to transform $\mathcal{T}$ and $\mathcal{T}'$ into isomorphic models, $\mathcal{T}^*$ and $\mathcal{T}'^*$, and hence be able to apply proposition 2.1.1. We shall proceed inductively on the levels of the trees, starting from the roots. We shall not be too explicit about the sequence of trees we construct: excessive notation will only serve to cloud an intuitively simple idea.

Define an equivalence relation $\sim_{\leftrightarroweq} \subseteq T^2 \cup T'^2$ as follows: for every $u, v \in T$ let $u \sim_{\leftrightarroweq} v$ if $u$ and $v$ are siblings (i.e. they are successors of the same node) and $u \leftrightarroweq_{\mathcal{L}_{1,\diamond}(\overline{q})} v$. Define $\sim_{\leftrightarroweq}$ on $T'$ similarly. Denote the $\sim_{\leftrightarroweq}$-equivalence class of a point $u$ by $[u]$. Thus $[u]$ is the set of all siblings of $u$ (including $u$) that are weak counting bisimilar to $u$, with local harmony taken with respect to the proposition letters in $\overline{q}$.

Simultaneous with the transformation of the trees, we shall define a isomorphism, $f$. We shall ensure throughout the construction that the $f$-image of any point is always weakly counting bisimilar to it, with respect to the language $\mathcal{L}(\overline{q})$. Proceed by induction on the levels of the trees. For level 0 we define $f(m) = m'$. Suppose the trees have been transformed up to level $n$ and that $f$ has been extended to a partial isomorphism up to and including level $n$. Now let $u$ and $v$ be points on level $n$ of the (transformed) trees $\mathcal{T}$ and $\mathcal{T}'$ respectively, such that $f(u) = v$. We consider the sets of successors of $u$ and $v$, say $S = \{s_1, s_2, \ldots\}$ and $S' = \{s'_1, s'_2, \ldots\}$ respectively. Note that as $u$ and $v$ are weakly counting bisimilar and we are working on countable models, $|S| = |S'|$. We consider the equivalence classes $[s_i]$ of the points in $S$. Note that to each $[s_i]$ of successors of $u$, there corresponds a $[s'_i]$ of successors of $v$, in the sense that every point in $[s_i]$ is weakly counting bisimilar to every point in $[s'_i]$ and to no other point in $S'$, and vice versa. Our task now, is to transform the trees such that these corresponding equivalence classes contain the same number of points. Suppose, without loss of generality, that $\alpha = |[s_i]| \leq |[s'_i]| = \beta$, $\alpha$ and $\beta$ natural numbers or $\omega$.

Distinguish two cases: either $S$ is infinite or finite. In the first we simply add $\beta - \alpha$ copies of $F^{s_i}$ as successors of $u$. (Recall that this notation was introduced in lemma 4.3.11.) Note that it is still the case that $u \leftrightarroweq_{\mathcal{L}(\overline{q})} v$, as they both still have the same number of successors, and modulo $\mathcal{L}(\overline{q})$-equivalence nothing new was added.

In the second case we need to juggle a bit more carefully. Note that both $\alpha$ and $\beta$ must be finite, suppose $\beta - \alpha = k$. As $|S| = |S'|$ there must be (distinct) equivalence classes of successors of $u$, $[s_{j_1}], [s_{j_2}], \ldots [s_{j_m}]$, all different from $[s_i]$, such that, $|[s_{j_l}]| > |[s'_{j_l}]|$ for all $1 \leq l \leq m$, and $\Sigma_{l=1}^{m}(|[s_{j_l}]| - |[s'_{j_l}]|) \geq k$. In particular, note that $|[s_{j_l}]| \geq 2$, for all $1 \leq l \leq m$. Now for some $w \in [s_{j_o}]$, ( for some $1 \leq o \leq m$ ) we take $\mathcal{T}[F^w/F^{s_i}]$, i.e. we replace the subtree generated by $w$ by the subtree generated by $s_i$. By lemma 4.3.11 we have that the new tree is weakly counting bisimilar (the roots in particular being weakly counting bisimilar) to the tree prior to the replacement of the subtree. We continue replacing generated subtrees in this way until $[s_i] = [s'_i]$, taking care that for all the equivalence classes $[s_{j_1}], [s_{j_2}], \ldots [s_{j_m}]$, we do not replace nodes (and the subtrees generated by them) past the point where $|[s_{j_l}]|$ become equal to $|[s'_{j_l}]|$.

We repeat this process for all different $[s_i]$, $s_i \in S$, and $[s'_i]$, $s'_i \in S$. In this way we end up with a tree in which $|[s_i]| = |[s'_i]|$ for all $i$, and it is still the case that $u \leftrightarroweq_{\mathcal{L}_{1,\diamond}(\overline{q})} v$. Hence for each $i$, there exists a bijection from $[s_i]$ onto $[s'_i]$. Extend $f$ with the union of these. Clearly this extended version of $f$ is still a partial isomorphism.

This completes the process for the successors of $u$ and $v$. It is now repeated for the other nodes on level $n$ and the sets of their successors, hence transforming the trees and extending $f$ up to level $n+1$.

Clearly the function $f$ obtained after this has been done for all levels is an isomorphism between the obtained trees $\mathcal{T}^*$ and $\mathcal{T}'^*$. Further

$$\mathcal{T}, m \leftrightarroweq_{\mathcal{L}_{1,\diamond}(\overline{q},p)} \mathcal{T}^*, m$$

and

$$\mathcal{T}', m' \leftrightarroweq_{\mathcal{L}_{1,\diamond}(\overline{q},p)} \mathcal{T}'^*, m'$$

But we have

$$\mathcal{T}^*, m \cong_{\mathcal{L}_{1,\diamond}(\overline{q})} \mathcal{T}'^*, m'$$

As $\mathcal{T}^*$ and $\mathcal{T}'^*$ are models of $\Gamma$ we may apply proposition 2.1.1 and obtain

$$\mathcal{T}^*, m \cong_{\mathcal{L}_{1,\diamond}(\overline{q},p)} \mathcal{T}'^*, m'$$

from which it follows that

$$\mathcal{M}, m \leftrightarroweq_{\mathcal{L}_{1,\diamond}(\overline{q},p)} \mathcal{M}', m'$$

as desired. This proves the Beth property for $\mathsf{G}^{\omega}_{1,\diamond}$, i.e. $\mathsf{G}_{1,\diamond}$ with $\mathsf{K}$ replaced by $\mathsf{K}^{\omega}$, the class of all models of at most countable cardinality. All that remains now is to appeal to the Löwenhein-Skolem theorem to extend the result to $\mathsf{G}_{1,\diamond}$, and the proof is complete. $\dashv$

## 4.4 Beth definability for $\mathsf{G}_2$ and extensions

Let us investigate Beth definability for $\mathsf{G}_2$. We shall begin by showing that $\mathsf{G}_2$ has the Beth property. The proof gives some nice glimpses of the expressive power of $\mathcal{L}_2$, as well as the strength of the corresponding generalized version of bisimulation we shortly define.

One needs a suitable version of bisimulation to obtain the similarity class for $\mathsf{G}_2$ that is needed for the application of theorem 2.2.4. This is what we now provide with the definition of counting bisimulation. Unlike weak counting bisimulations we do not simply stick counting clauses onto the definition of a bisimulation, but formulate a strictly stronger form of bisimulation. Note that this is different from the related notion to be found in [D'Agostino, 1998] under the same name. Various "simulations" may also be found in [Kurtonina and de Rijke, 1999], where they are presented along with the corresponding invariance results for a number of systems from the description logic hierarchies.

We introduce the following bit of notation. If $\mathcal{M} = (M, R, V)$ is a model and $m \in M$, then $R[m] = \{v \in M : Rmv\}$ denotes the set of $R$-successors of $m$.

**Definition 4.4.1** Let $\mathcal{M} = (M, R, V)$ and $\mathcal{M}' = (M', R', V')$ be models. A relation $Z \subseteq M \times M'$ is called a *counting bisimulation* if the following conditions are satisfied

**local harmony** if $uZu'$ then $u$ and $u'$ satisfy the same proposition letters

**counting forth** if $uZu'$ and $S \subseteq R[u]$ is a finite or countably infinite set, then there exists a set $S' \subseteq R'[u']$ such that (i) for each $s \in S$ there exists an $s' \in S'$ such that $sZs'$, (ii) for each $s' \in S'$ there exists an $s \in S$ such that $sZs'$ and (iii) $|S| = |S'|$.

