Monotonic Modal Logics

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Abstract

Monotonic modal logics form a generalisation of normal modal logics in which the additivity of the diamond modality has been weakened to monotonicity: $\Diamond p \vee \Diamond q \to \Diamond(p \vee q)$. This generalisation means that Kripke structures no longer form an adequate semantics. Instead monotonic modal logics are interpreted over monotonic neighbourhood structures, that is, neighbourhood structures where the neighbourhood function is closed under supersets. As specific examples of monotonic modal logics we mention Game Logic, Coalition Logic and the Alternating-Time Temporal Logic. This thesis presents results on monotonic modal logics in a general framework. The topics covered include model constructions and truth invariance, definability and correspondence theory, the canonical model construction, algebraic duality (for monotonic neighbourhood frames), coalgebraic semantics, Craig interpolation via superamalgamation, and simulations of monotonic modal logics by bimodal normal ones. The main contributions are: generalisations of the Sahlqvist correspondence and canonicity theorems, a detailed account of algebraic duality via canonical extensions, an analogue of the Goldblatt-Thomason theorem on definable frame classes, results on the relationship between bisimulation and coalgebraic notions of structural equivalence, Craig interpolation results, and a simulation construction which preserves descriptiveness of general frames.

Keywords: non-normal modal logic, neighbourhood semantics, definability, correspondence theory, algebraic duality, coalgebra, Craig interpolation, simulation.

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1 Introduction

Monotonic modal logics form a generalisation of normal modal logics in which the additivity principle, that is, the distribution of the diamond modality over disjunction, has been weakened to the monotonicity axiom: $\Diamond\varphi \vee \Diamond\psi \to \Diamond(\varphi \vee \psi)$. Why this axiom is called monotonicity comes out more clearly when we formulate it as a derivation rule: From $\vdash \varphi \to \psi$ infer $\vdash \Diamond\varphi \to \Diamond\varphi$.

Classical (or non-normal) modal logics, of which monotonic modal logics are a special case, are traditionally interpreted over neighbourhood structures. A neighbourhood model is a triple $\mathbb{M} = (W, \nu, V)$ where W is the set of worlds, V is a valuation, and $\nu : W \to \mathcal{P}(\mathcal{P}(W))$ is a neighbourhood function which associates a set of neighbourhoods with each world. A modal necessity operator ∇ is interpreted by ν as follows: $\mathbb{M}, w \Vdash \nabla\varphi$ iff $V(\varphi) \in \nu(w)$, where $V(\varphi) = \{x \in W \mid \mathbb{M}, x \Vdash \varphi\}$. When there are no restrictions on ν, the distributivity axiom and other principles of normal modal logics will generally not hold in a neighbourhood model. The class of *monotonic* (neighbourhood) frames in which ν is closed under supersets form the adequate semantics of monotonic modal logics.

In early history, classical modal logics are always mentioned but hardly used, and they seem to be studied mainly for their mathematical properties. However, in the past 15 years or so, applications have been found where the requirement of additivity turns out to be too strong or hold undesirable consequences. This is, for example, the case in Concurrent Propositional Dynamic Logic, see Goldblatt [32], and more recently, in Parikh's Game Logic [53], Pauly's Coalition Logic [58] and the Alternating-Time Temporal Logic of Alur et alii [2]. In the latter three cases, the (coalitional) ability of agents is formalised in languages containing modalities of the form $\nabla\varphi$. Loosely stated, $\nabla\varphi$ has the interpretation "the agent can bring about φ". This is made precise by the semantics which is defined in terms of strategic games, and thus, in game terms, $\nabla\varphi$ expresses that the agent has a strategy to achieve an outcome of the game where φ holds. In general, an agent cannot ensure that the outcome will be one particular state, as the outcome depends on which strategy the other agents choose; rather the agent can only ensure that the outcome falls within a certain set of states. Given this intended interpretation, it should be clear that additivity is not a valid principle. As an example, suppose that the set of agents is a group of friends who can all choose to go to the cinema or stay at home. Then an agent A has a strategy to ensure that the outcome is that she goes to the cinema with friends or she goes on her own, which we may specify in the formula $\nabla_A(\textit{cinema with friends} \vee \textit{cinema alone})$. But she cannot ensure one or the other, that is, $\nabla_A(\textit{cinema with friends}) \vee \nabla_A(\textit{cinema alone})$ is not the case, since the outcome will depend on what the other agents decide to do (assuming that if several of them go, then they go together). Similarly, with this interpretation, ∇ is not a 'box'-modality either, that is, the principle $\nabla p \wedge \nabla q \to \nabla(p \wedge q)$ is not valid, since the availability of strategies to achieve p and q separately need not imply a strategy where both p and q can be achieved simultaneously. However, monotonicity is clearly a valid principle: If a player has a strategy which ensures the outcome to be in X and $X \subseteq Y$, then the same strategy will ensure the outcome to be in Y.

Non-normal modal logics have also been suggested as adequate systems for deontic logic, where the formalisation of conditional obligation in terms of normal modal logic leads to paradoxes or counter-intuitive interpretations, see e.g. Chellas [14] and van der Torre [70]. Similar objections to normality are found when reasoning about knowledge and belief (the omni-science problem), see Fagin and Halpern [19], and Vardi [71, 72]. Furthermore, the topological models of Aiello and van Benthem [1] are specific examples of neighbourhood models.

Literature specifically devoted to monotonic modal logic is rather scarce. Early works on classical modal logics and neighbourhood semantics include Segerberg [62], and Chellas and McKinney [15], which mainly focus on completeness results. More recently, Gasquet [21] investigates completeness of monotonic multi-modal logics, and the relationship between neighbourhood completeness and Kripke completeness is treated in [64, 65, 12]. Bull and Segerberg [9] only mention neighbourhood semantics very briefly.

Chellas [14] is one of the few textbooks which treats non-normal modal logic and neighbourhood semantics in some detail. Pauly's work on Coalition Logic [57, 58] covers many aspects of monotonic modal logic and its semantics, including bisimulation invariance (see also Pauly [55]), and safety under bisimulation of the game constructions in Game Logic. To our knowledge, the only published work on algebraic duality for neighbourhood frames is by Došen [18], where full categorical duality is proved between neighbourhood frames and certain kinds of modal algebras. Although some of Došen's results easily adapt to the monotonic case, this is not entirely so when put into a unified framework of monotonic and algebraic semantics. On the algebraic side, Blok and Köhler [7] give an early account of algebraic semantics for non-normal modal logics in terms of so-called filtered modal algebras, and Gehrke and Jónsson [24] describe canonical extensions of algebras expanded with monotone operations.

As a more indirect way of gaining knowledge, some authors, including Gasquet and Herzig [22] and Kracht and Wolter [44], show how to simulate monotonic modal logics by bimodal normal ones. Simulations are a way of interpreting one logic in another, and by showing that properties such as decidability, completeness and canonicity are preserved or reflected by the simulation, one may transfer results on these properties between the two logics. In this way Kracht and Wolter [44] obtain a general completeness result for monotonic logics.

The emergence of modal systems such as Game Logic and Coalition Logic has given us motivation to study monotonic modal logics in more detail. Moreover, much of the existing knowledge is considered to be folklore, and as the above suggests, results are scattered and occur in different contexts. This indicates a clear need for a unified theory of monotonic modal logics. For normal modal logics, such a theory has over and over again proved its use: When dealing with normal modal logics, we immediately have a number of general results available such as the van Benthem Characterisation Theorem, definability via the Goldblatt-Thomason theorem, and Sahlqvist correspondence and canonicity. Thus the question naturally arises whether similar results hold for monotonic modal logics.

With this thesis, we hope to remedy part of the above problem by bringing together both known and new results pertaining to a general theory of monotonic modal logic. In particular, a detailed account of algebraic duality for monotonic structures will be presented, and in many situations we will see that it is due to this duality that we may generalise some important results on normal modal logic to monotonic modal logics. These results include some of the main contributions of this thesis which we list here: Theorem 5.4 (an analogue of the Goldblatt-Thomason theorem on definability of monotonic frame classes), Theorem 5.14 (an analogue of the Sahlqvist Correspondence theorem for monotonic modal logics), Theorems 8.35 and 8.37 (which link the notions of bisimulation and behavioural equivalence known from coalgebras to monotonic frame bisimulations), Theorem 9.10 (Craig interpolation), Theorems 10.34 and 10.44 (analogues of the Sahlqvist Canonicity theorem). Below is a more detailed description of the contents.

Outline

The basic definitions of classical and monotonic modal logics and their semantics are given in section 3.

In section 4 we define the notions of disjoint union, bounded morphism, bisimulation, generated submodel and unravelling of monotonic structures, and show that truth is invariant under these constructions. We also define filtrations and ultrafilter extensions of monotonic structures, and prove some standard technical results.

In section 5 we investigate the notions of frame definability and (first-order) correspondence, and present analogues of the Goldblatt-Thomason theorem (Theorem 5.4) and the Sahlqvist Correspondence theorem (Theorem 5.14). In the last subsection we present Pauly's [55] adaptation to monotonic modal logic of the van Benthem characterisation theorem.

In section 6, we define the canonical model, which, it should be noted, is different from the one found in Chellas [14] and Pauly [57]. Completeness for most of the standard monotonic logics is already known, but we give an alternative proof based on simulations and canonicity.

In section 7 we first describe the algebraisation of monotonic modal logics, and proceed to defining canonicity in terms of canonical extensions as in the tradition of Jónsson and Tarski [39]. Due to the lack of additivity of the function with which we have expanded our boolean algebras, the notion of canonicity splits into two variants: σ-canonicity and π-canonicity. In subsection 7.4 the basic duality between monotonic frames and algebras (Theorem 7.21) is shown, and in 7.5 we show full categorical duality between descriptive general monotonic frames and algebras (Theorem 7.36). Section 7 is concluded with a discussion of the relationship between the two notions of canonicity in subsection 7.6.

In section 8 we offer an alternative view on monotonic structures as coalgebras, and compare the coalgebraic notions of bisimulation and system equivalence with the model theoretic one. This section presents joint work with Clemens Kupke, ILLC, University of Amsterdam.

Section 9 centres around the relationship between the Craig interpolation property and superamalgamation of varieties, and we will see that our work on coalgebras and the relationship between the σ- and π-constructions of section 7 has further merit when we prove a monotonic analogue of the Zigzag Lemma in Marx [51], which in turn allows us to conclude that a large class of monotonic modal logics have the Craig interpolation property (Theorem 9.10).

Finally, in section 10, we show how to simulate monotonic modal logics by bimodal normal ones in such a way that (σ-)canonicity is reflected by the simulation. This is an improvement on the results in [22, 44], and is obtained by a simulation construction which preserves descriptiveness of general frames. We immediately obtain σ-canonicity for formulas which translate to bimodal Sahlqvist formulas (Theorem 10.34). We also briefly return to the relationship between the dual notions of σ- and π-canonicity which is captured by the dual simulation of subsection 10.6. The dual simulation provides us with π-canonicity of all formulas whose dual translation is σ-canonical (Theorem 10.44).

2 Preliminaries

The starting point for this paper was to collect and investigate results about monotonic modal logics which are well-known for normal modal logics. Therefore, it is assumed that the reader is familiar with the general theory of normal modal logics, more or less as presented in Blackburn et alii [6], which also accounts for most of the notation and terminology employed here. Other normal modal logic references include [11, 43]. For non-normal modal logic and neighbourhood semantics, see [14, 18, 63]. Furthermore, a large part of the theory presented here has been driven by interests in algebraic duality, hence a good knowledge of Stone spaces, boolean algebras with operators and (descriptive) general frames together with some basic category theory is particularly useful, see [29, 30, 37, 61, 18, 31, 43]. Also some exposure to the theory of canonical extensions [39, 40, 23, 24] will help. For the Algebra, Coalgebra and Interpolation sections, some familiarity with universal algebra [10] is expected, but no prior knowledge of coalgebra is assumed. For background knowledge on coalgebras, the reader may consult [59, 46, 54, 45].

3 Basic Concepts

3.1 Syntax

We will be working with the basic modal similarity type throughout most of this thesis, that is, a language $\mathcal{L}_\nabla$ which contains one unary modality ∇. For a fixed (countable) set of proposition letters $\textsc{prop}$, the well-formed formulas of $\mathcal{L}_\nabla$ ($\mathcal{L}_\nabla$-formulas) are given by,

$$\varphi := \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \nabla\varphi \qquad \text{where } p \in \textsc{prop}.$$

$\top, \wedge, \rightarrow$ and $\leftrightarrow$ are defined as the usual abbreviations, and Δ abbreviates $\neg\nabla\neg$. In most of the literature, including Chellas [14], the necessity-modality ∇ is denoted by $\Box$. However, the $\Box$-symbol which is also traditionally used for normal necessity-modalities, has an intuitive universal character, whereas the interpretation of a monotone necessity-modality has both a universal and an existential component, as we will see later (Remark 3.7). For notational convenience we will sometimes write $\varphi \in \mathcal{L}_\nabla$ instead of "φ is an $\mathcal{L}_\nabla$-formula", and $\Sigma \subseteq \mathcal{L}_\nabla$ instead of "Σ is a set of $\mathcal{L}_\nabla$-formulas".

Recall the following definition from Blackburn et alii [6]: A set of modal formulas Λ over a language $\mathcal{L}$ is a *modal $\mathcal{L}$-logic* if Λ contains all propositional tautologies and is closed under *modus ponens* and *uniform substitution.* In order to define classical and monotonic modal logics, consider the following inference rules:

$$(\mathrm{RE}_\nabla) \quad \frac{\varphi \leftrightarrow \psi}{\nabla\varphi \leftrightarrow \nabla\psi}$$

$$(\mathrm{RM}_\nabla) \quad \frac{\varphi \rightarrow \psi}{\nabla\varphi \rightarrow \nabla\psi}$$

Definition 3.1 Let $\mathcal{L}$ be a modal language with unary, primitive modalities ∇_i, $i \in I$, and let Λ be a modal $\mathcal{L}$-logic. Then ∇_i is *classical in* Λ if Λ is closed under RE_{∇_i}, and ∇_i is *monotone in* Λ if Λ is closed under RM_{∇_i}. Λ is *classical* if for all $i \in I$, ∇_i is classical in Λ, and Λ is *monotonic* if for all $i \in I$, ∇_i is monotone in Λ. $\dashv$

Remark 3.2 One can easily show that when ∇ is monotone in a modal $\mathcal{L}_\nabla$-logic Λ, then so is Δ, since for any modal $\mathcal{L}_\nabla$-logic Λ, if Λ is closed under the RM_∇ rule, then Λ is also closed under the rule RM_Δ:

$$(\mathrm{RM}_\Delta) \quad \frac{\varphi \to \psi}{\Delta\varphi \to \Delta\psi}$$

Furthemore, readers familiar with [6] and [14] will notice that we do not include the axiom (Dual) $\nabla p \leftrightarrow \neg\Delta\neg p$ in our definition of monotonic and classical modal logics. The reason for this is that we have chosen ∇ as our primitive symbol and Δ as an abbreviation.

An $\mathcal{L}_\nabla$-formula φ is a *theorem of* Λ (notation: $\vdash_\Lambda \varphi$) if $\varphi \in \Lambda$. Derivations in a modal logic are Hilbert-style proofs, and we define deducibility in terms of the local consequence relation: Let $\Sigma \cup \{\varphi\} \subseteq \mathcal{L}_\nabla$ and let Λ be a modal $\mathcal{L}_\nabla$-logic, then φ *is deducible from* Σ *in* Λ (notation: $\Sigma \vdash_\Lambda \varphi$) iff there are $\sigma_1, \ldots, \sigma_n \in \Sigma$ such that $\vdash_\Lambda \sigma_1 \wedge \ldots \wedge \sigma_n \to \varphi$. If φ is not deducible from Σ in Λ, we write $\Sigma \nvdash_\Lambda \varphi$. Σ is Λ-*consistent* iff $\Sigma \nvdash_\Lambda \bot$ and Λ-inconsistent otherwise. Λ is consistent iff $\nvdash_\Lambda \bot$ and consistent otherwise.

The smallest monotonic modal $\mathcal{L}_\nabla$-logic will be called **M** and later we shall be looking at various extensions of **M** with one or more of the following axioms:

M	$\nabla(p \wedge q) \to \nabla p$
N	$\nabla\top$
P	$\neg\nabla\bot$
C	$\nabla p \wedge \nabla q \to \nabla(p \wedge q)$
T	$\nabla p \to p$
4	$\nabla\nabla p \to \nabla p$
4’	$\nabla p \to \nabla\nabla p$
5	$\Delta p \to \nabla\Delta p$
B	$p \to \nabla\Delta p$
D	$\nabla p \to \Delta p$

If Σ is a set of $\mathcal{L}_\nabla$-formulas, then $\mathbf{M}.\Sigma$ denotes the smallest monotonic modal $\mathcal{L}_\nabla$-logic containing Σ. We will also say that $\mathbf{M}.\Sigma$ is the monotonic $\mathcal{L}_\nabla$-logic generated by Σ.

It is straightforward to show that a monotonic modal logic is also classical, and that a classical modal logic is monotonic iff it contains the axiom M iff it contains $\Delta p \vee \Delta q \to \Delta(p \vee q)$, which is the dual version of M. Normal modal logics are usually defined in terms of the K axiom $(\nabla(p \to q) \to (\nabla p \to \nabla q))$ and the Necessitation rule $(p/\nabla p)$: Λ is a normal modal $\mathcal{L}_\nabla$-logic if Λ contains K, and is closed under the Necessitation rule. Recall again that the axiom (Dual) is not needed, since ∇ is our primitive modality. However, one could equally well have defined normal modal logics to be the monotonic modal logics containing C and N, thus supporting our view of monotonic modal logic as a generalisation of normal modal logic. Note that the dual of axiom C, which is equivalent with $\Delta(p \vee q) \to \Delta p \vee \Delta q$, expresses that Δ is additive. The axiom N, which is equivalent with $\Delta\bot \leftrightarrow \bot$, expresses normality of Δ. In **M**, Δ only satisfies monotonicity: Defining $\varphi \leq \psi$ by $\varphi \vee \psi \leftrightarrow \psi$, we have $\varphi \leq \psi$ implies $\Delta\varphi \leq \Delta\psi$. See [14, 63] for more details on alternative characterisations of classical, monotonic and normal modal logics.

Example 3.3 As an example of a monotonic modal logic, we will consider *Coalition Logic* [57, 58, 36]. Coalition Logic formalises the ability of groups of agents to achieve certain outcomes in strategic games. The language of Coalition Logic contains modalities of the form $[C]$, and a formula $[C]\varphi$ is to be interpreted as "coalition C has a strategy to achieve an outcome state where φ holds". This will be made precise in the next section where we will look at the semantics of Coalition Logic. The language $\mathcal{L}_{CL(N)}$ of Coalition Logic is defined for a non-empty set (of agents) N and a fixed (countable) set of proposition letters PROP. The well-formed formulas of $\mathcal{L}_{CL(N)}$ are,

$$\varphi := \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid [C]\varphi \qquad \text{where } p \in \textsc{prop}, C \subseteq N.$$

In the context of Coalition Logic, we will always assume a fixed set of agents N and use C (possibly with subscripts) to denote a coalition of agents, i.e. $C \subseteq N$. A set of $\mathcal{L}_{CL(N)}$-formulas Λ is a *coalition logic for* N, if Λ contains all propositional tautologies, is closed under modus ponens, uniform substitution and for all $C \subseteq N$, Λ is closed under the inference rule

$$\frac{p \to q}{[C]p \to [C]q}\ \mathrm{RM}_{[C]}$$

and contains the following axioms:

$$\begin{array}{ll}
(\bot) & \neg[C]\bot \\
(\top) & [C]\top \\
(\mathtt{N}) & \neg[\emptyset]\neg p \to [N]p \\
(\mathtt{S}) & [C_1]p \wedge [C_2]q \to [C_1 \cup C_2](p \wedge q)) \quad \text{where } C_1 \cap C_2 = \emptyset.
\end{array}$$

Note that the axioms $(\bot)$ and $(\top)$ are the $\mathcal{L}_{CL(N)}$-versions of the axioms P and N. We will see later that the above axioms express exactly the properties of the semantic interpretations of the $[C]$-modalities which are needed to obtain a semantics in terms of strategic games.

3.2 Models and Frames

The generalisation of normal modal logic to monotonic modal logic means that Kripke frames no longer constitute an adequate semantics. For example, the C axiom, $\nabla p \wedge \nabla q \to \nabla(p \wedge q)$, is valid on all Kripke frames, but it is not a theorem of monotonic modal logics in general. The standard semantic tool used to interpret classical (non-normal) modal logics is *neighbourhood semantics* [26, 14, 18, 63]. In the possible world scenario, propositions are identified with sets of worlds, and in a neighbourhood model each world w is associated with a set of propositions ('neighbourhoods') via a neighourhood function ν. These are the propositions that are necessarily true at w. Hence a neighbourhood function is a map from the universe W to $\mathcal{P}(\mathcal{P}(W))$.

Definition 3.4 (Neighbourhood structures) A *neighbourhood frame* for the language $\mathcal{L}_\nabla$ is a pair $\mathbb{F} = (W, \nu)$ where W is a non-empty set (of worlds) and $\nu : W \to \mathcal{P}(\mathcal{P}(W))$ is a neighbourhood function.

If $\mathbb{F} = (W, \nu)$ is a neighbourhood frame and V: PROP$\to \mathcal{P}(W)$ a valuation on $\mathbb{F}$, then $\mathbb{M} = (W, \nu, V)$ is a *neighbourhood model* based on $\mathbb{F}$. ⊣

We will call a neighbourhood frame for the language $\mathcal{L}_\nabla$ an $\mathcal{L}_\nabla$-frame, and a neighbourhood model based on an $\mathcal{L}_\nabla$-frame will be referred to as an $\mathcal{L}_\nabla$-model. The particular class of $\mathcal{L}_\nabla$-frames that we will be working with are the ones in which ν is closed under supersets.

Definition 3.5 (Monotonic structures) A *monotonic $\mathcal{L}_\nabla$-frame* is an $\mathcal{L}_\nabla$-frame $\mathbb{F} = (W, \nu)$ in which ν is closed under supersets: $\forall w \in W,\ \forall X, Y \in \mathcal{P}(W) : X \subseteq Y, X \in \nu(w) \Rightarrow Y \in \nu(w)$. For convenience, we will also say that such a ν is monotone. A *monotonic $\mathcal{L}_\nabla$-model* is a model based on a monotonic $\mathcal{L}_\nabla$-frame. ⊣

The notion of a formula being true in an $\mathcal{L}_\nabla$-model is inductively defined for boolean connectives, the same way as for Kripke models. Only the interpretation by ν of the necessity operator ∇ is different. In accordance with the above interpretation of neighbourhoods as necessary propositions, $\nabla\varphi$ is true at a world $w \in W$, if the proposition expressed by φ is a neighbourhood of w.

Definition 3.6 (Truth conditions) Let $\mathbb{M} = (W, \nu, V)$ be an $\mathcal{L}_\nabla$-model. Truth of an $\mathcal{L}_\nabla$-formula at w in $\mathbb{M}$ is defined inductively as follows:

$$
\begin{array}{lll}
\mathbb{M}, w \Vdash \bot & & \text{never}, \\
\mathbb{M}, w \Vdash p & \text{iff} & w \in V(p),\ p \in \text{PROP}, \\
\mathbb{M}, w \Vdash \neg\varphi & \text{iff} & \text{not } \mathbb{M}, w \Vdash \varphi, \\
\mathbb{M}, w \Vdash \varphi \vee \psi & \text{iff} & \mathbb{M}, w \Vdash \varphi \text{ or } \mathbb{M}, w \Vdash \psi, \\
\mathbb{M}, w \Vdash \nabla\varphi & \text{iff} & V(\varphi) \in \nu(w).
\end{array}
$$

where $V(\varphi) = \{w \in W \mid \mathbb{M}, w \Vdash \varphi\}$. ⊣

Remark 3.7 In a monotonic $\mathcal{L}_\nabla$-model, $\mathbb{M}, w \Vdash \nabla\varphi$ iff $\exists X \in \nu(w)\, \forall x \in X : \mathbb{M}, x \Vdash \varphi$, and by definition of Δ, $\mathbb{M}, w \Vdash \Delta\varphi$ iff $\forall X \in \nu(w)\, \exists x \in X : \mathbb{M}, x \Vdash \varphi$. This combination of a universal and an existential quantification is the reason why we have chosen the symbol ∇ instead of $\Box$, since the $\Box$-symbol is thought of as having a universal meaning only, when interpreted in a Kripke model.

Global truth, satisfaction and frame validity are defined in the usual way: If φ is an $\mathcal{L}_\nabla$-formula and $\mathbb{M} = (W, \nu, V)$ is an $\mathcal{L}_\nabla$-model, then φ is *globally true in* $\mathbb{M}$ (notation: $\mathbb{M} \Vdash \varphi$) if for all $w \in W$, $\mathbb{M}, w \Vdash \varphi$, and φ is *satisfiable in* $\mathbb{M}$ if there is some $w \in W$ such that $\mathbb{M}, w \Vdash \varphi$. If $\mathbb{F} = (W, \nu)$ is an $\mathcal{L}_\nabla$-frame, then φ is *valid in* $\mathbb{F}$ (notation: $\mathbb{F} \Vdash \varphi$) if for all valuations V on $\mathbb{F}$ and all $w \in W$, $(\mathbb{F}, V), w \Vdash \varphi$. When Σ is a set of $\mathcal{L}_\nabla$-formulas, $(\mathbb{F}, V), w \Vdash \Sigma$ means $(\mathbb{F}, V), w \Vdash \sigma$ for all $\sigma \in \Sigma$ etc. We say φ is a *local semantic consequence* of Σ in $\mathbb{F}$ (notation: $\Sigma \Vdash_{\mathbb{F}} \varphi$), if for all valuations V on $\mathbb{F}$ and all $w \in W$, $(\mathbb{F}, V), w \Vdash \Sigma$ implies $(\mathbb{F}, V), w \Vdash \varphi$.

If K is a class of $\mathcal{L}_\nabla$-frames, then φ is satisfiable in K is φ is satisfied in some model based on a frame in K, and φ is *valid on* K (notation: $\mathsf{K} \Vdash \varphi$) if φ is valid in all $\mathbb{F} \in \mathsf{K}$. Furthermore, φ is a local semantic consequence of a set of formulas Σ in K (notation: $\Sigma \Vdash_{\mathsf{K}} \varphi$) if $\Sigma \Vdash_{\mathbb{F}} \varphi$ for all $\mathbb{F} \in \mathsf{K}$. We will write Λ_{K} or $\mathsf{Th}(\mathsf{K})$ for the set of $\mathcal{L}_\nabla$-formulas that are valid on K. For any class of neighbourhood (monotonic) frames K, Λ_{K} is a classical (monotonic) modal logic [14].

Similarly to Kripke semantics, a neighbourhood function ν defines a map $m_\nu : \mathcal{P}(W) \to \mathcal{P}(W)$:

(1) $\quad m_\nu(X) = \{w \in W \mid X \in \nu(w)\}$,

and we have $m_\nu(V(\varphi)) = V(\nabla\varphi)$. Note also that m_ν is monotone whenever ν is.

It will sometimes be convenient to think of a neighbourhood function $\nu : W \to \mathcal{P}(\mathcal{P}(W))$ as a relation R_ν between W and $\mathcal{P}(W)$ where $wR_\nu X$ iff $X \in \nu(w)$. This relational perspective on ν will be put to good use in sections 5 and 10 where we look at correspondence and simulations of monotonic modal logic.

Furthermore, for monotonic $\mathcal{L}_\nabla$-frames it is often useful to consider what we will call the *(non-monotonic) core of* ν denoted by ν^c.

Definition 3.8 (Non-monotonic core) For a monotonic $\mathcal{L}_\nabla$-frame $\mathbb{F} = (W, \nu)$, we define the *(non-monotonic) core of* ν, ν^c, as follows:

$$X \in \nu^c(w) \quad \text{iff} \quad X \in \nu(w) \;\&\; \forall X_0 \subsetneq X : X_0 \notin \nu(w). \qquad \dashv$$

It should be noted that it is not always the case that a neighbourhood contains a core neighbourhood as the following example shows.

Example 3.9 Let $\mathbb{R}$ be the set of real numbers, and $\mathbb{F} = (\mathbb{R}, \nu)$ the monotonic $\mathcal{L}_\nabla$-frame where $\nu(0) = \uparrow\{(0, \varepsilon) \mid 0 < \varepsilon\}$. That is, $\nu(0)$ contains all subsets X of $\mathbb{R}$ for which $(0, \varepsilon) = \{x \mid 0 < x < \varepsilon\} \subseteq X$ for some $\varepsilon > 0$. It should be easy to see that $\nu^c(0) = \emptyset$.

The frames in which this kind of infinite descending chain of neighbourhoods does not occur will be called *core-complete*.

Definition 3.10 (Core-complete) Let $\mathbb{F} = (W, \nu)$ be a monotonic $\mathcal{L}_\nabla$-frame. Then $\mathbb{F}$ is *core-complete* if for all $w \in W$ and all $X \subseteq W$ the following holds: If $X \in \nu(w)$, then there is a $C \in \nu^c(w)$ such that $C \subseteq X$. A monotonic $\mathcal{L}_\nabla$-model $\mathbb{M} = (\mathbb{F}, V)$ is core-complete if $\mathbb{F}$ is core-complete. $\dashv$

Remark 3.11 Finite models form an interesting class of core-complete structures, and since in a core-complete model, we may think of ν^c as an irredundant representation of ν, it seems relevant to study properties of the core when investigating complexity issues. In fact, Pauly [57] introduced the notion of non-monotonic core when analysing the size of finite coalition models in connection with model checking. Although, we will not be concerned with complexity here, we will give a number of results concerning the model constructions of section 4 for core-complete models.

Just as monotonic modal logics are a generalisation of normal ones, neighbourhood semantics can be seen as a generalisation of Kripke semantics. It is well-known (see e.g. Chellas [14]) that there is a 1-1 correspondence between the class of all Kripke models (for the basic modal similarity type) and the class of augmented $\mathcal{L}_\nabla$-models such that corresponding models are point-wise equivalent. An $\mathcal{L}_\nabla$-model is augmented whenever $X \in \nu(w)$ iff $\bigcap \nu(w) \subseteq X$ for all $w \in W$, or equivalently, $\bigcap \nu(w) \in \nu(w)$. Thus, augmented models are core-complete. The correspondence between Kripke models and augmented $\mathcal{L}_\nabla$-model is shown as follows. In one direction, given a Kripke frame (W, R) the neighbourhoods of a state w are defined by: $X \in \nu(w)$ iff $R[w] \subseteq X$, where $R[w] = \{s \in W \mid Rws\}$ is the set of R-successors of w. In the other direction, given an $\mathcal{L}_\nabla$-frame (W, ν), we define the R-successors of w by $R[w] = \bigcap \nu(w)$.

As we will be looking at extensions of $\mathbf{M}$ by the axioms listed in the previous section, we will also be interested in which properties these axioms impose on frames. For convenience, we will list a set of properties here.

(n) $\forall w \in W : W \in \nu(w)$.
(p) $\forall w \in W : \emptyset \notin \nu(w)$.
(c) $\forall w \in W\ \forall X_1, X_2 \subseteq W : X_1 \in \nu(w)\ \&\ X_2 \in \nu(w) \to X_1 \cap X_2 \in \nu(w)$.
(t) $\forall w \in W\ \forall X \subseteq W : X \in \nu(w) \to w \in X$.
(iv) $\forall w \in W\ \forall X, Y \subseteq W : (X \in \nu(w)\ \&\ \forall x \in X : Y \in \nu(x)) \to Y \in \nu(w)$.
(iv') $\forall w \in W\ \forall X \subseteq W : X \in \nu(w) \to m_\nu(X) \in \nu(w)$.
(v) $\forall w \in W\ \forall X \subseteq W : X \notin \nu(w) \to W \setminus m_\nu(X) \in \nu(w)$.
(b) $\forall w \in W\ \forall X \subseteq W : w \in X \to W \setminus m_\nu(W \setminus X) \in \nu(w)$
(d) $\forall w \in W\ \forall X \subseteq W : X \in \nu(w) \to W \setminus X \notin \nu(w)$.

Example 3.12 The semantics of Coalition Logic are given by *strategic games* and *effectivity functions*. A *strategic game form* $G = (N, \{\Sigma_i | i \in N\}, o, S)$ consists of the set of agents N, a non-empty set of strategies or actions Σ_i for every player $i \in N$, the set of states S and an outcome function $o : \Pi_{i \in N} \Sigma_i \to S$ which associates with every tuple of strategies of the players (strategy profile) an outcome state in S. An *effectivity function* is any function $E : \mathcal{P}(N) \to \mathcal{P}(\mathcal{P}(S))$ which is outcome-monotonic: $\forall C \subseteq N, X \subseteq Y \subseteq S$: $X \in E(C) \Rightarrow Y \in E(C)$. Thus E is a C-indexed collection of neighbourhood functions on S, and the neighbourhoods are the outcome sets for which C is effective.

A strategic game form defines an effectivity function as follows: Given a strategic game form $G = (N, \{\Sigma_i | i \in N\}, o, S)$ a coalition $C \subseteq N$ will be *α-effective* for a set $X \subseteq S$ iff C has a joint strategy which will result in an outcome in X no matter what strategies the other players choose. More formally, its *α-effectivity function* $E^\alpha_G : \mathcal{P}(N) \to \mathcal{P}(\mathcal{P}(S))$ is defined as follows:

$$X \in E^\alpha_G(C) \text{ iff } \exists \sigma_C \forall \sigma_{N \setminus C}\ \ o(\sigma_C, \sigma_{N \setminus C}) \in X.$$

Here $\sigma_C = (\sigma_i)_{i \in C}$ denotes the strategy tuple for coalition $C \subseteq N$ which consists of player i choosing strategy $\sigma_i \in \Sigma_i$, and $o(\sigma_C, \sigma_{N \setminus C})$ denotes the outcome state associated with the strategy profile induced by σ_C and $\sigma_{N \setminus C}$.

A strategic game form G is now said to *represent* an effectivity function $E : \mathcal{P}(N) \to \mathcal{P}(\mathcal{P}(S))$ if $E = E^\alpha_G$. Thus every strategic game form can be linked to an effectivity function, but not every effectivity function will be the α-effectivity function of some strategic game form. The properties required to obtain a precise characterisation result are the following.

An effectivity function $E : \mathcal{P}(N) \to \mathcal{P}(\mathcal{P}(S))$ is *playable* if it satisfies the following four conditions:

1. $\forall C \subseteq N :\ \emptyset \notin E(C)$.

2. $\forall C \subseteq N :\ S \in E(C)$.

3. E is *N-maximal*: for all X, if $S \setminus X \notin E(\emptyset)$ then $X \in E(N)$.

4. E is *superadditive*: for all X_1, X_2, C_1, C_2 such that $C_1 \cap C_2 = \emptyset$, if $X_1 \in E(C_1)$ and $X_2 \in E(C_2)$ then $X_1 \cap X_2 \in E(C_1 \cup C_2)$.

The main characterisation result (see [58]) states: E has a strategic game form representation iff E is playable. Note that the Coalition Logic axioms are a straightforward translation of the playability conditions into the language $\mathcal{L}_{CL(N)}$.

A *coalition model* for the language $\mathcal{L}_{CL(N)}$ is a triple $\mathbb{M} = (S, E, V)$, where S is a nonempty set of states, $V : \text{PROP} \to \mathcal{P}(S)$ is a valuation and

$$E : S \to (\mathcal{P}(N) \to \mathcal{P}(\mathcal{P}(S)))$$

is the playable effectivity structure of $\mathbb{M}$. That is, for all $s \in S$, $E(s)$ is a playable effectivity function. We will use the notation $sE_C X$ for $X \in E(s)(C)$. A coalition model may thus be seen as a generalisation of a monotonic model, as we no longer have just one neighbourhood function, but a whole family of neighbourhood functions for each state.

The truth of an $\mathcal{L}_{CL(N)}$-formula is defined as usual for atomic propositions and boolean connectives. For the modalities, truth is given by: $\mathbb{M}, s \Vdash [C]\varphi$ iff $sE_C\varphi^{\mathbb{M}}$, where $\varphi^{\mathbb{M}} = \{s \in S \mid \mathbb{M}, s \Vdash \varphi\}$. Due to the main characterisation result, we can associate a strategic game form $G(s)$ with each state $s \in S$ in a coalition model $\mathbb{M}$, which implies that $[C]\varphi$ holds at a state s iff the coalition C is effective for $\varphi^{\mathbb{M}}$ in $G(s)$.

3.3 General Frames

Although neighbourhood semantics allows for completeness results of many monotonic modal logics, it still suffers from the same so-called inadequacy as Kripke semantics. Namely, there are monotonic modal logics which are not complete with respect to any class of neighbourhood frames, see e.g. Gerson [26]. The analogue with Kripke semantics goes further, since this inadequacy does not occur at the algebraic level, and by considering a type of structures which are essentially set-theoretic representations of certain algebras, a general completeness result is possible. These structures are, of course, the neighbourhood versions of general frames, which we will introduce here.

Definition 3.13 (General Frames) A *general monotonic $\mathcal{L}_\nabla$-frame* is a pair $\mathbb{G} = (\mathbb{F}, A)$ where $\mathbb{F} = (W, \nu)$ is a monotonic $\mathcal{L}_\nabla$-frame, and A is a collection of *admissible* subsets of W which contains $\emptyset$ and is closed under finite unions, complementation in W and the modal operation m_ν. We will refer to $\mathbb{F}$ as the underlying frame of $\mathbb{G}$.

A model based on a general monotonic $\mathcal{L}_\nabla$-frame is a triple $\mathbb{M} = (\mathbb{F}, A, V)$ where $(\mathbb{F}, A)$ is a general monotonic $\mathcal{L}_\nabla$-frame, and $V : \text{PROP} \to A$ is an *admissible valuation* on $\mathbb{G}$.

The definition of *general $\mathcal{L}_\nabla$-frames* etc. is obtained by leaving out the requirement that the underlying neighbourhood frame $\mathbb{F}$ is monotonic. $\dashv$

Remark 3.14 In [18], Došen requires that in a general (neighbourhood) frame $\mathbb{G} = (W, \nu, A)$ all neighbourhoods must be admissible, i.e. $\nu : W \to \mathcal{P}(A)$. This requirement is also adopted by Kracht and Wolter [44] in their definition of general monotonic frames ("general N^h-frames"). But this has the undesirable consequence that the underlying frame of a general monotonic frame in most cases will not be monotonic. Little seems to depend on this extra requirement, and later, in the Algebra and Simulation sections, we will see that our definition is easier to work with, and we can still obtain full algebraic duality.

Validity in general monotonic frames is defined as usual: An $\mathcal{L}_\nabla$-formula φ is *valid in a general monotonic $\mathcal{L}_\nabla$-frame* $\mathbb{G}$ (notation: $\mathbb{G} \Vdash \varphi$) if for all admissible valuations V and all $w \in W$, $(\mathbb{G}, V), w \Vdash \varphi$. If G is a class of general monotonic $\mathcal{L}_\nabla$-frames, then φ is *valid on* G (notation: $\mathsf{G} \Vdash \varphi$) if φ is valid in all $\mathbb{G} \in \mathsf{G}$.

4 Models

Models are structures with which we can reason about truth and satisfaction. In Kripke semantics, the four main operations on Kripke models which leave truth invariant are *disjoint unions, generated submodels, bounded morphisms and ultrafilter extensions*, and, as is well-known, these operations may all be seen as special cases of *bisimulations*. In this section we will define their analogues and also look at the filtration technique.

Throughout this section, we assume that we are working in the basic modal similarity type unless otherwise stated. I.e. "model" should be read as "$\mathcal{L}_\nabla$-model" and "formula" as $\mathcal{L}_\nabla$-formula", etc.

4.1 Invariance Results

Disjoint unions

Given a collection of disjoint monotonic models, $\mathbb{M}_i = (W_i, \nu_i, V_i)$, $i \in I$, we wish to make a monotonic model $\mathbb{M} = (W, \nu)$ which contains the $\mathbb{M}_i$ as disjoint substructures. Note that we cannot simply take $\nu = \bigcup_{i \in I} \nu_i$, since then ν would not be closed under supersets. However, this problem is easily fixed, we simply add the supersets in W of the neighbourhoods from each ν_i. This leads to the following definition.

Definition 4.1 (Disjoint Unions) Let $\mathbb{M}_i = (W_i, \nu_i, V_i)$, $i \in I$ be a collection of disjoint models. Then we define their *disjoint union* as the model $\biguplus \mathbb{M}_i = (W, \nu, V)$ where $W = \bigcup_{i \in I} W_i$, $V(p) = \bigcup_{i \in I} V_i(p)$ and for $X \subseteq W$, $w \in W_i$,

$$X \in \nu(w) \text{ iff } X \cap W_i \in \nu_i(w). \qquad \dashv$$

Note that in Definition 4.1, even though ν is not exactly the disjoint union of the ν_i, that is the case when considering the core, i.e., $\nu^c = \bigcup_{i \in I} \nu_i^c$.

Proposition 4.2 *Let $\mathbb{M}_i = (W_i, \nu_i, V_i)$, $i \in I$, be a collection of disjoint models and $\biguplus \mathbb{M}_i = (W, \nu, V)$ their disjoint union. Then for each formula φ, for each $i \in I$ and each element $w \in W_i$, we have:*

$$\mathbb{M}_i, w \Vdash \varphi \text{ iff } \biguplus \mathbb{M}_i, w \Vdash \varphi.$$

Proof. It should be clear that the proposition states that $V(\varphi) = \bigcup_{i \in I} V_i(\varphi)$ for all modal formulas φ. The proof is by induction on φ. Let $i \in I$ and $w \in W_i$. The atomic case holds by definition of V, and the boolean cases are straight forward. For the modal case, i.e. $\varphi \equiv \nabla\psi$, we need to show that $V_i(\psi) \in \nu_i(w)$ iff $V(\psi) \in \nu(w)$. So suppose that $V_i(\psi) \in \nu_i(w)$. By the induction hypothesis $V_i(\psi) \subseteq V(\psi)$, hence by the definition of ν, we have $V(\psi) \in \nu(w)$. Now suppose $V(\psi) \in \nu(w)$, then by the definition of ν, $V(\psi) \cap W_i \in \nu_i(w)$. Now the induction hypothesis tells us that $V(\psi) \cap W_i = V_i(\psi)$, and hence $V_i(\psi) \in \nu_i(w)$. QED

Bounded morphisms

Bounded morphisms should be structure preserving and reflecting maps between monotonic models. More precisely, the structure which can be described in our modal language should be preserved and reflected.

Definition 4.3 (Bounded morphism) Let $\mathbb{M} = (W, \nu, V)$ and $\mathbb{M}' = (W', \nu', V)$ be monotonic models. A function $f : W \to W'$ is a *bounded morphism* from $\mathbb{M}$ to $\mathbb{M}'$ (notation: $f : \mathbb{M} \to \mathbb{M}'$) if

(BM0) w and $f(w)$ satisfy the same proposition letters.
(BM1) If $X \in \nu(w)$, then $f[X] \in \nu'(f(w))$.
(BM2) If $X' \in \nu'(f(w))$, then there is an $X \subseteq W$ such that $f[X] \subseteq X'$ and $X \in \nu(w)$.

If there is a surjective bounded morphism from $\mathbb{M}$ to $\mathbb{M}'$, then we say that $\mathbb{M}'$ is a *bounded morphic image* of $\mathbb{M}$ and write $\mathbb{M} \twoheadrightarrow \mathbb{M}'$. ⊣

Remark 4.4 In Definition 4.3 the conditions (BM1) and (BM2) taken together are equivalent with the following condition:

(2) $f^{-1}[X'] \in \nu(w)$ iff $X' \in \nu'(f(w))$

For suppose (BM1) and (BM2) hold, then we can show (2) as follows. Assume $f^{-1}[X'] \in \nu(w)$, then by (BM1), $X' \supseteq f[f^{-1}[X']] \in \nu'(f(w))$, hence by monotonicity of ν', $X' \in \nu'(f(w))$. Now assume that $X' \in \nu'(f(w))$, then by (BM2), there is an $X \subseteq W$ such that $f[X] \subseteq X'$ and $X \in \nu(w)$. From $f[X] \subseteq X'$ it follows that $X \subseteq f^{-1}[f[X]] \subseteq f^{-1}[X']$, and again by monotonicity of ν, $f^{-1}[X'] \in \nu(w)$.

To see that (2) implies (BM1) and (BM2), assume that (2) holds. For (BM1), suppose that $X \in \nu(w)$, then by monotonicity of ν, $f^{-1}[f[X]] \in \nu(w)$ and by (2) $f[X] \in \nu'(f(w))$. For (BM2), suppose $X' \in \nu'(f(w))$, then by (2), $f^{-1}[X'] \in \nu(w)$, and since $f[f^{-1}[X']] \subseteq X'$, we may take $f^{-1}[X']$ to be the required X.

Condition (2) ties up with the algebraic notion of bounded morphism and, as we will see in section 8, also with the coalgebraic one. However, the (BM1) and (BM2) conditions are chosen to reflect the fact that bounded morphisms are functional bisimulations, which we will define in the next subsection.

From (2) it follows more or less immediately that truth of modal formulas is invariant under bounded morphisms.

Proposition 4.5 *Let $\mathbb{M} = (W, \nu, V)$ and $\mathbb{M}' = (W', \nu', V')$ be monotonic models. If $f : W \to W'$ is a bounded morphism from $\mathbb{M}$ to $\mathbb{M}'$ then for each formula φ and each $w \in W$:*

$$\mathbb{M}, w \Vdash \varphi \quad \textit{iff} \quad \mathbb{M}', f(w) \Vdash \varphi.$$

Or equivalently, $f^{-1}[V'(\varphi)] = V(\varphi)$.

Proof. The proof is again by induction on φ. As before, the atomic case holds by definition, and the boolean cases are easily shown. The modal case is immediate by (2) and the induction hypothesis. QED

For core-complete models, we will now define the notion of a bounded morphism with respect to the core structure.

Definition 4.6 (Bounded core morphism) Let $\mathbb{M}_0 = (W_0, \nu_0, V_0)$ and $\mathbb{M}_1 = (W_1, \nu_1, V_1)$ be two core-complete, monotonic models. A function $f : W_0 \to W_1$ is a *bounded core morphism* from $\mathbb{M}_0$ to $\mathbb{M}_1$ if

$(\mathrm{BM0})_c$ w and $f(w)$ satisfy the same proposition letters.
$(\mathrm{BM1})_c$ If $X \in \nu_0^c(w)$, then $f[X] \in \nu_1^c(f(w))$.
$(\mathrm{BM2})_c$ If $Y \in \nu_1^c(f(w))$, then there is an $X \subseteq W_0$ such that $f[X] = Y$ and $X \in \nu_0^c(w)$.

⊣

When we talk of bounded core morphisms, we will always assume that we are dealing with core-complete models (or frames). It is easy to show that bounded core morphisms are also bounded morphisms, so truth is also invariant under bounded core morphisms. The other implication does not hold in general, as the following simple counter example shows.

Example 4.7 Consider the two models $\mathbb{M} = \{\{s, t, u, v\}, \nu, V\}$ where $\nu^c(s) = \{\{t, u\}, \{v\}\}$, $\nu(t) = \nu(u) = \nu(v) = \emptyset$, and $\mathbb{M}' = \{\{s', t', u'\}, \nu', V\}$ where $\nu'^c(s') = \{\{u'\}\}$, $\nu'(t') = \nu'(u') = \emptyset$, together with the function $f : s \mapsto s', t \mapsto t', u \mapsto u', v \mapsto u'$. The valuations are not important in this example, and we may assume that $V(p) = V'(p) = \emptyset$ for all $p \in \textsc{prop}$. It is easy to see that f is a bounded morphism from $\mathbb{M}$ to $\mathbb{M}'$, but f is not a bounded core morphism, since $\{t, u\} \in \nu^c(s)$, but $f[\{t, u\}] = \{t', u'\} \notin \nu'^c(s')$.

It is also not the case that $(\mathrm{BM1})_c$ and $(\mathrm{BM2})_c$ are equivalent with:

(3) $\quad f^{-1}[X] \in \nu_0^c(w)$ iff $X \in \nu_1^c(f(w))$.

Example 4.8 Consider the two models $\mathbb{M}_i = \{\{s_i, t_i, u_i\}, \nu_i, V_i\}$, for $i \in \{1, 2\}$, where $\nu_i^c(s_i) = \{\{t_i\}\}$, $\nu_i(t_i) = \nu_i(u_i) = \emptyset$. , together with the function $f : s_1 \mapsto s_2, t_1 \mapsto t_2, u_1 \mapsto t_2$. Then f is a bounded core morphism, but (3) does not hold, since $\{t_2\} \in \nu_2^c(s_2)$, but $f^{-1}[\{t_2\}] = \{t_1, u_1\} \notin \nu_1^c(s_1)$.

The equivalence between $(\mathrm{BM1})_c$ and $(\mathrm{BM2})_c$ together and (3) only seems to hold when f is a bijection. However, we do have the following result.

Proposition 4.9 *Let $\mathbb{M}_0 = (W_0, \nu_0, V_0)$ and $\mathbb{M}_1 = (W_1, \nu_1, V_1)$ be core-complete, monotonic models and $f : W_0 \to W_1$ a function. Then f is a bounded core morphism from $\mathbb{M}_0$ to $\mathbb{M}_1$ if f is an injective bounded morphism from $\mathbb{M}_0$ to $\mathbb{M}_1$.*

Proof. Assume that $f : \mathbb{M}_0 \to \mathbb{M}_1$ is an injective bounded morphism. We must now show that $(\mathrm{BM0\text{-}2})_c$ hold. $(\mathrm{BM0})_c$ is clear. For $(\mathrm{BM1})_c$, suppose $X \in \nu_0^c(w)$, then by (BM1) for f, we have $f[X] \in \nu_1(f(w))$, but we need to show that $f[X] \in \nu_1^c(f(w))$. Suppose for contradiction that there is an $X_1 \subsetneq f[X]$ such that $X_1 \in \nu_1(f(w))$. Then from (BM2) for f, there is an $X_0 \in \nu_0(w)$ and $f[X_0] \subseteq X_1 \subsetneq f[X]$. Applying the injectivity of f, we obtain $X_0 \subsetneq X$ from $f[X_0] \subsetneq f[X]$. But then $X_0 \in \nu_0(w)$ and $X_0 \subsetneq X$ which is a contradiction with $X \in \nu_0^c(w)$. To show that $(\mathrm{BM2})_c$ holds for f, assume that $Y \in \nu_1^c(f(w))$, then by (BM2) for f there is an $X \in \nu_0(w)$ such that $f[X] \subseteq Y$. So there is also an $X_0 \in \nu_0^c(w)$ such that $X_0 \subseteq X$, and hence $f[X_0] \subseteq f[X] \subseteq Y$. By (BM1) for f, we also have $f[X_0] \in \nu_1(f(w))$, but then $f[X_0] = Y$ since $f[X_0] \subseteq Y$ and $Y \in \nu_1^c(f(w))$. QED

Bisimulations

In modal logic, the central notion of model equivalence is that of bisimulation. Bisimulations for Kripke models were introduced in van Benthem [4], where one also finds the well-known characterisation result which states that modal logic is the bisimulation invariant fragment of first-order logic.

Bisimulation, being a much weaker notion of equivalence than isomorphism, may be seen as a measure of the expressivity of modal languages: If two states are bisimilar then they should not be distinguishable by a modal formula. Classes of models for which the converse implication holds are called Hennessy-Milner classes, and we will return to these in subsection 4.3.

For monotonic models, bisimulations have been presented by Pauly [55]. In [56, 57] Pauly generalises the definition to dynamic effectivity models (of which coalition models and models for Game Logic [53] are a special case), and analyses the expressivity of the language of Game Logic. Also in Pauly [55], results on the relationship between Kripke and monotonic bisimulations can be found, together with a version of the van Benthem characterisation theorem for monotonic modal logic. We will treat this result in subsection 5.3.

Furthermore, in coalgebra, various notions of equivalence exist, and in section 8, we will relate these with the model theoretic bisimulations of this section. But for the time being we will only concern ourselves with the invariance of truth under bisimulations.

Definition 4.10 (Bisimulation) Let $\mathbb{M} = (W, \nu, V)$ and $\mathbb{M}' = (W', \nu', V')$ be monotonic models. A non-empty binary relation $Z \subseteq W \times W'$ is a *bisimulation between* $\mathbb{M}$ *and* $\mathbb{M}'$ (notation: $Z : \mathbb{M} \leftrightarrows \mathbb{M}'$) if

(prop) If wZw' then w and w' satisfy the same proposition letters.
(forth) If wZw' and $X \in \nu(w)$, then there is an $X' \subseteq W'$ such that $X' \in \nu'(w')$ and $\forall x' \in X' \; \exists x \in X : xZx'$.
(back) If wZw' and $X' \in \nu'(w')$, then there is an $X \subseteq W$ such that $X \in \nu(w)$ and $\forall x \in X \; \exists x' \in X' : xZx'$.

If $w \in \mathbb{M}$ and $w' \in \mathbb{M}'$, then we will say that w and w' are *bisimilar states* (notation: $\mathbb{M}, w \leftrightarrows \mathbb{M}', w'$) if there is a bisimulation $Z : \mathbb{M} \leftrightarrows \mathbb{M}'$ such that wZw'.

If $dom(Z) = W$ and $ran(Z) = W'$, then we will call Z a *full bisimulation.* $\dashv$

Comparing Definitions 4.3 and 4.10, we see that bounded morphisms are the same as functional bisimulations (where $dom(Z) = W$). For example, for a bounded morphism f, the (forth) condition is satisfied by taking $X' = f[X]$, and the (back) condition by taking X equal to the set obtained by the (BM2) condition for f.

The following proposition states that truth of modal formulas is invariant under bisimulations.

Proposition 4.11 *Let* $\mathbb{M} = (W, \nu, V)$ *and* $\mathbb{M}' = (W', \nu', V)$ *be monotonic models. If* $Z \subseteq W \times W'$ *is a bisimulation between* $\mathbb{M}$ *and* $\mathbb{M}'$ *then for each formula* φ *and* $w \in W, w' \in W'$ *such that* wZw' *we have*

$$\mathbb{M}, w \Vdash \varphi \;\; \textit{iff} \;\; \mathbb{M}', w' \Vdash \varphi.$$

Proof. As usual, the modal case is the only nontrivial part of the proof: So assume that wZw'. For the direction from left to right, we have $\mathbb{M}, w \Vdash \nabla\varphi$ iff $V(\varphi) \in \nu(w)$, hence by the (forth) condition for Z there is an $X' \subseteq W'$ such that $X' \in \nu'(w')$ and for all $x' \in X'$ there is an $x \in V(\varphi)$ such that xZx'. By the induction hypothesis, it follows that $X' \subseteq V'(\varphi)$ and by upwards closure of $\nu'(w')$, $V'(\varphi) \in \nu'(w')$, hence $\mathbb{M}', w' \Vdash \nabla\varphi$. The other direction follows similarly from the (back) condition for Z. QED

When working with core-complete models, it will often be more convenient to show that the underlying core structure of two models are bisimilar. We therefore introduce the notion of core bisimulations.