**counting back** A similar condition for points in $\mathcal{M}'$ and sets of their successors.

$\lhd$

We will write $Z : \mathcal{M}, v \leftrightarroweq_{\mathcal{L}_2} \mathcal{N}, w$ to indicate that $Z$ is a counting bisimulation linking $v$ and $w$, specifying the set of proposition letters (say $\Phi$) with respect to which the local harmony condition is taken when necessary by writing $Z : \mathcal{M}, v \leftrightarroweq_{\mathcal{L}_2(\Phi)} \mathcal{M}, w$.

**Proposition 4.4.2** *Let $\mathcal{M} = (M, R, V)$ and $\mathcal{M}' = (M', R', V')$ be two (not necessarily distinct) models, $m \in M$, $m' \in M'$, and $Z : \mathcal{M}, m \leftrightarroweq_{\mathcal{L}_2} \mathcal{M}', m'$ a counting bisimulation linking $m$ and $m'$. Then $m \leftrightsquigarrow_{\mathcal{L}_2} m'$.*

PROOF: We have to show that for all $\varphi \in \mathcal{L}_2$, $\mathcal{M}, m \Vdash \varphi$ if and only if $\mathcal{M}', m' \Vdash \varphi$. We proceed by induction on the complexity of $\varphi$. If $\varphi$ is $\bot$ or a proposition letter it trivially follows by the local harmony-clause. The inductive step for the boolean connectives is trivial. So assume that $\varphi$ is of the form $\lhd_{\geq n}\psi$ for some $n \in \omega$[3]. Hence there is a set $S \subseteq M$ of successors of $m$ such that $\mathcal{M}, s \Vdash \psi$ for each $s \in S$, and $|S| = n$. By the counting forward-clause there is a set $S' \subseteq M'$ of successors of $m'$, $|S| = |S'|$ such that for each $s \in S$ there is an $s' \in S'$ such that $sZs'$ and vice versa. Now by the inductive hypothesis, $\mathcal{M}, s \Vdash \psi$ if and only if $\mathcal{M}', s' \Vdash \psi'$, for each of these linked pairs. Hence $\mathcal{M}', s' \Vdash \psi$ for each $s' \in S'$. So we have that $\mathcal{M}', m' \Vdash \lhd_{\geq n}\psi$. The converse follows by symmetry, and the proof is complete. $\dashv$

**Remark 4.4.3** Note that the truth of formulas of $G_2$ is invariant under the taking of generated submodels. Specifically, if $v \in \mathcal{M}$, then $\mathcal{M}, v \leftrightsquigarrow_{\mathcal{L}_2} \mathcal{M}(v), v$.

**Proposition 4.4.4** *If $\mathcal{M} = (M, R, V)$ is rooted at $r$, then $\mathcal{M}, r \leftrightarroweq_{\mathcal{L}_2} \vec{\mathcal{M}}, (r)$*

[3] Note that instead of checking the form $\lhd_{\leq n}\psi$ we check $\lhd_{\geq n}\psi$ (the definition of the language was given using $\lhd_{\leq n}$). This does not matter as $\lhd_{\leq n}$ and $\lhd_{\geq n}$ are definable in terms of each other, as we have demonstrated.

Proof: Let $Z = \{(w, \vec{s}) : \vec{s} = (w_0, w_1, \ldots, w_n)$ and $w_n = w\}$. Clearly $(r, (r)) \in Z$, and it is easy to check that $Z$ is indeed a counting bisimulation. $\dashv$

**Proposition 4.4.5** *Let $\mathcal{M}$ and $\mathcal{N}$ be countably saturated models. Then the relation $\leftrightsquigarrow_{\mathcal{L}_2}$ is a counting bisimulation between $\mathcal{M}$ and $\mathcal{N}$.*

Proof: The local harmony clause is obviously satisfied. We verify the counting forth-condition. To this end suppose that $\mathcal{M}, m \leftrightsquigarrow_{\mathcal{L}_2} \mathcal{N}, n$, and let $S = \{s_1, s_2, \ldots\}$ be a set of successors of $m$ in $\mathcal{M}$. Now if we interpret $\underline{n}$ as $n$ in $\mathcal{N}$, then the type $\{R(\underline{n}, x) \wedge \mathsf{ST}_x(\gamma)(x) : \mathcal{M}, s_1 \Vdash \gamma\}$ is finitely satisfied (in the set of successors of $n$) in $\mathcal{N}_{\{n\}}$. (This is the case, since for every finite set of $\mathcal{L}_2$-formulas, $\{\gamma_1, \ldots, \gamma_n\}$ such that $\mathcal{M}, s_1 \Vdash \gamma_i$, for $1 \leq i \leq n$, we have $\mathcal{M}, m \Vdash \Diamond \bigwedge_{i=1}^{n} \gamma_i$, and hence $\mathcal{N}, n \Vdash \Diamond \bigwedge_{i=1}^{n} \gamma_i$.) Since $\mathcal{N}$ is countably saturated $\{R(\underline{n}, x) \wedge \mathsf{ST}_x(\gamma)(x) : \mathcal{M}, s_1 \Vdash \gamma\}$ is satisfied in $\mathcal{N}_{\{n\}}$. (Recall that $\mathcal{N}_A$ is the model $\mathcal{N}$, extended to a model for the (first-order) language with a constant added for each $a \in A$.) Hence $n$ has a successor, $s'_1$ say, such that $s_1 \leftrightsquigarrow_{\mathcal{L}_2} s'_1$. Now we do the same for $s_2$, but to ensure we get a successor *different* from $s'_1$, we use the conjunct $(x \neq \underline{s}'_1)$ in our type as follows: $\{R(\underline{n}, x) \wedge (x \neq \underline{s}'_1) \wedge \mathsf{ST}(\gamma)(x) : \mathcal{M}, s_2 \Vdash \gamma\}$. Since this type is again finitely satisfiable[4] in $\mathcal{N}_{\{n, s'_1\}}$, it is satisfied in $\mathcal{N}_{\{n, s'_1\}}$, and we obtain the desired successor $s'_2$ of $n$ (different from $s'_1$) such that $s_2 \leftrightsquigarrow_{\mathcal{L}_2} s'_2$. We continue in this way obtaining the desired set $S' = \{s'_1, s'_2, \ldots\}$ of successors of $n$. The counting back-condition is proved symmetrically, and the proof is complete. $\dashv$

**Proposition 4.4.6** *Let $\mathcal{M} = (W, R, V)$, $\mathcal{N} = (W', R', V')$, $m \in W$ and $n \in W'$ such that $\mathcal{M}, m \leftrightsquigarrow_{\mathcal{L}_2} \mathcal{N}, n$, then*

1. *there exist ultrapowers $\mathcal{M}^I/U$ and $\mathcal{N}^I/U$ as well as a counting bisimulation $Z \subseteq W^I/U \times W'^I/U$ such that $Z : \mathcal{M}^I/U, id_m/U \leftrightarrows_{\mathcal{L}_2} \mathcal{N}^I/U, id_n/U$, and*
2. *if $\Gamma$ is a set of formulas such that $\mathcal{M} \Vdash \Gamma$ ($\mathcal{N} \Vdash \Gamma$), then $\mathcal{M}^I/U \Vdash \Gamma$ ($\mathcal{N}^I/U \Vdash \Gamma$).*

Proof: For (1) take $\mathbb{N}$ as index set and $U$ some countably incomplete ultrafilter on $\mathbb{N}$. We know that the ultrapowers $\mathcal{M}^{\mathbb{N}}/U$ and $\mathcal{N}^{\mathbb{N}}/U$ are countably saturated. We further know that $\mathcal{M}, m \leftrightsquigarrow_{\mathcal{L}_2} \mathcal{M}^{\mathbb{N}}/U, id_m/U$ and that $\mathcal{N}, n \leftrightsquigarrow_{\mathcal{L}_2} \mathcal{N}^{\mathbb{N}}/U, id_n/U$, hence $\mathcal{M}^{\mathbb{N}}/U, id_m/U \leftrightsquigarrow_{\mathcal{L}_2} \mathcal{N}^{\mathbb{N}}/U, id_n/U$. But by proposition 4.4.5 then

$$\mathcal{M}^{\mathbb{N}}/U, id_m/U \leftrightarrows_{\mathcal{L}_2} \mathcal{N}^{\mathbb{N}}/U, id_n/U.$$

As for two (2), this is another simple consequence of Łoś's theorem. $\dashv$

The next lemma follows from propositions 4.4.2, 4.4.5 and 4.4.6.

**Lemma 4.4.7** *The class of counting bisimulations is a similarity class for $G_2$.*

[4]If $s_1$ and $s_2$ are not equivalent this is easy to see. If they are equivalent however, note that for each $\gamma$ such that $\mathcal{M}, s_2 \Vdash \gamma$, we have that $\mathcal{M}, m \Vdash \lhd_{\geq 2}\gamma$ and hence also that $\mathcal{N}, n \Vdash \lhd_{\geq 2}\gamma$.

**Lemma 4.4.8** *Let $\mathcal{T} = (T, R, V)$ and $\mathcal{T}' = (T', R', V')$ be two at most countable tree models, rooted at $r$ and $r'$ respectively. If $\mathcal{T}, r \leftrightarroweq_{\mathcal{L}_2} \mathcal{T}', r'$ then $\mathcal{T}, r \cong_{\mathcal{L}_2} \mathcal{T}', r'$.*

PROOF: Define an isomorphism $f : T \to T'$, by induction on the levels of the trees, beginning from the roots, ensuring at every stage, that that which has been defined so far constitutes a partial isomorphism. Let $f(r) = r'$. We will ensure throughout the construction that if we define $f(v) = v'$ then $v \leftrightarroweq_{\mathcal{L}_2} v'$. Suppose that $f$ has been defined up to and including level $n$. Let $t$ and $t'$ be points on level $n$ in $\mathcal{T}$ and $\mathcal{T}'$ respectively, such that $f(t) = t'$. Now define $f$ for the set $R[t]$ of successors of $t$ (note that $R[t]$ is at most countable). Clearly every point in $R[t]$ is counting bisimilar to some point in $R'[t']$, and vice versa, as we have $t \leftrightarroweq_{\mathcal{L}_2} t'$. (See figure 4.3.) Also note that $|R[t]| = |R'[t']|$. Define an equivalence relation $\sim_{\leftrightarroweq} \subseteq R[t]^2 \cup R'[t']^2$ as follows: For $s_i, s_j \in R[t]$, $s_i \sim_{\leftrightarroweq} s_j$ if and only if $s_i \leftrightarroweq_{\mathcal{L}_2} s_j$, and similarly for points in $R'[t']$. (Note that when writing $s_i \leftrightarroweq_{\mathcal{L}_2} s_j$ we have no *particular* counting bisimulation in mind, but only the existence of some counting bisimulation linking $s_i$ and $s_j$.) Let $[s_i]$ represent the equivalence class of $s_i \in S_t$ and $[s'_i]$ that of $s'_i$ in $S_{t'}$. Each equivalence class of elements in $R[t]$, say $[s_i]$, corresponds to and equivalence class of elements in $R'[t']$, say $[s'_i]$ for simplicity's sake, in the sense that every point in $[s_i]$ is counting bisimilar to every point in $[s'_i]$ and not bisimilar to any point in $R'[t'] - [s'_i]$. Moreover $[s_i]$ and $[s'_i]$ have the same cardinality. This follows by virtue of the facts that $t \leftrightarroweq_{\mathcal{L}_2} t'$, that we can always find a counting bisimulation containing the union of two counting bisimulations (so obtaining a maximal counting bisimulation), and the application of the counting clauses. So if $[s_i] = \{v_1, v_2, \ldots\}$ and $[s'_i] = \{v'_1, v'_2, \ldots\}$, let $f(v_k) = v'_k$ for all $k$. We can continue this for every pair of equivalence classes of successors of $t$ and $t'$. Clearly this extension of $f$ is still a partial isomorphism.

Do this for all pairs $u$ and $f(u)$ on level $n$, thus extending $f$ to a partial isomorphism up to and including level $n + 1$.

Clearly $f$ obtained in this way is a isomorphism. ⊣

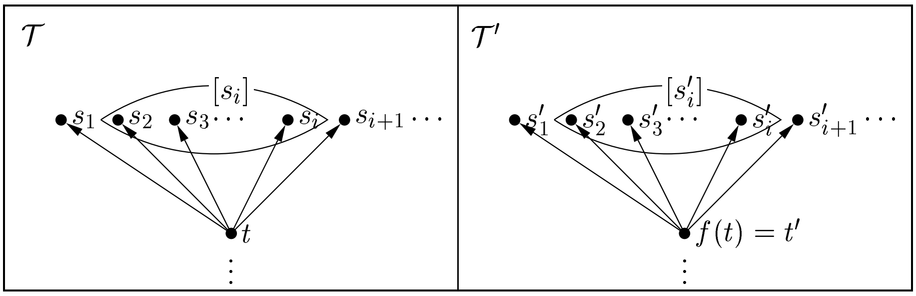


Figure 4.3: Lemma 4.4.8

**Definition 4.4.9** Let $\mathsf{G}_2^\omega = \langle \mathcal{L}_2, \mathsf{K}^\omega, \Vdash \rangle$ where $\mathsf{K}^\omega$ is the class of all models $\mathcal{M} = (W, R, V)$ of at most countable cardinality, and $\Vdash$ the consequence relation as defined for $\mathsf{G}_2$ ◁

**Theorem 4.4.10** *$G_2^\omega$ has the Beth property.*

Proof: Let $\Gamma(\overline{q}, p)$ be a set of formulas implicitly defining $p$. So suppose that $\mathcal{M} = (W, R, V)$, $\mathcal{M}' = (W', R', V')$, $m \in W$ and $m' \in W'$. By remark 4.4.3 we may assume, without loss of generality, that $\mathcal{M}$ and $\mathcal{M}'$ are rooted at $m$ and $m'$ respectively. Suppose $\mathcal{M}, m \leftrightarroweq_{\mathcal{L}_2(\overline{q})} \mathcal{M}', m'$. It is easy to check that the truth of formulas in $\mathsf{G}_2^\omega$ is preserved under the taking of disjoint unions of models and that (although not elementary) $\mathsf{G}_2^\omega$ is compact. Hence we may apply theorem 2.2.2, and hence it is sufficient to show that $\mathcal{M}, m \leftrightarroweq_{\mathcal{L}_2(\overline{q},p)} \mathcal{M}', m'$.

Consider the unravellings of $\vec{\mathcal{M}}$ and $\vec{\mathcal{M}}'$, of $\mathcal{M}$ and $\mathcal{M}'$ around $m$ and $m'$ respectively, and note that these are again models of $\Gamma$. By proposition 4.4.4 $\mathcal{M}, m \leftrightarroweq_{\mathcal{L}_2(\overline{q},p)} \vec{\mathcal{M}}, (m)$ and $\mathcal{M}', m' \leftrightarroweq_{\mathcal{L}_2(\overline{q},p)} \vec{\mathcal{M}}', (m')$. Hence $\vec{\mathcal{M}}, (m) \leftrightarroweq_{\mathcal{L}_2(\overline{q})} \vec{\mathcal{M}}', (m')$. But as $\mathcal{M}$ and $\mathcal{M}'$ are at most countable, so are $\vec{\mathcal{M}}$ and $\vec{\mathcal{M}}'$. Furthermore as $\vec{\mathcal{M}}$ and $\vec{\mathcal{M}}'$ are trees, we may apply lemma 4.4.8 and conclude that $\vec{\mathcal{M}}, (m) \cong_{\mathcal{L}_2(\overline{q})} \vec{\mathcal{M}}', (m')$. Now as $\Gamma$ implicitly defines $p$, we have, by proposition 2.1.1, that $\vec{\mathcal{M}}, (m) \cong_{\mathcal{L}_2(\overline{q},p)} \vec{\mathcal{M}}', (m')$. Hence $\mathcal{M}, m \leftrightarroweq_{\mathcal{L}_2(\overline{q},p)} \mathcal{M}', m'$ and the proof is complete. ⊣

We see that the notion of counting-bisimulation is strong enough to actually give us isomorphism in the case of countable trees, and consequently allows us to obtain a easy proof of Beth definability, but only if we restrict our attention to models of at most countable cardinality. But we may now apply the Löwenheim-Skolem theorem to extend this to the class of all models, and boldly assert:

**Corollary 4.4.11** *$\mathsf{G}_2$ has the Beth property.*

**Theorem 4.4.12** *$\mathsf{G}_{2,\mathsf{D}}$ does not have the Beth property.*

Proof: The same counterexample as was used in the proof of theorem 4.3.6 suffices. Note that all the formulas used in the implicit definition $\Gamma(q, r, p)$ are also formulas of $\mathcal{L}_{2,\mathsf{D}}$. The same proof goes through verbatim. ⊣

## 4.5 Beth definability for $\mathsf{G}_3$ and extensions

**Lemma 4.5.1** *Let $\mathcal{M} = (M, V)$ be a model and $m, m' \in M$. If $m$ and $m'$ have the same valuation, then $m \leftrightsquigarrow_{\mathcal{L}_3} m'$.*

Proof: Trivial. ⊣

**Theorem 4.5.2** *$\mathsf{G}_3$ has the Beth property.*

Proof: Let $\Gamma(\overline{q}, p)$ be a set of formulas that implicitly defines $p$. Let $\mathcal{M} = (M, V)$ and $\mathcal{M}' = (M', V')$ be models of $\Gamma$, $m \in M$ and $m' \in M'$. Suppose that $\mathcal{M}, m \leftrightsquigarrow_{\mathcal{L}_3(\overline{q})} \mathcal{M}', m'$. By proposition 2.2.1 it is sufficient to show that $\mathcal{M}, m \leftrightsquigarrow_{\mathcal{L}_3(\overline{q},p)} \mathcal{M}', m'$. Invoking lemma 4.5.1 the proof of the following claim is routine.