Definition 4.12 (Core Bisimulation) Let $\mathbb{M} = (W, \nu, V)$ and $\mathbb{M}' = (W', \nu', V')$ be core-complete, monotonic models. A non-empty binary relation $Z \subseteq W \times W'$ is a *core bisimulation between* $\mathbb{M}$ *and* $\mathbb{M}'$ (notation: $Z : \mathbb{M} \leftrightarroweq_c \mathbb{M}'$) if

(prop) If wZw' then w and w' satisfy the same proposition letters.
(forth)$_c$ If wZw' and $X \in \nu^c(w)$, then $\exists X' \subseteq W'$ such that $X' \in \nu'^c(w')$, and $\forall x' \in X'\ \exists x \in X : xZx'$.
(back)$_c$ If wZw' and $X' \in \nu'^c(w')$, then $\exists X \subseteq W$ such that $X \in \nu^c(w)$, and $\forall x \in X\ \exists x' \in X' : xZx'$.

$\dashv$

Just as for bounded morphisms, we will always assume that we are working with core-complete models when talking about core bisimulations. Unlike the case for bounded morphisms, there is no essential difference when considering bisimulations of the core and bisimulations of the entire neighbourhood structure. This also complies with the idea that the core neighbourhoods really characterise the structure of core-complete models.

Proposition 4.13 *Let* $\mathbb{M} = (W, \nu, V)$ *and* $\mathbb{M}' = (W', \nu', V')$ *be core-complete, monotonic models, and let* $Z \subseteq W \times W'$ *be a non-empty binary relation. Then* Z *is a core bisimulation between* $\mathbb{M}$ *and* $\mathbb{M}'$ *iff* Z *is a bisimulation between* $\mathbb{M}$ *and* $\mathbb{M}'$*.*

Proof. Assume first that Z is a core bisimulation between $\mathbb{M}$ and $\mathbb{M}'$. We will show that Z satisfies (forth). So suppose wZw' and $X \in \nu(w)$. Then there is an $X^c \in \nu^c(w)$ such that $X^c \subseteq X$. From (forth)$_c$ there is an $X'^c \in \nu'^c(w')$ such that $\forall x' \in X'^c\ \exists x \in X^c : xZx'$, and since $X^c \subseteq X$, also $\forall x \in X'^c\ \exists x \in X : xZx'$. I.e., X'^c satisfies the (forth) condition, since $X'^c \in \nu'^c(w')$ implies $X'^c \in \nu'(w')$. The (back) condition is shown analogously.

Now assume that Z is a bisimulation between $\mathbb{M}$ and $\mathbb{M}'$. We only show that (forth)$_c$ holds, since (back)$_c$ can be shown in a similar way. So suppose wZw' and $X \in \nu^c(w)$. It follows that $X \in \nu(w)$, and by (forth) there is an $X' \in \nu'(w')$ such that $\forall x' \in X'\ \exists x \in X : xZx'$. From $X' \in \nu'(w')$, we obtain an $X'^c \in \nu^c(w')$ for which $X'^c \subseteq X'$, and hence also $\forall x' \in X'^c\ \exists x \in X : xZx'$. QED

One might expect that functional core bisimulations and bounded core morphisms are also the same, but this is not so. It is easy to show that a bounded core morphism is also a core bisimulation, but a functional core bisimulation need not be a bounded core morphism. The above proposition also tells us so, since we know that bounded core morphisms really are a strict subset of the bounded morphisms.

Generated submodels

So far the model operations we have seen, have been fairly straightforward analogues of their Kripke counterparts. In this subsection, we will see that only core-complete models turn out to have a characterisation of generated submodels in terms of a heredity condition for core neighbourhoods. We start by defining submodels.

Definition 4.14 (Submodel) Let $\mathbb{M} = (W, \nu, V)$ be a monotonic model. Then $\mathbb{M}' = (W', \nu', V')$ is a *submodel of* $\mathbb{M}$ if $W' \subseteq W$, $V'(p) = V(p) \cap W'$ for all $p \in \textsc{prop}$, and $\nu' = \nu \cap (W' \times \mathcal{P}(W'))$. That is,

$$\forall s \in W' : \nu'(s) = \{X \subseteq W' \mid X \in \nu(s)\}. \qquad \dashv$$

Given a monotonic model $\mathbb{M} = (W, \nu, V)$ and $W' \subseteq W$, we can construct the submodel $\mathbb{M}' = (W', \nu', V')$ by taking V' and ν' as in Definition 4.14. In this situation, we will use the notation $\mathbb{M}{\upharpoonright_{W'}} = (W', \nu{\upharpoonright_{W'}}, V{\upharpoonright_{W'}})$ for $\mathbb{M}'$. Just as for Kripke models, submodels do not necessarily preserve the truth of modal formulas since a state in the submodel may have lost neighbourhoods when restricting ν to the submodel's universe.

In Kripke semantics, generated submodels provide the desired invariance result, and a similar notion can be defined for monotonic models in the obvious way by demanding that a submodel $\mathbb{M}' = (W', \nu', V')$ of $\mathbb{M} = (W, \nu, V)$ satisfies the following heredity condition: If $w' \in W'$ and $X' \in \nu(w')$ then $X' \subseteq W'$. However, due to the upwards closure of ν, the only generated submodels from some point $w \in W$ are the models $(\{w\}, \emptyset, V{\upharpoonright_{\{w\}}})$ and $\mathbb{M}$ itself. The first case being the result when $\nu(w) = \emptyset$, and the latter resulting when $\nu(w) \neq \emptyset$, since then $W \in \nu(w)$. Truth invariance is still obtained, but clearly in a rather trivial manner. The problem is caused by the neighbourhoods generated by the upwards closure of ν. Instead we will define generated submodels of monotonic models by requiring that the identity/inclusion map is a bounded morphism. With this definition, truth invariance is immediate.

Definition 4.15 (Generated Submodel) Let $\mathbb{M} = (W, \nu, V)$ be a monotonic model and $\mathbb{M}' = (W', \nu', V')$ a submodel of $\mathbb{M}$. Then $\mathbb{M}'$ is a *generated submodel of* $\mathbb{M}$ (notation: $\mathbb{M}' \rightarrowtail \mathbb{M}$) if the identity map $i : W' \to W$ is a bounded morphism. That is, for all $w' \in W'$ and all $X \subseteq W$,

$$i^{-1}[X] = X \cap W' \in \nu'(w') \text{ iff } X \in \nu(w').$$

Given a monotonic model $\mathbb{M} = (W, \nu, V)$ and a subset $X \subseteq W$, we define the *submodel generated by X in* $\mathbb{M}$ as the submodel $\mathbb{M}{\upharpoonright_{W'}}$ where W' is the intersection of all sets Y such that $X \subseteq Y$ and $\mathbb{M}{\upharpoonright_Y}$ is a generated submodel of $\mathbb{M}$. $\dashv$

Truth invariance now follows directly from Proposition 4.5.

Proposition 4.16 *Let $\mathbb{M}_0 = (W_0, \nu_0, V_0)$ be a generated submodel of $\mathbb{M} = (W, \nu, V)$. Then for all modal formulas φ and all $w_0 \in W_0$:*

$$\mathbb{M}_0, w_0 \Vdash \varphi \textit{ iff } \mathbb{M}, w_0 \Vdash \varphi.$$

The following Proposition should also be clear.

Proposition 4.17 *Let $\mathbb{M}_0 = (W_0, \nu_0, V_0)$ and $\mathbb{M}_1 = (W_1, \nu_1, V_1)$ be monotonic models. Then the following holds: If $f : \mathbb{M}_0 \to \mathbb{M}_1$ is an injective bounded morphism, then $\mathbb{M}_1 \restriction_{f[W_0]}$ is a generated submodel of $\mathbb{M}_1$.*

As mentioned above, for core-complete models we have an alternative characterisation of generated submodels, which resembles the definition of generated Kripke submodels.

Lemma 4.18 *Let $\mathbb{M} = (W, \nu, V)$ be a core-complete, monotonic model and $\mathbb{M}' = (W', \nu', V')$ a submodel of $\mathbb{M}$. Then $\mathbb{M}'$ is a generated submodel of $\mathbb{M}$ if and only if the following closure condition holds:*

(4) *If $w' \in W'$ and $X \in \nu^c(w')$, then $X \subseteq W'$.*

Proof. "$\Rightarrow$": Assume that $\mathbb{M}'$ is a generated submodel of $\mathbb{M}$, $w' \in W'$ and $X \in \nu^c(w')$. We must show that $X \subseteq W'$. But this follows almost immediately, since by the definition of generated submodel, $X \in \nu^c(w')$ implies $X \cap W' \in \nu'(w')$, hence by Definition 4.14, $X \cap W' \in \nu(w')$. From $X \cap W' \subseteq X \in \nu^c(w')$ we may then conclude that $X = X \cap W'$, thus $X \subseteq W'$.
"$\Leftarrow$": Assume that (4) holds, and let $w' \in W'$, $X \subseteq W$. Suppose first that $X \cap W' \in \nu'(w')$, then by Definition 4.14, $X \cap W' \in \nu(w')$ and by upwards closure, $X \in \nu(w')$. Suppose now that $X \in \nu(w')$, then as $\mathbb{M}$ is core-complete, there is a $C \subseteq X$ such that $C \in \nu^c(w')$, so by (4), $C \subseteq W'$, hence again by the definition of submodel and $C \in \nu(w')$, it follows that $C \in \nu'(w')$, and finally from $C \subseteq X \cap W'$ we conclude that $X \cap W' \in \nu'(w')$. QED

Thus for a core-complete, monotonic model $\mathbb{M} = (W, \nu, V)$ and $X \subseteq W$, the submodel generated by X in $\mathbb{M}$ may also be seen as the submodel obtained by restricting $\mathbb{M}$ to the subset $S_\omega(X)$, which we define now.

Definition 4.19 Let $\mathbb{M} = (W, \nu, V)$ be a core-complete, monotonic model. For $X \subseteq W$, we define $\nu^c_\omega(X)$ and $S_\omega(X)$ inductively by

$$\begin{array}{lcll} S_0(X) & = & X & , \quad \nu^c_0(X) = \bigcup_{x \in X} \nu^c(x) \\ S_{n+1}(X) & = & \bigcup_{Y \in \nu^c_n(X)} Y & , \quad \nu^c_{n+1}(X) = \bigcup_{x \in S_{n+1}(X)} \nu^c(x) \\ S_\omega(X) & = & \bigcup_{n \in \omega} S_n(X) & , \quad \nu^c_\omega(X) = \bigcup_{x \in S_\omega(X)} \nu^c(x) \end{array}$$

If $S_\omega(\{w\}) = W$, then we call $\mathbb{M}$ a *rooted* or *point-generated* model with root w. $\dashv$

Intuitively, at each stage n, $\nu^c_n(X)$ are the core neighbourhoods of the states found at stage n, and $S_n(X)$ is the set of states which are contained in a core neighbourhood of some state found at the previous stage. Thus $S_\omega(X)$ is the closure of X under the condition in (4), and $\nu^c_\omega(X)$ are all the core neighbourhoods which we will encounter by tracing through the entire model starting from a state in X. Due to this characterisation, we may think of generated submodels of core-complete models as being generated by the core neighbourhoods, and we shall use the term core generated submodel. For core generated submodels, the inclusion map is a bounded core morphism by Proposition 4.9.

Remark 4.20 The invariance result for disjoint unions may be seen as a special case of Proposition 4.16, since each of the components of the disjoint union is a generated submodel. Also, just as for Kripke models, we may assume that a satisfiable formula is satisfied on a generated submodel.

Unravelling monotonic models

Another well-known property of Kripke frames is that they can be *unravelled* into a tree-like structure that is bisimilar to the original model, which in turn shows that normal modal logic has the tree model property: If φ is satisfiable in some model, then φ is satisfiable at the root of a tree-like model.

We can do something similar for monotonic models, the only problem being that a tree-like neighbourhood model can never be a truly monotonic model. However, we can unravel a monotonic model into a model whose underlying core structure is tree-like in the following sense. First recall the definition of $S_n(\{w\})$ and $S_\omega(\{w\})$ in 4.19.

Definition 4.21 (Tree) Let $\mathbb{M} = (W, \nu, V)$ be a core-complete, monotonic model, and $root \in W$. Then $\mathbb{M}_{root}$ is a *tree-like monotonic model* if the following hold:

(i) $W = S_\omega(\{root\})$,

(ii) For all $w \in W$: $w \notin \bigcup_{n>0} S_n(\{w\})$,

(iii) For all w_0, w_1, v in W and for all $X_0, X_1 \subseteq W$:
If $v \in X_0 \in \nu^c(w_0)$ & $v \in X_1 \in \nu^c(w_1)$ then $X_0 = X_1$ & $w_0 = w_1$. ⊣

That is, we consider a monotonic model to be tree-like if it is core-complete, all states are reachable from the root through a sequence of core neighbourhoods, and when tracing through the core neighbourhoods starting in w, we will not encounter a core neighbourhood which contains w (no cycles), and furthermore, all core neighbourhoods must be unique and disjoint (total branching). We now wish to define the unravelling of a monotonic model $\mathbb{M}$ from a state w_0 in $\mathbb{M}$. The result should be a rooted tree-like monotonic model.

Let $\mathbb{M} = (W, \nu, V)$ be a monotonic model and $w_0 \in W$. The *unravelling of* $\mathbb{M}$ *from* w_0 is defined as the model $\vec{M}_{w_0} = (\vec{W}_{w_0}, \vec{\nu}_{w_0}, \vec{V}_{w_0})$, where $\vec{W}_{w_0}$, $\vec{\nu}_{w_0}$ and $\vec{V}_{w_0}$ are as follows. Let

(5) $$\vec{W}_{w_0} = \{(w_0 X_1 w_1 X_2 w_2 \dots X_n w_n) \mid n \geq 0 \;\&\; \forall i \in \{1, \dots, n\} : X_i \in \nu(w_{i-1}) \;\&\; w_i \in X_i\}.$$

That is, $\vec{W}_{w_0}$ consists of the sequences of states and neighbourhoods obtained by tracing through all non-empty neighbourhoods starting in w_0. Note that for each neighbourhood X in which an element w occurs, there will be a sequence $(w_0 \dots Xw)$.

For $(w_0 X_1 w_1 \dots X_n w_n) \in \vec{W}_{w_0}$, the maps *pre* and *last* are defined by

$$\begin{aligned} pre: &\quad (w_0 X_1 w_1 \dots X_n w_n) \mapsto (w_0 X_1 w_1 \dots X_{n-1} w_{n-1} X_n) \\ last: &\quad (w_0 X_1 w_1 \dots X_n w_n) \mapsto w_n. \end{aligned}$$

In particular, $pre : (w_0) \mapsto \epsilon$ where ϵ is the empty sequence, and $last : (w_0) \mapsto w_0$.

We now define a neighbourhood function $\mu : \vec{W}_{w_0} \to \mathcal{P}(\mathcal{P}(\vec{W}_{w_0}))$ as follows. Let $\overline{s} \in \vec{W}_{w_0}$ and $Y \subseteq \vec{W}_{w_0}$,

$$Y \in \mu(\overline{s}) \quad \text{iff} \quad \forall \overline{y} \in Y (pre(\overline{y}) = \overline{s}X) \;\&\; last[Y] = X \in \nu(last(\overline{s})), \quad \text{for some } X \in \mathcal{P}(W).$$

Thus every neighbourhood $X \in \nu(last(\overline{s}))$ gives rise to exactly one neighbourhood Y in $\mu(\overline{s})$, and all these Y are disjoint. Furthermore, $\emptyset \in \mu(\overline{s})$ if and only if $\emptyset \in \nu(last(\overline{s}))$.

Now, we simply define $\vec{\nu}_{w_0}$ to be the monotone neighbourhood function obtained by closing μ under supersets,

(6) $\quad Y \in \vec{\nu}_{w_0}(\overline{s}) \quad \text{iff} \quad \exists Y' \in \mu(\overline{s}) : Y' \subseteq Y.$

Note that $\mu = \vec{\nu}^{\,c}_{w_0}$. Finally, we define the valuation $\vec{V}_{w_0}$ by,

(7) $\quad \overline{s} \in \vec{V}_{w_0}(p) \quad \text{iff} \quad last(\overline{s}) \in V(p).$

It should now be clear that $\vec{M}_{w_0}$ is a tree-like monotonic model, and the map $last : \vec{M}_{w_0} \to \mathbb{M}$ is a surjective bounded morphism. Hence we have for all modal formulas φ:

$$\mathbb{M}, w_0 \Vdash \varphi \quad \text{iff} \quad \vec{M}_{w_0}, (w_0) \Vdash \varphi.$$

Proposition 4.22 (Tree model property) *Let φ be a modal formula. If φ is satisfiable in some monotonic model, then φ is satisfiable at the root of some tree-like motonone model.*

Proof. Given a monotonic model $\mathbb{M}$ and a point w_0 in $\mathbb{M}$, such that $\mathbb{M}, w_0 \Vdash \varphi$, $\mathbb{M}$ can be unravelled from w_0 to produce the rooted tree-like monotonic model $\vec{M}_{w_0} = (\vec{W}_{w_0}, \vec{\nu}_{w_0}, \vec{V}_{w_0})$, where $\vec{W}_{w_0}, \vec{\nu}_{w_0}$ and $\vec{V}_{w_0}$ are defined as in (5), (6) and (7) such that for all modal formulas ψ: $\mathbb{M}, w_0 \Vdash \psi$ iff $\vec{M}_{w_0}, (w_0) \Vdash \psi$. Hence φ is satisfied at the root (w_0) in $\vec{M}_{w_0}$. QED

Remark 4.23 The unravelling of a monotonic model $\mathbb{M} = (W, \nu, V)$ will generally be a much 'richer' structure than $\mathbb{M}$ in the sense that states in the unravelling will have a lot more neighbourhoods, since every neighbourhood in $\mathbb{M}$ produces a core neighbourhood in the unravelling. In particular, if the root w_0 has at least one neighbourhood, then the unravelling will have an infinite number of states because $w_0 \in W \in \nu(w_0)$. If $\mathbb{M}$ is core-complete then the above construction can be restricted to the core structure of $\mathbb{M}$. More precisely, we can replace ν with ν^c in the definition of $\vec{W}_{w_0}$ and μ, and still obtain a tree-like model which is bisimilar to $\mathbb{M}$. In fact, by doing so, the map *last* will be a bounded core morphism. Moreover, this kind of 'core-unravelling' will, in general, be more succinct than the unravelling of the entire neighbourhod structure. For example, consider the two kinds of unravellings of the model $(\{s, t\}, \nu, V)$ where $\nu(s) = \{\{t\}\}$ and $\nu(t) = \emptyset$.

4.2 Filtrations

Filtrations are a tool for obtaining finite models from infinite ones, but just as core generated submodels, they may also be seen as a means to reduce a model to what is essential when evaluating truth of a modal formula φ. The inductive definition of truth implies that we only need to know the truth of the subformulas of φ in order to say whether φ is itself true at some state. Hence, if a set of states in the model make exactly the same subformulas of φ true, then we may as well identify them. This gives rise to an equivalence relation, and a filtration of a model is a model over this equivalence relation, and thus a quotient of the original model. This is the idea behind filtrations of Kripke models as well as monotonic models. Many of the results in this section can be found in Chellas [14], although most of the proofs in [14] are left as exercises.

Let $\mathbb{M} = (W, \nu, V)$ be a monotonic model and Σ a subformula closed set of modal formulas. Then $\equiv_\Sigma$ is the equivalence relation induced by Σ on W which is defined as follows for all $w, v \in W$:

$$w \equiv_\Sigma v \text{ if and only if } \text{ for all } \varphi \in \Sigma \ (\mathbb{M}, w \Vdash \varphi \text{ iff } \mathbb{M}, v \Vdash \varphi).$$

Let $W_\Sigma = \{ |w| \mid w \in W\}$ be the set of equivalence classes induced by Σ on W. For $X \subseteq W$, denote by $|X|$ the set $\{|w| \mid w \in X\}$, and for $Y \subseteq W_\Sigma$ let $\dagger Y \dagger$ be the set $\{w \in W \mid |w| \in Y\}$. Then we have the following equivalences and identities which will be used without further reference:

$$\begin{array}{ll}
|V(\varphi)| \subseteq |V(\psi)| \Leftrightarrow V(\varphi) \subseteq V(\psi) & \text{for } \varphi, \psi \in \Sigma, \\
\dagger|V(\varphi)|\dagger = V(\varphi) & \text{for } \varphi \in \Sigma, \\
|\dagger X \dagger| = X & \text{for } X \subseteq W_\Sigma, \\
\dagger X \cap Y \dagger = \dagger X \dagger \cap \dagger Y \dagger & \text{for } X, Y \subseteq W_\Sigma, \\
W_\Sigma \setminus |V(\varphi)| = |V(\neg\varphi)| & \text{for } \varphi \in \Sigma, \\
W \setminus \dagger X \dagger = \dagger W_\Sigma \setminus X \dagger & \text{for } X \subseteq W_\Sigma.
\end{array}$$

Definition 4.24 (Filtration) Let $\mathbb{M} = (W, \nu, V)$ be a monotonic model and Σ a subformula closed set of formulas. A monotonic model $\mathbb{M}^f = (W^f, \nu^f, V^f)$ is a *filtration of* $\mathbb{M}$ *through* Σ if

(i) $W^f = W_\Sigma$.
(ii) For all $\nabla\varphi \in \Sigma$: if $V(\varphi) \in \nu(w)$ then $|V(\varphi)| \in \nu^f(|w|)$.
(iii) For all $\nabla\varphi \in \Sigma$: if $X \in \nu^f(|w|)$ and $\dagger X \dagger \subseteq V(\varphi)$ then $V(\varphi) \in \nu(w)$.
(iv) For all $p \in \textsc{prop}$: $V^f(p) = |V(p)|$.

⊣

Remark 4.25 In Definition 4.24(iii), the condition $\dagger X \dagger \subseteq V(\varphi)$ is equivalent with $X \subseteq |V(\varphi)|$, which may be the expected formulation. However, since most of the subsequent proofs rely on the properties of ν in the original model, it is more convenient to use the formulation in 4.24(iii).

Furthermore, taken together the conditions (ii) and (iii) of Definition 4.24 ensure that

(8) For all $\nabla\varphi \in \Sigma : V(\varphi) \in \nu(w)$ iff $|V(\varphi)| \in \nu^f(|w|)$

This means that for the neighbourhoods definable by modal formulas of the form $\nabla\varphi$, the natural map $|\cdot|$ is a bounded morphism, hence the truth of $\nabla\varphi$-formulas is guaranteed to be invariant under $|\cdot|$. Chellas [14], in fact, defines filtrations in terms of (8).

Theorem 4.26 (Filtration Theorem) *Let* $\mathbb{M} = (W, \nu, V)$ *be a monotonic model and* $\mathbb{M}^f = (W^f, \nu^f, V^f)$ *a filtration of* $\mathbb{M}$ *through* Σ. *Then for all formulas* $\varphi \in \Sigma$ *and all* $w \in W$ *we have:*

$$\mathbb{M}, w \Vdash \varphi \ \textit{ iff } \ \mathbb{M}^f, |w| \Vdash \varphi.$$

In other words, for all $\varphi \in \Sigma : V^f(\varphi) = |V(\varphi)|$.

Proof. The proof is by induction on the complexity of φ. The atomic case holds by condition (iv) of Definition 4.24. The boolean cases are straightforward, since we may apply the induction hypothesis using that Σ is subformula closed, and as remarked above, the modal case follows from (8):

$$\begin{aligned} \mathbb{M}, w \vDash \nabla\varphi &\iff V(\varphi) \in \nu(w) &&\stackrel{(8)}{\iff} |V(\varphi)| \in \nu^f(|w|) \\ &\stackrel{I.H.}{\iff} V^f(\varphi) \in \nu^f(|w|) &&\iff \mathbb{M}^f, |w| \Vdash \nabla\varphi. \end{aligned}$$

QED

We have seen that conditions (ii) and (iii) of Definition 4.24 are designed to make the modal induction step go through in the above proof, but they also provide us with concrete examples of filtrations. Condition (ii) tells us which neighbourhoods we *must* add, thus only adding these required neighbourhoods gives rise to the *smallest filtration* in which the neighbourhood function ν^s is given by

(9) $X \in \nu^s(|w|)$ iff there is a $\nabla\varphi \in \Sigma$ such that $V(\varphi) \subseteq \wr X \wr$ and $V(\varphi) \in \nu(w)$.

To see that (9) is well-defined, we should check that for all $\nabla\varphi \in \Sigma$, if $w \equiv_\Sigma w'$ then $V(\varphi) \in \nu(w)$ iff $V(\varphi) \in \nu(w')$, but this is clear since w and w' satisfy the same formulas in Σ, thus $V(\varphi) \in \nu(w)$ iff $\mathbb{M}, w \Vdash \nabla\varphi$ iff $\mathbb{M}, w' \Vdash \nabla\varphi$ iff $V(\varphi) \in \nu(w')$. It is also easy to see that ν^s is indeed upwards closed, since $X \subseteq Y$ implies $\wr X \wr \subseteq \wr Y \wr$, so if $V(\varphi) \subseteq \wr X \wr$ then also $V(\varphi) \subseteq \wr Y \wr$. The minimality of ν^s may be summarised in

(10) $X \in \nu^s(|w|)$ implies $\wr X \wr \in \nu(w)$, for all $X \subseteq W_\Sigma$,

which is immediate from (9).

Condition (iii) tells us which neighbourhoods we are *allowed* to add. Hence by adding all the allowed neighbourhoods we obtain the *largest filtration* where the neighbourhood function ν^l is given by

(11) $X \in \nu^l(|w|)$ iff for all $\nabla\varphi \in \Sigma$: If $\wr X \wr \subseteq V(\varphi)$ then $V(\varphi) \in \nu(w)$.

Again, ν^l is well-defined due to the properties of equivalent states, and it is also clear that ν^l is upwards closed, since $X \subseteq Y$ implies $\wr X \wr \subseteq \wr Y \wr$, hence if for all $\nabla\varphi$ in Σ, $\wr X \wr \subseteq V(\varphi)$ implies that $V(\varphi) \in \nu(w)$, then it is certainly also the case that $\wr Y \wr \subseteq V(\varphi)$ implies that $V(\varphi) \in \nu(w)$. The maximality of ν^l is expressed by

(12) $X \notin \nu^l(|w|)$ implies $\wr X \wr \notin \nu(w)$, for all $X \subseteq W_\Sigma$,

which follows directly from (11).

Lemma 4.27 *Let $\mathbb{M} = (W, \nu, V)$ be a monotonic model and Σ a subformula closed set of formulas. Then (W_Σ, ν^s, V^f) and (W_Σ, ν^l, V^f), where $V^f(p) = |V(p)|$ for all $p \in$ PROP, are both filtrations of $\mathbb{M}$ through Σ.*

Proof. We have already seen that ν^s and ν^l are both upwards closed. So for ν^s we only need to check that condition (iii) holds, since we have taken condition (ii) as the definition of ν^s. So assume that $X \in \nu^s(|w|)$. By (9), there is a $\nabla\varphi \in \Sigma$ such that $V(\varphi) \subseteq \wr X \wr$ and

$V(\varphi) \in \nu(w)$. Now take any $\nabla\psi \in \Sigma$ such that $\{\!\!|X|\!\!\} \subseteq V(\psi)$. Then $V(\varphi) \subseteq V(\psi)$, and by upwards closure of ν, we have $V(\psi) \in \nu(w)$.

For ν^l we only need to check condition (ii), and we will do that by contraposition: Let $\nabla\psi \in \Sigma$ be arbitrary and suppose that $|V(\psi)| \notin \nu^l(|w|)$. By (11), there must be a $\nabla\varphi \in \Sigma$ such that $\{\!\!|\,|V(\psi)|\,|\!\!\} \subseteq V(\varphi)$ and $V(\varphi) \notin \nu(w)$. From the upwards closure of ν and $V(\varphi) \notin \nu(w)$ it cannot be the case that $\{\!\!|\,|V(\psi)|\,|\!\!\} = V(\psi) \in \nu(w)$. QED

When we filtrate models, we are interested in preserving as many properties of the model as possible in the filtration. In particular, when we wish to show that a monotonic modal logic Λ has the finite model property, we often know that Λ is complete with respect to some class of models K. It is therefore interesting to know which classes of models *admit* filtrations.

Definition 4.28 A class K of models *admits filtrations*, if for all $\mathbb{M} \in \mathsf{K}$ and all subformula closed sets of formulas Σ, there is a filtration $\mathbb{M}^f$ of $\mathbb{M}$ through Σ such that $\mathbb{M}^f \in \mathsf{K}$. $\dashv$

Proposition 4.29 *The following model classes admit filtrations:*

- *(M) The class of all monotonic models.*
- *(N) The class of monotonic models satisfying*
  - *(n)* $\forall w \in W : W \in \nu(w)$.
- *(P) The class of monotonic models satisfying*
  - *(p)* $\forall w \in W : \emptyset \notin \nu(w)$.
- *(D) The class of monotonic models satisfying*
  - *(d)* $\forall w \in W : X \in \nu(w) \to W \setminus X \notin \nu(w)$.
- *(C) The class of monotonic models satisfying*
  - *(c)* $\forall w \in W\ \forall X_1, X_2 \subseteq W : (X_1 \in \nu(w)\ \&\ X_2 \in \nu(w)) \to X_1 \cap X_2 \in \nu(w)$.
- *(T) The class of monotonic models satisfying*
  - *(t)* $\forall w \in W\ \forall X \subseteq W : X \in \nu(w) \to w \in X$.
- *(4') The class of monotonic models satisfying*
  - *(iv')* $\forall w \in W\ \forall X \subseteq W : X \in \nu(w) \to m_\nu(X) \in \nu(w)$.
- *(5) The class of monotonic models satisfying*
  - *(v)* $\forall w \in W\ \forall X \subseteq W : X \notin \nu(w) \to W \setminus m_\nu(X) \in \nu(w)$.

Proof. Throughout the proof, $\mathbb{M} = (W, \nu, V)$ is a monotonic model, Σ is a subformula closed set of formulas, $\mathbb{M}^s = (W_\Sigma, \nu^s, V^f)$ is the smallest filtration of $\mathbb{M}$ and $\mathbb{M}^l = (W_\Sigma, \nu^l, V^f)$ is the largest filtration of $\mathbb{M}$. Recall from (10) and (12) that the following hold for all $w \in W$ and all $Y \subseteq W_\Sigma$:

$$Y \in \nu^s(|w|) \Rightarrow \{\!\!|Y|\!\!\} \in \nu(w)$$

$$Y \notin \nu^l(|w|) \Rightarrow \{\!\!|Y|\!\!\} \notin \nu(w)$$

These implications, also in their contraposed versions will be used without warning in the proofs below. We will also use that for all $\nabla\varphi \in \Sigma$ and all $w \in W$: $|\,m_\nu(V(\varphi))| = m_{\nu^f}(|V(\varphi)|)$, which is easily shown from the Filtration Theorem.

Proof of (M): Follows from Lemma 4.27.

Proof of (N): When $\mathbb{M}$ satisfies (n) then $\mathbb{M}^l$ also satisfies (n), since $W_\Sigma \notin \nu^l(|w|)$ implies $W \notin \nu(w)$.

Proof of (P): When $\mathbb{M}$ satisfies (p) then $\mathbb{M}^s$ also satisfies (p), since $\emptyset \in \nu^s(|w|)$ implies $\emptyset \in \nu(w)$.

Proof of (D): When $\mathbb{M}$ satisfies (d) then $\mathbb{M}^s$ also satisfies (d): $X \in \nu^s(|w|)$ implies $\nmid X \nmid \in \nu(w)$ and by (d) for $\mathbb{M}$, $W \setminus \nmid X \nmid = \nmid W_\Sigma \setminus X \nmid \notin \nu(w)$, hence $W_\Sigma \setminus X \notin \nu^s(|w|)$.

Proof of (C): Assume that $\mathbb{M}$ satisfies (c). Let ν^- be the neighbourhood function obtained by closing ν^s under finite intersections, that is,

$$X \in \nu^-(|w|) \quad \text{iff} \quad \begin{array}{l} \text{there are } \nabla\varphi_1, \ldots, \nabla\varphi_n \in \Sigma \text{ such that} \\ V(\varphi_i) \in \nu(w),\ i \in \{1, \ldots, n\} \text{ and } \bigcap_{i \in \{1,\ldots,n\}} V(\varphi) \subseteq \nmid X \nmid. \end{array}$$

Define $M^- = (W_\Sigma, \nu^-, V^f)$. Then $\mathbb{M}^-$ is clearly a monotonic model satisfying (c). We will now show that M^- is a filtration of $\mathbb{M}$ through Σ. Condition (ii) of Definition 4.24 holds because $\nu^s \subseteq \nu^-$ and ν^s satisfies (ii). For condition (iii) of Definition 4.24, assume that $X \in \nu^-(|w|)$ and let $\nabla\varphi \in \Sigma$ be arbitrary such that $\nmid X \nmid \subseteq V(\varphi)$. We must show that $V(\varphi) \in \nu(w)$. From $X \in \nu^-(|w|)$ we have $\nabla\varphi_1, \ldots, \nabla\varphi_n \in \Sigma$ such that $V(\varphi_i) \in \nu(w)$ for all $i \in \{1, \ldots, n\}$ and $V(\varphi_1) \cap \ldots \cap V(\varphi_n) \subseteq \nmid X \nmid$. As $\mathbb{M}$ satisfies (c), it follows that $V(\varphi_1) \cap \ldots \cap V(\varphi_n) \in \nu(w)$ and from $V(\varphi_1) \cap \ldots \cap V(\varphi_n) \subseteq \nmid X \nmid \subseteq V(\varphi)$ we may now conclude that $V(\varphi) \in \nu(w)$.

Proof of (T): When $\mathbb{M}$ satisfies (t), then $\mathbb{M}^s$ satisfies (t): $X \in \nu^s(|w|)$ implies $\nmid X \nmid \in \nu(w)$, and since $\mathbb{M}$ satisfies (t), $w \in \nmid X \nmid$, hence $|w| \in |\nmid X \nmid| = X$.

Proof of (4'): Assume that $\mathbb{M}$ satisfies (iv'). We will show that $\mathbb{M}^s$ also satisfies (iv'). Assuming $X \in \nu^s(|w|)$, then there is a $\nabla\varphi \in \Sigma$ such that $V(\varphi) \subseteq \nmid X \nmid$ and $V(\varphi) \in \nu(w)$. We need to prove $m_{\nu^s}(X) \in \nu^s(|w|)$. Since $\mathbb{M}$ satisfies (iv'), it follows that $m_\nu(V(\varphi)) \in \nu(w)$. From $m_\nu(V(\varphi)) = V(\nabla\varphi)$ and $\nabla\varphi \in \Sigma$ we obtain from the Filtration Theorem 4.26 that $|V(\nabla\varphi)| = |\,m_\nu(V(\varphi))| \in \nu^s(|w|)$, and since $|\,m_\nu(V(\varphi))| = m_{\nu^s}(|V(\varphi)|)$, also $m_{\nu^s}(|V(\varphi)|) \in \nu^s(|w|)$. Finally, from $V(\varphi) \subseteq \nmid X \nmid \Rightarrow |V(\varphi)| \subseteq X$ and the monotonicity of m^s_ν, we may conclude that $m_{\nu^s}(X) \in \nu^s(|w|)$.

Proof of (5): Assume that $\mathbb{M}$ satisfies (v). We will show that $\mathbb{M}^l$ also satisfies (v). Suppose $X \notin \nu^l(|w|)$. Then there is a $\nabla\varphi \in \Sigma$ such that $\nmid X \nmid \subseteq V(\varphi)$ and $V(\varphi) \notin \nu(w)$. As $\mathbb{M}$ satisfies (v), we have $W \setminus m_\nu(V(\varphi)) \in \nu(w)$. Suppose now for contradiction that $W_\Sigma \setminus m_{\nu^l}(X) \notin \nu^l(|w|)$. Then there is a $\nabla\psi \in \Sigma$ such that $\nmid W_\Sigma \setminus m_{\nu^l}(X) \nmid \subseteq V(\psi)$ and $V(\psi) \notin \nu(w)$. From $\nmid X \nmid \subseteq V(\varphi)$ it follows by monotonicity of m_ν that $W \setminus m_\nu(V(\varphi)) \subseteq W \setminus m_\nu(\nmid X \nmid)$. Using the Filtration Theorem, it is easy to show that $W \setminus m_\nu(V(\varphi)) = \nmid W_\Sigma \setminus m_{\nu^l}(V(\varphi)) \nmid$, hence $W \setminus m_\nu(V(\varphi)) \subseteq V(\psi)$. Now, since $V(\psi) \notin \nu(w)$, we have arrived at a contradiction with $W \setminus m_\nu(V(\varphi)) \in \nu(w)$ due to the upwards closure of ν. QED

Filtrations are not only good for showing the finite model property, they are also a way of transforming the canonical model into a model of the right kind. The best known example of this method is perhaps the completeness proof of propositional dynamic logic (`PDL`), but filtrations have also been the key tool in obtaining completeness for some better known monotonic modal logics: In Goldblatt's concurrent propositional dynamic logic (`CPDL`) [32], as in `PDL`, the canonical model is not regular, or standard, with respect to all program constructions, but the canonical model can be filtrated to produce a regular `CPDL` model. In [36], a characterisation of Nash-consistency for finite coalition models results in completeness with respect to the class of all Nash-consistent coalition models by filtrating the canonical coalition model of `CLNC`, the smallest coalition logic together with an added inference rule. Most recently, completeness for the alternating-time temporal logic (`ATL`) [2] has been shown in [33], where filtrations are used to obtain a model which is standard with respect to the effectivity functions interpreting the temporal fixed-point operators.

4.3 Hennessy-Milner Classes and Ultrafilter Extensions

The last model construction, we will treat is that of taking *ultrafilter extensions*. For Kripke models, ultrafilter extensions may be seen as a completion of the underlying frame structure, and in the Algebra section we will see that the same holds for ultrafilter extensions of monotonic models. But perhaps more interestingly, ultrafilter extensions of Kripke models are *modally saturated* structures, which implies that modally equivalent states are bisimilar. This result is a basis for the slogan "modal equivalence implies bisimilarity somewhere else", namely in the ultrafilter extensions, see e.g. Blackburn et alii [6]. We were hoping to obtain an analogous result for ultrafilter extensions of monotonic models, but unfortunately, we must leave this as an open problem. Nevertheless, the basic construction will be presented here, together with some results which will be needed or helpful when looking for a solution.

The classes of models for which modal equivalence implies bisimilarity are called *Hennessy-Milner classes*.

Definition 4.30 (Hennessy-Milner Classes) Let K be a class of monotonic models. K is a *Hennessy-Milner class*, or *has the Hennessy-Milner property*, if for every two models $\mathbb{M}$ and $\mathbb{M}'$ in K and any two states w, w' of $\mathbb{M}$ and $\mathbb{M}'$, respectively, if w and w' satisfy the same modal formulas (notation: $\mathbb{M}, w \leftrightsquigarrow \mathbb{M}', w'$), then $\mathbb{M}, w \bumpeq \mathbb{M}', w'$. ⊣

For Kripke models, image finiteness or finite branching, is sufficient for the Hennessy-Milner property. For monotonic models, the Hennessy-Milner property is ensured for classes of *locally core-finite monotonic models*, that is, core-complete models in which $\nu^c(w)$ consists of a finite collection of finite neighborhoods for each state w. Note that there may well be infinitely many core neighbourhoods in the entire model, if W is inifinite. The case where there are only finitely many core neighbourhoods has been proved in [57] (Theorem 3.5) in the context of Coalition Logic. Simply observe that Pauly's definition of a *uniformly finitary coalition model* is the same as requiring that the model is core-complete, has finitely many core neighbourhoods, and all core neighbourhoods are finite. The proof of the following proposition is essentially the same as in [57] (Theorem 3.5), but we include it for completeness' sake.

Proposition 4.31 *Let K be a class of locally core-finite monotonic models. Then K is a Hennessy-Milner class.*

Proof. Let $\mathbb{M} = (W, \nu, V)$ and $\mathbb{M}' = (W', \nu', V')$ be two models in K, $s \in W$ and $s' \in W'$. Assume that $\mathbb{M}, s \leftrightsquigarrow \mathbb{M}', s'$. We will show that the modal equivalence relation $\leftrightsquigarrow$ is a core bisimulation between $\mathbb{M}$ and $\mathbb{M}'$, which suffices by Proposition 4.13.

The (prop) clause is immediate. To show the $(\text{forth})_c$ condition, suppose that sZs' and $X \in \nu^c(s)$. We need an $X' \subseteq W'$ such that $X' \in \nu'^c(s')$ and for all $x' \in X'$ there is an $x \in X$ such that $\mathbb{M}, x \leftrightsquigarrow \mathbb{M}', x'$. Since both W and W' are assumed to be locally core-finite, $\nu'^c(s') = \{X'_1, \dots, X'_n\}$ and $X = \{x_1, \dots, x_k\}$ for some $n, k \in \omega$.

In order to derive a contradiction, suppose that for all $X'_i \in \nu'^c(s')$ there is an $x'_i \in X'_i$ such that for all $x_j \in X$, it is not the case that $\mathbb{M}, x_j \leftrightsquigarrow \mathbb{M}', x'_i$. Let these x'_i be fixed for

$i = 1, \ldots, n$. Now, there must be formulas witnessing this, i.e., for each x'_i we have for all $x_j \in X$ a formula φ_{ij} such that $\mathbb{M}', x'_i \Vdash \varphi_{ij}$ and $\mathbb{M}, x_j \nVdash \varphi_{ij}$. Consider the formula

$$\varphi = \bigvee_{i=1,\ldots,n} \bigwedge_{j=1,\ldots,k} \varphi_{ij}.$$

It should be clear that for each of the x'_i, $\mathbb{M}', x'_i \Vdash \bigwedge_{j=1,\ldots,k} \varphi_{ij}$. It follows that

$$\mathbb{M}', s' \Vdash \Delta\varphi.$$

However, the formula φ cannot be satisfied at any x_j in X, hence, as $X \in \nu(s)$,

$$\mathbb{M}, s \nVdash \Delta\varphi.$$

But this is a contradiction with the assumption that $\mathbb{M}, s \leftrightsquigarrow \mathbb{M}', s'$.

The $(\text{back})_c$ condition is shown in a similar way. QED

The next step would naturally be to define a notion of modal saturation for monotonic models, which implies that classes of modally saturated models have the Hennessy-Milner property. Inspecting the proof of Proposition 4.31, we see that if we are no longer ensured to work with locally core-finite models, then both the disjunction and the conjunction in the formula φ may be infinite. In practice, modal saturation should thus allow us to encode (lack of) modal equivalence in a formula, i.e., we must be able to reduce the possibly infinite number of witnessing formulas to a single one. The definition below is due to Pauly [55]. Independently, we also found a definition of m-saturation which we believe to be equivalent with the one below, but its formulation was rather baroque. Hence our choice to use Pauly's version.

Definition 4.32 (m-saturation) Let $\mathbb{M} = (W, \nu, V)$ be a monotonic model. Then $\mathbb{M}$ is *m-saturated* if the following conditions hold.

(m1) For any $\Gamma \subseteq \mathcal{L}_\nabla$, $w \in W$ and $X \subseteq W$ such that $X \in \nu(w)$, if Γ is finitely satisfiable at some state in X, then Γ is also satisfiable at some state in X.

(m2) For any $\Gamma \subseteq \mathcal{L}_\nabla$ and $w \in W$, if for every finite $\Gamma^0 \subseteq_\omega \Gamma$ there is an $X \in \nu(w)$ such that all $x \in X$ satisfy Γ^0, then there is an $Y \in \nu(w)$ such that all $y \in Y$ satisfy Γ.

⊣

With Definition 4.32 above, it is fairly straightforward to show that classes of modally saturated monotonic models have the Hennessy-Milner property.

Proposition 4.33 *Classes of m-saturated models are Hennessy-Milner classes.*

Proof. The proof, which may be found in [55], is similar to the proof of Proposition 4.31, and we leave out the details. QED

Before we define ultrafilter extensions, recall that for any set W we can construct the powerset algebra over W, $\mathfrak{P}(W) = (\mathcal{P}(W), \cup, -_W, W)$, and the dual Stone space of $\mathfrak{P}(W)$ is the zero-dimensional, compact and Hausdorff topological space $\mathcal{S}_{\mathfrak{P}(W)} = (Uf(W), \tau_S)$, where $Uf(W)$ is the set of ultrafilters over W, and the topology τ_S is generated by the clopen basis consisting of the sets $\widehat{a} = \{u \in Uf(W) \mid a \in u\}$ for each $a \in \mathcal{P}(W)$. The set of closed sets in $\mathcal{S}_{\mathfrak{P}(W)}$ will be denoted by $K(\mathcal{S}_{\mathfrak{P}(W)})$. If $w \in W$, then π_w denotes the principal ultrafilter generated by w.

Definition 4.34 (Ultrafilter Extension) Let $\mathbb{F} = (W, \nu)$ be a monotonic frame. We define the *ultrafilter extension* $\mathfrak{ue}\mathbb{F}$ *of* $\mathbb{F}$ to be

$$\mathfrak{ue}\mathbb{F} = (Uf(W), \nu_{ue})$$

where for $u \in Uf(W), X \subseteq Uf(W)$,

(13) $X \in \nu_{ue}(u)$ iff $\exists C \in K(\mathcal{S}_{\mathfrak{P}(W)}) : C \subseteq X \;\&\; \forall a \in \mathcal{P}(W) : C \subseteq \widehat{a} \to m_\nu(a) \in u$.

For a monotonic model $\mathbb{M} = (\mathbb{F}, V)$, the *ultrafilter extension* $\mathfrak{ue}\mathbb{M}$ *of* $\mathbb{M}$ is the model

$$\mathfrak{ue}\mathbb{M} = (\mathfrak{ue}\mathbb{F}, V_{ue}),$$

where

(14) $V_{ue}(p) = \widehat{V(p)}$.

⊣

Remark 4.35 For clopens $\widehat{a}$ in the Stone space $\mathcal{S}_{\mathfrak{P}(W)}$, i.e., for $a \in \mathcal{P}(W)$, (13) reduces to

(15) $\widehat{a} \in \nu_{ue}(u)$ iff $m_\nu(a) \in u$.

For closed elements C in $\mathcal{S}_{\mathfrak{P}(W)}$, (13) reduces to

(16) $C \in \nu_{ue}(u)$ iff $\forall a \in \mathcal{P}(W) : C \subseteq \widehat{a} \to m_\nu(a) \in u$.

The following truth lemma is one of the steps on the way to showing that modal equivalence implies bisimilarity in the ultrafilter extension, since it implies that: If $w \leftrightsquigarrow v$ then $\pi_w \leftrightsquigarrow \pi_v$, hence if ultrafilter extensions can be shown to be m-saturated, the result is immediate.

Lemma 4.36 *Let $\mathbb{M} = (W, \nu, V)$ be a monotonic model and $\mathfrak{ue}\mathbb{M} = (W_{ue}, \nu_{ue}, V_{ue})$. Then for any modal formula φ and any ultrafilter $u \in W_{ue}$, we have, $V(\varphi) \in u$ iff $\mathfrak{ue}\mathbb{M}, u \Vdash \varphi$. In other words, $V_{ue}(\varphi) = \widehat{V(\varphi)}$. Hence we have for all $w \in W$, $w \leftrightsquigarrow \pi_w$.*

Proof. The proof of the first part is as expected by induction on φ, and as usual, the atomic case holds by definition. The boolean cases follow easily from the defining properties of ultrafilters, so we only show the modal case:

$$\begin{aligned} V(\nabla\varphi) \in u &\iff m_\nu(V(\varphi)) \in u \\ &\overset{(15)}{\iff} \widehat{V(\varphi)} \in \nu_{ue}(u) \\ &\overset{\text{(I.H.)}}{\iff} V_{ue}(\varphi) \in \nu_{ue}(u) \\ &\iff \mathfrak{ue}\mathbb{M}, u \Vdash \nabla\varphi. \end{aligned}$$

The last part of the proposition follows from the first:

$$\mathbb{M}, w \Vdash \varphi \Leftrightarrow w \in V(\varphi) \Leftrightarrow V(\varphi) \in \pi_w \Leftrightarrow \pi_w \in \widehat{V(\varphi)} = V_{ue}(\varphi) \Leftrightarrow \mathfrak{ue}\mathbb{M}, \pi_w \Vdash \varphi.$$

QED

As mentioned already, we have to leave the question of whether modal equivalence implies bisimilarity in the ultrafilter extension as an open problem. The difficulty is to show that ultrafilter extensions of monotonic frames are m-saturated.

5 Definability and Correspondence

The topics of this section are definability and correspondence [5, 41], as the title says. But we will also lay some groundwork for the simulations of section 10, which, in the current context, may be thought of as offering correspondence between monotonic modal logic and normal modal logic.

In the last subsection 5.3, we present Pauly's [55] adaptation to monotonic modal logic of the van Benthem characterisation theorem.

5.1 Definability

When we wish to characterise a class of structures satisfying certain properties, we are often mainly interested in the frame theoretic properties, thus abstracting away from particular model instances. The notion of frame validity gives us a handle on definability of frame classes. We first extend the definition of frame validity and recall the definition of modal definability.

An $\mathcal{L}_\nabla$-formula φ is valid at a state w in an $\mathcal{L}_\nabla$-frame $\mathbb{F}$ (notation: $\mathbb{F}, w \Vdash \varphi$) if for every model $(\mathbb{F}, V)$ based on $\mathbb{F}$, $(\mathbb{F}, V), w \Vdash \varphi$. Similarly, if Γ is a set of $\mathcal{L}_\nabla$-formulas, then Γ is valid at a state w in an $\mathcal{L}_\nabla$-frame $\mathbb{F}$ (notation: $\mathbb{F}, w \Vdash \Gamma$) if for all $\varphi \in \Gamma$, $\mathbb{F}, w \Vdash \varphi$.

Let K be a class of $\mathcal{L}_\nabla$-frames and φ an $\mathcal{L}_\nabla$-formula. Then φ *defines* K if for all $\mathcal{L}_\nabla$-frames $\mathbb{F}$, $\mathbb{F} \in \mathsf{K}$ iff $\mathbb{F} \Vdash \varphi$. A set Γ of $\mathcal{L}_\nabla$-formulas defines K if for all $\mathcal{L}_\nabla$-frames $\mathbb{F}$, $\mathbb{F} \in \mathsf{K}$ iff $\mathbb{F} \Vdash \Gamma$. A class of frames K is *modally definable* if there is a set of $\mathcal{L}_\nabla$-formulas that defines K.

Proposition 5.1 *The following formulas define the class of monotonic $\mathcal{L}_\nabla$-frames satisfying the indicated condition.*

N $\nabla\top$	(n) $\forall w \in W : W \in \nu(w)$.
P $\neg\nabla\bot$	(p) $\forall w \in W : \emptyset \notin \nu(w)$.
C $\nabla p \wedge \nabla q \to \nabla(p \wedge q)$	(c) $\forall w \in W\ \forall X_1, X_2 \subseteq W :$ $(X_1 \in \nu(w)\ \&\ X_2 \in \nu(w)) \to X_1 \cap X_2 \in \nu(w)$.
T $\nabla p \to p$	(t) $\forall w \in W\ \forall X \subseteq W : X \in \nu(w) \to w \in X$.
4 $\nabla\nabla p \to \nabla p$	(iv) $\forall w \in W\ \forall X, Y \subseteq W :$ $(X \in \nu(w)\ \&\ \forall x \in X : Y \in \nu(x)) \to Y \in \nu(w)$.
4' $\nabla p \to \nabla\nabla p$	(iv') $\forall w \in W\ \forall X \subseteq W : X \in \nu(w) \to m_\nu(X) \in \nu(w)$.
5 $\Delta p \to \nabla\Delta p$	(v) $\forall w \in W\ \forall X \subseteq W : X \notin \nu(w) \to W \setminus m_\nu(X) \in \nu(w)$.
B $p \to \nabla\Delta p$	(b) $\forall w \in W\ \forall X \subseteq W : w \in X \to W \setminus m_\nu(W \setminus X) \in \nu(w)$
D $\nabla p \to \Delta p$	(d) $\forall w \in W\ \forall X \subseteq W : X \in \nu(s) \to W \setminus X \notin \nu(w)$.

Proof. Let $\mathbb{F} = (W, \nu)$ be a monotonic $\mathcal{L}_\nabla$-frame. For each of the listed formulas φ, we must show that $\mathbb{F} \Vdash \varphi$ if and only if $\mathbb{F}$ satisfies the indicated condition. The proof of the "if" direction is quite trivial in all cases, so we only show the "only if" direction, and the method for proving this is the same for all formulas φ: We assume that $\mathbb{F}$ does not satisfy the condition in question, and use this assumption to find a suitable valuation V such that we can refute φ at some state in $(\mathbb{F}, V)$. Some abuse of notation will simplify the formulations: $\mathbb{F} \not\models$ (n) will denote that $\mathbb{F}$ does not satisfy condition (n); similarly for the other frame conditions.

N: Assume $\mathbb{F} \not\models$ (n), then there is a $w \in W$ such that $W \notin \nu(w)$. It follows that for any valuation V, $V(\top) \notin \nu(w)$, hence $(\mathbb{F}, V), w \nVdash \nabla\top$, i.e., $\mathbb{F} \nVdash$ N.

P: Assume $\mathbb{F} \not\models$ (p), then there is a $w \in W$ such that $\emptyset \in \nu(w)$. Hence for any valuation V, $V(\bot) \in \nu(w)$, and $(\mathbb{F}, V), w \Vdash \nabla\bot$, i.e., $\mathbb{F} \nVdash$ P.

C: Assume $\mathbb{F} \not\models$ (c), then there are $w \in W$ and $X_1, X_2 \subseteq W$ such that $X_1 \in \nu(w)$ and $X_2 \in \nu(w)$, but $X_1 \cap X_2 \notin \nu(w)$. Let V be a valuation with $V(p) = X_1$ and $V(q) = X_2$, then $(\mathbb{F}, V), w \Vdash \nabla p \wedge \nabla q$, but $(\mathbb{F}, V), w \nVdash \nabla(p \wedge q)$, and hence $\mathbb{F} \nVdash$ C.

T: Assume $\mathbb{F} \not\models$ (t), then there are $w \in W$ and $X \subseteq W$ such that $X \in \nu(w)$ and $w \notin X$. Let V be a valuation with $V(p) = X$, then $(\mathbb{F}, V), w \Vdash \nabla p$ and $(\mathbb{F}, V), w \nVdash p$, hence $\mathbb{F} \nVdash$ T.

4: Assume $\mathbb{F} \not\models$ (iv), then there are $w \in W$ and $X, Y \subseteq W$ such that $X \in \nu(w)$, $\forall x \in X : Y \in \nu(x)$ and $Y \notin \nu(w)$. Let V be a valuation with $V(p) = Y$, then for all $x \in X$, $(\mathbb{F}, V), x \Vdash \nabla p$, so $X \subseteq V(\nabla p)$ and by upwards closure of $\nu(w)$, $(\mathbb{F}, V), w \Vdash \nabla\nabla p$. But $Y = V(p) \notin \nu(w)$, hence $(\mathbb{F}, V), w \nVdash \nabla p$, and it follows that $\mathbb{F} \nVdash$ 4.

4': Assume $\mathbb{F} \not\models$ (iv'), then there are $w \in W$ and $X \subseteq W$ such that $X \in \nu(w)$ and $m_\nu(X) \notin \nu(w)$. Let V be a valuation with $V(p) = X$, then $(\mathbb{F}, V), w \Vdash \nabla p$, but $V(\nabla p) = m_\nu(X) \notin \nu(w)$, thus $(\mathbb{F}, V), w \nVdash \nabla\nabla p$, and hence $\mathbb{F} \nVdash$ 4'.