Claim 1: If $\mathcal{N}$ is a model of $\Gamma$ and $n, n' \in \mathcal{N}$ have the same $\overline{q}$-valuation, then $n$ and $n'$

have the same $p$-valuation.

Define a relation $Z \subseteq M \times M'$ as follows

$$Z = \{(x, y) \in M \times M' : x \text{ and } y \text{ have the same } \overline{q}\text{-valuation.}\}$$

Note that $mZm'$. It is easy to see that for any finite subset of points $S \subseteq M$, there is a subset $S' \subseteq M'$, such that $|S| = |S'|$, every point in $S$ is $Z$-related to a point in $S'$ and vice versa (we use the fact that $\mathcal{M}, m \leftrightsquigarrow_{\mathcal{L}_3(\overline{q})} \mathcal{M}', m'$). Now define a new model $\mathcal{M}^* = (M, V^*)$ on the same frame (set) as $\mathcal{M}$ as follows. Let $V^*$ be such that each point $n \in \mathcal{M}^*$ has the same $(\overline{q}, p)$-valuation as some point $n' \in M'$ for which $nZn'$. By the claim 1 this is well defined. Note that $V(q) = V^*(q)$ for all $q \in \overline{q}$, and hence that $\mathcal{M} \cong_{\mathcal{L}_3(\overline{q})} \mathcal{M}^*$.

Claim 2: For all points $n \in M$ and $n' \in M'$, if $nZn'$, then $\mathcal{M}^*, n \leftrightsquigarrow_{\mathcal{L}_3(\overline{q},p)} \mathcal{M}', n'$.

Proof of Claim 2: Let $\varphi \in \mathcal{L}_3(\overline{q}, p)$. We show by induction that $\mathcal{M}^*, n \Vdash \varphi$ iff $\mathcal{M}', n' \Vdash \varphi$. The case for proposition letters follows from the construction of $\mathcal{M}^*$, and those for the boolean connectives are trivial. So suppose $\varphi$ is of the form $\mathsf{U}_{\geq n}\psi$ for some $n \in \omega$. (The definition of the language was in terms of modalities of the form $\mathsf{U}_{\leq n}$, but as $\mathsf{U}_{\leq n}$ and $\mathsf{U}_{\geq n}$ are interdefinable it suffices to check the latter.) If $\mathcal{M}^*, n \Vdash \mathsf{U}_{\geq n}\psi$ there is a set of $n$ points from $\mathcal{M}^*$, say $S$, that all satisfy $\psi$. But then we can find a set $S' \subseteq M'$ such that $n = |S| = |S'|$ and each point in $S'$ is $Z$-related to a point in $S$. As $\psi$ falls under the inductive hypothesis it follows that each point in $|S'|$ makes $\psi$ true. Hence $\mathcal{M}, n' \Vdash \mathsf{U}_{\geq n}\psi$. Showing the converse is symmetric. $\dashv$

It follows that $\mathcal{M}^* \Vdash \Gamma$. Applying proposition 2.1.1 to the fact that $\mathcal{M} \cong_{\mathcal{L}_3(\overline{q})} \mathcal{M}^*$ it follows that $\mathcal{M} \cong_{\mathcal{L}_3(\overline{q},p)} \mathcal{M}^*$. Now as $mZm'$ we have by claim 2 that $\mathcal{M}^*, m \leftrightsquigarrow_{\mathcal{L}_3(\overline{q},p)} \mathcal{M}', m'$. Combining this with $\mathcal{M} \cong_{\mathcal{L}_3(\overline{q},p)} \mathcal{M}^*$ we have $\mathcal{M}, m \leftrightsquigarrow_{\mathcal{L}_3(\overline{q},p)} \mathcal{M}', m'$ as desired, and the proof is complete. $\dashv$

**Lemma 4.5.3** *Let $\mathcal{M} = (M, R, V)$ be a model and $m, m' \in M$. Let $\Phi$ be a set of proposition letters. If $m \leftrightarroweq m'$ (bisimilar in the usual modal sense, taking the local harmony clause with respect to $\Phi$), then $m \leftrightsquigarrow_{\mathcal{L}_{3,\diamond}(\Phi)} m'$. Hence if $m \leftrightarroweq_{\mathcal{L}_2(\Psi)} m'$ then, $m \leftrightsquigarrow_{\mathcal{L}_{3,\diamond}(\Phi)} m'$.*

Proof: Note that, by the definitions, if $m \leftrightarroweq_{\mathcal{L}_2(\Psi)} m'$, then $m \leftrightarroweq m'$. Let $\varphi \in \mathcal{L}_{3,\diamond}$. We proceed by induction on $\varphi$. The cases for proposition letters and boolean connective are trivial. The case for $\varphi$ of the form $\diamond\psi$ is the same as in the standard proof of the fact that bisimulation implies modal equivalence. Suppose $\varphi$ is of the form $\mathsf{U}_{\leq n}\psi$. Then $m \Vdash \mathsf{U}_{\leq n}\psi$ iff there are at most $n$ states in $\mathcal{M}$ where $\psi$ is true iff $m' \Vdash \mathsf{U}_{\leq n}\psi$. This completes the induction. $\dashv$

**Theorem 4.5.4** *$G_{3,\diamond}$ does not have the Beth property.*

Proof: Let $q$ and $p$ be proposition letters and let

$$\Gamma(q,p) = \left\{ \begin{array}{l} \mathsf{U}_{=1}p \\ \mathsf{U}_{\geq 1}q \\ q \to \Diamond\top \\ q \to \Box p \end{array} \right\}$$

Claim: If $\mathcal{M} = (M, R, V)$ is any model of $\Gamma$, then

$$[\![p]\!]_{\mathcal{M}} = \{x \in M : \text{for all } y \in M,\ \text{if } \mathcal{M}, y \Vdash q \text{ then } Ryx\}$$

Proof of Claim: Let $S = \{x \in M : \text{for all } y \in M, \text{ if } \mathcal{M}, y \Vdash q \text{ then } Ryx\}$. Suppose $x \in [\![p]\!]_{\mathcal{M}}$. This means $x$ is the unique $p$-point in $M$. Suppose $y \in M$ and $\mathcal{M}, y \Vdash q$. As $\mathcal{M} \Vdash (q \to \Diamond\top) \wedge (q \to \Box p)$ it must be the case that $Ryx$. Thus $x \in S$, and $[\![p]\!]_{\mathcal{M}} \subseteq S$.

Conversely suppose $x \in S$. Let $y \in M$ such that $\mathcal{M}, y \Vdash q$. As $\mathcal{M} \Vdash \mathsf{U}_{\geq 1}q$ such a point is guaranteed to exist. As $x \in S$ it must be the case that $Ryx$. But as $\mathcal{M} \Vdash q \to \Box p$, it must be the case that $x \in [\![p]\!]_{\mathcal{M}}$. Hence $S \subseteq [\![p]\!]_{\mathcal{M}}$.

Combining the two inclusions we have $[\![p]\!]_{\mathcal{M}} = S$ as desired. $\dashv$

Now consider the model in figure 4.4, once again based on the by-now famous three point frame. It is easy to check that this is a model of $\Gamma$. We may take $\Phi = \{q\}$ and apply lemma 4.5.3 to see that the nodes $\alpha$ and $\beta$ are indistinguishable in the language $\mathcal{L}_{3,\Diamond}(q)$. Hence there can be no explicit definition of $p$ relative to the implicit definition $\Gamma$. $\dashv$

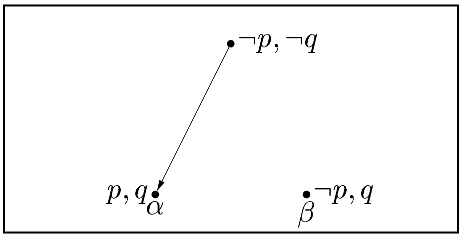


Figure 4.4: Theorem 4.5.4

**Remark 4.5.5** As the reader may check, the counterexample provided in the proof of theorem 3.3.4 can be used just as it is to give and alternative proof of theorem 4.5.4, where $p$ is not defined as singleton.