5: Assume $\mathbb{F} \not\models$ (v), then there are $w \in W$ and $X \subseteq W$ such that $X \notin \nu(w)$ and $W \setminus m_\nu(X) \notin \nu(w)$. If now V is a valuation with $V(p) = W \setminus X$, then $V(\neg p) = X \notin \nu(w)$ and $W \setminus V(\nabla\neg p) = V(\Delta p) \notin \nu(w)$, and hence $(\mathbb{F}, V), w \nVdash \nabla\neg p$, i.e., $(\mathbb{F}, V), w \Vdash \Delta p$ and $(\mathbb{F}, V), w \nVdash \nabla\Delta p$, thus we may conclude that $\mathbb{F} \nVdash$ 5.

B: Assume $\mathbb{F} \not\models$ (b), then there are $w \in W$ and $X \subseteq W$ such that $w \in X$ and $W \setminus m_\nu(W \setminus X) \notin \nu(w)$. If V is a valuation with $V(p) = X$, then $w \in V(p)$ and $W \setminus m_\nu(W \setminus V(p)) = V(\Delta p) \notin \nu(w)$. Hence $(\mathbb{F}, V), w \Vdash p$ and $(\mathbb{F}, V), w \nVdash \nabla\Delta p$, so $\mathbb{F} \nVdash$ B.

D: Assume $\mathbb{F} \not\models$ (d), then there are $w \in W$ and $X \subseteq W$ such that $X \in \nu(w)$ and $W \setminus X \in \nu(w)$. Let V be a valuation such that $V(p) = X$, then $V(p) \in \nu(w)$ and $V(\neg p) \in \nu(w)$, hence $(\mathbb{F}, V), w \Vdash \nabla p$ and $(\mathbb{F}, V), w \nVdash \Delta p$, and it follows that $\mathbb{F} \nVdash$ D. QED

The frame theoretic analogues of the model constructions of the previous section are obtained by simply leaving out the clauses concerning the valuation, and the truth invariance results for models translate more or less immediately into results on preservation of frame validity when forming disjoint unions, generated subframes, bounded morphic images and ultrafilter extensions. These in turn tell us about the limitations of definability via frame validity.

Definition 5.2 For a family of disjoint monotonic $\mathcal{L}_\nabla$-frames $\{\mathbb{F}_i = (W_i, \nu_i) \mid i \in I\}$ their *disjoint union* is the $\mathcal{L}_\nabla$-frame $\biguplus_{i \in I} \mathbb{F}_i = (W, \nu)$ where $W = \biguplus_{i \in I} W_i$ and for $X \subseteq W$, $X \in \nu(w)$ iff $X \cap W_i \in \nu_i(w)$.

A bounded morphism from a monotonic $\mathcal{L}_\nabla$-frame $\mathbb{F} = (W, \nu)$ to a monotonic $\mathcal{L}_\nabla$-frame $\mathbb{F}' = (W', \nu')$ is a function satisfying

(BM1) If $X \in \nu(w)$, then $f[X] \in \nu'(f(w))$.
(BM2) If $X' \in \nu'(f(w))$, then there is an $X \subseteq W$ such that $f[X] \subseteq X'$ and $X \in \nu(w)$.

If there is a surjective bounded morphism from $\mathbb{F}$ to $\mathbb{F}'$, then we say that $\mathbb{F}'$ is a *bounded morphic image* of $\mathbb{F}$ (notation: $\mathbb{F} \twoheadrightarrow \mathbb{F}'$).

For monotonic $\mathcal{L}_\nabla$-frames $\mathbb{F}' = (W', \nu')$ and $\mathbb{F} = (W, \nu)$, $\mathbb{F}'$ is a *generated subframe* of $\mathbb{F}$ (notation: $\mathbb{F}' \rightarrowtail \mathbb{F}$) if $W' \subseteq W$, and the identity map $i : W' \to W$ is a bounded morphism from $\mathbb{F}'$ to $\mathbb{F}$. For a subset X of W, the *subframe generated by X in* $\mathbb{F}$ is the subframe (W', ν') where W' is the intersection of all sets Y such that $X \subseteq Y$ and (Y, ν') is a generated subframe of $\mathbb{F}$. $\dashv$

The proof of the following proposition is standard, and we leave it to the reader.

Proposition 5.3 *Let φ be an $\mathcal{L}_\nabla$-formula. Then the following holds for monotonic $\mathcal{L}_\nabla$-frames.*

(i) If for all $i \in I$, $\mathbb{F}_i \Vdash \varphi$, then $\biguplus_{i\in I} \mathbb{F}_i \Vdash \varphi$.
(ii) If $\mathbb{F}' \rightarrowtail \mathbb{F}$, then $\mathbb{F} \Vdash \varphi$ implies $\mathbb{F}' \Vdash \varphi$.
(iii) If $\mathbb{F} \twoheadrightarrow \mathbb{F}'$, then $\mathbb{F} \Vdash \varphi$ implies $\mathbb{F}' \Vdash \varphi$.
(iv) If $\mathfrak{ue}\mathbb{F} \Vdash \varphi$, then $\mathbb{F} \Vdash \varphi$.

It is now clear that any modally definable frame class K must be closed under disjoint unions, generated subframes and bounded morphic images, and K must reflect ultrafilter extensions, that is, if $\mathfrak{ue}\mathbb{F}$ is in K, then $\mathbb{F}$ is in K. It turns out that within the class of frames which are closed under taking ultrafilter extensions, these (anti) closure conditions are also sufficient for definability. This is stated in the following analogue of the Goldblatt-Thomason theorem, which we will prove in section 7.

Theorem 5.4 (Monotonic frame definability) *Let K be a class of monotonic $\mathcal{L}_\nabla$-frames which is closed under taking ultrafilter extensions. Then K is modally definable iff K is closed under disjoint unions, generated subframes and bounded morphic images, and reflects ultrafilter extensions.*

Theorem 5.4 may be seen as a frame theoretic analogue of Birkhoff's characterisation of equationally definable classes of algebras as varieties, i.e. classes of algebras which are closed under taking direct products, subalgebras and homomorphic images. In section 7.4, we will see that disjoint unions, bounded morphic images and generated subframes are the dual notions of these algebraic constructions.

5.2 Correspondence

In correspondence theory, we compare modal languages with other languages such as first- and second-order logic for the purpose of expressing properties of models and frames. At first glance, there seems to be a problem when it comes to the correspondence theory of monotonic modal logic, namely, neighbourhood frames have a second-order character. However, we will see that we can treat monotonic frames as two-sorted relational structures for a suitable first-order language, by viewing the neighbourhood function as a relation R_ν between the universe W and $\mathcal{P}(W)$ (cf. the remark after (1)). This is also the idea behind the simulation of section 10, where we will simulate monotonic modal logics by normal bimodal ones. The observation, also made by Kracht and Wolter [44], that a certain class of $\mathcal{L}_\nabla$-formulas correspond to Sahlqvist formulas in a language which is interpreted over Kripke structures will allow us to apply the Sahlqvist Correspondence Theorem to obtain a similar result for monotonic modal logic (Theorem 5.14).

Model Correspondence

We will start by looking at correspondence on models, as this is interesting in its own right, but subsequently we will also use some of the definitions and results for models when treating frame correspondence.

As mentioned, the key to viewing a monotonic model as a first-order structure is to think of the neighbourhood function as a relation R_ν between the universe W and $\mathcal{P}(W)$. This means that the first-order language must contain variables which can be assigned to sets of states. One way to achieve this would be with a two-sorted first-order language, but we will instead use a unary relation symbol to distinguish the two kinds of variable interpretations. Furthermore, by turning a monotonic model into a relational structure we can view it as a Kripke model. Therefore, we also define a modal language which will be interpreted via Kripke semantics.

Definition 5.5 (Model Correspondence Languages) For a (countable) collection Φ of proposition letters, $\mathcal{L}^1_\nabla(\Phi)$ is the first-order language of $\mathcal{L}_\nabla$ which has equality $=$, first-order variables $x, y, z, \ldots$, unary predicates $Q_0, Q_1, Q_2, \ldots$ for each $q_0, q_1, q_2, \ldots$ in Φ, two binary relation symbols R_ν and $R_\ni$ and one unary relation symbol P.

$\mathcal{L}_\Diamond$ denotes the modal language (over Φ) which contains two unary modalities (diamonds), $\Diamond_\nu$ and $\Diamond_\ni$, and a nullary modality (constant) pt. $\dashv$

An $\mathcal{L}_\nabla$-model $\mathbb{M} = (W, \nu, V)$ may now be seen as a Kripke model by viewing ν as a relation R_ν, and interpreting $R_\ni$ by the element-of relation and P by W. That is,

$$(17)\quad \begin{array}{lcl} R_\nu wu & \text{iff} & u \in \nu(w), \\ R_\ni uw & \text{iff} & w \in u, \\ P & = & W. \end{array}$$

More precisely, we are viewing $\mathbb{M}$ as the Kripke model $\mathbb{M}^\bullet = (W \cup \mathcal{P}(W), R_\nu, R_\ni, P, V)$, where elements of $\mathcal{P}(W)$ have been added as new states, and R_ν, $R_\ni$ and P are defined as above. The truth definition in $\mathbb{M}^\bullet$ is as usual in Kripke models:

$$\begin{array}{lll} \mathbb{M}^\bullet, w \Vdash \bot & & \textit{never}, \\ \mathbb{M}^\bullet, w \Vdash q_i & \text{iff} & w \in V(q_i), \\ \mathbb{M}^\bullet, w \Vdash \Diamond_\nu \varphi & \text{iff} & \exists u (R_\nu wu \;\&\; \mathbb{M}^\bullet, u \Vdash \varphi), \\ \mathbb{M}^\bullet, w \Vdash \Diamond_\ni \varphi & \text{iff} & \exists u (R_\ni wu \;\&\; \mathbb{M}^\bullet, u \Vdash \varphi), \\ \mathbb{M}^\bullet, w \Vdash \mathsf{pt} & \text{iff} & w \in P (= W). \end{array}$$

When interpreting $\mathcal{L}^1_\nabla(\Phi)$-formulas on a monotonic model $\mathbb{M} = (W, \nu, V)$ (in which case we will write $\mathbb{M}^1$ for $\mathbb{M}$), we assume that an assignment θ on $\mathbb{M}^1$ assigns either an old state (in W) or a new state (in $\mathcal{P}(W)$) to each variable. The unary predicates Q_i are interpreted by $V(q_i)$, and R_ν, $R_\ni$ and P as above. It should be clear that $\mathbb{M}^1$ is just $\mathbb{M}^\bullet$ viewed as a first-order model.

We will see that the unary relation P is only needed to obtain global correspondence, therefore we will also define two translations into $\mathcal{L}^1_\nabla(\Phi)$, a local and a global one.

Definition 5.6 (Standard Translation) Let x be a first-order variable. The *local standard translation* $st_x : \mathcal{L}_\nabla \to \mathcal{L}^1_\nabla(\Phi)$ is defined inductively by

$$\begin{array}{lcl} st_x(\bot) & = & x \neq x, \\ st_x(q) & = & Qx, \\ st_x(\neg\varphi) & = & \neg st_x(\varphi), \\ st_x(\varphi \vee \psi) & = & st_x(\varphi) \vee st_x(\psi), \\ st_x(\nabla\varphi) & = & \exists y(R_\nu xy \wedge \forall z(R_\ni yz \to st_z(\varphi))). \end{array}$$

The *global standard translation* $ST_x : \mathcal{L}_\nabla \to \mathcal{L}^1_\nabla(\Phi)$ is defined by

$$ST_x(\varphi) \ = \ Px \to st_x(\varphi). \qquad \dashv$$

We also need a translation which will take an $\mathcal{L}_\nabla$-formula to an $\mathcal{L}_\diamond$-formula, and its definition should not come as a surprise.

Definition 5.7 (Diamond Translation) Define the translation $(\cdot)^t : \mathcal{L}_\nabla \to \mathcal{L}_\diamond$ inductively as follows:

$$\begin{array}{rcl} \bot^t & = & \bot \\ p^t & = & p \\ (\neg\varphi)^t & = & \neg\varphi^t \\ (\varphi \vee \psi)^t & = & \varphi^t \vee \psi^t \\ (\nabla\varphi)^t & = & \diamond_\nu\Box_\ni\varphi^t \ . \end{array}$$

Define the translation $(\cdot)^\diamond : \mathcal{L}_\nabla \to \mathcal{L}_\diamond$ by

$$\varphi^\diamond \ = \ \mathsf{pt} \to \varphi^t. \qquad \dashv$$

With these translations, the following proposition should be almost immediate. Observe that if $ST^\diamond_x(\cdot)$ denotes the standard translation for normal modal logic, then $st_x(\varphi) = ST^\diamond_x(\varphi^t)$, and $ST_x(\varphi) = ST^\diamond_x(\varphi^\diamond)$, when defining $ST^\diamond_x(\mathsf{pt}) = Px$.

Proposition 5.8 (Correspondence on Models) *Let φ be an $\mathcal{L}_\nabla$-formula. Then*

(i) *For all monotonic $\mathcal{L}_\nabla$-models $\mathbb{M} = (W, \nu, V)$ and all states w in $\mathbb{M}$:*

$$\mathbb{M}, w \Vdash \varphi \ \text{ iff } \ \mathbb{M}^\bullet, w \Vdash \varphi^t \ \text{ iff } \ \mathbb{M}^1 \models st_x(\varphi)[w]$$

(ii) *For all monotonic $\mathcal{L}_\nabla$-models $\mathbb{M} = (W, \nu, V)$:*

$$\mathbb{M} \Vdash \varphi \ \text{ iff } \ \mathbb{M}^\bullet \Vdash \varphi^\diamond \ \text{ iff } \ \mathbb{M}^1 \models \forall x ST_x(\varphi).$$

Proof. The proof of the first equivalence in (i) is by straightforward induction on φ. We only show the modal case:

$$\begin{array}{rll} \mathbb{M}, w \Vdash \nabla\varphi & \text{iff} & V(\varphi) \in \nu(w) \\ {\scriptstyle (\nu \text{ monotone})} & \text{iff} & \exists u \in W \cup \mathcal{P}(W) : u \in \nu(x) \ \&\ \forall y \in u : \mathbb{M}, y \Vdash \varphi \\ {\scriptstyle (\text{IH})} & \text{iff} & \exists u \in W \cup \mathcal{P}(W) : u \in \nu(x) \ \&\ \forall y \in u : \mathbb{M}^\bullet, y \Vdash \varphi^t \\ {\scriptstyle (\text{def. } \mathbb{M}^\bullet)} & \text{iff} & \exists u \in W^\bullet : R_\nu xu \ \&\ \forall y \in W^\bullet : R_\ni uy \to \mathbb{M}^\bullet, y \Vdash \varphi^t \\ & \text{iff} & \mathbb{M}^\bullet, w \Vdash \Diamond_\nu \Box_\ni \varphi^t \\ & \text{iff} & \mathbb{M}^\bullet, w \Vdash (\nabla\varphi)^t \ . \end{array}$$

The second equivalence in (i) follows from the first and local correspondence on Kripke models together with the observation that $st_x(\varphi) = ST^\diamond(\varphi^t)$. The proof of the first equivalence of (ii) is also easy:

$$\begin{array}{rll} \mathbb{M} \Vdash \varphi & \text{iff} & \text{for all } w \in W : \mathbb{M}, w \Vdash \varphi \\ & \text{iff} & \text{for all } w \in W \cup \mathcal{P}(W) : \mathbb{M}^\bullet, w \nVdash \mathsf{pt} \text{ or } \mathbb{M}^\bullet, w \Vdash \varphi^t \\ & \text{iff} & \text{for all } w \in W \cup \mathcal{P}(W) : \mathbb{M}^\bullet, w \Vdash \mathsf{pt} \to \varphi^t \\ & \text{iff} & \mathbb{M}^\bullet \Vdash \varphi^\diamond. \end{array}$$

Again, the second equivalence follows from the first together with global correspondence on Kripke models and the observation that $ST_x(\varphi) = ST_x^\diamond(\varphi^\diamond)$. QED

Frame Correspondence

As mentioned at the beginning of this section, frame validity is really a second-order property. This is well-known and unsurprising, since validity factors out valuations by quantifying over all possible subsets of the universe. The interesting cases are those in which the monadic second-order correspondent is equivalent to a first-order formula. We start by defining the first- and second-order languages for monotonic frames, and recall the definition of local and global frame correspondence.

Definition 5.9 (Frame Correspondence Languages) $\mathcal{L}^1_\nabla$ denotes the *first-order frame language* of $\mathcal{L}_\nabla$. $\mathcal{L}^1_\nabla$ has equality $=$, first-order variables $x, y, z, \ldots$, two binary relation symbols R_ν and $R_\ni$ and one unary relation symbol P.

The *monadic second-order frame language* $\mathcal{L}^2_\nabla$ is obtained from $\mathcal{L}^1_\nabla$ by allowing second-order quantification, that is, $\mathcal{L}^2_\nabla$ has everything $\mathcal{L}^1_\nabla$ has, and in addition $\mathcal{L}^2_\nabla$ contains monadic predicate variables $Q_0, Q_1, Q_2, \ldots$ over which may be quantified. $\dashv$

Definition 5.10 (Frame Correspondence) Let φ be an $\mathcal{L}_\nabla$-formula and $\alpha(x)$ a formula of the corresponding first- or second-order language (x is assumed to be the only free variable in α). Then φ and $\alpha(x)$ are each other's *local frame correspondents* if for any $\mathcal{L}_\nabla$-frame $\mathbb{F}$ and any state w in $\mathbb{F}$,

$$\mathbb{F}, w \Vdash \varphi \ \text{ iff } \ \mathbb{F} \vDash \alpha[w].$$

That is, for any valuation V, $(\mathbb{F}, V), w \Vdash \varphi$ iff $(\mathbb{F}, V)^1 \vDash \alpha[w]$.

If a class K of $\mathcal{L}_\nabla$-frames is definable by both φ and $\alpha(x)$, then we say that φ and $\alpha(x)$ are each other's *global frame correspondents*. $\dashv$

The second-order translation of an $\mathcal{L}_\nabla$-formula φ is obtained from the standard translation of φ by quantifying over the proposition letters occurring in φ. When interpreting an $\mathcal{L}^2_\nabla$-formula on a monotonic frame $\mathbb{F}$, we will write $\mathbb{F}^2$ for $\mathbb{F}$. And just as we can view a monotonic $\mathcal{L}_\nabla$-model as a Kripke model, we can view a monotonic $\mathcal{L}_\nabla$-frame $\mathbb{F}$ as a Kripke $\mathcal{L}_\diamond$-frame, $\mathbb{F}^\bullet = (W \cup \mathcal{P}(W), R_\nu, R_\ni, P)$, where R_ν, $R_\ni$ and P are interpreted as in (17). We can now show the following frame analogue of Proposition 5.8.

Proposition 5.11 (Correspondence on Frames) *Let φ be an $\mathcal{L}_\nabla$-formula. Then*

(i) *For all monotonic $\mathcal{L}_\nabla$-frames $\mathbb{F}$ and all states w in $\mathbb{F}$:*

$$\mathbb{F}, w \Vdash \varphi \text{ iff } \mathbb{F}^\bullet, w \Vdash \varphi^t \text{ iff } \mathbb{F}^2 \models \forall Q_1, \ldots, Q_n st_x(\varphi)[w]$$

(ii) *For all monotonic $\mathcal{L}_\nabla$-frames $\mathbb{F}$:*

$$\mathbb{F} \Vdash \varphi \text{ iff } \mathbb{F}^\bullet \Vdash \varphi^\diamond \text{ iff } \mathbb{F}^2 \models \forall Q_1, \ldots, Q_n \forall x ST_x(\varphi).$$

Proof. Let $\mathbb{F} = (W, \nu)$ be a monotonic $\mathcal{L}_\nabla$-frame. For the first equivalence in (i), note that a valuation on $\mathbb{F}$ is also a valuation on $\mathbb{F}^\bullet$, and a valuation V on $\mathbb{F}^\bullet$ induces a valuation $V{\restriction_W}$ on $\mathbb{F}$ where $V{\restriction_W}(q) = V(q) \cap W$. Moreover, these valuations agree on W, that is, for all $w \in W$, $w \in V(q)$ iff $w \in V{\restriction_W}(q)$.

Claim 1 If V and V' are valuations on $\mathbb{F}$ and $\mathbb{F}^\bullet$, respectively, and V and V' agree on W, then we have for all $w \in W$ and all $\mathcal{L}_\nabla$-formulas φ,

$$(\mathbb{F}, V), w \Vdash \varphi \text{ iff } (\mathbb{F}^\bullet, V'), w \Vdash \varphi^t.$$

Proof of Claim The proof is by induction on φ and is more or less the same as in the proof of item (i) of Proposition 5.8. We leave out the details. ◀

The direction from left to right now follows by contraposition: If $\mathbb{F}^\bullet, w \nVdash \varphi^t$ then there is a valuation V on $\mathbb{F}^\bullet$ such that $(\mathbb{F}^\bullet, V), w \nVdash \varphi^t$. By Claim 1 it follows that $(\mathbb{F}, V{\restriction_W}), w \nVdash \varphi$, hence $\mathbb{F}, w \nVdash \varphi$. The other direction is shown similarly, and the second equivalence in (i) follows from local frame correspondence on Kripke frames and the observation that $st_x(\varphi) = ST^\diamond_x(\varphi^t)$.

The first equivalence of (ii) is also shown by using Claim 1. For the direction from left to right, assume that $\mathbb{F}^\bullet \nVdash \varphi^\diamond$. Then there is a w in $W \cup \mathcal{P}(W)$ and a valuation V on $\mathbb{F}^\bullet$ such that $(\mathbb{F}^\bullet, V), w \nVdash \mathsf{pt} \to \varphi^t$, hence $(\mathbb{F}^\bullet, V), w \Vdash \mathsf{pt}$ and $(\mathbb{F}^\bullet, V), w \nVdash \varphi^t$. By the definition of P in $\mathbb{F}^\bullet$ it follows that $w \in W$ and so by Claim 1, $(\mathbb{F}, V{\restriction_W}), w \nVdash \varphi$, hence $\mathbb{F} \nVdash \varphi$.

The direction from right to left is also shown by contraposition. So suppose $\mathbb{F} \nVdash \varphi$, then there is a $w \in W$ and a valuation V on $\mathbb{F}$ such that $(\mathbb{F}, V), w \nVdash \varphi$. Since V is also a valuation on $\mathbb{F}^\bullet$ it follows from Claim 1 that $(\mathbb{F}^\bullet, V), w \nVdash \varphi^t$, hence $(\mathbb{F}^\bullet, V), w \Vdash \mathsf{pt} \wedge \neg\varphi^t$, and we may conclude that $\mathbb{F}^\bullet \nVdash \varphi^\diamond$. Again, the last equivalence in (ii) follows from global frame correspondence on Kripke frames and the observation that $ST_x(\varphi) = ST^\diamond_x(\varphi^\diamond)$. QED

Remark 5.12 Note that when we are looking for a first-order correspondent it suffices to find a local first-order correspondent, since if $\alpha(x)$ locally corresponds with φ, then $\forall x(Px \to \alpha(x))$ is a global correspondent of φ.

In the above, we have in fact established second-order frame correspondence by simulating monotonic $\mathcal{L}_\nabla$-frames by Kripke $\mathcal{L}_\Diamond$-frames, and using known correspondence results for Kripke frames. Even though, up until now, we could have left out all mentioning of Kripke frames and the translation $(\cdot)^t$ and simply shown the correspondence results directly, it is clear that this correspondence between neighbourhood semantics and Kripke semantics is useful. In section 10, we will return to the topic of simulations in more detail, but for now we will use the simulation of monotonic frames by Kripke frames to obtain an analogue of the Sahlqvist Correspondence Theorem for monotonic modal logic.

Recall that the Sahlqvist Correspondence Theorem gives us a syntactic characterisation of modal formulas which have first-order frame correspondents. In [44], Kracht and Wolter simulate monotonic modal logic with bimodal normal modal logic via the translation $(\cdot)^t$, and they use the observation that a certain class of $\mathcal{L}_\nabla$-formulas translate into bimodal Sahlqvist formulas via $(\cdot)^t$ to obtain results concerning their simulation. However, in the current context the usefulness of this $\mathcal{L}_\nabla$-formula fragment is to obtain automatic first-order correspondence.

Definition 5.13 (KW-formulas) An $\mathcal{L}_\nabla$-formula $\psi \to \chi$ is a *KW-formula* if ψ is of the form $\bigwedge_{i\leq n} \nabla p_i \wedge \bigwedge_{j\leq m} q_j$ and χ is built from propositional variables by using $\wedge, \vee, \nabla, \Delta$ only.

An $\mathcal{L}_\nabla$-formula $\psi \to \chi$ is a *dual KW-formula* if ψ is of the form $\bigwedge_{i\leq n} \Delta p_i \wedge \bigwedge_{j\leq m} q_j$ and χ is built from propositional variables by using $\wedge, \vee, \nabla, \Delta$ only. $\dashv$

Theorem 5.14 (First-Order Correspondence) *If φ is a KW-formula, then φ locally corresponds to an $\mathcal{L}^1_\nabla$-formula $a_\varphi(x)$ on monotonic frames, and $a_\varphi(x)$ is effectively computable from φ.*

Proof. By Proposition 5.11, φ locally corresponds to φ^t, and when φ is a KW-formula, then φ^t is a Sahlqvist $\mathcal{L}_\Diamond$-formula, hence it has a first-order correspondent $c_{\varphi^t}(x)$, so we can take $a_\varphi(x) = c_{\varphi^t}(x)$. Furthermore, using the Sahlqvist-van Benthem algorithm (see e.g. Blackburn et alii [6]) $c_{\varphi^t}(x)$ is effectively computable from φ^t and since φ^t is also effectively computable from φ, so is $a_\varphi(x)$. QED

Example 5.15 If we apply Theorem 5.14 to the KW-formula $q \to \nabla\Delta q$ (B), then we should obtain an $\mathcal{L}^1_\nabla$-formula which is equivalent with the frame condition (b) $\forall w \in W\ \forall X \subseteq W : w \in X \to W \setminus m_\nu(W \setminus X) \in \nu(w)$. We have,

$$\mathrm{B}^t = (q \to \nabla\Delta q)^t = q \to \Diamond_\nu \Box_\ni \Box_\nu \Diamond_\ni q.$$

Applying the Sahlqvist-van Benthem algorithm to B^t, we obtain the $\mathcal{L}^1_\nabla$-formula,

$$c(x) = \exists y(R_\nu(x,y) \wedge \forall z(R_\ni(y,z) \to \forall u(R_\nu(z,u) \to \exists r(R_\ni(u,r) \wedge r = x)))).$$

On a monotonic $\mathcal{L}_\nabla$-frame $\mathbb{F} = (W, \nu)$, $c(x)$ expresses the condition $\{z \mid \forall u \in \nu(z) : x \in u\} \in \nu(x)$ which by upwards closure of $\nu(x)$ is equivalent with

$$\{z \mid W \setminus \{x\} \notin \nu(z)\} \in \nu(x) \qquad (*)$$

Using the upwards closure of ν again, and the fact that $x \in \{x\}$, it is easy to show that $(*)$ is equivalent with the implication $x \in X \to W \setminus m_\nu(W \setminus X) \in \nu(x)$.

Example 5.16 Applying Theorem 5.14 to the KW-formula $\nabla q \to \Delta q$ (D), we first obtain the translation $\mathrm{D}^t = \Diamond_\nu \Box_\ni q \to \Box_\nu \Diamond_\ni q$, and from D^t we compute the first-order correspondent $c(x)$ with the Sahlqvist-van Benthem algorithm,

$$c(x) = \forall y \forall z (R_\nu(x,y) \wedge R_\nu(x,z) \to \exists u (R_\ni(y,u) \wedge R_\ni(z,u))).$$

On a monotonic $\mathcal{L}_\nabla$-frame $\mathbb{F} = (W, \nu)$, $c(x)$ expresses the condition,

$$X_1 \in \nu(x) \text{ and } X_2 \in \nu(x) \to X_1 \cap X_2 \neq \emptyset,$$

which is easily shown to be equivalent with $X \in \nu(x) \to W \setminus X \notin \nu(x)$.

5.3 Monotonic Modal Fragment of FOL

The characterisation due to J. van Benthem [4] of the modal fragment of first-order logic as the bisimulation invariant fragment, is one of the highlights in (normal) modal correspondence theory. Pauly [55] has adapted van Benthem's proof to achieve an analogous result for monotonic modal logic.

To avoid confusion between the notion of standard translation (as defined in 5.6) for monotonic model logic and the standard translation for normal modal logic, we will use the prefixes 'monotonic' and 'normal' explicitly in this subsection. Similarly, for bisimulation we will speak of monotonic bisimulations and Kripke bisimulations. If we refer to the fragment of first-order logic which consists of the monotonic standard translations of $\mathcal{L}_\nabla$-formulas, then Pauly's result can be stated as (Theorem 5.23): *The monotonic modal fragment of first-order logic is precisely the monotonic bisimulation invariant fragment.*

This subsection presents the main steps in Pauly's proof from [55], only slightly adapted to our setting, and we will leave out most technical details, of which several occur elsewhere in this thesis.

The reason why we have chosen to devote as much attention to the proof of an existing result, is partly because we find Pauly's characterisation theorem an interesting and elegant analogue of van Benthem's characterisation theorem, and partly because [55] only exists as an unpublished, not easily available, manuscript.

In the previous sections, we have seen how to view monotonic $\mathcal{L}_\nabla$-models as $\mathcal{L}^1_\nabla(\Phi)$-models, that is first-order models for the language $\mathcal{L}^1_\nabla(\Phi)$. We can also translate the definition of bisimulation to $\mathcal{L}^1_\nabla(\Phi)$-models.

Definition 5.17 (FO monotonic bisimulation) Let $\mathbb{M} = (W, R_\nu, R_\ni, P, V)$ and $\mathbb{M}' = (W', R'_\nu, R'_\ni, P', V')$ be two $\mathcal{L}^1_\nabla(\Phi)$-models. A non-empty binary relation $Z \subseteq W \times W'$ is a *monotonic bisimulation between* $\mathbb{M}$ *and* $\mathbb{M}'$ (notation: $Z : \mathbb{M} \leftrightarrows_1 \mathbb{M}'$) if

(pred) If wZw' then $\mathbb{M} \models Q(x)[w]$ iff $\mathbb{M}' \models Q(x)[w']$ for all unary predicates Q.

(P) If wZw' then $w \in P$ iff $w' \in P'$.

$(\text{forth})_1$ If wZw' and $wR_\nu u$, then there is a $u' \in W'$ such that $w'R'_\nu u'$ and for all $x' \in W'$, $u'R'_\ni x'$ implies that there is an $x \in W$ such that $uR_\ni x$ and xZx'.

$(\text{back})_1$ If wZw' and $w'R'_\nu u'$, then there is a $u \in W$ such that $wR_\nu u$ and for all $x \in W$, $uR_\ni x$ implies that there is an $x' \in W'$ such that $u'R'_\ni x'$ and xZx'.

$\dashv$

The above definition applies to all $\mathcal{L}^1_\nabla(\Phi)$-models, but as Pauly points out, monotonic bisimulation is a weaker notion than that of bimodal Kripke bisimulation, even for $\mathcal{L}^1_\nabla(\Phi)$-models which are obtained from monotonic $\mathcal{L}_\nabla$-models by the $(\cdot)^\bullet$ operation. However, it should be clear that the $(\cdot)^\bullet$ operation preserves monotonic bisimulations.

Lemma 5.18 *Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be monotonic $\mathcal{L}_\nabla$-models, and let w_1, w_2 be states in $\mathbb{M}_1$ and $\mathbb{M}_2$, respectively. Then*

$$\mathbb{M}_1, w_1 \leftrightarroweq \mathbb{M}_2, w_2 \quad \textit{iff} \quad \mathbb{M}_1^\bullet, w_1 \leftrightarroweq_1 \mathbb{M}_2^\bullet, w_2.$$

A first-order formula $\alpha(x)$ is now called *invariant for monotonic bisimulation* if for any $\mathcal{L}^1_\nabla(\Phi)$-models $\mathbb{M}$ and $\mathbb{M}'$,

(18) $\mathbb{M}_1, w_1 \leftrightarroweq_1 \mathbb{M}_2, w_2$ implies $\mathbb{M}_1^\bullet \models \alpha[w_1]$ iff $\mathbb{M}_2^\bullet \models \alpha[w_2]$

The following proposition states one direction of the characterisation theorem, namely, that the monotonic standard translations of $\mathcal{L}_\nabla$-formulas are invariant for monotonic bisimulation.

Proposition 5.19 *Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be $\mathcal{L}^1_\nabla(\Phi)$-models, and let w_1, w_2 be states in $\mathbb{M}_1$ and $\mathbb{M}_2$, respectively. Then for all $\mathcal{L}_\nabla$-formulas φ,*

$$\mathbb{M}_1, w_1 \leftrightarroweq_1 \mathbb{M}_2, w_2 \quad \textit{implies} \quad \mathbb{M}_1^\bullet \models st_x(\varphi)[w_1] \quad \textit{iff} \quad \mathbb{M}_2^\bullet \models st_x(\varphi)[w_2].$$

Proof. The proof is similar to that of Proposition 4.11, and we leave out the details. QED

In order to establish the converse of the above proposition, we will need the standard definitions of ω-saturation and elementary extensions from first-order model theory, together with the result that every first-order model has an ω-saturation elementary extension. We refer the reader to [13].

We should now recall the definition of m-saturation (Definition 4.32). Just as we have translated the notion of monotonic bisimulation to $\mathcal{L}^1_\nabla(\Phi)$-models, we can translate the notion of m-saturation in the obvious way, by using the definition of R_ν and $R_\ni$ in (17). Furthermore, when Γ is set of $\mathcal{L}_\nabla$-formulas, then we wil use the notation $\Gamma_{st}(x)$ for the set $\{st_x(\varphi) \mid \varphi \in \Gamma\}$ of $\mathcal{L}^1_\nabla(\Phi)$-formulas.

Definition 5.20 Let $\mathbb{M} = (W, R_\nu, R_\ni, P, V)$ be an $\mathcal{L}^1_\nabla(\Phi)$-model. Then $\mathbb{M}$ is *M-saturated* if the following conditions hold.

(M1) For any $\Gamma \subseteq \mathcal{L}_\nabla$ and any $w, u \in W$ such that $R_\nu wu$, if $\Gamma_{st}(x)$ is finitely satisfiable in the set of $R_\ni$-successors of u, then $\Gamma_{st}(x)$ is satisfiable in the set of $R_\ni$-successors of u.

(M2) For any $\Gamma \subseteq \mathcal{L}_\nabla$ and any $w \in W$, if for every finite $\Gamma^0 \subseteq_\omega \Gamma$ there is an R_ν-successor of w whose $R_\ni$-successors all satisfy $\Gamma^0_{st}(x)$, then there is an R_ν-successor of w whose $R_\ni$-successors all satisfy $\Gamma_{st}(x)$.

$\dashv$

One can now show the following lemma analogously to the case for m-saturation of Kripke models. See e.g. Blackburn et alii [6].

Lemma 5.21 *Any ω-saturated $\mathcal{L}^1_\nabla(\Phi)$-model is M-saturated*

The proof of the converse of Proposition 5.19 is now also completely analogous to the proof for normal modal logic. Again, the reader may consult [6] for the details. All we have to note is that from Proposition 4.33 it is more or less immediate that classes of M-saturated models have the Hennessy-Milner property with respect to monotonic bisimulation.

Proposition 5.22 *If an $\mathcal{L}^1_\nabla(\Phi)$-formula $\alpha(x)$ is invariant for monotonic bisimulation, then $\alpha(x)$ is equivalent to $st_x(\varphi)$ for some $\varphi \in \mathcal{L}_\nabla$.*

Taken together, the Propositions 5.19 and 5.22 establish the characterization theorem.

Theorem 5.23 (Pauly) *The monotonic modal fragment of $\mathcal{L}^1_\nabla(\Phi)$ is precisely the monotonic bisimulation invariant fragment.*

6 Completeness

In this section, we define the canonical model for monotonic modal logics, and prove the general completeness result of monotonic modal logics with respect to general monotonic frames.

First we recall the following definitions. Let Λ be a monotonic $\mathcal{L}_\nabla$-logic and S a class of monotonic $\mathcal{L}_\nabla$-structures. Then Λ is *sound with respect to* S if for all $\mathbb{S} \in \mathsf{S}$, $\mathbb{S} \Vdash \Lambda$. Λ is *weakly complete with respect to* S, if for all $\varphi \in \mathcal{L}_\nabla$, $\mathsf{S} \Vdash \varphi$ implies $\vdash_\Lambda \varphi$. For $\Sigma \cup \{\varphi\} \subseteq \mathcal{L}_\nabla$, we will use the notation $\Sigma \Vdash_{\mathsf{S}} \varphi$ to mean that for all $\mathbb{S} \in \mathsf{S}$, φ is a local semantic consequence of Σ in $\mathbb{S}$. Then Λ is *strongly complete with respect to* S, if for all $\Sigma \cup \{\varphi\} \subseteq \mathcal{L}_\nabla$, $\Sigma \Vdash_{\mathsf{S}} \varphi$ implies $\Sigma \vdash_\Lambda \varphi$. In practice, strong completeness is usually established by showing the contrapositive: Λ is strongly complete with respect to S, if for any Λ-consistent set of $\mathcal{L}_\nabla$-formulas Σ, Σ can be satisfied at a state in some $\mathbb{S} \in \mathsf{S}$. When showing strong completeness with respect to a class of monotonic frames K, this amounts to satisfying Σ at some state in a model based on a frame in K. Finally, Λ is (frame) complete if $\Lambda = \Lambda_{\mathsf{K}}$ for some frame class K, and Λ is strongly (frame) complete, if $\Lambda = \Lambda_{\mathsf{K}}$ for some frame class K, and Λ is also strongly complete with respect to K.

6.1 The Canonical Model Construction

The basic idea behind the construction of the canonical model $\mathbb{M}^\Lambda$ for a monotonic logic Λ, is, as usual, to build a model from maximally Λ-consistent sets. Via the standard argument of Lindenbaum's Lemma, a Λ-consistent set of formulas can be extended to a maximally Λ-consistent set (Λ-MCS). If φ is an $\mathcal{L}_\nabla$-formula, we denote the set of Λ-MCSs which contain φ as

$$\widehat{\varphi} = \{ \Gamma \mid \Gamma \text{ is } \Lambda\text{-MCS and } \varphi \in \Gamma \} .$$

The usual properties hold.

Lemma 6.1 (Properties of Λ-MCSs) *Let Λ be a modal $\mathcal{L}_\nabla$-logic and Γ a Λ-MCS, then:*

(i) $\perp \notin \Gamma$,
(ii) Γ *is closed under modus ponens: if* $\varphi, \varphi \to \psi \in \Gamma$ *then* $\psi \in \Gamma$,
(iii) *For all* $\mathcal{L}_\nabla$*-formulas* φ*: if* $\Gamma \vdash_\Lambda \varphi$ *then* $\varphi \in \Gamma$,
(iv) *For all* $\mathcal{L}_\nabla$*-formulas* φ*:* $\varphi \in \Gamma$ *or* $\neg\varphi \in \Gamma$,
(v) *For all* $\mathcal{L}_\nabla$*-formulas* φ, ψ*:* $\varphi \vee \psi \in \Gamma$ *iff* $\varphi \in \Gamma$ *or* $\psi \in \Gamma$.

Definition 6.2 (Canonical Model) Let Λ be a monotonic modal $\mathcal{L}_\nabla$-logic. The *canonical model for* Λ is the triple $\mathbb{M}^\Lambda = (W^\Lambda, \nu^\Lambda, V^\Lambda)$ where

(1) $W^\Lambda = \{\Gamma \subseteq \mathcal{L}_\nabla \mid \Gamma \text{ is } \Lambda\text{-MCS}\}$,

(2) For all $\Gamma \in W^\Lambda, X \subseteq W^\Lambda$: $X \in \nu^\Lambda(\Gamma)$ iff there is $\{\varphi_i \mid i \in I\} \subseteq \mathcal{L}_\nabla$ such that $\bigcap_{i\in I} \widehat{\varphi}_i \subseteq X$ and $\forall\psi \in \mathcal{L}_\nabla : \bigcap_{i\in I} \widehat{\varphi}_i \subseteq \widehat{\psi} \to \nabla\psi \in \Gamma$.

(3) For all $\Gamma \in W$, $p \in \textsc{prop} : \Gamma \in V^\Lambda(p) \Leftrightarrow p \in \Gamma$.

The pair $\mathbb{F}^\Lambda = (W^\Lambda, \nu^\Lambda)$ is called the *canonical frame for* Λ, and V^Λ will be called the *canonical valuation for* Λ. ⊣

Whereas the first and third clause of Definition 6.2 will be immediately clear, the second needs some more explanation. To begin with, the easiest way to understand 6.2(2) is to think of ν^Λ as defined in three stages. At the first stage, we say which subsets of the form $\widehat{\varphi}$ are in $\nu^\Lambda(\Gamma)$. Using the monotonicity rule (RM_∇), it is easy to show that for $\varphi \in \mathcal{L}_\nabla$ Definition 6.2(2) reduces to:

(19) $\widehat{\varphi} \in \nu^\Lambda(\Gamma)$ iff $\nabla\varphi \in \Gamma$,

Then we consider subsets of the form $C = \bigcap_{i\in I} \widehat{\varphi}_i$, which are the closed theories of Λ, and here we can show that 6.2(2) reduces to:

(20) $C \in \nu^\Lambda(\Gamma)$ iff $\forall\psi \in \mathcal{L}_\nabla : C \subseteq \widehat{\psi} \to \nabla\psi \in \Gamma$.

Finally, we add arbitrary supersets of the neighbourhoods from the first two stages, and this is exactly what Definition 6.2(2) states.

Readers familiar with Chellas [14] will have noticed that the above definition of the canonical model is not the same as Chellas' definition of the (supplementation of the) smallest canonical model (Def. 9.3 in [14]) in which the neighbourhood function ν_s^Λ is defined by,

(21) $X \in \nu_s^\Lambda(\Gamma)$ iff $\exists\nabla\varphi \in \Gamma : \widehat{\varphi} \subseteq X$.

The neighbourhoods of ν_s^Λ are exactly the neighbourhoods which are needed to make the Truth Lemma 6.3 hold. However, as Sergot [63] also points out, the Truth Lemma will hold for any neighbourhood function $\nu^{\Lambda'}$ if for each Λ-MCS Γ, $\nu^{\Lambda'}$ is of the form $\nu^{\Lambda'}(\Gamma) = \nu_s^\Lambda(\Gamma) \cup \mathcal{X}$, where $\mathcal{X}$ is a collection of non-definable subsets of W^Λ, i.e., $\forall\chi \in \mathcal{X}\ \forall\varphi \in \mathcal{L}_\nabla : \chi \neq \widehat{\varphi}$.

The difference between ν^Λ and ν_s^Λ lies in the addition of the neighbourhoods in ν^Λ of the form $C = \bigcap_{i\in I} \widehat{\varphi}_i$, which correspond to closed theories of Λ, or infinite conjunctions of $\mathcal{L}_\nabla$-formulas, hence the extra neighbourhoods of ν^Λ are indeed non-definable in the language $\mathcal{L}_\nabla$. The well-known bijection between ultrafilters and maximal consistent sets tells us that the closed theories of Λ correspond to the closed subsets of the dual topological space of the Lindenbaum-Tarski algebra $\mathbb{L}_\Lambda(\Phi)$. The analogy between the definition of ν_{ue} in ultrafilter extensions and ν^Λ should now be obvious, and both ν_{ue} and ν^Λ are, in fact, defined in this way based on motivations of algebraic duality, which we would not have obtained, had we defined $\nu^\Lambda = \nu_s^\Lambda$. In the sections 7 and 10 algebraic duality plays a key role, and we will see that this more sophisticated definition of the canonical frame fits better in our framework.

Pauly's [57, 58] definition of the canonical model for Coalition Logic is also a generalisation of Chellas' smallest canonical model, but neither Chellas nor Pauly have investigated algebraic duality. It should be said, though, that if one is merely interested in proving completeness, the smallest canonical model will often be more convenient to work with.

The key result needed to prove the Canonical Model Theorem 6.4 is the following lemma which lifts the "truth=membership" definition for proposition letters to arbitrary $\mathcal{L}_\nabla$-formulas.

Lemma 6.3 (Truth Lemma) *Let Λ be a monotonic $\mathcal{L}_\nabla$-logic, and let $\mathbb{M}^\Lambda = (W^\Lambda, \nu^\Lambda, V^\Lambda)$ be the canonical model for Λ. Then for all $\mathcal{L}_\nabla$-formulas φ and $\Gamma \in W^\Lambda$: $\mathbb{M}^\Lambda, \Gamma \Vdash \varphi$ iff $\varphi \in \Gamma$. In other words, $V^\Lambda(\varphi) = \widehat{\varphi}$.*

Proof. The proof is by induction on φ. The atomic case holds by definition of V^Λ, and the boolean cases follow easily from the properties of Λ-MCSs . So we only show the modal case: $\mathbb{M}^\Lambda, \Gamma \Vdash \nabla\varphi \Leftrightarrow V^\Lambda(\varphi) \in \nu^\Lambda(\Gamma) \Leftrightarrow_{(\mathrm{IH})} \widehat{\varphi} \in \nu^\Lambda(\Gamma) \Leftrightarrow_{(19)} \nabla\varphi \in \Gamma$. QED

The following theorem is now a direct consequence of the Truth Lemma. The proof is the same as for normal modal logics and is left to the reader.

Theorem 6.4 (Canonical Model Theorem) *Let Λ be a monotonic $\mathcal{L}_\nabla$-logic, and let $\mathbb{M}^\Lambda = (W^\Lambda, \nu^\Lambda, V^\Lambda)$ be the canonical model for Λ. Then Λ is sound and strongly complete with respect to $\{\mathbb{M}^\Lambda\}$.*

6.2 Applications

In Kripke semantics, strong completeness proofs are often completeness-via-canonicity arguments. That is, one shows for a normal modal logic Λ that Λ is valid on the canonical frame for Λ. The Canonical Model Theorem 6.4 ensures that the same argument may be applied to monotonic $\mathcal{L}_\nabla$-logics, however, there is one detail we should take care of. A monotonic $\mathcal{L}_\nabla$-logic Λ, and hence also $\mathbb{F}^\Lambda$, is assumed to be defined for a fixed and countable set of proposition letters Φ, thus $\mathbb{F}^\Lambda \Vdash \Lambda$ only means that Λ is valid on the canonical frame for this particular Φ. The definition of canonicity which we will employ (see Definition 7.11 of the Algebra section) is equivalent with saying that Λ is canonical if $\mathbb{F}^\Lambda \Vdash \Lambda$ where $\mathbb{F}^\Lambda$ is defined for Φ of arbitrary cardinality. It is an open problem whether the two notions are equivalent.

In order to formulate our notion of canonicity in logic terms, we will henceforth assume that $\mathbb{F}^\Lambda$ is defined for Φ of arbitrary cardinality. When we wish to emphasize that $\mathbb{F}^\Lambda$ depends on (the cardinality of) Φ, we will use the notation $\mathbb{F}^\Lambda(\Phi)$. Usually, Φ is assumed to be countable, simply because countably many proposition letters suffice, and the definitions and results of this section are not compromised by this generalisation.

We can now state our definition of canonicity as follows. An $\mathcal{L}_\nabla$-formula φ *is canonical* if for every monotonic $\mathcal{L}_\nabla$-logic Λ, $\varphi \in \Lambda$ implies that $\mathbb{F}^\Lambda(\Phi) \Vdash \varphi$, for any Φ. A monotonic $\mathcal{L}_\nabla$-logic Λ *is canonical* if $\mathbb{F}^\Lambda(\Phi) \Vdash \Lambda$, for any Φ. An $\mathcal{L}_\nabla$-formula φ *is canonical for a frame property* P if for every monotonic $\mathcal{L}_\nabla$-logic Λ, $\varphi \in \Lambda$ implies that $\mathbb{F}^\Lambda(\Phi)$ has P, for any Φ, and for every monotonic $\mathcal{L}_\nabla$-frame $\mathbb{F}$, if $\mathbb{F}$ has P then $\mathbb{F} \Vdash \varphi$.

The definability results of Proposition 5.1 tell us that for each of the mentioned formulas φ, we only need to show that φ is canonical in order to prove that φ is canonical for the frame property it defines. Instead of showing this directly, we will use Theorem 10.34 from section 10, which states that all KW-formulas (5.13) are, in fact, canonical.

Proposition 6.5 *If* $\varphi \in \{$N,C,T,4',B,D$\}$*, then* φ *is canonical for the frame property it defines.*

Proof. Let $\varphi \in \{$N,C,T,4',B,D$\}$ and let *FP* be the frame property defined by φ. Then φ is a KW-formula, and by Theorem 10.34, φ is canonical, hence for every monotonic $\mathcal{L}_\nabla$-logic Λ, $\varphi \in \Lambda$ implies that $\mathbb{F}^\Lambda(\Phi) \Vdash \varphi$, and by Proposition 5.1 it follows that $\mathbb{F}^\Lambda(\Phi)$ has *FP*, for any Φ. When $\mathbb{F}$ is a monotonic $\mathcal{L}_\nabla$-frame which has *FP*, $\mathbb{F} \Vdash \varphi$ follows immediately from Proposition 5.1. QED

Proposition 6.6 *If* $\Gamma \subseteq \{$N,C,T,4',B,D$\}$*, then* $\Lambda = \mathbf{M}.\Gamma$ *is sound and strongly complete with respect to the class of monotonic* $\mathcal{L}_\nabla$*-frames which have all the properties defined by the formulas in* Γ*.*

Proof. Let Γ and Λ be as stated, and let K be the class of monotonic $\mathcal{L}_\nabla$-frames which have all the properties defined by the formulas in Γ. Then by Theorem 10.34, Λ is canonical, hence Λ is valid on $\mathbb{F}^\Lambda(\Phi)$, for any Φ. From Proposition 5.1, it follows that $\mathbb{F}^\Lambda(\Phi) \in \mathsf{K}$, so by the Canonical Model Theorem Λ is strongly complete with respect to K. Soundness is likewise a consequence of Proposition 5.1. QED

The formulas P, 4 and 5 are not KW-formulas, but they are dual KW-formulas, and logics generated by these formulas are also strongly complete. However, the strong completeness relies on the notion of π-canonicity. Very briefly explained, when Λ is π-canonical, then Λ is valid on the π-canonical frame (see page 68), which can be thought of as a dual version of the canonical frame. The π-canonical frame is also canonical for Λ in the sense that any Λ-consistent set can be satisfied at some state. In section 10 we will see that dual KW-formulas are π-canonical (Theorem 10.44). However, if Λ is generated by a set of axioms where some are KW-formulas and others are dual KW-formulas, then we need not have strong completeness, since the π-canonical frame may not validate the KW-axioms, and vice versa, the canonical frame may not validate the dual KW-formulas. We refer to the sections 7 and 10 for more details.

Proposition 6.7 *If* $\Gamma \subseteq \{$P,4,5$\}$*, then* $\Lambda = \mathbf{M}.\Gamma$ *is sound and strongly complete with respect to the class of* Λ*-frames.*

Proof. Follows from Theorems 10.43 and 10.44. QED

6.3 General Completeness

As mentioned in subsection 3.3, neighbourhood semantics suffer from the same inadequacy as Kripke semantics, namely, there are monotonic modal logics which are not complete with respect to any class of monotonic frames, cf. Gerson [26]. However, similar to the case for normal modal logic and Kripke semantics, a general completeness result with respect to general monotonic frames does hold.

Recall that for a monotonic $\mathcal{L}_\nabla$-logic Λ, and a general monotonic $\mathcal{L}_\nabla$-frame $\mathbb{G}$, $\mathbb{G}$ is called a Λ-frame (or a frame for Λ), if $\mathbb{G} \Vdash \Lambda$, that is, for all admissible valuations V on $\mathbb{G}$, $(\mathbb{G}, V) \Vdash \Lambda$. And Λ is sound with respect to a class of general monotonic $\mathcal{L}_\nabla$-frames K, if for all $\mathbb{G} \in \mathsf{K}$, $\mathbb{G} \Vdash \Lambda$.

Definition 6.8 (Canonical general frame) Let Λ be a monotonic modal $\mathcal{L}_\nabla$-logic, and $\mathbb{F}^\Lambda$ the canonical frame for Λ. Then we define the *canonical general frame for* Λ as the pair $\mathbb{G}^\Lambda = (\mathbb{F}^\Lambda, \widehat{\Phi})$ where $\widehat{\Phi} = \{\widehat{\varphi} \mid \varphi \in \mathcal{L}_\nabla\}$. $\dashv$

It is easy to show that $\mathbb{G}^\Lambda$ is indeed a general monotonic $\mathcal{L}_\nabla$-frame, i.e., that $\widehat{\Phi}$ has the required closure properties. For example, to see that $\widehat{\Phi}$ is closed under the operation m_ν, simply recall that for any valuation V, $m_\nu(V(\varphi)) = V(\nabla\varphi)$, hence as V^Λ is admissible on $\mathbb{G}^\Lambda$, it follows by the Truth Lemma that $m_\nu(\widehat{\varphi}) = \widehat{\nabla\varphi}$.

Theorem 6.9 (General Completeness) *Let Λ be a monotonic $\mathcal{L}_\nabla$-logic. Then Λ is sound and strongly complete with respect to the class of general monotonic Λ-frames.*

Proof. Let F be the class of general monotonic Λ-frames. Soundness is clear by the definition of F. For strong completeness, we must show that any Λ-consistent $\Sigma \subseteq \mathcal{L}_\nabla$ is satisfiable in a model which is based on a general monotonic Λ-frame.

Let $\mathbb{G}^\Lambda = (W^\Lambda, \nu^\Lambda, \widehat{\Phi})$ be the general canonical frame for Λ. As V^Λ is an admissble valuation on $\mathbb{G}^\Lambda$, it follows that the Truth Lemma holds for the model $(\mathbb{G}^\Lambda, V^\Lambda)$. Hence every Λ-consistent set of formulas can be satisfied in the model $(\mathbb{G}^\Lambda, V^\Lambda)$. It remains to show that $\mathbb{G}^\Lambda \Vdash \Lambda$.

We will show that for all $\psi \in \Lambda : \mathbb{G}^\Lambda \Vdash \psi$. So let $\psi \in \Lambda$ and let V be an arbitrary admissible valuation on $\mathbb{G}^\Lambda$. Then we have for every proposition letter p_i, $i = 0, \dots, n$ occurring in ψ that $V(p_i) = \widehat{\psi}_i$ for some $\widehat{\psi}_i$, $i = 0, \dots, n$. Now, consider the formula $\psi' = \psi[\psi_0/p_0, \dots, \psi_n/p_n]$, that is, the formula ψ with ψ_i uniformly substituted for p_i.

Claim 1 $V(\psi) = V^\Lambda(\psi')$.

Proof of Claim By induction on the complexity of ψ:
Atomic case: Suppose $\psi = p$, then for some ψ_0, $V(p) = \widehat{\psi}_0$ and $p' = \psi_0$. So by the Truth Lemma for $(\mathbb{G}^\Lambda, V^\Lambda)$ it follows that, $V(p) = V^\Lambda(\psi_0) = V^\Lambda(p')$.
Induction step: The boolean cases are trivial. For the modal case, suppose $\psi = \nabla\delta$, then $\psi' = (\nabla\delta)' = \nabla\delta'$ and we have for all $\Gamma \in W^\Lambda$: $\Gamma \in V(\nabla\delta) \;\Leftrightarrow\; V(\delta) \in \nu^\Lambda(\Gamma) \;\Leftrightarrow_{(\mathrm{IH})}\; V^\Lambda(\delta') \in \nu^\Lambda(\Gamma) \;\Leftrightarrow\; \Gamma \in V^\Lambda(\nabla\delta')$. ◀

From the claim it follows immediately that

(22) $(\mathbb{G}^\Lambda, V) \Vdash \psi$ iff $(\mathbb{G}^\Lambda, V^\Lambda) \Vdash \psi'$.