**Theorem 4.5.6** *$G_{2,3}$ does not have the Beth property.*

Proof: The same counterexample provided in theorem 4.5.4 suffices. Note that all formulas used in the implicit definition are equivalent to formulas in $\mathcal{L}_{2,3}$ and that in figure 4.4 $\alpha \leftrightarrows_{\mathcal{L}_2(q)} \beta$. Hence we may apply lemma 4.5.3 and conclude that $\alpha \leftrightsquigarrow_{\mathcal{L}_{2,3}(q)} \beta$. $\dashv$

# Chapter 5

# Some Results on Hybrid Logic

## 5.1 Hybrid Logic

In previous chapter various extensions of modal logic were considered, each adding a different kind of expressivity to the basic logic. Here we consider another, namely hybrid logic. For a hybrid logic "manifesto", giving much detailed motivation and background, see [Blackburn, 2000]. Section 7.3. in [Blackburn et al., 2001] also provides a nice introduction. Although states (points, worlds, or whatever one prefers to call them) are central to modal logic, there is no mechanism within the logic to refer to them explicitly. Hybrid logic remedies this, by providing the tools to name states and to talk about these named states. This is accomplished in a natural way. The set of proposition letters is sorted into two kinds, the first of which is just the ordinary proposition letters that we are used to, and the second a special sort called nominals. We shall insist that a nominal must be true at precisely one state in any model, hence serving as a name for the state at which it holds. To take full advantage of these names we will introduce a new operator, the so-called "at"-operator, $@_n\varphi$, to be read "at $n$, $\varphi$" where $n$ is some nominal and $\varphi$ a formula. Let us make this precise. The language $\mathcal{L}_H$ is defined as follows

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \vee \psi \mid \Diamond\varphi \mid @_n\varphi$$

where $p$ is a proposition letter and $n$ a nominal. When the specific sets of proposition letters and nominals present in the language is important we shall write $\mathcal{L}_H(\Phi, \Psi)$ where $\Phi$ and $\Psi$ are sets of proposition letters and nominals respectively.

$\mathcal{L}_H$ is interpreted over models of the form $\mathcal{M} = (M, R, V)$, where $R$ is a binary relation, and for each nominal $n$, $V(n) = \{u\}$ for some $u \in M$. If $V(n) = \{u\}$ then $u$ is called the denotation of $n$ in $\mathcal{M}$, i.e the state $u$ is called $n$. The semantics is as follows

- $\bot$, proposition letters and boolean combinations are given the usual semantics
- $\mathcal{M}, u \Vdash n$ iff $u \in V(n)$ iff $V(u) = \{u\}$, where $n$ is a nominal
- $\mathcal{M}, u \Vdash \Diamond\varphi$ iff there exists a point $v \in M$ such that $Ruv$ and $\mathcal{M}, v \Vdash \varphi$

- $\mathcal{M}, u \Vdash @_n\varphi$ iff $[\![n]\!]_{\mathcal{M}} \subseteq [\![\varphi]\!]_{\mathcal{M}}$ iff if $v$ is the state in $M$ such that $\mathcal{M}, v \Vdash n$, it is the case that $\mathcal{M}, v \Vdash \varphi$.

The standard translation of $\mathcal{L}_H$ holds no surprises. Constants are introduced to accommodate the translation of formulas involving nominals in the obvious way. For a precise formulation see [Areces et al., 2001].

**Definition 5.1.1** Let $\mathsf{S}_H = \langle \mathcal{L}_H, \mathsf{K}, \Vdash \rangle$, where $\mathsf{K}$ is the class of all models and $\Vdash$ is as defined above. Let $\mathsf{S5}_H = \langle \mathcal{L}_H, \mathsf{K}_{\mathsf{S5}}, \Vdash \rangle$, where $\mathsf{K}_{\mathsf{S5}}$ is the class of all models on which the accessibility relation is an equivalence relation. ◁

**Remark 5.1.2** As we have a standard translation into first order logic of the language $\mathcal{L}_H$, and the classes $\mathsf{K}$ and $\mathsf{K}_{\mathsf{S5}}$ are first order definable, it follows that $\mathsf{S}_H$ and $\mathsf{S5}_H$ are elementary, and hence compact.

The presence of nominals in the language gives rise to a slight complication in the notion of a generated submodel. If the denotation of some nominal, say $n$, is not accessible via a finite path from some point in the generating set, it will not be included in the generated submodel, as defined in definition 1.1.6. But then $n$ will have no denotation in the generated submodel, violating the constraint on the valuation of nominals. We therefore define the notion of a generated hybrid submodel.

**Definition 5.1.3** Let $\mathcal{M} = (M, R, V)$ be a model and $S \subseteq M$. Let $\mathsf{Denote} = \{u \in M : \mathcal{M}, u \Vdash n$ for $n$ a nominal in $\mathcal{L}_H(\Phi, \Psi)\}$. Then the hybrid submodel of $\mathcal{M}$ generated by $S$, $\mathcal{M}(S)_{\mathcal{L}(\Phi,\Psi)}$, is $\mathcal{M}(S \cup \mathsf{Denote})$. ◁

Notice that this implies that the domain of a generated hybrid submodel depends on the nominals present in the language. For the definition of a version of bisimulation appropriate for hybrid logic, as well as a number of invariance results, see [Areces et al., 2001]. It follows from theorem 3.11 in [Areces et al., 2001] that $\mathcal{M}, u \leftrightsquigarrow_{\mathcal{L}_H} \mathcal{M}(u)_{\mathcal{L}_H}, u$. For the purposes of this chapter, however, a much simpler version of bisimulation will suffice. This will be nothing more than a ordinary modal bisimulation for which we require that the denotations of each nominal be linked.

**Definition 5.1.4** Let $\mathcal{M} = (M, R, V)$ and $\mathcal{M}' = (M', R', V')$ be models and $Z \subseteq M \times M'$. $Z$ is called a *hybrid bisimulation* if

- $Z$ is a modal bisimulation (with the local harmony clause extended to nominals)
- if $n$ is a nominal, $\mathcal{M}, u \Vdash n$ and $\mathcal{M}', u' \Vdash n$, then $uZu'$.

We write $\mathcal{M}, u \leftrightarroweq_{\mathcal{L}_H} \mathcal{M}', u'$ to indicate that $u$ and $u'$ are linked by a hybrid bisimulation, parameterizing with sets of proposition letters and nominals as the need arises, as usual. ◁

The reader may easily verify the following lemma, which we state without proof.

**Lemma 5.1.5** *If $\mathcal{M}, u \leftrightarroweq_{\mathcal{L}_H} \mathcal{M}', u'$ then $\mathcal{M}, u \leftrightsquigarrow_{\mathcal{L}_H} \mathcal{M}', u'$.*

## 5.2 Beth and Interpolation for Hybrid Logic

When talking about implicit- and explicit definitions in a hybrid logic setting, it makes sense to distinguish between definitions of nominals and definitions of ordinary proposition letters. If one considers (implicit- and explicit) definitions of nominals, it is very easy to come up with counterexamples to the Beth property, much in the spirit of those given for modal logic with the difference operator. We therefore choose to consider only the (implicit- and explicit) definitions of proposition letters as legitimate in our investigations of the Beth property for the logics discussed in this chapter.

### 5.2.1 Interpolation

**Theorem 5.2.1** *$S_H$ does not have interpolation.*

PROOF: The counterexample presented here is adapted from [Areces et al., 2001]. Let $p$, $q$ and $r$ be proposition letters and let $i$ and $j$ be nominals. Let

$$\varphi = \Diamond(p \wedge q) \wedge \Diamond(\neg p \wedge q) \wedge \Diamond(\neg p \wedge \neg q)$$

and

$$\psi = (\Diamond r \wedge \Box(r \to i)) \to (\Diamond(\neg r \wedge j) \to \Diamond(\neg r \wedge \neg j))$$

It is easy to see that $\Vdash \varphi \to \psi$. For let $\mathcal{M} = (M, R, V)$ be any model and $m \in M$. If $\mathcal{M}, m \Vdash \varphi$ it must be the case that $m$ has at least three distinct successors in $\mathcal{M}$. Now suppose $\mathcal{M}, m \Vdash (\Diamond r \wedge \Box(r \to i))$, hence $m$ has a unique $r$-successor, call this point $v$. If $\mathcal{M}, m \Vdash \Diamond(\neg r \wedge j)$ it means that some successor of $m$ other than $v$ is named by the nominal $j$. But as $m$ has at least three successors, it must be that the case that at one of these successors $(\neg r \wedge \neg j)$ holds. Hence $\mathcal{M}, m \Vdash \Diamond(\neg r \wedge \neg j)$, i.e. $\mathcal{M}, m \Vdash \psi$, i.e $\mathcal{M}, m \Vdash \varphi \to \psi$.