Furthermore, since Λ is closed under uniform substitution, $\psi \in \Lambda$ implies that $\psi' \in \Lambda$, hence by the Truth Lemma for $(\mathbb{G}^\Lambda, V^\Lambda)$ and Lemma 6.1(iii) we obtain $(\mathbb{G}^\Lambda, V^\Lambda) \Vdash \psi'$. Using (22) we may now conclude that $(\mathbb{G}^\Lambda, V) \Vdash \psi$. Since V was an arbitrary admissible valuation on $\mathbb{G}^\Lambda$, we have shown that $\mathbb{G}^\Lambda \Vdash \psi$ for all $\psi \in \Lambda$, so $\mathbb{G}^\Lambda \Vdash \Lambda$. QED

7 Algebra

In this section we will show that monotonic modal logics and their semantics can be algebraised much in the same manner as normal modal logic. In fact, the basic results needed for an algebraic completeness theorem (Theorem 7.5) are easy analogues of their versions for normal modal logic, and we will therefore be quite brief in the exposition of subsection 7.2.

Furthermore, we will present the algebraic notion of canonicity which we define in terms of canonical extensions in subsection 7.3. In subsections 7.4 and 7.5, we will treat duality between monotonic frames and their algebraic counterparts. By then it should be clear how the definitions of the canonical frame and ultrafilter extensions came to be, since, just as in the normal modal logic case, they are natural by-products of algebraic duality. In the final subsection 7.6, we will look at the relationship between the two dual notions of canonicity which result from our definitions in subsection 7.3.

7.1 Notation and Basic Notions

We denote a boolean algebra by $(A, +, -, 0)$ where A is the carrier, $+$ is the join operation, $-$ is complementation and 0 the lower bound.

A *boolean algebra expansion* (BAE) is a boolean algebra expanded with a collection of functions $f_i : A \to A$, $i \in I$. However, we will restrict ourselves to BAEs for the basic modal similarity type, thus a BAE will be a boolean algebra expanded with one unary function, and we will use the notation $\mathbb{A} = (A, +, -, 0, f)$ or $\mathbb{A} = (\mathbb{B}, f)$, where $\mathbb{B}$ is a boolean algebra. If $\mathbb{A} = (A, +, -, 0, f)$ is a BAE, then we denote by $Bl\mathbb{A}$ the boolean reduct $(A, +, -, 0)$ of $\mathbb{A}$.

If P is a property of boolean algebras, then we will say that a BAE $\mathbb{A}$ has P if the boolean reduct of $\mathbb{A}$ has P. In particular, we will say that $\mathbb{A}$ is atomic, or complete, whenever $Bl\mathbb{A}$ is. We also use the following (standard) abbreviations: $1 = -0$, $a \cdot b = -(-a + -b)$, $a - b = a \cdot -b$, $a \to b = -a + b$ and $a \leftrightarrow b = (a \to b) \cdot (b \to a)$.

If $\mathbb{A} = (A, +, -, 0, f)$ and $\mathbb{A}' = (A', +, -, 0, f')$ are BAEs, then a map $\eta : \mathbb{A} \to \mathbb{A}'$ is a BAE-*homomorphism* if η is a boolean homomorphism, and for all $a \in A$, $\eta(f(a)) = f'(\eta(a))$.

For a BAE $\mathbb{A}$ we define the standard partial ordering $\leq$ on $\mathbb{A}$ by: $a \leq b$ iff $-a + b = 1$ iff $a \cdot b = a$ iff $a + b = b$. A *monotonic boolean algebra expansion* (BAM) is a BAE in which f is monotone, i.e., $a \leq b$ implies $f(a) \leq f(b)$, or equivalently, $f(a \cdot b) \leq f(a)$ for all $a, b \in A$. A *boolean algebra with operators* (BAO) is a BAE in which f is additive, i.e., $f(a+b) = f(a) + f(b)$ for all $a, b \in A$.

As is usual, formulas may be viewed as terms. More precisely, for a given set of (propositional) variables Φ, we denote the *terms over* Φ by $Ter(\Phi)$. Then $\mathcal{L}_\nabla$-formulas are simply the elements of the term algebra $\mathbb{T}\mathsf{er}(\Phi) = (Ter(\Phi), +, -, 0, f)$ where $0 := \bot$, $s + t := s \vee t$, $-s := \neg s$ and $f(s) := \nabla s$. We denote the set of propositional variables occurring in a formula φ, by $\mathrm{FV}(\varphi)$.

Terms are interpreted in a BAE via assignments in the usual way. That is, for a given set of variables Φ and a BAE $\mathbb{A} = (A, +, -, 0, f)$, an *assignment in* $\mathbb{A}$ is a function $\theta : \Phi \to A$, and we can extend θ uniquely to a *meaning function* $\tilde{\theta} : Ter(\Phi) \to A$ satisfying:

$$\begin{aligned}
\tilde{\theta}(p) &= \theta(p), \text{ for all } p \in \Phi, \\
\tilde{\theta}(\bot) &= 0, \\
\tilde{\theta}(\neg s) &= -\tilde{\theta}(s), \\
\tilde{\theta}(s \vee t) &= \tilde{\theta}(s) + \tilde{\theta}(t), \\
\tilde{\theta}(\nabla s) &= f(\tilde{\theta}(s)).
\end{aligned}$$

An *equation* is a pair of terms (s, t), usually written as $s \approx t$, and $s \approx t$ is *valid in a* BAM $\mathbb{A}$ (notation: $\mathbb{A} \models s \approx t$) if for all assignments θ, $\tilde{\theta}(s) = \tilde{\theta}(t)$. If K is a class of BAMs, then $s \approx t$ is *valid on* K (notation: $\mathsf{K} \models s \approx t$) if $s \approx t$ is valid in all $\mathbb{A} \in \mathsf{K}$. Furthermore, we denote the *equational theory of* K by $\mathsf{EqTh}(\mathsf{K}) = \{s \approx t \mid \mathbb{A} \models s \approx t, \text{ for all } \mathbb{A} \in \mathsf{K}\}$.

For a set of formulas Γ, we use the notation $\Gamma^{\approx}$ for the set of equations $\{\varphi \approx \top \mid \varphi \in \Gamma\}$. If Σ is a set of equations, then we denote the class of BAMs in which all equations in Σ are valid, with V_Σ. By Birkhoff's theorem, V_Σ is a variety. In particular, when Λ is a monotonic logic, then V_Λ denotes the variety of BAMs in which $\Lambda^{\approx}$ is valid.

Furthermore, we will denote the dual Stone space of a BAM $\mathbb{A}$ by $S_{\mathbb{A}} = (Uf\mathbb{A}, \tau)$, where τ is the topology generated by the clopen basis consisting of the subsets of $Uf\mathbb{A}$ which are of the form $\{u \in Uf\mathbb{A} \mid a \in u\}$ for $a \in A$.

Since we will restrict ourselves to the basic modal similarity type, "frame" should be read as "$\mathcal{L}_\nabla$-frame" and "formula" as $\mathcal{L}_\nabla$-formula", etc.

7.2 Algebraisation

Algebraising monotonic semantics

Given a monotonic frame we may obtain a BAM in the following way.

Definition 7.1 (Complex algebra) Let $\mathbb{F} = (W, \nu)$ be a monotonic frame.

We define $\mathbb{F}^+$, the *(full) complex algebra of* $\mathbb{F}$, as $\mathbb{F}^+ = (\mathcal{P}(W), \cup, -, \emptyset, m_\nu)$, where $(\mathcal{P}(W), \cup, -, \emptyset)$ is the power set algebra over W, and m_ν is as defined in (1).

A *complex algebra* is a subalgebra of a full complex algebra. If K is a frame class, then $\mathbf{Cm}\mathsf{K}$ denotes the class of full complex algebras of frames in K. ⊣

In a monotonic frame $\mathbb{F}$, the map $m_\nu : \mathcal{P}(W) \to \mathcal{P}(W)$ is monotone, so it is clear that $\mathbb{F}^+$ is a BAM. Furthermore, since the elements of $\mathbb{F}^+$ are subsets of the universe of $\mathbb{F}$, an assignment in $\mathbb{F}^+$ is nothing but a valuation on $\mathbb{F}$. This observation leads to the following proposition.

Proposition 7.2 *Let $\mathbb{F} = (W, \nu)$ be a monotonic frame, θ an assignment in $\mathbb{F}^+$ (or valuation on $\mathbb{F}$), φ a formula and w a state in $\mathbb{F}$. Then*

$$\begin{array}{lrcl}
(i) & (\mathbb{F}, \theta), w \Vdash \varphi & \textit{iff} & w \in \tilde{\theta}(\varphi), \\
(ii) & \mathbb{F} \Vdash \varphi & \textit{iff} & \mathbb{F}^+ \models \varphi \approx \top, \\
(iii) & \mathbb{F}^+ \models \varphi \approx \psi & \textit{iff} & \mathbb{F} \Vdash \varphi \leftrightarrow \psi.
\end{array}$$

Proof. Let $\mathbb{F}$, θ, φ and w be as in the proposition. We prove (i) by induction on φ. The atomic case holds by definition of $\tilde{\theta}$, and the boolean cases are straightforward. For the modal case we have,

$$\begin{array}{rll} (\mathbb{F},\theta),w \Vdash \nabla\varphi & \text{iff} & \text{there is an } X \in \nu(w) \text{ such that for all } x \in X\text{: } (\mathbb{F},\theta),x \Vdash \varphi \\ & _{(\text{IH})}\ \text{iff} & \text{there is an } X \in \nu(w) \text{ such that for all } x \in X\text{: } x \in \tilde{\theta}(\varphi) \\ & \text{iff} & \tilde{\theta}(\varphi) \in \nu(w) \\ & \text{iff} & w \in m_\nu(\tilde{\theta}(\varphi)) = \tilde{\theta}(\nabla\varphi). \end{array}$$

(ii) now follows easily from (i):

$$\begin{array}{rll} \mathbb{F} \Vdash \varphi & \text{iff} & \forall\theta\forall w \in W : (\mathbb{F},\theta), w \Vdash \varphi \\ & \text{iff} & \forall\theta\forall w \in W : w \in \tilde{\theta}(\varphi) \\ & \text{iff} & \forall\theta : \tilde{\theta}(\varphi) = W = \tilde{\theta}(\top) \\ & \text{iff} & \mathbb{F}^+ \models \varphi \approx \top. \end{array}$$

We leave the proof of (iii) to the reader. QED

From Proposition 7.2 we immediately obtain the following theorem which states that classes of complex algebras algebraise monotonic modal semantics.

Theorem 7.3 *Let* K *be a class of monotonic frames, and let* φ *and* ψ *be formulas. Then*

$$\begin{array}{lrcl} (i) & \mathsf{K} \Vdash \varphi & \textit{iff} & \mathbf{Cm}\mathsf{K} \models \varphi \approx \top, \\ (ii) & \mathbf{Cm}\mathsf{K} \models \varphi \approx \psi & \textit{iff} & \mathsf{K} \Vdash \varphi \leftrightarrow \psi. \end{array}$$

Theorem 7.3 thus tells us that the modal logic Λ_{K} of a class of monotonic frames K may be identified with the equational theory of the class of complex algebras $\mathbf{Cm}\mathsf{K}$, that is, the set of equations $\mathsf{EqTh}(\mathbf{Cm}\mathsf{K}) = \{s \approx t \mid \mathbb{F}^+ \models s \approx t, \text{ for all } \mathbb{F} \in \mathsf{K}\}$.

Algebraising monotonic axiomatics

The algebraisation of monotonic modal axiomatics is also completely analogous to the case of normal modal logic. Given a set of propositional variables Φ and a monotonic modal logic Λ, we define the binary relation $\equiv_\Lambda$ on formulas by

$$\varphi \equiv_\Lambda \psi \ \text{ iff } \ \vdash_\Lambda \varphi \leftrightarrow \psi.$$

Then it is easy to show that $\equiv_\Lambda$ is an equivalence relation on the term algebra $\mathbb{T}\mathsf{er}(\Phi)$. Let $[\varphi]$ denote the equivalence class of φ under $\equiv_\Lambda$, and let $Ter(\Phi)/\equiv_\Lambda$ denote the set of all equivalence classes under $\equiv_\Lambda$. To see that $\equiv_\Lambda$ is also a congruence relation, we should check that if $[\varphi_1] = [\varphi_2]$ and $[\psi_1] = [\psi_2]$, then $[\neg\varphi_1] = [\neg\varphi_2]$, $[\varphi_1 \vee \psi_1] = [\varphi_2 \vee \psi_2]$ and $[\nabla\varphi_1] = [\nabla\varphi_2]$. The first two properties follow from propositional logic, and the property concerning the modality follows from the fact that a monotonic logic is also classical, i.e., closed under the rule

$$(\mathrm{RE}_\nabla) \quad \frac{\varphi \leftrightarrow \psi}{\nabla\varphi \leftrightarrow \nabla\psi}$$

The *Lindenbaum-Tarski algebra of* Λ *over the set of generators* Φ, is defined as the structure

(23) $\mathbb{L}_\Lambda(\Phi) = (Ter(\Phi)/\equiv_\Lambda, +, -, 0, f_\nabla)$

where

$$\begin{array}{rcl} 0 & := & [\bot], \\ [\varphi] + [\psi] & := & [\varphi \vee \psi], \\ -[\varphi] & := & [\neg\varphi], \\ f_{\nabla}([\varphi]) & := & [\nabla\varphi]. \end{array}$$

The operations $+, -, f_{\nabla}$ are well-defined since $\equiv_{\Lambda}$ is a congruence relation. Furthermore, for any monotonic logic Λ and any Φ, $\mathbb{L}_{\Lambda}(\Phi)$ is a BAM. It is easy to check that $\mathbb{L}_{\Lambda}(\Phi)$ is a boolean algebra expansion. To see that f_{∇} is monotone, let $[\varphi], [\psi] \in Ter(\Phi)/\equiv_{\Lambda}$ and assume that $[\varphi] \leq [\psi]$ in $\mathbb{L}_{\Lambda}(\Phi)$. That means, $-[\varphi] + [\psi] = [\varphi \to \psi] = [\top]$, so by definition of $\equiv_{\Lambda}$, $\vdash_{\Lambda} (\varphi \to \psi) \leftrightarrow \top$ which implies $\vdash_{\Lambda} \varphi \to \psi$. Applying the monotonicity rule RM_{∇}, we obtain $\vdash_{\Lambda} \nabla\varphi \to \nabla\psi$, from which it follows that $\vdash_{\Lambda} (\nabla\varphi \to \nabla\psi) \leftrightarrow \top$, and hence $[\nabla\varphi] \leq [\nabla\psi]$. It now follows from the definition of f_{∇} that $f_{\nabla}([\varphi]) \leq f_{\nabla}([\psi])$.

From the perspective of universal algebra, $\mathbb{L}_{\Lambda}(\Phi)$ is the V_{Λ}-free algebra over $\Phi/\equiv_{\Lambda}$, and as is well-known, free algebras only depend on the cardinality of the set of generators Φ. But more importantly in the current context, the Lindenbaum-Tarski algebra of a monotonic modal logic Λ is a canonical algebraic model in the following sense.

Theorem 7.4 *Let Λ be a monotonic logic, let φ be a formula and Φ a set of propositional variables with cardinality at least that of* FV(φ). *Then*

$$\vdash_{\Lambda} \varphi \quad \textit{iff} \quad \mathbb{L}_{\Lambda}(\Phi) \models \varphi \approx \top.$$

Proof. This theorem is shown in the same way as Proposition 5.14 of Blackburn et alii [6], and we leave out the details. QED

It should now be clear that the Lindenbaum-Tarski algebra of a monotonic logic Λ is in the variety V_{Λ} defined by Λ, and the following theorem is immediate.

Theorem 7.5 (Algebraic completeness) *Let Λ be a monotonic logic. Then Λ is sound and complete with respect to V_{Λ}, that is, for all formulas φ, we have*

$$\vdash_{\Lambda} \varphi \quad \textit{iff} \quad \mathsf{V}_{\Lambda} \models \varphi \approx \top.$$

Proof. The soundness direction is clear by the definition of V_{Λ}. Completeness follows from Theorem 7.4 and the fact that $\mathbb{L}_{\Lambda}(\Phi) \in \mathsf{V}_{\Lambda}$. For suppose $\nvdash_{\Lambda} \varphi$, then by Theorem 7.4, we have $\mathbb{L}_{\Lambda}(\Phi) \not\models \varphi \approx \top$ where Φ is a set of propositional variables of cardinality at least that of FV(φ). Now since $\mathbb{L}_{\Lambda}(\Phi)$ belongs to V_{Λ}, it follows that $\mathsf{V}_{\Lambda} \not\models \varphi \approx \top$. QED

Remark 7.6 For a set of axioms Σ, we have $\mathsf{V}_{\mathbf{M}.\Sigma} = \mathsf{V}_{\Sigma}$. The inclusion from left to right is trivial, and the other inclusion may easily be shown by induction on the length of proofs in $\mathbf{M}.\Sigma$. Thus if $\Lambda = \mathbf{M}.\Sigma$, then the equivalence in Theorem 7.5 may also be stated as

$$\vdash_{\mathbf{M}.\Sigma} \varphi \quad \text{iff} \quad \mathsf{V}_{\Sigma} \models \varphi \approx \top.$$

Theorem 7.5 tells us that any monotonic logic is complete with respect to the class of BAMs it defines, and this is in sharp contrast with the situation for frame completeness. Now, we would like to transform this abstract completeness result into a result which will provide us with an algebraic approach to proving frame completeness. First, completely analogous to the case for normal modal logic, a BAM-variety V is said to be *complete* if V is generated by full complex algebras, that is, $\mathsf{V} = \mathbf{HSPCm}\mathsf{K}$ for some frame class K, and we have

(24) $\Lambda = \Lambda_{\mathsf{K}}$ iff $\mathsf{V}_\Lambda = \mathbf{HSPCm}\mathsf{K}$.

In other words, a monotonic logic Λ is complete iff V_Λ is complete.

We should point out that the above notion of completeness is with respect to neighbourhood semantics. Usually, a BAO-variety is said to be complete if it is generated by full complex algebras of Kripke frames. In our setting we view BAOs as a special kind of BAMs, thus even if V is a BAO-variety, we will say that V is complete if it is generated by full complex algebras of monotonic frames. In particular, if Λ is a normal modal logic, then Λ is complete with respect to neighbourhood semantics iff V_Λ is generated by full complex algebras which satisfy $f(1) = 1$ and $f(a \cdot b) = f(a) \cdot f(b)$. But Λ is complete with respect to Kripke semantics iff V_Λ is generated by full complex algebras in which $f(1) = 1$, and f distributes over *arbitrary* meets (which is the case in all complex algebras of Kripke frames). See also Kracht and Wolter [44]. In our terminology, a normal modal logic will be called complete if it is complete with respect to neighbourhood semantics.

Suppose we want to use (24) to show that Λ is complete with respect to the class K of Λ-frames. Then $\Lambda \subseteq \Lambda_{\mathsf{K}}$ holds trivially, so we only need $\mathsf{V}_\Lambda \subseteq \mathbf{HSPCm}\mathsf{K}$, since then

$$\begin{aligned} \varphi \in \Lambda_{\mathsf{K}} \quad &\Rightarrow \quad \mathbf{HSPCm}\mathsf{K} \models \varphi \approx \top \\ &\Rightarrow \quad \mathsf{V}_\Lambda \models \varphi \approx \top \\ &\Rightarrow \quad \mathbb{L}_\Lambda(\Phi) \models \varphi \approx \top \\ &\Rightarrow \quad \varphi \in \Lambda. \end{aligned}$$

But how should we show $\mathsf{V}_\Lambda \subseteq \mathbf{HSPCm}\mathsf{K}$? This is where the algebraic notion of canonicity becomes relevant. The approach taken in Blackburn et alii [6] for normal modal logic is to use the representation theorem by Jónsson and Tarski [39] which shows that any BAO $\mathbb{A}$ is a subalgebra of the full complex algebra of its ultrafilter frame $\mathbb{A}_+$. The result then follows by showing that Λ is valid on $\mathbb{A}_+$, or equivalently, that $(\mathbb{A}_+)^+$ belongs to V_Λ. In [6], $(\mathbb{A}_+)^+$ is referred to as $\mathfrak{Em}\mathbb{A}$, the canonical embedding algebra of $\mathbb{A}$, thus the question becomes, which varieties are closed under taking canonical embedding algebras.

Here we will essentially do the same, but instead of defining $\mathfrak{Em}\mathbb{A}$ in terms of $\mathbb{A}_+$, we will start by defining $\mathfrak{Em}\mathbb{A}$ and derive our definition of $\mathbb{A}_+$ from there. The reason for this somewhat roundabout approach is that it allows us to describe $\mathfrak{Em}\mathbb{A}$ using the theory of canonical extensions, and in what follows, we will therefore use the term *canonical extension* instead of canonical embedding algebra.

7.3 Canonical Extensions

The study of canonical extensions originated with Jónsson and Tarski [39], where, among other things, it was shown that every BAO has a canonical extension which is unique up to isomorphism, and that the validity of certain types of equations is preserved under taking canonical extensions. More recently, Gehrke, Jónsson and others [38, 23, 24, 25] have generalised these results to various types of lattice expansions.

The subject of canonical extensions of BAMs falls in between the work of [39](BAOs) and [24](distributive bounded lattices expanded with monotone functions). This is very convenient, since it means that most of the results we need have already been shown, albeit in a slightly different context. The general approach when defining the canonical extension of an algebra expansion is to first consider the boolean or lattice structure, and then define the extension of the added functions. We refer to [39, 38, 23, 24] for the following definitions and results.

Definition 7.7 A *canonical extension of a boolean algebra* $\mathbb{A}$ is a complete and atomic boolean algebra $\mathbb{A}^\sigma$ which contains $\mathbb{A}$ as a subalgebra such that the following conditions hold.

(density)	Every atom of $\mathbb{A}^\sigma$ is a meet of elements of $\mathbb{A}$.
(compactness)	Every subset of $\mathbb{A}$ whose join in $\mathbb{A}^\sigma$ is 1 has a finite subset whose join is also 1.

$\dashv$

One way of obtaining a canonical extension of a boolean algebra is by way of the Stone representation theorem (as we will see in a moment). Thus canonical extensions exist. Furthermore, the requirements in Definition 7.7 are enough to guarantee the following for any boolean algebras $\mathbb{A}$ and $\mathbb{B}$, and any canonical extensions $\mathbb{A}^\sigma$ and $\mathbb{B}^\sigma$ of $\mathbb{A}$ and $\mathbb{B}$.

- If $h : \mathbb{A} \to \mathbb{B}$ is an isomorphism, then there is an isomorphism $g : \mathbb{A}^\sigma \to \mathbb{B}^\sigma$ such that $g\restriction_{\mathbb{A}} = h$. Thus canonical extensions are unique up to isomorphism.
- Every element of $\mathbb{A}^\sigma$ is a join of meets of elements of $\mathbb{A}$.
- Every element of $\mathbb{A}^\sigma$ is a meet of joins of elements of $\mathbb{A}$.

The first item above tells us that we do not have to worry about the exact details of the representation, and we will therefore speak of 'the' canonical extension. However, in order to make the present exposition less abstract and more familiar (especially to readers of Blackburn et alii [6]), we will use the Stone representation explicitly. That is, for a boolean algebra $\mathbb{A} = (A, +, -, 0)$, the canonical extension is $\mathbb{A}^\sigma = (\mathcal{P}(Uf\mathbb{A}), \cup, \setminus, \emptyset)$, where $\mathcal{P}(Uf\mathbb{A})$ is the powerset of the set of all ultrafilters of $\mathbb{A}$, and the map $\widehat{\cdot} : \mathbb{A} \to \mathbb{A}^\sigma$ defined by

(25) $\widehat{a} = \{u \in Uf\mathbb{A} \mid a \in u\}$

is a boolean embedding of $\mathbb{A}$ into $\mathbb{A}^\sigma$. Strictly speaking, $(\mathcal{P}(Uf\mathbb{A}), \cup, \setminus, \emptyset)$ is the canonical extension of the image $\widehat{\mathbb{A}}$ of $\mathbb{A}$ under the map $\widehat{\cdot}$, but due to the equivalence modulo isomorphism, we may think of $(\mathcal{P}(Uf\mathbb{A}), \cup, \setminus, \emptyset)$ as the canonical extension of $\mathbb{A}$. In the following, the notation $\widehat{a}$ will always stand for an element of $\widehat{\mathbb{A}}$.

Recall that the elements of $\widehat{\mathbb{A}}$ form a clopen basis of the dual Stone space $S_{\mathbb{A}}$ of $\mathbb{A}$. Thus using Stone duality, it is natural that we refer to elements of $\widehat{\mathbb{A}}$, the meet closure $K(\mathbb{A}^\sigma)$ and the join closure $O(\mathbb{A}^\sigma)$ of $\mathbb{A}^\sigma$ as the *clopen*, *closed*, respectively *open*, elements of $\mathbb{A}^\sigma$.

We now know how to extend the boolean part of a BAM. In order to extend the entire structure of a BAM $\mathbb{A} = (Bl\mathbb{A}, f)$, we may clearly think of f as a monotone function from the boolean algebra $Bl\mathbb{A}$ to itself. We can therefore define the extension of f in the slightly more general setting of extending a function f, where f is a monotone map from a boolean algebra $\mathbb{A}$ to a boolean algebra $\mathbb{B}$, to a function from $\mathbb{A}^\sigma$ to $\mathbb{B}^\sigma$.

Since $\mathbb{A}^\sigma$ is join generated by $K(\mathbb{A}^\sigma)$ as well as meet generated by $O(\mathbb{A}^\sigma)$, and similarly for $\mathbb{B}^\sigma$, this leads to two (dual) ways of extending a function $f : \mathbb{A} \to \mathbb{B}$.

Definition 7.8 (Extending maps) Let $\mathbb{A}$ and $\mathbb{B}$ be boolean algebras, and $f : A \to B$ a monotone function. Then we define $f^\sigma, f^\pi : \mathcal{P}(Uf\mathbb{A}) \to \mathcal{P}(Uf\mathbb{B})$ by

$$\begin{aligned} f^\sigma(X) &= \bigcup_{K(\mathbb{A}^\sigma)\ni C\subseteq X} \bigcap_{C\subseteq\widehat{a}} \widehat{f(a)}. \\ f^\pi(X) &= \bigcap_{X\subseteq O\in O(\mathbb{A}^\sigma)} \bigcup_{\widehat{a}\subseteq O} \widehat{f(a)}. \end{aligned}$$

⊣

For clopen, closed and open elements of $\mathbb{A}^\sigma$, Definition 7.8 reduces to the following.

(26) $f^\sigma(\widehat{a}) = f^\pi(\widehat{a}) = \widehat{f(a)}$, for all clopens $\widehat{a} \in \widehat{A}$.

(27) $f^\sigma(C) = \bigcap_{C\subseteq\widehat{a}} \widehat{f(a)}$, for all closed $C \in K(\mathbb{A}^\sigma)$.

(28) $f^\pi(O) = \bigcup_{\widehat{a}\subseteq O} \widehat{f(a)}$, for all open $O \in O(\mathbb{A}^\sigma)$.

It can be shown that f^σ and f^π are monotone, agree on closed, as well as on open elements, and both functions map closed elements to closed elements, and open elements to open elements. In general, f^σ and f^π will be different, but if $f^\sigma = f^\pi$ then f is called *smooth*. For a more detailed treatment of f^σ and f^π, we refer to [39, 24] and subsection 7.6.

We are now ready to define canonical extensions for BAMs.

Definition 7.9 (Canonical extensions) Let $\mathbb{A}$ be a BAM. Then we define the *σ-canonical extension of* $\mathbb{A}$ by

$$\mathbb{A}^\sigma = ((Bl\mathbb{A})^\sigma, f^\sigma),$$

and the *π-canonical extension of* $\mathbb{A}$ by

$$\mathbb{A}^\pi = ((Bl\mathbb{A})^\sigma, f^\pi).$$

⊣

The above canonical extensions clearly satisfy the (density) and (compactness) conditions from Definition 7.7, since these are properties of the underlying boolean algebras. Moreover, f^σ and f^π have been defined in such a way that their restriction to $\widehat{\mathbb{A}}$ in $\mathbb{A}^\sigma$, respectively $\mathbb{A}^\pi$, is the image of f under the Stone representation map. This is the essence of the following proposition.

Proposition 7.10 *Let $\mathbb{A}$ be a* BAM*, and let $\mathbb{B}$ be either the σ-canonical extension or the π-canonical extension of $\mathbb{A}$. Then $\mathbb{A}$ is isomorphic to a subalgebra of $\mathbb{B}$.*

Proof. The Stone representation map $r : a \mapsto \widehat{a}$ is an embedding of $Bl\mathbb{A}$ into $(Bl\mathbb{A})^\sigma$. From (26), it is clear that r is also BAE-homomorphism, that is, $r(f(a)) = f^\sigma(r(a))$ and $r(f(a)) = f^\pi(r(a))$. Hence r is an embedding of $\mathbb{A}$ into both $\mathbb{A}^\sigma$ and $\mathbb{A}^\pi$. QED

Returning to our discussion on completeness, we recall that for a monotonic logic Λ, the variety V_Λ is complete if V_Λ is closed undere taking canonical extensions. Since we have two ways of defining canonical extensions, we also have two kinds of canonicity.

Definition 7.11 (Canonicity) A formula φ *is σ-canonical (π-canonical)* if the validity of $\varphi \approx \top$ is preserved under taking σ-canonical extensions (π-canonical extensions). That is, if $\mathbb{A} \models \varphi \approx \top$ then $\mathbb{A}^\sigma \models \varphi \approx \top$ ($\mathbb{A}^\pi \models \varphi \approx \top$). A class K of BAMs is *σ-canonical (π-canonical)* if K is closed under taking σ-canonical extensions (π-canonical extensions). That is, if $\mathbb{A} \in \mathsf{K}$ then $\mathbb{A}^\sigma \in \mathsf{K}$ ($\mathbb{A}^\pi \in \mathsf{K}$). A monotonic logic Λ *is σ-canonical (π-canonical)* if the variety V_Λ defined by Λ is σ-canonical (π-canonical). ⊣

It may well be that a variety is σ-canonical, but not π-canonical, or vice versa. As easy examples of varieties which are both σ-canonical and π-canonical, we list the following.

Theorem 7.12 *The following* BAM*-varieties are both σ-canonical and π-canonical:*

$$\begin{array}{rcl}
\mathsf{V}_{\mathbf{M}} & = & \textit{the variety of all BAMs}, \\
\mathsf{V}_{\mathbf{N}} & = & \textit{the variety of BAMs validating } f(1) \approx 1, \\
\mathsf{V}_{\mathbf{P}} & = & \textit{the variety of BAMs validating } f(0) \approx 0, \\
\mathsf{V}_{\mathbf{T}} & = & \textit{the variety of BAMs validating } f(x) \leq x.
\end{array}$$

Proof. Throughout the proof, $\mathbb{A} = (A, +, -, 0, f)$ is a BAM, $\mathbb{A}^\sigma = (\mathcal{P}(Uf\mathbb{A}), \cup, \setminus, \emptyset, f^\sigma)$ is the σ-canonical extension of $\mathbb{A}$, and $\mathbb{A}^\pi = (\mathcal{P}(Uf\mathbb{A}), \cup, \setminus, \emptyset, f^\pi)$ is the π-canonical extension of $\mathbb{A}$.
$\mathsf{V}_{\mathbf{M}}$: When $\mathbb{A}$ is a BAM, then $\mathbb{A}^\sigma$ and $\mathbb{A}^\pi$ are clearly also BAMs, since f^σ and f^π are monotone.
$\mathsf{V}_{\mathbf{N}}$: Assume that $\mathbb{A} \in \mathsf{V}_{\mathbf{N}}$, i.e., $f(1) = 1$. We must show that $f^\sigma(Uf\mathbb{A}) = Uf\mathbb{A}$ and $f^\pi(Uf\mathbb{A}) = Uf\mathbb{A}$. For f^σ, we have $f^\sigma(Uf\mathbb{A}) = f^\sigma(\widehat{1}) = \widehat{f(1)} = \widehat{1} = Uf\mathbb{A}$. The case for f^π is shown in the same manner.
$\mathsf{V}_{\mathbf{P}}$: Assume that $\mathbb{A} \in \mathsf{V}_{\mathbf{P}}$, i.e., $f(0) = 0$. We must show that $f^\sigma(\emptyset) = \emptyset$ and $f^\pi(\emptyset) = \emptyset$. The proof is just as simple as the previous one, and we only show the case for f^π. We have $f^\pi(\emptyset) = f^\pi(\widehat{0}) = \widehat{f(0)} = \widehat{0} = \emptyset$.
$\mathsf{V}_{\mathbf{T}}$: Assume that $\mathbb{A} \in \mathsf{V}_{\mathbf{T}}$, i.e., $\forall a \in A : f(a) \leq a$. For f^σ, we must show that for all $X \subseteq Uf\mathbb{A} : f^\sigma(X) \subseteq X$. We have

$$\begin{array}{rcl}
& & u \in f^\sigma(X) = \displaystyle\bigcup_{K(\mathbb{A}) \ni C \subseteq X} \; \bigcap_{C \subseteq \widehat{a}} \widehat{f(a)} \\
& \iff & \exists C \in K(\mathbb{A}) : C \subseteq X \text{ and } \forall \widehat{a} \supseteq C : f(a) \in u \\
{\scriptstyle (f(a) \leq a,\, u \in Uf\mathbb{A})} & \implies & \exists C \in K(\mathbb{A}) : C \subseteq X \text{ and } \forall \widehat{a} \supseteq C : a \in u \\
& \iff & u \in \displaystyle\bigcup_{K(\mathbb{A}) \ni C \subseteq X} \; \bigcap_{C \subseteq \widehat{a}} \widehat{a} \;=\; X.
\end{array}$$

The case for f^π is shown similarly. QED

Theorem 7.13 *The following monotonic modal logics are both σ-canonical and π-canonical:*

$$\begin{array}{rcl}
\mathbf{M}, & & \\
\mathbf{M.N} & = & \mathbf{M}.\{\nabla\top\}, \\
\mathbf{M.P} & = & \mathbf{M}.\{\neg\nabla\bot\}, \\
\mathbf{M.T} & = & \mathbf{M}.\{\nabla p \to p\}.
\end{array}$$

We will take the notion of σ-canonicity to be our default, thus "canonical" will mean "σ-canonical". But in subsection 7.6 we will return to the dual relationship between the two notions of canonicity, and look closer at the connection with the duality between ∇ and Δ.

7.4 Basic Duality Theory

Duality theory in modal logic is concerned with the relationships between the frame based semantics and the algebraic or topological semantics. Frame semantics are often preferred due to their intuitive character, or simply because we have an application driven interest in describing certain structures. However, the level of abstraction provided by looking at problems from an algebraic angle may often be beneficial, and in some cases, we are able to give very short and elegant algebraic proofs of results whose frame theoretic proofs are rather long-winded. An example hereof, is the so-called Goldblatt-Thomason theorem, see e.g. Blackburn et alii [6], and we will also give a simple proof of our analogue, Theorem 5.4, by using the duality results of the present subsection.

In normal modal logic, duality theory is a well-studied field [67, 29, 30, 61, 31, 43]. But for non-normal modal logic, the only paper known to us so far which treats duality between neighbourhood semantics and BAEs is Došen [18]. Although many of Došen's results easily adapt to monotonic frames and BAMs, the duality we are interested in here is slightly more general. See Remarks 7.24 and 7.40 below for comments on how our duality relates to that of [18].

Some basic categorical concepts

We will not make extensive use of category theory in our treatment, but since duality is essentially a category theoretical notion, we will make the basic categorical content of our results explicit. In the interest of self-containment, we list the relevant definitions below. For more details, we refer to [47, 43].

A *category* is a structure $\mathsf{C} = (\mathsf{Ob}, \mathsf{Mor}, dom, cod, \circ, id)$ where Ob is called the class of *objects*; Mor the class of *morphisms*; $dom, cod : \mathsf{Mor} \to \mathsf{Ob}$ assign a *domain* and *codomain* to each morphism; $\circ : \mathsf{Mor} \times \mathsf{Mor} \to \mathsf{Mor}$ assigns the *composition* of a suitable pair of morphisms, and $id : \mathsf{Ob} \to \mathsf{Mor}$ assigns to each object A its *identity* morphism id_A. We write $f : A \to B$ to state that f is a morphism with domain A and codomain B. There are a number of natural requirements which must also be satisfied:

- For $f, g \in \mathsf{Mor}$, $f \circ g$ is defined iff $cod(g) = dom(f)$,
- For $A \in \mathsf{Ob}$, $dom(id_A) = cod(id_A) = A$,
- If $f : A \to B$, then $f \circ id_A = f$ and $id_B \circ f = f$,
- If $f : A \to B$, $g : B \to C$ and $h : C \to D$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

Let C and D be two categories, and F a map from the objects of C to the objects of D, as well as a map from the morphisms of C to the morphisms of D. Then F is called a *contravariant functor* if for all objects A and morphisms f and g of C,

(Contra) $F(cod(f)) = dom(F(f))$ and $F(dom(f)) = cod(F(f))$,
(Comp) $F(g \circ f) = F(f) \circ F(g)$,
(Id) $F(id_A) = id_{F(A)}$.

Two categories C and D are *dually equivalent* if there exist contravariant functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ such that for all objects A of C we have $A \cong G(F(A))$, and all objects B of D we have $B \cong F(G(B))$.

Duality for monotonic frames

Let MF be the category of monotonic frames with bounded morphisms, and let BAM be the category of BAMs with BAE-homomorphisms. The map $(\cdot)^+$ which sends a monotonic frame to its full complex algebra, is thus a map from the objects of MF to the objects of BAM. In the other direction, given a BAM $\mathbb{A}$ we will define the *ultrafilter frame* $\mathbb{A}_+$ of $\mathbb{A}$ as a monotonic frame such that $\mathbb{A}^\sigma = (\mathbb{A}_+)^+$. Using the representation of BAMs via Stone duality, where $\mathbb{A}^\sigma = (\mathcal{P}(Uf\,\mathbb{A}), \cup, \setminus, \emptyset, f^\sigma)$, we are lead immediately to the following definition.

Definition 7.14 (Ultrafilter frame) Let $\mathbb{A} = (A, +, -, 0, f)$ be a BAM. We define the *ultrafilter frame of* $\mathbb{A}$ by $\mathbb{A}_+ = (Uf\,\mathbb{A}, \nu_f)$ where $Uf\,\mathbb{A}$ is the set of ultrafilters of $\mathbb{A}$, and for all $u \in Uf\,\mathbb{A}, X \subseteq Uf\,\mathbb{A}$:

(29) $X \in \nu_f(u)$ iff $u \in f^\sigma(X)$.

$\dashv$

From Definition 7.14, it is clear that $\mathbb{A}_+$ is a monotonic frame and $\mathbb{A}^\sigma = (\mathbb{A}_+)^+$. For the different types of elements of $\mathcal{P}(Uf\,\mathbb{A})$, (29) reduces to the following:

(30) $\widehat{a} \in \nu_f(u) \Leftrightarrow f(a) \in u$, for all clopens $\widehat{a} \in \widehat{A}$.

(31) $C \in \nu_f(u) \Leftrightarrow \forall a \in A : C \subseteq \widehat{a} \to f(a) \in u$, for all closed $C \in K(\mathbb{A}^\sigma)$.

(32) $X \in \nu_f(u) \Leftrightarrow \exists C \in K(\mathbb{A}^\sigma) : C \subseteq X \;\&\; C \in \nu_f(u)$, for arbitrary $X \in \mathcal{P}(Uf\,\mathbb{A})$.

Remark 7.15 The three stages in the definition of ν_f should look familiar by now, and if we recall the definition of ν_{ue} in the ultrafilter extension of a monotonic frame $\mathbb{F}$ (Definition 4.34 on pg. 30), then we see that $\mathfrak{ue}\mathbb{F} = (\mathbb{F}^+)_+$.

Also recall Definition 6.2 of the canonical frame $\mathbb{F}^\Lambda = (W^\Lambda, \nu^\Lambda)$ and the definition of the Lindenbaum-Tarski algebra $\mathbb{L}_\Lambda(\Phi) = (Ter(\Phi)/\equiv_\Lambda, +, -, 0, f_\nabla)$ of a monotonic logic Λ for some countably infinite set of proposition letters Φ. Let L denote $Ter(\Phi)/\equiv_\Lambda$. It is well-known that the map $\theta : W^\Lambda \to \mathcal{P}(L)$ defined by

(33) $\theta(\Gamma) = \{[\varphi] \mid \varphi \in \Gamma\}$

is a bijection between maximal Λ-consistent sets and ultrafilters of $\mathbb{L}_\Lambda(\Phi)$, and it is a straightforward, but labourious, task to show that $\mathbb{F}^\Lambda(\Phi) \cong (\mathbb{L}_\Lambda(\Phi))_+$. Due to Proposition 7.2, it is thus clear that if Λ is a canonical monotonic modal logic, then $\mathbb{F}^\Lambda(\Phi) \Vdash \Lambda$. But as we have mentioned already, the question of whether $\mathbb{F}^\Lambda(\Phi) \Vdash \Lambda$ implies that Λ is canonical, in the sense of Definition 7.11, is still an open problem.

We have now defined the maps $(\cdot)^+$ and $(\cdot)_+$ on the objects of the two categories MF and BAM, but we still need to define $(\cdot)^+$ and $(\cdot)_+$ on the morphisms.

Definition 7.16 (Lifting morphisms) If $\mathbb{F} = (W, \nu)$ and $\mathbb{F}' = (W', \nu')$ are monotonic frames, and ϑ is a map from W to W', then we define the *lift of* ϑ *to complex algebras*, $\vartheta^+ : \mathcal{P}(W') \to \mathcal{P}(W)$, as

(34) $\vartheta^+(X') := \vartheta^{-1}[X'] = \{x \in W \mid \vartheta(x) \in X'\}$

If $\mathbb{A} = (A, +, -, 0, f)$ and $\mathbb{A}' = (A', +, -, 0, f')$ are BAMs, and $\eta : \mathbb{A} \to \mathbb{A}'$ a map from A to A', then we define the *lift of η to ultrafilters*, $\eta_+ : Uf\mathbb{A}' \to \mathcal{P}(A)$ as

(35) $\eta_+(u') := \eta^{-1}[u'] = \{a \in A \mid \eta(a) \in u'\}$ ⊣

It can easily be checked that when $\eta : \mathbb{A} \to \mathbb{A}'$ is a boolean homomorphism, then

(36) $(\eta_+)^+ = \eta^\sigma : \mathbb{A}^\sigma \to \mathbb{A}'^\sigma$.

We will now show that $(\cdot)^+$ and $(\cdot)_+$ are indeed maps between the morphisms of MF and BAM.

Proposition 7.17 *Let $\mathbb{F}$ and $\mathbb{F}'$ be monotonic frames, and $\vartheta : \mathbb{F} \to \mathbb{F}'$ a bounded morphism. Then the following hold:*

(i) ϑ^+ is a BAE*-homomorphism from $\mathbb{F}^+$ to $\mathbb{F}'^+$.*
(ii) If ϑ is injective, then ϑ^+ is surjective.
(iii) If ϑ is surjective, then ϑ^+ is injective.

Proof. Let $\mathbb{F} = (W, \nu)$ and $\mathbb{F}' = (W', \nu')$ and ϑ be as above. For the proof of (i), we leave out the details regarding the boolean part, which are standard. It remains to show that ϑ satisfies: $m_\nu(\vartheta^+(X')) = \vartheta^+(m_{\nu'}(X'))$:

$$\begin{array}{rll} x \in \vartheta^+(m_{\nu'}(X')) & \text{iff} & \vartheta(x) \in m_{\nu'}(X') \\ & \text{iff} & X' \in \nu'(\vartheta(x)) \\ \text{(Remark 4.4)} & \text{iff} & \vartheta^+(X') \in \nu(x) \\ & \text{iff} & x \in m_\nu(\vartheta^+(X')). \end{array}$$

The proofs of (ii) and (iii) are standard, and basically follow from the fact that $\vartheta^+(\vartheta[X]) = \vartheta^{-1}[\vartheta[X]] = X$ when ϑ is injective, and $\vartheta[\vartheta^+(X')] = \vartheta[\vartheta^{-1}[X']] = X'$ when ϑ is surjective. We leave out the details. QED

For the next proposition, we will need the following lemma.

Lemma 7.18 *Let $\mathbb{A} = (A, +, -, 0, f)$ be a* BAM *and D a downwards directed set of closed sets in $S_\mathbb{A}$. Then*

$$f^\sigma(\bigcap_{d \in D} d) = \bigcap_{d \in D} f^\sigma(d).$$

Proof. Let $C = \bigcap_{d \in D} d$, then C is closed and $f^\sigma(C) = \bigcap_{C \subseteq \widehat{a}} f^\sigma(\widehat{a})$. For the inclusion "$\subseteq$", assume $u \in f^\sigma(C)$. We must show that

$$\forall d \in D : u \in f^\sigma(d) = \bigcap_{d \subseteq \widehat{b}} f^\sigma(\widehat{b}).$$

So take an arbitrary $d \in D$ and an arbitrary b such that $d \subseteq \widehat{b}$. From $C \subseteq d \subseteq \widehat{b}$, it follows from the monotonicity of f^σ and the assumption, $u \in f^\sigma(C)$, that $u \in f^\sigma(\widehat{b})$.

We will show the other inclusion "$\supseteq$" by contraposition. So assume there is an $a_0 \in A$ such that $C \subseteq \widehat{a_0}$ and $u \notin f^\sigma(\widehat{a_0})$. We must now find a $d \in D$ such that $u \notin f^\sigma(d)$.

Claim 2 There is a $d_0 \in D$ such that $d_0 \subseteq \widehat{a_0}$.

Proof of Claim From $C = \bigcap_{d \in D} d \subseteq \widehat{a_0}$ it follows that

$$-\bigcap_{d \in D} d = \bigcup_{d \in D} -d \supseteq \widehat{-a_0}.$$

So $\{-d \mid d \in D\}$ is an open covering of $\widehat{-a_0}$. In Stone spaces, closed sets are compact, hence by compactness of $\widehat{-a_0}$, there is a $D_0 \subseteq_\omega D$ such that $\bigcup_{d \in D_0} -d \supseteq \widehat{-a_0}$ and we have $\bigcap_{d \in D_0} d \subseteq \widehat{a_0}$. By assumption D is downwards directed, so there is a $d_0 \in D$ such that $d_0 \subseteq \bigcap_{d \in D_0} d \subseteq a_0$ ◀

From the claim and the monotonicity of f^σ, we obtain that $f^\sigma(d_0) \subseteq f^\sigma(\widehat{a_0})$, and since $u \notin f^\sigma(\widehat{a_0})$ we may conclude that $u \notin f^\sigma(d_0)$. QED

Proposition 7.19 *Let $\mathbb{A}$, $\mathbb{A}'$ be* BAM*s and $\eta : \mathbb{A} \to \mathbb{A}'$ a* BAE*-homomorphism. Then the following hold:*

(i) η_+ maps ultrafilters to ultrafilters.
(ii) η_+ is a continuous map from $S_{\mathbb{A}'}$ to $S_{\mathbb{A}}$.
(iii) η_+ is a bounded morphism from $\mathbb{A}_+$ to $\mathbb{A}'_+$.
(iv) If η is injective, then η_+ is surjective.
(v) If η is surjective, then η_+ is injective.

Proof. Let $\mathbb{A} = (A, +, -, 1, f)$ and $\mathbb{A}' = (A', +, -, 1, f')$ and η be as in the assumption of the proposition.

To prove (i) and (ii), we only need η to be a boolean homomorphism. The proof of (i) is standard, and we leave out the details. Thus $\eta_+ : \mathit{Uf}\,\mathbb{A}' \to \mathit{Uf}\,\mathbb{A}$. In order to prove (ii), it suffices to prove that $\eta_+^{-1}[C]$ is closed in $S_{\mathbb{A}'}$ if C is closed in $S_{\mathbb{A}}$, but this follows from the observation that $\eta_+^{-1}[C] = (\eta_+)^+(C) = \eta^\sigma(C)$ (36), and the fact that η^σ maps closed sets to closed sets. In particular, we have for an arbitrary closed $C \subseteq \mathit{Uf}\,\mathbb{A}$:

$$(37) \quad \eta_+^{-1}[C] = \bigcap_{C \subseteq \widehat{a}} \widehat{\eta(a)},$$

For (iii), we will start by proving that η_+ satisfies the (BM1) condition of Definition 4.3. So assume that $u' \in \mathit{Uf}\,\mathbb{A}'$ and $\mathit{Uf}\,\mathbb{A}' \supseteq X' \in \nu_{f'}(u')$. In order to show that $\eta_+[X'] \in \nu_f(\eta_+(u'))$, we need a closed $C \subseteq \eta_+[X']$ such that $C \in \nu_f(\eta_+(u'))$. By definition of $\nu_{f'}$ there is a closed $C' \subseteq X'$ such that $C' \in \nu_{f'}(u')$. Take $C = \eta_+[C']$, then clearly $C \subseteq \eta_+[X']$ since $C' \subseteq X'$.

Claim 1 $\eta_+[C']$ is closed in $S_{\mathbb{A}}$ whenever C' is closed in $S_{\mathbb{A}'}$.

Proof of Claim Suppose that C' is closed in $S_{\mathbb{A}'}$. Since Stone spaces are compact and Hausdorff, we only need to prove that $\eta_+[C']$ is compact. Suppose $\{\widehat{a_i} \mid i \in I\}$ is a covering of $\eta_+[C']$.

$$\begin{array}{lll} \eta_+[C'] \subseteq \bigcup_{i \in I} \widehat{a_i} & \text{iff} & \forall u \in C'\, \exists i \in I : \eta_+(u) \in \widehat{a_i} \quad \text{iff} \\ \forall u \in C'\, \exists i \in I : u \in \widehat{\eta(a_i)} & \text{iff} & C' \subseteq \bigcup_{i \in I} \widehat{\eta(a_i)} \end{array}$$

C' is compact, since $S_{\mathbb{A}'}$ is compact and C' closed. Hence there is a finite $I_0 \subseteq_\omega I$ such that $C' \subseteq \bigcup_{i \in I_0} \widehat{\eta(a_i)}$, and it follows that $\eta_+[C'] \subseteq \bigcup_{i \in I_0} \widehat{a_i}$, i.e., $\eta_+[C']$ is compact. ◀

By the above claim, $C = \eta_+[C']$ is closed in $S_{\mathbb{A}}$. It remains to prove that $C \in \nu_f(\eta_+(u'))$, that is, $\forall a \in A : \eta_+[C'] \subseteq \widehat{a} \to f(a) \in \eta_+(u')$. We have

$$\eta_+[C'] \subseteq \widehat{a} \text{ iff } \forall v' \in C' : a \in \eta_+(v') \text{ iff } \forall v' \in C' : \eta(a) \in v' \text{ iff } C' \subseteq \widehat{\eta(a)},$$

which together with $C' \in \nu_{f'}(u')$ implies that if $\eta_+[C'] \subseteq \widehat{a}$, then $\widehat{\eta(a)} \in \nu_{f'}(u')$. By the definition of $\nu_{f'}$ this means that $f'(\eta(a)) \in u'$, and since we assumed that η is a BAE-homomorphism, we get $\eta(f(a)) \in u'$ and hence $f(a) \in \eta_+(u')$.

For the (BM2) condition, assume $X \in \nu_f(\eta_+(u'))$, i.e., there is a closed $C \subseteq X$ such that $C \in \nu_f(\eta_+(u'))$. It suffices to find a closed $C' \subseteq Uf\mathbb{A}'$ such that $\eta_+[C'] \subseteq C$ and $C' \in \nu_{f'}(u')$. Take $C' = \{u' \in Uf\mathbb{A}' \mid \eta_+(u') \in C\}$, then clearly $\eta_+[C'] \subseteq C$, and since η_+ is continuous by (ii), we have that C' is closed. $C' \in \nu_{f'}(u')$ follows from $C \in \nu_f(\eta_+(u'))$ and the following equivalences:

$$\begin{array}{rll}
C \in \nu_f(\eta_+(u')) & \text{iff} & \forall a \in A : C \subseteq \widehat{a} \to f(a) \in \eta_+(u') \\
 & \text{iff} & \forall a \in A : C \subseteq \widehat{a} \to \eta(f(a)) \in u' \\
{\scriptstyle (\eta \text{ is a BAE-hom.})} & \text{iff} & \forall a \in A : C \subseteq \widehat{a} \to f'(\eta(a)) \in u' \\
 & \text{iff} & u' \in \bigcap_{C \subseteq \widehat{a}} \widehat{f'(\eta(a))} = \bigcap_{C \subseteq \widehat{a}} f'^{\sigma}(\widehat{\eta(a)}) \\
{\scriptstyle (\text{Lemma 7.18})} & \text{iff} & u' \in f'^{\sigma}(\bigcap_{C \subseteq \widehat{a}} \widehat{\eta(a)}) \\
{\scriptstyle (37)} & \text{iff} & u' \in f'^{\sigma}(C') \\
 & \text{iff} & C' \in \nu_{f'}(u').
\end{array}$$

The proof of (iv) and (v) is identical to that of the normal BAO case in Proposition 5.52 in Blackburn et alii [6]. QED

Proposition 7.20 *$(\cdot)^+$ is a contravariant functor from* MF *to* BAM*, and $(\cdot)_+$ is a contravariant functor from* BAM *to* MF*.*

Proof. Contravariance is in both cases clear. To see that $(\cdot)^+$ is a functor from MF to BAM, it only remains to show that $(\cdot)^+$ satisfies (Comp) and (Id). In order to prove (Comp), let $\mathbb{F}_1, \mathbb{F}_2$ and $\mathbb{F}_3$ be monotonic frames, $\vartheta_1 : \mathbb{F}_1 \to \mathbb{F}_2$, $\vartheta_2 : \mathbb{F}_2 \to \mathbb{F}_3$ bounded morphisms and assume that X is a subset of the universe of $\mathbb{F}_3$. Then,

$$x \in (\vartheta_2 \circ \vartheta_1)^+(X) \iff \vartheta_2 \circ \vartheta_1(x) \in X \iff \vartheta_1(x) \in \vartheta^+(X) \iff x \in \vartheta_1^+ \circ \vartheta_2^+(X).$$

Hence $(\vartheta_2 \circ \vartheta_1)^+ = \vartheta_1^+ \circ \vartheta_2^+$.

To prove (Id), let $\mathbb{F} = (W, \nu)$ be a monotonic frame, then $id_{\mathbb{F}} : \mathbb{F} \to \mathbb{F}$, and for $X \subseteq W$,

$$(id_{\mathbb{F}})^+(X) = \{x \in W \mid id_{\mathbb{F}}(x) \in X\} = X,$$

hence $(id_{\mathbb{F}})^+ = id_{\mathbb{F}^+}$.

The proof for $(\cdot)_+$ is similar. QED

We can now summarise the results of Propositions 7.17 and 7.19 in terms of duality between frame constructions and algebraic notions. First, we recall (or define) the following notation for relations between structures.

Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be two monotonic frames, and let $\mathbb{A}_1$ and $\mathbb{A}_2$ be two BAMs. Then we write

- $\mathbb{F}_1 \rightarrowtail \mathbb{F}_2$ for $\mathbb{F}_1$ is isomorphic to a generated subframe of $\mathbb{F}_2$,
- $\mathbb{F}_1 \twoheadrightarrow \mathbb{F}_2$ for $\mathbb{F}_2$ is a bounded morphic image of $\mathbb{F}_1$,
- $\mathbb{A}_1 \rightarrowtail \mathbb{A}_2$ for $\mathbb{A}_1$ is isomorphic to a subalgebra of $\mathbb{A}_2$,
- $\mathbb{A}_1 \twoheadrightarrow \mathbb{A}_2$ for $\mathbb{A}_2$ is a homomorphic image of $\mathbb{A}_1$.

Theorem 7.21 (Basic duality for monotonic frames) *Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be two monotonic frames, and let $\mathbb{A}_1$ and $\mathbb{A}_2$ be two* BAM*s. Then we have*

(i) If $\mathbb{F}_1 \rightarrowtail \mathbb{F}_2$, then $\mathbb{F}_2^+ \twoheadrightarrow \mathbb{F}_1^+$,
(ii) If $\mathbb{F}_1 \twoheadrightarrow \mathbb{F}_2$, then $\mathbb{F}_2^+ \rightarrowtail \mathbb{F}_1^+$,
(iii) If $\mathbb{A}_1 \rightarrowtail \mathbb{A}_2$, then $\mathbb{A}_{2+} \twoheadrightarrow \mathbb{A}_{1+}$,
(iv) If $\mathbb{A}_1 \twoheadrightarrow \mathbb{A}_2$, then $\mathbb{A}_{2+} \rightarrowtail \mathbb{A}_{1+}$.

Proof. Follows from Propositions 7.17 and 7.19 above. QED

The following proposition states that the algebraic notion of direct product is the dual notion of disjoint unions of frames.

Proposition 7.22 *Let $\mathbb{F}_i = (W_i, \nu_i)$, $i \in I$, be a collection of disjoint frames. Then*

$$\left(\biguplus_{i\in I} \mathbb{F}_i\right)^+ \cong \prod_{i\in I} \mathbb{F}_i^+ .$$

Proof. Let $\mathbb{F} = (\biguplus_{i\in I} W_i, \nu) = \biguplus_{i\in I} \mathbb{F}_i$, then

$$\mathbb{F}^+ = (\mathcal{P}(\biguplus_{i\in I} W_i), +, -, 0, f).$$

Let

$$\mathbb{F}_i^+ = (\mathcal{P}(W_i), +, -, 0, f_i)$$

and

$$\prod_{i\in I} \mathbb{F}_i^+ = (\prod_{i\in I} \mathcal{P}(W_i), +, -, 0, f').$$

Note that for $\overline{X} = (X_i)_{i\in I} \in \prod_{i\in I} \mathcal{P}(W_i)$:

$$f'(\overline{X}) = (f_i(X_i))_{i\in I}.$$

Define the map

$$\begin{array}{rccc} \eta: & \mathcal{P}(\biguplus_{i\in I} W_i) & \to & \prod_{i\in I} \mathcal{P}(W_i) \\ & X & \mapsto & (X \cap W_i)_{i\in I} \end{array}$$

We claim that η is the desired isomorphism.

Injectivity: Let $X, Y \subseteq \biguplus_{i\in I} W_i$ and suppose $\eta(X) = \eta(Y)$. Then $\forall i \in I : X \cap W_i = Y \cap W_i$. Since $X = \biguplus_{i\in I} X \cap W_i$ and $Y = \biguplus_{i\in I} Y \cap W_i$, it follows that $X = Y$.

Surjectivity: Let $(Y_i)_{i\in I} \in \prod_{i\in I} \mathcal{P}(W_i)$, we then need an $X \subseteq \biguplus_{i\in I} W_i$ such that $\forall i \in I$: $X \cap W_i = Y_i$. But X is easily obtained by taking $X = \biguplus_{i\in I} Y_i$.

Homomorphism: We need to show for $X \subseteq \biguplus_{i\in I} W_i$ that $\eta(f(X)) = f'(\eta(X))$. We have,

$$f(X) = m_\nu(X) = \{x \in \biguplus_{i\in I} W_i \mid X \in \nu(x)\}.$$

By Definition 4.1 we have for $x_i \in W_i$,

$$X \in \nu(x_i) \text{ iff } X \cap W_i \in \nu_i(x_i).$$

Products are defined coordinatewise, so it suffices to show that for all $i \in I$,

$$f(X) \cap W_i = \eta(f(X))(i) = f'(\eta(X))(i) = f_i(X \cap W_i).$$

We have,

$$\begin{aligned} x \in f(X) \cap W_i \quad &\text{iff} \quad x \in W_i \;\&\; X \cap W_i \in \nu_i(x) \\ &\text{iff} \quad x \in f_i(X \cap W_i). \end{aligned}$$

QED

The results on preservation of frame validity in Proposition 5.3 can now be proved as simple consequences of the above duality results, and the fact that equational validity is preserved under the formation of subalgebras, homomorphic images and direct products of algebras. We leave it to the reader to work out the details, and instead we move on to the proof of Theorem 5.4, which we restate here.

Theorem 7.23 (Monotonic frame definability) *Let K be a class of monotonic frames which is closed under taking ultrafilter extensions. Then K is modally definable iff K is closed under disjoint unions, generated subframes and bounded morphic images, and reflects ultrafilter extensions.*

Proof. The direction from left to right follows from the previous results on preservation of validity in Proposition 5.3. For the other direction, we will show that if K has the mentioned closure properties, then $\mathsf{Th}(\mathsf{K})$ defines K. So assume $\mathbb{F} \Vdash \mathsf{Th}(\mathsf{K})$, we must then show that $\mathbb{F} \in \mathsf{K}$. From $\mathbb{F} \Vdash \mathsf{Th}(\mathsf{K})$ it follows that $\mathbb{F}^+ \models \mathsf{EqTh}(\mathbf{Cm}\mathsf{K})$, and by Birkhoff's theorem, $\mathbb{F}^+ \in \mathbf{HSPCm}\mathsf{K}$. Hence there is a collection $\mathbb{B}_i$, $i \in I$, of frames in K and a BAM $\mathbb{A}$ such that:

$$\mathbb{F}^+ \twoheadleftarrow \mathbb{A} \rightarrowtail \prod_{i \in I} \mathbb{B}_i^+ \cong \left(\biguplus_{i \in I} \mathbb{B}_i \right)^+,$$

using Proposition 7.22. Let $\mathbb{B} = \biguplus_{i \in I} \mathbb{B}_i$, then by the assumption that K is closed under disjoint unions, $\mathbb{B} \in \mathsf{K}$. Now apply the duality results from Propositions 7.17 and 7.19:

$$\mathfrak{ue}\mathbb{F} = (\mathbb{F}^+)_+ \rightarrowtail \mathbb{A}_+ \twoheadleftarrow (\mathbb{B}^+)_+ = \mathfrak{ue}\mathbb{B}$$

Since we assumed K is closed under taking ultrafilter extensions, $\mathfrak{ue}\mathbb{B} \in \mathsf{K}$. Closure of K under bounded morphic images yields $\mathbb{A}_+ \in \mathsf{K}$, and closure under generated subframes implies that $\mathfrak{ue}\mathbb{F} \in \mathsf{K}$. Finally, since K reflects ultrafilter extensions, we may conclude that $\mathbb{F} \in \mathsf{K}$. QED

Remark 7.24 It should perhaps be pointed out that we have not shown that the categories MF and BAM are dually equivalent via the functors $(\cdot)^+$ and $(\cdot)_+$, since $\mathbb{A} \ncong (\mathbb{A}_+)^+ = \mathbb{A}^\sigma$ and $\mathbb{F} \ncong (\mathbb{F}^+)_+ = \mathfrak{ue}\mathbb{F}$, for $\mathbb{F}$ with infinitely many states. In [18], Došen shows that the category of complete, atomic BAEs with complete BAE-homomorphisms is dually equivalent with the category of neighbourhood frames with bounded morphisms by taking atom structure when going from algebras to frames. This result easily adapts to the monotonic case, but fits less nicely into our context where we wish to be able to obtain a frame from a BAM even if the BAM is not atomic. This is necessary when describing the canonical extension of an arbitrary BAM $\mathbb{A}$ as $(\mathbb{A}_+)^+$. Our way of obtaining $\mathbb{A}_+$ is, actually, the same as taking the atom structure of $\mathbb{A}^\sigma$.

7.5 Full Duality for General Monotonic Frames

As we already know, general frames are structures which bring together frames and algebras, and we will now extend our constructions of the previous subsection to general frames. In the duality theory of normal modal logic, Goldblatt [29, 31] introduced the notion of descriptive general Kripke frames to obtain full duality for the category of BAOs and BAO-homomorphisms. Below we will define a notion of descriptiveness for general monotonic frames for the same purpose. That is, we will show that BAM is dually equivalent with the category of descriptive general monotonic frames (Theorem 7.36).

We start by defining maps which take general frames to BAMs or monotonic frames and back.

Definition 7.25 Let $\mathbb{G} = (W, \nu, A)$ be a general monotonic frame, then we define the *underlying* BAM *of* $\mathbb{G}$ as

$$\mathbb{G}^* = (A, \cup, \setminus, \emptyset, m_\nu).$$

For a BAM $\mathbb{A} = (A, +, -, 0, f)$, the *general ultrafilter frame* of $\mathbb{A}$ is defined as

$$\mathbb{A}_* = (\mathbb{A}_+, \widehat{A}).$$

where $\widehat{A} = \{\widehat{a} \mid a \in A\}$ is the image of A under the map $a \mapsto \widehat{a} = \{u \in Uf\mathbb{A} \mid a \in u\}$.

Furthermore, the *underlying monotonic frame of* $\mathbb{G}$ is defined as

$$\mathbb{G}_\sharp = (W, \nu),$$

and finally, for a monotonic frame $\mathbb{F} = (W, \nu)$, we define the *full general monotonic frame of* $\mathbb{F}$ as

$$\mathbb{F}^\sharp = (W, \nu, \mathcal{P}(W)).$$

⊣

There are some obvious relationships between the maps $(\cdot)_+$, $(\cdot)^+$ of the previous subsection, and the above. We list a few,

$$\begin{aligned} (\mathbb{A}_*)_\sharp &= \mathbb{A}_+, \\ ((\mathbb{G}_\sharp)^\sharp)^* &= (\mathbb{G}_\sharp)^+, \\ (((\mathbb{F}^\sharp)^*)_*)_\sharp &= \mathbb{F}_+. \end{aligned}$$

We have the following analogue of Proposition 7.2.

Proposition 7.26 *Let $\mathbb{G}$ be a general monotonic frame, and let $\mathbb{A}$ be a* BAM. *Then for every formula φ:*

$$\begin{array}{lrcl} (i) & \mathbb{G} \Vdash \varphi & \textit{iff} & \mathbb{G}^* \models \varphi \approx \top, \\ (ii) & \mathbb{A} \models \varphi \approx \top & \textit{iff} & \mathbb{A}_* \Vdash \varphi. \end{array}$$

Proof. Left to the reader. QED

A bounded morphism between general monotonic frames must not only preserve the underlying frame structure, but also the structure of the underlying BAM.

Definition 7.27 Let $\mathbb{G}_1 = (W_1, \nu_1, A_1)$ and $\mathbb{G}_2 = (W_2, \nu_2, A_2)$ be two general monotonic frames, and $\theta : W_1 \to W_2$ a map. Then θ is a *bounded morphism between the general frames* $\mathbb{G}_1$ *and* $\mathbb{G}_2$ (notation: $\theta : \mathbb{G}_1 \to \mathbb{G}_2$) if θ is a bounded morphism between the monotonic frames $\mathbb{F}_1 = (W_1, \nu_1)$ and $\mathbb{F}_2 = (W_2, \nu_2)$, and θ also satisfies the following condition,

(38) $\theta^{-1}[a_2] \in A_1$ for all $a_2 \in A_2$.

If θ is an injective bounded morphism between $\mathbb{G}_1$ and $\mathbb{G}_2$, and θ satisfies

(39) for all $a_1 \in A_1$ there is an $a_2 \in A_2$ such that $\theta[a_1] = \theta[W_1] \cap a_2$,

then θ is an *embedding* of $\mathbb{G}_1$ in $\mathbb{G}_2$ (notation: $\theta : \mathbb{G}_1 \rightarrowtail \mathbb{G}_2$).

$\mathbb{G}_2$ is called a *bounded morphic image of* $\mathbb{G}_1$ if there is a surjective bounded morphism from $\mathbb{G}_1$ to $\mathbb{G}_2$ (notation: $\mathbb{G}_1 \twoheadrightarrow \mathbb{G}_2$). Finally, $\mathbb{G}_1$ and $\mathbb{G}_2$ are *isomorphic* is there exists a surjective embedding from $\mathbb{G}_1$ to $\mathbb{G}_2$. $\dashv$

Lifting maps between general monotonic frames and BAMs is done in the same way as for monotonic frames. More precisely, if $\mathbb{G} = (W, \nu, A)$ and $\mathbb{G}' = (W', \nu', A')$ are general monotonic frames, and θ is a map from W to W', then we define $\theta^* : A' \to \mathcal{P}(W)$, as

(40) $\theta^*(a') := \theta^{-1}[a'] = \{x \in W \mid \theta(x) \in a'\}$

If $\mathbb{A} = (A, +, -, 0, f)$ and $\mathbb{A}' = (A', +, -, 0, f')$ are BAMs, and $\eta : \mathbb{A} \to \mathbb{A}'$ a map from A to A', then we define $\eta_* : Uf\mathbb{A}' \to \mathcal{P}(A)$ as

(41) $\eta_*(u') := \eta^{-1}[u'] = \{a \in A \mid \eta(a) \in u'\}$

The following propositions are now easy extensions of Propositions 7.17 and 7.19.

Proposition 7.28 *Let $\mathbb{G}_1 = (W_1, \nu_1, A_1)$ and $\mathbb{G}_2 = (W_2, \nu_2, A_2)$ be two general monotonic frames, and $\vartheta : \mathbb{G}_1 \to \mathbb{G}_2$ a bounded morphism. Then*

(i) ϑ^* *is a* BAE*-homomorphism from* $\mathbb{G}_2^*$ *to* $\mathbb{G}_1^*$.
(ii) *If* $\vartheta : \mathbb{G}_1 \rightarrowtail \mathbb{G}_2$, *then* $\vartheta^* : \mathbb{G}_2^* \twoheadrightarrow \mathbb{G}_1^*$.
(iii) *If* $\vartheta : \mathbb{G}_1 \twoheadrightarrow \mathbb{G}_2$, *then* $\vartheta^* : \mathbb{G}_2^* \rightarrowtail \mathbb{G}_1^*$.

Proof. (i) and (iii) may be shown as in the proof of Proposition 7.17. For (i), note that since ϑ is a bounded morphism between $\mathbb{G}_1$ and $\mathbb{G}_2$, then (38) ensures that $\vartheta^*(a_2) \in A_1$,

To prove (ii), assume that ϑ is an embedding of $\mathbb{G}_1$ into $\mathbb{G}_2$. From (i) we know that ϑ^* is a BAE-homomorphism, hence we only need to show surjectivity. So let $a_1 \in A_1$. As ϑ is an embedding, there is an $a_2 \in A_2$ such that $\vartheta[a_1] = \vartheta[W_1] \cap a_2$. It is now easy to show that $\vartheta^*(a_2) = a_1$:

$$x \in a_1 \;\Leftrightarrow\; \vartheta(x) \in \vartheta[a_1] \;\Leftrightarrow\; \vartheta(x) \in \vartheta[W_1] \cap a_2 \;\Leftrightarrow\; x \in \vartheta^*(a_2).$$

The first equivalence follows from the injectivity of ϑ. QED

Proposition 7.29 *Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be two* BAM*s, and $\eta : \mathbb{A}_1 \to \mathbb{A}_2$ a* BAE*-homomorphism. Then*

(i) η_* *is a bounded morphism from* ${\mathbb{A}_2}_*$ *to* ${\mathbb{A}_1}_*$.
(ii) *If* $\eta : \mathbb{A}_1 \rightarrowtail \mathbb{A}_2$, *then* $\eta_* : {\mathbb{A}_2}_* \twoheadrightarrow {\mathbb{A}_1}_*$.
(iii) *If* $\eta : \mathbb{A}_1 \twoheadrightarrow \mathbb{A}_2$, *then* $\eta_* : {\mathbb{A}_2}_* \rightarrowtail {\mathbb{A}_1}_*$.

Proof. Let $\mathbb{A}_1 = (A_1, +, -, 0, f_1)$, $\mathbb{A}_2 = (A_2, +, -, 0, f_2)$ and $\mathbb{A}_{1*} = (Uf\mathbb{A}_1, \nu_1, \widehat{A_1})$, $\mathbb{A}_{2*} = (Uf\mathbb{A}_2, \nu_2, \widehat{A_2})$.

For the proof of (i), we already know from Proposition 7.19 that η_* is a bounded morphism between the underlying monotonic frames of $\mathbb{A}_{2*}$ and $\mathbb{A}_{1*}$. Hence it only remains to show that (38) holds. But this is immediate, since for $\widehat{a_1} \in \widehat{A_1}$, ${\eta_*}^{-1}[\widehat{a_1}] = \eta^\sigma(\widehat{a_1}) = \widehat{\eta(a_1)} \in \widehat{A_2}$.

(ii) may be shown as in the proof of Proposition 7.19. For the proof of (iii), we only need to show that (39) holds, since η_* is a bounded morphism by (i), and injectivity follows as in Proposition 7.19. So let $\widehat{a_2} \in \widehat{A_2}$. Then by the surjectivity of η there is an $a_1 \in A_1$ such that $\eta(a_1) = a_2$, and hence $\widehat{\eta(a_1)} = \widehat{a_2}$. We will show that $\eta_*[\widehat{a_2}] = \eta_*[Uf\mathbb{A}_2] \cap \widehat{a_1}$:

$$
\begin{array}{ll}
 & u_1 \in \eta_*[\widehat{a_2}] \\
\text{iff} & \exists u_2 \in Uf\mathbb{A}_2 : u_2 \in \widehat{a_2} \;\&\; \eta_*(u_2) = u_1 \\
\text{iff} & \exists u_2 \in Uf\mathbb{A}_2 : u_2 \in \widehat{\eta(a_1)} \;\&\; \eta_*(u_2) = u_1 \\
\text{iff} & \exists u_2 \in Uf\mathbb{A}_2 : \eta(a_1) \in u_2 \;\&\; \eta_*(u_2) = u_1 \\
\text{iff} & \exists u_2 \in Uf\mathbb{A}_2 : a_1 \in \eta_*(u_2) \;\&\; \eta_*(u_2) = u_1 \\
\text{iff} & u_1 \in \eta_*[Uf\mathbb{A}_2] \cap \widehat{a_1}.
\end{array}
$$

QED

(Comp) and (Id) are shown for $(\cdot)^*$ and $(\cdot)_*$ in the same easy way as for $(\cdot)^+$ and $(\cdot)_+$, thus it is clear that $(\cdot)^*$ and $(\cdot)_*$ are contravariant functors between the category of general monotonic frames with bounded morphisms and the category BAM. Before we define what it means for a general monotonic frame to be descriptive, recall that in a general monotonic frame $\mathbb{G} = (W, \nu, A)$, the admissible sets A may be taken as the basis for the open sets of a topology τ_A on W. We will refer to $\mathbb{W} = (W, \tau_A)$ as the *topological space of* $\mathbb{G}$, the collections of closed and open subsets in $\mathbb{W}$ will be denoted by $K(\mathbb{W})$ and $O(\mathbb{W})$, respectively.

Definition 7.30 (Properties of general monotonic frames) Let $\mathbb{G} = (W, \nu, A)$ be a general monotonic frame. Then $\mathbb{G}$ is called

differentiated if for all $w, v \in W$:

$$w = v \;\text{ iff }\; \forall a \in A (w \in a \;\Leftrightarrow\; v \in a),$$

tight if for all $w \in W$, all $C \in K(\mathbb{W})$ and all $X \subseteq W$,

$$
\begin{array}{lll}
C \in \nu(w) & \text{iff} & \forall a \in A (C \subseteq a \to a \in \nu(w)), \\
X \in \nu(w) & \text{iff} & \exists C \in K(\mathbb{W}) (C \subseteq X \;\&\; C \in \nu(w)).
\end{array}
$$

compact if for all $A' \subseteq A$, $\bigcap A' \neq \emptyset$ if A' has the finite intersection property,

descriptive if $\mathbb{G}$ is differentiated, tight and compact,

full if $A = \mathcal{P}(W)$.

$\dashv$

For brevity, we will refer to descriptive general monotonic frames simply as descriptive monotonic frames. Note that only the condition of tightness differs from the corresponding properties of general Kripke frames, hence applying our knowledge about general Kripke frames, if $\mathbb{G}$ is differentiated and compact, then $\mathbb{W}$ is a Stone space where A forms a clopen basis, and the map $p : W \to Uf\mathbb{A}$, where $\mathbb{A} = \mathbb{G}^*$, defined by

$$p : w \mapsto U_w = \{a \in A \mid w \in A\}$$

is a bijection (see for example [29]). The tightness condition for a general Kripke frame (W, R, A) ensures that the accessibility relation R is point-closed. That is, the set $R[w]$ of R-successors of w is closed in $\mathbb{W}$, see for example Blackburn et alii [6]. The following lemma may be seen as the monotonic frame analogue hereof.

Lemma 7.31 *Let $\mathbb{G} = (W, \nu, A)$ be a tight monotonic frame, and $\mathbb{W} = (W, \tau_A)$ the topological space of $\mathbb{G}$. Then we have for all $w \in W$, $\nu^c(w) \subseteq K(\mathbb{W})$. In other words, all core neighbourhoods are closed in $\mathbb{W}$.*

Proof. Assume that $X \in \nu^c(w)$, that is, for all $Y \subsetneq X$, $Y \notin \nu(w)$. By the tightness of $\mathbb{G}$, there is a $C \in K(\mathbb{W})$ such that $C \subseteq X$ and $C \in \nu(w)$. Hence $X = C \in K(\mathbb{W})$. QED

Let DMF be the category of descriptive monotonic frames with bounded morphisms. It is clear that for any object $\mathbb{G}$ in DMF, $\mathbb{G}^*$ is a BAM. But it is perhaps less obvious that $\mathbb{A}_*$ is descriptive, when $\mathbb{A}$ is a BAM.

Proposition 7.32 *Let $\mathbb{A}$ be a* BAM. *Then $\mathbb{A}_*$ is a descriptive monotonic frame.*

Proof. Let $\mathbb{A} = (A, +, -, 0, f)$ and $\mathbb{A}_* = (Uf\mathbb{A}, \nu_f, \widehat{A})$. It is easy to see that $\mathbb{A}_*$ is indeed a general monotonic frame: Closure of $\widehat{A}$ under union and complement is immediate from the closure of A, and closure of $\widehat{A}$ under m_{ν_f} follows from

$$u \in m_{\nu_f}(\widehat{a}) \iff \widehat{a} \in \nu_f(u) \iff u \in f^\sigma(\widehat{a}) = \widehat{f(a)}.$$

To see that $\mathbb{A}_*$ is differentiated, let $v, u \in Uf\mathbb{A}$, and suppose $v \neq u$. Then we may assume that there is an $a \in v \setminus u$, and it follows that $v \in \widehat{a}$ and $u \notin \widehat{a}$. For compactness, let $\widehat{B} \subseteq \widehat{A}$, i.e, $B \subseteq A$, and assume that $\widehat{B}$ has the finite intersection property. Then it easily follows that B has the finite meet property and hence B can be extended to an ultrafilter $u \in Uf\mathbb{A}$. Thus we have for all $b \in B$, $u \in \widehat{b}$, i.e., $u \in \bigcap \widehat{B}$, so $\bigcap \widehat{B} \neq \emptyset$.

From the definition of ν_f and (30), (31), (32) it is immediate that $\mathbb{A}_*$ is tight. QED

Proposition 7.33 *$(\cdot)^*$ is a contravariant functor from* DMF *to* BAM*, and $(\cdot)_*$ is a contravariant functor from* BAM *to* DMF*.*

Proof. Follows from Propositions 7.28, 7.29 and 7.32. QED

From Propositions 7.26 and 7.32, we also immediately obtain the following.

Corollary 7.34 *Let $\mathbb{G}$ be a general monotonic frame. Then $(\mathbb{G}^*)_*$ is a descriptive monotonic frame equivalent to $\mathbb{G}$. That is for every formula φ,*

$$\mathbb{G} \Vdash \varphi \text{ iff } (\mathbb{G}^*)_* \Vdash \varphi.$$

The crucial property of the functors $(\cdot)^*$ and $(\cdot)_*$, which is needed to show that BAM and DMF are dually equivalent, is stated in the following theorem.

Theorem 7.35 *Let $\mathbb{A}$ be a* BAM *and let $\mathbb{G}$ be a descriptive monotonic frame. Then*

(i) $\mathbb{A} \cong (\mathbb{A}_*)^*$,
(ii) $\mathbb{G} \cong (\mathbb{G}^*)_*$.

Proof. The proof of (i) is standard, and we leave it to the reader to verify that the map $r : \mathbb{A} \to (\mathbb{A}_*)^*$ given by $r(a) = \widehat{a} = \{u \in Uf\,\mathbb{A} \mid a \in u\}$ is a BAE-isomorphism.

For (ii), let $\mathbb{G} = (W, \nu, A)$ be a descriptive monotonic frame, and let $\mathbb{A} = \mathbb{G}^* = (A, \cup, \setminus, \emptyset, m_\nu)$ and $(\mathbb{G}^*)_* = (Uf\,\mathbb{A}, \nu_{m_\nu}, \widehat{A})$. We will show that the map

$$\begin{array}{rccl} p: & \mathbb{G} & \to & (\mathbb{G}^*)_* \\ & x & \mapsto & U_x = \{a \in A \mid x \in a\} \end{array}$$

is the desired isomorphism. As already mentioned, differentiation and compactness of $\mathbb{G}$ ensure that p is a bijection. Thus it only remains to prove that p is a bounded morphism which satisfies (39). In order for p to be a bounded morphism of the underlying frames, it suffices to show for all $X \subseteq Uf\,\mathbb{A}$ (cf. Remark 4.4),

(42) $p^{-1}[X] \in \nu(w)$ iff $X \in \nu_{m_\nu}(U_w)$.

We first note that for $a \in A$, $p[a] = \widehat{a}$, since

$$U_x \in p[a] \iff x \in a \iff a \in U_x \iff U_x \in \widehat{a},$$

and hence also $p^{-1}[\widehat{a}] = a$, since p is a bijection. It is now easy to show (42) for clopen $X = \widehat{a}$:

$$p^{-1}[\widehat{a}] = a \in \nu(w) \iff w \in m_\nu(a) \iff m_\nu(a) \in U_w \iff \widehat{a} \in \nu_{m_\nu}(U_w).$$

It is also easy to verify that

$$\begin{array}{rcll} p[C] & = & \bigcap_{C \subseteq a} \widehat{a}, & \text{for all } C \in K(\mathbb{W}), \\ p^{-1}[D] & = & \bigcap_{D \subseteq \widehat{a}} a, & \text{for all } D \in K(\mathbb{A}^\sigma). \end{array}$$

And it follows that

(43) $p[C] \in K(\mathbb{A}^\sigma)$ iff $C \in K(\mathbb{W})$.

To show (42) for arbitrary $X \subseteq Uf\,\mathbb{A}$, we now have

$$\begin{array}{rll} & & p^{-1}[X] \in \nu(w) \\ \text{($\mathbb{G}$ tight)} & \text{iff} & \exists C \in K(\mathbb{W})(C \subseteq p^{-1}[X] \;\&\; \forall a \in A(C \subseteq a \to a \in \nu(w))) \\ \text{(clopen case)} & \text{iff} & \exists C \in K(\mathbb{W})(C \subseteq p^{-1}[X] \;\&\; \forall a \in A(C \subseteq a \to \widehat{a} \in \nu_{m_\nu}(U_w))) \\ \text{(p bijective)} & \text{iff} & \exists C \in K(\mathbb{W})(p[C] \subseteq X \;\&\; \forall a \in A(p[C] \subseteq \widehat{a} \to \widehat{a} \in \nu_{m_\nu}(U_w))) \\ \text{(43)} & \text{iff} & \exists D \in K(\mathbb{A}^\sigma)(D \subseteq X \;\&\; \forall a \in A(D \subseteq \widehat{a} \to \widehat{a} \in \nu_{m_\nu}(U_w))) \\ & \text{iff} & X \in \nu_{m_\nu}(U_w). \end{array}$$

Furthermore, from $p^{-1}[\widehat{a}] = a$, it follows that for all $\widehat{a} \in \widehat{A}$, $p^{-1}[\widehat{a}] \in A$, so p satisfies (38), and is hence a bounded morphism from $\mathbb{G}$ to $(\mathbb{G}^*)_*$. Finally, from $p[a] = \widehat{a}$ it is clear that p satisfies (39):

$$p[a] = \widehat{a} = Uf\,\mathbb{A} \cap \widehat{a} = p[W] \cap \widehat{a}.$$

Thus we may conclude that p is an embedding. QED

We can now state the announced dual equivalence result.

Theorem 7.36 *The categories* DMF *and* BAM *are dually equivalent via the functors* $(\cdot)^* : \mathsf{DMF} \to \mathsf{BAM}$ *and* $(\cdot)_* : \mathsf{BAM} \to \mathsf{DMF}$.

Proof. Follows from Propositions 7.28, 7.29 and Theorem 7.35. QED

Combining Propositions 7.28, 7.29 and Theorem 7.35, the duality for descriptive frames and BAMs is now summarised in the following.

Proposition 7.37 (Duality for descriptive frames) *Let* $\mathbb{G}_1$ *and* $\mathbb{G}_2$ *be two descriptive monotonic frames, and* $\mathbb{A}_1$ *and* $\mathbb{A}_2$ *be two* BAM*s. Then*

- *(i)* $\mathbb{G}_1 \rightarrowtail \mathbb{G}_2$ *iff* $\mathbb{G}_2^* \twoheadrightarrow \mathbb{G}_1^*$.
- *(ii)* $\mathbb{G}_1 \twoheadrightarrow \mathbb{G}_2$ *iff* $\mathbb{G}_2^* \rightarrowtail \mathbb{G}_1^*$.
- *(iii)* $\mathbb{A}_1 \rightarrowtail \mathbb{A}_2$ *iff* $\mathbb{A}_{2*} \twoheadrightarrow \mathbb{A}_{1*}$.
- *(iv)* $\mathbb{A}_1 \twoheadrightarrow \mathbb{A}_2$ *iff* $\mathbb{A}_{2*} \rightarrowtail \mathbb{A}_{1*}$.

Remark 7.38 In Remark 7.15, we mentioned that for a monotonic modal logic Λ, and a countably infinite set of proposition letters Φ, we have $\mathbb{F}^\Lambda(\Phi) \cong (\mathbb{L}_\Lambda(\Phi))_+$. In the same way, it is easy to see that the general canonical frame of Λ, $\mathbb{G}^\Lambda(\Phi)$, is isomorphic to $(\mathbb{L}_\Lambda(\Phi))_*$. From Proposition 7.32, we know that $(\mathbb{L}_\Lambda(\Phi))_*$ is descriptive, hence together with Theorem 6.9, the following theorem is immediate.

Theorem 7.39 *Let* Λ *be a monotonic modal logic. Then* Λ *is sound and strongly complete with respect to the class of descriptive monotonic* Λ*-frames.*

Remark 7.40 Došen's [18] definition of 'descriptive' equals 'differentiated+compact' in our terminology, that is, it does not include tightness. Thus Došen shows dual equivalence between BAEs and the category of differentiated and compact general neighbourhood frames. However, his construction does not immediately adapt to the monotonic case, because he only adds clopen neighbourhoods in his definition of general ultrafilter frame, which, consequently, is not monotonic by our definition, since there may be non-clopen sets of ultrafilters which should be neighbourhods according to the upwards closure of the neighbourhood function. Note that in Došen's definition of general frames, all neighbourhoods must be admissible, and the definition of a general monotonic frame in Kracht and Wolter [44] is simply obtained by only requiring upwards closure over clopen/admissible sets. Thus adapting Došen's results to the general monotonic frames of Kracht and Wolter leads immediately to dual equivalence. The restriction to clopens is also the reason why Došen's notion of descriptiveness does not include tightness, since the bijection between states of the original general frame $\mathbb{G}$ and ultrafilters suffices to show $\mathbb{G} \cong (\mathbb{G}^*)_*$ when all neighbourhoods of $(\mathbb{G}^*)_*$ are clopens.

An easy way of turning Došen's general ultrafilter frame into a general monotonic frame according to our definition, would be to just add all supersets of clopen neighbourhoods to the neighbourhood relation. But with this simple approach, we were unable to show the desired duality for the morphisms, and this might also be expected considering the fact that the canonical extension tells us that we must add arbitrary intersections of clopen neighbourhoods before taking the upwards closure. Our tightness condition for general montone frames captures exactly this structure of the neighbourhood function.

Persistence

Theorem 7.36 states that descriptive monotonic frames and BAMs may be thought of as having the same mathematical properties. Thus we may ask, what the dual notion of canonicity is in the category of descriptive monotonic frames. The answer is d-persistence.

Definition 7.41 (Persistence) A formula φ is *d-persistent* if for any descriptive monotonic frame $\mathbb{G}$, if $\mathbb{G} \Vdash \varphi$ then $\mathbb{G}_\sharp \Vdash \varphi$. $\dashv$

Proposition 7.42 *For any formula φ, φ is canonical if and only if φ is d-persistent.*

Proof. For the direction from left to right, assume that φ is canonical, and let $\mathbb{G}$ be a descriptive monotonic frame such that $\mathbb{G} \Vdash \varphi$. By Proposition 7.26(i), we then have $\mathbb{G}^* \vDash \varphi \approx \top$, and from the canonicity of φ, it now follows that $(\mathbb{G}^*)^\sigma \vDash \varphi \approx \top$. Since $(\mathbb{G}^*)^\sigma \cong ((\mathbb{G}^*)_+)^+$, Proposition 7.2(i) now tells us that $(\mathbb{G}^*)_+ \Vdash \varphi$. Recalling the definitions of $(\cdot)_*$, $(\cdot)_+$ and $(\cdot)_\sharp$, we see that $(\mathbb{G}^*)_+ = ((\mathbb{G}^*)_*)_\sharp$ and since $\mathbb{G}$ was assumed descriptive, we obtain from Theorem 7.35 that $(\mathbb{G}^*)_+ \cong \mathbb{G}_\sharp$, thus we may conclude that $\mathbb{G}_\sharp \Vdash \varphi$.

For the direction from right to left, assume that φ is d-persistent, and let $\mathbb{A}$ be a BAM such that $\mathbb{A} \vDash \varphi \approx \top$. We will show that $\mathbb{A}^\sigma \vDash \varphi \approx \top$. From $\mathbb{A} \vDash \varphi \approx \top$ and Proposition 7.26(ii) it follows that $\mathbb{A}_* \Vdash \varphi$. Now we use the assumption that φ d-persistent, and the fact that $\mathbb{A}_*$ is descriptive and $(\mathbb{A}_*)_\sharp = \mathbb{A}_+$ to deduce $\mathbb{A}_+ \Vdash \varphi$, whence by Proposition 7.2(i) we have, $\mathbb{A}^\sigma \cong (\mathbb{A}_+)^+ \vDash \varphi \approx \top$. QED

Note that the above proposition refers to our default notion of σ-canonicity. The reader will have noticed that our ultrafilter frame is defined in terms of σ-canonical extensions, and, consequently, the definitions of tightness and persistence are tailored to the notion of σ-canonicity. In the following subsection, we will show that the duality between σ and π is closely related to the duality between the modalities ∇ and Δ, and using this observation we can transform the above results on σ-canonicity into results on π-canonicity.

7.6 σ versus π

There is no particular reason why we have chosen σ-canonicity as our default, other than the desire to keep things simple and not work with both canonicity notions side by side. The same could be said about our choice of ∇ as our primitive modality; we could easily have taken both ∇ and Δ as primitives of our language, and simply defined the interpretation of the two in such a way that they become dual to each other. In Remark 3.2 we pointed out that in a monotonic $\mathcal{L}_\nabla$-logic, Δ is also a monotonic modality. Furthermore, it is easy to derive from the truth definitions in a monotonic model $\mathbb{M} = (W, \nu, V)$, that $\mathbb{M}, w \Vdash \Delta\varphi$ iff $W \setminus V(\varphi) \notin \nu(w)$. The idea behind 'dualising' monotonic frames is to interchange the interpretation of ∇ and Δ.

Definition 7.43 (Duals of monotonic frames) Let $\mathbb{F} = (W, \nu)$ be a monotonic frame. The *dual frame* of $\mathbb{F}$ is the monotonic frame $\mathbb{F}^d = (W, \nu^d)$ where for all $X \subseteq W$,

$$X \in \nu^d(w) \text{ iff } W \setminus X \notin \nu(w).$$

When $\mathbb{G} = (\mathbb{F}, A)$ is a general monotonic frame, then we define the *dual general frame* of $\mathbb{G}$ as $\mathbb{G}^d = (\mathbb{F}^d, A)$. $\dashv$

Clearly, we have the following identities, which will be used without warning:

$$\begin{array}{rcll} (\mathbb{F}^d)^d & = & \mathbb{F}, & \text{for all monotonic frames } \mathbb{F}. \\ (\mathbb{G}^d)^d & = & \mathbb{G}, & \text{for all general monotonic frames } \mathbb{G}. \\ (\mathbb{G}^d)_\sharp & = & (\mathbb{G}_\sharp)^d, & \text{for all general monotonic frames } \mathbb{G}. \end{array}$$

Proposition 7.44 *Let $\mathbb{G} = (\mathbb{F}, A)$ be a general monotonic frame, and $\mathbb{G}^d$ its dual frame. Then $\mathbb{G}$ is a general monotonic frame.*

Proof. To see that A is closed under the modal operation m_{ν^d}, all one has to observe is that $m_{\nu^d}(X) = W \setminus m_\nu(W \setminus X)$. Closure of A under complement and m_ν yields the result. QED

On the algebraic side, it should be clear why BAMs are dualised as follows.

Definition 7.45 (Duals of BAMs) Let $\mathbb{A} = (A, +, -, 0, f)$ be a BAM. Then we define the *dual of* $\mathbb{A}$ as $\mathbb{A}^d = (A, +, -, 0, f^d)$ where for all $a \in A$, $f^d(a) := -f(-a)$. $\dashv$

It is easy to show that when $\mathbb{A}$ is a BAM, then $\mathbb{A}^d$ is also a BAM, and $(\mathbb{A}^d)^d = \mathbb{A}$.

We now return to the definition of the ultrafilter frame of a BAM $\mathbb{A}$. As mentioned already, $\mathbb{A}_+$ is defined such that $(\mathbb{A}_+)^+ = \mathbb{A}^\sigma$. But we could just as well have defined $\mathbb{A}_+$ such that $(\mathbb{A}_+)^+ = \mathbb{A}^\pi$. In the rest of this subsection, we will use the notation $\mathbb{A}_\sigma = (\mathit{Uf}\,\mathbb{A}, \nu_{f^\sigma})$ for the ultrafilter frame as defined in Definition 7.14, that is, $(\mathbb{A}_\sigma)^+ = \mathbb{A}^\sigma$. Similarly, we will say that a general monotonic frame is *σ-tight* or *σ-descriptive* if $\mathbb{G}$ is tight or descriptive according to Definition 7.30, and a formula φ is d_σ persistent, if φ is persistent with respect to σ-descriptive monotonic frames. The category of σ-descriptive monotonic frames with bounded morphisms will be denoted DMF_σ.

We now define $\mathbb{A}_\pi$ as the monotonic frame $\mathbb{A}_\pi = (\mathit{Uf}A, \nu_{f^\pi})$ where

(44) $X \in \nu_{f^\pi}(u)$ iff $u \in f^\pi(X)$.

Then it is clear that $(\mathbb{A}_\pi)^+ = \mathbb{A}^\pi$. The π-canonical model $\mathbb{F}^\Lambda_\pi$ of a monotonic $\mathcal{L}_\nabla$-logic Λ is the structure $(W^\Lambda, \nu^\Lambda_\pi, V^\Lambda)$ where W^Λ and V^Λ are as in Definition 6.2 and ν^Λ_π is defined by

$$\begin{array}{rcl} \widehat{\varphi} \in \nu^\Lambda_\pi(\Gamma) & \text{iff} & \nabla\varphi \in \Gamma, \\ \bigcup_{i \in I} \widehat{\varphi}_i = O \in \nu^\Lambda_\pi(w) & \text{iff} & \exists \psi \in \mathcal{L}_\nabla(\widehat{\psi} \subseteq O \,\&\, \nabla\psi \in \Gamma), \\ X \in \nu^\Lambda_\pi(w) & \text{iff} & \forall O \subseteq W^\Lambda (X \subseteq O \to O \in \nu^\Lambda_\pi(w)). \end{array}$$

It can be shown that $\mathbb{F}^\Lambda_\pi(\Phi) \cong (\mathbb{L}_\Lambda(\Phi))_\pi$. The general π-ultrafilter frame of a BAM $\mathbb{A}$ is now defined as $\mathbb{A}_\star = (\mathbb{A}_\pi, \widehat{A})$, and the corresponding π-notions of tightness, descriptiveness and persistence are then defined as follows. A general monotonic frame $\mathbb{G} = (W, \nu, A)$ is *π-tight* if for all $w \in W$, all $O \in O(\mathbb{W})$ and all $X \subseteq \mathit{Uf}\,\mathbb{A}$,

$$\begin{array}{rcl} O \in \nu(w) & \text{iff} & \exists a \in A(a \subseteq O \,\&\, a \in \nu(w)), \\ X \in \nu(w) & \text{iff} & \forall O \in O(\mathbb{W})(X \subseteq O \to O \in \nu(w)). \end{array}$$

$\mathbb{G}$ is π-descriptive if $\mathbb{G}$ is differentiated, π-tight and compact. A formula φ is *d_π-persistent* if for any π-descriptive monotonic frame $\mathbb{G}$, $\mathbb{G} \Vdash \varphi$ implies $\mathbb{G}_\sharp \Vdash \varphi$. We denote the category of π-descriptive monotonic frames with bounded morphisms by DMF_π. In the following we will work towards showing that BAM is also dually equivalent with DMF_π.

First, we list some of the relationships between dual frames, dual algebras and σ/π-constructions.

Proposition 7.46 *Let $\mathbb{F}$ be a monotonic frame, $\mathbb{G}$ a general monotonic frame, and let $\mathbb{A}$ be a* BAM. *Then*

$$\begin{array}{rclcrcl}
(\mathbb{F}^d)^+ & = & (\mathbb{F}^+)^d, & \qquad & (\mathbb{G}^*)^d & = & (\mathbb{G}^d)^*, \\
(\mathbb{A}^d)^\sigma & = & (\mathbb{A}^\pi)^d, & & (\mathbb{A}^\sigma)^d & = & (\mathbb{A}^d)^\pi, \\
(\mathbb{A}^d)_\sigma & = & (\mathbb{A}_\pi)^d, & & (\mathbb{A}_\sigma)^d & = & (\mathbb{A}^d)_\pi, \\
(\mathbb{A}^d)_* & = & (\mathbb{A}_\star)^d, & & (\mathbb{A}_*)^d & = & (\mathbb{A}^d)_\star.
\end{array}$$

Proof. The first two items should be clear. To show that $(\mathbb{A}^d)^\sigma = (\mathbb{A}^\pi)^d$, we must prove that for all $X \subseteq Uf\,\mathbb{A}$, we have

(45) $(f^d)^\sigma(X) = (f^\pi)^d(X)$.

For clopens, (45) follows from:

$$(f^d)^\sigma(\widehat{a}) = \widehat{f^d(a)} = -\widehat{f(-a)} = -f^\pi(-\widehat{a}) = (f^\pi)^d(\widehat{a}).$$

For closed subsets C of $Uf\,\mathbb{A}$, we have

$$\begin{array}{rclcl}
(f^d)^\sigma(C) & = & \bigcap_{C\subseteq\widehat{a}} (f^d)^\sigma(\widehat{a}) & = & \bigcap_{C\subseteq\widehat{a}} -\widehat{f(-a)} \quad = \quad -\bigcup_{C\subseteq\widehat{a}} \widehat{f(-a)} \\
 & = & -\bigcup_{-\widehat{a}\subseteq -C} f(-\widehat{a}) & = & -\bigcup_{\widehat{b}\subseteq -C} f(\widehat{b}) \quad = \quad -f^\pi(-C) \\
 & = & (f^\pi)^d(C). & &
\end{array}$$

Finally, for arbitrary $X \subseteq Uf\,\mathbb{A}$,

$$\begin{array}{rl}
u \in (f^d)^\sigma(X) & \text{iff } \exists C \in K((\mathbb{A}^d)^\sigma) : C \subseteq X \;\&\; u \in (f^d)^\sigma(C) \\
 & \text{iff } \exists C \in K(\mathbb{A}^\sigma) : C \subseteq X \;\&\; u \in -f^\pi(-C) \\
 & \text{iff } \exists O \in O(\mathbb{A}^\sigma) : -X \subseteq O \;\&\; u \notin f^\pi(O) \\
 & \text{iff } u \notin f^\pi(-X) \\
 & \text{iff } u \in (f^\pi)^d(X).
\end{array}$$

We leave the proof of the other items to the reader. QED

Proposition 7.47 *Let $\mathbb{G} = (W, \nu, A)$ be a general monotonic $\mathcal{L}_\nabla$-frame. Then $\mathbb{G}$ is σ-descriptive iff $\mathbb{G}^d$ is π-descriptive.*

Proof. Let $\mathbb{G}$ be as above, and let $K(\mathbb{W})$ and $O(\mathbb{W})$ denote the closed, respectively open, subsets of W in the topological space $\mathbb{W}$ of $\mathbb{G}$. We only need to show that $\mathbb{G}$ is σ-tight iff $\mathbb{G}^d$ is π-tight. Assume first that $\mathbb{G}$ is σ-tight. We will first show that for all $w \in W$ and all $D \in O(\mathbb{W})$,

$$D \in \nu^d(w) \;\Rightarrow\; \exists a \in A(a \subseteq D \;\&\; a \in \nu^d(w)).$$

So suppose $D \in \nu^d(w)$, then it follows by the definition of ν^d that $W \setminus D \notin \nu(w)$. Since $W \setminus D \in K(\mathbb{W})$ and $\mathbb{G}$ is σ-tight, there is a $b \in A$ such that $W \setminus D \subseteq b$ and $b \notin \nu(w)$. Hence by taking $a = W \setminus b$, we have $a \subseteq D$ and $a \in \nu^d(w)$.

To see that for arbitrary $X \subseteq W$,

$$X \in \nu^d(w) \;\Rightarrow\; \forall O \in O(\mathbb{W})(X \subseteq O \to O \in \nu^d(w))$$

assume $X \in \nu^d(w)$. Then $W \setminus X \notin \nu(w)$, and by the σ-tightness of $\mathbb{G}$, we have for all $C \in K(\mathbb{W})$, if $C \subseteq X$ then $C \notin \nu(w)$, whence for all $O \in O(\mathbb{W})$, if $X \subseteq O$ then $W \setminus O \notin \nu(w)$, i.e., $O \in \nu^d(w)$.

The direction from right to left is shown is a similar way, and we leave out the details.

QED

In order to derive duality results for the map $(\cdot)_\pi$, we only need to show that the morphisms between monotonic frames, general monotonic frames and BAMs behave well with respect to dualisations.

Proposition 7.48 *Let $\mathbb{F}_i$, $\mathbb{G}_i$ and $\mathbb{A}_i$, $i \in \{1, 2\}$ be a pair of monotonic frames, general monotonic frames and* BAM*s, respectively. Then*

(i) *If $\vartheta : \mathbb{F}_1 \to \mathbb{F}_2$ is a bounded morphism, then ϑ is also a bounded morphism from $\mathbb{F}_1^d$ to $\mathbb{F}_2^d$.*
(ii) *If $\vartheta : \mathbb{G}_1 \to \mathbb{G}_2$ is a bounded morphism, then ϑ is also a bounded morphism from $\mathbb{G}_1^d$ to $\mathbb{G}_2^d$.*
(iii) *If $\eta : \mathbb{A}_1 \to \mathbb{A}_2$ is a* BAE*-homomorphism, then η is also a* BAE*-homomorphism from $\mathbb{A}_1^d$ to $\mathbb{A}_2^d$.*

Proof. Let $\mathbb{F}_i = (W_1, \nu_i)$, $\mathbb{G}_i = (W_i, \nu_i, A_i)$ and $\mathbb{A}_i = (A_i, +, -, 0, f_i)$, $i \in \{1, 2\}$. To prove (i), assume that $\vartheta : \mathbb{F}_1 \to \mathbb{F}_2$ is a bounded morphism. For the (BM1) condition for $\vartheta : \mathbb{F}_1^d \to \mathbb{F}_2^d$, let $X_1 \in \nu_1^d(w)$, and suppose for the sake of contradiction that $\vartheta[X_1] \notin \nu_2^d(\vartheta(w))$. That means $W_2 \setminus \vartheta[X_1] \in \nu_2(\vartheta(w))$. Applying the (BM2) condition for $\vartheta : \mathbb{F}_1 \to \mathbb{F}_2$, we obtain a $Y_1 \subseteq W_1$ such that $Y_1 \in \nu_1(w)$ and $Y_1 \subseteq W_2 \setminus \vartheta[X_1]$. It now follows that

$$Y_1 \subseteq \vartheta^{-1}[\vartheta[Y_1]] \subseteq \vartheta^{-1}[W_2 \setminus \vartheta[X_1]] \subseteq W_1 \setminus X_1,$$

and hence by monotonicity, $W_1 \setminus X_1 \in \nu_1(w)$ which is a contradiction with the assumption that $X_1 \in \nu^d(w)$.

For the (BM2) condition for $\vartheta : \mathbb{F}_1^d \to \mathbb{F}_2^d$, assume $X_2 \in \nu_2^d(\vartheta(w))$, i.e., $W_2 \setminus X_2 \notin \nu_2(\vartheta(w))$. We need an $X_1 \subseteq W_1$ such that $X_1 \in \nu_1^d(w)$ and $\vartheta[X_1] \subseteq X_2$. Take $X_1 := \vartheta^{-1}][X_2]$, then $\vartheta[X_1] \subseteq X_2$. Suppose now again for contradiction that $X_1 = \vartheta^{-1}[X_2] \notin \nu_1^d(w)$, that is, $W_1 \setminus \vartheta^{-1}[X_2] \in \nu_1(w)$. Then by the (BM1) condition for $\vartheta : \mathbb{F}_1 \to \mathbb{F}_2$, it follows that $\vartheta[W_1 \setminus \vartheta^{-1}[X_2]] \in \nu_2(\vartheta(w))$, and since $\vartheta[W_1 \setminus \vartheta^{-1}[X_2]] \subseteq W_2 \setminus X_2$, we have by monotonicity that $W_2 \setminus X_2 \in \nu_2(\vartheta(w))$, which is a contradiction with the assumption that $X_2 \in \nu_2^d(\vartheta(w))$.

The proof of (ii) follows immediately from (i) and the definition of the dual general frame. For (iii) we must show that $\eta(f_1^d(a)) = f_2^d(\eta(a))$ for all $a \in A_1$. But this follows easily from the assumption that η is a BAE-homomorphism:

$$\eta(f_1^d(a)) = \eta(-f_1(-a)) = -\eta(f_1(-a)) = -f_2(\eta(-a)) = -f_2(-\eta(a)) = f_2^d(\eta(a)).$$

QED

Proposition 7.49 *Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be two* BAM*s, and $\eta : \mathbb{A}_1 \to \mathbb{A}_2$ a* BAE*-homomorphism. Then*

(i) *η_+ is a bounded morphism from ${\mathbb{A}_2}_\pi$ to ${\mathbb{A}_1}_\pi$,*
(ii) *η_* is a bounded morphism from ${\mathbb{A}_2}_\star$ to ${\mathbb{A}_1}_\star$.*

Proof. Both items are easy consequences of Propositions 7.19, 7.29, 7.48 and 7.46, and we only show (i):

$$
\begin{array}{lll}
 & & \eta : \mathbb{A}_1 \to \mathbb{A}_2 \text{ is a BAE-homomorphism} \\
\text{Prop. 7.48(iii)} & \Rightarrow & \eta : \mathbb{A}_1^d \to \mathbb{A}_2^d \text{ is a BAE-homomorphism} \\
\text{Prop. 7.19} & \Rightarrow & \eta_+ : (\mathbb{A}_2^d)_\sigma \to (\mathbb{A}_1^d)_\sigma \text{ is a bounded morphism} \\
\text{Prop. 7.46} & \Rightarrow & \eta_+ : (\mathbb{A}_{2\,\pi})^d \to (\mathbb{A}_{1\,\pi})^d \text{ is a bounded morphism} \\
\text{Prop. 7.48(i)} & \Rightarrow & \eta_+ : \mathbb{A}_{2\,\pi} \to \mathbb{A}_{1\,\pi} \text{ is a bounded morphism.}
\end{array}
$$

QED

Proposition 7.50 *The map* $(\cdot)_\star$ *defined by* $\mathbb{A} \mapsto \mathbb{A}_\star$ *for all* BAMs $\mathbb{A}$, *and* $\eta \mapsto \eta_\star = \eta_*$ *for all* BAE*-homomorphisms* η, *is a contravariant functor from the category* BAM *to the category* DMF_π.

Proof. Proposition 7.49 above shows that $(\cdot)_\star$ maps BAE-homomorphisms contravariantly to bounded morphisms. So it only remains to show that $\mathbb{A}_\star$ is π-descriptive whenever $\mathbb{A}$ is a BAM. So let $\mathbb{A}$ be a BAM, then by Proposition 7.46, $\mathbb{A}_\star = ((\mathbb{A}^d)_*)^d$. Since $\mathbb{A}^d$ is a BAM, $(\mathbb{A}^d)_*$ is σ-descriptive by Proposition 7.32, and hence by Proposition 7.47 $((\mathbb{A}^d)_*)^d$ is π-descriptive. QED

Theorem 7.51 *The categories* BAM *and* DMF_π *are dually equivalent via the functors* $(\cdot)^* : \mathsf{DMF}_\pi \to \mathsf{BAM}$ *and* $(\cdot)_\star : \mathsf{BAM} \to \mathsf{DMF}_\pi$.

Proof. We must show the π-analogue of Theorem 7.35. So let $\mathbb{A}$ be a BAM, then we need ${\mathbb{A}_\star}^* \cong \mathbb{A}$. From Proposition 7.46 we have $(\mathbb{A}_\star)^* \cong (((\mathbb{A}^d)_*)^d)^*$, and also $(((\mathbb{A}^d)_*)^d)^* \cong (((\mathbb{A}^d)_*)^*)^d$, hence $(\mathbb{A}_\star)^* \cong (((\mathbb{A}^d)_*)^*)^d$. From Theorem 7.35(i), it follows that $(((\mathbb{A}^d)_*)^*)^d \cong (\mathbb{A}^d)^d$, and we have $(\mathbb{A}_\star)^* \cong (\mathbb{A}^d)^d \cong \mathbb{A}$.

Now let $\mathbb{G}$ be a π-descriptive monotonic frame. Then $\mathbb{G}^d$ is σ-descriptive, and by Theorem 7.35(ii), it follows that $((\mathbb{G}^d)^*)_* \cong \mathbb{G}^d$. From Proposition 7.46, we also have

$$((\mathbb{G}^d)^*)_* \cong ((\mathbb{G}^*)^d)_* \cong ((\mathbb{G}^*)_\star)^d.$$

Hence $\mathbb{G}^d \cong ((\mathbb{G}^*)_\star)^d$, and thus we may conclude that $\mathbb{G} \cong (\mathbb{G}^*)_\star$. QED

The following expected analogue of Proposition 7.42 should be clear from the above results and we leave the proof to the reader.