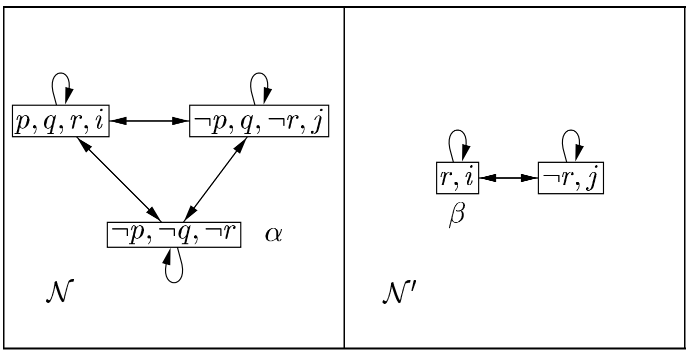


Figure 5.1: Theorem 5.2.1

So it remains to show that there can be no interpolant. Note that the common language of $\varphi$ and $\psi$ is $\mathcal{L}_H(\emptyset, \emptyset)$. Consider the models, $\mathcal{N}$ and $\mathcal{N}'$, depicted in figure 5.1. It is easy to check that $\alpha \Vdash \varphi$ and that $\alpha \Vdash \psi$, but that $\beta \Vdash \neg\psi$. Further $N \times N'$ is a hybrid bisimulation with respect to the empty language $\mathcal{L}_H(\emptyset, \emptyset)$. Hence $\mathcal{N}, \alpha \leftrightsquigarrow_{\mathcal{L}_H(\emptyset,\emptyset)} \mathcal{N}', \beta$. Hence every formula $\gamma \in \mathcal{L}_H(\emptyset, \emptyset)$ such that $\Vdash \varphi \to \gamma$ is consistent with both $\psi$ and $\neg\psi$. Hence there can be no formula $\gamma \in \mathcal{L}_H(\emptyset, \emptyset)$ such that $\Vdash \varphi \to \gamma$ and $\Vdash \gamma \to \psi$. $\dashv$

By simply noting that the models $\mathcal{N}$ and $\mathcal{N}'$ in figure 5.1 are based on S5-frames, we have the following corollary.

**Corollary 5.2.2** *$S5_H$ does not have interpolation.*

### 5.2.2 Beth Definability

**Proposition 5.2.3** *$S5_H$ has the Beth property.*

PROOF: By remark 5.1.2 we may apply theorem 2.2.1. Let $\overline{q} = \{q_1, q_2, \ldots, q_l\}$ be a set of proposition letters, $\overline{n} = \{n_1, n_2, \ldots, n_m\}$ a set of nominals and $p$ a proposition letter. Suppose $\Gamma(\overline{q}, \overline{n}, p)$ is a set of formulas that implicitly defines $p$. So by proposition 2.1.1 we have that for all models $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\Gamma$, if $\mathcal{M}_1 \cong_{\mathcal{L}_H(\overline{q},\overline{n})} \mathcal{M}_2$ then $\mathcal{M}_1 \cong_{\mathcal{L}_H(\overline{q},\overline{n},p)} \mathcal{M}_2$. In order to show that there exists an explicit definition of $p$ relative to $\Gamma$, it is enough to show that any two points on models of $\Gamma$, modally equivalent with respect to the language $\mathcal{L}_H(\overline{q}, \overline{n})$ are also modally equivalent with respect to the language $\mathcal{L}_H(\overline{q}, \overline{n}, p)$. This is what we will now proceed to do.

Let $\mathcal{M}_1 = (M_1, R_1, V_1)$ and $\mathcal{M}_2 = (M_2, R_2, V_2)$ be models of $\Gamma$, $m_1 \in M_1$, $m_2 \in M_2$ and $m_1 \leftrightsquigarrow_{\mathcal{L}_H(\overline{q},\overline{n})} m_2$. For convenience let us assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ are in fact the hybrid submodels generated by $m_1$ and $m_2$ respectively (recall that, since we are working in hybrid logic, these are the submodels generated by the mentioned points as well as the denotations of the nominals in $\overline{n}$). Let $m^1_{n_j}$ and $m^2_{n_j}$ denote the denotations of $n_j$, $1 \leq j \leq m$, in $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Further, for convenience of notation let $m^1_0 = m_1$ and $m^2_0 = m_2$. So $\mathcal{M}_1$ and $\mathcal{M}_2$ each consist of a (finite) number of equivalence classes, namely $[m^1_0], [m^1_{n_1}], [m^1_{n_2}], \ldots [m^1_{n_m}]$ and $[m^2_0], [m^2_{n_1}], [m^2_{n_2}], \ldots [m^2_{n_m}]$ respectively. Note that these are not necessarily distinct. We will call two equivalence classes $[\alpha_1]$ and $[\alpha_2]$, from $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively, "corresponding equivalence classes", if there is a $j \in \{0, n_1, \ldots, n_m\}$ such that $[\alpha_1] = [m^1_j]$ and $[\alpha_2] = [m^2_j]$.

CLAIM 1: Let $w \in [m^1_j]$, $j \in \{0, n_1, \ldots, n_m\}$, then there is a point $w' \in [m^2_j]$ with the same $(\overline{q}, \overline{n})$-valuation. In other words, the same valuation-combinations of the $\overline{q}$'s and $\overline{n}$'s are represented in the corresponding equivalence classes of $\mathcal{M}_1$ and $\mathcal{M}_2$.

PROOF OF CLAIM 1: Let $\varphi$ express the $(\overline{q}, \overline{n})$-valuation of $w$ (i.e. $\varphi$ is the appropriate conjunction of the symbols from $(\overline{q}, \overline{n})$ or their negations). If in fact $[m^1_j] = [m_1]$, then $\mathcal{M}_1, m_1 \Vdash \Diamond\varphi$, hence $\mathcal{M}_2, m_2 \Vdash \Diamond\varphi$, hence there is a point $w' \in [m_2]$ with the same $(\overline{q}, \overline{n})$-valuation as $w$.

If not, $\mathcal{M}_1, m_1 \Vdash @_j\Diamond\varphi$, hence $\mathcal{M}_2, m_2 \Vdash @_j\Diamond\varphi$, hence there is a point $w' \in [m^2_j]$ sharing the $(\overline{q}, \overline{n})$-valuation of $w \in [m^1_j]$. $\dashv$

CLAIM 2: If $w \in M_1$ and $w' \in M_2$ are in corresponding equivalence classes and share the same $(\overline{q}, \overline{n})$-valuation, then $M_1, w \leftrightsquigarrow_{\mathcal{L}_H(\overline{q},\overline{n})} M_2, w'$.

Proof of Claim 2: Suppose $w \in M_1$ and $w' \in M_2$ share the same $(\overline{q}, \overline{n})$-valuation. Further suppose that $w \in [m_j^1]$ and $w' \in [m_j^2]$, $j \in \{0, n_1, \ldots, n_m\}$. Let $\varphi \in \mathcal{L}_H(\overline{q}, \overline{n})$. We proceed by induction on $\varphi$ to show that $\mathcal{M}_1, w \Vdash \varphi$ iff $\mathcal{M}_2, w' \Vdash \varphi$. The cases for proposition letters, nominals and the boolean connectives are trivial.

So, suppose $\varphi$ is of the form $\Diamond\psi$. If $\mathcal{M}_1, w \Vdash \Diamond\psi$, then there exists a point $v \in [m_j^1]$ such that $\mathcal{M}_1, v \Vdash \psi$. Then by the above claim there exists a point $v' \in [m_j^2]$ with the same $(\overline{q}, \overline{n})$-valuation as $v$. By the inductive hypothesis $\mathcal{M}_2, v' \Vdash \psi$, hence $\mathcal{M}_2, w' \Vdash \Diamond\psi$. To prove that if $\mathcal{M}_2, w' \Vdash \Diamond\psi$ then $\mathcal{M}_1, w \Vdash \Diamond\psi$ is precisely symmetrical.