Proposition 7.52 *For all formulas* φ, φ *is* π*-canonical if and only if* φ *is* d_π*-persistent.*

We now have a good understanding of how the σ- and π-constructions relate to each other. But we have not yet brought strong completeness into the picture, which is where part of our interest in σ-canonicity stems from. We postpone the investigation of this matter to subsection 10.6, since we require a number of technical results on simulations, but in subsection 10.6 it will be shown that π-canonicity does, after all, imply strong completeness (Theorem 10.43).

8 Coalgebra

As an alternative way of placing monotonic modal logic in a mathematical context, we will now see that monotonic structures can be viewed as coalgebras. Briefly stated, a T-coalgebra for a functor T on the category of sets is given by a pair (X, γ) where X is a set and γ is a map from X to $T(X)$. The link between coalgebras and modal logic has been studied in the field of non-well-founded set theory [3, 17], and in computer science coalgebras are used to model infinite data types, such as streams, as well as dynamic systems, like automata and labelled transition systems. When considering coalgebras of this kind, it makes sense to think of the carrier set X as a state space and γ as a transition structure.

Compared with algebraic duality theory, the coalgebraic perspective on modal logic is a quite recent development, and will generally not be included in a first course in modal logic. Therefore, we present our material without assuming any knowledge of coalgebras and very little of category theory. However, this section is by no means an introduction to the general theory of coalgebras, and only the relevant definitions will be given. For more background knowledge, the reader is referred to [59], and for coalgebra and modal logic to [46, 54, 45].

The main purpose of this section is to show how coalgebras can be employed as a semantics for monotonic modal logic. In particular, we will see that the category of $Up\mathcal{P}$-coalgebras and $Up\mathcal{P}$-coalgebra morphisms (defined below) is isomorphic with the category of monotonic frames and bounded morphisms via the identity functor. Furthermore, we investigate how the coalgebraic notions of system equivalence relate to bisimulations between monotonic frames.

8.1 Basic Definitions and Notation

We will work in the category Set, which has sets as objects and set-theoretic functions as its morphisms.

Definition 8.1 A (Set-)*endofunctor* $T : \mathsf{Set} \to \mathsf{Set}$ maps Set-objects to Set-objects and Set-morphisms $f : X \to Y$ to Set-morphisms $Tf : TX \to TY$ such that

$$T(g \circ f) = Tg \circ Tf \quad \text{and} \quad T\mathrm{id}_X = \mathrm{id}_{TX}. \qquad \dashv$$

There are two endofunctors which the reader should be familiar with. The *covariant powerset functor* is denoted by $\mathcal{P}$, and maps a set X to its powerset $\mathcal{P}(X)$ and $\mathcal{P}f$ maps a subset C of X to the image of C under f:

$$\begin{array}{lrcllcl} \mathcal{P}: & \mathsf{Set} & \to & \mathsf{Set} & & & \\ & X & \mapsto & \mathcal{P}(X) & & & \\ & f : X \to Y & \mapsto & \mathcal{P}f = f[\cdot]: & \mathcal{P}(X) & \to & \mathcal{P}(Y) \\ & & & & C & \mapsto & f[C] \end{array}$$

The *contravariant powerset functor* is denoted by $2^{(\cdot)}$, and also maps a set to its powerset, but functions map to the inverse image operation:

$$\begin{array}{lrcllcl} 2^{(\cdot)}: & \mathsf{Set} & \to & \mathsf{Set} & & & \\ & X & \mapsto & 2^X = \mathcal{P}(X) & & & \\ & f : X \to Y & \mapsto & 2^f = f^{-1}[\cdot]: & 2^Y & \to & 2^X \\ & & & & D & \mapsto & f^{-1}[D] \end{array}$$

Definition 8.2 (Coalgebras) Let T be an endofunctor. A T-*coalgebra* (over the base category Set) is a pair (X, γ) where X is a set and $\gamma : X \to T(X)$ is a function. $\dashv$

We will mainly be thinking about T-coalgebras as systems, and we will therefore occasionally refer to them as T-systems, or simply systems if T is clear from the context. Thus, as stated in the introduction, given a T-coalgebra (X, γ), X will be called the state space and γ its transition structure. The notions of (homo)morphism and bisimulations amount to the following.

Definition 8.3 (Coalgebra morphisms) Let T be an endofunctor, (X, γ) and (Y, δ) two T-coalgebras. Then a function $f : X \to Y$ is a T-*coalgebra morphism* if: $Tf \circ \gamma = \delta \circ f$. That is, the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\gamma"'] & Y \arrow[d, "\delta"] \\ T(X) \arrow[r, "Tf"] & T(Y) \end{tikzcd}$$

$\dashv$

It can be checked that the composition of two T-coalgebra morphisms is again a T-coalgebra morphism, and the identity map on a set X is also a T-coalgebra morphism. Hence T-coalgebras and T-coalgebra morphisms form a category Set_T.

Definition 8.4 (Coalgebra bisimulations) Let T be an endofunctor and let (X, γ) and (Y, δ) be two T-coalgebras. Then a non-empty relation $Z \subseteq X \times Y$ is a T-*coalgebra bisimulation* between (X, γ) and (Y, δ) if there is a function $\mu : Z \to T(Z)$ such that: $\gamma \circ \pi_1 = T\pi_1 \circ \mu$ and $\delta \circ \pi_2 = T\pi_2 \circ \mu$, where $\pi_1 : Z \to X$ and $\pi_2 : Z \to Y$ are the projection maps. That is, π_1 and π_2 are T-coalgebra morphisms, and the following diagram commutes.

$$\begin{tikzcd} X \arrow[d, "\gamma"'] & Z \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\mu"'] & Y \arrow[d, "\delta"] \\ T(X) & T(Z) \arrow[l, "T\pi_1"'] \arrow[r, "T\pi_2"] & T(Y) \end{tikzcd}$$

Two states $x \in X$ and $y \in Y$ are called T-*coalgebra bisimilar* if there is a T-coalgebra bisimulation Z between (X, γ) and (Y, δ) such that $(x, y) \in Z$. Finally, (X, γ) and (Y, δ) are called T-*coalgebra bisimilar* if there is a T-coalgebra bisimulation between (X, γ) and (Y, δ). $\dashv$

The intuitive idea behind bisimulation is that bisimilar states should be seen to display the same behaviour. This may not be all that clear from Definition 8.4, and as we shall see in subsection 8.3, the coalgebraic notion of bisimulation is, in fact, stronger than the frame theoretic one. However, bisimilar states turn out to be exactly the states which are behaviourally equivalent.

Definition 8.5 (Behavioural equivalence) Let T be an endofunctor and let (X, γ) and (Y, δ) be T-coalgebras. Then $x \in X$ and $y \in Y$ are *behaviourally equivalent states* if there is a T-coalgebra (Z, μ) and T-coalgebra morphisms $f : X \to Z$ and $g : Y \to Z$ such that

$f(x) = g(y)$; and (X, γ) and (Y, δ) are *behaviourally equivalent systems* if there is a T-coalgebra (Z, μ) and surjective T-coalgebra morphisms $f : X \to Z$ and $g : Y \to Z$. ⊣

The concept of behavioural equivalence is derived from the notion of *final systems*. A T-coalgebra (P, ζ) is *final* if for any T-coalgebra (X, δ) there is a unique T-coalgebra morphism $f : (X, \delta) \to (P, \zeta)$. Final systems are viewed as the "system of observable behaviours", see [59, 46], and this means that behaviourally equivalent states are identified in the final system, if it exists. Definition 8.5 does not assume the existence of a final system such that behavioural equivalence may be applied to all coalgebras. We will not treat final systems any further.

Definition 8.6 (Natural transformation) Let T and S be endofunctors. Then a *natural transformation* τ between T and S (notation: $\tau : T \Rightarrow S$) is a map which takes a set X as argument and returns a function $\tau_X : T(X) \to S(X)$ which satisfies the following condition. For all sets X, Y and functions $f : X \to Y$, we have: $Sf \circ \tau_X = \tau_Y \circ Tf$. That is, the following diagram commutes.

$$\begin{tikzcd}
T(X) \arrow[r, "Tf"] \arrow[d, "\tau_X"'] & T(Y) \arrow[d, "\tau_Y"] \\
S(X) \arrow[r, "Sf"] & S(Y)
\end{tikzcd}$$

⊣

Given a natural transformation $\tau : T \Rightarrow S$, any T-coalgebra (X, γ) can be seen as an S-coalgebra (X, δ) where $\delta = \tau_X \circ \gamma : X \to S(X)$. Furthermore, natural transformations preserve coalgebra morphisms and bisimulations, since for $\tau : T \Rightarrow S$, the following diagrams commute.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\gamma"'] & Y \arrow[d, "\delta"] \\
T(X) \arrow[r, "Tf"] \arrow[d, "\tau_X"'] & T(Y) \arrow[d, "\tau_Y"] \\
S(X) \arrow[r, "Sf"] & S(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[d, "\gamma"'] & Z \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\mu"] & Y \arrow[d, "\delta"] \\
T(X) \arrow[d, "\tau_X"'] & T(Z) \arrow[l, "T\pi_1"'] \arrow[r, "T\pi_2"] \arrow[d, "\tau_Z"] & T(Y) \arrow[d, "\tau_Y"] \\
S(X) & S(Z) \arrow[l, "S\pi_1"'] \arrow[r, "S\pi_2"] & S(Y)
\end{tikzcd}$$

8.2 Coalgebraic Semantics for Monotonic Modal Logic

Coalgebra and monotonic frames

It is well-known that Kripke frames may be seen as $\mathcal{P}$-coalgebras (see e.g. Pattinson [54]), and we will now show that monotonic frames have a fairly simple interpretation as coalgebras for a functor which is essentially the same as $2^{(\cdot)} \circ 2^{(\cdot)}$, but restricted to upwards closed families of subsets. The functor $2^{(\cdot)} \circ 2^{(\cdot)}$ is mentioned in Rutten [59] as giving rise to so-called hyper systems.

Definition 8.7 The map $Up\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ is defined as follows. For a set X

$$Up\mathcal{P}(X) = \{Z \in \mathcal{P}(\mathcal{P}(X)) \mid Z \text{ is upwards closed}\}$$

and for a function $f : X \to Y$,

$$\begin{array}{rrcl} Up\mathcal{P}f: & Up\mathcal{P}(X) & \to & Up\mathcal{P}(Y) \\ & U & \mapsto & Up\mathcal{P}f(U) = (f^{-1})^{-1}[U] = 2^{2^f}(U) \\ & & & \quad = \{D \in \mathcal{P}(Y) \mid f^{-1}[D] \in U\} \end{array}$$ $\dashv$

It may be easier to think of $(f^{-1})^{-1}[U]$ as the set

(46) $\uparrow f^*[U] := \{D \in \mathcal{P}(Y) \mid \exists C \in U : f[C] \subseteq D\}$,

The notation $\uparrow f^*[U]$ is meant to indicate that this is the upwards closure of the set consisting of all images $f[C]$ for $C \in U$. To see that $(f^{-1})^{-1}[U] = \uparrow f^*[U]$, suppose $f^{-1}[D] \in U$, then taking $C = f^{-1}[D]$, we have $f[C] = f[f^{-1}[D]] \subseteq D$. On the other hand, if $C \in U$ and $f[C] \subseteq D$, then $C \subseteq f^{-1}[f[C]] \subseteq f^{-1}[D]$, and $f^{-1}[D] \in U$ follows from U being upwards closed.

Lemma 8.8 *$Up\mathcal{P}$ is an endofunctor.*

Proof. Let $g : X \to Y$ and $f : Y \to Z$. To see that $Up\mathcal{P}(f \circ g) = Up\mathcal{P}f \circ Up\mathcal{P}g$, let $U \in Up\mathcal{P}(X)$, then

$$\begin{array}{ll} & D \in (Up\mathcal{P}f \circ Up\mathcal{P}g)(U) \\ \text{iff} & D \in \uparrow f^*[\uparrow g^*[U]] \\ \text{iff} & \exists E \in \uparrow g^*[U] : f[E] \subseteq D \\ \text{iff} & \exists E \in \mathcal{P}(Y) : (\exists C \in U : g[C] \subseteq E) \ \&\ f[E] \subseteq D \\ \text{iff} & \exists C \in U : f[g[C]] \subseteq D \\ \text{iff} & D \in Up\mathcal{P}(f \circ g)(U). \end{array}$$

In the next to last step, the direction from left to right holds because $g[C] \subseteq E \Rightarrow f[g[C]] \subseteq f[E] \subseteq D$, and the direction from right to left follows by taking $E = g[C]$.

To see that $Up\mathcal{P}\,\mathrm{id}_X = \mathrm{id}_{Up\mathcal{P}X}$, let $U \in Up\mathcal{P}(X)$, then

$$\begin{array}{ll} & D \in Up\mathcal{P}\,\mathrm{id}_X(U) \\ \text{iff} & \exists C \in U : C \subseteq D \\ \text{iff} & D \in U \\ \text{iff} & D \in \mathrm{id}_{Up\mathcal{P}X}(U). \end{array}$$

Here the next to last step follows from the fact that U is upwards closed. QED

An $Up\mathcal{P}$-coalgebra is thus a pair $(W, \nu : W \to Up\mathcal{P}(W))$ where W is a set and ν is a function which maps elements of W to collections of upwards closed subcollections of $\mathcal{P}(W)$. It should now be clear that an $Up\mathcal{P}$-coalgebra can simply be seen as a monotonic $\mathcal{L}_\nabla$-frame, and vice versa. Due to this observation, we may simply formulate our results for $Up\mathcal{P}$-coalgebras. It will be clear from the context whether a pair (X, ν) should be seen as a monotonic $\mathcal{L}_\nabla$-frame or as an $Up\mathcal{P}$-coalgebra.

In order to extend the identity map to a categorical isomorphism, we must also show that $Up\mathcal{P}$-coalgebra morphisms and bounded morphisms really are the same mathematical objects. We first rephrase Definition 8.3 for $Up\mathcal{P}$-coalgebras.

Definition 8.9 ($Up\mathcal{P}$-coalgebra morphisms) Let (X,γ) and (Y,δ) be two $Up\mathcal{P}$-coalgebras. Then a function $f : X \to Y$ is an *$Up\mathcal{P}$-coalgebra morphism* if: $Up\mathcal{P}f \circ \gamma = \delta \circ f$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\gamma"'] & Y \arrow[d, "\delta"] \\ Up\mathcal{P}(X) \arrow[r, "Up\mathcal{P}f"] & Up\mathcal{P}(Y) \end{tikzcd}$$

⊣

Proposition 8.10 *Let (X,γ) and (Y,δ) be two $Up\mathcal{P}$-coalgebras. Then a function $f : X \to Y$ is an $Up\mathcal{P}$-coalgebra morphism if and only if f is a bounded morphism.*

Proof. Recall from Remark 4.4 that f is a bounded morphism from (X,γ) to (Y,δ) iff for all $x \in X$ and all $D \in \mathcal{P}(Y)$

(47) $f^{-1}[D] \in \gamma(x)$ iff $D \in \delta(f(x))$.

We also have for all $x \in X$ and all $D \in \mathcal{P}(Y)$

(48) $D \in Up\mathcal{P}f(\gamma(x))$ iff $D \in (f^{-1})^{-1}[\gamma(x)]$ iff $f^{-1}[D] \in \gamma(x)$.

The result is now immediate from (47) and (48). QED

Theorem 8.11 *The category $\mathsf{Set}_{Up\mathcal{P}}$ consisting of $Up\mathcal{P}$-coalgebras and $Up\mathcal{P}$-coalgebra morphisms is isomorphic with the category MF consisting of monotonic $\mathcal{L}_\nabla$-frames and bounded morphisms via the Set-identity map.*

Coalgebra and monotonic models

In order to get a coalgebraic notion of a monotonic model, we must add the equivalent of a valuation to the $Up\mathcal{P}$-functor. Given a monotonic frame (W,ν), we usually define a valuation V as a map from the set of atomic propositions Φ to the powerset of W, specifying at which states $V(p) \subseteq W$ an atomic proposition p is true. However, we could equally well have defined V as a map from W to the powerset of Φ, specifying which atomic propositions $V[w] \subseteq \Phi$ are true at a state w in W. Clearly, these two views on valuations are equivalent: Given $V : W \to \mathcal{P}(\Phi)$, define $V(p) = \{w \in W \mid p \in V[w]\}$, and given $V : \Phi \to \mathcal{P}(W)$, define $V[w] = \{p \in \Phi \mid p \in V(p)\}$.

Definition 8.12 ($Up\mathcal{P}_\Phi$) Let Φ be a set of atomic propositions. Then the map $Up\mathcal{P}_\Phi$: $\mathsf{Set} \to \mathsf{Set}$ is defined as follows. For a set X

$$Up\mathcal{P}_\Phi(X) = Up\mathcal{P}(X) \times \mathcal{P}(\Phi),$$

and for a function $f : X \to Y$,

$$\begin{array}{rrcl} Up\mathcal{P}_\Phi f : & Up\mathcal{P}_\Phi(X) & \to & Up\mathcal{P}_\Phi(Y) \\ & (U,A) & \mapsto & (Up\mathcal{P}f(U),A). \end{array}$$

⊣

Checking that $Up\mathcal{P}_\Phi$ is an endofunctor can be done in almost the same way as in Lemma 8.8, and we leave out the details. For notational convenience, when $\gamma : X \to Up\mathcal{P}_\Phi(X)$ and $x \in X$, we will write $\gamma_i(x)$ for the i'th projection of $\gamma(x)$, $i \in \{1, 2\}$.

An $Up\mathcal{P}_\Phi$-coalgebra (X, γ) then defines a monotonic $\mathcal{L}_\nabla$-model via the following map. Define $\mathbb{M}\mathsf{od}(X, \gamma) = (X, \nu_\gamma, V_\gamma)$ where for $x \in X$,

$$\begin{array}{rcl} \nu_\gamma(x) & = & \gamma_1(x), \\ V_\gamma[x] & = & \gamma_2(x). \end{array}$$

In the other direction, given a monotonic $\mathcal{L}_\nabla$-model (W, ν, V), define an $Up\mathcal{P}_\Phi$-coalgebra $\mathbb{C}\mathsf{oa}(W, \nu, V) = (W, \gamma)$ by taking

$$\begin{array}{rrcl} \gamma : & W & \to & Up\mathcal{P}(W) \times \mathcal{P}(\Phi) \\ & w & \mapsto & (\nu(w), V[w]) \end{array}$$

It is easy to see that for all $Up\mathcal{P}_\Phi$-coalgebras (X, γ),

$$\mathbb{C}\mathsf{oa}(\mathbb{M}\mathsf{od}(X, \gamma)) = (X, \gamma),$$

and for all monotonic $\mathcal{L}_\nabla$-models (W, ν, V),

$$\mathbb{M}\mathsf{od}(\mathbb{C}\mathsf{oa}(W, \nu, V)) = (W, \nu, V).$$

When $f : X \to Y$ is a function between the sets X and Y, we simply define $\mathbb{M}\mathsf{od} f = f$ and $\mathbb{C}\mathsf{oa} f = f$. To establish categorical isomorphism, we only need to show the following easy extensions of Proposition 8.10.

Proposition 8.13 *Let Φ be a set of atomic propositions, and $\mathbb{M} = (W, \nu, V)$ and $\mathbb{M}' = (W', \nu', V')$ two monotonic $\mathcal{L}_\nabla$-models. Then a function $f : W \to W'$ is a bounded morphism from $\mathbb{M}$ to $\mathbb{M}'$ if and only if f is an $Up\mathcal{P}_\Phi$-coalgebra morphism from $\mathbb{C}\mathsf{oa}(\mathbb{M})$ to $\mathbb{C}\mathsf{oa}(\mathbb{M}')$*

Proof. Let $\mathbb{C}\mathsf{oa}(\mathbb{M}) = (W, \gamma)$, and $\mathbb{C}\mathsf{oa}(\mathbb{M}') = (W', \delta)$. Then

$$\begin{array}{rcll} \gamma(x) & = & (\nu(x), V[x]), & \text{for all } x \in W, \\ \delta(y) & = & (\nu'(y), V'[y]), & \text{for all } y \in W'. \end{array}$$

We must show that f is a bounded morphism between $\mathbb{M}$ and $\mathbb{M}'$ if and only if for all $x \in W$, $Up\mathcal{P}_\Phi f(\gamma(x)) = \delta(f(x))$. We have,

$$\begin{array}{rcl} Up\mathcal{P}_\Phi f(\gamma(x)) & = & (Up\mathcal{P} f(\nu(x)), V[x]), \\ \delta(f(x)) & = & (\nu'(f(x)), V'[f(x)]). \end{array}$$

Since for all proposition letters $p \in \Phi$

$$\begin{array}{lll} \mathbb{M}, x \Vdash p & \text{iff} & p \in V[x], \\ \mathbb{M}', y \Vdash p & \text{iff} & p \in V'[y], \end{array}$$

it is clear that x and $f(x)$ satisfy the same proposition letters if and only if $V[x] = V'[f(x)]$. And just as in the proof of Proposition 8.10, we obtain that $Up\mathcal{P} f(\nu(x)) = \nu'(f(x))$ if and only if f is a bounded morphism of the underlying frames (W, ν) and (W', ν') of $\mathbb{M}$ and $\mathbb{M}'$, respectively. QED

Proposition 8.14 *Let Φ be a set of atomic propositions, and (X,γ) and (Y,δ) two $Up\mathcal{P}_\Phi$-coalgebras. Then a function $f : X \to Y$ is an $Up\mathcal{P}_\Phi$-coalgebra morphism from (X,γ) to (Y,δ) if and only if f is a bounded morphism from $\mathbb{M}\mathsf{od}(X,\gamma)$ to $\mathbb{M}\mathsf{od}(Y,\delta)$*

Proof. Let $\mathbb{M}\mathsf{od}(X,\gamma) = (X,\nu_\gamma,V_\gamma)$, and $\mathbb{M}\mathsf{od}(Y,\delta) = (Y,\nu_\delta,V_\delta)$. Then

$$\begin{array}{rclrcll} \nu_\gamma(x) & = & \gamma_1(x), & V_\gamma[x] & = & \gamma_2(x) & \text{for all } x \in X, \\ \nu_\delta(y) & = & \delta_1(y), & V_\delta[y] & = & \delta_2(y) & \text{for all } y \in Y. \end{array}$$

We only sketch the proof, since it is similar to that of the previous proposition. From the definition of V_γ and V_δ, we see that x and $f(x)$ satisfy the same proposition letters if and only if $\gamma_2(x) = \delta_2(f(x))$. Furthermore, we easily obtain that $Up\mathcal{P}f(\gamma_1(x)) = \delta_1(f(x))$ if and only if f is a bounded morphism of the underlying frames (W_γ,ν_γ) and (W_δ,ν_δ) of $\mathbb{M}\mathsf{od}(X,\gamma)$ and $\mathbb{M}\mathsf{od}(Y,\delta)$, respectively. Thus $Up\mathcal{P}_\Phi f(\gamma(x)) = (Up\mathcal{P}f(\gamma_1(x)),\gamma_2(x)) = (\delta_1(f(x)),\delta_2(f(x))) = \delta(f(x))$ if and only if f is bounded morphism from $\mathbb{M}\mathsf{od}(X,\gamma)$ to $\mathbb{M}\mathsf{od}(Y,\delta)$. QED

Theorem 8.15 *Let Φ be a set of atomic propositions. The category $\mathsf{Set}_{Up\mathcal{P}_\Phi}$ consisting of $Up\mathcal{P}_\Phi$-coalgebras and $Up\mathcal{P}_\Phi$-coalgebra morphisms is isomorphic with the category MM consisting of monotonic $\mathcal{L}_\nabla$-models and bounded morphisms via the functors $\mathbb{M}\mathsf{od} : \mathsf{Set}_{Up\mathcal{P}_\Phi} \to \mathsf{MM}$ and $\mathbb{C}\mathsf{oa} : \mathsf{MM} \to \mathsf{Set}_{Up\mathcal{P}_\Phi}$.*

The next step is to define truth of $\mathcal{L}_\nabla$-formulas in an $Up\mathcal{P}_\Phi$-coalgebra. Fix an $Up\mathcal{P}_\Phi$-coalgebra (W,γ). For the atomic propositions $p \in \Phi$, γ tells us what the *extension* $[\![p]\!]$ of p should be, namely, $[\![p]\!] = V(p)$, and for the boolean connectives truth may then inductively be defined in the obvious way.

For modal $\mathcal{L}_\nabla$-formulas of the form $\nabla\varphi$, we wish to specify for which elements w of W that $[\![\varphi]\!] \in \nu(w) = \gamma_1(w)$. This may be considered the same as specifying which neighbourhood collections $\nu(w)$ contain $[\![\varphi]\!]$. Thus we are lifting $[\![\varphi]\!]$, which is a predicate over W, to a predicate over $Up\mathcal{P}(W)$. In general, for a T-coalgebra (X,γ), a predicate lifting may be thought of as a map which lifts a predicate over X to a predicate over the 'type of observations' $T(X)$ in a natural way.

Definition 8.16 (Predicate lifting) Let T be an endofunctor. A *predicate lifting* λ for T is an order preserving natural transformation $\lambda : 2^{(\cdot)} \Rightarrow 2^T$, where $2^T = 2^{(\cdot)} \circ T$. That is, for all sets X, Y and functions $f : X \to Y$, we have:

- $\lambda_X : 2^X \to 2^{T(X)}$, $\lambda_Y : 2^Y \to 2^{T(Y)}$;
- If $C \subseteq D$, then $\lambda_X(C) \subseteq \lambda_X(D)$, idem λ_Y;
- $2^{Tf} \circ \lambda_Y = \lambda_X \circ 2^f$. $\dashv$

Predicate liftings give rise to modalities due to the naturality condition, which ensures invariance under bisimulation. See Kurz [46] for a more detailed treatment.

We now define a predicate lifting for $Up\mathcal{P}$ which will tell us how to interpret modal formulas of the form $\nabla\varphi$.

Definition 8.17 Define a family of maps $\lambda_{(\cdot)} : 2^{(\cdot)} \to 2^{Up\mathcal{P}(\cdot)}$ as follows. Let W be a set, then

$$\begin{array}{rrcl} \lambda_W : & 2^W & \to & 2^{Up\mathcal{P}(W)} \\ & C & \mapsto & \{U \in Up\mathcal{P}(W) \mid C \in U\} \end{array}$$

$\dashv$

Lemma 8.18 $\lambda : 2 \Rightarrow 2^{Up\mathcal{P}}$, *that is,* λ *is a predicate lifting for* $Up\mathcal{P}$.

Proof. Let W be a set and $C, D \subseteq W$. To see that $\lambda_W(C) \subseteq \lambda_W(D)$ whenever $C \subseteq D$, suppose that $U \in \lambda_W(C)$, i.e., $C \in U$, then by the upwards closure of U and $C \subseteq D$, we get $D \in U$, hence $U \in \lambda_W(D)$.

To see that λ also satisfies the naturality condition, we first observe that for $f : X \to Y$ and $D \in \mathcal{P}(Y)$:

$$\begin{array}{rcl} 2^{Up\mathcal{P}f} \circ \lambda_Y(D) & = & (Up\mathcal{P}f)^{-1}[\lambda_Y(D)] \\ & = & \{U \in Up\mathcal{P}(X) \mid Up\mathcal{P}f(U) \in \lambda_Y(D)\}. \end{array}$$

Now for $U \in Up\mathcal{P}(X)$, we have

$$\begin{array}{ll} & U \in 2^{Up\mathcal{P}f} \circ \lambda_Y(D) \\ \text{iff} & Up\mathcal{P}f(U) \in \lambda_Y(D) \\ \text{iff} & D \in Up\mathcal{P}f(U) \\ \text{iff} & f^{-1}[D] = 2^f(D) \in U \\ \text{iff} & U \in \lambda_X(2^f(D)). \end{array}$$

QED

Given an $Up\mathcal{P}$-coalgebra (W, ν), it is worth noticing the similarity between $\lambda_W : \mathcal{P}(W) \to Up\mathcal{P}(W)$ and the map $m_\nu : \mathcal{P}(W) \to \mathcal{P}(W)$. The only difference is that for an $X \subseteq W$, $\lambda_W(X)$ tells us which neighbourhood collections $\nu(w)$ contain X, whereas $m_\nu(X)$ gives us the states w for which $\nu(w)$ contains X. Put differently, $\lambda_W(X)$ is the set of 'observations of X' and $m_\nu(X)$ is the set elements that 'observe X'. Recall that in a monotonic model, $V(\nabla\varphi) = m_\nu(V(\varphi))$. Thus in an $Up\mathcal{P}_\Phi$-coalgebra (W, γ), we obtain the extension $[\![\nabla\varphi]\!]$ of $\nabla\varphi$ as the set of states w for which $\gamma_1(w) = \nu_\gamma(w) \in \lambda_w([\![\varphi]\!])$:

$$[\![\nabla\varphi]\!] = \gamma_1^{-1}[\lambda_W([\![\varphi]\!])] = \{w \in W \mid \gamma_1(w) \in \lambda_W([\![\varphi]\!])\}.$$

And we have

$$w \in [\![\nabla\varphi]\!] \quad \text{iff} \quad \gamma_1(w) = \nu_\gamma(w) \in \lambda_W([\![\varphi]\!]) \quad \text{iff} \quad [\![\varphi]\!] \in \nu(w).$$

We sum up the above in the following definition.

Definition 8.19 (Truth conditions) Let (W, γ) be an $Up\mathcal{P}_\Phi$-coalgebra for some set Φ of atomic propositions. Then we define the extension of $\mathcal{L}_\nabla$-formulas inductively by:

$$\begin{array}{rcl} [\![\bot]\!] & = & \emptyset, \\ [\![p]\!] & = & \{w \in W \mid p \in \gamma_2(w)\}, \\ [\![\neg\varphi]\!] & = & W \setminus [\![\varphi]\!], \\ [\![\varphi \vee \psi]\!] & = & [\![\varphi]\!] \cup [\![\psi]\!], \\ [\![\nabla\varphi]\!] & = & \gamma_1^{-1}[\lambda_W([\![\varphi]\!])]. \end{array}$$

$\dashv$

From Definition 8.19 it should be clear that: $\mathbb{M}\mathsf{od}(X, \gamma), x \Vdash \varphi$ iff $x \in [\![\varphi]\!]$. Thus the truth definition in 8.19 may be seen as equivalent with the usual one in monotonic $\mathcal{L}_\nabla$-models.

8.3 Bisimulations and Behavioural Equivalence

To keep notation and proofs simple, we will only consider bisimulations and behavioural equivalence between monotonic frames and $Up\mathcal{P}$-coalgebras. From the above treatment of bounded morphisms, it should be easy to see how to adapt the below results to monotonic models and $Up\mathcal{P}_\Phi$-coalgebras.

Bisimulations

Definition 8.20 ($Up\mathcal{P}$-coalgebra bisimulations) Let (X, γ) and (Y, δ) be two $Up\mathcal{P}$-coalgebras. Then a non-empty relation $Z \subseteq X \times Y$ is an *$Up\mathcal{P}$-coalgebra bisimulation* between (X, γ) and (Y, δ) if there is a function $\mu : Z \to Up\mathcal{P}(Z)$ such that: $\gamma \circ \pi_1 = Up\mathcal{P}\pi_1 \circ \mu$ and $\delta \circ \pi_2 = Up\mathcal{P}\pi_2 \circ \mu$.

$$\begin{tikzcd} X \arrow[d, "\gamma"'] & Z \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\mu"'] & Y \arrow[d, "\delta"] \\ Up\mathcal{P}(X) & Up\mathcal{P}(Z) \arrow[l, "Up\mathcal{P}\pi_1"'] \arrow[r, "Up\mathcal{P}\pi_2"] & Up\mathcal{P}(Y) \end{tikzcd}$$

⊣

The requirement that the projections are coalgebra morphisms turns out to be quite a strong, and we shall see that we need to strengthen the notion of bisimulation between monotonic frames in order to obtain an equivalent notion.

Definition 8.21 (Strong bisimulation) Let $\mathbb{F}_1 = (W_1, \nu_1)$ and $\mathbb{F}_2 = (W_2, \nu_2)$ be monotonic $\mathcal{L}_\nabla$-frames. A non-empty relation $Z \subseteq W_1 \times W_2$ is a *strong bisimulation* between F_1 and $\mathbb{F}_2$ if the following conditions are met:

$(\text{forth})_s$ If $(w_1, w_2) \in Z$ and $Y_1 \in \nu_1(w_1)$, then there is a $Y_2 \subseteq W_2$ such that

- $Y_2 \in \nu_2(w_2)$,
- for all $y_2 \in Y_2$ there is a $y_1 \in Y_1$ such that $(y_1, y_2) \in Z$, and
- $Z^{-1}[Y_2] \cap Y_1 \in \nu_1(w_1)$.

$(\text{back})_s$ If $(w_1, w_2) \in Z$ and $Y_2 \in \nu_2(w_2)$, then there is a $Y_1 \subseteq W_1$ such that

- $Y_1 \in \nu_1(w_1)$,
- for all $y_1 \in Y_1$ there is a $y_2 \in Y_2$ such that $(y_1, y_2) \in Z$, and
- $Z[Y_1] \cap Y_2 \in \nu_2(w_2)$. ⊣

It is obvious from the above definition that strong bisimulations are also bisimulations. The following example shows that strong bisimilarity really is a stronger concept.

Example 8.22 Consider the frames $\mathbb{F}_1 = (\{s_1, t_1, u_1, v_1\}, \nu_1)$ where $\nu_1^c(s_1) = \{\{t_1\}, \{u_1, v_1\}\}$, $\nu_1^c(u_1) = \{\{u_1\}\}$ and $\nu_1(t_1) = \nu_1(v_1) = \emptyset$; and $\mathbb{F}_2 = (\{s_2, t_2\}, \nu_2)$ where $\nu_2^c(s_2) = \{\{t_2\}\}$ and $\nu_2(t_2) = \emptyset$. It is straightforward to check that $Z = \{(s_1, s_2), (t_1, t_2), (v_1, t_2)\}$ is a bisimulation. In fact, Z is the maximal bisimulation on $\mathbb{F}_1$ and $\mathbb{F}_2$. But Z is not strong, since $\{u_1, v_1\} \in \nu_1(s_1)$, and the only neighbourhood in $\nu_2(s_2)$ which can satisfy the (forth) clause is $\{t_2\}$, but $Z^{-1}[\{t_2\}] \cap \{u_1, v_1\} = \{v_1\} \notin \nu_1(s_1)$. It is not too hard to see that this problem will occur for any bisimulation, thus s_1 and s_2 are bisimilar, but not strongly bisimilar. Similarly, $\mathbb{F}_1$ and $\mathbb{F}_2$ are bisimilar systems, but also not strongly bisimilar.

We will now show that the extra condition in Definition 8.21 is exactly what is needed in order to be able to obtain an $Up\mathcal{P}$-coalgebra bisimulation.

Proposition 8.23 *Let (W_1, ν_1) and (W_2, ν_2) be two $Up\mathcal{P}$-coalgebras, and $\emptyset \neq Z \subseteq W_1 \times W_2$ a relation. Then Z is an $Up\mathcal{P}$-coalgebra bisimulation if and only if Z is a strong bisimulation.*

Proof. By Definition 8.4, Z is a $Up\mathcal{P}$-coalgebra bisimulation if and only if there is a $\mu : Z \to Up\mathcal{P}(Z)$ such that for all $(x_1, x_2) \in Z$: $\nu_1(x_1) = \uparrow\pi_1^*[\mu(x_1, x_2)]$ and $\nu_2(x_2) = \uparrow\pi_2^*[\mu(x_1, x_2)]$. That is for all $(x_1, x_2) \in Z$,

(49) $\forall Y_1 \subseteq W_1 \, (Y_1 \in \nu_1(x_1) \iff \exists C \in \mu(x_1, x_2)(\pi_1[C] \subseteq Y_1))$,

and

(50) $\forall Y_2 \subseteq W_2 \, (Y_2 \in \nu_2(x_2) \iff \exists C \in \mu(x_1, x_2)(\pi_2[C] \subseteq Y_2))$.

Observe that (49) and (50) entail that

(51) $\forall C \in \mu(x_1, x_2)(\pi_1[C] \in \nu_1(x_1) \;\&\; \pi_2[C] \in \nu_2(x_2))$.

"$\Longrightarrow$": Assume that Z is an $Up\mathcal{P}$-coalgebra bisimulation with $\mu : Z \to Up\mathcal{P}(Z)$ satisfying (49) and (50). To show the $(\text{forth})_s$ condition for Z, suppose that $(w_1, w_2) \in Z$ and $Y_1 \in \nu_1(w_1)$. Then by (49) there is a $C \in \mu(w_1, w_2)$ such that $\pi_1[C] \subseteq Y_1$. Define $Y_2 := \pi_2[C]$, then it follows from (50) that $Y_2 \in \nu_2(w_2)$.

To show the second part of the $(\text{forth})_s$ clause, let $v_2 \in Y_2 = \pi_2[C]$. Then there is a $v_1 \in W_1$ such that $(v_1, v_2) \in C$. Since $\pi_1[C] \subseteq Y_1$ and $C \subseteq Z$, we have $v_1 \in Y_1$ and $(v_1, v_2) \in Z$. To see that $Z^{-1}[Y_2] \cap Y_1 \in \nu_1(w_1)$, we first note that since $C \subseteq Z$, we have

$$\pi_1[C] \subseteq Z^{-1}[\pi_2[C]] = Z^{-1}[Y_2].$$

Hence since $\pi_1[C] \subseteq Y_1$, we obtain $\pi_1[C] \subseteq Z^{-1}[Y_2] \cap Y_1$. From $C \in \mu(w_1, w_2)$ and (51) it follows that $\pi_1[C] \in \nu_1(w_1)$, so by upwards closure of $\nu_1(w_1)$, we may conclude that $Z^{-1}[Y_2] \cap Y_1 \in \nu_1(w_1)$.

The $(\text{back})_s$ condition is shown analogously; we leave it to the reader to work out the details.

"$\Longleftarrow$": Assume that Z is a strong bisimulation between the monotonic $\mathcal{L}_\nabla$-frames $\mathbb{F}_1 = (W_1, \nu_1)$ and $\mathbb{F}_2 = (W_2, \nu_2)$. We must now define a map $\mu : Z \to Up\mathcal{P}(Z)$ such that (49) and (50) hold. For $C \subseteq Z$ and $(x_1, x_2) \in Z$, we define,

(52) $C \in \mu_l(x_1, x_2) \quad \text{iff} \quad \pi_1[C] \in \nu_1(x_1) \;\&\; \pi_2[C] \in \nu_2(x_2)$.

Then μ_l is clearly upwards closed. To see that (49) holds, first note that from (52) and the upwards closure of $\nu_1(x_1)$, it is clear that the inclusion from right to left holds. For the inclusion from left to right, suppose $(x_1, x_2) \in Z$ and $Y_1 \in \nu_1(w_1)$. We need a $C \subseteq Z$ such that for $i \in \{1, 2\}$, $\pi_i[C] \in \nu_i(x_1)$, and $\pi_1[C] \subseteq Y_1$. From the $(\text{forth})_s$ clause, we obtain a $Y_2 \in \nu_2(x_2)$ such that

(53) $Z^{-1}[Y_2] \cap Y_1 \in \nu_1(x_1)$,

and

(54) $\forall y_2 \in Y_2 \, \exists y_1 \in Y_1 : (y_1, y_2) \in Z$.

We have $\pi_1[(Y_1 \times Y_2) \cap Z] = Z^{-1}[Y_2] \cap Y_1$, so by (53), $\pi_1[(Y_1 \times Y_2) \cap Z] \in \nu_1(x_1)$. From (54) it follows that $\pi_2[(Y_1 \times Y_2) \cap Z] = Y_2 \in \nu_2(x_2)$. Hence if we take $C = (Y_1 \times Y_2) \cap Z$, we have just shown that $C \in \mu_l(x_1, x_2)$, and it is also clear that $\pi_1[C] \subseteq Y_1$.

(50) is shown analogously. QED

Remark 8.24 Rutten [59] shows that a T-coalgebra morphism is also a T-coalgebra bisimulation for arbitrary T. Thus by Propositions 8.10 and 8.23 we may conclude that a bounded morphism between monotonic frames is a functional strong bisimulation. We can also show this directly. Assume that $f : W_1 \to W_2$ is a bounded morphism between the monotonic $\mathcal{L}_\nabla$-frames $\mathbb{F}_1 = (W_1, \nu_1)$ and $\mathbb{F}_2 = (W_2, \nu_2)$. For the (forth)$_s$ condition, suppose $x \in W_1$ and $Y_1 \in \nu_1(x)$. Then by the (BM1) condition for f, we have $Y_2 := f[Y_1] \in \nu_2(f(x))$, and for all $y_2 \in Y_2 = f[Y_1]$ there is a $y_1 \in Y_1$ such that $f(y_1) = y_2$, i.e., $(y_1, y_2) \in f$. To see that $f^{-1}[Y_2] \cap Y_1 \in \nu_1(x)$, note that $Y_1 \subseteq f^{-1}[f[Y_1]] = f^{-1}[Y_2]$, hence $f^{-1}[Y_2] \cap Y_1 = Y_1 \in \nu_1(x)$.

For the (back)$_s$ condition, suppose that $x \in W_1$ and $Y_2 \in \nu_2(f(x))$. Then by the (BM2) condition for f there is a $Y_1 \in \nu_1(x)$ such that $f[Y_1] \subseteq Y_2$. Again, it is clear that for all $y_1 \in Y_1$ there is a $y_2 \in Y_2$ (namely $y_2 = f(y_1)$) such that $(y_1, y_2) \in f$. Now, we still need to show that $f[Y_1] \cap Y_2 \in \nu_2(f(x))$. Since $f[Y_1] \subseteq Y_2$, we have $f[Y_1] \cap Y_2 = f[Y_1]$. From $Y_1 \in \nu_1(x)$ and the (BM1) condition for f, we also obtain $f[Y_1] \in \nu_2(f(x))$, hence $f[Y_1] \cap Y_2 \in \nu_2(f(x))$.

The extra condition that is required to make a bisimulation Z strong, will fail if, for example, $Y_1 \in \nu_1^c(x_1)$ and Y_1 contains a state which is not in $dom(Z)$ in which case $Z^{-1}[Y_2] \cap Y_1 \subsetneq Y_1$ for any Y_2, and since Y_1 is a core neighbourhood, $Z^{-1}[Y_2] \cap Y_1 \notin \nu_1(x_1)$.

This failure can be eliminated if we consider full bisimulations, that is bisimulations $Z \subseteq W_1 \times W_2$ where $dom(Z) = W_1$ and $ran(Z) = W_2$. As a consequence we can show that full bisimulations are also $Up\mathcal{P}$-coalgebra bisimulations.

Proposition 8.25 *Let (W_1, ν_1) and (W_2, ν_2) be two $Up\mathcal{P}$-coalgebras, and $\emptyset \neq Z \subseteq W_1 \times W_2$. Then the following holds: If Z is a full bisimulation between (W_1, ν_1) and (W_2, ν_2), then Z is an $Up\mathcal{P}$-bisimulation between (W_1, ν_1) and (W_2, ν_2) .*

Proof. Assume that Z is a full bisimulation between $\mathbb{F}_1 = (W_1, \nu_1)$ and $\mathbb{F}_2 = (W_2, \nu_2)$ viewed as monotonic $\mathcal{L}_\nabla$-frames. We must then define a map $\mu : Z \to Up\mathcal{P}(Z)$ such that for all $(x_1, x_2) \in Z$: $\nu_1(x_1) = (\pi_1^{-1})^{-1}[\mu(x_1, x_2)]$ and $\nu_2(x_2) = (\pi_2^{-1})^{-1}[\mu(x_1, x_2)]$. These two identities are equivalent with the following conditions:

(55) $\forall Y_1 \subseteq W_1\, (Y_1 \in \nu_1(x_1) \iff \pi_1^{-1}[Y_1] \in \mu(x_1, x_2))$,

and

(56) $\forall Y_2 \subseteq W_2\, (Y_2 \in \nu_2(x_2) \iff \pi_2^{-1}[Y_2] \in \mu(x_1, x_2))$.

Define μ_s as follows: For all $C \subseteq Z, (x_1, x_2) \in Z$,

(57) $C \in \mu_s(x_1, x_2)$ iff $\exists C_1 \in \nu_1(x_1) : \pi_1^{-1}[C_1] \subseteq C$ or $\exists C_2 \in \nu_2(x_2) : \pi_2^{-1}[C_1] \subseteq C$.

Clearly, μ_s is upwards closed. To see that (55) holds, note that the $\Rightarrow$-direction is trivially fulfilled by the definition of μ_s. For the $\Leftarrow$-direction, let $Y_1 \subseteq W_1$ and assume $\pi_1^{-1}[Y_1] \in \mu_s(x_1, x_2)$. Then by (57), there is either a $C_1 \in \nu_1(x_1)$ such that $\pi_1^{-1}[C_1] \subseteq \pi_1^{-1}[Y_1]$ or there is a $C_2 \in \nu_2(x_2)$ such that $\pi_2^{-1}[C_2] \subseteq \pi_1^{-1}[Y_1]$.

We first treat the C_1-case: $\pi_1^{-1}[C_1] \subseteq \pi_1^{-1}[Y_1]$ implies $\pi_1[\pi_1^{-1}[C_1]] \subseteq \pi_1[\pi_1^{-1}[Y_1]]$, and since $dom(Z) = W_1$, this is equivalent with $C_1 \subseteq Y_1$. As $C_1 \in \nu_1(x_1)$, we obtain $Y_1 \in \nu_1(x_1)$ by upwards closure.

Now, for the C_2-case: By the (back) condition for Z, there is a $C_1 \in \nu_1(x_1)$ such that $\forall c_1 \in C_1 \exists c_2 \in C_2 : (c_1, c_2) \in Z$, i.e., $C_1 \subseteq \pi_1[\pi_2^{-1}[C_2]]$. Together with $\pi_2^{-1}[C_1] \subseteq \pi_1^{-1}[Y_1]$, this implies

$$C_1 \subseteq \pi_1[\pi_2^{-1}[C_2]] \subseteq \pi_1[\pi_1^{-1}[Y_1] \subseteq Y_1.$$

Hence, as $C_1 \in \nu_1(x_1)$, also $Y_1 \in \nu_1(x_1)$.

(56) is shown in a similar manner. QED

Remark 8.26 As an alternative to the above proof, we could have shown directly that full bisimulations are strong bisimulations. We only sketch the proof: When $(x_1, x_2) \in Z$ and $Y_1 \in \nu_1(x_1)$ then by the (forth) clause there is a $Y_2' \in \nu_2(x_2)$ such that $Y_2' \subseteq Z[Y_1]$. Taking $Y_2 := Z[Y_1]$, then $Y_1 \subseteq Z^{-1}[Y_2]$ since $Y_1 \subseteq W_1 = dom(Z)$, hence $Z^{-1}[Y_2] \cap Y_1 = Y_1 \in \nu_1(x_1)$. Similarly for the $(\text{back})_s$ condition.

The following easy example shows that strong bisimulations need not be full bisimulations.

Example 8.27 Let $\mathbb{F}_1 = (\{s_1, t_1, u_1\}, \nu_1)$ and $\mathbb{F}_2 = (\{s_2, t_2\})$ where $\nu_i^c(s_i) = \{\{t_i\}\}$, $\nu_i(t_i) = \emptyset$, for $i \in \{1, 2\}$, and $\nu_1^c(u_1) = \{\{u_1\}\}$. Then $Z = \{(s_1, s_2), (t_1, t_2)\}$ is a (maximal) strong bisimulation, but clearly not a full bisimulation.

Remark 8.28 The reason why we gave the proof of Proposition 8.25, was to demonstrate that the function $\mu : Z \to Up\mathcal{P}(Z)$ can be defined in a number of ways. Since $\uparrow\pi_i^*[\mu(x_1, x_2)] = (\pi_i^{-1})^{-1}[\mu(x_1, x_2)]$, the conditions (49) and (50) are, of course, equivalent with (55) and (56), and the latter clearly express that the projections are bounded morphisms, as Definition 8.4 demands.

Observe that μ_l is the map we obtain by taking the (BM1) condition as our definition, and μ_s is obtained by taking (BM2) as the definition. There is an obvious analogy here with the smallest and largest filtrations (cf. section 4.2), and we have indeed chosen our notation to reflect this.

Lemma 8.29 *Let (W_1, ν_1) and (W_2, ν_2) be $Up\mathcal{P}$-coalgebras, $Z \subseteq W_1 \times W_2$ a non-empty relation, and let $\mu : Z \to Up\mathcal{P}(Z)$ be a map satisfying (55) and (56). Then for all $(x_1, x_2) \in Z$,*

$$\mu_s(x_1, x_2) \subseteq \mu(x_1, x_2) \subseteq \mu_l(x_1, x_2).$$

Proof. Let $(x_1, x_2) \in Z$ and $C \subseteq Z$. For the first inclusion, assume $C \in \mu_s(x_1, x_2)$. Then for some $i \in \{1, 2\}$ there is a $C_i \in \nu_i(x_i)$ such that $\pi_i^{-1}[C_i] \subseteq C$. Depending on i, it follows from (55) or (56) together with $C_i \in \nu_i(x_i)$ that $\pi_i^{-1}[C_i] \in \mu(x_1, x_2)$, and since $\pi_i^{-1}[C_i] \subseteq C$, by upwards closure we have $C \in \mu(x_1, x_2)$.

The second inclusion is easily seen to hold, since $C \in \mu(x_1, x_2)$, (55) and (56) imply that $\pi_i[C] \in \nu_i(x_i)$ for all $i \in \{1, 2\}$. QED

The following example shows that μ_s and μ_l are not always the same.

Example 8.30 Consider the frames $\mathbb{F}_i = (W_i = \{s_i, t_i\}, \nu_i)$ where $\nu_i(s_i) = \nu_i(t_i) = \{W_i\}$ for $i \in \{1, 2\}$. Then $Z = W_1 \times W_2$ is a full bisimulation, and $\mu_l^c(w_1, w_2) = \{D, E\}$ where $D = \{(s_1, s_2), (t_1, t_2)\}$ and $E = \{(s_1, t_2), (t_1, s_2)\}$, for all $(w_1, w_2) \in Z$, since $\pi_i[D] = \pi_i[E] = W_i \in \nu_i(w_i)$ for $i \in \{1, 2\}$. However, $\mu_s(w_1, w_2) = \{Z\}$, for all $(w_1, w_2) \in Z$, since $\pi_i^{-1}[W_i] = Z$ for $i \in \{1, 2\}$.

The proof of Proposition 8.23 shows that when Z is a strong bisimulation, then we can always take μ_l to show that Z is an $Up\mathcal{P}$-coalgebra bisimulation. On the other hand, the proof of Proposition 8.25 shows that when Z is a full bisimulation, then we can endow Z with the structure μ_s to show that Z is an $Up\mathcal{P}$-coalgebra bisimulation. Since we also know that full bisimulations form a strict subset of the strong bisimulations, we may ask whether μ_s will also work for strong, but not full bisimulations. We will have to leave this questions open.

Behavioural equivalence

As it turns out, the concept of behavioural equivalence ties in better with the frame theoretic notion of bisimulation.

Recall from Definition 8.5 that for two $Up\mathcal{P}$-coalgebras (X_1, ν_1) and (X_2, ν_2), two states $s_1 \in X_1$ and $s_2 \in X_2$ are behaviourally equivalent if they can be identified via two $Up\mathcal{P}$-coalgebra morphisms $f_i : X_i \to Y$ in some $Up\mathcal{P}$-coalgebra (Y, δ).

$$\begin{tikzcd}
(X_1,\nu_1) \arrow[dr, "f_1"'] & & (X_2,\nu_2) \arrow[dl, "f_2"] \\
& (Y,\delta) &
\end{tikzcd}$$

Suppose that in the above diagram, we consider the relation

$$\mathrm{INSEP}(f_1, f_2) = \{(x_1, x_2) \in X_1 \times X_2 \mid f_1(x_1) = f_2(x_2)\}.$$

Then the fact that f_1 and f_2 are $Up\mathcal{P}$-coalgebra morphisms implies that $\mathrm{INSEP}(f_1, f_2)$ is a bisimulation between the monotonic $\mathcal{L}_\nabla$-frames (X_1, ν_1) and (X_2, ν_2).

Proposition 8.31 *Let (X_1, ν_1), (X_2, ν_2) and (Y, δ) be $Up\mathcal{P}$-coalgebras, and assume that $f_i : X_i \to Y$, $i \in \{1, 2\}$, are $Up\mathcal{P}$-coalgebra morphisms and* $\mathrm{INSEP}(f_1, f_2) = \{(x_1, x_2) \in X_1 \times X_2 \mid f_1(x_1) = f_2(x_2)\}$ *is non-empty. Then* $\mathrm{INSEP}(f_1, f_2)$ *is a bisimulation between (X_1, ν_1) and (X_2, ν_2).*

Proof. For notational convenience, let $\mathrm{INSEP} := \mathrm{INSEP}(f_1, f_2)$. To show the bisimulation (forth) condition for INSEP, assume $(s_1, s_2) \in \mathrm{INSEP}$ and $X_1 \in \nu_1(s_1)$. From the (BM1) condition for f_1, we have $f_1[X_1] \in \nu_0(f_1(s_1))$, and since $(s_1, s_2) \in \mathrm{INSEP}$, $f_1(s_1) = f_2(s_2)$, hence $f_1[X_1] \in \nu_0(f_2(s_2))$. Now we apply the (BM2) condition for f_2, to obtain an $X_2 \subseteq W_2$ such that $f_2[X_2] \subseteq f_1[X_1]$ and $X_2 \in \nu_2(s_2)$. From $f_2[X_2] \subseteq f_1[X_1]$ it follows that $\forall x_2 \in X_2\, \exists x_1 \in X_1 : f_1(x_1) = f_2(x_2)$. The bisimulation (*back*) condition for INSEP is shown in a similar way. QED

When the exact definition of a diagram $(X_1, \nu_1) \xrightarrow{f_1} (Y, \delta) \xleftarrow{f_2} (X_2, \nu_2)$ is assumed or irrelevant, we will refer to $\mathrm{INSEP}(f_1, f_2)$ simply as an INSEP-relation. The question that comes to mind now, is whether INSEP-relations are also strong bisimulations. The frames of Example 8.22 show that the answer is no.

Example 8.32 Recall the frames from Example 8.22:

$$\begin{array}{rcl} \mathbb{F}_1 & = & (W_1, \nu_1) \text{ where} \\ W_1 & = & \{s_1, t_1, u_1, v_1\}, \\ \nu_1^c(s_1) & = & \{\{t_1\}, \{u_1, v_1\}\}, \\ \nu_1^c(u_1) & = & \{\{u_1\}\}, \\ \nu_1(t_1) = \nu_1(v_1) & = & \emptyset. \end{array} \qquad \begin{array}{rcl} \mathbb{F}_2 & = & (W_2, \nu_2) \text{ where} \\ W_2 & = & \{s_2, t_2\}, \\ \nu_2^c(s_2) & = & \{\{t_2\}\}, \\ \nu_2(t_2) & = & \emptyset. \end{array}$$

Also consider the following isomorphic copy of $\mathbb{F}_2$:

$$\begin{array}{rcl} \mathbb{G} & = & (Y, \mu) \text{ where} \\ Y & = & \{x, y\}, \\ \mu^c(x) & = & \{\{y\}\}, \\ \mu(y) & = & \emptyset. \end{array}$$

Then the bisimulation $Z = \{(s_1, s_2), (t_1, t_2), (v_1, t_2)\}$ is the INSEP(f_1, f_2)-relation of $f_i : W_i \to Y$, $i \in \{1, 2\}$, where $f_1(s_1) = f_2(s_2) = x$ and $f_1(u_1) = f_1(v_1) = f_2(t_2) = y$. But as we already know, Z is not a strong bisimulation, and there is no strong bisimulation linking s_1 and s_2.