Suppose now that $\varphi$ is of the form $@_n\psi$, $n \in \overline{n}$. Then $\mathcal{M}_1, w \Vdash @_n\psi$ iff $\mathcal{M}_1, m_n^1 \Vdash \psi$ iff $\mathcal{M}_2, m_n^2 \Vdash \psi$ (since $m_n^1$ and $m_n^2$ have the same $(\overline{q}, \overline{n})$-valuation, are obviously in corresponding equivalence classes, and $\psi$ falls under the inductive hypothesis) iff $\mathcal{M}_2, v' \Vdash @_n\psi$. $\dashv$

Claim 3: If $w, w' \in M_1$ are in the same equivalence class and have the same $(\overline{q}, \overline{n})$-valuation then $w$ and $w'$ have the same $(\overline{q}, \overline{n}, p)$-valuation.

Proof of Claim 3: Suppose to the contrary that $w, w' \in M_1$ are in the same equivalence class and have the same $(\overline{q}, \overline{n})$-valuation but differ on $p$. Without loss of generality we assume that $\mathcal{M}_1, w \Vdash p$ and $\mathcal{M}_1, w' \Vdash \neg p$. Let $f : M_1 \to M_1$ be defined by $f(x) = x$ for all $x \in M_1 - \{w, w'\}$ and $f(w) = w'$ and $f(w') = w$, i.e. $f$ fixes all points except $w$ and $w'$ which it "swaps". It is easy to check that $f : \mathcal{M}_1 \cong_{\mathcal{L}_H(\overline{q}, \overline{n})} \mathcal{M}_2$, but obviously $f : \mathcal{M}_1 \not\cong_{\mathcal{L}_H(\overline{q}, \overline{n}, p)} \mathcal{M}_2$. Since $\mathcal{M}_1 \Vdash \Gamma$ this contradicts the fact that $\Gamma$ implicitly defines $p$. $\dashv$

We will define a binary relation, $Z$, between $M_1$ and $M_2$. Let $Z = \{(x, y) \in M_1 \times M_2 :$ $x$ and $y$ are in corresponding equivalence classes and share the same $(\overline{q}, \overline{n})$-valuation$\}$.

Now on the frame $(M_2, R_2)$ we will construct a new model $\mathcal{M}' = (M_2, R_2, V')$ such that $\mathcal{M}_1, m_1 \leftrightsquigarrow_{\mathcal{L}_H(\overline{q}, \overline{n}, p)} \mathcal{M}', m_2$ and $\mathcal{M}_2 \cong_{\mathcal{L}_H(\overline{q}, \overline{n})} \mathcal{M}'$. We need to define the $(\overline{q}, \overline{n}, p)$-valuation $V'$. Let $m' \in M_2$ and $m \in M_1$ such that $mZm'$. (Note that such a point $m$ is guaranteed to exist, since by the first subclaim in this proof corresponding equivalence classes realize precisely the same valuations.) Let $V'$ be such that $m'$ has the same $(\overline{q}, \overline{n}, p)$-valuation as $m$. Note that, by the claim 3, this is indeed well defined, and that, as far as the $(\overline{q}, \overline{n})$-valuation is concerned, $\mathcal{M}_2$ and $\mathcal{M}'$ are in fact the same model. The only possible difference lies in the $p$-valuation. In particular note that we may apply claim 1, mutatis mutandis, to $\mathcal{M}_1$ and $\mathcal{M}'$.

Claim 4: Points related by $Z$ (in $\mathcal{M}_1$ and $\mathcal{M}'$) are modally equivalent with respect to the language $\mathcal{L}_H(\overline{q}, \overline{n}, p)$.

Proof of Claim 4: Let $m \in M_1$ and $m' \in M_2$ and $mZm'$, specifically suppose that $m \in [m_j^1]_{\mathcal{M}_1}$ and $m' \in [m_j^2]_{\mathcal{M}'}$, for some $j \in \{0, n_1, \ldots, n_m\}$. Let $\varphi \in \mathcal{L}_H(\overline{q}, \overline{n}, p)$. We proceed by induction on $\varphi$ to show that $\mathcal{M}_1, m \Vdash \varphi$ if and only if $\mathcal{M}', m' \Vdash \varphi$. The cases for proposition letters, nominals and the boolean connectives are trivial.

So suppose $\varphi$ is of the form $\Diamond\psi$. If $\mathcal{M}_1, m \Vdash \Diamond\psi$ there is a point $w \in [m_j^1]_{\mathcal{M}_1}$ such that $\mathcal{M}_1, w \Vdash \psi$. By claim 1 and the construction of $\mathcal{M}'$ there is a point $w' \in [m_j^2]_{\mathcal{M}'}$ such that $wZw'$. Then by the inductive hypothesis $\mathcal{M}', w' \Vdash \psi$, hence $\mathcal{M}', m' \Vdash \Diamond\psi$. To show that if

$\mathcal{M}', m' \Vdash \Diamond\psi$ then $\mathcal{M}_1, m \Vdash \Diamond\psi$ is precisely symmetrical.

Now suppose $\psi$ is of the form $@_{n_j}\psi$ for some $1 \leq j \leq m$. Then $\mathcal{M}_1, m \Vdash @_{n_j}\psi$ iff $\mathcal{M}_1, m^1_{n_j} \Vdash \psi$ iff $\mathcal{M}', m^2_{n_j} \Vdash \psi$ (note that $m^1_{n_j} Z m^2_{n_j}$, and we may apply the inductive hypothesis to $\psi$) iff $\mathcal{M}', m \Vdash @_{n_j}\psi$. $\dashv$

Claim 5: $\mathcal{M}' = (M_2, R_2, V')$ is a model of $\Gamma$.

Proof of Claim 5: By claim 4 every point in $\mathcal{M}'$ is modally equivalent (with respect to the language $\mathcal{L}_H(\overline{q}, \overline{n}, p)$) to a point in $\mathcal{M}_1$. As every point in $\mathcal{M}_1$ satisfies $\Gamma$, every point in $\mathcal{M}'$ satisfies $\Gamma$. $\dashv$

Obviously the identity function $\mathsf{ID} : \mathcal{M}' \to \mathcal{M}_2$ is a $\mathcal{L}_H(\overline{q}, \overline{n})$-isomorphism between two models of $\Gamma$. As $\Gamma$ is an implicit definition and $\mathcal{M}', m_2 \cong_{\mathcal{L}_H(\overline{q},\overline{n})} \mathcal{M}_2, m_2$ we have that $\mathcal{M}', m_2 \cong_{\mathcal{L}_H(\overline{q},\overline{n},p)} \mathcal{M}_2, m_2$, and hence $\mathcal{M}', m_2 \leftrightsquigarrow_{\mathcal{L}_H(\overline{q},\overline{n},p)} \mathcal{M}_2, m_2$. Hence we have that $\mathcal{M}_1, m_1 \leftrightsquigarrow_{\mathcal{L}_H(\overline{q},\overline{n},p)} \mathcal{M}', m_2$ and $\mathcal{M}', m_2 \leftrightsquigarrow_{\mathcal{L}_H(\overline{q},\overline{n},p)} \mathcal{M}_2, m_2$, and so that $\mathcal{M}_1, m_1 \leftrightsquigarrow_{\mathcal{L}_H(\overline{q},\overline{n},p)} \mathcal{M}_2, m_2$ as desired. $\dashv$

Now a negative result. By restricting attention to the class of all models with at least four points, we are able to provide and an implicit definition which can not be matched by an explicit definition.

**Definition 5.2.4** Let $\mathsf{S}^4_H = \langle \mathcal{L}_H, \mathsf{K}^4, \Vdash \rangle$, where $\mathsf{K}^4$ is the class of all models with at least four points. $\lhd$

**Theorem 5.2.5** *$S^4_H$ does not have the Beth property.*

Proof: Let $p$ and $q$ be proposition letters and $n_1$, $n_2$, and $n_3$ nominals. Let

$$\Gamma(q, n_1, n_2, n_3, p) = \left\{ \begin{array}{l} p \to (n_1 \vee n_2) \\ @_{n_1}p \vee @_{n_2}p \\ \neg(@_{n_1}p \wedge @_{n_2}p) \\ @_{n_1}(\neg q) \wedge @_{n_2}(\neg q) \\ \neg n_1 \wedge \neg n_2 \wedge \neg n_3 \to q \\ q \to \Diamond\top \\ q \to \Box p \end{array} \right\}$$

Claim: Let $\mathcal{M} = (M, R, V) \in \mathsf{K}^4$. If $\mathcal{M} \Vdash \Gamma$, then

$$[\![p]\!]_{\mathcal{M}} = \{u \in M : \text{for all } v \in M, \text{ if } \mathcal{M}, v \Vdash q \text{ then } Rvu\}$$

Proof of Claim: Note that there is precisely one $p$-point in $M$ ($p$ holds at either $n_1$ or $n_2$ but not at both). Also, as $\mathcal{M} \Vdash \neg n_1 \wedge \neg n_2 \wedge \neg n_3 \to q$ and $|M| \geq 4$, there must be at least one $q$-point in $M$. Let $S = \{u \in M : \text{for all } v \in M, \text{ if } \mathcal{M}, v \Vdash q \text{ then } Rvu\}$. We will show that $[\![p]\!]_{\mathcal{M}} = S$.