Thus, like bisimilarity, behavioural equivalence of states is a weaker notion than $Up\mathcal{P}$-coalgebra bisimilarity of states, and we will show that the two notions are equivalent. However, first we will show that we can take quotients over maximal bisimulations. This is needed in the proof of Proposition 8.34, but is also interesting in its own right. Before we proceed we should remark that bisimulations are closed under unions (this is easy to show, and we leave out the proof), hence for bisimilar frames there is always a maximal bisimulation. Also, any monotonic frame is obviously bisimilar to itself.

Lemma 8.33 (Bisimulation Quotients) *Let $\mathbb{F} = (W, \nu)$ be a monotonic $\mathcal{L}_\nabla$-frame, and $Z \subseteq W \times W$ the maximal bisimulation on $\mathbb{F}$. Then Z defines an equivalence relation $\equiv_Z$ on W, and there is a map $\mu : |W| \to \mathcal{P}(\mathcal{P}(|W|))$ where $|W|$ denotes $W/\equiv_Z$ such that the natural map $\varepsilon : W \to W/\equiv_Z$ is a bounded morphism.*

Proof. In order to prove that Z is an equivalence relation, one can show that the reflexive, symmetric and transitive closure of Z is again a bisimulation, hence contained in Z. We leave out the details.

Now we define a neighbourhood function μ on $W/\equiv_Z$ by,

$$X \in \mu(\varepsilon(w)) \text{ iff } \varepsilon^{-1}[X] \in \nu(w).$$

Then μ is upwards closed, and once we have have shown that μ is well-defined, the lemma is immediate. To do so, we must show that for all $x, y \in W$ and all $X \subseteq |W|$, if $\varepsilon(x) = \varepsilon(y)$ then

(58) $\varepsilon^{-1}[X] \in \nu(x)$ iff $\varepsilon^{-1}[X] \in \nu(y)$.

So assume $\varepsilon(x) = \varepsilon(y)$. Then in particular, $(x, y) \in Z$. We only show the direction from left to right in (58). Suppose $\varepsilon^{-1}[X] \in \nu(x)$. Then by the (forth) condition for Z, there is a $C \in \nu(y)$ such that for all $c \in C$ there is a $d \in \varepsilon^{-1}[X]$ such that $(d, c) \in Z$. When $d \in \varepsilon^{-1}[X]$, we have $\varepsilon(d) \in X$, and $(d, c) \in Z$ implies that $\varepsilon(c) = \varepsilon(d) \in X$. Thus the (forth) condition says that we have a $C \in \nu(y)$ such that $\varepsilon[C] \subseteq X$, and hence $C \subseteq \varepsilon^{-1}[X]$, so by upwards closure of $\nu(y)$, we may conclude that $\varepsilon^{-1}[X] \in \nu(y)$. QED

Proposition 8.34 *Let (X_1, ν_1) and (X_2, ν_2) be two $Up\mathcal{P}$-coalgebras, and assume that $Z \subseteq X_1 \times X_2$ is a bisimulation such that $(s_1, s_2) \in Z$. Then s_1 and s_2 are behaviourally equivalent.*

Proof. We must find an $Up\mathcal{P}$-coalgebra (Y, μ) and $Up\mathcal{P}$-coalgebra morphisms $f_i : X_i \to Y$ such that $f_1(s_1) = f_2(s_2)$.

Since (X_1, ν_1) and (X_2, ν_2) may simply be seen as monotonic $\mathcal{L}_\nabla$-frames, we can form their disjoint union $(X_1 \uplus X_2, \nu)$ (see Definition 4.1). Then the inclusion maps $k_i : X_i \to X_1 \uplus X_2$, $i \in \{1, 2\}$, are bounded morphisms, or equivalently, $Up\mathcal{P}$-coalgebra morphisms, and Z is a bisimulation on $(X_1 \uplus X_2, \nu)$. Let Z_M denote the maximal bisimulation on $(X_1 \uplus X_2, \nu)$, then $Z \subseteq Z_M$, and by Lemma 8.33 we can take the quotient (Y, μ) of $(X_1 \uplus X_2, \nu)$ with Z_M. That is, $Y = (X_1 \uplus X_2)/\equiv_{Z_M}$ and $\mu : Y \to Up\mathcal{P}(Y)$ is defined such that the natural map $\varepsilon : (X_1 \uplus X_2) \to Y$ is an $Up\mathcal{P}$-coalgebra morphism. Thus if we take $f_i := \varepsilon \circ k_i : X_i \to Y$, then f_i is an $Up\mathcal{P}$-coalgebra morphism from (X_i, ν_i) to (Y, μ), $i \in \{1, 2\}$, and since $(s_1, s_2) \in Z \subseteq Z_M$ it follows that $f_1(s_1) = \varepsilon(s_1) = \varepsilon(s_2) = f_2(s_2)$.

$$
\begin{tikzcd}
(X_1, \nu_1) \arrow[dr, "k_1"] \arrow[ddr, bend right, "f_1"'] & & (X_2, \nu_2) \arrow[dl, "k_2"'] \arrow[ddl, bend left, "f_2"] \\
& (X_1 \uplus X_2, \nu) \arrow[d, "\varepsilon"] & \\
& (Y, \mu) &
\end{tikzcd}
$$

QED

From Propositions 8.31 and 8.34 we may conclude the following.

Theorem 8.35 (State equivalence) *Let (X_1, ν_1) and (X_2, ν_2) be two $Up\mathcal{P}$-coalgebras. Then two states $s_1 \in X_1$ and $s_2 \in X_2$ are behaviourally equivalent if and only if s_1 and s_2 are bisimilar.*

Remark 8.36 If Z is the INSEP(f_1, f_2)-relation of two $Up\mathcal{P}$-coalgebra morphisms f_1 and f_2, then (Z, π_1, π_2) is the pullback in Set of f_1 and f_2. Rutten [59] shows that when a functor T preserves weak pullbacks, then INSEP-relations are also T-coalgebra bisimulations; in other words, INSEP-relations are weak pullbacks in the category Set_T of T-coalgebras and T-coalgebra morphisms. Since we know from Example 8.32 that INSEP-relations are not necessarily $Up\mathcal{P}$-coalgebra bisimulations, we can conclude that $Up\mathcal{P}$ does not preserve weak pullbacks. This may not be so surprising since $2^{(\cdot)} \circ 2^{(\cdot)}$ does not preserve weak pullbacks (see also [59]), and $Up\mathcal{P}$ and $2^{(\cdot)} \circ 2^{(\cdot)}$ work in the same way on functions.

We end this section with a theorem which sums up the relationships between the various system equivalence notions.

Theorem 8.37 (System equivalence) *Let (X_1, ν_1) and (X_2, ν_2) be two $Up\mathcal{P}$-coalgebras. Then the following are equivalent.*

(i) (X_1, ν_1) *and* (X_2, ν_2) *are behaviourally equivalent systems,*
(ii) *there exists a full bisimulation between* (X_1, ν_1) *and* (X_2, ν_2)*,*
(iii) *there exists a full $Up\mathcal{P}$-coalgebra bisimulation between* (X_1, ν_1) *and* (X_2, ν)*.*

Proof. (i) $\Rightarrow$ (ii): Assume that (X_1, ν_1) and (X_2, ν_2) are behaviourally equivalent. Then there is an $Up\mathcal{P}$-coalgebra (Y, δ) and $Up\mathcal{P}$-coalgebra morphisms f_1 and f_2 such that $f_i : (X_1, \nu_1) \twoheadrightarrow (Y, \delta)$, $i \in \{1, 2\}$. From Proposition 8.31 we know that $\mathrm{INSEP}(f_1, f_2)$ is a bisimulation., and due to the surjectivity of f_1 and f_2, it is easy to see that $\mathrm{INSEP}(f_1, f_2)$ is also a full bisimulation.
(ii) $\Rightarrow$ (i): This can be proved in the same way as Proposition 8.34. All we need to do is observe that when Z is a full bisimulation between (X_1, ν_1) and (X_2, ν_2), then every equivalence class in $|X_1 \uplus X_2|$ will contain elements from both X_1 and X_2, and this clearly implies that the $f_i = \varepsilon \circ k_i$, defined as in the proof of Proposition 8.34 are surjective, $i \in \{1, 2\}$.
(ii) $\Leftrightarrow$ (iii): Clear by Propositions 8.25 and 8.23. QED

The diagram below illustrates the relationships between the various model theoretic and coalgebraic notions of bisimulations. An arrow from one property P1 to another P2 indicates that if a relation is P1, then it is also P2. The diagram is of course transitive.

$$\begin{tikzcd}
 & \text{full bisimulation} \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{bisimulation} & \text{strong bisimulation} \arrow[l] \arrow[r, leftrightarrow] & Up\mathcal{P}\text{-coalgebra bisimulation} \\
 & \text{bounded morphism} \arrow[ul] \arrow[u] \arrow[r, leftrightarrow] & Up\mathcal{P}\text{-coalgebra morphism} \arrow[u]
\end{tikzcd}$$

9 Interpolation

In this section, we will investigate the relationship between interpolation in a modal logic Λ and superamalgamation of the corresponding variety V_Λ. Since we are working with the local consequence relation, we will be concerned with the Craig Interpolation Property (CIP). Superamalgamation (SUPAP) of varieties has provided algebraic characterizations of CIP for a large class of modal logics, where it is possible to show that: Λ has CIP iff V_Λ has SUPAP. However, we have found only little in the literature regarding interpolation in monotonic modal logics or superamalgamation in BAM-varieties, although Madarász [48, 49] generalises results for BAO-varieties to BAE-varieties in which the added operation f is non-normal, i.e. $f(0) \neq 0$, but still additive, and she also provides some results on the limitations of the CIP-SUPAP relationship.

9.1 General Definitions and Results

Definition 9.1 (Interpolation) A modal logic Λ over a language $\mathcal{L}$ has the *Craig Interpolation Property (CIP)* if for any formulas $\varphi, \psi \in \mathcal{L}$ such that $\vdash_\Lambda \varphi \to \psi$, there is a formula $\theta \in \mathcal{L}$ such that $\mathrm{FV}(\theta) \subseteq \mathrm{FV}(\varphi) \cap \mathrm{FV}(\psi)$ and $\vdash_\Lambda \varphi \to \theta$, $\vdash_\Lambda \theta \to \psi$. θ is called an *interpolant.* $\dashv$

Definition 9.2 (Superamalgamation) Let K be a class of algebras such that each $\mathbb{A} \in \mathsf{K}$ has a partial ordering. K has the *superamalgamation property (SUPAP)* if, for any $\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2 \in$

K and embeddings e_1, e_2 such that $\mathbb{A}_1 \overset{e_1}{\leftarrowtail} \mathbb{A}_0 \overset{e_2}{\rightarrowtail} \mathbb{A}_2$, there exists an $\mathbb{A} \in \mathsf{K}$ and embeddings g_1, g_2 such that

1. $\mathbb{A}_1 \overset{g_1}{\rightarrowtail} \mathbb{A} \overset{g_2}{\leftarrowtail} \mathbb{A}_2$.
2. $g_1 \circ e_1 = g_2 \circ e_2$.
3. $\forall x_1 \in \mathbb{A}_1, \forall x_2 \in \mathbb{A}_2$,
$$\begin{array}{lcl} g_1(x_1) \leq g_2(x_2) & \Rightarrow & \exists x_0 \in \mathbb{A}_0 : x_1 \leq e_1(x_0) \text{ and } e_2(x_0) \leq x_2 \\ g_2(x_2) \leq g_1(x_1) & \Rightarrow & \exists x_0 \in \mathbb{A}_0 : x_2 \leq e_2(x_0) \text{ and } e_1(x_0) \leq x_1. \end{array}$$

$$\begin{tikzcd} & \mathbb{A} & \\ \mathbb{A}_1 \arrow[ur, tail, "g_1"] & & \mathbb{A}_2 \arrow[ul, tail, "g_2"'] \\ & \mathbb{A}_0 \arrow[ul, tail, "e_1"] \arrow[ur, tail, "e_2"'] & \end{tikzcd}$$

$\dashv$

Showing that Λ has CIP under the assumption that V_Λ has SUPAP can be done under very general circumstances. In most of the encountered literature [11, 43, 50] this implication is proved by showing that $\nvdash_\Lambda \varphi \to \psi$ whenever V_Λ has SUPAP and φ, ψ have no interpolant. However, this proof method is rather involved. Below we show the desired result directly, and it relies on the simple observation that when Λ has CIP then the free algebras of V_Λ can be superamalgamated.

Theorem 9.3 (SUPAP $\Rightarrow$ CIP) *Let Λ be a modal $\mathcal{L}_\nabla$-logic, and V_Λ the variety of* BAE*s defined by Λ. Then Λ has CIP if V_Λ has SUPAP.*

Proof. Assume that V_Λ has SUPAP, and suppose $\vdash_\Lambda \varphi_1 \to \varphi_2$ for $\varphi_1, \varphi_2 \in \mathcal{L}_\nabla$. Let $\Phi_i = \text{FV}(\varphi_i)$, $i = 1, 2$, $\Phi_0 = \Phi_1 \cap \Phi_2$, $\Phi_{12} = \Phi_1 \cup \Phi_2$, and let $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_0$ and $\mathcal{L}_{12}$ denote the corresponding languages.

Let $\mathbb{F}_\iota$ be the V_Λ-free algebra generated by $[\Phi_\iota]$, $\iota \in \{0, 1, 2, 12\}$. For $\varphi \in \mathcal{L}_\iota$, $[\varphi]_\iota$ denotes the equivalence class of φ in $\mathbb{F}_\iota$, and $[\Phi_\iota] = \{[p]_\iota \mid p \in \Phi_\iota\}$, $\iota \in \{0, 1, 2, 12\}$. Then $\mathbb{F}_\iota \in \mathsf{V}_\Lambda$, and $\mathbb{F}_\iota$ has a partial ordering $\leq_\iota$ defined by: $[\varphi]_\iota \leq_\iota [\psi]_\iota$ iff $\vdash_\Lambda \varphi \to \psi$, $\iota \in \{0, 1, 2, 12\}$. Furthermore, we have embeddings e_1, e_2, e_1', e_2' defined by $e_i([\varphi]_0) = [\varphi]_i$, $e_i'([\varphi]_i) = [\varphi]_{12}$, $i = 1, 2$ such that $\mathbb{F}_1 \overset{e_1}{\leftarrowtail} \mathbb{F}_0 \overset{e_2}{\rightarrowtail} \mathbb{F}_2$ and $\mathbb{F}_1 \overset{e_1'}{\rightarrowtail} \mathbb{F}_{12} \overset{e_2'}{\leftarrowtail} \mathbb{F}_2$. Hence by the assumption that V_Λ has SUPAP there are $\mathbb{A} \in \mathsf{V}_\Lambda$ and embeddings g_1, g_2 such that the three conditions of Definition 9.2 hold.

Claim 1 $g_1([\varphi_1]_1) \leq_{12} g_2([\varphi_2]_2)$.

Proof of Claim Define a map $h : [\Phi_{12}] \to \mathbb{A}$ by

$$h([p]_{12}) = \begin{cases} g_1([p]_1) & \text{if } p \in \Phi_1 \\ g_2([p]_2) & \text{if } p \in \Phi_2 \end{cases}$$

In case $p \in \Phi_0 = \Phi_1 \cap \Phi_2$, we must check that h is well-defined, but this is clear from Definition 9.2,2.

By the universal mapping property for free algebras, there is a unique extension $\hat{h}$ of h such that $\hat{h} : \mathbb{F}_{12} \to \mathbb{A}$ is a homomorphism. Since in $\mathbb{F}_{12}$, $[\varphi_1]_{12} \leq_{12} [\varphi_2]_{12}$, it follows from monotonicity of homomorphisms that $\hat{h}([\varphi_1]_{12}) \leq_{\mathbb{A}} \hat{h}([\varphi_2]_{12})$ $(*)$.

Note also that $\hat{h} \circ e_i'\restriction_{[\Phi_i]} = g_i\restriction_{[\Phi_i]}$ is a map from $[\Phi_i]$ to $\mathbb{A}$, $i = 1, 2$. Hence, again by the universal mapping property, $g_i\restriction_{[\Phi_i]}$ can be uniquely extended to a homomorphism from $\mathbb{F}_i$ to $\mathbb{A}$, and since $\hat{h} \circ e_i'$ and g_i are both homomorphisms extending $g_i\restriction_{[\Phi_i]}$ it follows that $\hat{h} \circ e_i' = g_i$. Together with $(*)$ this implies $g_1([\varphi_1]_1) = \hat{h}(e_1'([\varphi_1]_1)) \leq_{12} \hat{h}(e_2'([\varphi_2]_2)) = g_2([\varphi_2]_2)$. ◀

From the claim and SUPAP for V_Λ we obtain a $\theta \in \mathcal{L}_0$ such that

$$[\varphi_1]_1 \leq_1 e_1([\theta]_0) = [\theta]_1 \text{ and } [\theta]_2 = e_2([\theta]_0) \leq_2 [\varphi_2]_2,$$

hence $\text{FV}(\theta) \subseteq \text{FV}(\varphi_1) \cap \text{FV}(\varphi_2) = \Phi_0$, and

$$\vdash_\Lambda \varphi_1 \to \theta \text{ and } \vdash_\Lambda \theta \to \varphi_2$$

We have shown that Λ has CIP.

$$\begin{tikzcd}
\mathbb{A} & & \\
& \mathbb{F}_{12} \arrow[ul, "\hat{h}"'] & \mathbb{F}_2 \arrow[l, "e_2'"'] \arrow[ull, bend right, "g_2"'] \\
& \mathbb{F}_1 \arrow[u, "e_1'"] \arrow[uul, bend left, "g_1"] & \mathbb{F}_0 \arrow[l, "e_1"] \arrow[u, "e_2"']
\end{tikzcd}$$

QED

9.2 Bisimulation Products

Marx [51] provides sufficient conditions for SUPAP formulated in terms of Kripke frames, and here we will prove a version for monotonic frames in Lemma 9.8. The construction involves duality and *bisimulation products* which are a generalisation of the $Up\mathcal{P}$-coalgebras, or equivalently, the monotonic $\mathcal{L}_\nabla$-frames, (Z, μ_s) we constructed in the proof of Proposition 8.25, where we showed that when Z is a full bisimulation then Z is an $Up\mathcal{P}$-coalgebra bisimulation. The results will enable us to prove that a number of monotonic modal logics, including **M**, have CIP in Theorem 9.10.

Definition 9.4 (Direct product) Let $\mathbb{F}_i = (W_i, \nu_i)$, $i \in I$, be a collection of monotonic $\mathcal{L}_\nabla$-frames. The *direct product of* $(\mathbb{F}_i)_{i \in I}$ is defined as $\Pi_{i \in I}\mathbb{F}_i = (W, \nu)$ where

$$W = \Pi_{i \in I} W_i \quad \text{and}$$

$$\forall \overline{s} = (s_i)_{i \in I} \in W, \forall X \subseteq W : X \in \nu(\overline{s}) \text{ iff } \exists i \in I \exists C_i \in \nu_i(s_i) : \pi_i^{-1}[C_i] \subseteq X.$$

Here $\pi_j : \Pi_{i \in I}\mathbb{F}_i \to \mathbb{F}_j$ denotes the projection map from W to W_j. When $|I| = 2$, we will use the infix notation: $\mathbb{F}_1 \times \mathbb{F}_2$ instead of $\Pi_{i \in \{1,2\}}\mathbb{F}_i$. ⊣

From the above definition, it is clear that the direct product of monotonic $\mathcal{L}_\nabla$-frames is also monotonic.

Definition 9.5 (Bisimulation product) Let $(\mathbb{F}_i)_{i\in I}$ be a collection of monotonic $\mathcal{L}_\nabla$-frames. A monotonic frame $\mathbb{G} = (W, \nu)$ is a *bisimulation product of* $(\mathbb{F}_i)_{i\in I}$ if $\mathbb{G}$ is a subframe of $\Pi_{i\in I}\mathbb{F}_i$ and the projections $\pi_i : \mathbb{G} \to \mathbb{F}_i$, $i \in I$, are surjective bounded morphisms. $\dashv$

We will use the same notation for subframes as for submodels (recall Definition 4.14). Thus if $\mathbb{F} = (W, \nu)$ is a frame and and $X \subseteq W$ then $\mathbb{F}{\restriction}_X$ denotes the subframe $(X, \nu \cap (X \times \mathcal{P}(X)))$.

From Theorem 8.37, the following results are immediate.

Proposition 9.6 *Let $\mathbb{F}_i = (W_i, \nu_i)$, $i \in \{1, 2\}$, be monotonic $\mathcal{L}_\nabla$-frames, and $Z \subseteq W_1 \times W_2$. Then Z is a full bisimulation between $\mathbb{F}_1$ and $\mathbb{F}_2$ if and only if $(\mathbb{F}_1 \times \mathbb{F}_2){\restriction}_Z$ is a bisimulation product of $\mathbb{F}_1$ and $\mathbb{F}_2$.*

Proposition 9.7 *If $\mathbb{F}_i = (W_i, \nu_i)$, $i \in \{0, 1, 2\}$ are monotonic $\mathcal{L}_\nabla$-frames, and $f_i : W_i \twoheadrightarrow W_0$, are surjective bounded morphisms, $i \in \{1, 2\}$, then*

- $\mathrm{INSEP} = \{(x, y) \in W_1 \times W_2 \mid f_1(x) = f_2(y)\}$ *is a full bisimulation between $\mathbb{F}_1$ and $\mathbb{F}_2$.*
- $(\mathbb{F}_1 \times \mathbb{F}_2){\restriction}_{\mathrm{INSEP}}$ *is a bisimulation product of $\mathbb{F}_1$ and $\mathbb{F}_2$.*
- $f_1 \circ \pi_1 = f_2 \circ \pi_2$, *where $\pi_i : \mathrm{INSEP} \to W_i$, $i \in \{1, 2\}$, are the projection maps.*

$$\begin{tikzcd}
(\mathbb{F}_1 \times \mathbb{F}_2){\restriction}_{\mathrm{INSEP}} \arrow[r, two heads, "\pi_2"] \arrow[d, two heads, "\pi_1"'] & \mathbb{F}_2 \arrow[d, two heads, "f_2"] \\
\mathbb{F}_1 \arrow[r, two heads, "f_1"'] & \mathbb{F}_0
\end{tikzcd}$$

Bisimulation products of the form $(\mathbb{F}_1 \times \mathbb{F}_2){\restriction}_{\mathrm{INSEP}}$ will be called INSEP-products. Before we state the main technical result, recall that $\mathbb{A}_\sigma$ denotes the usual (σ-)ultrafilter frame of a BAM $\mathbb{A}$ (see Definition 7.14), and $\mathbb{A}_\pi$ denotes the π-ultrafilter frame defined in subsection 7.6.

Lemma 9.8 (Bisimulation product lemma) *Let K be a class of BAMs and F a class of monotonic $\mathcal{L}_\nabla$-frames. Then K has SUPAP if the following three conditions are satisfied:*

(i) *F is closed under taking finite bisimulation products.*
(ii) *For all $\mathbb{F}$ in F: $\mathbb{F}^+ \in \mathsf{K}$.*
(iii) *One of the following holds:*
  (σ) *For all $\mathbb{A}$ in K: $\mathbb{A}_\sigma \in \mathsf{F}$, or*
  (π) *For all $\mathbb{A} \in \mathsf{K}$: $\mathbb{A}_\pi \in \mathsf{F}$.*

Note that if K is a σ-canonical or π-canonical variety and $\mathsf{F} = \{\mathbb{F} \mid \mathbb{F}^+ \in \mathsf{K}\}$, then conditions (ii) and (iii) in the bisimulation product lemma always hold. Suppose, for example, that K is π-canonical. Then $\mathbb{A} \in \mathsf{K}$ implies that $\mathbb{A}^\pi = (\mathbb{A}_\pi)^+ \in \mathsf{K}$, and hence $\mathbb{A}_\pi \in \mathsf{F}$.

Proof of Lemma 9.8. The proof of this lemma is virtually identical to that of Lemma 5.2.6 in Marx [51], which relies only on the Kripke version of Proposition 9.7 and the duality between Kripke frames and BAOs. We have shown the analogues for monotonic $\mathcal{L}_\nabla$-frames and BAMs in Propositions 7.17, 7.19 and 7.49. QED

Proposition 9.9 *The following frame classes are closed under taking finite bisimulation products:*

(M) *The class* M *of all monotonic* $\mathcal{L}_\nabla$*-frames.*
(N) *The class* N *of monotonic* $\mathcal{L}_\nabla$*-frames satisfying*
  (n) $\forall w \in W : W \in \nu(w)$.
(P) *The class* P *of monotonic* $\mathcal{L}_\nabla$*-frames satisfying*
  (p) $\forall w \in W : \emptyset \notin \nu(w)$.
(4') *The class* $\mathsf{4}'$ *of monotonic* $\mathcal{L}_\nabla$*-frames satisfying*
  (iv') $\forall w \in W\ \forall X \subseteq W : X \in \nu(w) \to m_\nu(X) \in \nu(w)$.
(T) *The class* T *of monotonic* $\mathcal{L}_\nabla$*-frames satisfying*
  (t) $\forall w \in W\ \forall X \subseteq W : X \in \nu(w) \to w \in X$.
(D) *The class* D *of monotonic* $\mathcal{L}_\nabla$*-frames satisfying*
  (d) $\forall w \in W : X \in \nu(w) \to W \setminus X \notin \nu(w)$.

Proof. Throughout the proof we assume that $\mathbb{F}_i = (W_i, \nu_i)$, $i \in I$, are monotonic $\mathcal{L}_\nabla$-frames, and $\mathbb{G} = (W, \nu)$ is a bisimulation product of $\{\mathbb{F}_i \mid i \in I\}$. Recall from Definition 9.5 that for $\overline{s} = (s_i)_{i \in I} \in W, X \subseteq W$:

(59) $X \in \nu(\overline{s}) \iff X \subseteq W$ and $\exists i \in I \exists C_i \in \nu_i(s_i) : \pi_i^{-1}[C_i] \subseteq X$.

Proof of (M): Clear by the definition of direct product and subframes.

Proof of (N): Assume that $\mathbb{F}_i \in \mathsf{N}$, $\forall i \in I$. We must show that $W \in \nu(\overline{s})$ for all $\overline{s} \in W$. But this is clear, simply take any $i \in I$, then $W_i \in \nu_i(s_i)$ and $\pi_i^{-1}[W_i] \subseteq W$.

Proof of (P): Assume that $\mathbb{F}_i \in \mathsf{P}$, $\forall i \in I$. We must show that $\emptyset \notin \nu(\overline{s})$ for all $\overline{s} \in W$. Suppose for contradiction that there is a $\overline{s} \in W$, such that $\emptyset \in \nu(\overline{s})$. Then there is an $i \in I$ and a $C_i \in \nu_i(s_i)$ such that $\pi_i^{-1}[C_i] = \emptyset$. Hence by the surjectivity of π_i, $C_i = \emptyset$, which is a contradiction with $C_i \in \nu_i(s_i)$ and $\mathbb{F}_i \in \mathsf{P}$.

Proof of (4'): Assume that $\mathbb{F}_i \in \mathsf{4}'$, $\forall i \in I$, and $X \in \nu(\overline{s})$, we then wish to show that $m_\nu(X) \in \nu(\overline{s})$. From $X \in \nu(\overline{s})$, we obtain an $i \in I$ and a $C_i \in \nu_i(s_i)$ such that $\pi_i^{-1}[C_i] \subseteq X$ $(*)$. Since $\mathbb{F}_i \in \mathsf{4}'$, we have $m_{\nu_i}(C_i) \in \nu_i(s_i)$. If we can show that $\pi_i^{-1}[m_{\nu_i}(C_i)] \subseteq m_\nu(X)$, then $m_\nu(X) \in \nu(\overline{s})$ follows. So take $\overline{t} \in \pi_i^{-1}[m_{\nu_i}(C_i)]$, then $t_i \in m_{\nu_i}(C_i)$, hence $C_i \in \nu_i(t_i)$ and by (59) and $(*)$, it now follows that $X \in \nu(\overline{t})$, so $\overline{t} \in m_\nu(X)$.

Proof of (T): Assume that $\mathbb{F}_i \in \mathsf{T}$, $\forall i \in I$, and $X \in \nu(\overline{s})$. We then wish to show that $\overline{s} \in X$. From $X \in \nu(\overline{s})$, it follows that there is an $i \in I$ and a $C_i \in \nu_i(s_i)$ such that $\pi_i^{-1}[C_i] \subseteq X$. Then by our assumption that $\mathbb{F}_i \in \mathsf{T}$, it must be the case that $s_i \in C_i$, which implies that $\overline{s} \in \pi_i^{-1}[C_i] \subseteq X$.

Proof of (D): Assume that $\mathbb{F}_i \in \mathsf{D}$, $\forall i \in I$, and $X \in \nu(\overline{s})$. We wish to show that $W \setminus X \notin \nu(\overline{s})$. Suppose for contradiction that $W \setminus X \in \nu(\overline{s})$, i.e., there is an $i \in I$ and a $C_i \in \nu_i(s_i)$ such that $\pi_i^{-1}[C_i] \subseteq W \setminus X$. Let i and C_i be fixed, then $X \subseteq W \setminus \pi_i^{-1}[C_i]$, hence by upwards closure, $W \setminus \pi_i^{-1}[C_i] \in \nu(\overline{s})$, whence $\pi_i[W \setminus \pi_i^{-1}[C_i]] \in \nu_i(s_i)$. From $C_i \in \nu_i(s_i)$ and our assumption that $\mathbb{F}_i \in \mathsf{D}$, we obtain $W_i \setminus C_i \notin \nu_(s_i)$. Hence if we can show $\pi_i[W \setminus \pi_i^{-1}[C_i]] \subseteq W_i \setminus C_i$ $(**)$, we have a contradiction with the upwards closure of $\nu_i(s_i)$. To see that $(**)$ is the case, take $t_i \in \pi_i[W \setminus \pi_i^{-1}[C_i]]$, then there is a $\overline{u} \in W \setminus \pi_i^{-1}[C_i]$ such that $u_i = t_i$, i.e., there is a $\overline{u} \notin \pi_i^{-1}[C_i]$ such that $u_i = t_i$. It then follows that $t_i \notin C_i$ (since otherwise $\overline{u} \in \pi_i^{-1}[C_i]$). QED

Theorem 9.10 *If* $\Gamma \subseteq \{\text{N,P,T,4',D}\}$ *then* $\Lambda = \mathbf{M}.\Gamma$ *has CIP.*

Proof. By Theorem 9.3 it suffices to show that the conditions (i) - (iii) of the bisimulation product lemma hold. As remarked, conditions (ii) and (iii) of the bisimulation product lemma always hold for σ-canonical or π-canonical varieties K with $\mathsf{F} = \{\mathbb{F} \mid \mathbb{F}^+ \in \mathsf{K}\}$.

When $\Gamma \subseteq \{\text{N,P,T,4',D}\}$, then $\Lambda = \mathbf{M}.\Gamma$ is σ-canonical by Theorem 7.13 and Corollary 10.35, and hence V_Λ is σ-canonical. Furthermore, each of the formulas in $\{\text{N,P,T,4',D}\}$ defines the corresponding frame property. Let F_Γ be the class of monotonic $\mathcal{L}_\nabla$-frames defined by Γ. Then $\mathbb{F} \in \mathsf{F}_\Gamma$ if and only if $\mathbb{F}^+ \in \mathsf{V}_\Lambda$. Combining this with the results in Proposition 9.9, we find that F_Γ is closed under finite bisimulation products. Thus V_Λ and F_Γ satisfy the conditions of the bisimulation product lemma and we may conclude that Λ has CIP. QED

Remark 9.11 As we have mentioned, the bisimulation products defined in Definition 9.5 are generalisations of the 'smallest' neighbourhood function μ_s with which we showed that full bisimulations are $Up\mathcal{P}$-coalgebras bisimulations, cf. Remark 8.28. Alternatively, we could have chosen the 'largest' neighbourhood function μ_l, and obtained the same general results, including Lemma 9.8. Let us refer to the two kinds of bisimulation products as the 'smallest' and the 'largest' depending on whether we choose μ_s or μ_l for our definition of direct product.

One can show that the frame classes defined by the formulas N, P and 4 are closed under largest bisimulation products, but simple counter examples to this can be found for C and T. For the formulas 4', D, B and 5, we found no answer to the closure under largest bisimulation products.

10 Simulation

In subsection 5.2 we introduced the idea of viewing monotonic $\mathcal{L}_\nabla$-frames as Kripke frames for the language $\mathcal{L}_\diamond$, and in this way we obtained various correspondence results between monotonic modal logic and first-order logic. In the current section we take this idea further and investigate how to simulate monotonic modal logics with normal ones.

Briefly stated, if Λ_1 is a modal $\mathcal{L}_1$-logic, Λ_2 is a modal $\mathcal{L}_2$-logic and $(\cdot)^f$ is an interpretation of $\mathcal{L}_1$-formulas in $\mathcal{L}_2$ which satisfies certain uniformity requirements, then Λ_2 simulates Λ_1 with respect to $(\cdot)^f$ if for all $\varphi \in \mathcal{L}_1$: $\varphi \in \Lambda_1$ iff $\varphi^f \in \Lambda_2$.

Simulations occur as early as the 1930's with Gödel's translation of intuitionistic logic in Grzegorczyk's logic [27]. In the mid-1970's, Thomason [66, 69] showed how to simulate normal polymodal logics with monomodal normal ones, and used this to prove a number of negative results on monomodal logics. In [68], Thomason extends his results and simulates monadic second-order logic with polymodal normal ones. More recently, Kracht and Wolter [42, 44] have refined Thomason's technique to achieve further results on transfer of properties between polymodal logics and monomodal ones, and Goranko et alii [34, 20] present transfer results on a simulation involving the universal modality. As one of the latest contributions, Goguadze et alii [28] simulate normal modal logics with polyadic operatos by normal monadic ones, and also prove a great number of transfer results concerning their simulation.

In the line of simulations of monotonic modal logics, Parikh [53] sketches how to simulate Game Logic in the modal μ-calculus, and Pauly [57] elaborates on this simulation, thereby obtaining results on the complexity and expressivity of Game Logic. Gasquet and Herzig [22] simulate (complete) classical modal logics by bimodal normal ones with the motivation to apply proof methods for normal modal logics to a number of non-normal logics. Their results are improved by Kracht and Wolter in [44], where they show how to simulate incomplete monotonic modal logics by translating general monotonic frames into general Kripke frames. However, whereas most simulations of monotonic modal logics translate frames and models in the manner introduced in section 5, the construction in [44] is fairly complicated.

In this section we will show that we can achieve the same and improve on Kracht and Wolter's results on simulations of monotonic modal logics by using the more intuitive simulation of section 5. We do so by simulating descriptive monotonic frames. The main technical results and contributions are found in Propositions 10.19, 10.32 and 10.33.

Parts of this section previously occurred in the author's mini-thesis [35], however, the simulation construction has been much improved in order to obtain the abovementioned results.

10.1 Interpretations and Simulations

Kracht and Wolter [44] formalise the notion of simulations via interpretations. The following is an adaptation of their definitions to our setting.

Definition 10.1 (Interpretation) Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be modal languages. An *interpretation* $(\cdot)^f$ of $\mathcal{L}_1$ in $\mathcal{L}_2$ is a mapping from $\mathcal{L}_1$-formulas to $\mathcal{L}_2$-formulas which satisfies the following uniformity conditions:

$$\begin{array}{lcll} q^f & = & p^f[q/p] & \\ (\neg\varphi)^f & = & (\neg p)^f[\varphi^f/p] & \\ (\varphi_1 \vee \varphi_2)^f & = & (p_1 \vee p_2)^f[\varphi_i^f/p_i]\ , & \text{i=1,2} \\ (\nabla\varphi)^f & = & (\nabla p)^f[\varphi^f/p]\ . & \end{array}$$

If Θ is a set of $\mathcal{L}_1$-formulas, then we define $\Theta^f := \{\varphi^f \mid \varphi \in \Theta\}$. ⊣

The above uniformity conditions ensure that an interpretation is 'well-behaved'. It can be checked that for an interpretation $(\cdot)^f$, no new variables can occur in φ^f if they were not already present in φ. If for all $p \in \text{PROP}$, $p^f = p$, we say that $(\cdot)^f$ is *atomic*.

Definition 10.2 (Simulation) Let Λ and Λ' be modal logics over the languages $\mathcal{L}$ respectively $\mathcal{L}'$ and $(\cdot)^f$ an interpretation of $\mathcal{L}$ in $\mathcal{L}'$. Then Λ' *simulates* Λ *with respect to* $(\cdot)^f$ if for all $\Sigma \cup \{\varphi\} \subseteq \mathcal{L}$:

$$\Sigma \vdash_\Lambda \varphi \quad \text{iff} \quad \Sigma^f \vdash_{\Lambda'} \varphi^f.$$

Thus Λ' simulates Λ with respect to an interpretation $(\cdot)^f$ if their consequence relations are equivalent under $(\cdot)^f$. A *simulation* is a map from $\mathcal{L}$-logics to $\mathcal{L}'$-logics. If Q is a property of logics, then a simulation $(\cdot)^s$ *preserves* Q if Λ^s has Q whenever Λ has Q, $(\cdot)^s$ *reflects* Q if Λ has Q whenever Λ^s has Q, and $(\cdot)^s$ *transfers* Q if it both preserves and reflects Q. ⊣

10.2 Simulating monotonic $\mathcal{L}_\nabla$-structures

When simulating a modal logic, we must not only translate from one language to another, but also simulate the semantic structures. The basic ideas of simulating monotonic frames with Kripke frames have already been introduced in subsection 5.2, and we will now recall those definitions and results in a more formal setting.

The basic idea

The aim is to simulate monotonic $\mathcal{L}_\nabla$-logics with normal modal logics over the language $\mathcal{L}_\diamond$, which was introduced in subsection 5.2.

Definition 10.3 (Diamond Language $\mathcal{L}_\diamond$) The modal language $\mathcal{L}_\diamond$ contains two diamonds, $\diamond_\nu$ and $\diamond_\ni$, and a nullary modality pt. More precisely, for a fixed set of proposition letters PROP, the well-formed formulas of the language $\mathcal{L}_\diamond$ are given by:

$$\mathcal{L}_\diamond : \qquad \varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \diamond_\nu\varphi \mid \diamond_\ni\varphi \mid \mathsf{pt} \qquad \text{where } p \in \text{PROP}.$$

$\top, \wedge, \to$ and $\leftrightarrow$ are defined as the usual abbreviations; $\Box_\nu \equiv \neg\diamond_\nu\neg$ and $\Box_\ni \equiv \neg\diamond_\ni\neg$. ⊣

We may think of $\mathcal{L}_\diamond$ as a bimodal language since pt behaves like a proposition letter. The minimal normal modal $\mathcal{L}_\diamond$-logic will be denoted by $\mathbf{K}_\diamond$. Note that since $\diamond_\nu$ and $\diamond_\ni$ are the primitives of the language $\mathcal{L}_\diamond$, the (Dual) axiom for each modality is needed in the axiomatisation of $\mathbf{K}_\diamond$. For a set Γ of $\mathcal{L}_\diamond$-formulas, $\mathbf{K}_\diamond.\Gamma$ denotes the smallest normal modal $\mathcal{L}_\diamond$-logic which contains Γ.

The diamond translation of subsection 5.2, will be the interpretation from $\mathcal{L}_\nabla$ to $\mathcal{L}_\diamond$. We recall its definition.

Definition 10.4 (Diamond Translation) Define the *local diamond translation* $(\cdot)^t : \mathcal{L}_\nabla \to \mathcal{L}_\diamond$ inductively as follows:

$$\begin{aligned} \bot^t &= \bot \\ p^t &= p \\ (\neg\varphi)^t &= \neg\varphi^t \\ (\varphi \vee \psi)^t &= \varphi^t \vee \psi^t \\ (\nabla\varphi)^t &= \diamond_\nu\Box_\ni\varphi^t \ . \end{aligned}$$

Define the *diamond translation* $(\cdot)^\diamond : \mathcal{L}_\nabla \to \mathcal{L}_\diamond$ by

$$\varphi^\diamond = \mathsf{pt} \to \varphi^t.$$

⊣

Remark 10.5 It should be clear that $(\cdot)^t$ is an atomic interpretation, but $(\cdot)^\diamond$ is not. In fact, $(\cdot)^\diamond$ is not even an interpretation according to Definition 10.1, since it can easily be checked that $(\nabla q)^\diamond \neq (\nabla p)^\diamond[q^\diamond/p]$. However, as we will see later, the role of the pt-antecedent is to ensure truth invariance of translated formulas in the simulation structure at base points of the original monotonic structure. The pt-antecedent and the structure of translated formulas ensure that atomic propositions of a translated formula are only evaluated at base points, thus we may, and will, think of $(\cdot)^\diamond$ as an atomic interpretation.

As previously mentioned, a neighbourhood function ν in an $\mathcal{L}_\nabla$-frame $\mathbb{F} = (W, \nu)$ can be seen as a relation R_ν between W and $\mathcal{P}(W)$. Thus viewing a monotonic $\mathcal{L}_\nabla$-frame as a two-sorted Kripke frame with universe $W \cup \mathcal{P}(W)$ gives rise to two relations, R_ν and $R_\ni$, which will interpret the modalities $\diamond_\nu$ and $\diamond_\ni$ respectively. However, we also need to be able to distinguish between old states (W) and new states ($\mathcal{P}(W)$), and this we will do by interpreting the old states with a unary relation (set) P, which in turn interprets the nullary (constant) modality pt.

Definition 10.6 (Simulating monotonic $\mathcal{L}_\nabla$-frames) Let $\mathbb{F} = (W, \nu)$ be a monotonic $\mathcal{L}_\nabla$-frame. The *simulation frame* of $\mathbb{F}$ is the Kripke $\mathcal{L}_\diamond$-frame $\mathbb{F}^\bullet = (W^\bullet, R_\nu, R_\ni, P)$ where

$$\begin{array}{lcl} W^\bullet & = & W \cup \mathcal{P}(W), \\ R_\nu & = & \{(x,u) \in W \times \mathcal{P}(W) \mid u \in \nu(x)\}, \\ R_\ni & = & \{(u,x) \in \mathcal{P}(W) \times W \mid x \in u\}, \\ P & = & W. \end{array}$$

$\dashv$

The following two propositions are simply reformulations of Claim 1 and item(ii) of Proposition 5.11 (frame correspondence).

Proposition 10.7 *Let $\mathbb{F} = (W, \nu)$ be a monotonic $\mathcal{L}_\nabla$-frame, $\mathbb{F}^\bullet$ its simulation frame, and V and V', valuations on $\mathbb{F}$ and $\mathbb{F}^\bullet$, respectively, such that V and V' agree on W. Then for all $x \in W$ and all $\mathcal{L}_\nabla$-formulas φ:*

$$(\mathbb{F}, V), x \Vdash \varphi \quad \textit{iff} \quad (\mathbb{F}^\bullet, V'), x \Vdash \varphi^t .$$

Proposition 10.8 *Let $\mathbb{F}$ be a monotonic $\mathcal{L}_\nabla$-frame and $\mathbb{F}^\bullet$ its simulation frame. Then for all $\mathcal{L}_\nabla$-formulas φ:*

$$\mathbb{F} \Vdash \varphi \quad \textit{iff} \quad \mathbb{F}^\bullet \Vdash \varphi^\diamond .$$

Propositions 10.7 and 10.8 pave the way for simulating complete monotonic logics. But in order to be able to simulate incomplete logics, we will now extend the simulation to general monotonic frames.

Simulating general monotonic $\mathcal{L}_\nabla$-frames

Before we define simulations of general monotonic frames, we recall the following. Given a general monotonic frame $\mathbb{G} = (W, \nu, A)$, A induces a topology τ_A on W by taking A as a subbasis for τ_A. In the case that $\mathbb{G}$ is descriptive, the *topological space of* $\mathbb{G}$, $\mathbb{W} = (W, \tau_A)$, is a Stone space, where A is a clopen basis for τ_A. Furthermore, all core neighbourhoods of a descriptive $\mathbb{G}$ are in $K(\mathbb{W})$, the collection of closed subsets in $\mathbb{W}$.

We will see that for general monotonic frames it suffices to add $K(\mathbb{W})$ as new points instead of the entire $\mathcal{P}(W)$. We now need to define the admissible sets of the simulation frame, and we may view this problem as defining a topology on $W \cup K(\mathbb{W})$. The admissible sets of $\mathbb{G}$ will take care of the W-part, and for the $K(\mathbb{W})$-part, we will apply the Vietoris construction [52, 37, 45] to $\mathbb{W}$.

The following definition and lemma are presented more or less as in [45], which may also be consulted for proofs.

Definition 10.9 (Vietoris topology) Let $\mathbb{X} = (X, \tau)$ be a topological space, and let $K(\mathbb{X})$ denote the collection of closed subsets of X. Define the operations $[\ni], \langle\ni\rangle : \mathcal{P}(X) \to \mathcal{P}(K(\mathbb{X}))$ by

$$\begin{array}{lcl} [\ni]U & = & \{F \in K(\mathbb{X}) \mid F \subseteq U\}, \\ \langle\ni\rangle U & = & \{F \in K(\mathbb{X}) \mid F \cap U \neq \emptyset\} . \end{array}$$

Given a subset $Q \subseteq \mathcal{P}(X)$, define

$$V_Q := \{[\ni]U \mid U \in Q\} \cup \{\langle\ni\rangle U \mid U \in Q\}.$$

The Vietoris space of $\mathbb{X}$ is given by $\mathbb{V}(\mathbb{X}) = (K(\mathbb{X}), \upsilon_{\mathbb{X}})$, where $\upsilon_{\mathbb{X}}$ is the topology on $K(\mathbb{X})$ generated by the subbasis V_τ. ⊣

The $[\ni], \langle\ni\rangle$-notation reflects the fact that $[\ni]$ and $\langle\ni\rangle$ are the box- and diamond maps of $R_\ni \subseteq K(\mathbb{X}) \times X$. Note that $[\ni]$ and $\langle\ni\rangle$ are each other's dual in the sense that,

$$\begin{array}{rcl} [\ni]U &=& K(\mathbb{X}) - \langle\ni\rangle\,(W - U), \\ \langle\ni\rangle U &=& K(\mathbb{X}) - [\ni]\,(W - U). \end{array}$$

Lemma 10.10 *Let $\mathbb{X} = (X, \tau)$ be a topological space, and let $\mathsf{Clp}_{\mathbb{X}}$ denote the collection of clopen subsets of X, and $\mathbb{V}(\mathbb{X}) = (K(\mathbb{X}), \upsilon_{\mathbb{X}})$ the Vietoris space of $\mathbb{X}$. Then we have the following:*

1. *If $\mathbb{X}$ is compact, then $\mathbb{V}(\mathbb{X})$ is compact.*

2. *If $\mathbb{X}$ is a Stone space, then $\mathbb{V}(\mathbb{X})$ is a Stone space, and the collection*

$$V_{\mathsf{Clp}_{\mathbb{X}}} = \{[\ni]U \mid U \in \mathsf{Clp}_{\mathbb{X}}\} \cup \{\langle\ni\rangle U \mid U \in \mathsf{Clp}_{\mathbb{X}}\}$$

forms a clopen subbasis of $\upsilon_{\mathbb{X}}$. Hence the clopen subsets of $K(\mathbb{X})$ form a basis of $\upsilon_{\mathbb{X}}$.

We are now ready to define simulations of general monotonic frames.

Definition 10.11 (Simulating general monotonic $\mathcal{L}_\nabla$-frames) Let $\mathbb{G} = (W, \nu, A)$ be a general monotonic $\mathcal{L}_\nabla$-frame, and let $K(\mathbb{W})$ denote the collection of closed subsets in the topological space $\mathbb{W} = (W, \tau_A)$ of $\mathbb{G}$.

Let V_A^+ be those subsets b of $K(\mathbb{W})$ for which there are finite subsets $I_1, \ldots, I_k$ and $J_1, \ldots, J_k$ of A such that $b = b_1 \cup \ldots \cup b_k$ where for $i = 1, \ldots, k$,

$$b_i = \bigcap_{a \in I_i} [\ni]a \;\cap\; \bigcap_{a \in J_i} \langle\ni\rangle a\,.$$

The *simulation frame of* $\mathbb{G}$ is then defined as $\mathbb{G}^\bullet = (W^\bullet, R_\nu, R_\ni, P, A^\bullet)$, where

$$\begin{array}{rcl} W^\bullet &=& W \cup K(\mathbb{W}), \\ R_\nu &=& \{(x, u) \in W \times K(\mathbb{W}) \mid u \in \nu(x)\}, \\ R_\ni &=& \{(u, x) \in K(\mathbb{W}) \times W \mid x \in u\}, \\ P &=& W, \\ A^\bullet &=& \{a \cup b \mid a \in A,\ b \in V_A^+\}. \end{array}$$

⊣

The essential closure properties of V_A^+ which ensure that $\mathbb{G}^\bullet$ is a general Kripke frame are shown in the following lemma.

Lemma 10.12 *V_A^+ is closed under finite unions, finite intersections and complementation with respect to $K(\mathbb{W})$.*

Proof. Clearly, V_A^+ is closed under finite unions. Let $b = b_1 \cup \ldots \cup b_k$ and $b' = b'_1 \cup \ldots \cup b'_l$ be elements of V_A^+ defined in terms of $I_1, \ldots, I_k, J_1, \ldots, J_k$, respectively $I'_1, \ldots, I'_l, J'_1, \ldots, J'_l$.

To show closure under finite intersections, we have $b \cap b' = \bigcup_{1 \leq i \leq k,\, 1 \leq j \leq l} b_i \cap b'_j$, and

$$\begin{aligned} b_i \cap b'_j &= \bigcap_{a \in I_i} [\ni] a \cap \bigcap_{a \in J_i} \langle \ni \rangle a \cap \bigcap_{a \in I'_j} [\ni] a \cap \bigcap_{a \in J'_j} \langle \ni \rangle a \\ &= \bigcap_{a \in I_i \cup I'_j} [\ni] a \cap \bigcap_{a \in J_i \cup J'_j} \langle \ni \rangle a. \end{aligned}$$

I_i, I'_j, J_i and J'_j are all finite subsets of A, therefore, $I_i \cup I'_j$ and $J_i \cup J'_j$ are also finite subsets of A which proves that $b_i \cap b'_j \in V_A^+$, and hence $b \cap b' \in V_A^+$, since V_A^+ is closed under finite unions.

To show that $K(\mathbb{W}) - b$ is in V_A^+, we have

$$K(\mathbb{W}) - b = K(\mathbb{W}) - \bigcup_{i=1}^{k} b_i = \bigcap_{i=1}^{k} (K(\mathbb{W}) - b_i) .$$

We have just seen in the above that V_A^+ is closed under finite intersections, so it suffices to show that $K(\mathbb{W}) - b_i \in B$ for $i = 1, \ldots, k$.

$$\begin{aligned} K(\mathbb{W}) - b_i &= K(\mathbb{W}) - (\bigcap_{a \in I_i} [\ni] a \cap \bigcap_{a \in J_i} \langle \ni \rangle a) \\ &= \bigcup_{a \in I_i} (K(\mathbb{W}) - [\ni] a) \cup \bigcup_{a \in J_i} (K(\mathbb{W}) - \langle \ni \rangle a) \\ &= \bigcup_{a \in I_i} \langle \ni \rangle (W - a) \cup \bigcup_{a \in J_i} [\ni] (W - a)) \end{aligned}$$

This is a finite union of elements in V_A^+ and hence belongs to V_A^+. QED

Proposition 10.13 *If $\mathbb{G} = (W, \nu, A)$ is a general monotonic $\mathcal{L}_\nabla$-frame, then its simulation frame $\mathbb{G}^\bullet = (W^\bullet, R_\nu, R_\ni, P, A^\bullet)$ is a general Kripke $\mathcal{L}_\diamond$-frame.*

Proof. $(W^\bullet, R_\nu, R_\ni, P)$ is clearly a Kripke $\mathcal{L}_\diamond$-frame. So it remains to show that $A^\bullet$ contains $\emptyset$ and is closed under union, complementation with respect to $W^\bullet$ and the modal operations m_{R_ν}, $m_{R_\ni}$ and m_P.

$\emptyset \in A^\bullet$ follows from $\emptyset \in A$ and $\emptyset = \langle \ni \rangle \emptyset \in V_A^+$. Closure under finite unions and complementation follow from the closure properties of A and V_A^+.

$m_{R_\ni}$: Let $a \cup b \in A^\bullet$. We have $m_{R_\ni}(a \cup b) = m_{R_\ni}(a) \cup m_{R_\ni}(b)$. Recall that $b \in V_A^+$ and that $R_\ni$-successors are always elements of W, so $m_{R_\ni}(b) = \emptyset$. Hence it suffices to show that $m_{R_\ni}(a) \in A^\bullet$.

$$m_{R_\ni}(a) = \{w \in W^\bullet \mid \exists x \in a : w R_\ni x\} = \{F \in K(\mathbb{W}) \mid F \cap a \neq \emptyset\} = \langle \ni \rangle a \in V_A^+.$$

m_{R_ν}: Let $a \cup b \in A^\bullet$. As always, $m_{R_\nu}(a \cup b) = m_{R_\nu}(a) \cup m_{R_\nu}(b)$. Points from W can never be R_ν-successors, so $m_{R_\nu}(a \cup b) = m_{R_\nu}(b)$. Now, assume $b = \bigcup_{i=1}^k b_i$, then $m_{R_\nu}(b) = \bigcup_{i=1}^k m_{R_\nu}(b_i)$, and we have,

$$\begin{array}{rll} & & x \in m_{R_\nu}(b_i) \\ & \text{iff} & \text{there is a } u \in b_i : xR_\nu u \\ {\scriptstyle (x\in W, \text{def.}\, b_i)} & \text{iff} & \text{there is a } u \in \nu(x) : (\forall a \in I_i : u \subseteq a \text{ and } \forall a \in J_i : u \cap a \neq \emptyset) \\ & \text{iff} & \text{there is a } u \in \nu(x) : (u \subseteq \bigcap I_i \text{ and } \forall a \in J_i : u \cap a \neq \emptyset) \\ & \text{iff} & \bigcap I_i \in \nu(x) \text{ and } \forall a \in J_i : \bigcap I_i \cap a \neq \emptyset\ . \end{array}$$

There are two cases to consider: If there is an $a \in J_i$ such that $\bigcap I_i \cap a = \emptyset$, then $m_{R_\nu}(b_i) = \emptyset \in A$.
Otherwise, $m_{R_\nu}(b_i) = \{x \in W \mid \bigcap I_i \in \nu(x)\} = m_\nu(\bigcap I_i)$. Note also, that $\bigcap I_i \subseteq W$, so $m_\nu(\bigcap I_i)$ is well-defined. Furthermore, I_i is a finite subset of A, therefore $\bigcap I_i \in A$, hence $m_\nu(\bigcap I_i) \in A$ since A is closed under the m_ν-operation. This shows that for all $i = 1, \ldots, k$, $m_{R_\nu}(b_i) \in A$. So $m_{R_\nu}(b)$ is a finite union of elements in A, hence $m_{R_\nu}(b)$ is itself an element of $A \subseteq A^\bullet$.

m_P: A nullary modality behaves like a propositional letter and we have

$$m_P = \{x \in W \cup K(\mathbb{W}) \mid Px\} = W \in A \subseteq A^\bullet\ .$$

QED

Proposition 10.14 *Let $\mathbb{G} = (W, \nu, A)$ be a general monotonic $\mathcal{L}_\nabla$-frame, $\mathbb{G}^\bullet$ its simulation frame and V, respectively V', admissible valuations on $\mathbb{G}$, respectively $\mathbb{G}^\bullet$, such that V and V' agree on W. Then for all $x \in W$ and all $\mathcal{L}_\nabla$-formulas φ:*

$$(\mathbb{G}, V), x \Vdash \varphi \quad \textit{iff} \quad (\mathbb{G}^\bullet, V'), x \Vdash \varphi^t\ .$$

Proof. The proof is by induction on φ, and we only show the modal case.

$$\begin{array}{rll} & & (\mathbb{G}, V), x \Vdash \nabla\varphi \\ & \text{iff} & V(\varphi) \in \nu(x) \\ {\scriptstyle (V(\varphi)\in A\subseteq K(\mathbb{W}))} & \text{iff} & \exists F \in K(\mathbb{W}) : F \in \nu(x)\ \&\ \forall y \in F : (\mathbb{G}, V), y \Vdash \varphi \\ {\scriptstyle (IH)} & \text{iff} & \exists F \in K(\mathbb{W}) : F \in \nu(x)\ \&\ \forall y \in F : (\mathbb{G}^\bullet, V'), y \Vdash \varphi^t \\ {\scriptstyle (def.\mathbb{G}^\bullet)} & \text{iff} & \exists F \in W^\bullet : R_\nu xF\ \&\ \forall y \in W^\bullet : R_\ni Fy \to (\mathbb{G}^\bullet, V'), y \Vdash \varphi^t \\ & \text{iff} & (\mathbb{G}^\bullet, V'), x \Vdash \Diamond_\nu \Box_\ni \varphi^t \\ & \text{iff} & (\mathbb{G}^\bullet, V'), x \Vdash (\nabla\varphi)^t\ . \end{array}$$

QED

Proposition 10.15 *Let $\mathbb{G} = (\mathbb{F}, A)$ be a general monotonic $\mathcal{L}_\nabla$-frame and $\mathbb{G}^\bullet$ its general simulation frame. Then for all $\mathcal{L}_\nabla$-formulas φ,*

$$\mathbb{G} \Vdash \varphi \ \textit{iff}\ \mathbb{G}^\bullet \Vdash \varphi^\diamond\ .$$

Proof. This proof is completely analogous to the proof of item (ii) in Proposition 5.11. By construction $\{a \cap W \mid a \in A^\bullet\} = A$, hence, if V' is an admissible valuation on $\mathbb{G}^\bullet$ then taking $V = V'\!\restriction_W$, i.e, $V(p) = V'(p) \cap W$ for all $p \in \text{PROP}$, V is admissible on $\mathbb{G}$. Also analogously, if V is admissible on $\mathbb{G}$ then V is also admissible on $\mathbb{G}^\bullet$ since $A \subseteq A^\bullet$. Claim 1 in the proof of Proposition 5.11 yields the result. QED

Remark 10.16 When $\mathbb{G}$ is descriptive, then $V_A^+ = \mathsf{Clp}_{\mathbb{V}(\mathbb{W})}$, the collection of clopens in the Vietoris space: $V_A^+ \subseteq \mathsf{Clp}_{\mathbb{V}(\mathbb{W})}$ follows from $V_A \subseteq \mathsf{Clp}_{\mathbb{V}(\mathbb{W})}$, which is an easy consequence of the dual relationship $[\ni]$ and $\langle\ni\rangle$. The inclusion $V_A^+ \supseteq \mathsf{Clp}_{\mathbb{V}(\mathbb{W})}$ follows since V_A is a clopen subbasis of $\mathbb{V}(\mathbb{W})$ when $\mathbb{W}$ is a Stone space.

If we had chosen to define simulations for descriptive frames only, we would therefore not have needed to prove Lemma 10.12, since $\mathsf{Clp}_{\mathbb{V}(\mathbb{W})}$ clearly has the desired closure properties. However, Definition 10.11 allows us to consider simulations of monotonic frames as a special case. First recall the definition of the operations $(\cdot)^\sharp$ and $(\cdot)_\sharp$ in Definition 7.25: For a monotonic $\mathcal{L}_\nabla$-frame $\mathbb{F} = (W, \nu)$, $\mathbb{F}^\sharp = (W, \nu, \mathcal{P}(W))$ is the full general frame of $\mathbb{F}$, and for a general monotonic frame $\mathbb{G} = (W, \nu, A)$, $\mathbb{G}_\sharp = (W, \nu)$ is the underlying monotonic frame of $\mathbb{G}$. The collection of closed subsets $K(\mathbb{W})$ of the topological space of $\mathbb{F}^\sharp$ is now $K(\mathbb{W}) = \mathcal{P}(W)$, since $\mathcal{P}(W)$ induces the discrete topology.

Note, however, that for a general monotonic frame $\mathbb{G} = (W, \nu, A)$, in most cases, $(\mathbb{G}^\bullet)_\sharp$ and $(\mathbb{G}_\sharp)^\bullet$ are distinct:

$$(\mathbb{G}^\bullet)_\sharp = (W \cup K(\mathbb{W}), R_\nu, R_\ni, P) \neq (W \cup \mathcal{P}(W), R_\nu, R_\ni, P) = (\mathbb{G}_\sharp)^\bullet.$$

But we do have the following invariance results for descriptive frames.

Proposition 10.17 *Let $\mathbb{G} = (W, \nu, A)$ be a descriptive general monotonic $\mathcal{L}_\nabla$-frame and $\mathbb{G}^\bullet = (W^\bullet, R_\nu, R_\ni, P, A^\bullet)$ its simulation frame. If V and V' are valuations on $\mathbb{G}$ and $\mathbb{G}^\bullet$, respectively, such that V and V' agree on $W = P$, then for all $w \in W$ and all $\mathcal{L}_\nabla$-formulas φ, we have*

$$((\mathbb{G}^\bullet)_\sharp, V), w \Vdash \varphi^t \quad \textit{iff} \quad ((\mathbb{G}_\sharp)^\bullet, V'), w \Vdash \varphi^t.$$

Proof. The proof is by induction on φ, but as usual, only the modal case deserves attention.

$$\begin{array}{rll} & & ((\mathbb{G}^\bullet)_\sharp, V), w \Vdash \Diamond_\nu \Box_\ni \varphi^t \\ & \text{iff} & \exists u \in K(\mathbb{W}) : wR_\nu u \;\&\; \forall v \in W^\bullet (uR_\ni v \to ((\mathbb{G}^\bullet)_\sharp, V), w \Vdash \varphi^t) \\ \text{(IH \& } R_\ni[u] \subseteq W) & \text{iff} & \exists u \in K(\mathbb{W}) : wR_\nu u \;\&\; \forall v \in W^\bullet (uR_\ni v \to ((\mathbb{G}_\sharp)^\bullet, V'), w \Vdash \varphi^t) \\ (\mathbb{G} \text{ descriptive}) & \text{iff} & \exists u \in \mathcal{P}(W) : wR_\nu u \;\&\; \forall v \in W^\bullet (uR_\ni v \to ((\mathbb{G}_\sharp)^\bullet, V'), w \Vdash \varphi^t) \\ & \text{iff} & ((\mathbb{G}_\sharp)^\bullet, V'), w \Vdash \Diamond_\nu \Box_\ni \varphi^t. \end{array}$$

QED

The following proposition is an easy consequence of Proposition 10.17, and we omit the proof.

Proposition 10.18 *If $\mathbb{G} = (W, \nu, A)$ is a descriptive general monotonic $\mathcal{L}_\nabla$-frame and $\mathbb{G}^\bullet = (W^\bullet, R_\nu, R_\ni, P, A^\bullet)$ its simulation frame, then for all $\mathcal{L}_\nabla$-formulas φ, we have*

$$(\mathbb{G}^\bullet)_\sharp \Vdash \varphi^\diamond \quad \textit{iff} \quad (\mathbb{G}_\sharp)^\bullet \Vdash \varphi^\diamond.$$

The main motivation behind changing the simulation construction from the original one in [35], was to obtain the following results on preservation of properties of general frames.

Proposition 10.19 (Preservation of descriptiveness) *If $\mathbb{G} = (W, \nu, A)$ is a general monotonic $\mathcal{L}_\nabla$-frame and $\mathbb{G}^\bullet = (W^\bullet, R_\nu, R_\ni, P, A^\bullet)$ its simulation frame, then we have,* *(i) If $\mathbb{G}$ is differentiated, then $\mathbb{G}^\bullet$ is differentiated.*

(ii) If $\mathbb{G}$ is compact, then $\mathbb{G}^\bullet$ is compact.

(iii) If $\mathbb{G}$ is tight, then $\mathbb{G}^\bullet$ is tight.

(iv) If $\mathbb{G}$ is descriptive, then $\mathbb{G}^\bullet$ is descriptive.

Proof. Recall that differentiation and compactness of $\mathbb{G}$ together imply that the topological space $\mathbb{W}$ of $\mathbb{G}$ is a Stone space. Items (i) and (ii) more or less follow from the theory of Stone spaces, but we spell out the details here in the current context.

(i) Differentiation: We must show that for all $w, v \in W^\bullet$,

$$w \neq v \quad \Rightarrow \quad \exists c \in A^\bullet (w \in c \;\;\&\;\; v \notin c) \qquad (*).$$

So let $w, v \in W^\bullet$, and assume that $w \neq v$. If $w, v \in W$, then $(*)$ follows from the differentiation of $\mathbb{G}$. If $w \in W$ and $v \in K(\mathbb{W})$, then we can take $c = W \in A \subseteq A^\bullet$. Vice versa, if $w \in K(\mathbb{W})$ and $v \in W$, then we can take $c = K(\mathbb{W}) \in V_A^+ \subseteq A^\bullet$.

If w and v are both in $K(\mathbb{W})$, then $w = \bigcap_{w \subseteq a \in A} a$ and $v = \bigcap_{v \subseteq b \in A} b$, and we may assume without loss of generality that there is an $x \in w \setminus v$. As $x \notin v$, there must be a $b \in A$ such that $v \subseteq b$ and $x \notin b$. Taking $b' = W \setminus b$, it follows that $v \cap b' = \emptyset$ and $x \in w \cap b'$, which in turn tells us that $v \notin \langle\ni\rangle a'$ and $w \in \langle\ni\rangle a'$. Since $\langle\ni\rangle a' \in V_A^+ \subseteq A^\bullet$, we have separated w from v with $c = \langle\ni\rangle a'$.

(ii) Compactness: This item basically follows from the fact that the Vietoris construction preserves compactness.

We must now show that for all $C \subseteq A^\bullet$,

$$\bigcap C = \emptyset \quad \Rightarrow \quad \exists C_0 \subseteq_\omega C : \bigcap C_0 = \emptyset.$$

Let $C = \{a_i \cup b_i \mid i \in I\} \subseteq A^\bullet$, and assume that $\bigcap C = \emptyset$. As the a_i and b_i are disjoint, this is equivalent with,

$$\bigcap_{i \in I} a_i = \emptyset \;\;\&\;\; \bigcap_{i \in I} b_i = \emptyset.$$

When $\mathbb{G}$ is compact, then $\mathbb{V}(\mathbb{W})$ is compact (cf. Lemma 10.10). So by the compactness of $\mathbb{G}$ and $\mathbb{V}(\mathbb{W})$ there are finite subsets $I_0, J_0 \subseteq_\omega I$ such that

$$\bigcap_{i \in I_0} a_i = \emptyset \;\;\&\;\; \bigcap_{j \in J_0} b_j = \emptyset.$$

Taking $C_0 = \{a_i \cup b_i \mid i \in I_0 \cup J_0\}$, it follows that C_0 is a finite subset of C, and

$$\bigcap C_0 = (\bigcap_{i \in I_0} a_i \cup \bigcap_{i \in I_0} b_i) \cap (\bigcap_{j \in J_0} a_j \cup \bigcap_{j \in J_0} b_j) = \emptyset.$$

(iii) Tightness: To see that $\mathbb{G}^\bullet$ is tight with respect to the relation $R_\ni$, we must prove that for all $w, v \in W^\bullet$,

$$v \notin R_\ni[w] \quad \Rightarrow \quad \exists c \in A^\bullet (v \in c \;\&\; w \notin m_{R_\ni}(c)).$$

There are three cases to consider, (i) $w \in W$, (ii) $v \in K(\mathbb{W})$ and (iii) $w \in K(\mathbb{W}), v \in W$ & $v \notin w$. If (i) $w \in W$, then we can take $c = W^\bullet \in A^\bullet$, since $m_{R_\ni}(W^\bullet) \subseteq K(\mathbb{W})$, hence $w \notin m_{R_\ni}(W^\bullet)$. If (ii) $v \in K(\mathbb{W})$, then we can take $c = K(\mathbb{W}) \in V_A^+ \subseteq A^\bullet$, since then $w \notin m_{R_\ni}(K(\mathbb{W})) = \emptyset$. If (iii) $w \in K(\mathbb{W}), v \in W$ & $v \notin w$, then $w = \bigcap_{w \subseteq a \in A} a$ and it follows that there is an $a \in A$ such that $w \subseteq a$ and $v \notin a$. Hence by taking $c = W \setminus a \in A \subseteq A^\bullet$, then $v \in c$ and $w \cap c = \emptyset$ from which it follows that $w \notin \langle \ni \rangle c = m_{R_\ni}(c)$.

To see that $\mathbb{G}^\bullet$ is tight with respect to the relation R_ν, we must prove that for all $w, v \in W^\bullet$,

$$v \notin R_\nu[w] \ \Rightarrow\ \exists c \in A^\bullet (v \in c \,\&\, w \notin m_{R_\nu}(c)).$$

Again, there are three cases to consider, (i) $w \in K(\mathbb{W})$, (ii) $v \in W$, and (iii) $w \in W, v \in K(\mathbb{W})$ & $v \notin \nu(w)$. If (i) $w \in K(\mathbb{W})$ then we can take $c = W^\bullet \in A^\bullet$, since $w \notin m_{R_\nu}(W^\bullet) \subseteq W$. If (ii) $v \in W$ then we can take $c = W$, since then $w \notin m_{R_\nu}(W) = \emptyset$.

If (iii) $w \in W, v \in K(\mathbb{W})$ & $v \notin \nu(w)$, then by the tightness of $\mathbb{G}$ there is an $a \in A$ such that $v \subseteq a$ and $a \notin \nu(w)$. From $v \subseteq a$, it follows that $v \in [\ni]a$. We will show that if we take $c = [\ni]a$, then $w \notin m_{R_\nu}(c)$. Suppose for contradiction that $w \in m_{R_\nu}([\ni]a)$, then there is an $F \in K(\mathbb{W})$ such that $F \in [\ni]a$ and $R_\nu wF$. This implies that $F \subseteq a$ and $F \in \nu(w)$, hence by upwards closure of $\nu(w)$, we have $a \in \nu(w)$, a contradiction, since $a \notin \nu(w)$.

For tightness of $\mathbb{G}^\bullet$ with respect to the unary relation (set) P, we need to show that for all $w \in W^\bullet$, $w \notin P \ \Rightarrow\ w \notin m_P$, but this is trivial since $m_P = P$.

(iv) Descriptiveness: Follows from item (i) to (iii). QED

10.3 Unsimulating $\mathcal{L}_\Diamond$-structures

When proving the Simulation Theorem 10.30, we will see that it is useful to also have an operation which transforms a Kripke $\mathcal{L}_\Diamond$-frame into a monotonic $\mathcal{L}_\nabla$-frame.

In other words, we wish to be able to unsimulate $\mathcal{L}_\Diamond$-frames. However, the unsimulation operation will not be defined for any $\mathcal{L}_\Diamond$-frame, but only for the ones whose structure is similar to that of simulation frames.

Axiomatising simulation frames

The $\mathcal{L}_\Diamond$-frames which are the simulation frame of some monotonic $\mathcal{L}_\nabla$-frame may be characterised by the following first-order axioms.

(S1) $\forall x \forall y (R_\nu xy \ \rightarrow\ \neg P(y))$
(S2) $\forall x \forall y (R_\ni xy \ \rightarrow\ P(y))$
(S3) $\forall x \forall y (R_\nu xy \ \rightarrow\ P(x))$
(S4) $\forall x \forall y (R_\ni xy \ \rightarrow\ \neg P(x))$
(S5) $\exists x\, P(x)$
(S6) $\forall x \forall y \forall z (R_\nu xy \wedge (\forall w : R_\ni yw \rightarrow R_\ni zw) \ \rightarrow\ R_\nu xz)$

The axioms (S1-4) express that in a simulation frame, $R_\nu \subseteq W \times K(\mathbb{W})$ and $R_\ni \subseteq K(\mathbb{W}) \times W$, (S5) expresses that $P \neq \emptyset$, and (S6) expresses that R_ν is the relation of an upwards closed ν. Unfortunately, these first-order axioms do not all immediately translate into the modal language $\mathcal{L}_\Diamond$. Nevertheless, we can find a modal axiomatisation for the $\mathcal{L}_\Diamond$-frames which satisfy (S1-4). We will call this class of $\mathcal{L}_\Diamond$-frames, the class of Sim^--frames.

Definition 10.20 Consider the following set of modal axioms:

(A1) $\Box_\nu \neg \mathsf{pt}$
(A2) $\Box_\ni \mathsf{pt}$
(A3) $\Diamond_\nu \top \ \rightarrow\ \mathsf{pt}$
(A4) $\Diamond_\ni \top \ \rightarrow\ \neg\mathsf{pt}$

Define the normal modal $\mathcal{L}_\Diamond$-logic $\mathbf{Sim}^- := \mathbf{K}_\Diamond.\{A1, A2, A3, A4\}$. $\dashv$

Proposition 10.21 *The axioms (A1-4) modally characterise the class of Sim$^-$-frames, and* $\mathbf{Sim}^-$ *is canonical, and sound and strongly complete with respect to the class of* $\mathbf{Sim}^-$*-frames.*

Proof. Immediate by the syntactic shape of the modal axioms, and the properties of closed formulas: Each of the axioms (Ai) is a closed formula, i.e., (Ai) does not contain any propositional variables, and its first-order correspondent is (Si), $i = 1, \ldots, 4$. QED

Clearly, $\mathbf{Sim}^-$ is valid on any simulation frame, and although, the $\mathbf{Sim}^-$-axioms are too weak to axiomatise the class of simulation frames, they almost suffice to ensure that the unsimulation operation is well-defined.

Unsimulating $\mathcal{L}_\Diamond$-frames

The $\mathbf{Sim}^-$-axioms clearly capture the type of the relations R_ν and $R_\ni$, but in order to be able to "undo" the simulation operation, we should ensure that the set of points which will play the role of base points, that is P, is not empty. We will call a $\mathcal{L}_\Diamond$-frame $\mathbb{F} = (W, R_\nu, R_\ni, P)$ *P-regular* if $P \neq \emptyset$.

Definition 10.22 Let $\mathbb{F} = (W, R_\nu, R_\ni, P)$ be a P-regular $\mathbf{Sim}^-$-frame. Then we define a neighbourhood function $\mu : P \to \mathcal{P}(\mathcal{P}(P))$ as follows: For all $w \in P$ and $X \subseteq P$,

$$X \in \mu(w) \ \text{ iff } \ \exists u \in W(R_\nu wu \ \&\ R_\ni[u] \subseteq X).$$

Then μ is monotone, and we define the *unsimulation frame* of $\mathbb{F}$ to be the monotonic $\mathcal{L}_\nabla$-frame $\mathbb{F}_\bullet = (P, \mu)$. $\dashv$

Note that in a $\mathbf{Sim}^-$-frame, $R_\nu[w] \subseteq W \setminus P$ and $R_\ni[w] \subseteq P$ for any $w \in W$. We also need to show that the unsimulation preserves truth and validity with respect to our interpretation.

Proposition 10.23 *Let $\mathbb{F} = (W, R_\nu, R_\ni, P)$ be a P-regular* $\mathbf{Sim}^-$*-frame, and $\mathbb{F}_\bullet = (P, \mu)$ its unsimulation frame. If V and V' are valuations on $\mathbb{F}$, respectively $\mathbb{F}_\bullet$, such that V and V' agree on P, then we have for all $w \in P$ and all $\mathcal{L}_\nabla$-formulas φ:*

$$(\mathbb{F}, V), w \Vdash \varphi^t \ \text{ iff } \ (\mathbb{F}_\bullet, V'), w \Vdash \varphi.$$

Proof. The proof is, of course, by induction on φ and again the modal case is the only nontrivial part of the proof.

$$\begin{array}{rll} & & (\mathbb{F}, V), w \Vdash \Diamond_\nu \Box_\ni \varphi^t \\ & \text{iff} & \exists u(R_\nu wu \ \&\ \forall v(R_\ni uv \to (\mathbb{F}, V), v \Vdash \varphi^t)) \\ \text{(IH \& } R_\ni[u] \subseteq P) & \text{iff} & \exists u(R_\nu wu \ \&\ \forall v(R_\ni uv \to (\mathbb{F}_\bullet, V'), v \Vdash \varphi)) \\ & \text{iff} & \exists u(R_\nu wu \ \&\ R_\ni[u] \subseteq V'(\varphi)) \\ (\mu \text{ monotone}) & \text{iff} & (\mathbb{F}_\bullet, V'), w \Vdash \nabla\varphi. \end{array}$$

QED

Proposition 10.24 *Let $\mathbb{F} = (W, R_\nu, R_\ni, P)$ be a P-regular* **Sim**$^-$*-frame, and $\mathbb{F}_\bullet = (P, \mu)$ its unsimulation frame. Then we have for all $\mathcal{L}_\nabla$-formulas φ:*

$$\mathbb{F} \Vdash \varphi^\diamond \quad \textit{iff} \quad \mathbb{F}_\bullet \Vdash \varphi.$$

Proof. For the proof from left to right, suppose $\mathbb{F}_\bullet \nVdash \varphi$. Then there is a valuation V on $\mathbb{F}_\bullet$ and a $w \in P$ such that $(\mathbb{F}_\bullet, V), w \nVdash \varphi$. Since V is also a valuation on $\mathbb{F}$, it follows from Proposition 10.23 that $(\mathbb{F}, V), w \nVdash \varphi^t$, and hence also $(\mathbb{F}, V), w \nVdash \mathsf{pt} \to \varphi^t = \varphi^\diamond$, and we may conclude that $\mathbb{F} \nVdash \varphi^\diamond$.

For the direction from right to left, suppose $\mathbb{F} \nVdash \varphi^\diamond$. Then there is a valuation V on $\mathbb{F}$ and a $w \in W$ such that $(\mathbb{F}, V), w \nVdash \varphi^\diamond$. It follows that $(\mathbb{F}, V), w \Vdash \mathsf{pt}$ and $(\mathbb{F}, V), w \nVdash \varphi^t$, so $w \in P$ and by Proposition 10.23 $(\mathbb{F}_\bullet, V{\restriction}_P), w \nVdash \varphi$, hence $\mathbb{F}_\bullet \nVdash \varphi$. QED

Unsimulating general $\mathcal{L}_\diamond$-frames

We will now extend the unsimulation operation to general $\mathcal{L}_\diamond$-frames, and this can be done in a very simple manner. First some terminology: A *P-regular general* **Sim**$^-$*-frame* is a general $\mathcal{L}_\diamond$-frame based on a P-regular **Sim**$^-$-frame.

Definition 10.25 Let $\mathbb{G} = (\mathbb{F}, A)$ be a P-regular general **Sim**$^-$-frame. Then its *unsimulation frame* $\mathbb{G}_\bullet$ is defined as $\mathbb{G}_\bullet = (\mathbb{F}_\bullet, A_\bullet)$ where $A_\bullet = \{a \cap P \mid a \in A\}$. ⊣

Proposition 10.26 *Let $\mathbb{G}$ be a P-regular general* **Sim**$^-$*-frame, then $\mathbb{G}_\bullet$ is a general monotonic $\mathcal{L}_\nabla$-frame.*

Proof. We need to show that $A_\bullet$ contains $\emptyset$, is closed under finite unions, complementation with respect to P and the map m_μ. $\emptyset \in A_\bullet$ is clear since $\emptyset \in A$. Observe now that $P \in A$, since $\mathbb{G}$ is a general $\mathcal{L}_\diamond$-frame, hence $A_\bullet \subseteq A$, so closure of $A_\bullet$ under finite unions follows from the closure properties of A. For closure under complement in P, we have for all $a \cap P \in A_\bullet$, $P \setminus (a \cap P) = P \setminus a = P \cap (W \setminus a) \in A_\bullet$.

For the closure of $A_\bullet$ under m_μ, we have for all $a \cap P \in A_\bullet$:

$$\begin{array}{rll} & & w \in m_\mu(a \cap P) \\ & \text{iff} & a \cap P \in \mu(w) \\ (\text{def. } \mu) & \text{iff} & \exists u \in W\,(R_\nu wu \;\&\; R_\ni[u] \subseteq a \cap P) \\ & \text{iff} & w \in m_{R_\nu}(l_{R_\ni}(a \cap P)) \\ (\forall X \subseteq W(m_{R_\nu}(X) \subseteq P)) & \text{iff} & w \in m_{R_\nu}(l_{R_\ni}(a \cap P)) \cap P \end{array}$$

By the closure properties of A, $m_{R_\nu}(l_{R_\ni}(a \cap P)) \in A$, hence $m_\mu(a \cap P) \in A_\bullet$. QED

Proposition 10.27 *Let $\mathbb{G}$ be a P-regular general* **Sim**$^-$*-frame, and $\mathbb{G}_\bullet$ its unsimulation frame. Then for all $\mathcal{L}_\nabla$-formulas φ,*

$$\mathbb{G} \Vdash \varphi^\diamond \quad \textit{iff} \quad \mathbb{G}_\bullet \Vdash \varphi.$$

Proof. This proof is similar to the proof of Propositions 10.15 and 10.24. Just note that since $A_\bullet \subseteq A$, if V is an admissible valuation on $\mathbb{G}_\bullet$, then V is also admissible on $\mathbb{G}$. And if V is an admissible valuation on $\mathbb{G}$, then $V{\restriction}_P$ is admissble on $\mathbb{G}_\bullet$. QED

10.4 Simulation Results

Definition 10.28 (Simulation map) For a set of $\mathcal{L}_\nabla$-formulas Γ and an $\mathcal{L}_\nabla$-logic $\Lambda = \mathbf{M}.\Gamma$, we define

$$\Lambda^{sim} = (\mathbf{M}.\Gamma)^{sim} := \mathbf{Sim}^-.\Gamma^\diamond. \qquad \dashv$$

$(\cdot)^{sim}$ is a map from $\mathcal{L}_\nabla$-logics to $\mathcal{L}_\diamond$-logics, and in Theorem 10.30 we will see that $(\cdot)^{sim}$ is indeed a simulation. First, we prove the following lemma which will simplify the proof of 10.30.

Lemma 10.29 *Let Λ be a modal $\mathcal{L}_\nabla$-logic and Λ' a modal $\mathcal{L}_\diamond$-logic, then Λ' simulates Λ with respect to $(\cdot)^\diamond$ iff $\varphi \in \Lambda \Leftrightarrow \varphi^\diamond \in \Lambda'$. That is, for all $\Sigma \cup \{\varphi\} \subseteq \mathcal{L}_\nabla$ the following are equivalent:*

(i) $\Sigma \vdash_\Lambda \varphi \Leftrightarrow \Sigma^\diamond \vdash_{\Lambda'} \varphi^\diamond$

(ii) $\varphi \in \Lambda \Leftrightarrow \varphi^\diamond \in \Lambda'$

Proof. The implication (i) $\Rightarrow$ (ii) is trivial, since we may take $\Sigma = \emptyset$. To show the implication (ii) $\Rightarrow$ (i), assume that (ii) holds. We first show the following: Let $\sigma_1, \ldots, \sigma_n, \varphi$ be $\mathcal{L}_\nabla$-formulas, then we have

$$\begin{array}{rll} & & \vdash_\Lambda (\sigma_1 \wedge \ldots \wedge \sigma_n) \to \varphi \\ \text{(ii)} & \text{iff} & \vdash_{\Lambda'} ((\sigma_1 \wedge \ldots \wedge \sigma_n) \to \varphi)^\diamond \\ \text{(def. } (\cdot)^\diamond) & \text{iff} & \vdash_{\Lambda'} \mathsf{pt} \to ((\sigma_1^t \wedge \ldots \wedge \sigma_n^t) \to \varphi^t) \\ \text{(prop.logic)} & \text{iff} & \vdash_{\Lambda'} ((\mathsf{pt} \to \sigma_1^t) \wedge \ldots \wedge (\mathsf{pt} \to \sigma_n^t)) \to (\mathsf{pt} \to \varphi^t) \\ \text{(def. } (\cdot)^\diamond) & \text{iff} & \vdash_{\Lambda'} (\sigma_1{}^\diamond \wedge \ldots \wedge \sigma_n{}^\diamond) \to \varphi^\diamond \end{array}$$

Using the above equivalences, we can now show that (i) holds:

$$\begin{array}{ll} & \Sigma \vdash_\Lambda \varphi \\ \text{iff} & \text{there are } \sigma_1, \ldots, \sigma_n \in \Sigma \text{ such that } \vdash_\Lambda (\sigma_1 \wedge \ldots \wedge \sigma_n) \to \varphi \\ \text{iff} & \text{there are } \sigma_1, \ldots, \sigma_n \in \Sigma \text{ such that } \vdash_{\Lambda'} (\sigma_1{}^\diamond \wedge \ldots \wedge \sigma_n{}^\diamond) \to \varphi^\diamond \\ \text{iff} & \Sigma^\diamond \vdash_{\Lambda'} \varphi^\diamond. \end{array}$$

QED

Theorem 10.30 (Simulation Theorem) *Let $\Lambda = \mathbf{M}.\Gamma$ be a monotonic modal $\mathcal{L}_\nabla$-logic. Then $\Lambda^{sim} = \mathbf{Sim}^-.\Gamma^\diamond$ simulates Λ with respect to $(\cdot)^\diamond$.*

Proof. By Lemma 10.29 it suffices to show for all $\mathcal{L}_\nabla$-formulas φ: $\varphi \in \Lambda$ iff $\varphi^\diamond \in \Lambda^{sim}$.
For the direction from right to left, assume $\varphi \notin \Lambda$. By completeness of Λ with respect to general monotonic Λ-frames, there is a general monotonic $\mathcal{L}_\nabla$-frame $\mathbb{G}$ such that $\mathbb{G} \Vdash \Lambda$ and $\mathbb{G} \nVdash \varphi$. From $\Gamma \subseteq \Lambda$ and Proposition 10.15 it follows that $\mathbb{G}^\bullet \Vdash \mathbf{Sim}^-.\Gamma^\diamond$ and $\mathbb{G}^\bullet \nVdash \varphi^\diamond$, hence $\varphi^\diamond \notin \Lambda^{sim}$.

For the direction from left to right, assume $\varphi^\diamond \notin \Lambda^{sim}$. Then by the general completeness theorem of Kripke $\mathcal{L}_\diamond$-frames, there is a general $\mathcal{L}_\diamond$-frame $\mathbb{G}$ such that $\mathbb{G} \Vdash \Lambda^{sim}$ and $\mathbb{G} \nVdash \varphi^\diamond$. Hence $\mathbb{G}$ is a $\mathbf{Sim}^-$-frame and there is a $w \in \mathbb{G}$ and an admissible valuation V on $\mathbb{G}$ such that $(\mathbb{G}, V), w \Vdash \mathsf{pt} \wedge \neg\varphi^t$. So $w \in P \neq \emptyset$, and $\mathbb{G}_\bullet$ is well-defined. By Proposition 10.27, $\mathbb{G}_\bullet \Vdash \mathbf{M}.\Gamma$ and $\mathbb{G}_\bullet \nVdash \varphi$, so we may conclude that $\varphi \notin \Lambda = \mathbf{M}.\Gamma$. QED

Now that we know $(\cdot)^{sim}$ is a simulation, we are interested in which properties of logics $(\cdot)^{sim}$ preserves and/or reflects. With the theory we have available so far, the following three results are easily shown.

Corollary 10.31 *The simulation map $(\cdot)^{sim}$ preserves finite and recursive axiomatisability.*

Proof. This result is a direct consequence of the syntactic definition of $(\cdot)^{sim}$. QED

Proposition 10.32 *The simulation map $(\cdot)^{sim}$ reflects strong and weak completeness. More precisely, let $\Lambda = \mathbf{M}.\Gamma$ be a monotonic $\mathcal{L}_\nabla$-logic. If $\Lambda^{sim} = \mathbf{Sim}^-.\Gamma^\diamond$ is strongly (weakly) complete with respect to some class of Kripke $\mathcal{L}_\diamond$-frames, then Λ is strongly (weakly) complete with respect to some class of monotonic $\mathcal{L}_\nabla$-frames.*

Proof. We only show that strong completeness is reflected, since the case for weak completeness may be proved in a similar manner. So let $\Lambda = \mathbf{M}.\Gamma$ be a monotonic $\mathcal{L}_\nabla$-logic, and assume that $\Lambda^{sim} = \mathbf{Sim}^-.\Gamma^\diamond$ is strongly complete with respect to a class K of $\mathcal{L}_\diamond$-frames.

We will show that Λ is strongly complete with respect to the class of $\mathcal{L}_\nabla$-frames $\mathsf{K}_\bullet := \{\mathbb{F}_\bullet \mid \mathbb{F} \in \mathsf{K}, \mathbb{F} \text{ is } P\text{-regular}\}$.

Suppose now that $\Sigma \nvdash_\Lambda \varphi$, then by the Simulation Theorem 10.30, $\Sigma^\diamond \nvdash_{\Lambda^{sim}} \varphi^\diamond$, and from the strong completeness of Λ^{sim}, we obtain an $\mathcal{L}_\diamond$-frame $\mathbb{F} \in \mathsf{K}$, a valuation V on $\mathbb{F}$ and a w in $\mathbb{F}$ such that $(\mathbb{F}, V), w \Vdash \Sigma^\diamond$ and $(\mathbb{F}, V), w \nVdash \varphi^\diamond = \mathsf{pt} \to \varphi^t$. Hence $w \in P$ and we have $(\mathbb{F}, V), w \Vdash \Sigma^t$ and $(\mathbb{F}, V), w \nVdash \varphi^t$. Furthermore, $\mathbb{F}$ is P-regular, since $P \neq \emptyset$ so $\mathbb{F}_\bullet$ is well-defined, and by Proposition 10.23, $(\mathbb{F}_\bullet, V{\restriction_P}), w \Vdash \Sigma$ and $(\mathbb{F}_\bullet, V{\restriction_P}), w \nVdash \varphi$, which concludes the proof. QED

Proposition 10.33 *The simulation map $(\cdot)^{sim}$ reflects canonicity. More precisely, let $\Lambda = \mathbf{M}.\Gamma$ be a monotonic $\mathcal{L}_\nabla$-logic. If $\Lambda^{sim} = \mathbf{Sim}^-.\Gamma^\diamond$ is canonical, then Λ is canonical.*

Proof. Let $\mathbb{G}$ be an arbitrary descriptive monotonic $\mathcal{L}_\nabla$-frame such that $\mathbb{G} \Vdash \Lambda$. It suffices to show that for all $\varphi \in \Gamma$, $\mathbb{G}_\sharp \Vdash \varphi$. So let $\varphi \in \Gamma$. From $\mathbb{G} \Vdash \varphi$ and Proposition 10.15, it follows that $\mathbb{G}^\bullet \Vdash \varphi^\diamond$. Since Λ^{sim} is canonical and $\varphi^\diamond \in \Lambda^{sim}$, we have $(\mathbb{G}^\bullet)_\sharp \Vdash \varphi^\diamond$. From Proposition 10.18, it now follows that $(\mathbb{G}_\sharp)^\bullet \Vdash \varphi^\diamond$, and by Proposition 10.8, $\mathbb{G}_\sharp \Vdash \varphi$. QED

10.5 Applications

The two reflection results of the last subsection imply that for a syntactically specified $\mathcal{L}_\nabla$-logic, questions of canonicity and completeness (with respect to monotonic $\mathcal{L}_\nabla$-frames) may be reduced to the same questions for a normal modal $\mathcal{L}_\diamond$-logic (and Kripke $\mathcal{L}_\diamond$-frames). When trying to show completeness of a normal modal logic, a whole array of techniques is available. For example, one may try to transform the canonical model using filtrations or other model constructions, or apply the mosaic method or the step-by-step method, see e.g. Blackburn et alii [6]. Even though we have seen how to generalise some of these techniques to monotonic structures, the simulation results tell us that in many cases, we do not even have to work that hard. As the reader may already have anticipated, there is a class of monotonic $\mathcal{L}_\nabla$-logics for which we obtain canonicity (and hence strong completeness) virtually for free via simulation, namely those which are generated by KW-formulas (see Definition 5.13).

Theorem 10.34 (KW-Canonicity) *Let Γ be a set of KW-formulas over the language $\mathcal{L}_\nabla$. Then the monotonic modal $\mathcal{L}_\nabla$-logic generated by Γ, $\Lambda = \mathbf{M}.\Gamma$, is canonical.*

Proof. For all formulas $\varphi \to \psi \in \Gamma$, $(\varphi \to \psi)^\diamond$ is equivalent to the Sahlqvist $\mathcal{L}_\diamond$-formula $\mathsf{pt} \wedge \varphi^t \to \psi^t$. As $\mathbf{Sim}^-$ is a normal modal Sahlqvist-logic, so is $\Lambda^{sim} = \mathbf{Sim}^-.\Gamma^\diamond$, hence Λ^{sim} is canonical. From Proposition 10.33 it now follows that Λ is canonical. QED

Corollary 10.35 *If $\Gamma \subseteq \{$N,C,T,4',B,D$\}$, then $\Lambda = \mathbf{M}.\Gamma$ is canonical.*

The formulas P, 4 and 5 are not KW-formulas. However, we already know from Theorem 7.13 that P is both σ- and π-canonical. As it turns out, the formulas 4 and 5 are π-canonical, but in order to see that we must return to the dual frame constructions of subsection 7.6.

10.6 Dual Simulation

In Remark 3.2, we mentioned that in a monotonic $\mathcal{L}_\nabla$-logic Λ, Δ is also a monotone modality. This observation was also made in Kracht and Wolter [44] in the context of simulations, where they sketch how to simulate Λ by translating ∇ as Δ. We will now work out the details of this *dual simulation*. The results in this subsection are easy to show, so we will be rather brief in most of the proofs.

Definition 10.36 (Dual Translation) Define the *dual translation* $(\cdot)^d : \mathcal{L}_\nabla \to \mathcal{L}_\nabla$ inductively as follows:

$$\begin{array}{rcl} \bot^d & = & \bot \\ p^d & = & p \\ (\neg\varphi)^d & = & \neg\varphi^d \\ (\varphi \vee \psi)^d & = & \varphi^d \vee \psi^d \\ (\nabla\varphi)^d & = & \Delta\varphi^d. \end{array}$$

When Σ is a set of $\mathcal{L}_\nabla$-formulas, then $\Sigma^d = \{\varphi^d \mid \varphi \in \Sigma\}$. $\dashv$

The semantic part of the dual translation is simply given by the dual frame constructions of Definition 7.43, where the interpretation of ∇ and Δ are interchanged. Thus for a monotonic $\mathcal{L}_\nabla$-frame $\mathbb{F}$, the *dual simulation frame* of $\mathbb{F}$ is simply defined as $\mathbb{F}^d$, and for a general monotonic $\mathcal{L}_\nabla$-frame, the *dual general simulation frame* of $\mathbb{G}$ is $\mathbb{G}^d$. It is straighforward to show that truth and validity is preserved by the dual simulation, and we leave the proof to the reader.

Proposition 10.37 *Let $\mathbb{F} = (W, \nu)$ be a monotonic $\mathcal{L}_\nabla$-frame. Then*

(i) For all valuations $V : \text{PROP} \to \mathcal{P}(W)$, all $w \in W$ and all $\varphi \in \mathcal{L}_\nabla$,

$$(\mathbb{F}, V), w \Vdash \varphi \quad \textit{iff} \quad (\mathbb{F}^d, V), w \Vdash \varphi^d.$$

(ii) For all $\varphi \in \mathcal{L}_\nabla$:

$$\mathbb{F} \Vdash \varphi \quad \textit{iff} \quad \mathbb{F}^d \Vdash \varphi^d.$$

Let $\mathbb{G} = (W, \nu, A)$ be a general monotonic $\mathcal{L}_\nabla$-frame. Then

(iii) For all admissible valuations $V : \text{PROP} \to A$, all $w \in W$ and all $\varphi \in \mathcal{L}_\nabla$,

$$(\mathbb{G}, V), w \Vdash \varphi \;\text{ iff }\; (\mathbb{G}^d, V), w \Vdash \varphi^d.$$

(iv) For all $\varphi \in \mathcal{L}_\nabla$:

$$\mathbb{G} \Vdash \varphi \;\text{ iff }\; \mathbb{G}^d \Vdash \varphi^d.$$

Theorem 10.38 (Dual Simulation) *The map $(\cdot)^{dual} : \Lambda \mapsto \Lambda^d$ is a simulation with respect to $(\cdot)^d$*

Proof. Let Λ be a monotonic $\mathcal{L}_\nabla$-logic. By Lemma 10.29 we only need to show that for all $\mathcal{L}_\nabla$-formulas φ: $\varphi \in \Lambda$ iff $\varphi^d \in \Lambda^d$. The direction from left to right is trivial. For the other direction, suppose $\varphi \notin \Lambda$. Then by the general completeness result of monotonic $\mathcal{L}_\nabla$-logics, there is a general monotonic $\mathcal{L}_\nabla$-frame $\mathbb{G}$ such that $\mathbb{G} \Vdash \Lambda$ and $\mathbb{G} \nVdash \varphi$. It follows from Proposition 10.37(iv) that $\mathbb{G}^d \Vdash \Lambda^d$ and $\mathbb{G}^d \nVdash \varphi^d$. Hence $\varphi^d \notin \Lambda^d$. QED

Proposition 10.39 *Let Γ be a set of $\mathcal{L}_\nabla$-formulas. Then $(\mathbf{M}.\Gamma)^d = \mathbf{M}.\Gamma^d$. As a consequence, $(\cdot)^{dual}$ transfers finite and recursive axiomatisability.*

Proof. It suffices to show that $(\mathbf{M}.\Gamma)^d \subseteq \mathbf{M}.\Gamma^d$, since then $(\mathbf{M}.\Gamma^d)^d \subseteq \mathbf{M}.\Gamma$ and hence $\mathbf{M}.\Gamma^d \subseteq (\mathbf{M}.\Gamma)^d$. The proof is by induction on the length n of proofs in $\Lambda = \mathbf{M}.\Gamma$. So assume that $\vdash_\Lambda \varphi$. If $n = 0$ then φ is either a propositional tautology or $\varphi \in \Gamma$. In both cases it is clear that $\varphi^d \in \mathbf{M}.\Gamma^d$. For the induction step, we must show that $\mathbf{M}.\Gamma^d$ contains the dual translation of all formulas which can be derived with the rules modus ponens, uniform substitution and RM_∇ in Λ. The cases for modus ponens and uniform substitution follow from the corresponding rules in $\mathbf{M}.\Gamma^d$, thus we only show the case for RM_∇. Assume $\vdash_\Lambda \varphi \to \psi$. Then by the induction hypothesis $\vdash_{\mathbf{M}.\Gamma^d} (\varphi \to \psi)^d$ which implies that $\vdash_{\mathbf{M}.\Gamma^d} \varphi^d \to \psi^d$. Since any monotonic $\mathcal{L}_\nabla$-logic is also closed under the rule RM_Δ, it follows that $\vdash_{\mathbf{M}.\Gamma^d} \Delta\varphi^d \to \Delta\psi^d$, hence $\vdash_{\mathbf{M}.\Gamma^d} (\nabla\varphi \to \nabla\psi)^d$. QED

Proposition 10.40 *Let Λ be a monotonic $\mathcal{L}_\nabla$-logic. If Λ is weakly (strongly) complete with respect to a class K of monotonic $\mathcal{L}_\nabla$-frames, then Λ^d is weakly (strongly) complete with respect to $\mathsf{K}^d = \{\mathbb{F}^d \mid \mathbb{F} \in \mathsf{K}\}$. As a consequence, $(\cdot)^{dual}$ transfers weak and strong completeness.*

Proof. Follows easily from Proposition 10.37. Details are left to the reader. QED

As for canonicity, we recall from subsection 7.6 that the dualisation of (general) frames and BAMs corresponds with the duality between the two kinds of canonical extension, σ and π. The dual translation is, of course, simply the syntactic counterpart of this duality, and we can now combine the above results on the dual simulation with the results of subsection 7.6 in the following.

Proposition 10.41 *Let φ be an $\mathcal{L}_\nabla$-formula, and let Λ be a monotonic $\mathcal{L}_\nabla$-logic. Then*

(i) φ is σ-canonical iff φ^d is π-canonical.
(ii) Λ is σ-canonical iff Λ^d is π-canonical.

Proof. For the proof from left to right in (i), assume that φ is σ-canonical. Then φ is d_σ-persistent. We will show that φ^d is d_π-persistent. So let $\mathbb{G}$ be a π-descriptive general monotonic $\mathcal{L}_\nabla$-frame, and suppose $\mathbb{G} \Vdash \varphi^d$, then by Proposition 10.37(iv), $\mathbb{G}^d \Vdash (\varphi^d)^d$, and since φ and $(\varphi^d)^d$ are logically equivalent also $\mathbb{G}^d \Vdash \varphi$.

By Proposition 7.47 and $(\mathbb{G}^d)^d = \mathbb{G}$, $\mathbb{G}^d$ is σ-descriptive, hence by the d_σ-persistence of φ, $(\mathbb{G}^d)_\sharp \Vdash \varphi$, and since $(\mathbb{G}^d)_\sharp = (\mathbb{G}_\sharp)^d$, we obtain $(\mathbb{G}_\sharp)^d \Vdash \varphi$, whence by Proposition 10.37(ii) and $(\varphi^d)^d \leftrightarrow \varphi$ we can conclude that $\mathbb{G}_\sharp \Vdash \varphi^d$. The other direction of (i) is shown in a similar way.

(ii) follows from (i) and the fact that $\mathbf{M}$ is both σ- and π-canonical, QED

We will now show that π-canonicity implies strong completeness. Recall from Remark 7.15 that for a monotonic logic Λ, the canonical frame $\mathbb{F}^\Lambda(\Phi)$, as defined in section 6, is isomorphic to the ultrafilter frame $(\mathbb{L}_\Lambda(\Phi))_+$ of the Lindenbaum-Tarski algebra of Λ. However, when Λ is π-canonical, then have $(\mathbb{L}_\Lambda(\Phi))_\pi \Vdash \Lambda$, where $(\mathbb{L}_\Lambda(\Phi))_\pi$ is the π-ultrafilter frame of $\mathbb{L}_\Lambda(\Phi)$, as defined in subsection 7.6. The question is now, how can we describe $(\mathbb{L}_\Lambda(\Phi))_\pi$ in terms of the dual map $(\cdot)^d$. To start with, we have the following.

Proposition 10.42 *Let Λ be a monotonic $\mathcal{L}_\nabla$-logic, and let Φ be a set of propositional variables. Then*

$$\mathbb{L}_{\Lambda^d}(\Phi) \cong (\mathbb{L}_\Lambda(\Phi))^d.$$

Proof. Let $\mathbb{L}_\Lambda(\Phi) = (\mathit{Ter}(\Phi)/\equiv_\Lambda, +, -, 0, f_\nabla))$ and $\mathbb{L}_{\Lambda^d}(\Phi) = (\mathit{Ter}(\Phi)/\equiv_{\Lambda^d}, +, -, 0, g_\nabla))$. In order to avoid confusion, we will denote elements of $\mathbb{L}_{\Lambda^d}(\Phi)$ by $[\varphi]_{\Lambda^d}$, and elements of $\mathbb{L}_\Lambda(\Phi)$ by $[\varphi]_\Lambda$. We will show that the map $\theta : [\varphi]_{\Lambda^d} \mapsto [\varphi^d]_\Lambda$ is the desired isomorphism. First of all, we must check that θ is well-defined:

$$\varphi \equiv_{\Lambda^d} \psi \ \text{ iff } \ \vdash_{\Lambda^d} \varphi \leftrightarrow \psi \ \text{ iff }_{(Thm.10.38)} \vdash_\Lambda \varphi^d \leftrightarrow \psi^d \ \text{ iff } \ \varphi^d \equiv_\Lambda \psi^d.$$

Surjectivity of θ is clear, and injectivity follows easily from Theorem 10.38. To see that θ is an isomorphism, we must show that

$$\begin{array}{rcl} \theta([\varphi \vee \psi]_{\Lambda^d}) & = & [\varphi^d \vee \psi^d]_\Lambda, \\ \theta([\neg\varphi]_{\Lambda^d}) & = & [\neg\varphi^d]_\Lambda, \\ \theta([\nabla\varphi]_{\Lambda^d}) & = & [\neg\nabla\neg\varphi^d]_\Lambda. \end{array}$$

But this is clear from the definition of φ^d. QED

Theorem 10.43 *Let Λ be a monotonic $\mathcal{L}_\nabla$-logic. If Λ is π-canonical, then Λ is sound and strongly complete with respect to the class of Λ-frames.*

Proof. As already mentioned, $(\mathbb{L}_\Lambda(\Phi))_\pi \Vdash \Lambda$, and we also have $\mathbb{F}^\Lambda_\pi(\Phi) \cong (\mathbb{L}_\Lambda(\Phi))_\pi$, where $\mathbb{F}^\Lambda_\pi(\Phi)$ is the π-canonical frame (see page 68). Thus it suffices to show that any Λ-consistent set of $\mathcal{L}_\nabla$-formulas can be satisfied on $\mathbb{F}^\Lambda_\pi(\Phi)$. So let Σ be a Λ-consistent set of formulas, then by the Dual Simulation Theorem 10.38, Σ^d is Λ^d-consistent, hence there is a maximal Λ^d-consistent set Γ such that $(\mathbb{F}^{\Lambda^d}(\Phi), V^{\Lambda^d}), \Gamma \Vdash \Sigma^d$, where $\mathbb{F}^{\Lambda^d}(\Phi)$ and V^{Λ^d} are the canonical frame and canonical valuation for Λ^d, respectively. It then follows from Proposition 10.37(i) that $((\mathbb{F}^{\Lambda^d}(\Phi))^d, V^{\Lambda^d}), \Gamma \Vdash \Sigma$. From Propositions 7.46 and 10.42, we get

$$\mathbb{F}^\Lambda_\pi(\Phi) \cong (\mathbb{L}_\Lambda(\Phi))_\pi \cong ((\mathbb{L}_\Lambda(\Phi)^d)_\sigma)^d \cong (\mathbb{L}_{\Lambda^d}(\Phi)_\sigma)^d \cong (\mathbb{F}^{\Lambda^d}(\Phi))^d.$$

Thus, since truth is preserved under isomorphism, Σ can be satisfied in $\mathbb{F}^\Lambda_\pi(\Phi)$. QED

We can now finally extend the canonicity result for KW-formulas to dual KW-formulas (recall Definition 5.13). Note that an $\mathcal{L}_\nabla$-formula φ is a dual KW-formula iff φ^d is logically equivalent with a KW-formula.

Theorem 10.44 *Let Γ be a set of dual KW-formulas. Then the monotonic modal $\mathcal{L}_\nabla$-logic $\Lambda = \mathbf{M}.\Gamma$ is π-canonical, and sound and strongly complete with respect to the class of Λ-frames.*

Proof. For each $\varphi \in \Gamma$, φ^d is equivalent with a KW-formula, hence by Proposition 10.39 and Theorem 10.34, $\Lambda^d = \mathbf{M}.\Gamma^d$ is σ-canonical, and by Proposition 10.41, Λ is π-canonical. Soundness and completeness follow from Theorem 10.43. QED

Corollary 10.45 *If $\Gamma \subseteq \{\text{P,4,5}\}$, then $\Lambda = \mathbf{M}.\Gamma$ is π-canonical, and sound and strongly complete with respect to the class of Λ-frames.*

11 Conclusion and Further Research

We hope to have shown that the framework of normal modal logics can be extended to monotonic modal logics in a natural and useful way in the sense that most of the known constructions and techniques can be generalised to monotonic modal logics and their semantics. We have cashed out on this by obtaining a number of results for monotonic modal logic which are analogues of known results for normal ones. Below is a list of some issues which we either left unsolved, or which we believe present interesting research directions.

m-saturation We had to leave open the problem of showing that ultrafilter extensions are m-saturated.

Definability and Correspondence We did not present any non-first-order definability results. In normal modal logic we have gradations: first-order, then fixed-point definable (like Löb's axiom $\nabla(\nabla p \to p) \to \nabla p$), and still worse, like McKinsey's axiom $\nabla\Delta p \to \Delta\nabla p$. Is the situation similar for monotonic modal logics?

The translation of monotonic modal logic into normal bimodal logic tells us that monotonic modal logic can be seen as a guarded fragment of a first-order language. Does this fragment extend naturally to richer guarded fragments while preserving nice properties, such as decidability and an analogue of the tree model property?

Can we generalise our results on KW-formulas to broader formula classes?

Completeness and Incompleteness Our treatment of completeness was rather brief, and our results were all derived from other parts of the theory. A more systematic study of complete and incomplete monotonic modal logics would be relevant. For instance, does our definition of the canonical frame lead to different results than the canonical frame of Chellas [14]? Which (in)completeteness results with respect to neighbourhood semantics can we obtain for the notorious Löb and McKinsey axioms?

Algebra and Duality Duality for normal modal logic provides many results which we have not mentioned in the text. For example, there are interesting connections between subdirect irreducibility on the algebraic side and various notions of rootedness on the dual side of Kripke frames and descriptive general frames, see for instance Sambin [60] and Venema [73]. We would be interested in a further exploration of the duality of section 7 to see if we can obtain similar results.

Interpolation In section 9, we only used the implication that SUPAP $\Rightarrow$ CIP. But it is natural to ask whether the other direction also holds when considering classes of BAMs and monotonic modal logics. Our initial attempt to show this along the lines of Madarász's [48] proof for additive, but not necessarily normal, modal logic was unsuccessful. The problem seems to revolve around the notion of a BAM-filter, that is, a boolean filter F for which $a \to b \in F$ implies $f(a) \to f(b) \in F$. BAM-filters turn out to be less convenient to work with than BAO-filters which are the boolean filters satisfying that $a \in F$ implies $f(a) \in F$. The difficulty with the BAM-filters also appears to be related to the seeming lack of a deduction theorem. The work of Madarász [48, 49], Czelakowski, Blok and Pigozzi [16, 8] may provide some further insights.

Which (general) results on properties related to Craig interpolation, such as the Beth definability property, global interpolation and Lyndon interpolation can be shown for monotonic modal logics?

Bisimulation products The frame classes defined by the axioms 4, 5, B and C could not be shown to be closed under bisimulation products, and neither have we been able to find any counterexamples. Also for a number of the other standard axioms, closure could only be shown under smallest or largest bisimulation products, not both. A complete description of the (positive) closure properties would be of interest.

Simulation It would be good to have a more comprehensive list of the properties which are preserved and/or reflected by the simulation described in section 10. Negative results are also welcome. Our results were mainly on reflection of properties. In order to establish preservation of e.g. completeness, one would also need an axiomatisation of the bimodal Kripke frames which are the simulation frame of some monotonic frame.

Fixed-point operations Extended Coalition Logic (Pauly [57]) and the Alternating-Time Temporal Logic (Alur et alii [2]) are both examples of monotonic modal logics expanded with fixed-point operations. It would be interesting to study these kind of extensions in more detail.

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