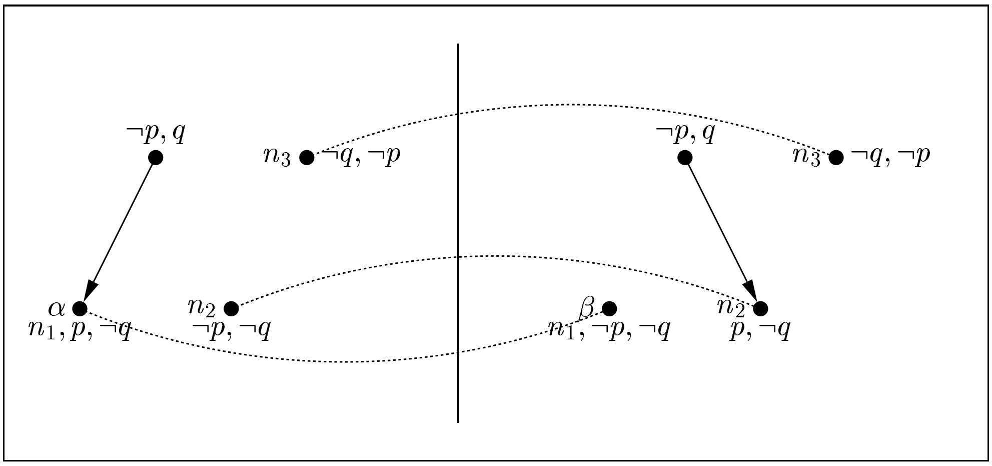


Figure 5.2: Theorem 5.2.5

Suppose $x \in [\![p]\!]_{\mathcal{M}}$, i.e. $[\![p]\!]_{\mathcal{M}} = \{x\}$. Let $v \in M$ arbitrarily. Suppose $\mathcal{M}, v \Vdash q$ then as $\mathcal{M} \Vdash (q \to \Diamond\top) \wedge (q \to \Box p)$ it must be the case that $Rvx$. Hence $x \in S$ and $[\![p]\!]_{\mathcal{M}} \subseteq S$.

Now suppose $x \in S$. We have at least one $q$-point, say $v \in M$. As $x \in S$ , $Rvx$ and since $\mathcal{M} \Vdash q \to \Box p$ it must be the case that $x \in [\![p]\!]_{\mathcal{M}}$. Hence $S \subseteq [\![p]\!]_{\mathcal{M}}$, and $S = [\![p]\!]_{\mathcal{M}}$. $\dashv$

So $\Gamma$ implicitly defines $p$. But now consider the models depicted in figure 5.2 as well as the indicated hybrid bisimulation (with respect to the language $\mathcal{L}_H(q, n_1, n_2, n_3)$). It is easy to check that these are indeed models of $\Gamma$. So $\alpha \leftrightsquigarrow_{\mathcal{L}_H(q,n_1,n_2,n_3)} \beta$ but $\alpha \not\leftrightsquigarrow_{\mathcal{L}_H(q,p,n_1,n_2,n_3)} \beta$. Hence there can be no explicit definition of $p$ relative to $\Gamma$. $\dashv$

One might feel that this counterexample could be easily generalized to the hybrid logic of the class of all models, $\mathsf{S}_H$. After all, it is easy to use $k$ nominals to force the existence of at least $k$ points. Once one starts experimenting however, it quickly becomes clear that there is no obvious way to do this while simultaneously having the crucial $q$-point unnamed by a nominal. As soon as some $q$-point is named, it becomes possible to distinguish two points like $\alpha$ and $\beta$ in figure 5.2, derailing any attempt at a counterexample along these lines.

### 5.2.3 Other Hybrid Logics

In closing this chapter let us briefly remark on some other hybrid logics, having quite different behavior as far as interpolation is concerned. One may take the hybridization of the language still further, adding "state variables" which act much like variables is first-order logic. Taking advantage of these one also adds a binder ($\downarrow$), binding a variable to the current state, in so doing adding great expressive power to the language. As it turns out a significantly large class of these hybrid logics have interpolation. [Blackburn and Marx, 2002] provide tableaux systems to constructively compute interpolants.

# Conclusions and Directions for Further Research

The Beth property in general is very "fragile" – it is quite sensitive to seemingly minor changes in the language. Enriching a langauge certainly results in more candidate formulas for explicit definitions, but on the other hand it also offers more possibilities for implicit definitions. It is not easy to make generalizations about which kinds of logics have Beth definability and which do not, neither is it our intention to do so here. We merely observe that, at least in the cases considered in this text, it seems that the mixing of modalities and operators offering local and global perspectives on structures, often leads to negative Beth results.

We began by introducing the difference operator, proving that although difference logic lacks interpolation, it does still enjoy Beth definability. Adding the difference operator to modal logic and tense logic we lost the Beth property. This was the first indication that the mixing of global- and local perspectives might bode ill for definability.

Looking at three types of graded modalities we were able to prove the Beth property for a number of logics, notably the logics with only type 1 modalities, only type 2 modalities as well as the logic with type 1 modalities and the modal diamond. Another interesting positive result here is that for the logic with the type 1 graded modalities and the difference operator. From the results in the previous chapter one would suspect that the mixing of perspectives here would again lead to a negative answer. An intuitive explanation might be, however, that the local perspective provided by type 1 graded modalities is weak enough that combining it with the global perspective of the difference operator does not lead to the loss of Beth definability. Combining the strong global perspective of the type 3 graded modalities with local perspectives, the results were once again negative.

As far as hybrid logic with nominals and the @-operator is concerned, we were able to give a positive Beth result by restricting attention to the class of models based on $\mathsf{S5}$ frames. A rather artificial (albeit first-order definable) restriction to the class of all models with at least four points gave a negative result. Although it is known that the hybrid logic (with nominals and the @-operator) of the class of all models lacks interpolation, it is at present unknown whether it has the Beth property. Many of the methods used to obtain positive results for other logics in this thesis break down when applied to this logic. To give but one example, hybrid logic lacks the tree model property: we are unable to unravel models.

Take as a counterexample the model consisting of two points, each named by a nominal and where the accessibility relation is the total relation.

Some of the results mentioned above would seem to indicate that it might be interesting to investigate just how operators offering local- and global perspectives interact. One might try to characterize the types of combinations that lead to positive definability results.

There exist versions of the graded modalities in the literature, different from those treated here, see for example [D'Agostino, 1998]. These might still hold possibilities for similar research. Although taking its cue from description logics, chapter 4 does not adhere to the hierarchy of logics considered in the description logic literature. These logics may not contain all boolean operators (notably omitting negation or allowing only restricted negation), or may contain certain other operators and constructs. We have not undertaken a thorough study of this area, but indications are that there are possibilities for Beth investigations here.

Obtaining the necessary tools for a more thoroughgoing study of the Beth property for hybrid logics with nominals and the @-operator is definitely an important direction that merits further attention.

A summary of the results obtained is given in table 5.2.3.

| Logic | Beth Property | Proof |
|---|---|---|
| $\mathsf{Dif}$ | Yes | 3.2.7 |
| $\mathsf{ML_D}$ | No | 3.3.4 |
| $\mathsf{TL_D}$ | No | 3.4.4 |
| $\mathsf{G}_1$ | Yes | 4.3.3 |
| $\mathsf{G}_{1,\mathsf{D}}$ | Yes | 4.3.5 |
| $\mathsf{G}_{1,\mathsf{D},\diamond}$ | No | 4.3.6 |
| $\mathsf{G}_{1,\diamond}$ | Yes | 4.3.12 |
| $\mathsf{G}_2^\omega$ | Yes | 4.4.10 |
| $\mathsf{G}_2$ | Yes | 4.4.11 |
| $\mathsf{G}_{2,\mathsf{D}}$ | No | 4.4.12 |
| $\mathsf{G}_3$ | Yes | 4.5.2 |
| $\mathsf{G}_{3,\diamond}$ | No | 4.5.4 |
| $\mathsf{G}_{2,3}$ | No | 4.5.6 |
| $\mathsf{S5}_H$ | Yes | 5.2.3 |
| $\mathsf{S}_H^4$ | No | 5.2.5 |

Table of Results

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